IMPROVED SOLUTION TECHNIQUES FOR THE EIGENSTRUCTURE OF FRACTIONAL ORDER SYSTEMS

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Abstract

The structural problem of a viscoelastically damped rod is considered. A four parameter fractional derivative viscoelastic model rather than the traditional viscous model is used to describe the stress-strain relationship. The introduction of fractional order derivatives leads to high order matrix equations, which are cumbersome and time consuming to solve. Thus, there exists a motivation to seek alternate solution techniques. An existing technique, modified matrix iteration, is presented, and a new one, employing spectrum shift concepts, is proposed. The spectrum shift technique is shown to be significantly more efficient.

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Nomenclature

$[\ ]$	square matrix
$[\ ]^T$	transpose of matrix
$[\ ]^{-1}$	inverse of matrix
$\{\ \}$	column vector
A	cross-sectional area
b_m	parameters of viscoelastic model
$[D(\lambda)]$	dynamical matrix
$D^\alpha[\ ]$	generalized derivative of order α
E	Young's modulus
E_n	parameters of viscoelastic model
$F[\ ]$	Fourier transform operator
$\{F(s)\}$	Laplace transform of the vector of forcing functions
i	square root of negative one
$[K(s)]$	viscoelastic stiffness matrix
$[\widetilde{K}]$	pseudo stiffness matrix of expanded equations of motion
$L[\ ]$	Laplace transform operator
L	length of a rod element
$[M]$	mass matrix
$[\widetilde{M}]$	pseudo mass matrix of expanded equations of motion
s	Laplace parameter
$\{x(t)\}$	column vector of structural displacements
$\{X(s)\}$	Laplace transform of $\{x(t)\}$
α_n	parameters of viscoelastic model
β_m	parameters of viscoelastic model
$\epsilon(t)$	strain history
$\Gamma(\alpha)$	gamma function of α
λ	eigenvalue associated with expanded equations of motion
$\{\phi\}$	mode shape
μ	shift factor
$\sigma(t)$	stress history
ω	Fourier parameter and frequency

Introduction

The fractional derivative viscoelastic model has its earliest roots in Nutting's observations that fractional powers of time could model the stress relaxation phenomenon [5]. Gemant later noted that stiffness and damping properties of viscoelastic materials seemed proportional to fractional powers of frequency, implying that fractional order time differentials might be used to model the behavior [14]. Scott-Blair combined the ideas of Nutting and Gemant by proposing

the use of fractional order time derivatives [2]. Caputo applied the concept to the viscoelastic behavior of geological strata [4]. Then he and Minardi showed that constitutive relationships employing the fractional calculus described the mechanical properties of some metals and glasses [5]. Bagley proposed incorporating fractional derivatives into finite element models of viscoelastically damped structures. Since then, he and Torvik have jointly published several papers demonstrating the feasibility and benefits of using fractional calculus. Of particular note is "A Theoretical Basis for the Application of Fractional Calculus to Viscoelasticity" [5], which uses molecular theory to derive the existence of generalized derivatives. Their efforts have shown that fractional calculus is an attractive approach to modelling viscoelastically damped structures. The resulting model requires very few parameters and is often accurate over six decades of frequency [2].

Generalized calculus is not a new concept — mathematicians have dealt with it for some time [10, 115-118]. A generalized derivative is represented in this paper as

$$D^{\alpha}[x(t)].$$

The generalized derivative can be defined for complex α, but only real values will be considered here. Fractional derivatives are generalized derivatives with rational α. The term "fractional calculus" implies the use of fractional derivatives.

This paper reviews the properties of generalized derivatives and the expanded equations of motion for a fractional order system describing a viscoelastically damped rod. The technique proposed by Bagley to solve for the eigenstructure is presented. A more efficient method is presented later, along with some examples.

Generalized Derivatives Applied to Viscoelastic Materials

Before applying generalized derivatives to structural problems, it is necessary to understand the properties of generalized derivatives and their use in viscoelastic theory. As will be shown, generalized derivatives behave in much the same way as conventional derivatives. When used to model viscoelastic materials, generalized derivatives typically provide an excellent model over a broad range of frequencies [4]. To show how generalized derivatives can be used to model viscoelastic materials, it is appropriate to present first the properties of generalized derivatives, especially the Laplace and Fourier transforms. The generalized derivative is defined as [1, 2]

$$D^{\alpha}[x(t)] \equiv \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(\tau)}{(t-\tau)^{\alpha}} d\tau \qquad \text{for } 0 \leq \alpha < 1 \tag{1}$$

Note that this definition is only valid for $\alpha < 1$. However, the definition requires only a slight modification for a generalized derivative of order greater than one.

