PRACTICAL DESIGN AND ANALYSIS OF SYSTEMS WITH FRACTIONAL DERIVATIVE MATERIALS AND ACTIVE CONTROLS

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ABSTRACT

The fractional derivative model of viscoelasticity is considered to be the most exact representation of viscoelastic material behavior, as it is based on the molecular theory of polymers. The classical fractional derivative expanded equations of motion, however, result in extremely large eigenproblems which are intractable for typical damped structural systems.

This paper discusses a procedure and numerical algorithms which can be used in the design and analysis of structures incorporating viscoelastic materials. Modal strain energy methods are used during the preliminary design phases, permitting inexpensive design iterations and structural modifications. After a satisfactory design is achieved, the solution to the problem is then generated using the frequency-dependent complex impedance matrix implied by the fractional derivative model. The eigensolution is generated using an accelerated complex subspace iteration procedure with spectral shifting. This technique provides the accurate solution to the fractional derivative eigenproblem with minimal computational requirements. When the complex open-loop modes are placed in an appropriate state-space form, active controls can then be directly applied to the reduced-order model.

The application of the method to an example problem with many degrees of freedom demonstrates that the method provides accurate closed-loop results, and can be implemented inexpensively on large-scale structural systems. Most importantly, the results show that the technique will be required for the application of sophisticated modern control algorithms to damped systems, and that the use of the modal strain energy technique to generate the open-loop system model for use in closed-loop analyses can provide results which are significantly in error.

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1.0 Introduction

Vibration control through the combined use of passive and active means has become an accepted method of performance enhancement for space systems with requirements for dimensional precision and stability. The Passive and Active Control of Space Structures (PACOSS) Program has shown that very accurate models of damped or undamped structures are required for successful implementation of modern control strategies. The Modal Strain Energy (MSE) method is a well known analytic method of approximating the behavior of damped systems with viscoelastic damping treatments, and this method serves as an outstanding tool for the design of complex structural systems with damping. As the MSE method is an approximation of the behavior of the damped system, the results of analyses using the MSE solution must be used cautiously.

The representation of the behavior of viscoelastic materials is most accurately described using fractional derivative models [1-5]. These models have their origin in the molecular behavior of polymers, and have been shown to accurately describe the behavior of many materials which lose energy in cyclic vibration. However, the fractional derivative representation of material behavior in the modeling of large-scale structural systems has generally been disregarded, due to the absence of numerical procedures which solve the equations of motion in an efficient manner.

This paper discusses a procedure which has been developed to efficiently design and analyze structural systems with materials which can be described by fractional derivative models, and provides an accurate reduced-order state-space form which can be used to design high-authority modern control systems and predict system performance. This method relies on an iterative solution of the differential equations of motion in the Laplace domain, which is termed spectral iteration. This method is used in conjunction with the subspace iteration eigensolution procedure to develop an efficient numerical algorithm for the solution of large fractional derivative eigenproblems typical of those which may be encountered in realistic structural applications.

The MSE method is the first step in this iterative process. Therefore, the new technique can be incorporated into an efficient design and analysis methodology which uses the MSE approximation during preliminary design stages, and improves the accuracy of the analysis as the design matures.

Through a sufficiently complex example problem, the new technique is shown to provide a system representation which may be used in the design of high-authority control systems, and to predict the closed-loop performance of passive/active systems. The number of the degrees of freedom of the example problem is large enough to demonstrate that the procedure can be used for the solution of realistic problems with viscoelastic damping treatments. It is also shown, however, that control designs which are generated based on a MSE plant model and exercised on the fractional derivative plant may have performance which is seriously degraded when compared with analytic predictions, and may even be unstable.

The new technique can be successfully used for materials represented using any order fractional derivative constitutive model, and indeed for any representation of the material behavior in the Laplace domain. The procedure may even prove to be more efficient in the eigensolution of large-scale problems which incorporate classical viscous damping than those which are presently available in many finite element codes. Further development of this and similar methods should result in techniques which can be effectively used on large-scale systems of the future with vibration control requirements.

2.0 The Fractional Derivative Representation of Material Behavior

The fractional derivative model of viscoelasticity is developed, based on a fractional derivative representation of the relationship between stress and strain within a viscoelastic material [1,2].

$$\sum_{m=1}^{M} b_m D^{\beta_m}\{\tau(t)\} + \tau(t) = G_0\gamma(t) + \sum_{n=1}^{N} G_n D^{\alpha_n}\{\gamma(t)\} \tag{1}$$

where $\tau(t)$ is the material stress, $\gamma(t)$ is the material strain, the b_m and G_n are real constants, and D^k is the fractional derivative operator of order k.

A 5-parameter model can be developed which includes a single fractional derivative of both stress and strain. In the Laplace domain, this provides a Young's modulus and shear modulus which are the ratios of the Laplace transforms of stress and strain, and depend on the Laplace variable (frequency). Using the 5-parameter model, the shear modulus can be expressed:

$$G^*(s) = \frac{\bar{\tau}(\omega)}{\bar{\gamma}(\omega)} = \frac{G_0 + G_1 s^{\alpha}}{1 + b\, s^{\beta}} \tag{2}$$

An additional constraint on the representation in equation (2) is that the values of the powers α and β must be equal to be consistent with thermodynamic considerations [3]. This representation of material behavior is consistent with the macroscopic behavior of many rubbery and glassy materials, and is based on the molecular theory of polymers. Experimental data of the frequency-dependent behavior of a material can be fit using the fractional derivative model to allow the description of the material behavior in the frequency domain.

As an example, consider the viscoelastic material DYAD-606 from Soundcoat. Experimental data previously gathered for use on the PACOSS program was available for this material at 68°F in the frequency range from 1 to 46 Hz. This raw experimental data was fit using the 5-parameter model and a nonlinear error-norm minimization process. Using this technique, the five parameters of the model which best fit the data were:

$$G_0 = 246.45 \text{ psi}, G_1 = 534.22 \text{ psi}, b = 0.1043 (\text{sec})^{0.7049}, \alpha = 0.7049, \beta = 0.4885$$

Graphs of the experimental data and the resulting fractional derivative representation of the frequency-dependent shear modulus and material loss factor are given in Figures 1 and 2. Notice that the agreement between the experimental data and the fractional derivative model are exceptional for this material. However, the two fractional powers (α and β) differ for this fit of the material properties. The cause of this anomaly is unknown, and a higher-order fractional derivative model may be required. The above parameters were used in the subsequent example problem which included this viscoelastic.

A similar fit was performed for the viscoelastic material 3M-966, and 68°F experimental data was also available for this material from previous PACOSS work. The optimum model parameters for this material were:

$$G_0 = 7.9856 \text{ psi}, G_1 = 7.6992 \text{ psi}, b = 5.29 \times 10^{-4} (\text{sec})^{0.6053}, \alpha = 0.6053, \beta = 0.6053$$

The agreement between the experimental data and the fractional derivative model were equivalent to those found for DYAD-606. In this case, however, even though no constraints were imposed on the model parameters, the optimum values of the fractional derivative orders were the same. This is consistent with the thermodynamic requirements of the 5-parameter model. For both of these materials, an outstanding representation of the frequency-dependent material properties was achieved using the 5-parameter model. This agreement demonstrates the applicability of the fractional derivative model to many viscoelastic materials.

3.0 The Modal Strain Energy Method in the Approximate Solution of the Open-Loop System

The MSE method is a well known method of approximating the eigenvalues and eigenvectors of a dynamic system which includes viscoelastic material damping treatments. This method assumes that the real modes associated with the real part of the system stiffness matrix evaluated in the neighborhood of the eigenvalue are a sufficiently accurate approximation to the complex system eigenvectors. The importance of the MSE method as a design tool cannot be overemphasized, and the method allows the economical design of damping treatments for complex structures. This method approximates the solution to the frequency-dependent complex eigenvalue problem, and provides insight to facilitate effective structural modifications.

