INVESTIGATION OF CONSTRAINTS IN THERMAL SIMILITUDE

VOLUME I

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FOREWORD

This report has been prepared by the Mechanical Engineering Department, Kansas State University, Manhattan, Kansas as part of U. S. Air Force Contract F 33615-68-C-1017, "Investigation of Constraints in Thermal Similitude." The work was administered under the direction of Air Force Flight Dynamics Laboratory, Air Force Systems Command, with Mr. Carl J. Feldmanis as Project Engineer. The work was performed between September 1967 and September 1969, with Dr. P. L. Miller as Principal Investigator.

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ABSTRACT

The studies described in this report clarify the effects of some of the limitations imposed by the laws of thermal similitude, and determine the thermal modeling laws for a heat pipe.

Solutions were presented for the steady-state temperature distribution and heat transfer in a radiating fin having temperature dependent thermal conductivity. Using these solutions, modeling prediction errors were determined for fin type prototype/model systems with dimensional distortions, with material having temperature dependent thermal conductivity, and with low prototype temperatures. These prediction discrepancies ranged from very small errors to errors in heat transfer rates as high as 75% in a severely distorted model.

The thermal modeling laws for a heat pipe were derived and experimentally verified. It was observed that prototype thermal behavior could be predicted, from model data, to within 10°F over the temperature range tested (140 to 330°F). Heat pipe failure due to capillary failure was also predictable to within ±10%.

A flexible heat pipe was also designed and experimentally tested. Performance was not degraded under conditions of bending.

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a = a constant in the equation $k = k_1 T^a$

B = $2\varepsilon\sigma\alpha^{\gamma}/k_1\delta\beta$

C = a constant in Equation (8)

G = $2\varepsilon\sigma k_1/\beta\delta$

k = thermal conductivity, btu/(hr sqft F)

k_1 = a constant in the equation $k = k_1 T^a$

L = length, ft

p = temperature gradient

q'' = heat flux, btu/(hr sqft)

T = absolute temperature, °R

x = variable distance, ft

<u>Greek</u>

α = $a + 1$

β = $a + 5$

γ = β/α

Γ = θ/θ_m

δ = fin half-thickness, ft

ε = fin emittance

θ = defined in Equation (2)

ξ = θ/θ_∞

σ = Stefan-Boltzmann constant, btu/(hr sqft R^4)

$\uparrow$ = defined in Equation (20)

ϕ = θ/θ_L

ψ = $\theta/\theta_L \uparrow$

Subscripts

L = property value at the outer end of the fin

m = property value at the minimum fin temperature

o = property value at the fin root

∞ = thermal sink property

SECTION I

INTRODUCTION

Thermal scale modeling is a valuable tool in the design of aerospace systems. Significant savings can be achieved if pre-flight testing can be performed in small facilities with reduced size models. Extensive efforts have been made to derive the basic thermal scaling criteria for transient and steady-state conditions. The general laws of thermal similitude have been derived and some experimental work performed which supports these derivations. There are, however some exceptions that cause difficulties.

The studies performed in this contract attempted to clarify some of the limitations imposed by the thermal modeling laws. The general areas of investigation include:

1. Determination of the limits of the dimensional ratios between model and prototype for both similar and distorted models.
2. Modeling errors caused by material property variations with temperature, or by large temperature differences between model and prototype.
3. Validity of scaling when prototype temperatures are very low.
4. Validity of scaling active heat transfer devices such as a heat pipe.
5. Effects encountered in complex radiative/conductive interchanges when geometry is distorted in the model.

The first three of these studies were accomplished analytically using an exact analysis of a radiating fin as the prototype/model system. These studies are the subject of Volume 1 of this report.

The fourth study was accomplished both analytically and experimentally by deriving the thermal scaling laws for a heat pipe and applying them to the design of a heat pipe prototype/model system which experimentally verified the results. Incidental to this study was the experimental development of a flexible heat pipe. The flexible heat pipe work was undertaken preliminary to the heat pipe modeling program and served to provide initial guidance and experience in the technology of heat pipes. These studies are the subject of Volume 2 of this report.

The fifth study was not successfully accomplished.

SECTION II

THERMAL SCALE MODELING

The basis for the thermal modeling of a physical system is the maintenance of thermal similitude between model and prototype. Thermal similitude is maintained by preserving the numerical equality of certain dimensionless groups of properties of the prototype and model. These dimensionless groups, or similarity parameters, must contain all the physical quantities which interact to determine the thermal behavior of the system.

The derivation of the similarity parameters has been performed by several authors (1,2,3,4). Perhaps the most useful to this study was the derivation by Miller and Wiebelt (4) who derived the set of similarity parameters given in Table I. The "starred" values represent the model/prototype value of the variable ($T^* = T_{model}/T_{prototype}$, etc).

The assumptions necessary for the derivation of this set of parameters are as follows:

1. Steady state heat transfer.
2. System shape is a solid cylinder.
3. Surface radiation properties of model and prototype are uniform and equal.
4. There is no simulation of solar radiation.
5. Model and prototype are made of the same material.
6. Material thermal conductivity varies with temperature according to the relation (with a = 0 conductivity is constant).

TABLE I

Similarity Parameters For Solid Cylindrical Systems

Parameter	Method 1	Method 2	Method 3
q*	$\left[\frac{R^{*(7-a)}}{L^{*(5+a)}}\right]^{\frac{1}{3-a}}$	L^{*3}	$\left[\frac{R^{*7}}{L^{*5}}\right]^{\frac{1}{3}}$
T*	$\left[\frac{R^{*}}{L^{*2}}\right]^{\frac{1}{3-a}}$	1	$\left[\frac{R^{*}}{L^{*2}}\right]^{\frac{1}{3}}$
L*	- - -	$R^{*\frac{1}{2}}$	- - -

$$k = k_1 T^a \tag{1}$$

The method 1 parameters take into account property variations with temperature and permits the prototype and model to be at different temperatures if desired. The method 2 parameters are a special case of method 1 where the model is distorted ($R^* = L^{*2}$) so that model and prototype temperatures are the same, and method 3 parameters are the special case, with $a = 0$, of constant thermal conductivity.

As described, the thermal modeling parameters presented in Table I are written for a rather restricted set of circumstances. It is quite possible to present a much more general set of modeling parameters. Unfortunately it has not been possible to utilize these general parameters to obtain a general method of error prediction for thermal modeling. However, through a study of the prediction errors of the more restricted fin systems it will be possible to obtain a better insight into the errors involved in the use of the general modeling parameters.

Section III of this report includes a derivation of equations which describe the steady state temperature distribution and heat transfer rates in a radiating fin. These equations are used in Section IV to determine errors involved in the thermal modeling of a fin using the parameters listed in Table I.

SECTION III

STEADY-STATE ANALYSIS OF A RADIATIVE FIN HAVING TEMPERATURE-DEPENDENT CONDUCTIVITY

1. INTRODUCTION

This section presents a steady-state analysis of radiating fins which have uniform cross section, constant emittance, temperature-dependent thermal conductivity, and either zero or non-zero thermal sink boundary conditions. The analysis is presented as a series of eight different cases of boundary conditions applied to the fin.

An analysis not previously presented in the literature, even for a fin with constant thermal conductivity, is presented as Case VII. Case VI describes a fin of finite length which is radiating to a non-zero sink and which has fixed but unequal temperatures on the ends. This solution is particularly useful in calculating energy losses from a thermal model in a space simulation chamber to its instrumentation and support wires.

Steady-state analyses of radiating fins with constant thermal conductivity have been presented by several authors [6, 7]. One previous paper considered a finite length fin with insulated end and a conductivity which varied linearly with temperature, similar to Case VI of this article [8].

2. ANALYSIS

Let us consider the steady-state behavior of a fin of variable thermal conductivity having a uniform root temperature of T_o, protruding into an evacuated "black" space having an effective uniform radiating temperature of T_∞. Figure 1 shows the dimensions of a typical uniform cross section rectangular fin, infinitely wide in the "z" direction. We consider only a unit width portion of the fin so that edge effects may be neglected. We

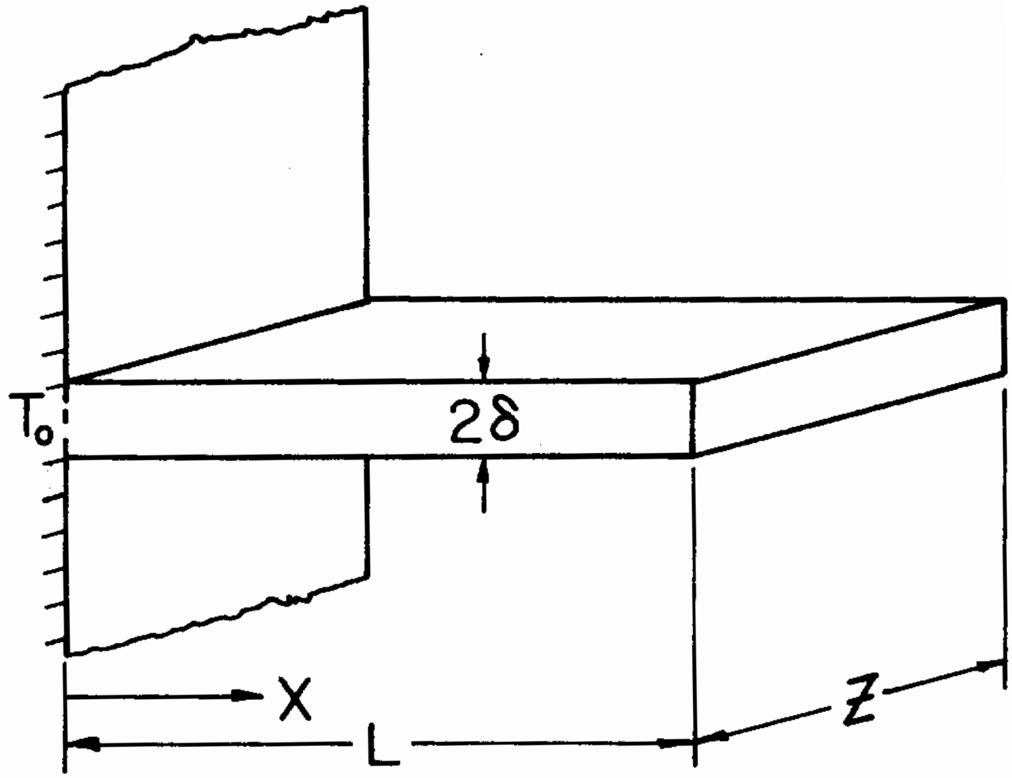

FIGURE 1. RADIATING FIN

further consider the fin to be "thin" relative to the length and width so that the heat transfer is, for practical purposes, one dimensional. An energy balance on an elemental section of the fin results in the following differential equation describing the temperature distribution in the fin as a function of distance from the root.

$$\frac{d}{dx}\left(k\,\frac{dT}{dx}\right) - \frac{\varepsilon\sigma}{\delta}\left(T^4 - T_\infty^4\right) = 0 \qquad (1)$$

This analysis assumes there is no radiant interchange between the fin and its base and that there is no external source of radiant flux such as a solar simulator. If, however, these fluxes are present and are not functions of x, they may be included in the T_∞^4 term by an appropriate additive factor.

Equation (1) applies also to a fin of circular cross-section (pin-fin) if one-fourth of the diameter of the fin is used instead of the rectangular fin half-thickness (δ).

For convenience, let us define a new variable, θ, as

$$\theta = \frac{1}{k_1}\int_0^T k\,dT \qquad (2)$$

where k_1 is a constant, then

$$k_1\,\frac{d\theta}{dx} = k\,\frac{dT}{dx} \qquad (3)$$

The variable θ obviously depends upon the relationship between the thermal conductivity and the temperature. One relationship which has proved useful over a reasonable range of temperatures, for various metals, is of the form described by the equation

$$k = k_1 T^a \qquad (4)$$

where k_1 and a are constants [4,5].

By substitution of Equation (4) into Equation (2) a relationship between θ and T is obtained.

$$T = (\alpha\theta)^{\frac{1}{\alpha}} \qquad \alpha>0 \tag{5}$$

where $\alpha = a + 1$

Equation (1) then becomes

$$\frac{d^2\theta}{dx^2} - \frac{\varepsilon\sigma\alpha^{(\gamma-1)}}{k_1\delta}\left[\theta^{(\gamma-1)} - \theta_\infty^{(\gamma-1)}\right] = 0 \tag{6}$$

where $\gamma = \dfrac{a + 5}{a + 1}$

A first indefinite integration of this equation leads to

$$\frac{d\theta}{dx} = \pm\left[B\left(\theta^\gamma - \gamma\theta\theta_\infty^{(\gamma-1)}\right) + C\right]^{\frac{1}{2}} \tag{7}$$

where $B = \dfrac{2\varepsilon\sigma\alpha^\gamma}{k_1\delta\beta}$

$\beta = a + 5$

Equation (7) may be integrated from θ_o to θ as x goes from 0 to x to give

$$x = \pm\int_{\theta_o}^{\theta} \frac{d\theta}{\left[B\left(\theta^\gamma - \gamma\theta\theta_\infty^{(\gamma-1)}\right) + C\right]^{\frac{1}{2}}} \tag{8}$$

Equation (8) requires the application of a boundary condition to determine the constant C and the proper sign before it may be integrated. The situation to be presented as Case I allows Equation (8) to be formally integrated, and for Case II, a solution in terms of complete and incomplete Beta functions has been given by Chen [6]. For most cases to be considered in this article, however, there is apparently no closed form solution to Equation (8). The use of digital computers for numerical integration, together with, in some cases, an iteration technique, makes possible a solution to the various forms of Equation (8) which result from the application of boundary conditions from a particular problem.

Table I presents a summary of eight sets of boundary conditions and the corresponding values of the constant C to be substituted into Equation (8) before integration. For all cases, it is required that the fin root temperature T_o be greater than T_∞, and in Cases V, VI and VIIa it is required that $T_\infty \leq T_L < T_o$.

Case I

Case I considers an infinite length fin radiating to a zero temperature sink. For this case the temperature and the temperature gradient must both approach zero as x approaches infinity. For these boundary conditions the constant C is zero and Equation (8) may be integrated. The result is, in terms of T and T_o,

$$x = \left(\frac{2k_1\delta\beta}{\varepsilon\sigma(3-a)^2}\right)^{\frac{1}{2}}\left(T^{-\frac{3-a}{2}} - T_o^{-\frac{3-a}{2}}\right) \qquad (a<3) \qquad (9)$$

The heat flux, per unit fin width, at the fin root is, in terms of T and T_o,

$$q_o'' = [G\, T_o^{\beta}]^{\frac{1}{2}} \qquad (10)$$

$$\text{where } G = \frac{2\varepsilon\sigma k_1}{\delta\beta}$$

Case II

For Case II, the fin is of finite length with the end insulated. The temperature at the end of the fin, T_L, is unknown and must be determined by an iteration procedure. By first applying the boundary conditions from Table I to Equation (7) to determine C, Equation (8) may be written as follows.

$$x = -\int_{\theta_o}^{\theta} \frac{d\theta}{[B(\theta^{\gamma} - \theta_L^{\gamma})]^{\frac{1}{2}}} \tag{11}$$

Let us define Φ as follows, recognizing that θ_L has a fixed value for a particular problem.

$$\Phi = \frac{\theta}{\theta_L} \tag{12}$$

By substitution of this relation into Equation (11), inverting the limits and expanding the integral, there results

$$x = \left(\frac{1}{B\theta_L^{(\gamma-2)}}\right)^{\frac{1}{2}} \left\{\int_1^{\Phi_o} \frac{d\Phi}{(\Phi^{\gamma} - 1)^{\frac{1}{2}}} - \int_1^{\Phi} \frac{d\Phi}{(\Phi^{\gamma} - 1)^{\frac{1}{2}}}\right\} \tag{13}$$

By noting that $\Phi \rightarrow \Phi_L = 1$ as $x \rightarrow L$, Equation (13) may be reduced to

$$L = \left(\frac{1}{B\theta_L^{(\gamma-2)}}\right)^{\frac{1}{2}} \int_1^{\Phi_o} \frac{d\Phi}{(\Phi^{\gamma} - 1)^{\frac{1}{2}}} \tag{14}$$

The integral in Equation (14) may be evaluated numerically without great difficulty. Caution must be exercised in evaluating the function

$$f(\Phi) = \frac{1}{(\Phi^{\gamma} - 1)^{\frac{1}{2}}} \tag{15}$$

near $\Phi = 1$ so that sufficient accuracy is obtained.

Once θ_L is determined from Equation (14), the temperature at the end of the fin, T_L, is known and x may be determined for any desired temperature between T_o and T_L from Equation (13). The heat flux, per unit fin width, at the fin root may be shown to be, in terms of T_o and T_L,

$$q_o'' = [G(T_o^\beta - T_L^\beta)]^{\frac{1}{2}} \tag{16}$$

<u>Case III</u>

The constant C for the fin of finite length with $T_\infty = 0$ and a radiating end, Case III, may be determined by equating the radiative flux to the conductive flux at the end of the fin, thus obtaining an expression for the thermal gradient at $x = L$.

$$q_L'' = -k_1 \left.\frac{d\theta}{dx}\right|_{x=L} = \varepsilon\sigma T_L^4 = \varepsilon\sigma[\alpha\theta_L]^{(\gamma-1)} \tag{17}$$

Using this expression and Equation (7), the result is:

$$C = -B\theta_L^\gamma\left(1 - \frac{\varepsilon\sigma\delta\beta}{2k_1}[\alpha\theta_L]^{(\gamma-2)}\right) \tag{18}$$

By making the substitution of C into Equation (8), and the additional substitutions of

$$\psi = \frac{\theta}{\theta_L \uparrow} \tag{19}$$

$$\uparrow = \left(1 - \frac{\delta\varepsilon\sigma\beta}{2k_1}[\alpha\theta_L]^{(\gamma-2)}\right)^{\frac{1}{\gamma}} \tag{20}$$

and expanding the integral, there results

$$x = \left(\frac{\uparrow^{(2-\gamma)}}{B\theta_L^{(\gamma-2)}}\right)^{\frac{1}{2}}\left(\int_1^{\psi_o}\frac{d\psi}{(\psi^\gamma - 1)^{\frac{1}{2}}} - \int_1^{\psi}\frac{d\psi}{(\psi^\gamma - 1)^{\frac{1}{2}}}\right) \tag{21}$$

By noting that as $x \to L$, $\psi \to \frac{1}{\uparrow}$ and $\theta \to \theta_L$, Equation (21) may be reduced to

$$L = \left(\frac{\uparrow^{(2-\gamma)}}{B\theta_L^{(\gamma-2)}}\right)^{\frac{1}{2}}\left(\int_1^{\psi_o}\frac{d\psi}{(\psi^\gamma - 1)^{\frac{1}{2}}} - \int_1^{\psi_L = \frac{1}{\uparrow}}\frac{d\psi}{(\psi^\gamma - 1)^{\frac{1}{2}}}\right) \tag{22}$$

from which θ_L and thus $\uparrow$ may be determined by iteration. Once θ_L is determined T_L is known and the location x of any temperature between T_o and

T_L may be determined from Equation (21).

The heat flux, per unit fin width, at the root of the fin may be shown to be

$$q_o'' = [G(T_o^\beta - T_L^\beta) + (\epsilon\delta T_L^4)^2]^{\frac{1}{2}} \tag{23}$$

Case IV

The temperature distribution in a fin radiating to a zero temperature sink and having both end temperatures fixed at unequal values ($T_o > T_L$) is considered next. The gradient at x = L is fixed by the boundary conditions but is unknown and may be either positive or negative, depending upon the fin properties and the end temperatures.

Applying the condition $\theta_\infty = 0$, Equation (7) may be written as

$$\left.\frac{d\theta}{dx}\right|_{x=L} = p_L = \pm(B\theta_L^\gamma + C)^{\frac{1}{2}} \tag{24}$$

from which

$$C = p_L^2 - B\theta_L^\gamma \tag{25}$$

Equation (8) may then be written as

$$x = \pm\left(\frac{1}{B\theta_L^{(\gamma-2)}}\right)^{\frac{1}{2}} \int_\phi^{\phi_o} \frac{d\phi}{\left(\phi^\gamma - 1 + \frac{p_L^2}{B\theta_L^\gamma}\right)^{\frac{1}{2}}} \tag{26}$$

$$\text{where } \phi = \frac{\theta}{\theta_L} \tag{27}$$

We shall consider three special cases of boundary conditions for which this equation must be solved.

