

Copies

**SERIES SOLUTION OF LAMINAR INCOMPRESSIBLE MIXING
OF TWO STREAMS WITH DIFFERENT VELOCITIES**

L. B. ELDRENKAMP

J. XERIKOS

K. D. RESIDE

T. P. TORDA

UNIVERSITY OF ILLINOIS

MAY 1954



AUG 4 1955

AERONAUTICAL RESEARCH LABORATORY

CONTRACT No. AF 33(038)-21251

PROJECT No. 1363

TASK No. 70125

WRIGHT AIR DEVELOPMENT CENTER
AIR RESEARCH AND DEVELOPMENT COMMAND
UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

Carpenter Litho & Prtg. Co., Springfield, O.
200 - 20 May 1955

Contracts

FOREWORD

This report was prepared in the Department of Aeronautical Engineering, University of Illinois, Urbana, Illinois, on Contract AF 33(038)-21251, Project No. 1363, Series Solution of Laminar Incompressible Mixing of Two Streams With Different Velocities, Task No. 70125, Theoretical Studies of the Mixing Process of a Jet Impinging Into a Stream of Large Mass. The work was administered under the direction of the Aeronautical Research Laboratory, Wright Air Development Center, Mr. Emil Walk as project engineer. The persons responsible for the research work presented and for the preparation of the technical report are: Mr. L. B. Eldrenkamp and Mr. J. Kerikos, Research Assistants, Mr. K. D. Reside, Undergraduate Assistant, and Dr. T. P. Torda, Professor, Project Director. Several graduate and undergraduate computers also worked on this project.

WADC TR 54-275

Contrails

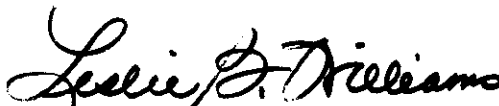
ABSTRACT

The analysis of laminar, incompressible, viscous mixing of two streams of different velocities is presented. The influence of the upstream boundary layers, which develop along the plate, separating the streams, is taken into account. The von Karman integral concept is applied. The width of the mixing region and the velocity distribution in the region are evaluated through the use of power series. An illustrative numerical example is presented.

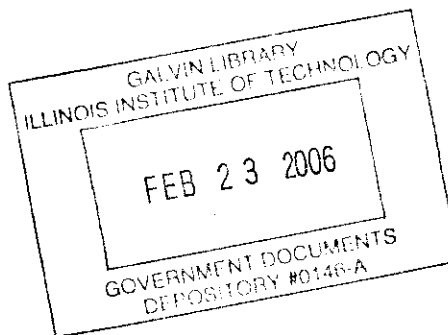
PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:



LESLIE B. WILLIAMS, Colonel, USAF
Chief, Aeronautical Research Laboratory
Directorate of Research



WADC TR 54-275

111

Contrails

TABLE OF CONTENTS

Introduction	1
Analysis	3
Numerical Example	10
Discussion	13
Conclusions	15
Figures	17
Tables	23
Bibliography	25
Appendix I	26
Appendix II	29

Contrails

INTRODUCTION

The first investigation of the influence of the upstream boundary layers on a symmetric mixing region behind a flat plate was completed in 1951 by T. P. Torda^{(1)*}. A quartic polynomial was used to approximate the velocity profile in the mixing region. In 1952, W. O. Ackermann⁽²⁾ used cosine curves to approximate the velocity profiles for the mixing of two streams of different velocities. Neither of the approximations give accurate results for the mixing region close to the trailing edge of the separating plate, since, in the physical case, a cusp in the velocity profile exists at the end of the plate.

A cosine curve approximation is made in the present work in preference to a quartic polynomial, since the former yields a velocity distribution with a sharper curvature at the beginning of mixing.

Power series in terms of the axial coordinate are used to determine the boundaries of and the velocities in the mixing region. These are substituted into the integrated forms of the boundary layer momentum and energy equations. The first terms in these series are known from the initial conditions. The beginning of the mixing region corresponds to the end of the plate and the boundary layer thicknesses there are given by the Blasius solution⁽³⁾. Each additional term of the series is then determined from the previous one. However, recurrence relationships cannot be set up due to the complicated nature of the

*Superscripts denote references.

Central
expanded forms of the differential equations. Consequently, the convergence of the series cannot be formally established and can only be indicated by successively taking additional terms in the series and observing their influence on the solution.

Contrails

ANALYSIS

The laminar, incompressible mixing of two streams of different velocities is investigated. The boundary layers developed along the flat plate separating the two streams are taken into account. The von Karman integral concept is used, i.e., the equations are integrated across the mixing region. This process eliminates one of the independent variables.

The boundary layer momentum equation is

$$u \left[\frac{\partial u}{\partial x} \right] + v \left[\frac{\partial u}{\partial y} \right] = \nu \left[\frac{\partial^2 u}{\partial y^2} \right] \quad (1)$$

The energy equation is

$$u^2 \left[\frac{\partial u}{\partial x} \right] + uv \left[\frac{\partial u}{\partial y} \right] = \nu u \left[\frac{\partial^2 u}{\partial y^2} \right] \quad (2)$$

The continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3)$$

The boundary conditions are

for

$$\begin{array}{lll} y = \delta_1 & u = U_1 & \frac{\partial u}{\partial y} = 0 \\ y = \delta_0 & u = U_0 & \frac{\partial u}{\partial y} = 0 \\ y = \delta_2 & u = U_2 & \frac{\partial u}{\partial y} = 0 \end{array} \quad (4)$$

Contrails

In the above

x, y are the cartesian coordinates

u, v are the velocities in the mixing region in the x, y directions respectively.

U is the velocity of the potential flow (outside the mixing region)

δ is the thickness of the mixing region (from the x axis to the boundary of the mixing region)

subscripts

1 refers to the faster stream

0 refers to the line δ_0

2 refers to the slower stream

Integration of equations (1) and (2) is carried out in detail in the Appendix I.

The mixing zone is divided into region A, where $\delta_0 \leq y \leq \delta_1$ and region B, where $\delta_2 \leq y \leq \delta_0$, see Fig. 1. Two cosine curves were used to approximate the velocity distribution in these regions. These curves meet and are tangent to each other at the line δ_0 . The boundary conditions in region A are satisfied by the non-dimensional expression

$$\left[\frac{u}{U_1} \right]_A = \frac{1}{2} (1 - Q) \cos \left[\frac{\pi}{1-k} (1 - \eta) \right] + \frac{1}{2} (1 + Q) \quad (5)$$

and the boundary conditions in region B are satisfied by the non-dimensional expression

$$\left[\frac{u}{U_1} \right]_B = \frac{1}{2} (m - Q) \cos \left[\frac{\pi}{k-n} (\eta - n) \right] + \frac{1}{2} (m + Q) \quad (6)$$

Contrails

where

$$\eta = \frac{U_2}{\delta_1}$$

$$k = \frac{\delta_0}{\delta_1}$$

$$n = \frac{\delta_2}{\delta_1}$$

$$Q = \frac{U_0}{U_1}$$

$$m = \frac{U_2}{U_1}$$

Introducing equations (5) and (6) into the integrated momentum and energy equations and evaluating the integrals (see Appendix I) results in the following four equations in δ , Q , k and n . (The momentum equations for region A and region B are added resulting in a simplified equation).