In general, the frequency-dependent complex stiffness matrix (complex impedance matrix) of a system which includes fractional derivative materials can be written as a sum of contributions from elastic elements and from each type of viscoelastic material. If the Poisson's ratio of the material is frequency independent, the total stiffness matrix can be written:

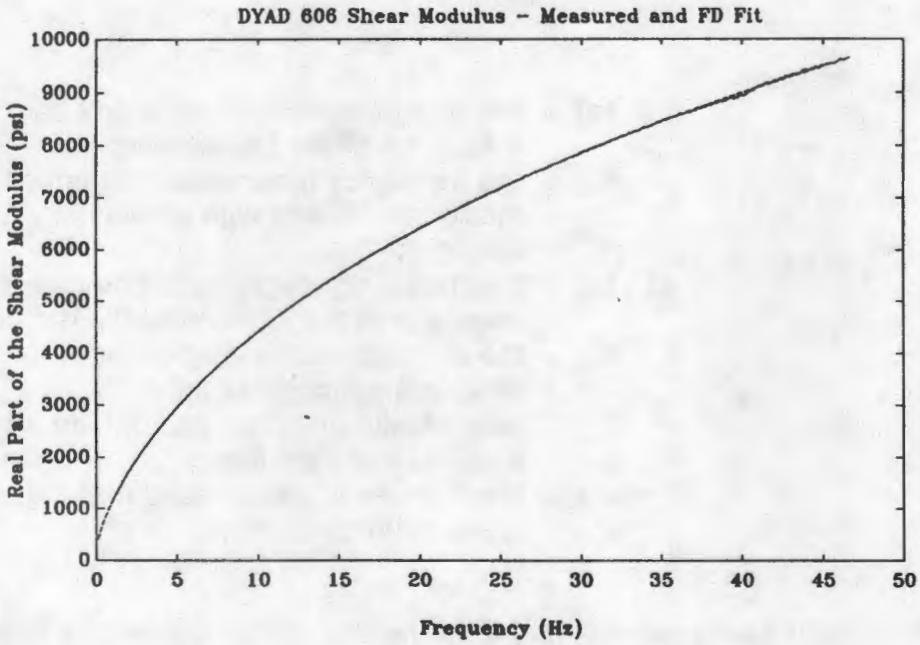

Figure 1 - Experimental DYAD - 606 Shear Modulus and Fractional Derivative Fit

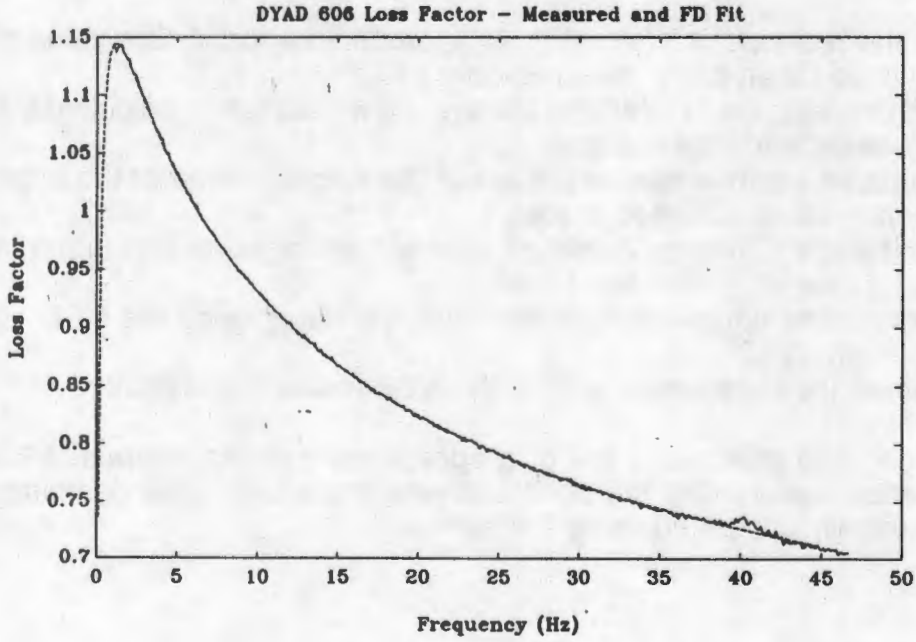

Figure 2 - Experimental DYAD - 606 Loss Factor and Fractional Derivative Fit

$$K(s) = K_{el} + \sum_{i=1}^{nvm} G^*_i(s) \cdot K_{vi} \tag{3}$$

where:

$K(s)$ = the complex impedance matrix as a function of the Laplace variable

K_{el} = the frequency independent stiffness matrix associated with all elastic elements

$G^*_i(s)$ = the frequency-dependent complex shear modulus of the ith viscoelastic material

K_{vi} = the stiffness matrix associated with all elements made of the ith viscoelastic material, assembled with a unit shear modulus

nvm = the number of viscoelastic material types in the system

The MSE method then assumes that if the modes of the system are found using the real part of the complex stiffness matrix which is assembled using viscoelastic material properties on the imaginary axis (at $s = i\omega$), that these vectors are sufficiently "close" to the actual system eigenvectors, and that these approximate vectors are uncoupled through both the real and imaginary parts of the stiffness matrix.

Therefore, the typical sequence of steps in an MSE analysis are:

1) Form the real part of **K(s)** using an appropriate approximate value of the eigenvalue taken along the imaginary axis.
2) Using the real part of **K(s)** and the system mass matrix, calculate the real eigenvalues and eigenvectors.
3) Store those eigenvectors which are in the neighborhood of the approximate frequency value assumed in step 1.
4) Repeat steps 1 through 3 until all approximate eigenvectors in the frequency range of interest have been found.
5) Determine the approximate modal damping ratios using the MSE distribution.
6) Construct the state-space form of the MSE model of the plant.

This procedures provides a set of **q** approximate mode shapes, frequencies, and damping ratios found using the MSE analysis procedure. The damping ratios of the system are usually computed using the formula:

$$\zeta_j = \frac{1}{2} \sum_{i=1}^{NE} S\, E_{ij} \cdot \eta_i \tag{4}$$

where:

ζ_j = the approximate modal damping ratio of the jth mode

SE_{ij} = the percentage of modal strain energy in the ith element in the jth mode

η_i = the loss factor of the ith element in the neighborhood of the jth modal frequency

NE = the number of system finite elements

This MSE procedure is equivalent to the following matrix operations:

1) Using an approximate eigenvalue on the imaginary axis, construct the stiffness matrix **K(s)** using equation (3) and the system mass matrix.
2) Decompose the stiffness matrix into its real and imaginary parts, $\mathbf{K_I}$ and $\mathbf{K_R}$.
3) Solve the real eigenvalue problem $(\mathbf{K_R} - \omega^2 \cdot \mathbf{M})\,\Phi = \mathbf{0}$ for the **q** lowest eigenvalues and eigenvectors.
4) Mass normalize the eigenvectors from step 3.
5) Using the modes which are near the approximate frequency, construct the matrix products:

$$\Phi^T \cdot \mathbf{M} \cdot \Phi = I$$
$$\Phi^T \cdot \mathbf{K_R} \cdot \Phi = \omega^2$$
$$\Phi^T \cdot \mathbf{K_I} \cdot \Phi = \mathbf{K_{mI}} \qquad (5)$$

6) Form the second-order modal equations:

$$\ddot{\mathbf{p}} + \mathbf{I} \cdot \mathbf{K_{mI}} \cdot \mathbf{p} + \omega^2 \cdot \mathbf{p} = \Phi^T \cdot \mathbf{f} \qquad (6)$$

7) Assume that the generalized velocities are equal to $I\omega$ times the generalized displacements, and neglect the off-diagonal terms in $\mathbf{K_{mI}}$:

$$\ddot{\mathbf{p}} + \mathbf{D_{vI}} \cdot \dot{\mathbf{p}} + \omega^2 \cdot \mathbf{p} = \Phi^T \cdot \mathbf{f} \qquad (7)$$

where:

$$\mathbf{D_{vI}}(j,j) = \mathbf{K_{mI}}(j,j)/\omega(j,j)$$

8) Under the above assumptions, the terms in the diagonal matrix $\mathbf{D_{vI}}$ are $2 \cdot \zeta_j \cdot \omega_j$.
9) Construct the state-space form of the plant using an assemblage of the appropriate modes, natural frequencies, and damping ratios:

$$\dot{x} = A \cdot x + B \cdot u$$
$$y = C \cdot x$$

or:

$$\begin{bmatrix} \ddot{p} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} -2 \cdot \zeta \cdot \omega & -\omega^2 \\ I & 0 \end{bmatrix} \cdot \begin{bmatrix} \dot{p} \\ p \end{bmatrix} + \begin{bmatrix} \Phi^T \\ 0 \end{bmatrix} \cdot f$$

$$\begin{bmatrix} \dot{x} \\ x \end{bmatrix} = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \cdot \begin{bmatrix} \dot{p} \\ p \end{bmatrix} \tag{8}$$

This state-space form of the open-loop system can be used to design a control system to provide desired closed-loop performance characteristics of the MSE plant. However, as numerous assumptions are used in the computation of the open-loop plant, the quality of the state-space model cannot be assessed. It will be demonstrated through an example problem that the effects of these assumptions can result in significant errors in the behavior of the closed-loop system, if the MSE plant is used in control design and performance evaluation.