Figure 2 depicts the temperature distribution in a radiating fin with all problem conditions fixed except fin length. In Figure 2a the fin is

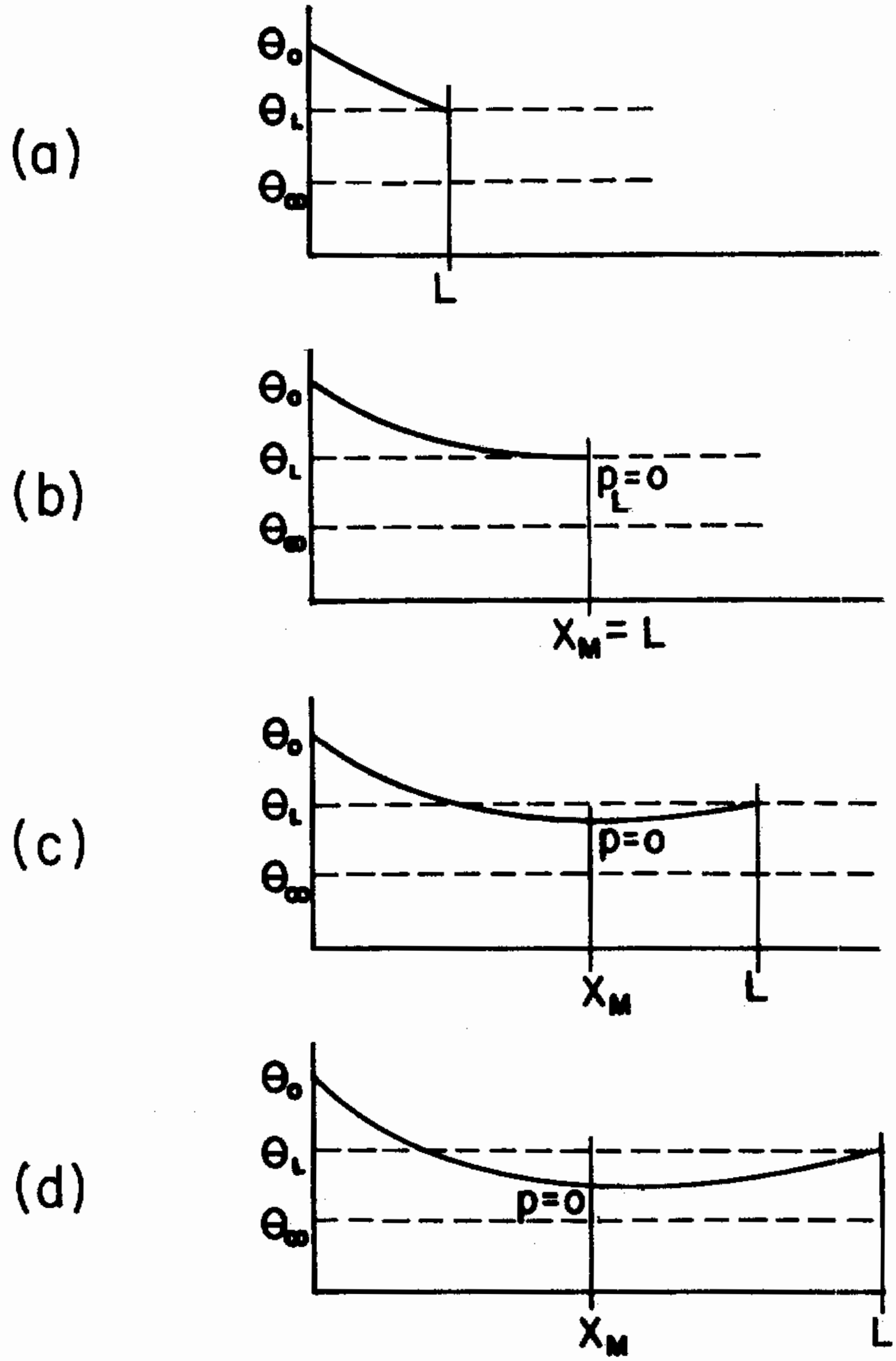

FIGURE 2. FIN TEMPERATURE DISTRIBUTION

short enough so that for any x, $\frac{d\theta}{dx} < 0$. Figure 2b depicts the precise length such that $\left.\frac{d\theta}{dx}\right|_{x=L} = 0$, which is identical to Case II. As the fin is progressively lengthened, the temperature passes through some minimum value less than θ_L and begins to increase with x, rather than decrease. These three cases will be called case IVa, IVb and IVc respectively.

To determine which case is actually present under specified conditions, the following procedure is used:

1. Solve Case II, given θ_o and θ_L for the length of an insulated end fin (L_2).

2. If L_2 is greater than the specified length of the fin for the Case IV problem (L_4), then $\frac{d\theta}{dx}$ is everywhere negative, and Equation (26) may be written

$$L = \left(\frac{1}{B\theta_L^{(\gamma-2)}}\right)^{\frac{1}{2}} \int_1^{\phi_o} \frac{d\phi}{\left(\phi^\gamma - 1 + \frac{p_L{}^2}{B\theta_L^\gamma}\right)^{\frac{1}{2}}} \tag{28}$$

This equation may be used with an integration-iteration process to solve for the unknown p_L, and then Equation (26) is used to solve for the location x of any desired temperature $\theta_o < \theta < \theta_L$. The heat flux at the root of the fin is

$$q_o'' = [G(T_o^\beta - T_L^\beta) + k_1^2 p_L^2]^{\frac{1}{2}} \tag{29}$$

The heat flux at the outer end of the fin is

$$q_L'' = k_1 p_L \tag{30}$$

3. If $L_2 = L_4$ then Case II exists.

4. If $L_2 < L_4$ then a minimum temperature exists at some location in the fin. At that minimum, $\left.\frac{d\theta}{dx}\right|_{x=x_m} = 0$. Using Equation (7) and giving the temperature at the minimum the subscript m, then

$$C = -B\theta_m^{\gamma} \tag{31}$$

Introducing the limits to equation (8) and expanding it to include the effects of both positive and negative gradients, there results

$$L = \left(\frac{1}{B\theta_m^{(\gamma-2)}}\right)^{\frac{1}{2}} \left[\int_1^{\Gamma_o} \frac{d\Gamma}{(\Gamma^{\gamma} - 1)^{\frac{1}{2}}} + \int_1^{\Gamma_L} \frac{d\Gamma}{(\Gamma^{\gamma} - 1)^{\frac{1}{2}}}\right] \tag{32}$$

where

$$\Gamma = \frac{\theta}{\theta_m} \tag{33}$$

The unknown θ_m may be solved for by an integration-iteration process as previously described. Once θ_m is known, the location of the minimum temperature is determined from

$$x_m = \left(\frac{1}{B\theta_m^{(\gamma-2)}}\right)^{\frac{1}{2}} \int_1^{\Gamma_o} \frac{d\Gamma}{(\Gamma^{\gamma} - 1)^{\frac{1}{2}}} \tag{34}$$

and then the temperature distribution along the fin may be determined from the appropriate equation below.

for $\begin{pmatrix} 0 < x < x_m \\ 1 < \Gamma < \Gamma_o \end{pmatrix}$

$$x = \left(\frac{1}{B\theta_m^{(\gamma-2)}}\right)^{\frac{1}{2}} \int_{\Gamma}^{\Gamma_o} \frac{d\Gamma}{(\Gamma^{\gamma} - 1)^{\frac{1}{2}}} \tag{35}$$

for $\begin{pmatrix} x_m < x < L \\ 1 < \Gamma < \Gamma_L \end{pmatrix}$

$$x = x_m + \left(\frac{1}{B\theta_m^{(\gamma-2)}}\right)^{\frac{1}{2}} \int_1^{\Gamma} \frac{d\Gamma}{(\Gamma^{\gamma} - 1)^{\frac{1}{2}}} \tag{36}$$

The heat flux at the root of the fin may be determined from

$$q_o'' = [G(T_o^\beta - T_m^\beta)]^{\frac{1}{2}} \tag{37}$$

and the heat flux at the outer end may be written

$$q_L'' = [G(T_L^\beta - T_m^\beta)]^{\frac{1}{2}} \tag{38}$$

Case V

For the case of infinite length fin where $T_\infty \neq 0$, let us define a new variable ξ as

$$\xi = \frac{\theta}{\theta_\infty} \tag{39}$$

and use the boundary conditions $\theta \rightarrow \theta_\infty$ and $\frac{d\theta}{dx} \rightarrow 0$ as $x \rightarrow \infty$ in Equation (7) to obtain:

$$C = -B\theta_\infty^\gamma(1 - \gamma) \tag{40}$$

Equation (8) may then be reduced to

$$x = \left(\frac{1}{B\theta_\infty^{(\gamma-2)}}\right)^{\frac{1}{2}} \int_\xi^{\xi_o} \frac{d\xi}{[\xi^\gamma - 1 - \gamma(\xi - 1)]^{\frac{1}{2}}} \tag{41}$$

Both θ_∞ and ξ_o are known so that this equation may be solved numerically for the location x of any temperature between T_o and T_∞. The heat flux at the base may be shown to be, in terms of T_o and T_∞

$$q_o'' = \left\{G[T_o^\beta - \gamma T_\infty^4 T_o^\alpha + (\gamma - 1)T_\infty^\beta]\right\}^{\frac{1}{2}} \tag{42}$$

Case VI

Case VI is for a fin of finite length, radiating to a non-zero temperature sink and having the free end insulated. By letting

$$\xi = \frac{\theta}{\theta_\infty} \tag{43}$$

and using the boundary condition $\left.\frac{d\theta}{dx}\right|_{x=L} = 0$ we obtain the constant of integration C in terms of an unknown ξ_L. The resulting relation is given in Table I. This result may be substituted into Equation (8) to obtain

$$x = \left[\frac{1}{B\theta_\infty^{(\gamma-2)}}\right]^{\frac{1}{2}} \int_{\xi}^{\xi_o} \frac{d\xi}{[\xi^\gamma - \xi_L^\gamma - \gamma(\xi - \xi_L)]^{\frac{1}{2}}} \tag{44}$$

The limiting conditions are $\xi \rightarrow \xi_L$ as $x \rightarrow L$, with which we obtain

$$L = \left[\frac{1}{B\theta_\infty^{(\gamma-2)}}\right]^{\frac{1}{2}} \int_{\xi_L}^{\xi} \frac{d\xi}{[\xi^\gamma - \xi_L^\gamma - \gamma(\xi - \xi_L)]^{\frac{1}{2}}} \tag{45}$$

The unknown ξ_L may be determined from Equation (45) by a numerical iterative technique. The second law of thermodynamics requires that ξ_L exceed unity.

$$q_o'' = \{G[T_L^\beta - \gamma T_\infty^4(T_o^\alpha - T_L^\alpha)]\}^{\frac{1}{2}} \tag{46}$$

Case VII

Case VII presents a more difficult problem and will be considered in three separate parts. First, however, we derive the temperature distribution equation by again applying the boundary conditions to Equations (7) and (8). The temperature and the temperature gradient at the end of the finite length fin are not zero and the gradient is unknown. Equation (7) may be used to obtain a relation for the constant C. By defining:

$$\xi = \frac{\theta}{\theta_\infty} \tag{47}$$

the result is

$$C = p_L^2 - B\theta_\infty^\gamma[\xi_L^\gamma - \gamma\xi_L \tag{48}$$

Equation (8) then may be written

$$x = \left(\frac{1}{B\theta_\infty^{(\gamma-2)}}\right)^{\frac{1}{2}} \int_{\xi}^{\xi_o} \frac{d\xi}{\left(\xi^\gamma - \xi_L^\gamma - \gamma(\xi - \xi_L) + \frac{p_L^2}{B\theta_\infty^\gamma}\right)^{\frac{1}{2}}} \qquad (49)$$

The first situation we shall consider is for $\theta_\infty > \theta_L$, designated Case VIIa. The variable ξ ranges in value from greater than unity to less than unity as the temperature decreases along the fin. When the fin temperature is above T_∞ the fin loses heat to the surroundings and the opposite is true when the fin temperature is below T_∞.

At some value of x along the fin there is an inflection point in the temperature vs. length curve, at which point the absolute value of the slope of the curve is a minimum, but not zero. Temperature is thus a single-valued function of distance. For this situation, the integral limits of Equation (49) are ξ_o and ξ_L at x = 0 and x = L, respectively.

$$L = \left(\frac{1}{B\theta_\infty^{(\gamma-2)}}\right)^{\frac{1}{2}} \int_{\xi_L}^{\xi_o} \frac{d\xi}{\left(\xi^\gamma - \xi_L^\gamma - \gamma(\xi - \xi_L) + \frac{p_L^2}{B\theta_\infty^\gamma}\right)^{\frac{1}{2}}} \qquad (50)$$

The value of p_L may thus be determined, by an iteration process, from Equation (50). Once p_L is known, Equation (49) may be used to determine the location of any temperature which exists along the fin.

The heat flux at the fin rcot, per unit of fin width, may be written as follows:

$$q_o'' = \left(G[T_o^\beta - T_L^\beta - T_\infty^{\ 4}\gamma(T_o^{\ \alpha} - T_L^{\ \alpha}) + k_1^2\ p_L^2]\right)^{\frac{1}{2}} \qquad (51)$$

For the situation of Case VIIb where $\theta_L = \theta_\infty$, the governing Equations (48) through (51) are changed only to the extent that $\xi_L = 1$. The procedure for a solution is the same as previously discussed under Case VIIa. This situation approaches Case V if the fin is long and the conductance is low.

If $\theta_L > \theta_\infty$, Case VIIc, the situation becomes more complex because the temperature distribution along the fin may be double-valued in x. Figure 2 shows the situation of an imaginary fin of fixed thermal properties but various lengths. In Figure 2a, the fin is short enough that for any x, $\frac{d\theta}{dx} < 0$. Figure 2b depicts the precise length such that $\left.\frac{d\theta}{dx}\right|_{x=L} = 0$, which is identical to Case VI. As the fin is progressively lengthened the temperature passes through some minimum value less than θ_L and begins to increase again.

The series of steps which will lead to a solution to this problem are as follows:

1. Solve Case VI, given ξ_L, for the length of an insulated end fin (L_6).
2. If L_6 is greater than the specified length of the fin for the Case VII problem (L_7), then the situation in Figure 2a exists and Equations (49) and (50) may be used to determine the temperature distribution.
3. If it happens that $L_6 = L_7$ then Case VI exists and Equations (44) and (46) are used to determine the temperature and heat flux.
4. If $L_6 < L_7$ then the fin temperature goes below θ_L somewhere along the fin. The location of the point of minimum temperature is unknown but the temperature gradient at that value of x must be zero. Using Equation (7) and giving the temperature at that point a subscript of m, then

$$C = -B\theta_\infty^\gamma(\xi_m^\gamma - \gamma\xi_m) \tag{52}$$

Introducing the limits to Equation (8) and expanding it to include the effects of both positive and negative gradients, there results

$$L = \left[\frac{1}{B\theta_\infty^{(\gamma-2)}}\right]^{\frac{1}{2}} \int_{\xi_m}^{\xi_o} \frac{d\xi}{\left[\xi^\gamma - \xi_m^\gamma - \gamma(\xi - \xi_m)\right]^{\frac{1}{2}}} + \int_{\xi_m}^{\xi_L} \frac{d\xi}{\left[\xi^\gamma - \xi_m^\gamma - \gamma(\xi - \xi_m)\right]^{\frac{1}{2}}} \tag{53}$$

wherein

$$\xi_L > \xi_m \geq 1$$

Equation (53) must be solved for ξ_m by iteration. Once ξ_m is determined, x_m, the location of the minimum temperature may be determined from:

$$x_m = \left[\frac{1}{B\theta_\infty^{(\gamma-2)}}\right]^{\frac{1}{2}} \int_{\xi_m}^{\xi_o} \frac{d\xi}{\left[\xi^\gamma - \xi_m^\gamma - \gamma(\xi - \xi_m)\right]^{\frac{1}{2}}} \tag{54}$$

The temperature distribution along the fin may then be determined from the appropriate equation below.

$$\text{for} \quad \begin{bmatrix} 0 < x < x_m \\ \xi_m < \xi < \xi_o \end{bmatrix} \qquad x = \left[\frac{1}{B\theta_\infty^{(\gamma-2)}}\right]^{\frac{1}{2}} \int_{\xi}^{\xi_o} \frac{d\xi}{\left[\xi^\gamma - \xi_m^\gamma - \gamma(\xi - \xi_m)\right]^{\frac{1}{2}}} \tag{55}$$

$$\text{for} \quad \begin{bmatrix} x_m < x < L \\ \xi_m < \xi < \xi_L \end{bmatrix} \qquad x = x_m + \left[\frac{1}{B\theta_\infty^{(\gamma-2)}}\right]^{\frac{1}{2}} \int_{\xi_m}^{\xi} \frac{d\xi}{\left[\xi^\gamma - \xi_m^\gamma - \gamma(\xi - \xi_m)\right]^{\frac{1}{2}}} \tag{56}$$

For Case VIIc, the equation for the heat flux at the fin root, per unit fin width, is

$$q_o'' = \left(G[(T_o^{\beta} - T_m^{\beta}) - T_\infty^4 \gamma (T_o^{\alpha} - T_m^{\alpha})]\right)^{\frac{1}{2}} \quad (57)$$

and the heat flux at the outer end of the fin is

$$q_L'' = -\left(G[(T_L^{\beta} - T_m^{\beta}) - T_\infty^4 \gamma (T_L^{\alpha} - T_m^{\alpha})]\right)^{\frac{1}{2}} \quad (58)$$

Case VIII

This case is similar to Case VI except that the end of the fin is radiating instead of being insulated. As before, let

$$\xi = \frac{\theta}{\theta_\infty} \quad (59)$$

and use the radiating boundary condition at the end of the fin

$$\left.\frac{d\theta}{dx}\right|_{x=L} = -\frac{\varepsilon\sigma[\alpha\theta_\infty]^{(\gamma-1)}}{k_1}[\xi_L^{(\gamma-1)} - 1] \quad (60)$$

to obtain the constant C in terms of an unknown ξ_L. The equation for C is given in Table I.

This result may be substituted into Equation (9) to obtain

$$x = \left(\frac{1}{B\theta_\infty^{(\gamma-2)}}\right)^{\frac{1}{2}} \int_{\xi}^{\xi_o} \frac{d\xi}{\left\{\xi^{\gamma} - \xi_L^{\gamma} - \gamma(\xi - \xi_L) + \frac{\varepsilon\sigma\delta\beta}{2k_1}(\alpha\theta_\infty)^{(\gamma-2)}(\xi_L^{(\gamma-1)} - 1)^2\right\}^{\frac{1}{2}}} \quad (61)$$

The limiting conditions are $\xi \to \xi_L$ as $x \to L$, with which we obtain

$$L = \left(\frac{1}{B\theta_\infty^{(\gamma-2)}}\right)^{\frac{1}{2}} \int_{\xi_L}^{\xi_o} \frac{d\xi}{\left\{\xi^{\gamma} - \xi_L^{\gamma} - \gamma(\xi - \xi_L) + \frac{\varepsilon\sigma\delta\beta}{2k_1}(\alpha\theta_\infty)^{(\gamma-2)}(\xi_L^{(\gamma-1)} - 1)^2\right\}^{\frac{1}{2}}} \quad (62)$$

The unknown ξ_L may be determined from Equation (62) by a numerical iterative technique. As before, ξ_L must exceed unity.

The heat flux at the fin base may be written as

$$q_o'' = \left(G[T_o^{\beta} - T_L^{\beta} - \gamma T_\infty^4 (T_o^{\alpha} - T_L^{\alpha})] + [\epsilon\sigma(T_L^4 - T_\infty^4]^2\right)^{\frac{1}{2}} \quad (63)$$

and the heat flux at the end of the fin is

$$q_L'' = \epsilon\sigma(T_L^4 - T_\infty^4) \quad (64)$$

4. COMPUTER PROGRAMS

The solution of the equations for temperature distribution, root and outer end heat transfer rates, for all of the eight cases has been programmed for numerical computation on a digital computer. Then programs and their input and output format are given in the appendix.