$$M_{A+B}: \frac{d}{dx} \left\{ \delta_1 [Q^2(-3n+3) + Q(\alpha_1 k + \alpha_2 n + \alpha_3) + (\alpha_4 k + \alpha_5 n + \alpha_6)] \right\} = 0 \quad (7)$$

$$M_A: \frac{d}{dx} \left\{ \delta_1 [Q^2(-3k+3) + Q(-2k + 4\alpha_7 n + \alpha_8) + (\alpha_9 k + 4\alpha_{10} n + \alpha_{11})] \right\} - Q \frac{d}{dx} \left\{ 4\delta_1 [Q(-k + \alpha_7 n + \alpha_{12}) + (\alpha_{13} k + \alpha_{10} n - \alpha_{12})] \right\} = 0 \quad (8)$$

$$E_A: (1-k) \delta_1 \frac{d}{dx} \left\{ \delta_1 [Q^3(-5k+5) + Q^2(-3k+3) + Q(-3k + 8\alpha_7 n + \alpha_{14}) + (\alpha_{15} k + 8\alpha_{10} n + \alpha_{16})] \right\} - (1-k) \delta_1 Q^2 \frac{d}{dx} \left\{ 8\delta_1 [Q(-k + \alpha_7 n + \alpha_{12}) + (\alpha_{13} k + \alpha_{10} n - \alpha_{12})] \right\} + A(1-Q^2) = 0 \quad (9)$$

$$E_B: (k-n) \delta_1 \frac{d}{dx} \left\{ \delta_1 [Q^3(5k-5n) + Q^2(\alpha_{17} k - \alpha_{17} n) + Q(\alpha_{18} k - \frac{\alpha_{18} \alpha_6}{3} n + 8\alpha_{10}) + (\alpha_{19} k + \alpha_{20} n - 8\alpha_{10})] \right\} - (k-n) \delta_1 Q^2 \frac{d}{dx} \left\{ 8\delta_1 [Q(k - \alpha_7 n - \alpha_{12}) + (-\alpha_{13} k - \alpha_{10} n + \alpha_{12})] \right\} + A(m-Q)^2 = 0 \quad (10)$$

Contrails

where

$$\alpha_1 = 2(m-1)$$

$$\alpha_{11} = -5 + \frac{4m}{m+1}$$

$$\alpha_2 = -2m + \frac{8m}{m+1}$$

$$\alpha_{12} = \frac{1}{m+1}$$

$$\alpha_3 = 2 - \frac{8m}{m+1}$$

$$\alpha_{13} = -1 + \frac{m}{m+1} - \frac{m^2}{m+1}$$

$$\alpha_4 = -3 + 3m^2 + \frac{8m}{m+1} - \frac{8m^2}{m+1}$$

$$\alpha_{14} = 3 - \frac{8m}{m+1}$$

$$\alpha_5 = -m^2 \alpha_{14}$$

$$\alpha_{15} = -5 + \frac{8m}{m+1} - \frac{8m^2}{m+1}$$

$$\alpha_6 = -5 + \frac{8m}{m+1}$$

$$\alpha_{16} = -11 + \frac{8m}{m+1}$$

$$\alpha_7 = \frac{m}{m+1}$$

$$\alpha_{17} = 3m$$

$$\alpha_8 = -\frac{\alpha_1}{m+1}$$

$$\alpha_{18} = m \alpha_{17}$$

$$\alpha_9 = -3 + \frac{4m}{m+1} - \frac{4m^2}{m+1}$$

$$\alpha_{19} = 5m^3 + \frac{8m^2}{m+1} - \frac{8m^3}{m+1}$$

$$\alpha_{10} = -m \alpha_7$$

$$\alpha_{20} = 11m^3 - \frac{8m^3}{m+1}$$

$$A = \frac{4\nu\pi^2}{U_1}$$

(11)

Contrails

The following power series for each of the four variables are assumed

$$\begin{aligned}\delta_1 &= \delta_1(\xi) = \sum_{r=0}^{\infty} a_r \xi^r \\ Q &= Q(\xi) = \sum_{r=0}^{\infty} b_r \xi^r \\ k &= k(\xi) = \sum_{r=0}^{\infty} c_r \xi^r \\ n &= n(\xi) = \sum_{r=0}^{\infty} d_r \xi^r\end{aligned}\tag{12}$$

where $\xi = \frac{x}{L}$ and L is the length of the mixing region, defined as the distance from the trailing edge of the separating plate to the point at which $k=n$.

The transformed conditions at the beginning and at the end of the region are

$$\text{for } \xi = 0, \quad Q = 0 \quad \text{and} \quad k = 0$$

$$\xi = 1, \quad Q = m \quad k = n\tag{13}$$

therefore

$$m = \sum_{r=0}^{\infty} b_r, \quad \sum_{r=0}^{\infty} c_r = \sum_{r=0}^{\infty} d_r$$

The series are substituted into equations (7), (8), (9) and (10) and the following transformation of variables is applied

$$\frac{d}{dx} = \frac{d}{d\xi} \frac{d\xi}{dx} = \frac{1}{L} \frac{d}{d\xi}\tag{14}$$

Contrails

This yields

$$M_{A+B} : \frac{1}{L} \frac{d}{d\xi} \left[\sum_{r=0}^{\infty} a_r \xi^r \left\{ \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} b_s b_{r-s} \xi^r [-3 \sum_{r=0}^{\infty} d_r \xi^r + 3] + \sum_{r=0}^{\infty} b_r \xi^r \right. \right. \\ \left. \left. [\alpha_1 \sum_{r=0}^{\infty} c_r \xi^r + \alpha_2 \sum_{r=0}^{\infty} d_r \xi^r + \alpha_3] + [\alpha_4 \sum_{r=0}^{\infty} c_r \xi^r + \alpha_{10} \sum_{r=0}^{\infty} d_r \xi^r + \alpha_6] \right\} \right] = 0 \quad (15)$$

$$M_A : \frac{1}{L} \frac{d}{d\xi} \left[\sum_{r=0}^{\infty} a_r \xi^r \left\{ \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} b_s b_{r-s} \xi^r [-3 \sum_{r=0}^{\infty} d_r \xi^r + 3] + \sum_{r=0}^{\infty} b_r \xi^r [-2 \sum_{r=0}^{\infty} c_r \xi^r \right. \right. \\ \left. \left. + 4 \alpha_7 \sum_{r=0}^{\infty} d_r \xi^r + \alpha_8] + [\alpha_9 \sum_{r=0}^{\infty} c_r \xi^r + 4 \alpha_{10} \sum_{r=0}^{\infty} d_r \xi^r + \alpha_{11}] \right\} \right] - \left(\sum_{r=0}^{\infty} b_r \xi^r \right) \\ \frac{1}{L} \frac{d}{d\xi} \left[4 \sum_{r=0}^{\infty} d_r \xi^r \left\{ \sum_{r=0}^{\infty} b_r \xi^r [- \sum_{r=0}^{\infty} c_r \xi^r + \alpha_7 \sum_{r=0}^{\infty} d_r \xi^r + \alpha_{12}] + [\alpha_{13} \sum_{r=0}^{\infty} c_r \xi^r \right. \right. \\ \left. \left. + \alpha_{10} \sum_{r=0}^{\infty} d_r \xi^r - \alpha_{12}] \right\} \right] = 0 \quad (16)$$