4.0 The Eigenstructure of the Fractional Derivative Eigenvalue Problem

It is necessary to develop the definition of the eigenvalues and eigenvectors of systems which include viscoelastics modeled using fractional derivatives, to allow a comparison with the approximate values derived from the MSE method and their improvement. The transformed equations of motion for the system can, in general, be written in the Laplace domain as:

$$[M \cdot s^2 + K(s)] \cdot X(s) = F(s) \tag{9}$$

where:

- M = the system mass matrix
- $K(s)$ = the complex frequency-dependent impedance matrix of equation (3)
- $X(s)$ = the Laplace transform of the system displacements
- $F(s)$ = the Laplace transform of the applied forces

The system dynamical matrix can now be defined as a function of the Laplace variable. This matrix is:

$$Z(s) = M \cdot s^2 + K(s) \tag{10}$$

For non-trivial solutions of the homogeneous differential equations, the dynamical matrix must become singular. Therefore, similar to the definition for classical undamped or viscously damped systems, the definition of an eigenvalue λ of the system with fractional derivative materials is:

$$\left| \mathbf{M}\, s^2 + \mathbf{K}(s) \right|_{s=\lambda} = 0 \tag{11}$$

The eigenvalues can be determined by expanding the determinant of the dynamical matrix and finding zeros of the characteristic polynomial. However, the matrix **K(s)** involves the Laplace variable raised to fractional powers. Therefore, the roots of this complex fractional-order characteristic equation are generally difficult to obtain. A surprising result of this development is that there are, in general, infinitely many roots to this equation for a finite number of degrees of freedom, if the fractional powers cannot be expressed as a rational fraction. If the fractional powers can be expressed as a rational fraction **r/m**, then there are **N(2m+r)** eigenvalues where **N** is the number of system degrees of freedom. The additional eigenvalues are located on branches of the multi-valued fractional power function and contribute to response of the system by an integral term along a branch cut [3].

All the eigenvalues of the system can be found by expanding the equations of motion to clear the fractional powers [3,4,5], if the powers are expressible as rational fractions and all viscoelastic materials in the system have the same denominator **m**. A state-space model of the expanded system equations of motion may then be constructed in either physical or generalized coordinates [6].

The associated eigenvector for any known eigenvalue can be found by solving the homogeneous form of equation (9) for the mode shapes. To find a selected eigenvalue and eigenvector, a procedure which will be termed spectral iteration may be used. In this procedure, an approximate value for an eigenvalue is used to construct the complex impedance matrix, and a complex eigenvalue problem is solved. This procedure is given in equation (12).

Solve equation (7) for Φ and λ_{new} using an assumed value of λ:

$$\left[\mathbf{M} \cdot \lambda_{new}^2 + \mathbf{K}(\lambda) \right] \cdot \Phi = 0 \tag{12}$$

A simple method of obtaining a selected eigenpair is to iterate using equation (12). Using an approximate eigenvalue λ, the matrix function **K(λ)** is evaluated, and the eigenvalues using this complex matrix are found. If λ is a good approximate value, then one of the eigenvalues found will be "close" to the initial guess. This new guess is then used to recalculate the complex stiffness matrix. By performing this process iteratively, the procedure will converge to an exact eigenvalue and eigenvector of the system.

This is similar to the inverse power method developed in [7] for the solution of the fractional derivative eigenproblem, where a solution procedure based on the inverse power method with spectral shifting is used to evaluate the eigenpairs of the system. This procedure of spectral iteration can be further developed and improved to provide computationally inexpensive solutions. It will be shown that the MSE method is a form of the spectral iteration procedure, and further improvements to the MSE solution are possible at minimal computational expense.

For the solution of large-scale dynamic systems with fractional derivative materials, the major objective is to construct a reduced-order state-space model of the structure from its finite element representation. Typically, only a small subset of the system eigenvalues and eigenvectors will be required; and the solution of the expanded fractional derivative equations of motion for a system with many degrees of freedom would be computationally infeasible or even impossible if several materials are used. Therefore, for a typical structural application, the concern is to locate eigenvalues and eigenvectors of the system within a selected spectral radius from the origin of the Laplace domain (i.e., the **q** smallest eigenvalues). Expansion of the ideas of the MSE method and spectral iteration into a procedure consistent with subspace iteration allows the evaluation of the desired eigenpairs in an efficient manner.

5.0 The MSE Method and Standard Subspace Iteration

The MSE method provides an approximation to the **q** lowest eigenvalues of a damped system. In the solution of the fractional derivative equations, it is advantageous to consider the mathematical basis of the MSE method and means to improve the accuracy of the approximations. Therefore, consider the form of the stiffness matrix developed in equation (3), and the reduction of the mass and stiffness matrices in equation (5) using a set of **q** MSE approximate vectors as a vector basis (subspace):

$$\left[\Phi^T \cdot M \cdot \Phi \cdot s^2 + \Phi^T \cdot K(s) \cdot \Phi\right] \cdot P(s) = \Phi^T \cdot F(s) \tag{13}$$

These reduced equations can be written:

$$\left[M^R \cdot s^2 + K_{el}^R + \sum_{l=1}^{nvm} G_l^*(s) \cdot K_{v_l}^R\right] \cdot P(s) = F^R(s) \tag{14}$$

where the reduced matrices are of size **q x q** and are formed by matrix triple products, and the forcing vector is a **q x 1** generalized forcing vector:

$$\begin{aligned} M^R &= \Phi^T \cdot M \cdot \Phi \\ K_{el}^R &= \Phi^T \cdot K_{el} \cdot \Phi \\ K_{v_l}^R &= \Phi^T \cdot K_{v_l} \cdot \Phi \\ F^R(s) &= \Phi^T \cdot F(s) \end{aligned} \tag{15}$$

It is obvious that since the eigenvectors computed using the MSE method will generally be computed using several real stiffness matrices, the reduced mass matrix will not be an identity matrix, and the real part of the reduced stiffness matrix will not be

diagonal. However, these matrix equations are similar in form to those in equation (12) and can be solved using spectral iteration in a similar manner,

i.e., solve:

$$\left[\mathbf{M}^{R} \cdot \lambda_{new}^{2} + \mathbf{K}^{R}(\lambda)\right] \cdot \Psi = 0 \tag{16}$$

for the solution vectors Ψ and the eigenvalues λ using spectral iteration.

This iteration process is a Ritz analysis with the MSE method approximate MSE vectors Φ_{MSE} as the solution space [8]. The approximate eigenvalues of the full system are the eigenvalues λ of the reduced system, and the approximate eigenvectors are the complex vectors:

$$\Phi_{C} = \Phi_{MSE} \cdot \Psi \tag{17}$$

If the range of the approximate MSE vectors spans the solution space of the exact eigenvectors, the eigenpairs generated in this manner are exact solutions for the system. Notable cases where this will occur are when the damping in the modes is negligible, or if the entire system is composed of damped elements with an identical fractional derivative representation (i.e., proportional damping). In these cases, a set of real vectors can be found which provides an invariant subspace of the fractional derivative eigenvalue problem. The MSE approximation for the natural frequency and damping will be in error, however, even for a system of all damped elements.

In this manner, the MSE method can be shown to be an uncoupled Ritz analysis with the MSE vectors as a subspace. In fact, it is a Ritz analysis using uncoupled vectors (the Rayleigh Quotient) which was first used to derive the modal strain energy method [9]. It is assumed in the MSE method that these vectors are uncoupled in equation (16); therefore, the assumed reduced basis eigenvectors form an identity matrix.

The improved solution of the eigenvalue problem with a single spectral iteration is an inexpensive means of improving the quality of the solution, as it merely requires the generation of a real reduced mass matrix, a real reduced elastic stiffness matrix, and as many real reduced viscoelastic matrices as there are types of viscoelastic materials. The reduced complex impedance matrix is then formed by simply adding the reduced elastic stiffness matrix and the reduced viscoelastic matrices multiplied by their respective complex shear moduli. Therefore, to generate the initial reduced problem, products which involve only real matrices and real vectors must be formed. The solution of the **q x q** reduced-order problem by spectral iteration will be inexpensive due to the small order of the system.