Let m be a nonnegative integer, and α defined as before. Then [1, 11]

$$D^{m+\alpha}[x(t)] \equiv \frac{1}{\Gamma(1-\alpha)} \frac{d^{m+1}}{dt^{m+1}} \int_0^t \frac{x(\tau)}{(t-\tau)^\alpha} d\tau \tag{2}$$

Although imposing in the time domain, in the Laplace (or Fourier) domain, the generalized derivative manifests itself as a fractional power of s (or ω). To calculate the Laplace transform, let $\tau = t - \eta$. Then,

$$D^\alpha[x(t)] = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(t-\eta)}{\eta^\alpha} d\eta \tag{3}$$

Applying Leibnitz's rule,

$$D^\alpha[x(t)] = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{\eta^\alpha} \frac{\partial}{\partial t} x(t-\eta)\, d\eta + \frac{x(0)}{\Gamma(1-\alpha)t^\alpha} \tag{4}$$

Noting that the integral is a time convolution, and that

$$L\left[\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right] = \frac{1}{s^{1-\alpha}} \tag{5}$$

the Laplace transform is

$$L[D^\alpha[x(t)]] = \frac{1}{s^{1-\alpha}} (sL[x(t)] - x(0)) + \frac{x(0)}{s^{1-\alpha}} \tag{6}$$

or, more simply,

$$L[D^\alpha[x(t)]] = s^\alpha L[x(t)] \tag{7}$$

where

$$L[x(t)] = \int_0^\infty x(t)\, e^{-st}\, dt \tag{8}$$

Notice that for initial conditions equal to zero, the Laplace transform of a generalized derivative of order α has the same property as the conventional derivative: the transform is s^α times the transform of the function. In fact, the generalized derivative satisfies many of the same properties as the conventional derivative, particularly linearity and the composition property [1, 8-10]

$$D^\alpha[y(t) + x(t)] = D^\alpha[y(t)] + D^\alpha[x(t)] \tag{9}$$

$$D^\alpha[D^\beta[x(t)] = D^{\alpha+\beta}[x(t)] \tag{10}$$

The Fourier transform is defined as

$$F[x(t)] \equiv \int_{-\infty}^\infty x(t)\, e^{-i\omega t}\, dt \tag{11}$$

If $x(t) = 0$ for $t < 0$, then the Fourier transform can be written as

$$F[x(t)] = \int_0^\infty x(t)\, e^{-i\omega t}\, dt \tag{12}$$

It is easily seen that the Fourier transform of a generalized derivative is

$$F[D^\alpha[x(t)]] = (i\omega)^\alpha F[x(t)] \tag{13}$$

In the preceding discussion, the only restriction placed on α was that it be a nonnegative real number less than one. However, for engineering applications, an irrational number can be approximated by a rational number. So α will now be restricted to be rational as well. Using the term "fractional derivative" will indicate this additional restriction.

To illustrate the use of fractional derivatives in viscoelastic theory, consider the standard linear viscoelastic model relating stress and strain [2]

$$\sigma(t) + \sum_{m=1}^{M} b_m \frac{d^m \sigma(t)}{dt^m} = E_0 \epsilon(t) + \sum_{n=1}^{N} E_n \frac{d^n \epsilon(t)}{dt^n} \tag{14}$$

Recalling Scott-Blair's proposal, replace the conventional derivatives by derivatives of fractional order. The result is the general form of the fractional derivative viscoelastic model [2]

$$\sigma(t) + \sum_{m=1}^{M} b_m D^{\beta_m}[\sigma(t)] = E_0 \epsilon(t) + \sum_{n=1}^{N} E_n D^{\alpha_n}[\epsilon(t)] \tag{15}$$

A large number of materials can be modelled by replacing each sum in Equation 15 by a single term involving a fractional derivative

$$\sigma(t) + b\, D^\beta[\sigma(t)] = E_0 \epsilon(t) + E_1 D^\alpha[\epsilon(t)] \tag{16}$$

Invoking the Second Law of Thermodynamics requires that [3]

$$\begin{array}{rcl@{\qquad}rcl} E_0 & \geq & 0 & E_1 & \geq & bE_0 \\ E_1 & \geq & 0 & \alpha & = & \beta \\ b & > & 0 & & & \end{array} \tag{17}$$

These constraints ensure nonnegative energy dissipation and nonnegative work. The stress-strain relation in the Laplace domain is

$$\frac{\sigma(s)}{\epsilon(s)} = \frac{E_0 + E_1 s^\alpha}{1 + b s^\alpha} \tag{18}$$

This is known as the four parameter model, and has been shown to be very accurate over several decades of frequency [4, 14, 15].