$$E_A : (1 - \sum_{r=0}^{\infty} c_r \xi^r) \left(\sum_{r=0}^{\infty} a_r \xi^r \right) \frac{1}{L} \frac{d}{d\xi} \left[\sum_{r=0}^{\infty} a_r \xi^r \left\{ \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} b_p b_{s-p} b_{r-s} \xi^r \right. \right. \\ \left. \left. [-5 \sum_{r=0}^{\infty} c_r \xi^r + 5] + \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} b_s b_{r-s} \xi^r [-3 \sum_{r=0}^{\infty} c_r \xi^r + 3] + \sum_{r=0}^{\infty} b_r \xi^r [-3 \sum_{r=0}^{\infty} c_r \xi^r \right. \right. \\ \left. \left. + 8 \alpha_7 \sum_{r=0}^{\infty} d_r \xi^r + \alpha_5] + [\alpha_{15} \sum_{r=0}^{\infty} c_r \xi^r + 8 \alpha_{10} \sum_{r=0}^{\infty} d_r \xi^r + \alpha_{16}] \right\} \right] \\ - (1 - \sum_{r=0}^{\infty} c_r \xi^r) \left(\sum_{r=0}^{\infty} a_r \xi^r \right) \left(\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} b_s b_{r-s} \xi^r \right) \frac{1}{L} \frac{d}{d\xi} \left[8 \sum_{r=0}^{\infty} a_r \xi^r \left\{ \sum_{r=0}^{\infty} b_r \xi^r [- \sum_{r=0}^{\infty} c_r \xi^r \right. \right. \\ \left. \left. + \alpha_7 \sum_{r=0}^{\infty} d_r \xi^r + \alpha_{12}] + [\alpha_{13} \sum_{r=0}^{\infty} c_r \xi^r + \alpha_{10} \sum_{r=0}^{\infty} d_r \xi^r - \alpha_{12}] \right\} \right] \\ + A (1 - \sum b_r \xi^r)^2 = 0 \quad (17)$$

$$E_B : \left(\sum_{r=0}^{\infty} c_r \xi^r - \sum_{r=0}^{\infty} d_r \xi^r \right) \left(\sum_{r=0}^{\infty} a_r \xi^r \right) \frac{1}{L} \frac{d}{d\xi} \left[\sum_{r=0}^{\infty} a_r \xi^r \left\{ \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} b_p b_{s-p} b_{r-s} \xi^r \right. \right. \\ \left. \left. [5 \sum_{r=0}^{\infty} c_r \xi^r - 5 \sum_{r=0}^{\infty} d_r \xi^r] + \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} b_s b_{r-s} \xi^r [\alpha_{17} \sum_{r=0}^{\infty} c_r \xi^r - \alpha_{17} \sum_{r=0}^{\infty} d_r \xi^r] \right. \right. \\ \left. \left. + \sum_{r=0}^{\infty} b_r \xi^r [\alpha_{18} \sum_{r=0}^{\infty} c_r \xi^r - \frac{\alpha_{18} \alpha_6}{3} \sum_{r=0}^{\infty} d_r \xi^r + 8 \alpha_{10}] + [\alpha_{19} \sum_{r=0}^{\infty} c_r \xi^r \right. \right. \\ \left. \left. + \alpha_{20} \sum_{r=0}^{\infty} d_r \xi^r - 8 \alpha_{10}] \right\} \right] - \left(\sum_{r=0}^{\infty} c_r \xi^r - \sum_{r=0}^{\infty} d_r \xi^r \right) \left(\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} b_p b_{s-p} b_{r-s} \xi^r \right)$$

$$\frac{1}{L} \frac{d}{d\xi} \left[\delta \sum_{r=0}^{\infty} a_r \xi^r \left\{ \sum_{r=0}^{\infty} b_r \xi^r \left[\sum_{r=0}^{\infty} c_r \xi^r - \alpha_7 \sum_{r=0}^{\infty} d_r \xi^r - \alpha_{12} \right] + \left[-\alpha_{13} \sum_{r=0}^{\infty} c_r \xi^r - \alpha_{10} \sum_{r=0}^{\infty} d_r \xi^r + \alpha_{12} \right] \right\} \right] + A \left(m - \sum_{r=0}^{\infty} b_r \xi^r \right)^2 = 0$$

(18)

As an illustration, differentiation of equations (15), (16), (17) and (18) yields the equations for the determination of the flow variables in final form. The differentiated form of the combined momentum equation is given below

$$M_{A+B}: \frac{1}{L} \left[-3 \sum_{r=0}^{\infty} \sum_{s=0}^r \sum_{p=0}^s \sum_{q=0}^p a_q b_{p-q} b_{s-p} d_{r-s} \xi^{r-1} + 3 \sum_{r=0}^{\infty} \sum_{s=0}^r \sum_{p=0}^s a_p b_{s-p} b_{r-s} \xi^{r-1} + \alpha_1 \sum_{r=0}^{\infty} \sum_{s=0}^r \sum_{p=0}^s a_p b_{s-p} c_{r-s} \xi^{r-1} + \alpha_2 \sum_{r=0}^{\infty} \sum_{s=0}^r \sum_{p=0}^s a_p b_{s-p} d_{r-s} \xi^{r-1} + \alpha_3 \sum_{r=0}^{\infty} \sum_{s=0}^r a_s b_{r-s} \xi^{r-1} + \alpha_4 \sum_{r=0}^{\infty} \sum_{s=0}^r a_s c_{r-s} \xi^{r-1} + \alpha_5 \sum_{r=0}^{\infty} \sum_{s=0}^r a_s d_{r-s} \xi^{r-1} + \alpha_6 \sum_{r=0}^{\infty} a_r \xi^{r-1} \right] = 0 \quad (19)$$

By expanding and collecting powers of ξ , the coefficients of the various series can be evaluated. This procedure is illustrated in the following numerical example.

Centra
NUMERICAL EXAMPLE

Three coefficients are evaluated in each of the power series.

Flow parameters used in this evaluation include the kinematic viscosity (ν), the length of the plate (l), the velocity of the faster stream (U_1), the velocity ratio ($m = U_2/U_1$), and the thickness of the boundary layer (of the faster stream) at the trailing edge of the plate (C_1). These values are

$$\nu = .0001566 \text{ ft}^2/\text{sec}$$

$$l = .5 \text{ ft}$$

$$U_1 = 100 \text{ ft/sec}$$

$$m = .10, .25, .50, .75, \text{ and } .90$$

$$C_1 = 0.0044 \text{ ft}$$

Use of these values and the conditions at the beginning of the mixing region enables the determination of the first coefficients as

$$a_0 = c_1 = 5\sqrt{\frac{lv}{U_1}} = .0044 \text{ ft. (} = \delta_1 \text{ at the trailing edge of the plate)}$$

$$b_0 = 0$$

$$c_0 = 0$$

$$d_0 = 5\sqrt{\frac{lv}{U_1}} / 5\sqrt{\frac{lv}{U_2}} = \sqrt{\frac{U_1}{U_2}} = \frac{1}{m} \quad (\text{values of } d_0 \text{ are tabulated in Table 2}).$$

The other a_r, b_r, c_r , and d_r 's (where $r = 1, 2, 3, \dots$) are found by setting the factors of ξ in the four expanded equations equal to zero for each power of ξ . For each value of r , this yields a group of four simultaneous equations determining a set of coefficients (a, b, c and d). These coefficients are expressed in terms of the length of the mixing region, L . Coefficients up to the second power of ξ are determined in this numerical example.