In general, the real MSE vectors will not provide an invariant subspace of the fractional derivative problem. Therefore, a method is required to improve the subspace and allow a more accurate representation of the solution. To examine

methods of subspace improvement, consider the subspace iteration eigensolution method for real, constant stiffness and mass matrices.

Subspace Iteration Steps:

1) Select an initial subspace of vectors Φ_k of size **N x p** where **N** is the system order, and **p** is larger than the number of desired vectors, **q**.
2) Decompose the stiffness matrix into its $L\,D\,L^T$ factorization
3) Perform a simultaneous power iteration on the approximate vectors using forward elimination and back substitution,
   i.e., solve for Φ_{k+1} using:

$$K \cdot \Phi_{k+1} = M \cdot \Phi_k \cdot \lambda_k^2 \tag{18}$$

4) Create reduced mass and stiffness matrices:

$$\begin{aligned} M_{k+1}^R &= \Phi_{k+1}^T \cdot M \cdot \Phi_{k+1} \\ K_{k+1}^R &= \Phi_{k+1}^T \cdot K \cdot \Phi_{k+1} \end{aligned} \tag{19}$$

5) Solve the reduced eigenproblem for λ_{k+1} and Ψ_{k+1}.

$$\left[M_{k+1}^R \cdot \lambda_{k+1}^2 + K_{k+1}^R \right] \cdot \Psi_{k+1} = 0 \tag{20}$$

6) Orthogonalize the current subspace:

$$\Phi_{k+1} = \Phi_{k+1} \cdot \Psi_{k+1} \tag{21}$$

7) Return to step 3 with Φ_{k+1} as new subspace, and iterate until convergence.
8) Perform a Sturm sequence check to determine if all desired eigenvalues and eigenvectors have been found.

This procedure may be used for the fractional derivative eigenvalue problem, with spectral iteration performed at step 5 for each major iteration. The greatest expense of this procedure, however, would be the computation which improves the subspace at step 3. For the fractional derivative eigenvalue problem, the "stiffness" matrix is complex and a function of the eigenvalue; and it is infeasible to factor the full-system size complex matrix and perform several complex matrix/complex vector products at each iteration. This is the major deficiency of the inverse power method with spectral shifting presented in [7], as a complex factorization was used for each power iteration.

To alleviate these difficulties, a method of accelerated subspace iteration which does not include the inversion of a matrix was developed for use in the solution of the fractional derivative eigenvalue problem. This procedure allows the improvement of the subspace with a minimum computational burden, and rapid convergence to the **q** smallest system eigenvalues.

5.0 Accelerated Subspace Iteration for the Solution of the Open-Loop Eigenvalue Problem

Subspace iteration was first developed by Bathe in the early 1970s [10]. Further advances in the technique were subsequently developed and were designated accelerated subspace iteration [11]. In this procedure, Lanczos vectors are used to generate the initial subspace, and spectral shifting during the power iterations is performed using the approximate inversion method of successive overrelaxation. This method allows fewer than **q** vectors to be used as the **p** size subspace, while in the standard subspace iteration method usually the minimum of **2•q** or **q+8** vectors are selected as a subspace. Using several of the ideas of the accelerated subspace iteration, along with the spectral iteration technique and several new developments, an accelerated subspace iteration procedure for the fractional derivative eigenvalue problem was developed.

The basis of this technique is a preconditioned conjugate gradient procedure developed specifically to allow the iterative solution of linear equations with a symmetric coefficient matrix which is complex,

i.e., the solution technique was developed to solve the linear equations:

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b} \tag{22}$$

where the matrix **A** is an **N x N** symmetric complex matrix, and the complex vectors **x** and **b** are of size **N**. The derivation and the numerical algorithm will not be discussed here, however, the use of the technique will be described. The technique splits a shifted dynamical matrix into two components: $\mathbf{K_0}$ and $\Delta\mathbf{K}+(2\alpha\mu+\mu^2)\mathbf{M}$, where μ is an appropriate spectral shift at each step selected to allow rapid convergence to a particular eigenvalue, and α is a shift used in the generation of the matrix $\mathbf{K_0}$. The updating procedure which is used to replace step 3 in the standard subspace iteration is then the formula:

$$\mathbf{K}_0 \cdot \Phi_{j+1} = -(\Delta\mathbf{K} + 2\alpha\mu_j\mathbf{M} + \mu_j^2\mathbf{M}) \cdot \Phi_j \tag{23}$$

where the residual error after **j** iterations is defined as:

$$\mathbf{r}_j = \mathbf{K}_0 \cdot \Phi_{j+1} + (\Delta\mathbf{K} + 2\alpha\mu_{j+1}\mathbf{M} + \mu_{j+1}^2\mathbf{M}) \cdot \Phi_{j+1} \tag{24}$$

The search directions are generated by conjugate gradients, and are selected to minimize the residual while being orthogonal through the dynamical matrix to all previous residuals and approximate modal vectors. This method is, therefore, similar

to the preconditioned method of conjugate gradients [12].; however, it was designed to allow complex matrices. The search vectors can be interpreted as Lanczos vectors selected to provide the greatest reduction in the norm of the residual vector on each step. With a proper selection of the matrix $\mathbf{K_0}$, this procedure will converge in few iterations to an eigenvector which is closest to the shift point $\alpha+\mu$. An excellent selection of the preconditioner is the shifted real stiffness matrix (shifted by a value of α) computed during the modal strain energy procedure (it can be assumed that a shifted real stiffness matrix ($\mathbf{K_{MSE}} + \alpha\,\mathbf{M}$) was factored during the solution of the MSE real eigenvalue problem):

$$\mathbf{K}_0^{-1} = \left(\mathbf{K}_{MSE} + \alpha \cdot \mathbf{M}\right)^{-1} \tag{25}$$

This selection results in low rank of the $\Delta\mathbf{K}$ matrix, as this matrix contains only terms from viscoelastic elements and, therefore, has many zero rows and columns. The eigenvalues of the matrix $\mathbf{K_0}^{-1} \cdot \Delta\mathbf{K}$ will be small, as the MSE stiffness matrix is "close" to $\mathbf{K(s)}$. These properties of $\Delta\mathbf{K}$ and $\mathbf{K_0}^{-1} \cdot \Delta\mathbf{K}$ provide rapid convergence of the iterations [12].

Convergence of this conjugate gradient procedure results in an eigenvector/ eigenvalue of the system. Very importantly, the iterations need not be performed until convergence in the accelerated subspace procedure, as the objective of step 3 in the standard method is simply to improve the subspace. The linear combination of Lanczos vectors generated as search vectors provide a good set of basis vectors with which the subspace can be improved, even though the iterations have not converged. Therefore, in the accelerated subspace procedure, the iterations implied by equation (23) are only performed once, and these vectors are used as a new vector basis. After orthogonalization, a further basis improvement is performed.

In summary, the steps which comprise the accelerated subspace iteration procedure for systems which include materials modeled with fractional derivatives are:

1) Select the MSE solution vectors as the initial subspace.
2) Create the reduced mass, elastic stiffness, and viscoelastic stiffness matrices as per equation (15). Store all matrix/vector products such as $\mathbf{K_{el}} \cdot \Phi_{k+1}$.
3) Perform spectral iteration within the subspace to compute new approximate eigenvalues and eigenvectors.
4) Update matrix products using reduced basis modal transformation.
5) Improve the basis vectors, and update matrix/vector products using the complex conjugate gradient procedure.
6) Return to step 3 using new basis vectors and updated matrix/vector products.
7) Iterate from steps 3 to 6 until convergence of the subspace.

Notice that the only matrix/vector products required in the solution procedure are contained in the conjugate gradient algorithm, with two plus the number of viscoelastic materials matrix/vector multiplications required for each basis vector per conjugate gradient iteration. Also, the total solution subspace need not be updated on

every iteration; and only the vectors which correspond to the eigenvalues which have not yet converged need be updated each time.

The above procedure can be considered a hybrid of several highly successful eigenvalue extraction methods: the power method with spectral shifting; Ritz analysis; and Lanczos methods, which are combined with spectral iteration to allow for the frequency dependence of the stiffness matrix. Experience on example problems shows that this method is very effective for solving the fractional derivative eigenvalue equations for the invariant subspace corresponding to the lowest system eigenvalues. A complex Sturm sequence check can be used to verify that all eigenvalues/vectors within a given spectral radius from the origin have been found by factoring a shifted impedance matrix; however, it may be assumed that the initial MSE solution provided approximations to all eigenvalues within the search region.