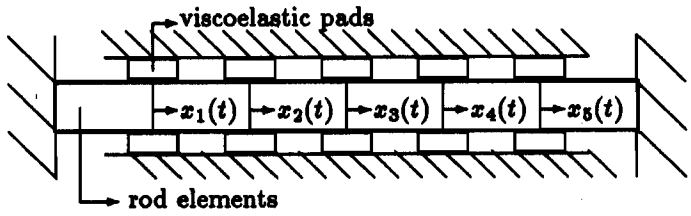

Figure 1: Finite Elements of Rod

Expanded Equations

Although the fractional derivative viscoelastic model may provide an excellent description of a material's properties, in order for it to be useful, its application to a structure must lead to a solvable problem. This section illustrates the existence of a solution by examining the finite element model of a viscoelastically damped rod. The equations of motion are developed using the elastic-viscoelastic correspondence principle, which states that a viscoelastic problem is equivalent to an elastic problem with the elastic moduli replaced by the appropriate viscoelastic moduli [7, 42]. This section develops the finite element model of a viscoelastically damped rod, constrained at each end. Figure 1 shows a five degree-of-freedom rod, constrained at each end, with viscoelastic damping pads at each node. Assume the rod is uniform and purely elastic. Using standard finite element techniques, the stiffness matrix for the elastic rod is of the form [8, 300]

$$[K_E] = \frac{EA}{L}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \tag{19}$$

where E is the Young's modulus for the material in the rod, A is the cross-sectional area, and L is the length of one element. Assume the modulus of the viscoelastic material is

$$E(s) = \frac{\sigma(s)}{\epsilon(s)} = \frac{E_0 + E_1 s^\alpha}{1 + b s^\alpha} \tag{20}$$

as derived in the previous section. The damping pads provide an out of phase shear stress to the rod. The shear stress is partially elastic and partially viscous.

due to the real and imaginary parts of the modulus. As an example, let $\alpha = 1/2$, $b = 0$, and $s = i\omega$, where ω is an observed frequency of the system. Then

$$\begin{aligned} E(\omega) &= E_0 + E_1(i\omega)^{1/2} \\ &= E_0 + (\omega)^{1/2} E_1 e^{i\pi/4} \\ &= (E_0 + (\omega)^{1/2} E_1 \cos\tfrac{\pi}{4}) + i(\omega)^{1/2} E_1 \sin\tfrac{\pi}{4}. \end{aligned} \tag{21}$$

The real part represents the elastic component of the shear stress, and the imaginary part represents the viscous component, which is ninety degrees out of phase.

The contribution to the structure's stiffness matrix due to the viscoelastic pads is

$$G(s)[K_V] = \frac{G_0 + G_1 s^\alpha}{1 + bs^\alpha} \begin{bmatrix} A_1/t_1 & 0 & 0 & 0 & 0 \\ 0 & A_2/t_2 & 0 & 0 & 0 \\ 0 & 0 & A_3/t_3 & 0 & 0 \\ 0 & 0 & 0 & A_4/t_4 & 0 \\ 0 & 0 & 0 & 0 & A_5/t_5 \end{bmatrix} \tag{22}$$

where A_i is the area of the pad attached to the rod at i^{th} degree of freedom and t_i is the pad's thickness. The ratios A_i/t_i are the stiffness coefficients for the damping material at the corresponding degree of freedom. Then the stiffness matrix for the total structure is

$$[K(s)] = [K_E] + \frac{G_0 + G_1 s^\alpha}{1 + bs^\alpha}[K_V] \tag{23}$$

The mass matrix for the rod is [8, 301-302]

$$[M] = \frac{\rho AL}{6} \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \tag{24}$$

where ρ is the density of the rod, and A and L are defined as above.

The equations of motion in the Laplace domain are

$$[s^2[M] + [K(s)]]\ \{X(s)\} = \{F(s)\} \tag{25}$$

where $\{F(s)\}$ is the Laplace transform of the forcing function. Setting $\{F(s)\} = 0$ yields the homogeneous equation, from which the eigenstructure is found.

To clear the denominator in $[K(s)]$, multiply through by $(1+bs^{\alpha})$. Defining

$$[A_0] = G_0[K_V] + [K_E] \tag{26}$$

$$[A_q] = G_1[K_V] + b[K_E] \tag{27}$$

and expressing α as a ratio in lowest terms, q/m, gives

$$\left(s^{(2m+q)/m}\, b[M] + s^{2m/m}[M] + s^{q/m}[A_q] + [A_0]\right)\{X(s)\}$$

$$= (1 + bs^{q/m})\{F(s)\} \tag{28}$$

In order to obtain an orthogonal transformation and decouple the equations of motion, cast the equations of motion in the following format

$$s^{1/m}[\widetilde{M}]\{\tilde{X}(s)\} + [\tilde{K}]\{\tilde{X}(s)\} = \{\tilde{F}(s)\} \tag{29}$$

$$[\widetilde{M}] = \begin{vmatrix} [0] & [0] & \cdots & [0] & b[M] \\ [0] & [0] & \cdots & b[M] & \vdots \\ \vdots & \vdots & \vdots & \vdots & [A_q] \\ [0] & b[M] & \cdots & [A_q] & \vdots \\ b[M] & \cdots & [A_q] & \cdots & [0] \end{vmatrix}$$