The general expressions for these coefficients are

$$d_1^* = \frac{\gamma_1 \beta_1 A (\omega_3 - \omega_1 m^2)}{(\omega_1 \omega_4 - \omega_2 \omega_3)}$$

$$c_1^* = \frac{-\gamma_1 \beta_1 A - \omega_2 d_1^*}{\omega_1}$$

$$b_1^* = \frac{-\gamma_2 c_1^* - \gamma_3 d_1^*}{\gamma_1}$$

$$a_1^* = \frac{-(\beta_2 b_1^* + \beta_3 c_1^* + \beta_4 d_1^*)}{\beta_1}$$

$$d_2^* = \frac{\omega_5 \omega_{10} + 2 \gamma_{10} \beta_{17} b_1^* A (\omega_5 m - \omega_8) - \omega_7 \omega_8}{(\omega_5 \omega_9 - \omega_6 \omega_8)}$$

$$c_2^* = \frac{\omega_7 + 2 \gamma_{10} \beta_{17} b_1^* A - \omega_6 d_2^*}{\omega_5}$$

$$b_2^* = \frac{\gamma_{13} - \gamma_{11} c_2^* - \gamma_{12} d_2^*}{\gamma_{10}}$$

$$a_2^* = \frac{B - \beta_{18} b_2^* - \beta_{19} c_2^* - \beta_{20} d_2^*}{\beta_{17}}$$

where

$$a_1^* = \frac{a_1}{l} \quad b_1^* = \frac{b_1}{l} \quad c_1^* = \frac{c_1}{l} \quad d_1^* = \frac{d}{l}$$

$$a_2^* = \frac{a_2}{l^2} \quad b_2^* = \frac{b_2}{l^2} \quad c_2^* = \frac{c_2}{l^2} \quad d_2^* = \frac{d_2}{l^2}$$

$$\omega = \omega(\gamma)$$

$$\gamma = \gamma(\beta)$$

$$\beta = \beta(a_r, b_r, c_r, d_r, \alpha_r)$$

$$A = \frac{4\nu\pi^2}{U_1}$$

$$B = B(a_r, b_r, c_r, d_r, \alpha_r)$$

Continued
L is found from the boundary condition that $k = n$ at $\xi = 1$.

Resulting L's are tabulated in Table 1 for various values of m .

Table 1 gives values of $a_1, a_2, b_1, b_2, c_1, c_2, d_0, d_1, d_2$ for various values of m .

The character of the final equations is such that recurrence relations for the coefficients of the series are unobtainable. Evaluation of each new coefficient requires the knowledge of all previous ones. Since the analytical and numerical work increases several-fold in the determination of a new set of coefficients, it was decided to work a numerical example using only the first three coefficients of each series. As can be expected, the results show the correct trend in the first part of the region only. Towards the end of the region, better results may be expected with the introduction of higher order terms. However, no great change in the results is anticipated at the beginning of the region, even if greater number of terms are used.

Fig. 2 shows the development of the boundary of the mixing region on the side of the faster stream. The boundary is plotted for various velocity ratios of the main streams. (Note: All figures are prescribed in non-dimensional form.) It may be seen that the correct trend in the development of the boundary is obtained up to twenty-five percent of the length of the region ($\xi = 0.25$) for the velocity ratio $m = 0.9$, and up to sixty percent, for $m = 0.1$. Fig. 3 presents the locus of the minimum axial velocity within the mixing region. Fig. 4 shows the boundary of the mixing region on the side of the slower stream. These curves show the correct trend for the first fifteen to twenty percent of the region. In Fig. 5, the minimum value of the axial velocity component at the end of the region ($\xi = 1$) is plotted against the velocity ratio. The "exact" values represent the boundary condition that at the end of the region

Contrails

($\xi = 1$), the minimum value of the axial velocity component (Q) should equal the velocity ratio (m). The "approximate" values of Q were calculated with the first three terms of the series. It should be recalled, that the end of the region ($x = L$, or $\xi = 1$) was determined from the condition that here k should be equal to n . It may be seen from this figure, that the deviation between the "exact" and "approximate" values of Q is not large, even with the small number of terms used. In Table 3, the "exact" and "approximate" values of Q are tabulated together with the percentage error. Fig. 6 shows the development of the profile of the axial velocity component at various distances from the beginning of mixing for a velocity ratio of 0.9.

Additional work was done in order to establish the complexity involved in evaluating additional terms in the series. The expressions for the coefficients of the third power of ξ were determined (see Appendix II). The numerical values of the coefficients seem to establish an alternating pattern in the signs, odd powers of ξ apparently being negative. No definite conclusions as to the validity of this pattern for higher powers of ξ may be drawn at this time. It seems to be appropriate to break off computations after terms with even powers of ξ . Further work was discontinued on the determination of coefficients of higher powers of ξ , because the work involved is disproportionately large, if desk calculating machines are used.

Conclusions

CONCLUSIONS

The primary aim of the present work was to establish the feasibility of use of infinite series in problems of this type. In addition, it was sought to determine whether a relatively small number of terms are sufficient to describe the phenomena in the mixing region. It appears that the first three terms yield reasonable results in the early part of the mixing region, the accuracy of the results being dependent on the velocity ratio of the mixing streams. Since no recurrence relations could be found for the coefficients, the determination of each additional set of coefficients involves an increasingly greater amount of both analytical and numerical work. Also, it seems necessary to keep the number of significant figures large in the numerical computations.

It is thought that the aim of this work has been attained by the present extent of the analysis. If a more complete solution of specific mixing problems is desired, then it is recommended that the expressions for additional sets of coefficients be computed and IBM-type calculating machines be used for the numerical evaluation of the coefficients.

For the time being, no comparison of the results can be made with experiments since there are no experimental data available

Contrails

presenting this type of mixing. However, comparison of these results will be possible with the experiments carried out in our high-speed wind tunnel on the present contract, once these latter are checked and evaluated.

It should be pointed out that the method of analysis is valid for any arbitrary velocity profile at the beginning of the mixing region. In the present work for convenience, a cosine curve approximation for velocity profile was used. However, the analysis presented is not valid immediately behind the trailing edge of the plate which separates the streams upstream of the mixing region. For this very short domain, the complete Navier-Stokes equations must be used. This work is underway and will be reported at a later date.