TABLE II

Boundary Condition and Equation Constants for the Radiating Fin

Case	θ_∞	L	$\left.\frac{d\theta}{dx}\right\|_{x=max.}$	$\theta_{x=max.}$	C
I	0	∞	0	0	0
II	0	Finite	0	Unknown	$-B\theta_L^\gamma$
III	0	Finite	$-\frac{\epsilon\sigma}{k_1}[\alpha\ \theta_L]^{(\gamma-1)}$	Unknown	$-B\theta_L^\gamma\left(1-\frac{\epsilon\sigma\delta\beta}{2k_1}[\alpha\theta_L]^{(\gamma-2)}\right)$
IVa	0	Finite	Unknown	θ_L	$p_L^2 - B\theta_L^\gamma$
IVb	0	Finite	0	Unknown	$-B\theta_L^\gamma$
IVc	0	Finite	Unknown (+)	θ_L	$-B\theta_m^\gamma$
V	>0	∞	0	θ_∞	$-B\theta_\infty^\gamma(1-\gamma)$
VI	>0	Finite	0	Unknown	$-B\theta_\infty^\gamma[\xi_L^\gamma - \gamma\xi_L]$
VIIa	>0	Finite	Unknown	$\theta_L < \theta_\infty$	$-B\theta_\infty^\gamma[\xi_L^\gamma - \gamma\xi_L] + p_L^2$
VIIb	>0	Finite	Unknown	$\theta_L = \theta_\infty$	$-B\theta_\infty^\gamma[1-\gamma] + p_L^2$
VIIc	>0	Finite	Unknown	$\theta_L > \theta_\infty$	$-B\theta_\infty^\gamma[\xi_m^\gamma - \gamma\xi_m]$
VIII	>0	Finite	$-\frac{\epsilon\sigma[\alpha\theta_\infty]^{(\gamma-1)}}{k_1}[\xi_L^{(\gamma-1)}-1]$	θ_L	$-B\theta_\infty^\gamma[\xi_L^\gamma-\gamma\xi_L+\frac{\epsilon\sigma\delta\beta}{2k_1}(\alpha\theta_\infty)^{(\gamma-2)}(\xi_L^{(\gamma-1)}-1)^2]$

TABLE III

Temperature Distribution and Root Heat Transfer Equations for the Radiating Fin

Case	Variable	x	q''_o
I	T	$\left(\frac{2k_1\delta\beta}{\epsilon\sigma(3-a)^2}\right)^{\frac{1}{2}}\left(T^{\frac{a-3}{2}} - T_o^{\frac{a-3}{2}}\right)$	$[GT_o^{\beta}]^{\frac{1}{2}}$
II	$\theta = \frac{T^{\alpha}}{\alpha}$ $\phi = \frac{\theta}{\theta_L}$	$\left(\frac{1}{B\theta_L^{(\gamma-2)}}\right)^{\frac{1}{2}}\int_{\phi}^{\phi_o}\frac{d\phi}{(\phi^{\gamma}-1)^{\frac{1}{2}}}$	$[G(T_o^{\beta} - T_L^{\beta})]^{\frac{1}{2}}$
III	$\psi = \frac{\theta}{\theta_L\dagger}$ $\dagger = [1 - \frac{\epsilon\sigma\delta\beta}{2k_1}(\alpha\theta_L)^{(\gamma-2)}]^{\frac{1}{2}}$	$\left(\frac{\dagger(2-\gamma)}{B\theta_L^{(\gamma-2)}}\right)^{\frac{1}{2}}\int_{\psi}^{\psi_o}\frac{d\psi}{(\psi^{\gamma}-1)^{\frac{1}{2}}}$	$[G(T_o^{\beta} - T_L^{\beta}) + (\epsilon\sigma T_L^4)^2]^{\frac{1}{2}}$
IVa	$\phi = \frac{\theta}{\theta_L}$	$\left(\frac{1}{B\theta_L^{(\gamma-2)}}\right)^{\frac{1}{2}}\int_{\phi}^{\phi_o}\frac{d\phi}{\left(\phi^{\gamma} - 1 + \frac{p_L^2}{B\theta_L^{\gamma}}\right)^{\frac{1}{2}}}$	$[G(T_o^{\beta} - T_L^{\beta}) + k_1^2 p_L^2]^{\frac{1}{2}}$
IVb	Same as II		
IVc	$\Gamma = \frac{\theta}{\theta_m}$	$\left.\begin{matrix} 0 \le x \le x_m \\ 1 \le \Gamma \le \Gamma_o \end{matrix}\right\}$ $\left(\frac{1}{B\theta_m^{(\gamma-2)}}\right)^{\frac{1}{2}}\int_{\Gamma}^{\Gamma_o}\frac{d\Gamma}{(\Gamma^{\gamma}-1)^{\frac{1}{2}}}$ $\left.\begin{matrix} x_m \le x \le L \\ 1 \le \Gamma \le \Gamma_L \end{matrix}\right\}$ $x_m + \left(\frac{1}{B\theta_m^{(\gamma-2)}}\right)^{\frac{1}{2}}\int_{1}^{\Gamma}\frac{d\Gamma}{(\Gamma^{\gamma}-1)^{\frac{1}{2}}}$	$[G(T_o^{\beta} - T_m^{\beta})]^{\frac{1}{2}}$

Table III (continued)

Case	Variable	x	q''_o
V	$\xi = \frac{\theta}{\theta_\infty}$	$\left(\frac{1}{B\theta_\infty(\gamma-2)}\right)^{\frac{1}{2}} \int_{\xi}^{\xi_o} \frac{d\xi}{[\xi^\gamma - 1 - \gamma(\xi - 1)]^{\frac{1}{2}}}$	$[G(T_o^\beta - \gamma T_\infty^4 T_o^\beta + (\gamma-1)T_\infty^\beta)]^{\frac{1}{2}}$
VI	$\xi = \frac{\theta}{\theta_\infty}$	$\left(\frac{1}{B\theta_\infty(\gamma-2)}\right)^{\frac{1}{2}} \int_{\xi}^{\xi_o} \frac{d\xi}{[\xi^\gamma - \xi_L^\gamma - \gamma(\xi - \xi_L)]^{\frac{1}{2}}}$	$(G[T_o^\beta - T_L^\beta - T_\infty^4 \gamma(T_o^\alpha - T_L^\alpha)])^{\frac{1}{2}}$
VIIa	$\xi = \frac{\theta}{\theta_\infty}$	$\left(\frac{1}{B\theta_\infty(\gamma-2)}\right)^{\frac{1}{2}} \int_{\xi}^{\xi_o} \frac{d\xi}{\left(\xi^\gamma - \xi_L^\gamma - \gamma(\xi - \xi_L) + \frac{P_L^2}{B\theta_\infty^\gamma}\right)^{\frac{1}{2}}}$	$(G[T_o^\beta - T_L^\beta - T_\infty^4 \gamma(T_o^\alpha - T_L^\alpha)]$ $+ k_1^2 p_L^2)^{\frac{1}{2}}$
VIIb	$\xi = \frac{\theta}{\theta_\infty}$ $\xi_L = 1$	$\left(\frac{1}{B\theta_\infty(\gamma-2)}\right)^{\frac{1}{2}} \int_{\xi}^{\xi_o} \frac{d\xi}{\left(\xi^\gamma - 1 - \gamma(\xi - 1) + \frac{P_L^2}{B\theta_\infty^\gamma}\right)^{\frac{1}{2}}}$	
VIIc	$\xi = \frac{\theta}{\theta_\infty}$	$\left.\begin{matrix} 0 \le x \le x_m \\ \xi_m \le \xi \le \xi_o \end{matrix}\right\}$ $\left(\frac{1}{B\theta_\infty(\gamma-2)}\right)^{\frac{1}{2}} \int_{\xi}^{\xi_o} \frac{d\xi}{[\xi^\gamma - \xi_m^\gamma - \gamma(1 - \xi_m)]^{\frac{1}{2}}}$ $\left.\begin{matrix} x_m \le x \le L \\ \xi_m \le \xi \le \xi_L \end{matrix}\right\}$ $x_m + \left(\frac{1}{B\theta_\infty(\gamma-2)}\right)^{\frac{1}{2}} \int_{\xi_m}^{\xi} \frac{d\xi}{[\xi^\gamma - \xi_m^\gamma - \gamma(\xi - \xi_m)]^{\frac{1}{2}}}$	$(G[T_o^\beta - T_m^\beta - T_\infty^4 \gamma(T_o^\alpha - T_m^\alpha)])^{\frac{1}{2}}$

Table III (continued)

Case	Variable	x	q''_o
VIII	$\xi = \frac{\theta}{\theta_\infty}$	$\left(\frac{1}{B\theta_\infty^{(\gamma-2)}}\right)^{\frac{1}{2}} \int_{\xi}^{\xi_o} \frac{d\xi}{\left(\xi^\gamma - \xi_L^\gamma - \gamma(\xi - \xi_L) + \frac{\varepsilon\sigma\delta\beta}{2k_1}(\alpha\theta_\infty)^{(\gamma-2)}(\xi_L^{(\gamma-1)} - 1)^2\right)^{\frac{1}{2}}}$	$\left(G[T_o^\beta - T_L^\beta - \gamma T_\infty^4(T_o^\alpha - T_L^\alpha)] + [\varepsilon\sigma(T_L^4 - T_\infty^4)]^2\right)^{\frac{1}{2}}$

SECTION IV

LIMITATIONS ON THERMAL SCALE MODELING

1. DIMENSIONAL LIMITATIONS

The thermal modeling parameters listed as the method 1 parameters of Table I of Section II are exact. From a theoretical standpoint there are no limitations on the size of the model used to predict prototype parameters. From a practical standpoint however, there are limitations to the accuracy to which a model may be constructed.

To obtain information on the magnitude of prototype prediction errors which may be caused by model dimensional errors, a detailed study of the temperatures and heat fluxes in a radiating cylindrical fin will be made. The fin has a fixed root temperature and radiates to black surroundings from both the cylindrical surface and the outer end area. This is case VIII of Section III. Although the equations of Section III are derived for a rectangular fin, it may be shown that they apply to a cylindrical fin if one-fourth of the fin diameter is used in place of the rectangular fin half-thickness.

The fin will be considered to be made of 2024 annealed aluminum, for which Lucks and Deem (5) have reported values of thermal conductivity as a function of temperature. The reported conductivity of this material at 528 R is 103 Btu/hr-ft-R and an analysis of the data at other temperatures indicates that the equation

$$k = 29.37(T)^{0.1980} \tag{65}$$

will predict the conductivity within 2% over the temperature range of 210 to 860°R. See Appendix B for complete conductivity data.

The exact model of the prototype fin is to be one-fourth size and not distorted, that is the length and diameter are to be scaled in the same ratio. Prototype and model dimensions, temperatures, heat fluxes and other properties are given in Table IV.

Using the model data and the method 1 parameters of Table I it is possible to calculate prototype temperatures and fluxes. These results are given in Table V. By inspection it is seen that the prototype quantities calculated from the model results are negligibly different from those directly calculated for the prototype.

To demonstrate the prototype prediction errors caused by errors in the dimensions of the model, calculations of temperatures and fluxes were made for models which had length and diameter errors of 1,2,5,8 and 10 percent, both above and below the exact dimension. The prototype prediction results are shown in Figures 3 and 4. These figures show that zero prototype prediction error occurs with + 1% error in model diameter and-1% error in model length. If the modeling laws are exact this should not occur. The solutions were obtained by numerical methods, however, and this apparent discrepancy is caused by the numerical integration scheme used in obtaining the solution.

These figures show that a $\pm 5\%$ error in either of the model dimension will result in less that ± 4R error in the fin tip temperature and less than 5% errors in the fin root and tip heat transfer rates. The exact model is 0.500 inches in diameter and 6 inches in length so this percentage

TABLE IV

Prototype and Exact Fin Data

Parameter	Prototype	Model
Input Data		
Root temperature, T_o (°R)	530.0	869.2
Fin length, L (ft)	2.000	0.500
Fin diameter, D (in)	2.000	0.500
Emittance, ε	0.950	0.950
Exponent, a	0.1980	0.1980
Constant, k_1	29.37	29.37
Surrounds Temp., T_∞ (°R)	160.0	160.0
Calculated Data		
Outer end temp., T_L (°R)	483.5	792.3
Outer end heat transfer q_L (Btu/hr)	1.941	.8750
Root end heat transfer q_o (Btu/hr)	107.0	48.73

TABLE V

Calculated Data for Prototype Fin

Parameter	Prototype	Calculated Prototype	Error
T_o	530.0 (°R)	530.0 (°R)	0.0 (°R)
T_L	483.5 (°R)	483.2 (°R)	-0.3 (°R)
q_L	1.941 (Btu/hr)	1.935 (Btu/hr)	-0.3 (%)
q_o	107.0 (Btu/hr)	107.7 (Btu/hr)	+0.7 (%)

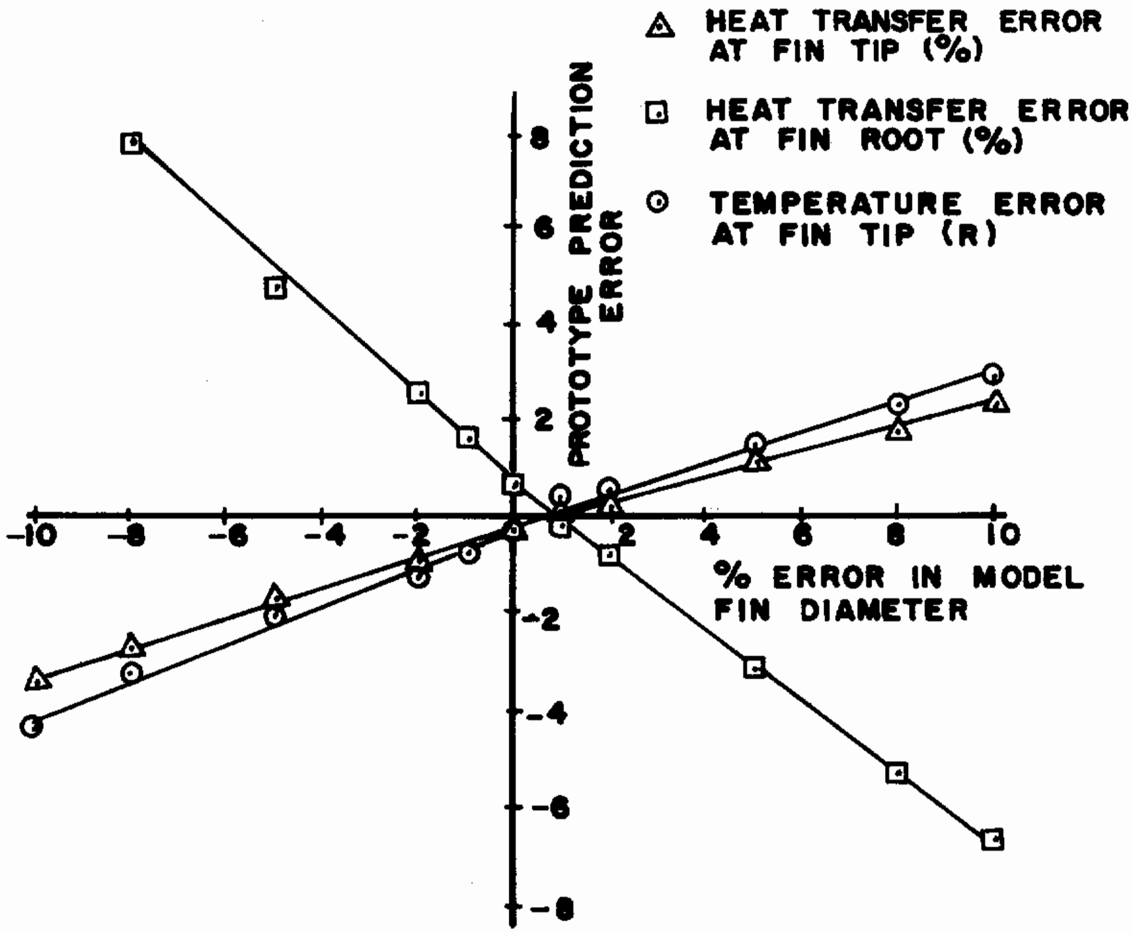

FIGURE 3. PROTOTYPE PREDICTION ERRORS CAUSED BY MODEL DIMENSION ERRORS (EXACT MODEL)

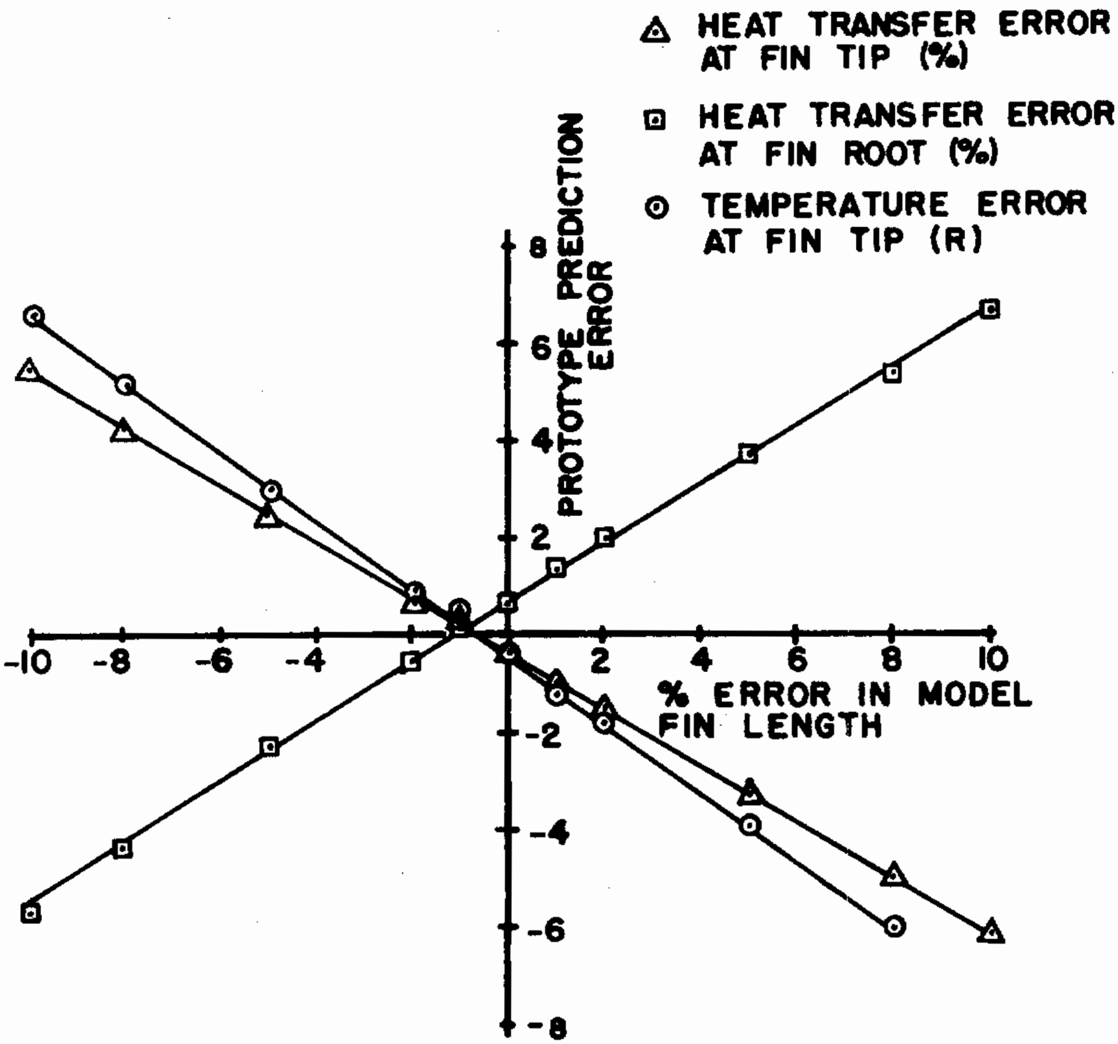

FIGURE 4. PROTOTYPE PREDICTION ERRORS CAUSED BY MODEL DIMENSION ERRORS (EXACT MODEL)

error amounts to 0.025 inches on the diameter and 0.30 inches on the length. From a manufacturing tolerance standpoint, these are very large errors, and the probability of their occurrence should be very small.

Combinations of length and diameter errors could occur and could either tend to null out the effects of each other if they were in opposite directions or reinforce each other if in the same direction.

2. DISTORTED MODEL ERRORS

To simplify the thermal modeling procedure, it would be convenient to have the model and prototype operate at the same temperature at homologous points. For models made of the same material as their prototype, this is possible only by distorting the model dimension, that is, not scaling by the same fraction in all directions. This procedure is not exact however, and the resulting prototype predictions are in error, the amount depending upon the degree of distortion of the model. The method 2 parameters of Table I were derived for the specific case under study here; a solid radiating cylinder. To obtain temperature equality between model and prototype at homologous locations the radius is scaled in proportion to the square of the length. To determine the amount of prototype prediction errors caused by model distortion, calculations were made for the distorted model-prototype system described in Table VI. The prototype is the same as used in the previous case of exact modeling. The model length is one-fourth of the prototype length and the model diameter is one-sixteenth of the prototype diameter. This is also a case VIII problem as previously described.

TABLE VI

Prototype and Distorted Model Data

Parameter	Prototype	Model	
Input Data			
T_o (°R)	530.0	530.0	
L (ft.)	2.000	0.500	
D (in.)	2.000	0.125	
ε	0.950	0.950	
a	0.1980	0.1980	
k_1	29.37	29.37	
T_∞ (°R)	160.0	160.0	
Calculated Data			
T_L (°R)	483.5	484.3	+0.8 (°R)
q_L (Btu/hr)	1.941	0.489	-74.8 (%)
q_o (Btu/hr)	107.0	106.2	-0.75 (%)

Figures 5 and 6 show prototype prediction errors for two of the fin parameters, the temperature at the fin tip and the heat transfer rate at the fin root. These two predictions are in error by approximately the same amounts as those for the exact model.

The heat transfer rate at the tip of the fin is approximately -75% in error, as shown in Table VI. The variation with model dimension errors is not so signigicant as the absolute error so the data are not shown graphically.

The cause of the error in the heat transfer at the fin tip is the distortion in the model. When the fin diameter is scaled as the square of the length, the radiation area at the tip of the fin is scaled by the fourth power of the length, resulting in the heat transfer rate at the fin tip also being scaled as the fourth power of the length since the temperature and emittance of model and prototype are equal. The cylindrical surface area of the fin, however, is scaled to the third power of the length. This results in the cylindrical surface radiation heat transfer being scaled to the third power of the length.