$$[\tilde{K}] = \begin{vmatrix} [0] & [0] & \cdots & [0] & -b[M] & [0] \\ [0] & [0] & \cdots & -b[M] & \cdots & [0] \\ \vdots & \vdots & \vdots & \vdots & [-A_q] & \vdots \\ [0] & -b[M] & \cdots & [-A_q] & \cdots & [0] \\ -b[M] & \cdots & [-A_q] & \cdots & [0] & [0] \\ [0] & [0] & \cdots & [0] & [0] & [A_0] \end{vmatrix}$$

$$\{\tilde{X}(s)\} = \begin{Bmatrix} s^{(2m-1)/m} & \{X(s)\} \\ s^{(2m-2)/m} & \{X(s)\} \\ \vdots & \\ s^{1/m} & \{X(s)\} \\ 1 & \{X(s)\} \end{Bmatrix}$$

$$\{\tilde{F}(s)\} = \begin{Bmatrix} [0] \\ [0] \\ \vdots \\ [0] \\ (1 + bs^{q/m})\{F(s)\} \end{Bmatrix}$$

With $\{F(s)\} = 0$, the problem is now in terms of real, square, symmetric matrices. Thus, the eigenvalues will be distinct and either real or occur in complex conjugate pairs. Also, the eigenvectors will be orthogonal to one another. It is a straightforward matter to decouple the expanded equations of motion using standard techniques [1, 67-68] Notice that for an n degree-of-freedom structure, the order of the expanded equations is $n(2m + q)$. From Equation 28, it can be seen that there are $(2m + q)$ branches to the problem, with n eigenvalues on each, resulting in $n(2m + q)$ eigenvalues. In a standard viscous formulation of the problem, only $2n$ eigenvalues would be found. The additional ones are due to the use of the fractional order derivatives. For a large structure, the higher order of the equations of motion represents a significant computational burden. Now that the existence of the solution has been proved, it will be beneficial to consider solution techniques that avoid solving the expanded equations of motion.

Modified Matrix Iteration Solution

The current method of determining the eigenstructure of the fractional order system developed in the previous section is to use a modified matrix iteration scheme on the homogeneous form of the original equation. Matrix iteration avoids computing and solving the characteristic polynomial of the matrix. Unlike using a Hessenberg matrix, which requires knowing the eigenvalue before the eigenvector can be calculated, matrix iteration determines both at the same time.

Matrix iteration is typically used to find the eigenstructure of undamped systems. With some modification, the concept can be applied to damped systems. Two different algorithms will be needed to find all $n(2m + q)$ modes. For convenience, the modes on a given branch will be numbered beginning with the one corresponding to the eigenvalue with the smallest magnitude. A mode corresponding to an eigenvalue with larger magnitude will be referred to as a higher mode. Lower modes are defined in the same way.

For an undamped system, the homogeneous form of the equations of motion in the Fourier domain is

$$-\omega^2[M]\{\phi\} + [K]\{\phi\} = 0 \tag{30}$$

or

$$[K]^{-1}[M]\{\phi\} = \frac{1}{\omega^2}\{\phi\} \tag{31}$$

To demonstrate matrix iteration, select a trial vector,$\{\psi\}$, and express it as a linear combination of the eigenvectors of $[K]^{-1}[M]$:

$$\{\psi\} = \sum_{i=1}^{n} c_i \{\phi_i\} \tag{32}$$

This is possible since the eigenvectors of $[K]^{-1}[M]$ span n-space. The only restriction on the c_i's is that $c_1 \neq 0$. Premultiplying both sides of Equation 32 by $[K]^{-1}[M]$ produces

$$[K]^{-1}[M]\{\psi\} = \sum_{i=1}^{n} \frac{c_i}{\omega_i^2} \{\phi_i\} \tag{33}$$

Subsequent multiplications produce

$$\left([K]^{-1}[M]\right)^k \{\psi\} = \sum_{i=1}^{n} \frac{c_i}{\omega_i^{2k}} \{\phi_i\} \tag{34}$$

Since for large k,

$$\omega_1^{2k} \ll \omega_2^{2k} \ll \cdots \ll \omega_n^{2k} \tag{35}$$

it is clear that Equation 34 converges to the lowest mode [11, 124- 125]. If Equation 34 is normalized with respect to the same element between premultiplications by $[K]^{-1}[M]$, the the normalization factor reaches a constant value, equal to $1/\omega_1^2$ (since $c_1 \neq 0$), and the normalized vectors converge to the first mode. To find higher modes, subtract off lower modes using Turner's method [6, 168-269]. Letting

$$[D] = [K]^{-1}[M] - \sum_{i=1}^{j-1} \frac{1}{\omega_i^2} \{\phi_i\}\{\phi_i\}^T [M] \tag{36}$$

then

$$[D]\{\phi\} = \frac{1}{\omega^2}\{\phi\} \tag{37}$$

converges to the j^{th} mode. Note that the lower modes must be normalized such that $\{\phi_i\}^T[M]\{\phi_i\} = 1$.