Contrails

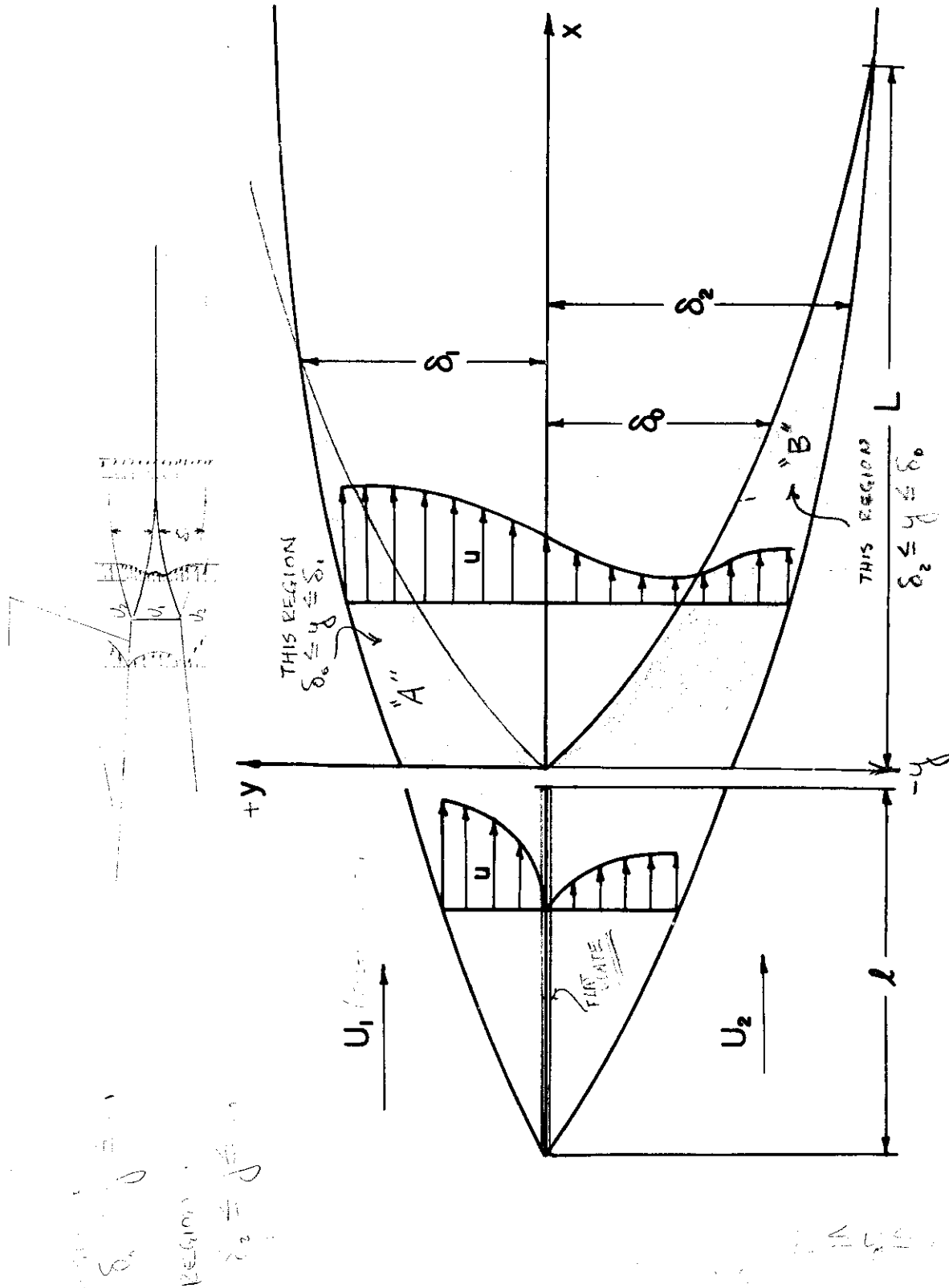


Fig. 1

Contrails

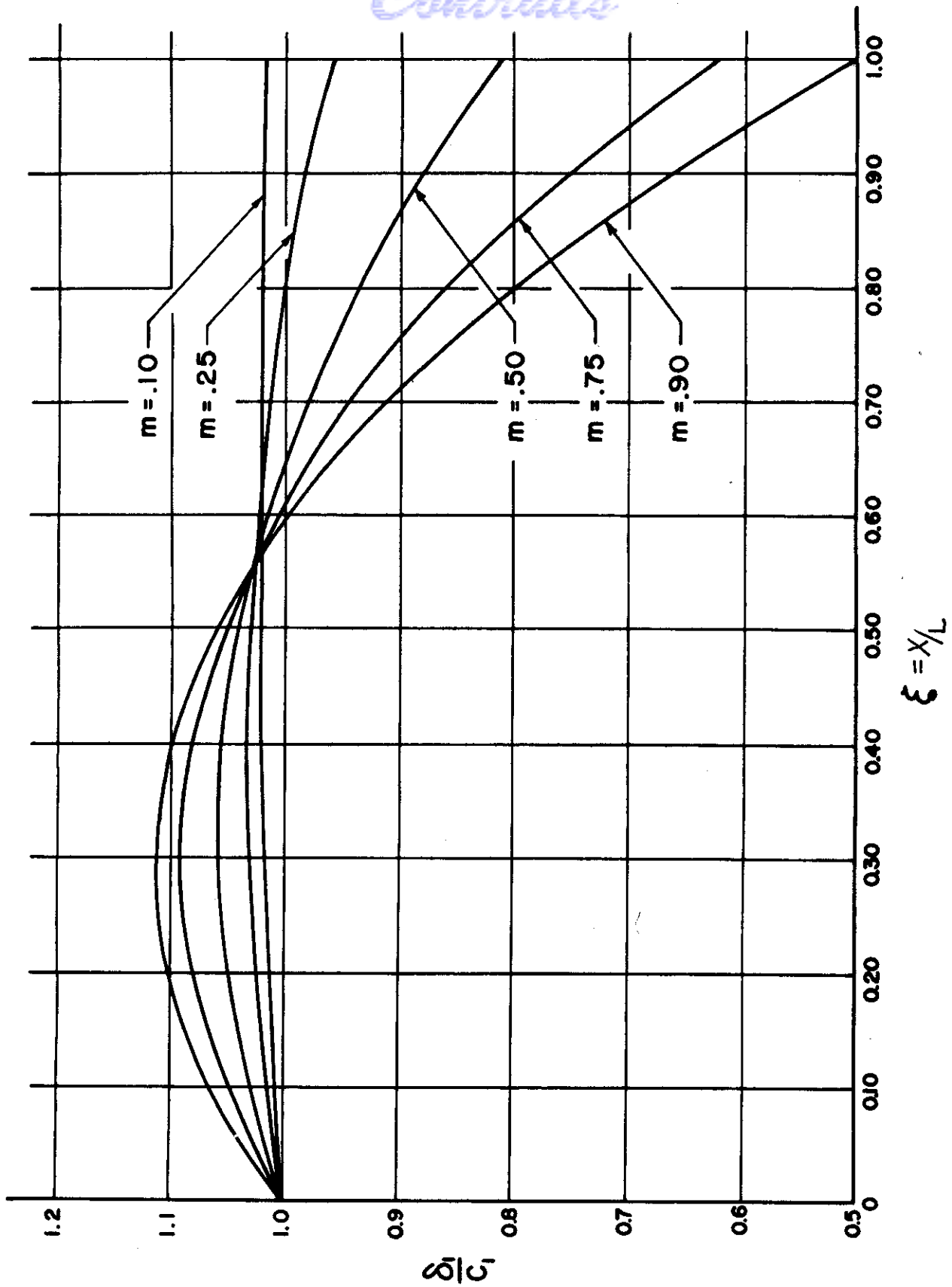


Fig. 2

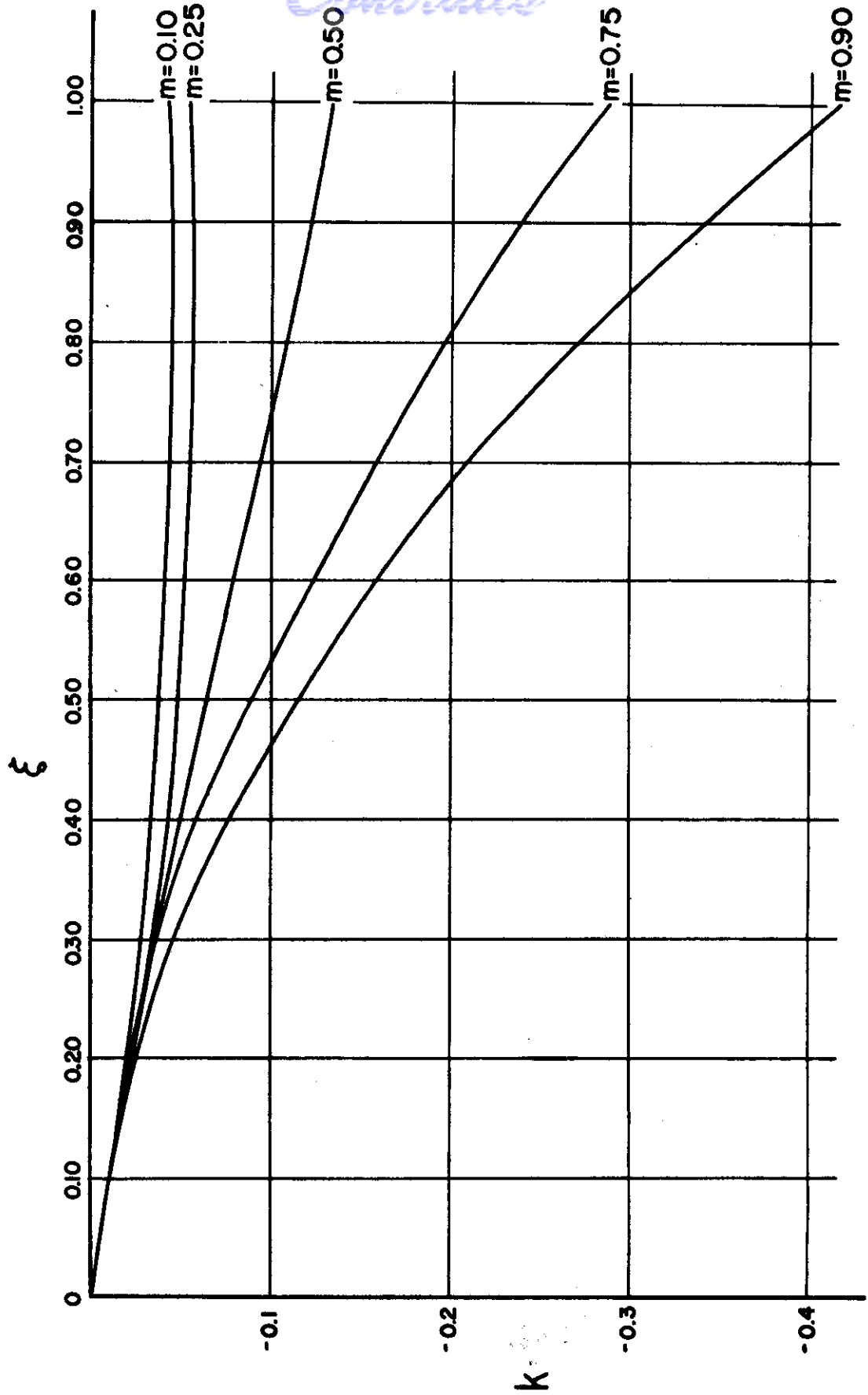


Fig. 3

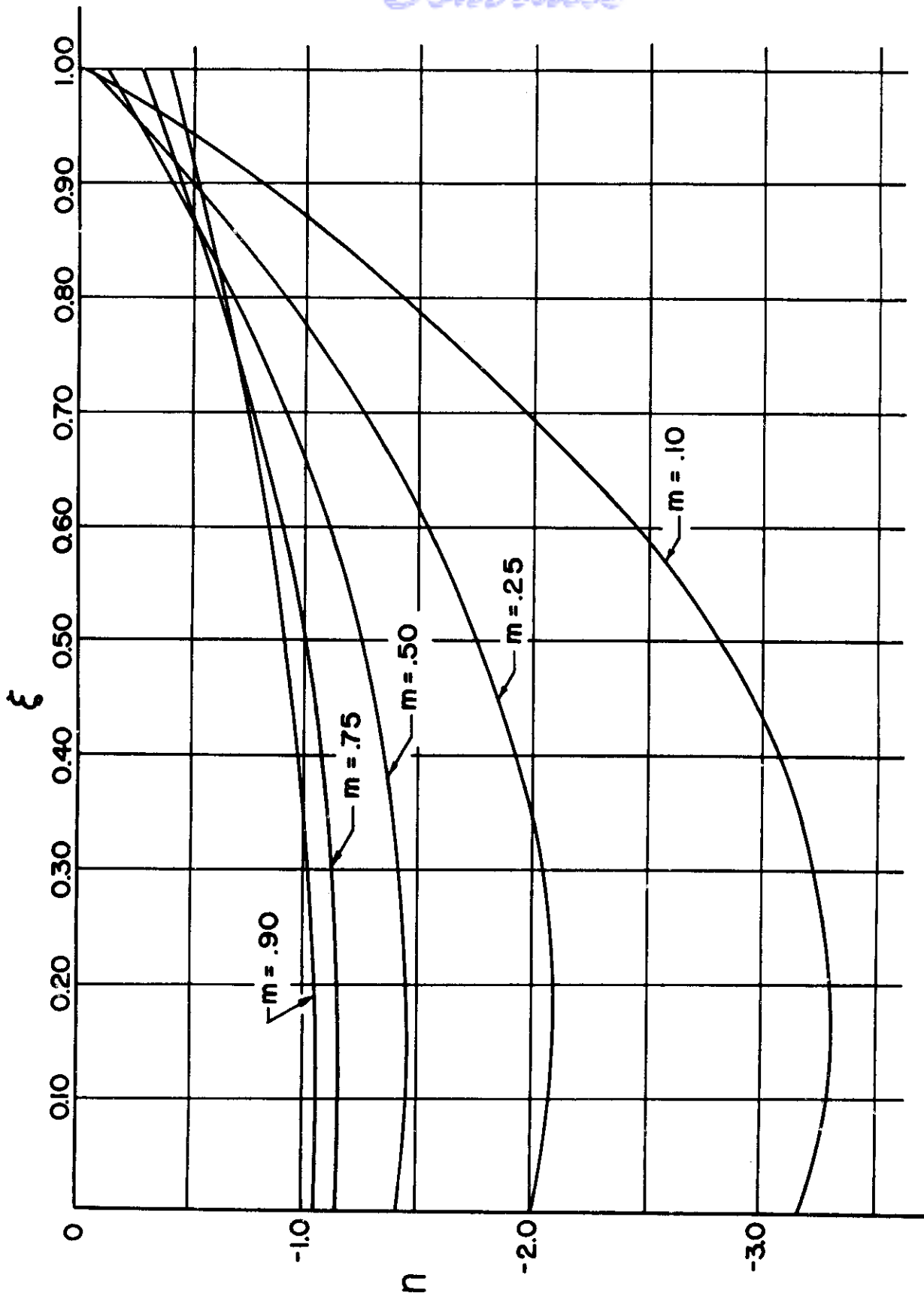


Fig. 4

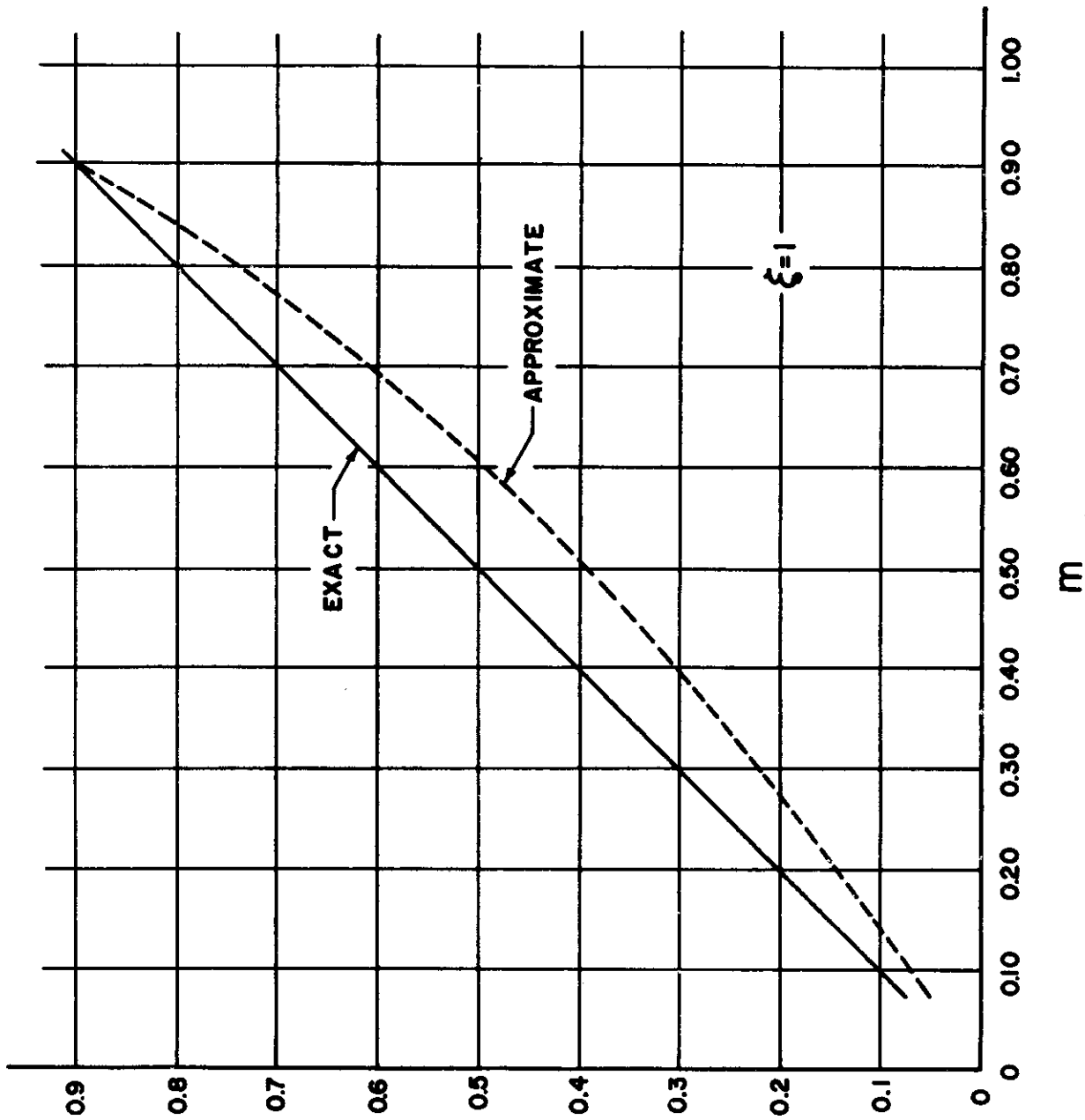


Fig. 5

Contrails

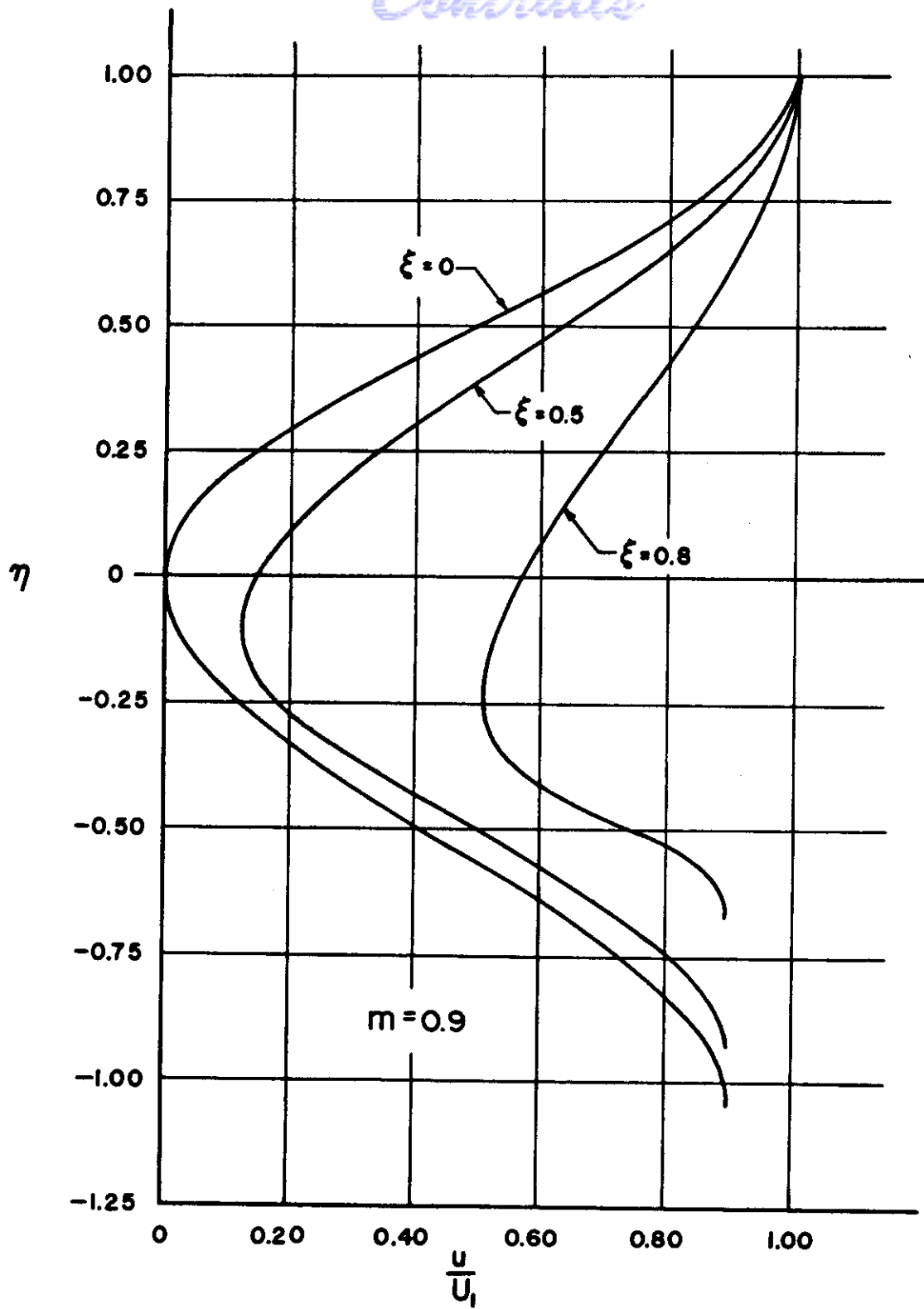


Fig. 6

Contrails TABLE 1

m	L (ft)
.10	.04547
.25	.06642
.50	.09001
.75	.1124
.90	.1277

TABLE 2

m	a_1	a_2	b_1	b_2
.10	.0003366	-.0002614	-.04543	.1207
.25	.0007234	-.0009005	-.1049	.2894
.50	.001558	-.002386	-.2010	.5900
.75	.002610	-.004249	-.3076	.9731
.90	.003350	-.005563	-.3833	1.282