7.0 Generation of the State-Space Plant Model

After a selected number of modes of the open-loop system have been generated by the above procedure, an appropriate state-space description of the plant is required for performance evaluation and the generation of vibration control systems, if needed. This state-space model should allow the use of available modern control algorithms to be used to generate a compensator which will provide desired performance of the closed-loop system. Therefore, a complex modal formulation of the plant was developed to obtain this state-space description. This formulation finds a viscous representation of the plant which has identical eigenvectors and eigenvalues to the fractional derivative system. This is an approximation, however, it provides the most accurate viscous representation possible.

The generation of the modal state-space equations begins with the equations of motion of the full-size open-loop system in the Laplace domain. An appropriate general form of these equations is:

$$\begin{bmatrix} \mathbf{M} & 0 \\ 0 & -\mathbf{K_R(s)} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{X(s)} \cdot \mathbf{s} \\ \mathbf{X(s)} \end{bmatrix} \cdot \mathbf{s} = \begin{bmatrix} -\mathbf{C_R(s)} & -\mathbf{K_R(s)} \\ -\mathbf{K_R(s)} & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{X(s)} \cdot \mathbf{s} \\ \mathbf{X(s)} \end{bmatrix} + \begin{bmatrix} \mathbf{F(s)} \\ 0 \end{bmatrix} \tag{26}$$

$$\text{where:} \quad \mathbf{K_R(s)} = \mathbf{real(K(s))} - \frac{\mathbf{real(s)}}{\mathbf{imag(s)}} \cdot \mathbf{imag(K(s))}$$

$$\mathbf{C_R(s)} = \frac{\mathbf{imag(K(s))}}{\mathbf{imag(s)}}$$

Notice that these equations provide an identical impedance matrix for the fractional derivative system and the viscous system. By solving the equations in the Laplace domain for a system eigenpair, a modal substitution which uncouples the equations with **K(s)** evaluated at an eigenvalue can be constructed. An appropriate modal substitution is:

$$\begin{bmatrix} \mathbf{X(s)} \cdot \mathbf{s} \\ \mathbf{X(s)} \end{bmatrix} = \begin{bmatrix} \Phi \cdot \lambda \\ \Phi \end{bmatrix} \cdot \mathbf{P(s)} \tag{27}$$

or $\chi = \psi P$

where:

Φ = A system eigenvector found using spectral iteration

λ = A system eigenvalue

P(s) = the Laplace transform of the generalized coordinate

With the substitution of the transformation as given in equation (27), the single coordinate which corresponds to an eigenvector of the fractional derivative system coordinate can be uncoupled from all other coordinates. This is achieved by noting that the matrix on the left-hand side is symmetric; and that if all the eigenvalues of the complex system were found using this constant value of **K(s)**, the full-size matrix ψ would be orthogonal to this matrix. Therefore, with the correct normalization, the left-hand side can be transformed to an identity matrix. This yields an uncoupled equation for a single generalized coordinate and also for its complex conjugate. The correct normalization for the individual eigenvectors is such that:

$$\Phi^T \cdot \mathbf{M} \cdot \Phi \cdot \lambda^2 - \Phi^T \cdot \mathbf{K}(\lambda) \cdot \Phi = 1 \tag{28}$$

where Φ is a single eigenvector, and λ is its associated eigenvalue.

Notice that the complex conjugate modes and natural frequencies and their normalization are found using this method. This is consistent with the fractional derivative material representation, as the fractional derivative description provides a complex conjugate shear modulus at a complex conjugate value of the Laplace variable:

K(s) and $\mathbf{K}(\tilde{\mathbf{s}})$ are related by:

$$\mathbf{K}(\tilde{\mathbf{s}}) = \tilde{\mathbf{K}}(\mathbf{s}) \tag{29}$$

where ~ denotes complex conjugation.

Therefore, the complex conjugate eigenvalue and eigenvector of any solutions found in the accelerated subspace procedure are also eigenvectors/eigenvalues of the fractional derivative system. This is required to provide real, stable solutions in the time domain. An additional requirement on the complex impedance matrix **K(s)** is that it must smoothly become a real matrix at the origin of the Laplace domain. This ensures causality of time domain solutions, and it is obvious from the 5-parameter model that this requirement is met.

Finally, a normalized modal matrix can then be assembled which provides the transformation of the equations into truncated modal form in the Laplace domain:

$$\bar{\Psi} = \begin{bmatrix} \Phi \cdot \lambda & \tilde{\Phi} \cdot \tilde{\lambda} \\ \Phi & \tilde{\Phi} \end{bmatrix} \tag{30}$$

where Φ is an **N x q** matrix of normalized eigenvectors, and λ is a **q x q** matrix of eigenvalues. Using this modal transformation and taking the inverse Laplace transform of these equations yields the final form of the state-space equations in the time domain:

$$\begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\tilde{\mathbf{p}}} \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \tilde{\lambda} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{p} \\ \tilde{\mathbf{p}} \end{bmatrix} + \bar{\Psi}^{T} \cdot \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{x} \end{bmatrix} = \bar{\Psi} \cdot \begin{bmatrix} \mathbf{p} \\ \tilde{\mathbf{p}} \end{bmatrix} \tag{31}$$

These equations are in standard first-order state-space form and, therefore, can be used in conjunction with modern control algorithms to design a control system or predict system performance.

6.0 Application of the Solution Procedure to the PACOSS Multi-Actuator Control Experiment

To show the applicability of the procedure to the solution of realistic dynamics and control problems which incorporate viscoelastic damping treatments, an example system was selected. This system was the PACOSS Multi-Actuator Control Experiment (MACE) shown in Figure 3. This structure was previously constructed under the PACOSS Program to verify the performance of the control system hardware using modern control algorithms in a multi-actuator digital control application.

The original MACE hardware consisted of three proof mass actuators mounted to a series of flat aluminum beams. The system was hung from steel cables at three points with 60.9-lb/in. springs located at the top of the suspension. Constrained layer damping treatments using DYAD-606 damping material with steel constraining layers were applied to six locations on the beam members. These damping treatments provided from 0.5% to 2% critical damping in the modes of the system below 15 Hz.

While the original design of the MACE was satisfactory for the purposes of validating the successful operation of the PACOSS control system hardware, the relatively low damping levels are not characteristic of those which can be achieved in damped systems. Therefore, several modifications to the original design of the MACE structure were made for this example problem. First, the thickness of the DYAD damping material was increased from 0.050 in. to 0.120 in. to increase the damping performance of the constrained layer treatments. Second, viscoelastic dampers were designed which were placed in parallel with the suspension springs. These dampers use 3M-966 material in a configuration such that the spring constants in units of lb/in.

Figure 3 - Photograph of the PACOSS Multi-Actuator Control Experiment

were 0.25 multiplied by the material shear modulus in psi. Finally, four ideal actuators with ideal inertial velocity sensors were used instead of the three proof mass actuators used on the actual hardware. Four actuators were used to remove the actuator from the symmetric axis of the structure and to increase the control authority for anti-symmetric modes (Figure 4).

A finite element model of the damped structure was created using MSC/NASTRAN (Figure 5), which included the typical plate and solid element modeling of the constrained layer damping treatments as well as the pendulum behavior due to the suspension. The full-system mass matrix, the elastic/differential stiffness matrix excluding the damping materials, the stiffness matrix associated with the DYAD-606 material, and the stiffness matrix associated with the 3M-966 shear dampers were then assembled. Compatible stiffness matrices were easily formed by altering the material properties of the various components to be a small number. For example, to form the DYAD-606 stiffness matrix, the moduli of the main members and constraining layers, the suspension spring constants, and the 3M-966 moduli were set to extremely low values; and the modulus of the DYAD was set to unity. The full-stiffness matrix for any value of the Laplace variable could then be easily constructed by addition of the constitutive matrices, as per equation (3).

A performance metric was selected for the system, which was the vertical motion of a single point on the structure for noise inputs at the actuator locations. The objective to be achieved was a factor of 100 decrease in root mean square (RMS) motion of the performance point for white noise inputs from 0 to 30 Hz, as compared to a system without added damping treatments or active controls.

A modal strain energy analysis was performed on the system to approximate the open-loop modes, natural frequencies, and damping ratios in the frequency range from 0 to 30 Hz. The real stiffness matrix was assembled at six selected frequencies which were known to be "close" to system eigenvalues. The standard MSE method was used, with the modes nearest the corresponding frequency used to construct a state-space model of the plant as detailed in Section 6.0. Table 1 provides the open-loop frequencies and damping of the system computed using the MSE method.