To apply this technique to a fractional order system, let $\lambda = s^{1/m}$. Then Equation 30 can be written as

$$\lambda^{2m}[M]\{\phi\} + [K(\lambda)]\{\phi\} = 0 \tag{38}$$

or

$$[K(\lambda)]^{-1}[M]\{\phi\} = \frac{-1}{\lambda^{2m}}\{\phi\} \tag{39}$$

where $[K(\lambda)]$ is equivalent to $[K(s)]$ in Equation 25. Each time the estimate of λ is updated, $[K(\lambda)]$ must be recomputed. Notice that for λ^{2m}, there are $2m$

possible values of λ. The different values arise because $z^{1/2m}$ is a multivalued function and has $2m$ branches. The value of λ on the k^{th} branch is computed using DeMoivre's Theorem [13, 22]. Using the form $\lambda^{2m} = re^{i\theta}$,

$$\lambda = r^{1/2m}\left(\cos\frac{\theta+2k\pi}{2m} + i\sin\frac{\theta+2k\pi}{2m}\right) \tag{40}$$

The primary branch is assigned the number "0", so $k = 0, 1, 2, \ldots, 2m-1$.

Since the stiffness matrix is a function of λ, to find the higher modes Equation 36 must be modified:

$$[D(\lambda)] = [K(\lambda)]^{-1}[M] - \sum_{i=1}^{j-1}\frac{1}{\Lambda_i^{2m}}\{\Phi_i\}\{\Phi_i\}^T[M] \tag{41}$$

The quantities Λ_i and $\{\Phi_i\}$ are called pseudoeigenvalues and pseudoeigenvectors. They are computed from the eigenvector problem:

$$[K(\lambda)]^{-1}[M]\{\Phi\} = \frac{-1}{\Lambda^{2m}}\{\Phi\} \tag{42}$$

It is important to realize that the pseudoeigenvalues and pseudoeigenvectors are not modes of the system. Their computation is merely an intermediate step in calculating the solutions of the equations of motion. In computing the j^{th} mode of the system, only the first $j-1$ pseudomodes of Equation 42 are needed. Then Equation 41 is used to converge on the j^{th} mode of the system. Notice that for each new guess of λ, $j-1$ pseudoeigenvalues and pseudoeigenvectors must be recalculated. This represents a significant computational burden. The next section proposes a technique to reduce the amount of computation required.

Note that this technique produces $2mn$ eigenvalues, but Equation 28 predicted $n(2m+q)$ eigenvalues. The remaining qn of the $n(2m+q)$ eigenvalues and eigenvectors are found using a scheme very similar to the one above [1, 80-83]. After clearing the denominator of Equation 38, it can be written as

$$\begin{aligned}\lambda^{2m}(1+b\lambda^q)[M]\{\phi\} + (1+b\lambda^q)[K_E]\{\phi\} + \\ (E_0 + E_1\lambda^q)[K_V]\{\phi\} \quad &= \quad 0\end{aligned} \tag{43}$$

Writing the equation in this form allows λ^q to appear explicitly in the equation, making it possible to find the remaining roots. Notice that these additional roots only exist for $b \neq 0$.

The solution method used to find the additional roots is somewhat subtle. By defining

$$Q = b\lambda^{2m+q} + \lambda^{2m} \tag{44}$$

$$[K'(\lambda)] = (1+b\lambda^q)[K_E] + (E_0 + E_1\lambda^q)[K_V] \tag{45}$$

Equation 43 can be written in the more recognizable form

$$[K'(\lambda)]^{-1}[M]\{\phi\} = \frac{-1}{Q}\{\phi\} \tag{46}$$

Matrix iteration is applied to this equation, with the i^{th} estimate of λ determined from

$$\lambda_i = \left[\left(\frac{Q - \lambda_{i-1}^{2m}}{b\lambda_{i-1}^{2m+q-1}}\right)^q\right]^{1/q} \tag{47}$$

The k^{th} branch of the q^{th} root of the quantity in brackets is used to determine the eigenvalue on that branch.

Turner's method is again employed to find the higher modes on each branch, as in Equation 41, with $Q(\lambda_i)$ replacing ω_i^2.

This section has shown that it is possible to find all $n(2m+q)$ eigenvalues and eigenvectors without solving the expanded equations of motion. However, the technique still requires a substantial amount of computation. In the next section, a technique is proposed which greatly reduces the computational burden.