The conduction heat transfer rate is scaled according to the ratio, cross-sectional area/length since the temperatures are the same at homologous locations, making the thermal conductivity the same in model and prototype. The ratio, cross-sectional area/length reduces to a conduction heat transfer scaling of length to the third power, the same as the cylindrical surface area heat transfer but different than the radiating tip heat transfer.

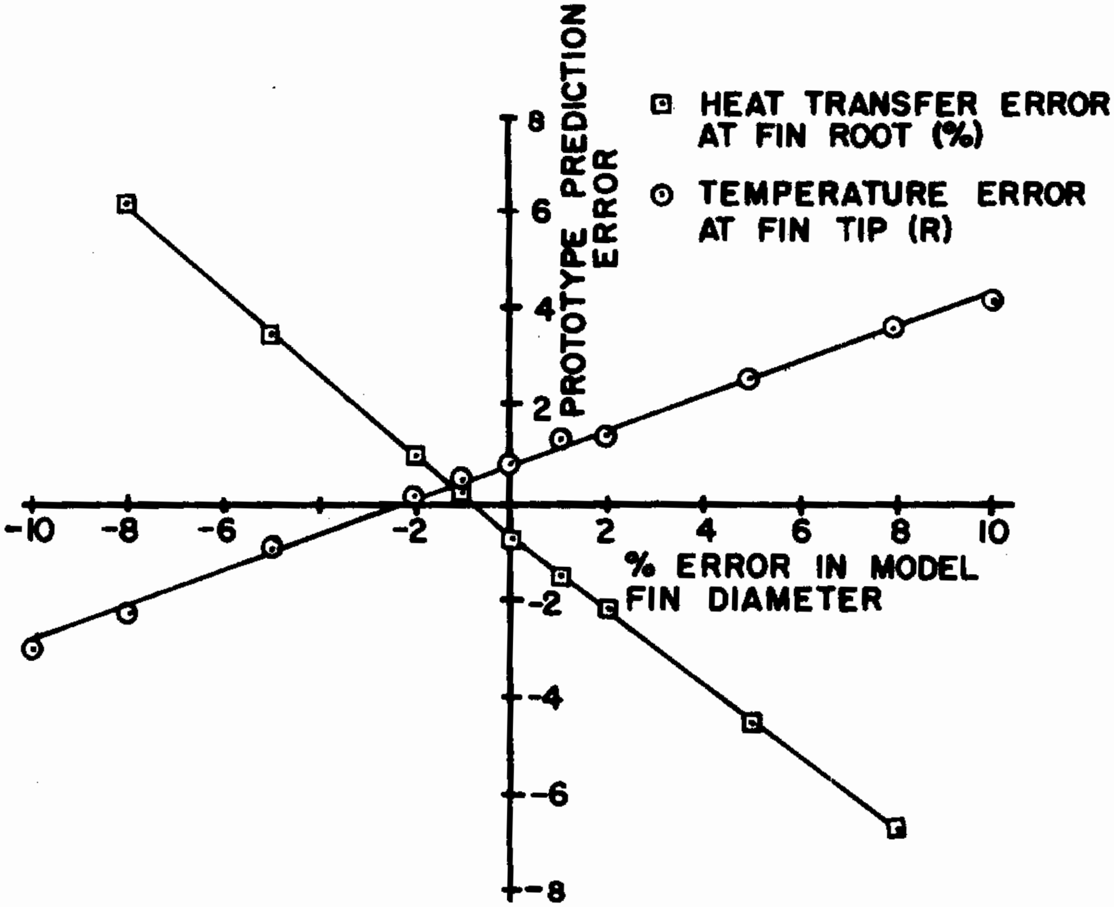

FIGURE 5. PROTOTYPE PREDICTION ERRORS CAUSED BY MODEL DIMENSION ERRORS (DISTORTED MODEL)

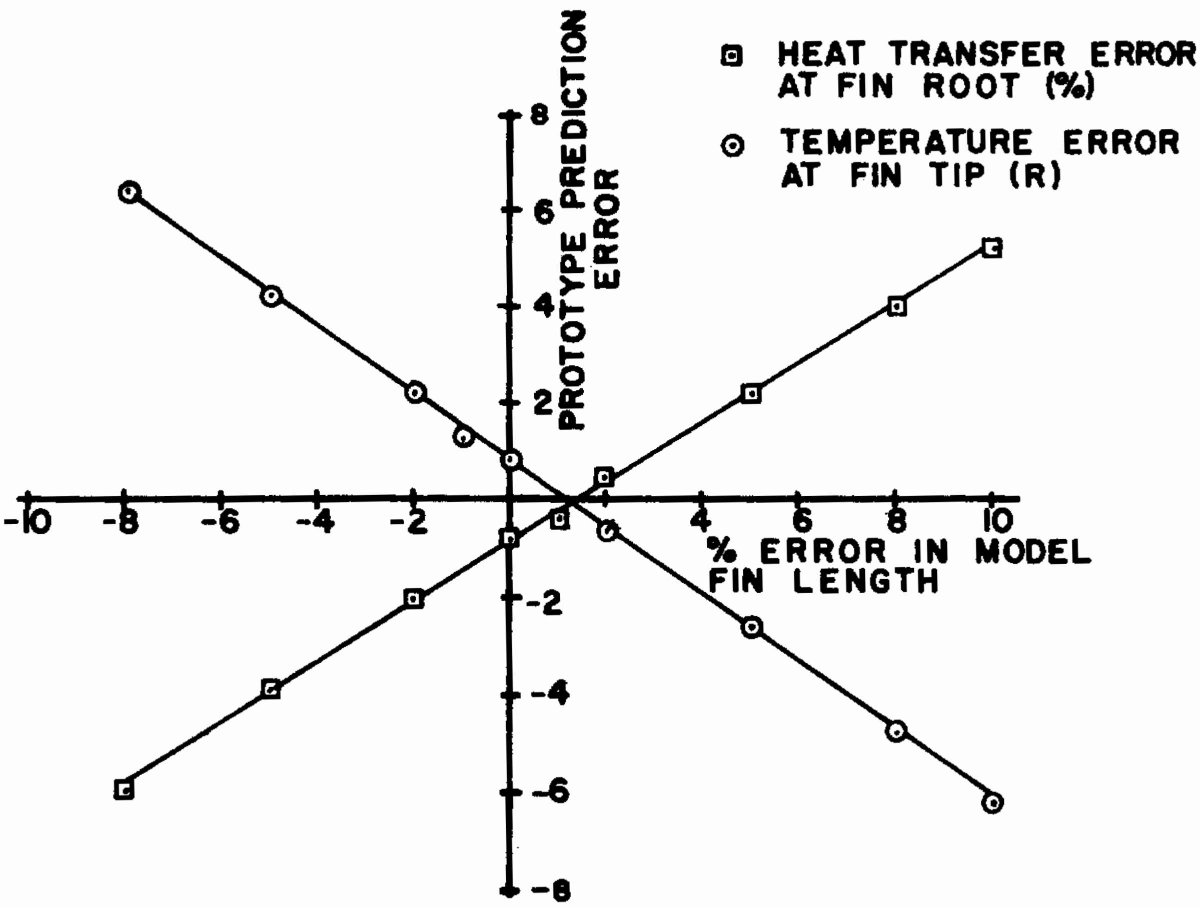

FIGURE 6. PROTOTYPE PREDICTION ERRORS CAUSED BY MODEL DIMENSION ERRORS (DISTORTED MODEL)

The effect of the radiating tip area is very small at the root of the fin, thus the error in predicted prototype heat transfer at the fin root is also very small. As the tip of the fin is approached, however, the error in the prediction of conduction heat transfer will increase to a maximum of -75% error at the tip of the fin.

For the prediction of fin temperature, the distorted model technique is valuable and sufficiently accurate. Large errors in heat transfer rates can occur, however, which is very undesirable.

3. EFFECTS OF CONDUCTIVITY VARIATION WITH TEMPERATURE

For this situation we will consider that the experimenter will assume that both model and prototype have thermal conductivities which do not vary with temperature, while in fact both the model and the prototype do have thermal conductiviites which vary with temperature. We will investigate the prediction errors the experimenter would make for the same prototype used in the previous two cases. For this situation the prediction would be made on the basis of the method 3 parameters of Table I, again assuming a one-fourth scale model. The model root temperature is calculated from dimensional ratios to be 841.3°R, based on a prototype root temperature of 530°R. Table VII gives a comparison of the actual prototype and model data and the errors incurred by the assumption of constant thermal conductivity.

Generally speaking, these errors are not large. However, the magnitude of such errors will increase with the degree of dependence of thermal conductivity on temperature. Table VIII shows the same type data for a

TABLE VII

Prototype and Model Data for Constant Thermal Conductivity

Parameter	Prototype	Model	Predicted Prototype	Errors
T_o (°R)	530.0	841.3	530.0	0.0
T_L (°R)	483.5	772.1	486.4	+2.9 (°R)
q_o (B/hr)	107.0	43.5	109.5	+2.3 (%)
q_L (B/hr)	1.941	0.789	1.988	+2.4 (%)

TABLE VIII

Prototype and Model Data for Constant Thermal Conductivity

Parameter	Prototype	Model	Predicted Prototype	Errors
T_o (°R)	530.0	841.3	530.0	0.0
T_L (°R)	483.5	779.4	491.0	+7.5 (°R)
q_o (Btu/hr)	107.1	44.5	98.5	-8.0 (%)
q_L (Btu/hr)	1.941	0.819	1.812	-6.7 (%)

prototype-model system made of a fictional material which has a thermal conductivity related by the following equation

$$k = 4.482\ T^{0.5} \tag{66}$$

This material has a conductivity of 103 Btu/hr-ft-R at 528°R, the same as the 2024 aluminum previously considered, but a larger variation of conductivity with temperature, and the prediction errors are significantly larger.

4. EFFECTS OF THE RADIATION ENVIRONMENT TEMPERATURE

In the previous cases we have considered the prototype and models to be situated in a radiation environment of 160°R. To determine the effects of the radiation environment we will consider again the same prototype and alter the temperature of the surroundings from 4°R to 350°R and calculate the errors in fin end temperatures and root and outer end heat transfers. The errors will be based on the differences from the 4°R values.

Figure 7 shows these results. With an environment temperature of 160 °R, the errors in the heat transfer rates are less than 2 percent and the fin tip temperature is in error by 1.3°R. These are negligible errors in most cases and generally speaking would not be large enough to warrant using any cooling means but liquid nitrogen to attempt to obtain a lower environmental temperature.

5. MODELING WITH LOW PROTOTYPE TEMPERATURES

As prototype operating temperatures are decreased, the heat transfer

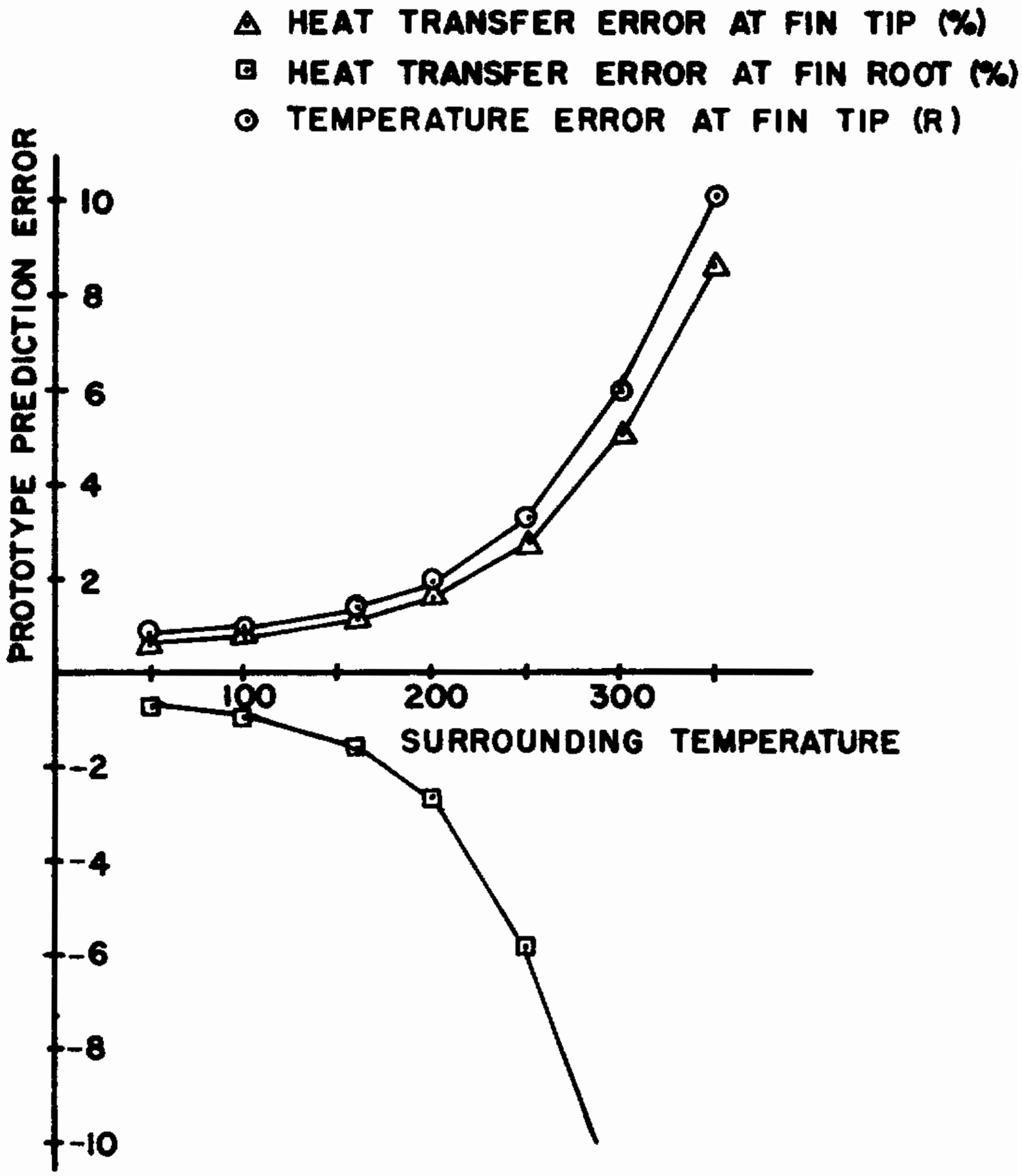

FIGURE 7. SURROUNDINGS TEMPERATURE EFFECT ON PROTOTYPE PREDICTION ERRORS

quantities decrease and there is a greater possibility of prediction errors from model studies. To obtain some idea of the magnitude of the errors involved in such a situation, a series of computer calculated temperature distributions and heat transfer rates were obtained for a prototype fin with root temperatures ranging from 200°R to 530°R were obtained. The prototype fin was the same size and material as that considered previously. It was a 2 inch diameter solid cylinder, 2 ft. long, and made of 2024 aluminum. The model was one-fourth scale before, and made of the same material. The corresponding model root temperatures varied from 328.0°R to 869.2°R. For both prototype and model the radiation environment temperature was 4 R.

The results are shown in Figure 8. The errors in predicted prototype temperature and fin tip heat transfer rate decreased as the fin root temperature decreased, reaching essentially zero error at 200°R. The percentage errors in predicted prototype root heat transfer rate increased as the prototype temperatures were decreased, from 1% at 530°R to 7.4% at 200°R.

6. COMPLEX RADIATIVE/CONDUCTIVE INTERCHANGES

The thermal scale modeling of structures which have multiple heat transfer paths utilizing different modes of heat transfer is not a difficult process if exact scale modeling is used. However, if it is necessary to distort some dimensions in the model, the prediction of prototype parameters from model data becomes increasingly difficult as the degree of distortion increases.

One of the objectives of this research was to investigate the effects of model distortion when complex radiative/conductive interchanges were in-

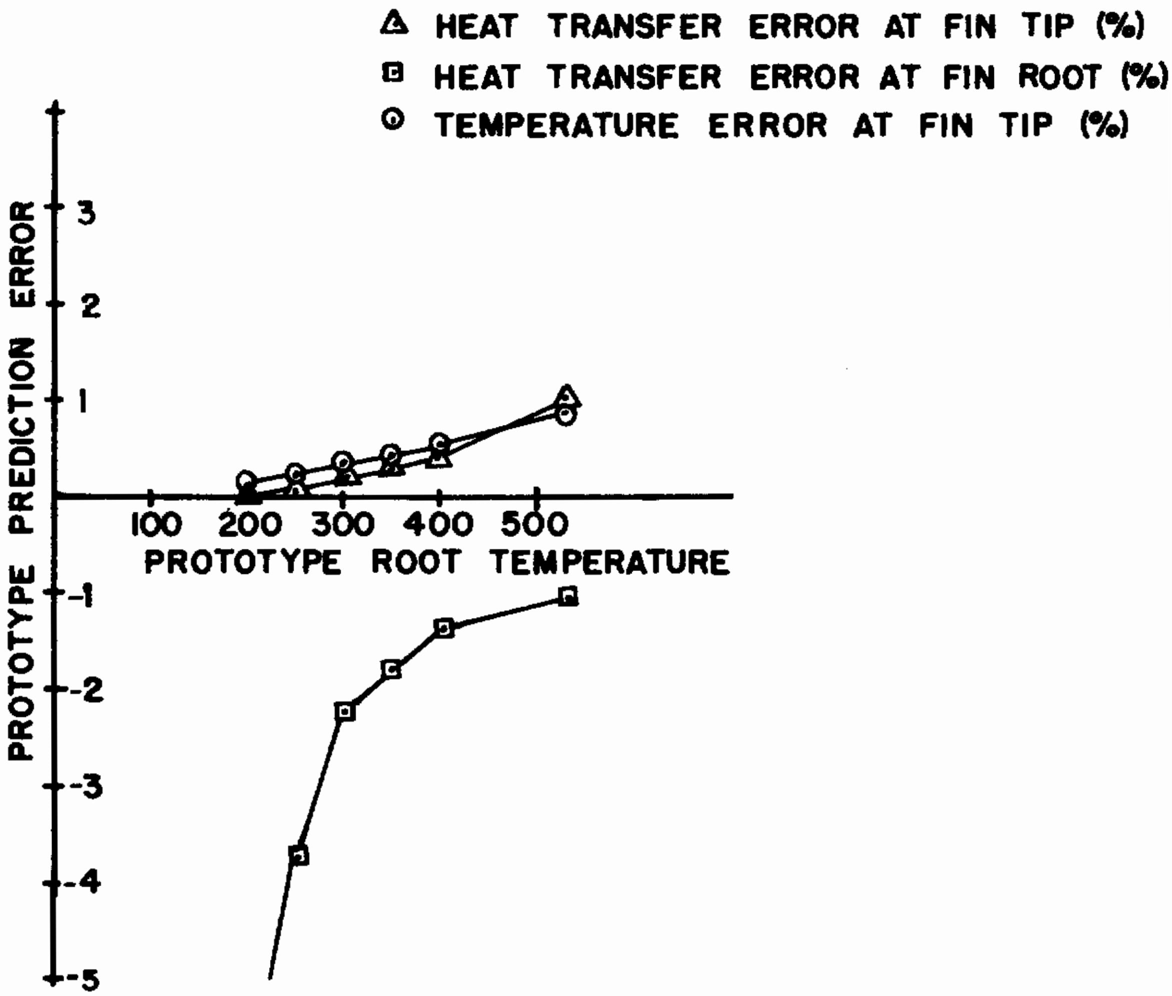

FIGURE 8. PROTOTYPE TEMPERATURE EFFECTS ON PROTOTYPE PREDICTION ERRORS

volved. This problem has been discussed by several authors, notably Barcus in 1966(12). The scheme which Barcus proposes is a "computer-assisted" technique using distorted models. The model is distorted as necessary and several tests at different conditions are made on the model. These results are then utilized as input data to a computer which solves many simultaneous equations to calculate prototype temperatures and heat transfers for a single prototype condition. Several model tests are necessary for each prototype prediction to account for the distortion in the model.

Although considerable effort has been expended during the term of this contract, it has not been possible to accomplish the objective of determining the effects of complex heat exchange paths when using distorted models.

SECTION V

SUMMARY OF RESULTS

1. Solutions were presented for the steady-state temperature distribution and heat transfer rates in a radiating fin having temperature dependent thermal conductivity. There were solutions for eight different cases of boundary conditions applied to the fin. The solutions were exact but numerical means were used for integration purposes so computer programs, in Fortran language, for obtaining the solutions were also presented. This analysis, and the computer programs, were used to determine the limitations on accuracy in thermal scale modeling.

2. It has not been possible to obtain a generalized method of error prediction for thermal modeling. It was, however, possible to determine the magnitude of prototype prediction errors for the case of steady-state radiating fins. Through a study of the prediction errors of these fin systems it was possible to obtain a better insight into the magnitude of errors involved in the use of thermal modeling techniques.