Figure 6 provides the frequency response of the performance point motion for inputs at actuator #2, for both the MSE system and also for the system with 0.2% critical damping in the modes. Notice that the system has high modal density in the 0 to 30-Hz frequency range. The addition of passive damping treatments to the system lowers the RMS response by approximately a factor 5, so that active control is required to further reduce the system RMS response by a factor of 20.

Two active control algorithms were considered for the MACE example problem: local velocity feedback, and Linear Quadratic Gaussian with Loop Transfer Recovery (LQG/LTR). For local velocity feedback, a feedback gain of 0.25 lb-sec/in. was used for the inertial velocity at each of the actuators. The closed-loop response of the system was generated using the MSE state-space model, and compared with the open-loop system and the exact frequency response of the closed-loop fractional derivative model (Figure 7). Notice from the Figure that the agreement between the exact solution and the MSE solution is relatively good, differing only in some frequency

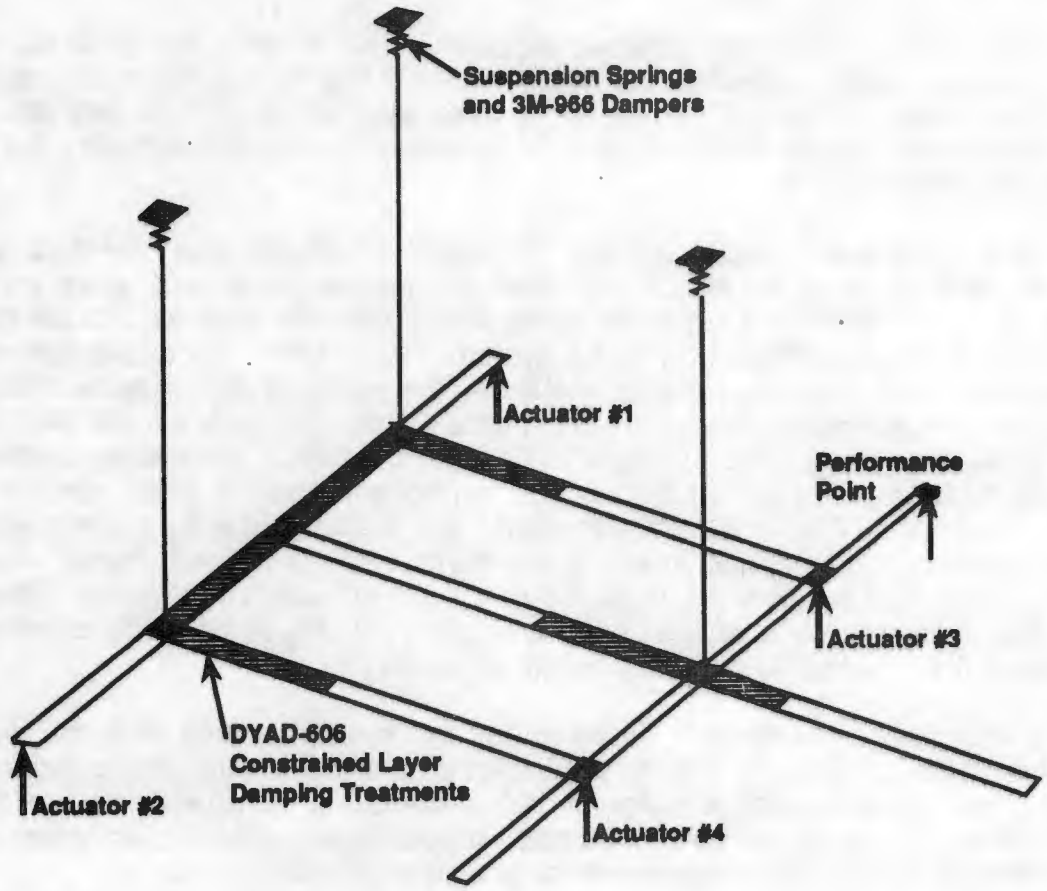

Figure 4 - Schematic of MACE Example Problem

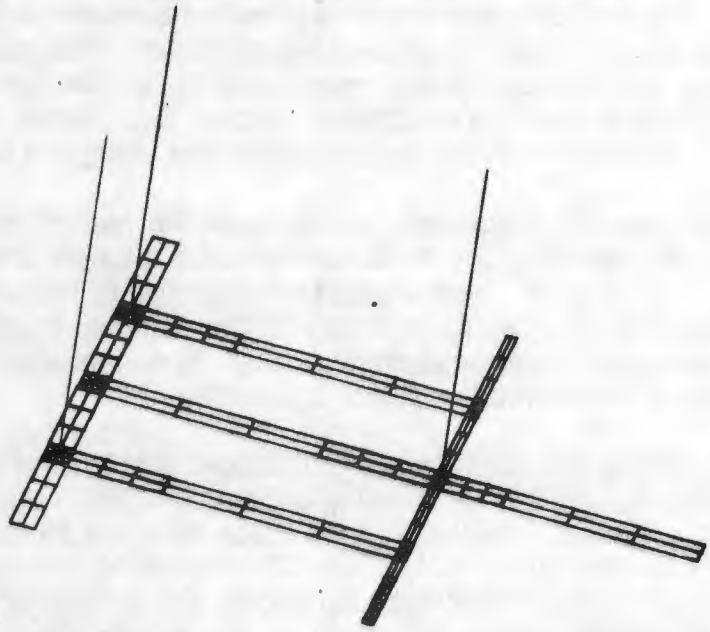

Figure 5 - NASTRAN Model of the MACE Example Problem

Table 1 - Natural Frequencies and Damping Ratios of the MACE Computed Using MSE

Mode #	Frequency (Hz)	Damping Ratio (%)	Mode #	Frequency (Hz)	Damping Ratio (%)
1	0.28	0.0	11	12.6	7.9
2	0.28	0.0	12	13.1	4.1
3	0.28	0.0	13	15.4	3.3
4	2.83	5.0	14	21.8	1.2
5	2.98	4.0	15	24.3	5.1
6	3.46	3.8	16	25.7	2.5
7	4.82	5.4	17	30.5	3.9
8	6.18	4.2	18	34.3	3.3
9	8.17	7.0	19	37.1	3.8
10	10.3	2.7	20	37.8	3.7

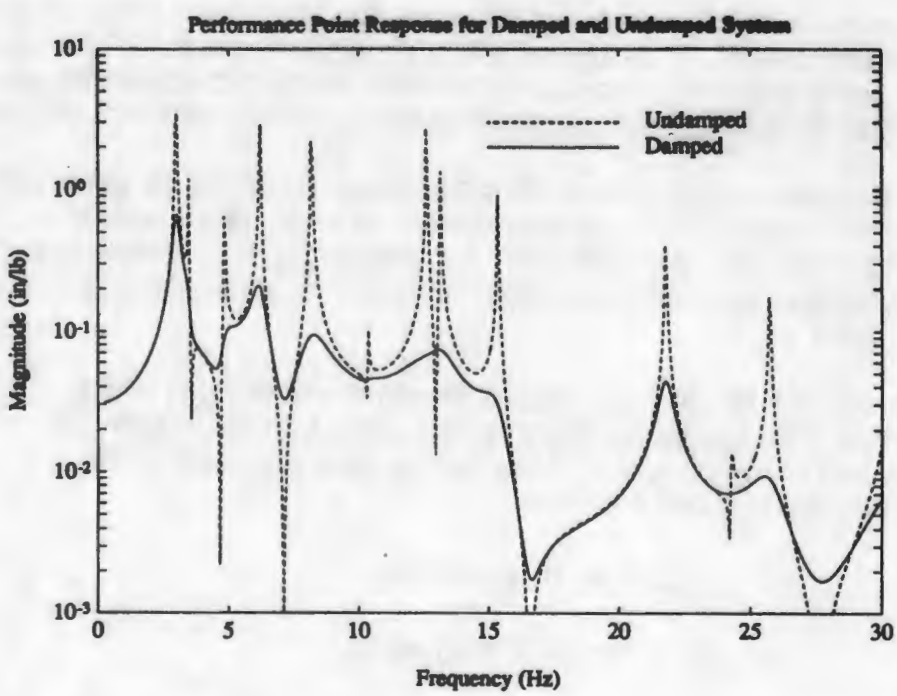

Figure 6 - Performance Point Frequency Response for Undamped System and MSE Damped Solution