Spectrum Shift Technique

While the modified matrix iteration technique is effective, it is not very efficient. In this section, spectrum shift methods will be combined with the matrix iteration technique, reducing the amount of computation required. The purpose of spectrum shift is to shift the eigenvalues of the system so that the desired eigenvalue becomes the fundamental one. Matrix iteration will then produce the desired eigenvalue. If spectrum shift methods could be used to compute the higher modes in the viscoelastic model, the pseudoeigenvalues and pseudoeigenvectors of the corresponding $[K(\lambda)]^{-1}[M]$ would not have to be computed. Determining the appropriate spectrum shifts is not easy, and requires certain precautions, which will be presented later.

The spectrum shift technique is usually used in elastic systems when a particular frequency and corresponding mode shape are of interest. To illustrate the theory behind the spectrum shift technique, consider again an undamped system

$$[[K] - \omega^2[M]]\{\phi\} = 0 \tag{48}$$

Picking the shift factor, μ, close to the desired ω_i^2 gives the shifted equations [8, 330]

$$[[K] - \mu[M] - (\omega^2 - \mu)[M]]\{\phi\} = 0 \tag{49}$$

Letting

$$[\hat{K}] = [K] - \mu[M] \quad \text{and} \quad \widehat{\omega^2} = \omega^2 - \mu \tag{50}$$

Then

$$\left[[\hat{K}] - \widehat{\omega^2}[M]\right]\{\phi\} = 0 \quad \text{or} \quad [\hat{K}]^{-1}[M]\{\phi\} = \frac{1}{\widehat{\omega^2}}\{\phi\} \tag{51}$$

Applying matrix iteration to this equation produces the mode closest to μ.

Now consider the matrix $[K(\lambda_i)]^{-1}[M]$ of the viscoelastic model. Only the i^{th} eigenvalue and eigenvector are desired. By letting $\lambda = s^{1/m} = (i\omega)^{1/m}$, Equations 50 and 51 can be written as

$$\begin{aligned} \widehat{\lambda^{2m}} &= \lambda_i^{2m} + \mu \\ [\hat{K}(\lambda_i)] &= [K(\lambda_i)] - \mu[M] \\ [\hat{K}(\lambda_i)]^{-1}[M]\{\phi\} &= \frac{1}{\widehat{\lambda^{2m}}}\{\phi\} \end{aligned} \tag{52}$$

As a first guess of the appropriate shift factor for the i^{th} mode, the eigenvalue of $[K(\lambda_{i-1})]^{-1}[M]$ closest to λ_{i-1} is used. It is computed by using Turner's method. The dynamical matrix is

$$\begin{aligned} [D(\lambda_{i-1})] &= [\hat{K}(\lambda_{i-1})]^{-1}[M] - \frac{1}{\lambda_{i-1}^{2m}}\{\phi_{i-1}\}\{\phi_{i-1}\}^T[M] \\ [D(\lambda_{i-1})]\{\phi\} &= \frac{1}{\widehat{\lambda^{2m}}}\{\phi\} \end{aligned} \tag{53}$$

If μ_{i-1} was the shift used to find λ_{i-1}, then by Equation 52, the new shift factor is

$$\mu_i = \mu_{i-1} - \widehat{\lambda^{2m}} \tag{54}$$

Since the magnitude of the i^{th} eigenvalue must be larger than the magnitude of λ_{i-1}, if

$$|\mu_i| < |\mu_{i-1}| \tag{55}$$

then $\widehat{\lambda^{2m}}$ was in the wrong direction. The shift is recomputed as

$$\mu_i = \mu_{i-1} + \widehat{\lambda^{2m}} \tag{56}$$

Notice that matrix iteration on

$$[\hat{K}(\hat{\lambda}_i, \mu_i)]^{-1}[M] = \frac{1}{\widehat{\lambda_i^{2m}}}\{\phi_i\} \tag{57}$$

will converge to the $i-1$ mode if the magnitude of μ_i is not large enough. If this occurs, μ_i is adjusted by adding the new $\widehat{\lambda^{2m}}$ (as in Equation 56).

For undamped systems, the j^{th} eigenvalues on all $2m$ branches have the same magnitude and are evenly spaced on a circle about the origin. For lightly damped systems, the j^{th} eigenvalues lie near the those for an undamped system. This is portrayed graphically for a single degree of freedom system with $\alpha = 1/2$ in Figure 2a. Since λ^{2m} is roughly the same magnitude for all the eigenvalues, the method can be modified slightly to use the λ_i^{2m} values on the principal branch to calculate shifts for the other branches. This modification is valid for systems with less than a 0.01 damping ratio.