3. It was determined that, for an exact thermal model, model length or diameter errors of $\pm 5\%$ caused less than 4° R error at the tip of a radiating end fin and less than $\pm 5\%$ error in the fin root and tip heat transfer rates.

4. For a severely distorted model, $\pm 5\%$ errors of model diameter or length caused approximately the same magnitude error in fin tip temperature and root heat transfer as in the exact model. The error in heat transfer rate increased along the length of the fin, however, to -75% at the fin tip. The cause of this prediction error was the distortion in the model.

5. For fin systems which were constructed of material having temperature dependent thermal conductivity, but for which the experimenter assumed constant thermal conductivity, the prediction errors increased with increasing dependence of thermal conductivity on temperature. Examples of prediction errors were given for a fin made of aluminum and another (fictional) material having greater changes of conductivity with temperature.

6. Liquid nitrogen, at 160°R, is often used in space simulators to simulate the low temperature of outer space. Depending upon the circumstances, this simulation may not be adequate. To determine the approximate magnitude of the errors involved in such a situation, a determination of fin temperatures and heat transfer rates was made for environmental temperatures between 50°R and 350°R. These were compared with values calculated for an environmental temperature of 4°R. The errors increase in magnitude with the environmental temperature, of course, and typical results with an environmental temperature of 160°R show that the fin tip temperature error was 1.3°R and the heat transfer rates at the fin root and tip were in error by less than 2%.

7. For a prototype fin with a low operating temperature (fin root temperature from 530°R down to 200°R) the errors in predicted prototype tip temperature and heat transfer rate decreased as the fin root temperature decreased and were always below 1°R and 1% respectively. The errors in root heat transfer rate increased from 1% at 530°R to 7% at 200°R.

8. Attempts were made to determine the effects of distorted models when complex radiative/conductive heat exchange paths were involved but this objective was not accomplished.

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APPENDIX A

COMPUTER PROGRAMS

The eight cases of boundary conditions which apply to the radiating fin are, for convenience, presented as eight separate computer programs. The input data for all programs uses exactly the same format specifications so the separate program could easily be incorporated as subroutines in one large program if desired. The input data format and definition of variables is presented in Table IX and the program listings are presented in Table X through Table XVII.

The output data has virtually the same form for all programs. Typical output data forms for each of the eight programs are shown in Table XVIII through Table XXV. All input data is printed as part of the output as well as calculated data, including fin distances as a function of temperature, minimum temperatures if they occur, and heat fluxes at various locations on the fin, such as at the root and at the outer end of the fin.

TABLE IX

Input Data Format

Radiating Fin Computer Programs

Fortran IV Language

A. The first input card states the number of data sets (2 cards per set) which follow. The field is I 10.

B. Input to each program consists of two cards, the first having eight F 10.4 fields, the second having two I 10 fields, two F 10.4 fields and one I 10 field.

C. Input data names for the first card are:

TO: Fin root temperature (°R)

TINF: Surroundings temperature (°R)

DEL: Fin half thickness, if rectangular or one-fourth of fin diameter if circular (FT)

E: Fin surface emittance

A: Exponent in the equation $k = k_o T^a$

XKO: Constant k_o in the previous equation. k must have units of (Btu/hr-ft.R).

XL: Fin length (FT)

TL: Fin temperature at outer end. Leave blank if not applicable. (R)

D. Input data names for the second card are:

IXNO: Temperature intervals at which fin distances from the root are calculated (°R)

N: The number of points at which the subroutine RIEMAN estimates the second derivative of the function to decide on new sub interval lengths (N>3). For most functions N = 10 is sufficient.

EPS: The absolute limit of error for the RIEMAN subroutine integration. Because of roundoff 1×10^{-6} is the recommended lower limit of EPS.

TABLE IX (cont)

ACC: The accuracy to which specified and calculated fin length must agree before iteration is ended. 1×10^{-3} is usually sufficient (FT).

ITYPE: Dummy data name. Not used in the program.

TABLE X

CASE I COMPUTER PROGRAM

```
C********SOLUTION TO CASE 1 PROBLEM**********
    1 FORMAT(8F10.4)
    2 FORMAT(2I10,2F10.4,I10)
  101 FORMAT(1H1,10X,'CASE 1 PROBLEM'//)
  112 FORMAT(//,20X,'HEAT FLUX AT ROOT =',F10.2,' BTU/HR-SQFT')
  114 FORMAT(20X,4HX = ,F8.3,4H FT.,3X,4HT = ,F8.2,' DEG.R')
  121 FORMAT(11X,'INFINITE LENGTH FIN')
  122 FORMAT(11X,'T INFINITY =',F8.2,' R')
  123 FORMAT(11X,'EMITTANCE =',F6.3)
  124 FORMAT(11X,'FIN HALF THICKNESS =',F12.7,' FT')
  125 FORMAT(11X,'EXPONENT =',F7.4//)
  126 FORMAT(11X,'ROOT TEMP. =',F8.2,' R')
  127 FORMAT(11X,'CONDUCTIVITY =',F7.2)
      READ(1,2) ITIMES
    4 DO 1000 IDA=1,ITIMES
      READ(1,1) TO,TINF,DEL,E,A,XKO,XL,TL
      READ(1,2) IXNO,N,EPS,ACC,ITYPE
      WRITE(3,101)
      WRITE(3,121)
      WRITE(3,126)TO
      WRITE(3,122)TINF
      WRITE(3,123)E
      WRITE(3,124)DEL
      WRITE(3,127)XKO
      WRITE(3,125)A
      SIGMA=0.1714E-08
      GAMA=(A+5.)/(A+1.)
      B=(E*SIGMA*((A+1.)**GAMA)/(XKO*DEL*(A+5.)))*2.
      G=2.*E*SIGMA*XKO/(DEL*(A+5.))
      WRITE(3,114) TINF,TO
      ITO = IFIX((TO+100)/100.)
      TTO = 100*ITO
      XNO = FLOAT(IXNO)
      L = 1
  343 T = TTO-L*XNO
      L = L + 1
      IF(T.GT.(TO-10)) GO TO 343
      IF(T.LT.(TINF+10)) GO TO 370
      X = SQRT((4.*(A+1.)**GAMA)/(B*(3.-A)**2))
      Y = -1./SQRT(TO**(3.-A)) + 1./SQRT(T**(3.-A))
      X = X*Y
      WRITE(3,114)X,T
      GO TO 343
  370 CONTINUE
      QO = SQRT(G*(TO**(A+5.)))
      WRITE(3,112) QO
 1000 CONTINUE
      STOP
      END
```

TABLE XI

CASE II COMPUTER PROGRAM

```
C********SOLUTION TO CASE 2 PROBLEM**********
      THETA(X,Y) = (Y**(X + 1.))/(X + 1.)
    1 FORMAT(8F10.4)
    2 FORMAT(2I10,2F10.4,I10)
  112 FORMAT(//,20X,'HEAT FLUX AT ROOT =',F10.2,' BTU/HR-SQFT')
  114 FORMAT(20X,4HX = ,F8.3,4H FT.,3X,4HT = ,F8.2,' DEG.R')
  122 FORMAT(11X,'T INFINITY =',F8.2,' R')
  123 FORMAT(11X,'EMITTANCE =',F6.3)
  124 FORMAT(11X,'FIN HALF THICKNESS =',F12.7,' FT')
  125 FORMAT(11X,'EXPONENT =',F7.4//)
  126 FORMAT(11X,'ROOT TEMP. =',F8.2,' R')
  127 FORMAT(11X,'CONDUCTIVITY =',F7.2)
  201 FORMAT(1H1,10X,'CASE 2 PROBLEM'//)
  206 FORMAT(11X,'FIN LENGTH =',F8.3,' FT.')
  207 FORMAT(11X,'TEMPERATURE AT OUTER END =',F8.2,' DEG. R'/)
  221 FORMAT(11X,'FINITE LENGTH FIN WITH INSULATED END')
 1001 FORMAT(//,11X,'N  =',I4)
 1002 FORMAT(11X,'EPS=',F9.7)
 1003 FORMAT(11X,'ACC=',F9.7)
      READ(1,2) ITIMES
    4 DO 1000 IDA=1,ITIMES
      READ(1,1) TO,TINF,DEL,E,A,XKO,XL,TL
      READ(1,2) IXNO,N,EPS,ACC,ITYPE
      WRITE(3,201)
      WRITE(3,221)
      WRITE(3,206)XL
      WRITE(3,126)TO
      WRITE(3,122)TINF
      WRITE(3,123)E
      WRITE(3,124)DEL
      WRITE(3,127)XKO
      WRITE(3,125)A
      SIGMA=0.1714E-08
      GAMA=(A+5.)/(A+1.)
      B=(E*SIGMA*((A+1.)**GAMA)/(XKO*DEL*(A+5.)))*2.
      G=2.*E*SIGMA*XKO/(DEL*(A+5.))
      A1 = 1.
      A2 = 1.01
      A3 = 0.
      A4 = 0.
      A5 = -1.
      A6 = GAMA
      R = RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
      A1 = A2
      X = SQRT(B*THETA(A,TO)**(GAMA-2.))
      TL = 0
      W = .5*TO
  251 TL = TL + W
      A2 = THETA(A,TO)/THETA(A,TL)
      IF(A2.LT.1.015) R = 0
      IF(A2.LT.1.015) A1 = 1
  219 Y = R + RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
```

```
      Y = Y*SQRT(A2**(GAMA-2.))/X
      XL1 = Y
      IF(ABS(XL-Y).LT.ACC) GO TO 6
      IF(Y-XL) 220,6,222
  220 TL = TL - W
      W = .5*W
      GO TO 251
  222 W = .5*W
      GO TO 251
    6 WRITE(3,207) TL
      WRITE(3,114) A3,TO
      XX = 1.01
      ITO = IFIX((TO+100)/100.)
      TTO = 100*ITO
      XNO = FLOAT(IXNO)
      L = 1
      A2 = THETA(A,TO)/THETA(A,TL)
      Z =  1./SQRT(B*THETA(A,TL)**(GAMA-2.))
  343 T = TTO-L*XNO
      L = L + 1
      IF(T.GT.(TO-10)) GO TO 343
      IF(T.LT.(TL+10)) GO TO 370
      A1 = THETA(A,T)/THETA(A,TL)
      Y1 = 0
      IF(A1.GT.1.005) GO TO 252
      Y1 = RIEMAN(A1,XX,A3,A4,A5,A6,N,EPS)
      A1 = XX
  252 Y = Y1 + RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
      X = Z*Y
  205 WRITE(3,114) X,T
      GO TO 343
  370 CONTINUE
      WRITE(3,114) XL1,TL
      QO = SQRT(G*(TO**(A+5.)-TL**(A+5.)))
      WRITE(3,112) QO
      WRITE(3,1001) N
      WRITE(3,1002) EPS
      WRITE(3,1003) ACC
 1C00 CONTINUE
      STOP
      END
      FUNCTION RIEMAN (A1,A2,A3,A4,A5,A6,N,EPS)
      DIMENSION A(3),F(3)
      VALUE(A1,A2,A3,A4,A5,A6,X)=1./(SQRT(X**A6-A3**A6-A4*A6*(X-A3)+A5))
  100 DIVA=(A2-A1)/FLOAT(N)
  101 DO 103 I=1,3
  102 A(I)=A1+(FLOAT(2*I-1)/2.0)*DIVA
  103 F(I) =  VALUE(A1,A2,A3,A4,A5,A6,A(I))
  104 END=A1
  105 RIEMAN=0.0
  199 DO 233 I=1,N
 1991 RESULT=0.0
```

TABLE XI (cont)

```
200 ERROR= (((F(3)-2.0*F(2)+F(1))*DIVA)/24.0)*FLOAT(N)
201 ERR=ABS(ERROR/EPS)
202 IF(ERR-1.0) 218,218,203
203 K=SQRT(ERR)+1.0
204 L=K/2
205 K=2*L+1
206 DIVR=DIVA/FLOAT(K)
207 DIVR2=DIVR/2.0
208 DO 211  J=1,L
209 ADD =    VALUE(A1,A2,A3,A4,A5,A6,END+DIVR2)*DIVR
210 RESULT=RESULT+ADD
211 END=END+DIVR
212 END=END+DIVR
213 DO 216  J=1,L
214 ADD =    VALUE(A1,A2,A3,A4,A5,A6,END+DIVR2)*DIVR
215 RESULT=RESULT+ADD
216 END=END+DIVR
217 GO TO 220
218 END=END+DIVA
219 DIVR=DIVA
220 IF(I-1) 222,222,221
221 IF(I-(N-1)) 224,229,229
222 RESULT=RESULT+F(1)*DIVR
223 GO TO 233
224 RESULT=RESULT+F(2)*DIVR
225 F(1)=F(2)
226 F(2)=F(3)
227 F(3) =  VALUE(A1,A2,A3,A4,A5,A6,END+(3./2.)*DIVA
228 GO TO 233
229 IF(I-N) 230,232,232
230 RESULT=RESULT+F(2)*DIVR
231 GO TO 233
232 RESULT=RESULT+F(3)*DIVR
233 RIEMAN=RIEMAN+RESULT
234 RETURN
    END
```

TABLE XII

CASE III COMPUTER PROGRAM

```
C********SOLUTION TO CASE 3 PROBLEM**********
      THETA(X,Y) = (Y**(X + 1.))/(X + 1.)
    1 FORMAT(8F10.4)
    2 FORMAT(2I10,2F10.4,I10)
  112 FORMAT(//,20X,'HEAT FLUX AT ROOT =',F10.2,' BTU/HR-SQFT')
  114 FORMAT(20X,4HX = ,F8.3,4H FT.,3X,4HT = ,F8.2,' DEG.R')
  122 FORMAT(11X,'T INFINITY =',F8.2,' R')
  123 FORMAT(11X,'EMITTANCE =',F6.3)
  124 FORMAT(11X,'FIN HALF THICKNESS =',F12.7,' FT')
  125 FORMAT(11X,'EXPONENT =',F7.4//)
  126 FORMAT(11X,'ROOT TEMP. =',F8.2,' R')
  127 FORMAT(11X,'CONDUCTIVITY =',F7.2)
  206 FORMAT(11X,'FIN LENGTH =',F8.3,' FT.')
  207 FORMAT(11X,'TEMPERATURE AT OUTER END =',F8.2,' DEG. R'/)
  221 FORMAT(11X,'FINITE LENGTH FIN WITH RADIATING END')
  301 FORMAT(1H1,10X,'CASE 3 PROBLEM'//)
  413 FORMAT(20X,'FLUX AT OUTER END =',F10.2,' BTU/HR-SQFT')
 1001 FORMAT(//,11X,'N  =',I4)
 1002 FORMAT(11X,'EPS=',F9.7)
 1003 FORMAT(11X,'ACC=',F9.7)
      READ(1,2) ITIMES
    4 DO 1000 IDA=1,ITIMES
      READ(1,1) TO,TINF,DEL,E,A,XKO,XL,TL
      READ(1,2) IXNO,N,EPS,ACC,ITYPE
      WRITE(3,301)
      WRITE(3,221)
      WRITE(3,206)XL
      WRITE(3,126)TO
      WRITE(3,122)TINF
      WRITE(3,123)E
      WRITE(3,124)DEL
      WRITE(3,127)XKO
      WRITE(3,125)A
      SIGMA=0.1714E-08
      GAMA=(A+5.)/(A+1.)
      B=(E*SIGMA*((A+1.)**GAMA)/(XKO*DEL*(A+5.)))*2.
      G=2.*E*SIGMA*XKO/(DEL*(A+5.))
      A3 = 0.
      A4 = 0.
      A5 = -1.
      A6 = GAMA
      W = .5*TO
      TL = 0.
      R = 1.01
      Z = 1.005
  303 TL = TL + W
      TAU = (1.-E*SIGMA*DEL*(A+5)*TL**(3-A)/(2*XKO))**(1./A6)
      A2 = THETA(A,TO)/(THETA(A,TL)*TAU)
      X = 1./SQRT(B*(THETA(A,TL)*TAU)**(GAMA-2.))
      A1 = 1./TAU
      Y1 = 0
      IF(A1.GT.Z) GO TO 304
```

```
      Y1 = X*RIEMAN(A1,R,A3,A4,A5,A6,N,EPS)
      A1 = R
 304  Y2 = X*RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
      Y = Y1+Y2
      Y5 = Y
      IF(ABS(XL-Y).LT.ACC)GO TO 340
      IF(XL-Y) 306,340,307
 307  TL = TL - W
      W = .5*W
      GO TO 303
 306  W = .5*W
      GO TO 303
 340  WRITE(3,207) TL
      WRITE(3,114) A3,TO
      ITO = IFIX((TO+100.)/100.)
      TTO = 100*ITO
      XNO = FLOAT(IXNO)
      L = 1
 343  T = TTO - L*XNO
      L = L + 1
      IF(T.GT.(TO-10.)) GO TO 343
      IF(T.LT.(TL+10)) GO TO 370
      A1 = THETA(A,T)/(THETA(A,TL)*TAU)
      Y1 = 0
      IF(A1.GT.Z) GO TO 341
      Y1 = X*RIEMAN(A1,R,A2,A3,A4,A5,A6,N,EPS)
      A1 = R
 341  Y2 = X*RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
      Y = Y1+Y2
      WRITE(3,114) Y,T
      GO TO 343
 370  WRITE(3,114) Y5,TL
      TAU = 1.-E*SIGMA*DEL*(A+5.)*TL**(3.-A)/(2*XKO)
      QO = SQRT(G*(TO**(A+5.)-TL**(A+5)*TAU))
      QL = E*SIGMA*TL**4
      WRITE(3,112)QO
      WRITE(3,413)QL
      WRITE(3,1001) N
      WRITE(3,1002) EPS
      WRITE(3,1003) ACC
1000  CONTINUE
      STOP
      END
      FUNCTION RIEMAN (A1,A2,A3,A4,A5,A6,N,EPS)
      DIMENSION A(3),F(3)
      VALUE(A1,A2,A3,A4,A5,A6,X)=1./(SQRT(X**A6-A3**A6-A4*A6*(X-A3)+A5))
 100  DIVA=(A2-A1)/FLOAT(N)
 101  DO 103 I=1,3
 102  A(I)=A1+(FLOAT(2*I-1)/2.0)*DIVA
 103  F(I) =  VALUE(A1,A2,A3,A4,A5,A6,A(I))
 104  END=A1
 105  RIEMAN=0.0
```

TABLE XII (cont)

```
 199 DO 233 I=1,N
1991 RESULT=0.0
 200 ERROR= (((F(3)-2.0*F(2)+F(1))*DIVA)/24.0)*FLOAT(N)
 201 ERR=ABS(ERROR/EPS)
 202 IF(ERR-1.0) 218,218,203
 203 K=SQRT(ERR)+1.0
 204 L=K/2
 205 K=2*L+1
 206 DIVR=DIVA/FLOAT(K)
 207 DIVR2=DIVR/2.0
 208 DO 211 J=1,L
 209 ADD =   VALUE(A1,A2,A3,A4,A5,A6,END+DIVR2)*DIVR
 210 RESULT=RESULT+ADD
 211 END=END+DIVR
 212 END=END+DIVR
 213 DO 216 J=1,L
 214 ADD =   VALUE(A1,A2,A3,A4,A5,A6,END+DIVR2)*DIVR
 215 RESULT=RESULT+ADD
 216 END=END+DIVR
 217 GO TO 220
 218 END=END+DIVA
 219 DIVR=DIVA
 220 IF(I-1) 222,222,221
 221 IF(I-(N-1)) 224,229,229
 222 RESULT=RESULT+F(1)*DIVR
 223 GO TO 233
 224 RESULT=RESULT+F(2)*DIVR
 225 F(1)=F(2)
 226 F(2)=F(3)
 227 F(3) =  VALUE(A1,A2,A3,A4,A5,A6,END+(3./2.)*DIVA)
 228 GO TO 233
 229 IF(I-N) 230,232,232
 230 RESULT=RESULT+F(2)*DIVR
 231 GO TO 233
 232 RESULT=RESULT+F(3)*DIVR
 233 RIEMAN=RIEMAN+RESULT
 234 RETURN
     END
```