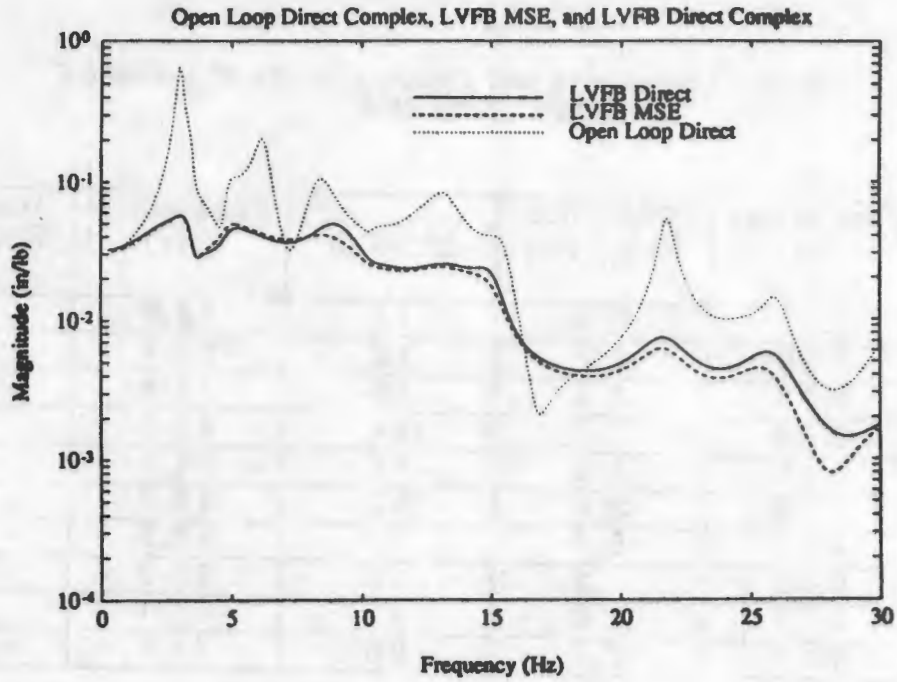

Figure 7 - Local Velocity Feedback Closed-Loop Response Exact and MSE Solutions

ranges. The reduction in RMS response in this case is a factor of 3, when compared to the passively damped system. This agreement is consistent with PACOSS experience on the Dynamic Test Article [13], which used the MSE method to accurately predict the closed-loop response of a damped system with a local velocity feedback controller.

The exact frequency response of the closed-loop system was generated, using direct inversion of the closed-loop impedance matrix at each frequency point. This can be accomplished for an arbitrary controller by converting the compensator into an equivalent **N x N** frequency-dependent impedance matrix and adding it to the open-loop impedance matrix,

i.e., determine the **N x N** frequency-dependent matrix **G(s)** which describes the Laplace transform of the control forces in terms of the motion of the structure. Then the Laplace transform of the closed-loop structural motion is:

$$\mathbf{X}_{cl}(s) = \mathbf{H}_{cl}(s) \cdot \mathbf{F}(s)$$

$$\mathbf{H}_{cl}(s) = \left[\mathbf{M} \cdot s^2 + \mathbf{K}(s) - \mathbf{G}(s)\right]^{-1} \tag{32}$$

Using this relationship, the closed-loop frequency response between applied noise at the actuators to the performance point motion can be computed at $s=i\omega$ by direct matrix inversion.

A compensator was also designed to control the MSE state-space model using the LQG/LTR algorithm. In this algorithm, a linear quadratic regulator is designed which provides the desired performance characteristics using full-state feedback. A linear quadratic estimator is then designed which approaches the full-state feedback performance by increasing a loop-transfer recovery parameter in the estimator design process. The regulator was designed so that the desired factor of 20 reduction in RMS response over the damped open-loop plant was obtained, and the loop-transfer recovery parameter was selected to the minimum value which provided acceptable performance of the estimator.

The exact closed-loop frequency response of the system was then generated using the LQG/LTR compensator designed for the MSE plant. A comparison of the open-loop response, the MSE prediction, and the exact closed-loop frequency response are given in Figure 8. Notice that there are large variations between the frequency responses predicted using the MSE plant and those found using direct inversion of the closed-loop impedance matrix. Although the closed-loop system does perform better than the open-loop system, in the 20-Hz region the closed-loop performance is actually amplified over the open-loop response. Furthermore, the reduction in RMS response is only a factor of 11, whereas the MSE plant predicts a reduction of a factor of nearly 21. This large discrepancy shows that for sophisticated active control algorithms, the MSE state-space model does not adequately describe the dynamic characteristics of the relatively simple MACE plant. A more accurate description of the open-loop system is required for a control design model.

A state-space model of the open-loop system was then generated using the accelerated subspace iteration procedure as described in the previous sections. This model was then used to design an LQG/LTR compensator to reduce the RMS response by a factor of 20, similar to the MSE design. The closed-loop frequency response predicted using this state-space model was then compared with the exact closed-loop solution. Figure 9 shows a comparison of the damped open-loop system frequency response, the exact open-loop frequency response, and the modal closed-loop response generated using the exact method and the reduced-order state-space model. Notice that there is good agreement between the open-loop performance predicted using the 40-state modal plant model and the exact response found by direct inversion.

Most important for the closed-loop system, the reduction in RMS response using the exact method is a factor of 21, which is the predicted reduction. The closed-loop performance predicted with the state-space model agrees well with the exact solution, although there are some differences in the frequency response. While the open-loop plant model accurately predicts the closed-loop response in this case, it may not in general. This is due to the assumption that the plant has viscous damping, which provides a plant which is locally accurate but may not be accurate if the poles are significantly altered by the controller. This is to be expected, as in the formation of the open-loop plant, the poles which describe the variation of the viscoelastic properties with frequency have been truncated. The introduction of the controller alters the eigenvalues of the plant and, therefore, a closed-loop eigenvalue problem must be solved with spectral iteration to achieve the best agreement with a modal model. However, the closed-loop performance shows that the described procedure provides