m	c_1	c_2	d_0	d_1	d_2
.10	-.1130	.07457	-3.162	-1.764	4.887
.25	-.1512	.09735	-2.000	-1.083	3.029
.50	-.1331	-.002593	-1.414	-.6056	1.884
.75	-.07288	-.2158	-1.155	-.2754	1.141
.90	-.02959	-.3886	-1.054	-.1067	.7426

Controls
TABLE 3

$\xi = 1$

$m = Q_{\text{exact}}$	$Q_{\text{approximate}}$	$\frac{m - Q}{m} \times 100\%$
.10	.07527	24.73%
.25	.1845	26.20%
.50	.3889	22.21%
.75	.6656	11.26%
.90	.8991	0.11%

Contrails
BIBLIOGRAPHY

1. Torda, T. P. Symmetric Laminar Mixing of Two Parallel Streams. Air Force Technical Report M-T-1, University of Illinois, October 1951.

Torda, T. P. Wake of a Flat Plate in Laminar Incompressible Flow. Proceedings of the Third Midwestern Conference on Fluid Mechanics. The University of Minnesota, Minneapolis, Minnesota, 1953.
2. Ackermann, W. O. Laminar Incompressible Mixing of Two Streams of Different Velocities. Air Force Technical Report M-T-4, University of Illinois, August, 1952.
3. Blasius, H. Grenzschichten in Flüssigkeiten mit kleiner Reibung. Zeitschrift für Mathematik und Physik, Vol. 56, 1908 (Translated as NACA T.M. No. 1256, February 1950).
4. Torda, T. P. Laminar and Turbulent Mixing of Streams. Air Force Technical Report M-TR-6, University of Illinois, June 1953.