TABLE XIII

CASE IV COMPUTER PROGRAM

```
C********SOLUTION TO CASE 4 PROBLEM**********
      THETA(X,Y) = (Y**(X + 1.))/(X + 1.)
    1 FORMAT(8F10.4)
    2 FORMAT(2I10,2F10.4,I10)
  112 FORMAT(//,20X,'HEAT FLUX AT ROOT =',F10.2,' BTU/HR-SQFT')
  114 FORMAT(20X,'X =',F8.3,' FT.',4X,'T =',F8.2,' R')
  121 FORMAT(11X,'FIN LENGTH =',F8.3,' FT')
  122 FORMAT(11X,'T INFINITY =',F8.2,' R')
  123 FORMAT(11X,'EMITTANCE =',F6.3)
  124 FORMAT(11X,'FIN HALF THICKNESS =',F12.7,' FT')
  125 FORMAT(11X,'EXPONENT =',F7.4//)
  126 FORMAT(11X,'ROOT TEMP. =',F8.2,' R')
  127 FORMAT(11X,'CONDUCTIVITY =',F7.2)
  401 FORMAT(1H1,10X,'CASE 4A PROBLEM'//)
  402 FORMAT(1H1,10X,'CASE 4B PROBLEM'//)
  403 FORMAT(1H1,10X,'CASE 4C PROBLEM'//)
  404 FORMAT(11X,'MINIMUM TEMP. OF',F8.2,' R OCCURS AT',F8.3,' FT'/)
  405 FORMAT(11X,'FINITE LENGTH FIN WITH BOTH END TEMPERATURES SPECIFIED
     1')
  406 FORMAT(11X,'TEMP. AT OUTER END =',F8.2,' DEG.R')
  413 FORMAT(20X,'FLUX AT OUTER END =',F10.2,' BTU/HR-SQFT')
 1001 FORMAT(//,11X,'N  =',I4)
 1002 FORMAT(11X,'EPS=',F9.7)
 1003 FORMAT(11X,'ACC=',F9.7)
      READ(1,2) ITIMES
    4 DO 1C00 IDA=1,ITIMES
      READ(1,1) TO,TINF,DEL,E,A,XKO,XL,TL
      READ(1,2) IXNO,N,EPS,ACC,ITYPE
      SIGMA=0.1714E-08
      GAMA=(A+5.)/(A+1.)
      B=(E*SIGMA*((A+1.)**GAMA)/(XKO*DEL*(A+5.)))*2.
      G=2.*E*SIGMA*XKO/(DEL*(A+5.))
      A1 = 1.
      A2 = 1.01
      A3 = 0
      A4 = 0
      A5 = -1.
      A6 = GAMA
      R = RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
      A1 = A2
      A2 = THETA(A,TO)/THETA(A,TL)
      DUM = 1./SQRT(B*THETA(A,TL)**(GAMA-2.))
      IF(A2.LT.1.01) R = 0
      IF(A2.LT.1.01) A1 = 1.
      IF(ABS(A2-A1).LT..001) GO TO 435
      Y = R + RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
      XL2 = DUM*Y
      IF(ABS(XL2-XL).LT.ACC) GO TO 450
      IF(XL2.GT.XL) GO TO 420
      IF(XL2.LT.XL) GO TO 435
  450 WRITE(3,402)
      WRITE(3,405)
```

```
      WRITE(3,121)XL
      WRITE(3,126)TO
      WRITE(3,406) TL
      WRITE(3,122)TINF
      WRITE(3,123)E
      WRITE(3,124)DEL
      WRITE(3,127)XKO
      WRITE(3,125)A
      GO TO 460
  420 WRITE(3,401)
      WRITE(3,405)
      WRITE(3,121)XL
      WRITE(3,126)TO
      WRITE(3,406) TL
      WRITE(3,122)TINF
      WRITE(3,123)E
      WRITE(3,124)DEL
      WRITE(3,127)XKO
      WRITE(3,125)A
      A1 = 1.
      A5 = 0
      A7 = 1.01
      W = (A2-1)**2/(XL**2*B*THETA(A,TL)**(GAMA-2))-1
  421 W = .5*W
      A5 = A5 + W
  424 R = RIEMAN(A1,A7,A3,A4,A5,A6,N,EPS)
      IF(A2.LT.1.01)A7 = 1.
      IF(A2.LT.1.01) R = 0
      XL2 = DUM*(R + RIEMAN(A7,A2,A3,A4,A5,A6,N,EPS))
      IF(ABS(XL2-XL).LT.ACC) GO TO 460
      IF(XL2-XL) 423,460,421
  423 A5 = A5 - W
      GO TO 421
  460 WRITE(3,114)A3,TO
      ITO = IFIX((TO+100)/100.)
      TTO = 100*ITO
      XNO = FLOAT(IXNO)
      L = 1
  343 T = TTO-L*XNO
      L = L + 1
      IF(T.GT.(TO-10)) GO TO 343
      IF(T.LT.(TL+10)) GO TO 370
      A1 = THETA(A,T)/THETA(A,TL)
      A7 = 1.01
      R = 0
      IF(A1.GT.1.005) GO TO 451
      R = RIEMAN(A1,A7,A3,A4,A5,A6,N,EPS)
      A1 = A7
  451 Y = R + RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
      X = DUM*Y
      WRITE(3,114) X,T
      GO TO 343
```

```
370 CONTINUE
    WRITE(3,114) XL2,TL
    PL2 = (B*THETA(A,TL)**GAMA)*(A5 + 1.)
    QO = SQRT(G*(TO**(A+5.)-TL**(A+5.)) + XKO**2*PL2)
    WRITE(3,112)QO
    QL = XKO*SQRT(PL2)
    WRITE(3,413) QL
    WRITE(3,1001) N
    WRITE(3,1002) EPS
    WRITE(3,1003) ACC
    GO TO 1000
435 WRITE(3,403)
    WRITE(3,405)
    WRITE(3,121)XL
    WRITE(3,126)TO
    WRITE(3,406) TL
    WRITE(3,122)TINF
    WRITE(3,123)E
    WRITE(3,124)DEL
    WRITE(3,127)XKO
    WRITE(3,125)A
    TM = 0
    W = TL
    A7 = 1.01
439 W = .5*W
    TM = TM + W
    X = 1./SQRT(B*THETA(A,TM)**(GAMA-2))
    A1 = 1
    R1 = 0
    R2 = 0
    A2 = THETA(A,TO)/THETA(A,TM)
    A8 = THETA(A,TL)/THETA(A,TM)
    IF(A2.LT.1.015) GO TO 452
    R1 = RIEMAN(A1,A7,A3,A4,A5,A6,N,EPS)
    A1 = A7
452 Y = X*(R1 + RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS))
    A1 = 1
    IF(A8.LT.1.015) GO TO 453
    R2 = RIEMAN(A1,A7,A3,A4,A5,A6,N,EPS)
    A1 = A7
453 Z = X*(R2 + RIEMAN(A1,A8,A3,A4,A5,A6,N,EPS))
    Y = Y + Z
    IF(ABS(XL-Y).LT.ACC) GO TO 436
    IF(Y-XL) 437,436,439
437 TM = TM - W
    GO TO 439
436 XM = XL - Z
    WRITE(3,404) TM,XM
    WRITE(3,114) A3,TO
    ITO = IFIX((TO+100)/100.)
    TTO = 100*ITO
    XNO = FLOAT(IXNO)
```

```
      L = 1
  373 T = TTO-L*XNO
      L = L + 1
      IF(T.GT.(TO-10)) GO TO 373
      IF(T.LT.(TM+10)) GO TO 372
      A1 = THETA(A,T)/THETA(A,TM)
      R1 = 0
      IF(A1.GT.1.005) GO TO 457
      R1 = RIEMAN(A1,A7,A3,A4,A5,A6,N,EPS)
      A1 = A7
  457 Z = X*(R1 + RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS))
      WRITE(3,114) Z,T
      GO TO 373
  372 CONTINUE
      WRITE(3,114) XM,TM
      IF(TM.GT.100.) GO TO 455
      ITO = 0
      GO TO 456
  455 ITO = IFIX((TM-100)/100.)
  456 TTO = 100*ITO
      XNO = FLOAT(IXNO)
      L = 1
  454 T = TTO + L*XNO
      L = L + 1
      IF(T.LT.(TM+10)) GO TO 454
      IF(T.GT.(TL-10)) GO TO 371
      A1 = 1
      A2 = THETA(A,T)/THETA(A,TM)
      R1 = 0
      IF(A2.LT.1.015) GO TO 458
      R1 = RIEMAN(A1,A7,A3,A4,A5,A6,N,EPS)
      A1 = A7
  458 Z = X*(R1 + RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS))
      Z = Z + XM
      WRITE(3,114) Z,T
      GO TO 454
  371 WRITE(3,114) Y,TL
  445 QO = SQRT(G*(TO**(A+5.)-TM   **(A+5.)))
      QL =-SQRT(G*(TL**(A+5.)-TM   **(A+5.)))
      WRITE(3,112) QO
      WRITE(3,413)QL
 1000 CONTINUE
      WRITE(3,1001) N
      WRITE(3,1002) EPS
      WRITE(3,1003) ACC
      STOP
      END
      FUNCTION RIEMAN (A1,A2,A3,A4,A5,A6,N,EPS)
      DIMENSION A(3),F(3)
      VALUE(A1,A2,A3,A4,A5,A6,X)=1./(SQRT(X**A6-A3**A6-A4*A6*(X-A3)+A5))
  100 DIVA=(A2-A1)/FLOAT(N)
  101 DO 103 I=1,3
```

```
 102 A(I)=A1+(FLOAT(2*I-1)/2.0)*DIVA
 103 F(I) =   VALUE(A1,A2,A3,A4,A5,A6,A(I))
 104 END=A1
 105 RIEMAN=0.0
 199 DO 233 I=1,N
1991 RESULT=0.0
 200 ERROR= (((F(3)-2.0*F(2)+F(1))*DIVA)/24.0)*FLOAT(N)
 201 ERR=ABS(ERROR/EPS)
 202 IF(ERR-1.0) 218,218,203
 203 K=SQRT(ERR)+1.0
 204 L=K/2
 205 K=2*L+1
 206 DIVR=DIVA/FLOAT(K)
 207 DIVR2=DIVR/2.0
 208 DO 211 J=1,L
 209 ADD =    VALUE(A1,A2,A3,A4,A5,A6,END+DIVR2)*DIVR
 210 RESULT=RESULT+ADD
 211 END=END+DIVR
 212 END=END+DIVR
 213 DO 216 J=1,L
 214 ADD =    VALUE(A1,A2,A3,A4,A5,A6,END+DIVR2)*DIVR
 215 RESULT=RESULT+ADD
 216 END=END+DIVR
 217 GO TO 220
 218 END=END+DIVA
 219 DIVR=DIVA
 220 IF(I-1) 222,222,221
 221 IF(I-(N-1)) 224,229,229
 222 RESULT=RESULT+F(1)*DIVR
 223 GO TO 233
 224 RESULT=RESULT+F(2)*DIVR
 225 F(1)=F(2)
 226 F(2)=F(3)
 227 F(3) =   VALUE(A1,A2,A3,A4,A5,A6,END+(3./2.)*DIVA)
 228 GO TO 233
 229 IF(I-N) 230,232,232
 230 RESULT=RESULT+F(2)*DIVR
 231 GO TO 233
 232 RESULT=RESULT+F(3)*DIVR
 233 RIEMAN=RIEMAN+RESULT
 234 RETURN
     END
```

TABLE XIV

CASE V COMPUTER PROGRAM

```
C********SOLUTION TO CASE 5 PROBLEM**********
      THETA(X,Y) = (Y**(X + 1.))/(X + 1.)
    1 FORMAT(8F10.4)
    2 FORMAT(2I10,2F10.4,I10)
  501 FORMAT(1H1,10X,'CASE 5 PROBLEM',//)
  112 FORMAT(//,20X,'HEAT FLUX AT ROOT =',F10.2,' BTU/HR-SQFT')
  114 FORMAT(20X,4HX = ,F8.3,4H FT.,3X,4HT = ,F8.2,' DEG.R')
  121 FORMAT(11X,'INFINITE LENGTH FIN')
  122 FORMAT(11X,'T INFINITY =',F8.2,' R')
  123 FORMAT(11X,'EMITTANCE =',F6.3)
  124 FORMAT(11X,'FIN HALF THICKNESS =',F12.7,' FT')
  125 FORMAT(11X,'EXPONENT =',F7.4//)
  126 FORMAT(11X,'ROOT TEMP. =',F8.2,' R')
  127 FORMAT(11X,'CONDUCTIVITY =',F6.1)
      READ(1,2) ITIMES
    4 DO 1C00 IDA=1,ITIMES
      READ(1,1) TO,TINF,DEL,E,A,XKO,XL,TL
      READ(1,2) IXNO,N,EPS,ACC,ITYPE
      WRITE(3,501)
      WRITE(3,121)
      WRITE(3,126)TO
      WRITE(3,122)TINF
      WRITE(3,123)E
      WRITE(3,124)DEL
      WRITE(3,127)XKO
      WRITE(3,125)A
      SIGMA=0.1714E-08
      GAMA=(A+5.)/(A+1.)
      B=(E*SIGMA*((A+1.)**GAMA)/(XKO*DEL*(A+5.)))*2.
      G=2.*E*SIGMA*XKO/(DEL*(A+5.))
      X = 1./SQRT(B*(THETA(A,TINF)**(GAMA-2.)))
      A2= THETA(A,TO)/THETA(A,TINF)
      A3 = 1.
      A4 = 1.
      A5 = 0.
      A6 = GAMA
      WRITE(3,114) A5,TO
      ITO = IFIX((TO+100)/100.)
      TTO = 100*ITO
      XNO = FLOAT(IXNO)
      L = 1
  343 T = TTO-L*XNO
      L = L + 1
      IF(T.GT.(TO-10)) GO TO 343
      IF(T.LT.(TINF+10)) GO TO 370
      A1 = THETA(A,T)/THETA(A,TINF)
      Y = RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
      XL1 = X*Y
      WRITE(3,114) XL1,T
      GO TO 343
  370 CONTINUE
      QO=SQRT(G*(TO**(A+5)-(A+5)/(A+1)*TINF**4*TO**(A+1)+4/(A+1)*TINF**(
```

```
     1A+5)))
      WRITE(3,112) QQ
      WRITE(3,1001) N
      WRITE(3,1002) EPS
      WRITE(3,1003) ACC
 1001 FORMAT(//,11X,'N   =',I4)
 1002 FORMAT(11X,'EPS=',F9.7)
 1003 FORMAT(11X,'ACC=',F9.7)
 1C00 CONTINUE
      STOP
      END
      FUNCTION RIEMAN (A1,A2,A3,A4,A5,A6,N,EPS)
      DIMENSION A(3),F(3)
      VALUE(A1,A2,A3,A4,A5,A6,X)=1./(SQRT(X**A6-A3**A6-A4*A6*(X-A3)+A5))
  100 DIVA=(A2-A1)/FLOAT(N)
  101 DO 103 I=1,3
  102 A(I)=A1+(FLOAT(2*I-1)/2.0)*DIVA
  103 F(I) =  VALUE(A1,A2,A3,A4,A5,A6,A(I))
  104 END=A1
  105 RIEMAN=0.0
  199 DO 233 I=1,N
 1991 RESULT=0.0
  200 ERROR= (((F(3)-2.0*F(2)+F(1))*DIVA)/24.0)*FLOAT(N)
  201 ERR=ABS(ERROR/EPS)
  202 IF(ERR-1.0) 218,218,203
  203 K=SQRT(ERR)+1.0
  204 L=K/2
  205 K=2*L+1
  206 DIVR=DIVA/FLOAT(K)
  207 DIVR2=DIVR/2.0
  208 DO 211 J=1,L
  209 ADD =   VALUE(A1,A2,A3,A4,A5,A6,END+DIVR2)*DIVR
  210 RESULT=RESULT+ADD
  211 END=END+DIVR
  212 END=END+DIVR
  213 DO 216 J=1,L
  214 ADD =   VALUE(A1,A2,A3,A4,A5,A6,END+DIVR2)*DIVR
  215 RESULT=RESULT+ADD
  216 END=END+DIVR
  217 GO TO 220
  218 END=END+DIVA
  219 DIVR=DIVA
  220 IF(I-1) 222,222,221
  221 IF(I-(N-1)) 224,229,229
  222 RESULT=RESULT+F(1)*DIVR
  223 GO TO 233
  224 RESULT=RESULT+F(2)*DIVR
  225 F(1)=F(2)
  226 F(2)=F(3)
  227 F(3) =  VALUE(A1,A2,A3,A4,A5,A6,END+(3./2.)*DIVA)
  228 GO TO 233
  229 IF(I-N) 230,232,232
```

TABLE XIV (cont)

```
230 RESULT=RESULT+F(2)*DIVR
231 GO TO 233
232 RESULT=RESULT+F(3)*DIVR
233 RIEMAN=RIEMAN+RESULT
234 RETURN
    END
```

TABLE XV

CASE VI COMPUTER PROGRAM

```
C*********SOLUTION TO CASE 6 PROBLEM**********
      THETA(X,Y) = (Y**(X + 1.))/(X + 1.)
    1 FORMAT(8F10.4)
    2 FORMAT(2I10,2F10.4,I10)
  112 FORMAT(//,20X,'HEAT FLUX AT ROOT =',F10.2,' BTU/HR-SQFT')
  114 FORMAT(20X,4HX = ,F8.3,4H FT.,3X,4HT = ,F8.2,' DEG.R')
  122 FORMAT(11X,'T INFINITY =',F8.2,' R')
  123 FORMAT(11X,'EMITTANCE =',F6.3)
  124 FORMAT(11X,'FIN HALF THICKNESS =',F12.7,' FT')
  125 FORMAT(11X,'EXPONENT =',F7.4//)
  126 FORMAT(11X,'ROOT TEMP. =',F8.2,' R')
  127 FORMAT(11X,'CONDUCTIVITY =',F7.2)
  206 FORMAT(11X,'FIN LENGTH =',F8.3,' FT.')
  207 FORMAT(11X,'TEMPERATURE AT OUTER END =',F8.2,' DEG. R'/)
  221 FORMAT(11X,'FINITE LENGTH FIN WITH INSULATED END')
  601 FORMAT(1H1,10X,'CASE 6 PROBLEM'//)
      READ(1,2) ITIMES
    4 DO 1000 IDA=1,ITIMES
      READ(1,1) TO,TINF,DEL,E,A,XKO,XL,TL
      READ(1,2) IXNO,N,EPS,ACC,ITYPE
      WRITE(3,601)
      WRITE(3,221)
      WRITE(3,206)XL
      WRITE(3,126)TO
      WRITE(3,122)TINF
      WRITE(3,123)E
      WRITE(3,124)DEL
      WRITE(3,127)XKO
      WRITE(3,125)A
      SIGMA=0.1714E-08
      GAMA=(A+5.)/(A+1.)
      B=(E*SIGMA*((A+1.)**GAMA)/(XKO*DEL*(A+5.)))*2.
      G=2.*E*SIGMA*XKO/(DEL*(A+5.))
      A4 = 1.
      A5 = 0
      A6 = GAMA
      W = .5*(TO-TINF)
      X = 1./SQRT(B*THETA(A,TINF)**(GAMA-2.))
      TL = TINF + W
      A2 = (TO/TINF)**(A + 1.)
  666 A1 = (TL/TINF)**(A + 1.)
      A3 = A1
      XL1 = X*RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
      IF(ABS(XL-XL1).LT.ACC) GO TO 650
      IF(XL1-XL) 620,650,630
  620 TL = TL - W
      W = .5*W
      TL = TL + W
      GO TO 666
  630 W = .5*W
      TL = TL + W
      GO TO 666
```

TABLE XV (cont)

```
 650 WRITE(3,207) TL
     WRITE(3,114) A5,TO
     ITO = IFIX((TO+100)/100.)
     TTO = 100*ITO
     XNO = FLOAT(IXNO)
     L = 1
 343 T = TTO-L*XNO
     L = L + 1
     IF(T.GT.(TO-10)) GO TO 343
     IF(T.LT.(TL+10)) GO TO 370
     A1 = (T/TINF)**(A+1)
     XL2 = X*RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
     WRITE(3,114) XL2,T
     GO TO 343
 370 CONTINUE
     WRITE(3,114) XL1,TL
     QO = SQRT(G*(TO**(A+5.)-TL**(A+5.)-(A+5.)/(A+1.)*TINF**4*(TO**(A+1
    1.)-TL**(A+1.)))) 
     WRITE(3,112) QO
     WRITE(3,1001) N
     WRITE(3,1002) EPS
     WRITE(3,1003) ACC
1001 FORMAT(//,11X,'N  =',I4)
1002 FORMAT(11X,'EPS=',F9.7)
1003 FORMAT(11X,'ACC=',F9.7)
1C00 CONTINUE
     STOP
     END
     FUNCTION RIEMAN (A1,A2,A3,A4,A5,A6,N,EPS)
     DIMENSION A(3),F(3)
     VALUE(A1,A2,A3,A4,A5,A6,X)=1./(SQRT(X**A6-A3**A6-A4*A6*(X-A3)+A5))
 100 DIVA=(A2-A1)/FLOAT(N)
 101 DO 103 I=1,3
 102 A(I)=A1+(FLOAT(2*I-1)/2.0)*DIVA
 103 F(I) =  VALUE(A1,A2,A3,A4,A5,A6,A(I))
 104 END=A1
 105 RIEMAN=0.0
 199 DO 233 I=1,N
1991 RESULT=0.0
 200 ERROR= (((F(3)-2.0*F(2)+F(1))*DIVA)/24.0)*FLOAT(N)
 201 ERR=ABS(ERROR/EPS)
 202 IF(ERR-1.0) 218,218,203
 203 K=SQRT(ERR)+1.0
 204 L=K/2
 205 K=2*L+1
 206 DIVR=DIVA/FLOAT(K)
 207 DIVR2=DIVR/2.0
 208 DO 211 J=1,L
 209 ADD =  VALUE(A1,A2,A3,A4,A5,A6,END+DIVR2)*DIVR
 210 RESULT=RESULT+ADD
 211 END=END+DIVR
 212 END=END+DIVR
```