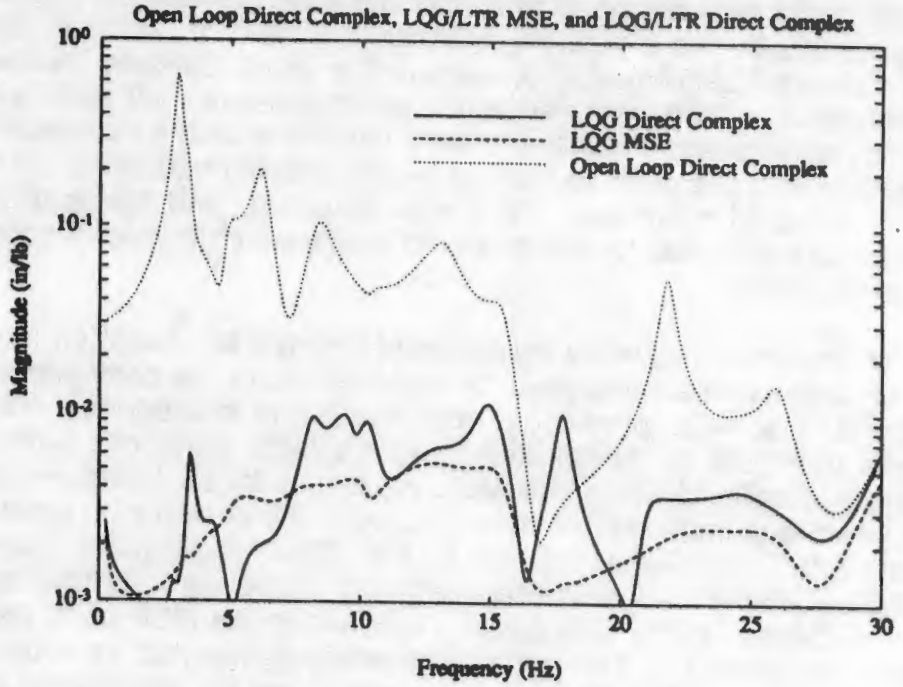

Figure 8 - LQG/LTR Closed-Loop Frequency Response Exact and MSE Solutions

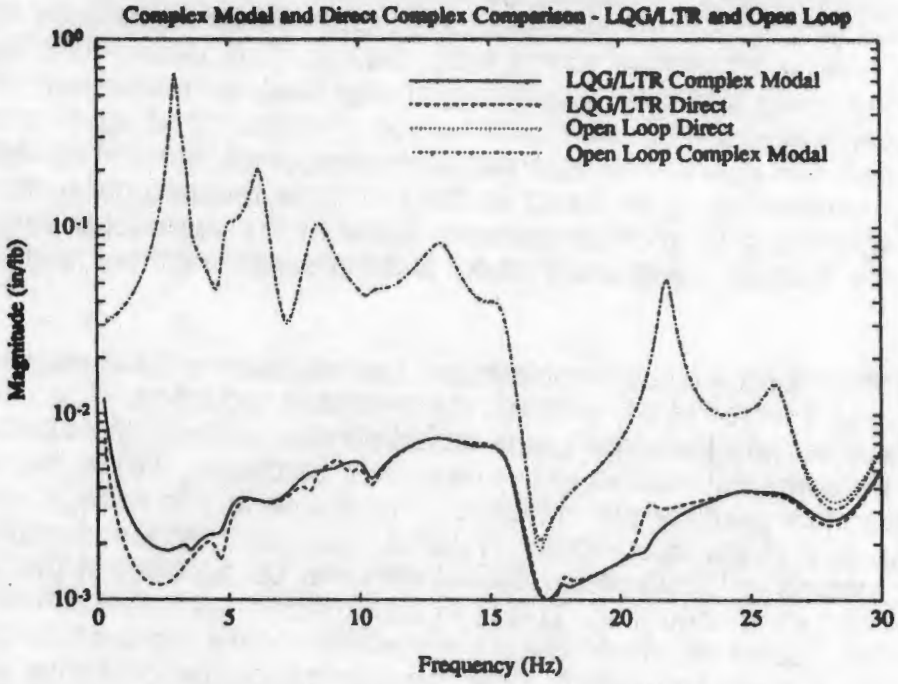

Figure 9 - LQG/LTR Closed-Loop Frequency Response Exact and Truncated Modal Solutions

an accurate reduced-order plant model which allows the design of a controller which provides the desired closed-loop performance.

The variation between the MSE and complex modal open-loop plant models can also be seen in the comparison of the open-loop frequency responses. The greatest variation between the models is typically in the phase of the frequency response. As an example, consider the comparison of the phase of the frequency response of the MSE model, the complex modal model, and the exact solution given in Figure 10. Notice that there is nearly exact agreement between the complex modal model and the exact solution, but there is a large discrepancy between the phase of the MSE frequency response and the exact frequency response in several frequency ranges. At a frequency of 25 Hz, the phase of the MSE frequency response is actually 180° out of phase with the exact solution. This phase difference can cause significant performance degradation or even instabilities of the closed-loop system.

For the accelerated subspace procedure to be used for the analysis of actual systems, the cost of the procedure must not be excessive. Therefore, a comparison of the computer requirements for the MSE solution and for the improvement of the MSE solution using accelerated subspace iteration was made. Table 2 provides a comparison of the solution times for the two methods.

These computations were performed on a SUN 3/50 workstation, with the plant model having 279 degrees of freedom. The computer times represent elapsed time in seconds. To facilitate these computations, the full-size system matrices were reduced from nearly 1100 degrees of freedom by a Guyan reduction using viscoelastic properties in the middle of the desired bandwidth. This reduced model was then considered the exact model description. The MSE solution was generated using standard subspace iteration, and the MSE method to find the modes, natural frequencies, and damping ratios of the real system. Six frequency values were used to compute the MSE modes with the appropriate viscoelastic shear moduli. The complex subspace iteration procedure was applied using the inverse of the shifted MSE stiffness matrix with properties in the middle of the frequency band.

Notice that the accelerated subspace procedure is not excessive in terms of computer time, requiring only 125% of the MSE solution time, no matrix inversions, and 973 real matrix/complex vector multiplications. Therefore, the improvement of the MSE solution using the procedure defined in this paper can be used economically on large-scale structural systems, and will provide an accurate model for damped systems with minimal additional computational expense.

7.0 Conclusions

The above development and example problem allows several conclusions to be made in connection with the modeling of damped structural systems and active controls. The most important of these are:

1) The MSE method provides a tool which is very effective for use in the design of damped structures; however, it may not have sufficient accuracy for use in the final design of modern control systems for damped structures.

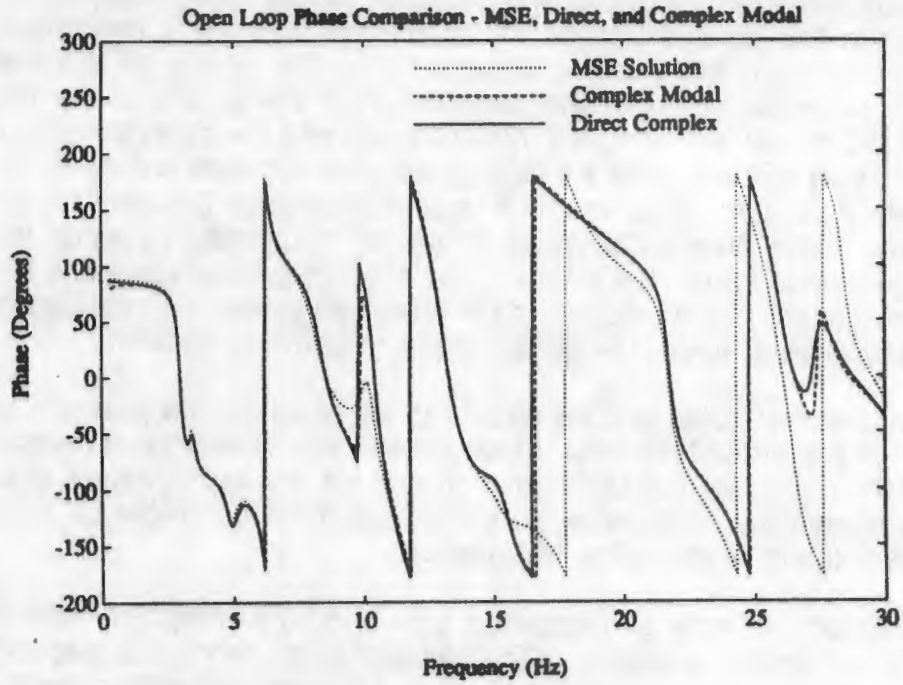

Figure 10 - Open-Loop Phase of MSE and Truncated Modal Solution Compared to Exact Solution

Table 2 - Solution Times for MSE Analyses and Accelerated Subspace Iteration with MSE Starting Vectors

Analysis Type	Number of Analyses	Number of Matrix Decomps	Number of Matrix/ Vector Products	Total Elapsed Solution Time (sec)
MSE Using Subspace Iteration	7	7	1108	2815
Acclerated Complex Subspace Iteration	1	0	973	3530

2) The MSE method can be shown to be a low-cost approximation to a subspace/spectral iteration. Therefore, the developed procedure can be used in concert with the MSE method to improve the solution accuracy as a design cycle progresses.

3) The developed design and analysis procedure can be used efficiently in the solution of large-scale dynamics problems with viscoelastic damping treatments and active controls.

4) Although not shown here, the technique of subspace/spectral iteration can be used for the solution of problems with viscous damping or combined viscous/viscoelastic damping, and for closed-loop damped systems. The method is as at least efficient as the techniques for solving complex eigenvalue problems currently available in many finite element codes. In fact, the conjugate gradient inverse power iterations are very similar to those performed in MSC/NASTRAN [14], but convergence is accelerated using the subspace procedure and conjugate gradients.

5) The solution procedure obviates the necessity for the description of viscoelastic materials or members using networks of springs and dashpots (Maxwell elements), as the solution using these descriptions will inevitably be more expensive and less accurate than using the fractional derivative representation. These methods typically add a number of degrees of freedom to the system matrices and increase solution costs; and the method of solution of these eigenvalue problems in most finite element codes can be shown to be similar to the eigenvalue procedure developed here for systems with fractional derivative material representations. Damping element properties can be written as a function of frequency using the fractional derivative representation, and the solution to the equations can efficiently be solved as developed previously. This frequency-dependent reduction can be considered the equivalent of static condensation for systems which contain damped elements that have negligible internal mass effects.

6) The cost of the eigenvalue solution procedure is independent of the description of the damping phenomenon in the system elements, as long as the element impedance properties can be described as a function of the Laplace variable. Therefore, if a higher-order fractional derivative representation is appropriate for a particular material, the solution procedure is unaltered, and the cost is effectively unchanged.

7) The accelerated subspace solution method can be used for eigenvalue problems which contain combinations of viscous dampers, viscoelastic dampers, and even for closed-loop systems. This facilitates an accurate modal representation of actively controlled damped systems.

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