To understand the location of the λ^2 values in the s-plane, it is necessary to realize that the Riemann surface for the function $w = z^{1/2}$ consists of two

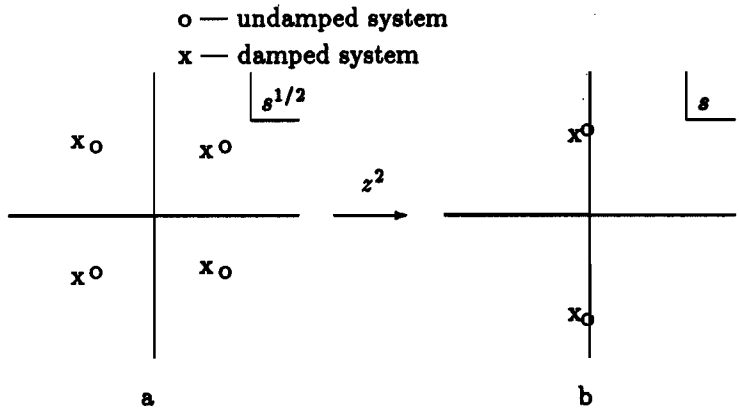

Figure 2: Locations of λ and λ^2 values

Riemann sheets, joined together at the branch cut. Taking the branch cut along the negative real axis, the sheets can be defined by

$$\begin{aligned} S_0 &= \{z | -\pi \leq \arg(z) < \pi\} \\ S_1 &= \{z | \pi \leq \arg(z) < 3\pi\} \end{aligned} \tag{58}$$

So the eigenvalues in the first and fourth quadrants of the $s^{1/2}$-plane map into the second and third quadrants, respectively, of S_0. These are shown in Fiqure 2b. But the eigenvalues in the second and third quadrants of the $s^{1/2}$-plane map into the fourth and first quadrants, respectively, of S_1. To see this let $re^{i(3\pi/4+\delta)}$ represent the second quadrant eigenvalue, where δ is an small angle. Then

$$\arg(\lambda^2) = \frac{3\pi}{2} + 2\delta \tag{59}$$

Since this angle is greater than π, λ^2 is on S_1 at the angle given by Equation 59.

The third quadrant eigenvalue is a little more subtle. Its angle is $-(3\pi/4+\delta)$, so

$$\arg(\lambda^2) = -\frac{3\pi}{2} - 2\delta \tag{60}$$

But neither sheet contains values with this angle. When the value crossed the negative real axis in the negative direction, its angle experienced a 4π jump discontinuity from $-\pi$ to 3π. Therefore the angle is really

$$\arg(\lambda^2) = -\frac{3\pi}{2} - 2\delta + 4\pi = \frac{5\pi}{2} - 2\delta \tag{61}$$

Technique	$t = 0.1$m	$t = 0.05$m
Modified matrix iteration	0:52.11	1:14.51
Spectrum shift	0:21.78	0:32.43
Modified spectrum shift	0:12.06	0:15.15

Table 1: Computation Times (in CPU minutes)

This angle is in the first quadrant of S_1. Notice that for undamped systems, the λ^2 values in S_1 lie directly above those in S_0. To map back into the $s^{1/2}$-plane, the 4π must be subtracted off before taking the square root.

For a ten degree-of-freedom system, the spectrum shift technique more than halved the computation time required by the modified matrix iteration technique. Storing the principal branch's λ^{2m} values reduced the computation time by another 50%. (Exact computation times are given in Table 1.) Computed eigenvalues were accurate to at least five significant figures.

Example Problem

To demonstrate the efficiency of this technique, a ten degree-of-freedom model was considered. The rod was similar to the one in Figure 1, and its equations of motion had the same form. The rod was assumed to be pure aluminum, with Butyl B252 damping pads. The values of the parameters were [4](all values are in compatible mks SI units)

$$\begin{array}{lcllcl} \rho & = & 2.71 \cdot 10^3 & E & = & 5.516 \cdot 10^{10} \\ A & = & 0.0625 & G_0 & = & 7.6 \cdot 10^5 \\ A_i & = & 0.0625 & G_1 & = & 2.95 \cdot 10^5 \\ L & = & 0.909 & b & = & 0.001 \\ t_i & = & 0.1 & & & \end{array}$$

These parameters resulted in low damping, on the order of 10^{-2}, so it could be solved using the modified spectrum shift technique, as well as by using modified matrix iteration or spectrum shift. The computation times for two different pad thicknesses are given in Table 1. The solution took longer than for the thinner pad due to the increased damping.

The damping in the system was increased by decreasing the thickness of the viscoelastic pads to 0.01m. For this case, the equivalent damping ratio was 0.069, as computed from the fundamental mode. For the principal branch, the complex frequencies and mode shapes were found to be

$$\begin{Bmatrix} -107 + 1545i \\ -77 + 2962i \\ -66 + 4459i \\ -61 + 6051i \\ -60 + 7762i \\ -61 + 9606i \\ -64 + 11566i \\ -69 + 13567i \\ -74 + 15415i \\ -79 + 16779i \end{Bmatrix}$$