TABLE XV (cont)

```
213 DO 216 J=1,L
214 ADD =    VALUE(A1,A2,A3,A4,A5,A6,END+DIVR2)*DIVR
215 RESULT=RESULT+ADD
216 END=END+DIVR
217 GO TO 220
218 END=END+DIVA
219 DIVR=DIVA
220 IF(I-1) 222,222,221
221 IF(I-(N-1)) 224,229,229
222 RESULT=RESULT+F(1)*DIVR
223 GO TO 233
224 RESULT=RESULT+F(2)*DIVR
225 F(1)=F(2)
226 F(2)=F(3)
227 F(3) =   VALUE(A1,A2,A3,A4,A5,A6,END+(3./2.)*DIVA)
228 GO TO 233
229 IF(I-N) 230,232,232
230 RESULT=RESULT+F(2)*DIVR
231 GO TO 233
232 RESULT=RESULT+F(3)*DIVR
233 RIEMAN=RIEMAN+RESULT
234 RETURN
    END
```

TABLE XVI

CASE VII COMPUTER PROGRAM

```
C********SOLUTION TO CASE 7 PROBLEM**********
      THETA(X,Y) = (Y**(X + 1.))/(X + 1.)
    1 FORMAT(8F10.4)
    2 FORMAT(2I10,2F10.4,I10)
  112 FORMAT(//,20X,'HEAT FLUX AT ROOT =',F10.2,' BTU/HR-SQFT')
  114 FORMAT(20X,4HX = ,F8.3,4H FT.,3X,4HT = ,F8.2,' DEG.R')
  121 FORMAT(11X,'FIN LENGTH =',F8.3,' FT')
  122 FORMAT(11X,'T INFINITY =',F8.2,' R')
  123 FORMAT(11X,'EMITTANCE =',F6.3)
  124 FORMAT(11X,'FIN HALF THICKNESS =',F12.7,' FT')
  125 FORMAT(11X,'EXPONENT =',F7.4//)
  126 FORMAT(11X,'ROOT TEMP. =',F8.2,' R')
  127 FORMAT(11X,'CONDUCTIVITY =',F7.2)
  404 FORMAT(11X,'MINIMUM TEMP. OF',F8.2,' R OCCURS AT',F8.3,' FT'/)
  405 FORMAT(11X,'FINITE LENGTH FIN WITH BOTH END TEMPERATURES SPECIFIED
     1')
  406 FORMAT(11X,'TEMP. AT OUTER END =',F8.2,' DEG. R')
  413 FORMAT(20X,'FLUX AT OUTER END =',F10.2,' BTU/HR-SQFT')
  701 FORMAT(1H1,10X,'CASE 7A PROBLEM'//)
  704 FORMAT(1H1,10X,'CASE 7B PROBLEM'//)
  705 FORMAT(11X,'TEMP AT END OF FIN EQUALS T INF')
  708 FORMAT(1H1,10X,'CASE 7C PROBLEM'//)
  777 FORMAT(20X,'HEAT FLUX AT INFLECTION POINT =',F10.2,' BTU/HR-SQFT')
  778 FORMAT(20X,'INFLECTION POINT OCCURS AT ',F8.3,' FT')
 1001 FORMAT(//,11X,'N  =',I4)
 1002 FORMAT(11X,'EPS=',F9.7)
 1003 FORMAT(11X,'ACC=',F9.7)
      READ(1,2) ITIMES
    4 DO 1000 IDA=1,ITIMES
      READ(1,1) TO,TINF,DEL,E,A,XKO,XL,TL
      READ(1,2) IXNO,N,EPS,ACC,ITYPE
      SIGMA=0.1714E-08
      GAMA=(A+5.)/(A+1.)
      B=(E*SIGMA*((A+1.)**GAMA)/(XKO*DEL*(A+5.)))*2.
      G=2.*E*SIGMA*XKO/(DEL*(A+5.))
      IF(ABS(TL-TINF).LT.1.)GOTO 740
      IF(TL.LT.TINF)GOTO 702
      IF(TL.GT.TINF)GOTO 760
  702 WRITE(3,701)
      B2=THETA(A,TL)/THETA(A,TINF)
      Z=1-GAMA+GAMA*B2-B2**GAMA
      IF(Z.LT.0)R=ABS(Z)
      IF(Z.GE.0)R=0.
      LMN=0
  703 CONTINUE
      IF(LMN.EQ.1.)WRITE(3,705)
      WRITE(3,405)
      WRITE(3,121)XL
      WRITE(3,126)TO
      WRITE(3,406) TL
      WRITE(3,122)TINF
      WRITE(3,123)E
```

```
      WRITE(3,124)DEL
      WRITE(3,127)XKO
      WRITE(3,125)A
      W=1000.
      A6=GAMA
      B1=THETA(A,TO)/THETA(A,TINF)
      B3=1.
      D=1./SQRT(B*THETA(A,TINF)**(GAMA-2.))
  710 R=R+W
      XL1=D*(RIEMAN(B2,B3,B2,B3,R,A6,N,EPS)+RIEMAN(B3,B1,B2,B3,R,A6,N,EP
     1S))
      IF(ABS(XL1-XL).LT.ACC)GOTO 720
      IF(XL1-XL)715,720,710
  715 R=R-W
      W=.5*W
      GOTO 710
  720 ALFA = 0
      WRITE(3,114) ALFA,TO
      ITO = IFIX((TO+100)/100.)
      TTO = 100*ITO
      XNO = FLOAT(IXNO)
      L = 1
  343 T = TTO - L*XNO
      L = L + 1
      IF(T.GT.(TO-10)) GO TO 343
      IF(T.LT.(TINF + 10)) GO TO 370
      B4 = THETA(A,T)/THETA(A,TINF)
      XL2=D* RIEMAN(B4,B1,B2,B3,R,A6,N,EPS)
      WRITE(3,114) XL2,T
      GO TO 343
  370 B4 = 1
      XL2=D* RIEMAN(B4,B1,B2,B3,R,A6,N,EPS)
      XI=XL2
      WRITE(3,114) XL2,TINF
      IF(LMN.EQ.1) GO TO 373
  371 ITO = IFIX((TINF+100)/100.)
      TTO = 100*ITO
      XNO = FLOAT(IXNO)
      L = 1
  345 T = TTO - L*XNO
      L = L + 1
      IF(T.GT.(TINF - 10)) GO TO 345
      IF(T.LT.(TL + 10)) GO TO 372
      B4 = THETA(A,T)/THETA(A,TINF)
      XL2=XI+D*RIEMAN(B4,B3,B2,B3,R,A6,N,EPS)
      WRITE(3,114) XL2,T
      GO TO 345
  372 B4 = THETA(A,TL)/THETA(A,TINF)
      XL2=XI+D*RIEMAN(B4,B3,B2,B3,R,A6,N,EPS)
      WRITE(3,114) XL2,TL
  373 CONTINUE
  730 PL2=R*B*THETA(A,TINF)**GAMA
```

```
      QO=SQRT(G*(TO**(A+5.)-TL**(A+5.)-TINF**4*(A+5.)/(A+1.)*(TO**(A+1.)
     1-TL**(A+1.)))+XKO*XKO*PL2)
      WRITE(3,112)QO
      QL= XKO*SQRT(PL2)
      IF(LMN.EQ.1) GO TO 776
      Z=THETA(A,TINF)
      ZZ=THETA(A,TL)
      Z2=Z**A6
      ZZ2 = ZZ**A6
      ZZZ=Z**(4./(A+1.))
      PI2=B*Z2*(1.-A6)+PL2-B*ZZ2+B*A6*ZZ*ZZZ
      QI=XKO*SQRT(PI2)
      WRITE(3,778)XI
      WRITE(3,777)QI
  776 WRITE(3,413)QL
      WRITE(3,1001) N
      WRITE(3,1002) EPS
      WRITE(3,1003) ACC
      GOTO 1000
  740 WRITE(3,704)
      R=0.
      B2=1.
      LMN=1
      GOTO 703
  760 WRITE(3,708)
      WRITE(3,405)
      WRITE(3,121)XL
      WRITE(3,126)TO
      WRITE(3,406) TL
      WRITE(3,122)TINF
      WRITE(3,123)E
      WRITE(3,124)DEL
      WRITE(3,127)XKO
      WRITE(3,125)A
      A2=THETA(A,TO)/THETA(A,TINF)
      A1=THETA(A,TL)/THETA(A,TINF)
      A3=A1
      A4=1.
      A5=0.
      A6=GAMA
      D=1./SQRT(B*THETA(A,TINF)**(GAMA-2.))
      IF(ABS(A2-A1).LT..001) GO TO 781
      Z6=D*RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
      IF(Z6.GT.XL+ACC)GOTO 761
      IF(Z6.LT.XL-ACC)GOTO 781
      WRITE(3,114) A5,TO
      ITO = IFIX((TO+100)/100.)
      TTO = 100*ITO
      XNO = FLOAT(IXNO)
      L = 1
  346 T = TTO - L*XNO
      L = L + 1
```

```
      IF(T.GT.(TO-10)) GO TO 346
      IF(T.LT.(TL + 10)) GO TO 374
      A1 = THETA(A,T)/THETA(A,TINF)
      XL2=D*RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
      WRITE(3,114)XL2,T
      GO TO 346
  374 A1 = THETA(A,TL)/THETA(A,TINF)
      XL2=D*RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
      WRITE(3,114)XL2,TL
      QO=SQRT(G*(TO**(A+5.)-TL**(A+5.)-(A+5.)/(A+1.)*TINF**4*(TO**(A+1.)
     1-TL**(A+1.))))
      WRITE(3,112)QO
      QL = 0.
      WRITE(3,413) QL
      WRITE(3,1001) N
      WRITE(3,1002) EPS
      WRITE(3,1003) ACC
      GOTO 1000
  761 W=((TO-TL)/XL)**2/(B*THETA(A,TINF)**GAMA)
  762 A5=A5+W
      XL2=D*RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
      IF(ABS(XL2-XL).LT.ACC)GOTO 771
      IF(XL2-XL)763,771,762
  763 A5=A5-W
      W=.5*W
      GOTO 762
  771 ALFA = 0
      WRITE(3,114) ALFA,TO
      ITO = IFIX((TO+100)/100.)
      TTO = 100*ITO
      XNO = FLOAT(IXNO)
      L = 1
  350 T = TTO - L*XNO
      L = L + 1
      IF(T.GT.(TO-10)) GO TO 350
      IF(T.LT.(TL + 10)) GO TO 380
      A1 = THETA(A,T)/THETA(A,TINF)
      XL1=D*RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
      WRITE(3,114) XL1,T
      GO TO 350
  380 A1 = THETA(A,TL)/THETA(A,TINF)
      XL1=D*RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
      WRITE(3,114) XL1,TL
      PL2=A5*B*THETA(A,TINF)**GAMA
      QO=SQRT(G*(TO**(A+5.)-TL**(A+5.)-(A+5.)/(A+1.)*TINF**4*(TO**(A+1.)
     1-TL**(A+1.)))+XKO*XKO*PL2)
      QL = XKO*SQRT(PL2)
      WRITE(3,112)QO
      WRITE(3,413) QL
      WRITE(3,1001) N
      WRITE(3,1002) EPS
      WRITE(3,1003) ACC
```

```
      GOTO 1000
  781 TM=TINF
      W = TL-TINF
  786 W = .5*W
      TM = TM + W
      IF(ABS(TM-TINF).LT..01) TM = TINF
      A3=THETA(A,TM)/THETA(A,TINF)
      XL2=D*RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)+2*D*RIEMAN(A3,A1,A3,A4,A5,A6
     1,N,EPS)
      IF(ABS(XL2-XL).LT.ACC)GOTO 790
      IF(XL2-XL)785,790,786
  785 TM=TM-W
      GOTO786
  790 XM=D*RIEMAN(A3,A2,A3,A4,A5,A6,N,EPS)
      WRITE(3,404)TM,XM
      ALFA = 0
      WRITE(3,114) ALFA,TO
      ITO = IFIX((TO+100)/100.)
      TTO = 100*ITO
      XNO = FLOAT(IXNO)
      L = 1
  351 T = TTO - L*XNO
      L = L + 1
      IF(T.GT.(TO-10)) GO TO 351
      IF(T.LT.(TM + 10)) GO TO 381
      A7 = THETA(A,T)/THETA(A,TINF)
      XL1=D*RIEMAN(A7,A2,A3,A4,A5,A6,N,EPS)
      WRITE(3,114) XL1,T
      GO TO 351
  381 A7 = THETA(A,TM)/THETA(A,TINF)
      XL1=D*RIEMAN(A7,A2,A3,A4,A5,A6,N,EPS)
      WRITE(3,114) XL1,TM
      ITO = IFIX((TM-100)/100.)
      TTO = 100*ITO
      XNO = FLOAT(IXNO)
      L = 1
  352 T = TTO + L*XNO
      L = L + 1
      IF(T.LT.(TM+10)) GO TO 352
      IF(T.GT.(TL - 10)) GO TO 382
      A1 = THETA(A,T)/THETA(A,TINF)
      XL1=XM+D*RIEMAN(A7,A1,A3,A4,A5,A6,N,EPS)
      WRITE(3,114) XL1,T
      GO TO 352
  382 A1 = THETA(A,TL)/THETA(A,TINF)
      XL1=XM+D*RIEMAN(A7,A1,A3,A4,A5,A6,N,EPS)
      WRITE(3,114) XL1,TL
  796 QO=SQRT(G*(TO**(A+5.)-TM**(A+5.)-(A+5.)/(A+1.)*TINF**4*(TO**(A+1.)
     1-TM**(A+1.))))
      QL=SQRT(G*(TL**(A+5.)-TM**(A+5.)-(A+5.)/(A+1.)*TINF**4*(TL**(A+1.)
     1-TM**(A+1.))))*(-1.)
      WRITE(3,112)QO
```

```
      WRITE(3,413)QL
      WRITE(3,1001) N
      WRITE(3,1002) EPS
      WRITF(3,1003) ACC
 1C00 CONTINUE
      STOP
      END
      FUNCTION RIEMAN (A1,A2,A3,A4,A5,A6,N,EPS)
      DIMENSION A(3),F(3)
      VALUE(A1,A2,A3,A4,A5,A6,X)=1./(SQRT(X**A6-A3**A6-A4*A6*(X-A3)+A5))
  100 DIVA=(A2-A1)/FLOAT(N)
  101 DO 103 I=1,3
  102 A(I)=A1+(FLOAT(2*I-1)/2.0)*DIVA
  103 F(I) =   VALUE(A1,A2,A3,A4,A5,A6,A(I))
  104 END=A1
  105 RIEMAN=0.0
  199 DO 233 I=1,N
 1991 RESULT=0.0
  200 ERROR= (((F(3)-2.0*F(2)+F(1))*DIVA)/24.0)*FLOAT(N)
  201 ERR=ABS(ERROR/EPS)
  202 IF(ERR-1.0) 218,218,203
  203 K=SQRT(ERR)+1.0
  204 L=K/2
  205 K=2*L+1
  206 DIVR=DIVA/FLOAT(K)
  207 DIVR2=DIVR/2.0
  208 DO 211 J=1,L
  209 ADD =   VALUE(A1,A2,A3,A4,A5,A6,END+DIVR2)*DIVR
  210 RESULT=RESULT+ADD
  211 END=END+DIVR
  212 END=END+DIVR
  213 DO 216 J=1,L
  214 ADD =   VALUE(A1,A2,A3,A4,A5,A6,END+DIVR2)*DIVR
  215 RESULT=RESULT+ADD
  216 END=END+DIVR
  217 GO TO 220
  218 END=END+DIVA
  219 DIVR=DIVA
  220 IF(I-1) 222,222,221
  221 IF(I-(N-1)) 224,229,229
  222 RESULT=RESULT+F(1)*DIVR
  223 GO TO 233
  224 RESULT=RESULT+F(2)*DIVR
  225 F(1)=F(2)
  226 F(2)=F(3)
  227 F(3) =   VALUE(A1,A2,A3,A4,A5,A6,END+(3./2.)*DIVA)
  228 GO TO 233
  229 IF(I-N) 230,232,232
  230 RESULT=RESULT+F(2)*DIVR
  231 GO TO 233
  232 RESULT=RESULT+F(3)*DIVR
  233 RIEMAN=RIEMAN+RESULT
```

TABLE XVI (cont)

```
234 RETURN
    END
```

TABLE XVII

CASE VIII COMPUTER PROGRAM

```
C********SOLUTION TO CASE 8 PROBLEM**********
      THETA(X,Y) = (Y**(X + 1.))/(X + 1.)
    1 FORMAT(8F10.4)
    2 FORMAT(2I10,2F10.4,I10)
  112 FORMAT(//,20X,'HEAT FLUX AT ROOT =',F10.2,' BTU/HR-SQFT')
  114 FORMAT(20X,4HX = ,F8.3,4H FT.,3X,4HT = ,F8.2,' DEG.R')
  122 FORMAT(11X,'T INFINITY =',F8.2,' R')
  123 FORMAT(11X,'EMITTANCE =',F6.3)
  124 FORMAT(11X,'FIN HALF THICKNESS =',F12.7,' FT')
  125 FORMAT(11X,'EXPONENT =',F7.4//)
  126 FORMAT(11X,'ROOT TEMP. =',F8.2,' R')
  127 FORMAT(11X,'CONDUCTIVITY =',F7.2)
  206 FORMAT(11X,'FIN LENGTH =',F8.3,' FT.')
  207 FORMAT(11X,'TEMPERATURE AT OUTER END =',F8.2,' DEG. R'/)
  221 FORMAT(11X,'FINITE LENGTH FIN WITH RADIATING END')
  413 FORMAT(20X,'FLUX AT OUTER END =',F10.2,' BTU/HR-SQFT')
  801 FORMAT(1H1,10X,'CASE 8 PROBLEM'//)
 1001 FORMAT(//,11X,'N  =',I4)
 1002 FORMAT(11X,'EPS=',F9.7)
 1003 FORMAT(11X,'ACC=',F9.7)
      READ(1,2) ITIMES
    4 DO 1000 IDA=1,ITIMES
      READ(1,1) TO,TINF,DEL,E,A,XKO,XL,TL
      READ(1,2) IXNO,N,EPS,ACC,ITYPE
      WRITE(3,801)
      WRITE(3,221)
      WRITE(3,206)XL
      WRITE(3,126)TO
      WRITE(3,122)TINF
      WRITE(3,123)E
      WRITE(3,124)DEL
      WRITE(3,127)XKO
      WRITE(3,125)A
      SIGMA=0.1714E-08
      GAMA=(A+5.)/(A+1.)
      B=(E*SIGMA*((A+1.)**GAMA)/(XKO*DEL*(A+5.)))*2.
      G=2.*E*SIGMA*XKO/(DEL*(A+5.))
      Z = 0.
      A4 = 1.
      A2 = THETA(A,TO)/THETA(A,TINF)
      A6 = GAMA
      W = .5*(TO-TINF)
      TL = TINF
      X = 1./SQRT(B*THETA(A,TINF)**(GAMA-2))
      XX = E*SIGMA*DEL*(A+5.)*TINF**(3.-A)/(2*XKO)
  821 TL = TL + W
      A1 = THETA(A,TL)/THETA(A,TINF)
      A3 = A1
      A5 = XX*(A1**(4./(A+1.))-1.)**2
      Y = X*RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
      IF(ABS(XL-Y).LT.ACC) GO TO 820
      IF(Y-XL) 810,820,830
```

```
 810 TL = TL - W
     W = .5*W
     GO TO 821
 830 W = .5*W
     GO TO 821
 820 WRITE(3,207) TL
     WRITE(3,114) Z,TO
     ITO = IFIX((TO+100.)/100.)
     TTO = 100*ITO
     XNO = FLOAT(IXNO)
     L = 1
 823 T = TTO - L*XNO
     L = L + 1
     IF(T.GT.(TO-10)) GO TO 823
     IF(T.LT.(TL+10)) GO TO 870
     A1 = THETA(A,T)/THETA(A,TINF)
     XL1 = X*RIEMAN(A1,A2,A3,A4,A5,A6,N,EPS)
     WRITE(3,114) XL1,T
     GO TO 823
 870 WRITE(3,114) Y,TL
     QO = SQRT(G*(TO**(A+5.)-TL**(A+5.)-A6*TINF**4*(TO**(A+1.)-TL**(A+1
    1.)))+(E*SIGMA*(TL**4-TINF**4))**2)
     QL = E*SIGMA*(TL**4-TINF**4)
     WRITE(3,112) QO
     WRITE(3,413) QL
     WRITE(3,1001) N
     WRITE(3,1002) EPS
     WRITE(3,1003) ACC
1000 CONTINUE
     STOP
     END
     FUNCTION RIEMAN (A1,A2,A3,A4,A5,A6,N,EPS)
     DIMENSION A(3),F(3)
     VALUE(A1,A2,A3,A4,A5,A6,X)=1./(SQRT(X**A6-A3**A6-A4*A6*(X-A3)+A5))
 100 DIVA=(A2-A1)/FLOAT(N)
 101 DO 103 I=1,3
 102 A(I)=A1+(FLOAT(2*I-1)/2.0)*DIVA
 103 F(I) =  VALUE(A1,A2,A3,A4,A5,A6,A(I))
 104 END=A1
 105 RIEMAN=0.0
 199 DO 233 I=1,N
1991 RESULT=0.0
 200 ERROR= (((F(3)-2.0*F(2)+F(1))*DIVA)/24.0)*FLOAT(N)
 201 ERR=ABS(ERROR/EPS)
 202 IF(ERR-1.0) 218,218,203
 203 K=SQRT(ERR)+1.0
 204 L=K/2
 205 K=2*L+1
 206 DIVR=DIVA/FLOAT(K)
 207 DIVR2=DIVR/2.0
 208 DO 211 J=1,L
 209 ADD =   VALUE(A1,A2,A3,A4,A5,A6,END+DIVR2)*DIVR
```