20172
20173

1. T. P. Torda
2. W. O. Ackermann
3. H. Blasius
4. T. P. Torda

Integration of the Momentum and Energy Equations

The equations for region A will be integrated.

The momentum equation in the integral form for region A is

$$\int_{\delta_0}^{\delta_1} u \frac{\partial u}{\partial x} dy + \int_{\delta_0}^{\delta_1} v \frac{\partial u}{\partial y} dy - \nu \int_{\delta_0}^{\delta_1} \frac{\partial^2 u}{\partial y^2} dy = 0$$

Integrating each term separately yields

$$\int_{\delta_0}^{\delta_1} u \frac{\partial u}{\partial x} dy = \frac{d}{dx} \int_{\delta_0}^{\delta_1} \frac{u^2}{2} dy + \left[\frac{u^2}{2} \right]_{\delta_0} \frac{d\delta_0}{dx} - \left[\frac{u^2}{2} \right]_{\delta_1} \frac{d\delta_1}{dx}$$

$$\int_{\delta_0}^{\delta_1} v \frac{\partial u}{\partial y} dy = [uv]_{\delta_0}^{\delta_1} + \int_{\delta_0}^{\delta_1} u \frac{\partial u}{\partial x} dy$$

$$\int_{\delta_0}^{\delta_1} \nu \frac{\partial^2 u}{\partial y^2} dy = \nu \left[\frac{\partial u}{\partial y} \right]_{\delta_0}^{\delta_1}$$

Then the region A momentum equation takes the form

$$\frac{d}{dx} \int_{\delta_0}^{\delta_1} u^2 dy + [u^2]_{\delta_0} \frac{d\delta_0}{dx} - [u^2]_{\delta_1} \frac{d\delta_1}{dx} + [uv]_{\delta_0}^{\delta_1} - \nu \left[\frac{\partial u}{\partial y} \right]_{\delta_0}^{\delta_1} = 0$$

The first two terms of the energy equation for region A are integrated in a similar manner to the corresponding ones of the momentum equation.

The last term is:

$$-\nu \int_{\delta_0}^{\delta_1} u \frac{\partial^2 u}{\partial y^2} dy = -\nu \left[u \frac{\partial u}{\partial y} \right]_{\delta_0}^{\delta_1} + \nu \int_{\delta_0}^{\delta_1} \left[\frac{\partial u}{\partial y} \right]^2 dy$$

Controls

Therefore, the region A energy equation takes the form:

$$\frac{d}{dx} \int_{\delta_0}^{\delta_1} u^3 dy + 2v \int_{\delta_0}^{\delta_1} \left[\frac{\partial u}{\partial y} \right]^2 dy + [u^3]_{\delta_0} \frac{d\delta_0}{dx} - [u^3]_{\delta_1} \frac{d\delta_1}{dx} + [u^2 v]_{\delta_0}^{\delta_1} - 2v \left[u \frac{\partial u}{\partial y} \right]_{\delta_0}^{\delta_1} = 0 \quad \text{O.K.}$$

Integration of the continuity equation yields:

$$v_0 = v_1 + \frac{d}{dx} \int_{\delta_0}^{\delta_1} u_A dy + u_0 \frac{d\delta_0}{dx} - u_1 \frac{d\delta_1}{dx}$$

and

$$v_1 = -\frac{d}{dx} \int_{\delta_1}^{\delta_0} u dy - U_2 \frac{d\delta_2}{dx} + v_2 - \frac{d}{dx} \int_{\delta_0}^{\delta_1} u dy + U_1 \frac{d\delta_1}{dx}$$

The transformation $\eta = \frac{y}{\delta_1}$ is made in the momentum and energy equations. The resulting integrals in the momentum and energy equations are evaluated as:

Handwritten notes: $\int_{\delta_0}^{\delta_1} u_A dy = u_A y$, $\eta = \frac{y}{\delta_1}$, $dm = \frac{dy}{\delta_1}$, $\frac{1}{2}(\delta_1 - \delta_0) \left(\frac{u_1 + u_0}{\delta_1} \right)$

$$\int_K^1 \left[\frac{u}{U_1} \right]_A d\eta = \frac{1}{2} (1-K)(1+Q) \quad \int_{\frac{\delta_0}{\delta_1}}^1 \left[\frac{u}{U_1} \right]_A \frac{dy}{\delta_1}$$

$$\int_K^1 \left[\frac{u}{U_1} \right]_A^2 d\eta = \frac{1}{8} (1-K)(3 + 2Q + 3Q^2) \quad \int_{\frac{\delta_0}{\delta_1}}^1 \left[\frac{u}{U_1} \right]_A^2 \frac{dy}{\delta_1} = \frac{1}{8} (\delta_1 - \delta_0) \left(2u_1 \frac{u_0}{\delta_1} + 3 \frac{u_0^2}{\delta_1} \right) = \frac{1}{8} (\delta_1 - \delta_0) \frac{u_0(u_1 + 2u_0)}{\delta_1}$$

$$\int_K^1 \left[\frac{u}{U_1} \right]_A^3 d\eta = \frac{1}{16} (1-K)(1+Q)(5 - 2Q + 5Q^2)$$

$$\int_K^1 \left[\frac{\partial u}{\partial \eta} / \partial \eta \right]_A^2 d\eta = \frac{\pi^2}{8} \frac{(1-Q)^2}{(1-K)}$$

$$\int_n^k \left[\frac{u}{U_1} \right]_B d\eta = \frac{1}{2} (k-n)(m+Q)$$

Contrails

$$\int_n^k \left[\frac{u}{U_1} \right]_B^2 d\eta = \frac{1}{8} (k-n)(3m^2 + 2mQ + 3Q^2)$$

$$\int_n^k \left[\frac{u}{U_1} \right]_B^3 d\eta = \frac{1}{16} (k-n)(m+Q)(5m^2 - 2mQ + 5Q^2)$$

$$\int_n^k \left[\frac{\partial u}{\partial \eta} \right]_B^2 d\eta = \frac{\pi^2}{8} \frac{(m-Q)^2}{(k-n)}$$

It is suggested in Ref. 4 that an additional boundary condition for the mixing of two streams is a force equilibrium between the two regions of mixing in the normal direction. This may be expressed as $U_1 v_1 = -U_2 v_2$ and it is used to eliminate v_2 from the preceding equation. Applying these values of v and the boundary conditions to the momentum equations and evaluating the resulting integrals yields equations (7) and (9). Equations (8) and (10) for region B are obtained in a similar manner.

The expressions for the coefficients of third power terms of ξ are

$$d_3^* = \frac{\omega_{11}\omega_{16} - \omega_{13}\omega_{14} + \{\gamma_{22}\beta_{13}A[b_1^*(\omega_{14} - \omega_{11}) - 2b_2^*(\omega_{14} - \omega_{11}m)]\}}{(\omega_{11}\omega_{15} - \omega_{12}\omega_{14})}$$

$$c_3^* = \frac{\omega_{13} + \gamma_{22}\beta_{33}A(2b_2^* - b_1^{*2}) - \omega_{12}d_3^*}{\omega_{11}}$$

$$b_3^* = \frac{\gamma_{25} - \gamma_{23}c_3^* - \gamma_{24}d_3^*}{\gamma_{22}}$$

$$a_3^* = \frac{F - (\beta_{34}b_3^* + \beta_{35}c_3^* + \beta_{36}d_3^*)}{\beta_{33}}$$

where

$$a_3^* = \frac{a_3}{l^3}, \quad b_3^* = \frac{b_3}{l^3}, \quad c_3^* = \frac{c_3}{l^3}, \quad d_3^* = \frac{d_3}{l^3}$$

$$F = F(a_r, b_r, c_r, d_r, \alpha_r)$$