and

$$\begin{bmatrix} 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 1.9 & 1.7 & 1.3 & 0.8 & 0.3 & -0.3 & -0.8 & -1.3 & -1.7 & -1.9 \\ 2.7 & 1.8 & 0.7 & -0.3 & -0.9 & -0.9 & -0.3 & 0.7 & 1.8 & 2.7 \\ 3.2 & 1.4 & -0.4 & -1.1 & -0.5 & 0.6 & 1.1 & 0.4 & -1.4 & -3.2 \\ 3.5 & 0.5 & -1.2 & -0.6 & 0.8 & 0.8 & -0.6 & -1.2 & 0.5 & 3.5 \\ 3.5 & -0.5 & -1.2 & 0.6 & 0.8 & -0.8 & -0.6 & 1.2 & 0.5 & -3.5 \\ 3.2 & -1.4 & -0.4 & 1.1 & -0.5 & -0.6 & 1.1 & -0.4 & -1.4 & 3.2 \\ 2.7 & -1.8 & 0.7 & 0.3 & -0.9 & 0.9 & -0.3 & -0.7 & 1.8 & -2.7 \\ 1.9 & -1.7 & 1.3 & -0.8 & 0.3 & 0.3 & -0.8 & 1.3 & -1.7 & 1.9 \\ 1.0 & -1.0 & 1.0 & -1.0 & 1.0 & -1.0 & 1.0 & -1.0 & 1.0 & -1.0 \end{bmatrix}$$

The first three mode shapes are plotted in Figures 3 to 5. The magnitude of the complex frequencies for the first five modes is less than 10% higher than those for an undamped continuum model, but the higher frequencies differ by up to 20%.

The spectrum shift method complements the finite element model. With spectrum shift, finite element problems with viscoelastic damping can be solved much faster than with modified matrix iteration. For a ten degree-of-freedom model, the savings was more than 50% of the CPU time.

Conclusions and Recommendations

The spectrum shift technique is more efficient than the matrix iteration technique. The computational burden does not increase as drastically with increasing degrees of freedom. For lightly damped systems, the modified spectrum shift technique represents even greater computational savings.

The existing program (presented in [9]) can be made more efficient by realizing that in real systems eigenvalues and eigenvectors appear in complex pairs, and by taking advantage of the symmetry of the stiffness and mass matrices. Also, for larger systems, it would be beneficial to examine matrix inversion techniques that are designed to handle large matrices.

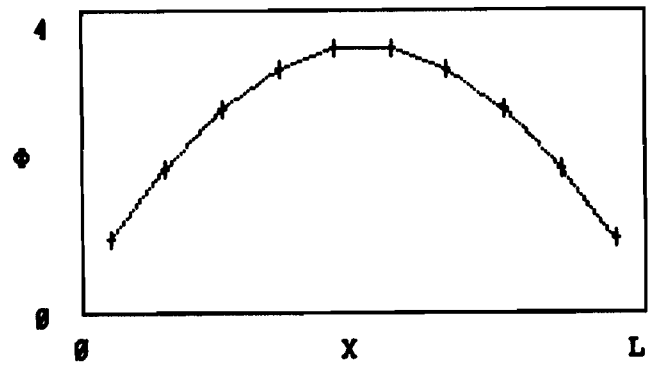

Figure 3: First Mode Shape for Damped Rod

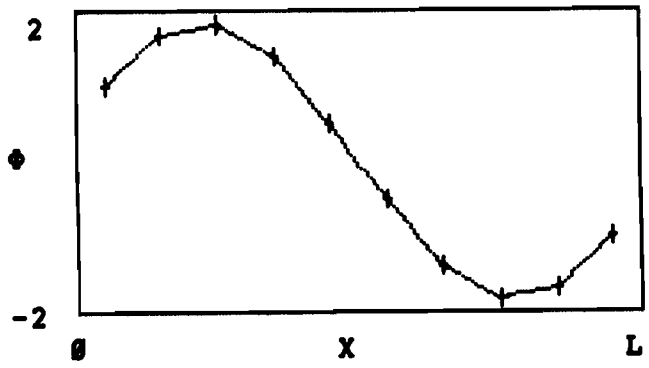

Figure 4: Second Mode Shape for Damped Rod

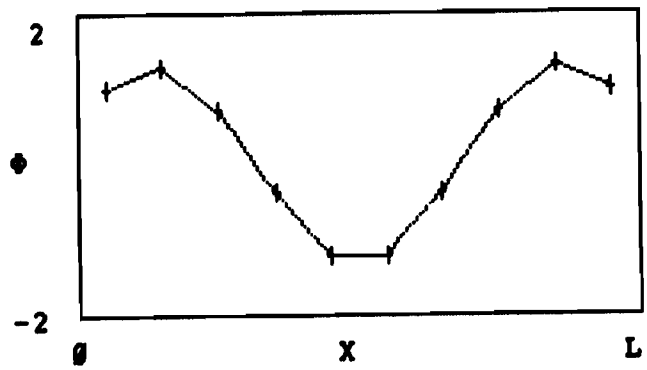

Figure 5: Third Mode Shape for Damped Rod

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