TABLE XVII (cont)

```
210 RESULT=RESULT+ADD
211 END=END+DIVR
212 END=END+DIVR
213 DO 216 J=1,L
214 ADD =    VALUE(A1,A2,A3,A4,A5,A6,END+DIVR2)*DIVR
215 RESULT=RESULT+ADD
216 END=END+DIVR
217 GO TO 220
218 END=END+DIVA
219 DIVR=DIVA
220 IF(I-1) 222,222,221
221 IF(I-(N-1)) 224,229,229
222 RESULT=RESULT+F(1)*DIVR
223 GO TO 233
224 RESULT=RESULT+F(2)*DIVR
225 F(1)=F(2)
226 F(2)=F(3)
227 F(3) =   VALUE(A1,A2,A3,A4,A5,A6,END+(3./2.)*DIVA)
228 GO TO 233
229 IF(I-N) 230,232,232
230 RESULT=RESULT+F(2)*DIVR
231 GO TO 233
232 RESULT=RESULT+F(3)*DIVR
233 RIEMAN=RIEMAN+RESULT
234 RETURN
    END
```

TABLE XVIII

CASE I SAMPLE OUTPUT

```
CASE 1 PROBLEM

INFINITE LENGTH FIN
ROOT TEMP. =  530.00 R
T INFINITY =    0.00 R
EMITTANCE = 0.950
FIN HALF THICKNESS =   0.0208333 FT
CONDUCTIVITY =  29.37
EXPONENT = 0.1980

        X =     0.000 FT.    T =    530.00 DEG.R
        X =     0.289 FT.    T =    500.00 DEG.R
        X =     0.876 FT.    T =    450.00 DEG.R
        X =     1.644 FT.    T =    400.00 DEG.R
        X =     2.682 FT.    T =    350.00 DEG.R
        X =     4.149 FT.    T =    300.00 DEG.R
        X =     6.346 FT.    T =    250.00 DEG.R
        X =     9.924 FT.    T =    200.00 DEG.R
        X =    16.538 FT.    T =    150.00 DEG.R
        X =    31.789 FT.    T =    100.00 DEG.R
        X =    89.532 FT.    T =     50.00 DEG.R

        HEAT FLUX AT ROOT =  11309.14 BTU/HR-SQFT
```

TABLE XX

CASE III SAMPLE OUTPUT

```
CASE 3 PROBLEM


FINITE LENGTH FIN WITH RADIATING END
FIN LENGTH =    3.000 FT.
ROOT TEMP. = 1000.00 R
T INFINITY =     0.00 R
EMITTANCE = 0.950
FIN HALF THICKNESS =    0.0208333 FT
CONDUCTIVITY = 100.00
EXPONENT = 0.0000


TEMPERATURE AT OUTER END =  530.76 DEG. R

         X =     0.000 FT.    T =  1000.00 DEG.R
         X =     0.098 FT.    T =   950.00 DEG.R
         X =     0.210 FT.    T =   900.00 DEG.R
         X =     0.341 FT.    T =   850.00 DEG.R
         X =     0.494 FT.    T =   800.00 DEG.R
         X =     0.678 FT.    T =   750.00 DEG.R
         X =     0.904 FT.    T =   700.00 DEG.R
         X =     1.191 FT.    T =   650.00 DEG.R
         X =     1.583 FT.    T =   600.00 DEG.R
         X =     2.237 FT.    T =   550.00 DEG.R
         X =     2.999 FT.    T =   530.76 DEG.R


         HEAT FLUX AT ROOT =  54723.61 BTU/HR-SQFT
         FLUX AT OUTER END =    129.22 BTU/HR-SQFT


N   =  25
EPS=0.0000010
ACC=0.0010000
```

TABLE XXI

CASE IVa SAMPLE OUTPUT

```
CASE 4A PROBLEM


FINITE LENGTH FIN WITH BOTH END TEMPERATURES SPECIFIED
FIN LENGTH =    3.000 FT
ROOT TEMP. = 1000.00 R
TEMP. AT OUTER END =   400.00 DEG.R
T INFINITY =      0.00 R
EMITTANCE = 0.400
FIN HALF THICKNESS =    0.0208333 FT
CONDUCTIVITY = 100.00
EXPONENT = 0.0000


          X =    0.000 FT.     T = 1000.00 R
          X =    0.137 FT.     T =  950.00 R
          X =    0.293 FT.     T =  900.00 R
          X =    0.461 FT.     T =  850.00 R
          X =    0.653 FT.     T =  800.00 R
          X =    0.867 FT.     T =  750.00 R
          X =    1.106 FT.     T =  700.00 R
          X =    1.368 FT.     T =  650.00 R
          X =    1.656 FT.     T =  600.00 R
          X =    1.966 FT.     T =  550.00 R
          X =    2.296 FT.     T =  500.00 R
          X =    2.642 FT.     T =  450.00 R
          X =    2.999 FT.     T =  400.00 R


          HEAT FLUX AT ROOT =   38644.31 BTU/HR-SQFT
          FLUX AT OUTER END =   13802.48 BTU/HR-SQFT


N  =  25
EPS=0.0000010
ACC=0.0010000
```

TABLE XXII

CASE IVb SAMPLE OUTPUT

```
CASE 4B PRCBLEM


FINITE LENGTH FIN WITH BCTH ENC TEMPERATLRES SPECIFIED
FIN LENGTH =    3.000 FT
RCCT TEMP. = 1000.00 R
TEMP. AT CLTER END =   532.84 DEG.R
T INFINITY =     0.00 R
EMITTANCE = 0.950
FIN HALF THICKNESS =    0.0208333 FT
CCNDUCTIVITY = 100.00
EXPONENT = 0.0000


         X =     0.000 FT.     T = 1000.00 R
         X =     0.098 FT.     T =  950.00 R
         X =     0.210 FT.     T =  900.00 R
         X =     0.341 FT.     T =  850.00 R
         X =     0.495 FT.     T =  800.00 R
         X =     0.679 FT.     T =  750.00 R
         X =     0.905 FT.     T =  700.00 R
         X =     1.194 FT.     T =  650.00 R
         X =     1.589 FT.     T =  600.00 R
         X =     2.261 FT.     T =  550.00 R
         X =     3.000 FT.     T =  532.84 R


         HEAT FLUX AT RCOT =   54699.71 BTU/HR-SQFT
         FLLX AT OUTER END =       0.00 BTU/HR-SQFT


N   =   25
EPS=0.0000010
ACC=0.0010000


N   =   25
EPS=0.0000010
ACC=0.0010000
```

TABLE XXIII

CASE IVc SAMPLE OUTPUT

```
CASE 4C PRCBLEM

FINITE LENGTH FIN WITH BCTH END TEMPERATURES SPECIFIED
FIN LENGTH =    2.000 FT
RCOT TEMP. =  800.00 R
TEMP. AT OUTER END =  800.00 DEG.R
T INFINITY =     0.00 R
EMITTANCE = 1.000
FIN HALF THICKNESS =   0.0208333 FT
CONDUCTIVITY =  50.00
EXPONENT = 0.0000

MINIMUM TEMP. OF  639.84 R OCCURS AT   1.000 FT

        X =    0.000 FT.    T =   800.00 R
        X =    0.149 FT.    T =   750.00 R
        X =    0.354 FT.    T =   700.00 R
        X =    0.725 FT.    T =   650.00 R
        X =    1.000 FT.    T =   639.84 R
        X =    1.274 FT.    T =   650.00 R
        X =    1.645 FT.    T =   700.00 R
        X =    1.850 FT.    T =   750.00 R
        X =    2.000 FT.    T =   800.00 R

        HEAT FLUX AT ROOT =   19045.10 BTU/HR-SQFT
        FLUX AT OUTER END = -19045.10 BTU/HR-SQFT
```

TABLE XXIV

CASE V SAMPLE OUTPUT

```
CASE 5 PROBLEM

INFINITE LENGTH FIN
ROOT TEMP. = 1000.00 R
T INFINITY =  160.00 R
EMITTANCE = 0.400
FIN HALF THICKNESS =    0.0208333 FT
CONDUCTIVITY = 100.0
EXPONENT = 0.0000

        X =     0.000 FT.    T =   1000.00 DEG.R
        X =     0.147 FT.    T =    950.00 DEG.R
        X =     0.315 FT.    T =    900.00 DEG.R
        X =     0.508 FT.    T =    850.00 DEG.R
        X =     0.732 FT.    T =    800.00 DEG.R
        X =     0.994 FT.    T =    750.00 DEG.R
        X =     1.304 FT.    T =    700.00 DEG.R
        X =     1.675 FT.    T =    650.00 DEG.R
        X =     2.127 FT.    T =    600.00 DEG.R
        X =     2.685 FT.    T =    550.00 DEG.R
        X =     3.389 FT.    T =    500.00 DEG.R
        X =     4.301 FT.    T =    450.00 DEG.R
        X =     5.521 FT.    T =    400.00 DEG.R
        X =     7.231 FT.    T =    350.00 DEG.R
        X =     9.787 FT.    T =    300.00 DEG.R
        X =    14.056 FT.    T =    250.00 DEG.R
        X =    23.198 FT.    T =    200.00 DEG.R

        HEAT FLUX AT ROOT =  36229.72 BTU/HR-SQFT

N  =  25
EPS=0.0000010
ACC=0.0010000
```

TABLE XXV

CASE VI SAMPLE OUTPUT

```
CASE 6 PROBLEM


FINITE LENGTH FIN WITH INSULATED END
FIN LENGTH =    6.000 FT.
ROOT TEMP. = 1000.00 R
T INFINITY =  160.00 R
EMITTANCE = 0.400
FIN HALF THICKNESS =    0.0208333 FT
CONDUCTIVITY = 100.00
EXPONENT = 0.0000


TEMPERATURE AT OUTER END =  467.00 DEG. R

         X =    0.000 FT.    T =  1000.00 DEG.R
         X =    0.149 FT.    T =   950.00 DEG.R
         X =    0.320 FT.    T =   900.00 DEG.R
         X =    0.517 FT.    T =   850.00 DEG.R
         X =    0.747 FT.    T =   800.00 DEG.R
         X =    1.020 FT.    T =   750.00 DEG.R
         X =    1.348 FT.    T =   700.00 DEG.R
         X =    1.752 FT.    T =   650.00 DEG.R
         X =    2.266 FT.    T =   600.00 DEG.R
         X =    2.959 FT.    T =   550.00 DEG.R
         X =    4.033 FT.    T =   500.00 DEG.R
         X =    6.001 FT.    T =   467.00 DEG.R

         HEAT FLUX AT ROOT =  35844.32 BTU/HR-SQFT


N  =  25
EPS=0.0000010
ACC=0.0010000
```

TABLE XXVI

CASE VIIa SAMPLE OUTPUT

```
CASE 7A PROBLEM


FINITE LENGTH FIN WITH BOTH END TEMPERATURES SPECIFIED
FIN LENGTH =  15.000 FT
ROOT TEMP. = 1000.00 R
TEMP. AT OUTER END =  300.00 DEG. R
T INFINITY =  500.00 R
EMITTANCE = 0.400
FIN HALF THICKNESS =   0.0208333 FT
CONDUCTIVITY = 100.00
EXPONENT = 0.0000


          X =     0.000 FT.    T =  1000.00 DEG.R
          X =     0.165 FT.    T =   950.00 DEG.R
          X =     0.357 FT.    T =   900.00 DEG.R
          X =     0.586 FT.    T =   850.00 DEG.R
          X =     0.862 FT.    T =   800.00 DEG.R
          X =     1.204 FT.    T =   750.00 DEG.R
          X =     1.643 FT.    T =   700.00 DEG.R
          X =     2.235 FT.    T =   650.00 DEG.R
          X =     3.103 FT.    T =   600.00 DEG.R
          X =     4.594 FT.    T =   550.00 DEG.R
          X =     7.954 FT.    T =   500.00 DEG.R
          X =    11.383 FT.    T =   450.00 DEG.R
          X =    13.076 FT.    T =   400.00 DEG.R
          X =    14.172 FT.    T =   350.00 DEG.R
          X =    15.000 FT.    T =   300.00 DEG.R


          HEAT FLUX AT ROOT =   32722.12 BTU/HR-SQFT
          INFLECTION POINT OCCURS AT     7.954 FT
          HEAT FLUX AT INFLECTION POINT =    1095.39 BTU/HR-SQFT
          FLUX AT OUTER END =    6747.94 BTU/HR-SQFT


N  =  25
EPS=0.0000010
ACC=0.0010000
```

TABLE XXVII

CASE VIIb SAMPLE OUTPUT

```
CASE 7B PROBLEM

TEMP AT END OF FIN EQUALS T INF
FINITE LENGTH FIN WITH BOTH END TEMPERATURES SPECIFIED
FIN LENGTH =  15.000 FT
ROOT TEMP. = 1000.00 R
TEMP. AT OUTER END =  500.00 DEG. R
T INFINITY =  500.00 R
EMITTANCE = 0.400
FIN HALF THICKNESS =   0.0208333 FT
CONDUCTIVITY = 100.00
EXPONENT = 0.0

         X =    0.0     FT.    T =   1000.00 DEG.R
         X =    0.165   FT.    T =    950.00 DEG.R
         X =    0.358   FT.    T =    900.00 DEG.R
         X =    0.586   FT.    T =    850.00 DEG.R
         X =    0.863   FT.    T =    800.00 DEG.R
         X =    1.206   FT.    T =    750.00 DEG.R
         X =    1.647   FT.    T =    700.00 DEG.R
         X =    2.244   FT.    T =    650.00 DEG.R
         X =    3.128   FT.    T =    600.00 DEG.R
         X =    4.719   FT.    T =    550.00 DEG.R
         X =   15.001   FT.    T =    500.00 DEG.R

         HEAT FLUX AT ROOT =   32703.72 BTU/HR-SQFT
         FLUX AT OUTER END =      60.13 BTU/HR-SQFT

N  =  25
EPS=0.0000100
ACC=0.0010000
```

TABLE XXVIII

CASE VIIc SAMPLE OUTPUT

```
CASE 7C PRCBLEM


FINITE LENGTH FIN WITH BOTH END TEMPERATURES SPECIFIED
FIN LENGTH =  15.000 FT
RCCT TEMP. = 1000.00 R
TEMP. AT CUTER END =  200.00 DEG. R
T INFINITY =  160.00 R
EMITTANCE = 0.400
FIN HALF THICKNESS =   0.0208333 FT
CONDUCTIVITY = 100.00
EXPONENT = 0.0000


        X =     0.000 FT.    T =  1000.00 DEG.R
        X =     0.147 FT.    T =   950.00 DEG.R
        X =     0.315 FT.    T =   900.00 DEG.R
        X =     0.508 FT.    T =   850.00 DEG.R
        X =     0.731 FT.    T =   800.00 DEG.R
        X =     0.992 FT.    T =   750.00 DEG.R
        X =     1.301 FT.    T =   700.00 DEG.R
        X =     1.671 FT.    T =   650.00 DEG.R
        X =     2.118 FT.    T =   600.00 DEG.R
        X =     2.669 FT.    T =   550.00 DEG.R
        X =     3.360 FT.    T =   500.00 DEG.R
        X =     4.243 FT.    T =   450.00 DEG.R
        X =     5.397 FT.    T =   400.00 DEG.R
        X =     6.935 FT.    T =   350.00 DEG.R
        X =     9.003 FT.    T =   300.00 DEG.R
        X =    11.716 FT.    T =   250.00 DEG.R
        X =    15.000 FT.    T =   200.00 DEG.R


        HEAT FLUX AT RCCT =  36256.70 BTU/HR-SQFT
        FLUX AT OUTER END =   1437.65 BTU/HR-SQFT


N   =  25
EPS=0.0000010
ACC=0.0010000
```

TABLE XXIX

CASE VIIc SAMPLE OUTPUT

```
CASE 7C PRCBLEM


FINITE LENGTH FIN WITH BOTH END TEMPERATURES SPECIFIED
FIN LENGTH =  15.000 FT
RCOT TEMP. = 1000.00 R
TEMP. AT CUTER END =  6CC.CO DEG. R
T INFINITY =  500.CO R
EMITTANCE = 0.400
FIN HALF THICKNESS =   0.0208333 FT
CONDUCTIVITY = 100.00
EXPONENT = C.0000


MINIMUM TEMP. OF  516.C7 R OCCURS AT   9.039 FT

        X =     0.000 FT.   T =  1000.00 DEG.R
        X =     0.165 FT.   T =   950.00 DEG.R
        X =     0.358 FT.   T =   900.00 DEG.R
        X =     0.586 FT.   T =   850.00 DEG.R
        X =     0.863 FT.   T =   800.00 DEG.R
        X =     1.207 FT.   T =   750.00 DEG.R
        X =     1.649 FT.   T =   700.00 DEG.R
        X =     2.247 FT.   T =   650.00 DEG.R
        X =     3.139 FT.   T =   600.00 DEG.R
        X =     4.774 FT.   T =   550.00 DEG.R
        X =     9.039 FT.   T =   516.07 DEG.R
        X =    13.335 FT.   T =   550.00 DEG.R
        X =    14.969 FT.   T =   600.00 DEG.R


        HEAT FLUX AT RCCT =  32697.C7 BTU/HR-SQFT
        FLUX AT OUTER END =  -4432.69 BTU/HR-SQFT


N  =  25
EPS=0.000CC10
ACC=0.0010CC0
```

TABLE XXX

CASE VIII SAMPLE OUTPUT

```
CASE 8 PRCBLEM

FINITE LENGTH FIN WITH RADIATING END
FIN LENGTH =    2.000 FT.
RCOT TEMP. =  530.00 R
T INFINITY =  160.00 R
EMITTANCE = 0.950
FIN HALF THICKNESS =   0.0416666 FT
CONDUCTIVITY =  29.37
EXPONENT = 0.1980

TEMPERATURE AT OUTER END =  483.48 DEG. R

         X =    0.000 FT.    T =    530.00 DEG.R
         X =    0.803 FT.    T =    500.00 DEG.R
         X =    2.001 FT.    T =    483.48 DEG.R

         HEAT FLUX AT ROOT =   4903.88 BTU/HR-SQFT
         FLUX AT OUTER END =     87.90 BTU/HR-SQFT

N  =  25
EPS=0.0000010
ACC=0.0010000
```

APPENDIX B

THERMAL CONDUCTIVITY OF ALUMINUM

In 1958, Lucks and Deem (5) reported values for the thermal properties of several engineering materials as a function of temperatures. One of these materials was 2024-T4 aluminum which had been heated to 600° F for one hour and then air cooled before testing.

The thermal conductivity of this material varies considerably with temperature but the variation may be approximated by the equation

$$k = 29.37\ (T)^{0.1980}$$

In this equation T has units of degrees Rankine and k has units of btu/hr sqft R. This equation will predict the thermal conductivity of the material within 2% over the temperature range 210°R to 860°R. Tabulated values of actual and calculated values of the thermal conductivity are given in Table XXXI.

TABLE XXXI

Thermal Conductivity of
2024 Aluminum

Temperature (°R)	True Conductivity (Btu/hr sqft R)	Calculated Conductivity (Btu/hr sqft R)
210	84	85
260	88	88
360	95	94
528	103	102
660	107	106
860	110	112

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13. ABSTRACT

The studies described in this report clarify the effects of some of the limitations imposed by the laws of thermal similitude, and determine the thermal modeling laws for a heat pipe.

In Volume 1 solutions were presented for the steady-state temperature distribution and heat transfer in a radiating fin having temperature dependent thermal conductivity. Using these solutions, modeling prediction errors were determined for fin type prototype/model systems with dimensional distortions, with material having temperature dependent thermal conductivity, and with low prototype temperatures. These prediction discrepancies ranged from very small errors to errors in heat transfer rate as high as 75% in a severely distorted model.

In Volume 2 the thermal modeling laws for a heat pipe were derived and experimentally verified. It was observed that prototype thermal behavior could be predicted, from model data, to within 10 F over the temperature range tested (140 to 330 F). Heat pipe failure due to capillary failure was also predictible to within ±10%.

A flexible heat pipe was also designed and experimentally tested. Performance was not degraded under conditions of bending.

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