FOREWORD

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The following personnel of the University of Minnesota contributed to this work: Dr. A. R. Robinson served as project engineer and Mr. S. T. Chow proofread the manuscript. The manuscript preparation was by Miss Carol Sherman.

An English Translation of "Kolebaniia uprugukh sistem c uchetom rasseianiia energii v materiale," by G. S. Pisarenko, Kiev, 1955, Academy of Sciences of the Ukrainian SSR

ABSTRACT

This monograph is devoted to an analytical and experimental investigation of vibrations of non-conservative elastic systems in which the sources of energy dissipation is irreversible cyclic straining of the material.

Modern methods of analysis of non-linear vibrating systems are extended to treat problems of the flexural vibrations of long bars of constant and variable cross section, short bars and turbine blades. Torsional vibrations of rods are also considered.

Considerable attention is given to the experimental investigation of energy dissipation in the material. Several apparatuses are described and some of the experimental results presented.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:

W. J. TRAPP
Chief, Strength and Dynamics Branch
Metals and Ceramics Laboratory
Directorate of Materials and Processes

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PREFACE

At the present time there is no need to justify the importance of the study of vibrations in almost all fields of contemporary technology. In particular, in machine design the investigation of the influence of damping factors on the development of oscillation is a pressing problem.

Up to now, however, only the influence of friction proporational to speed has been studied in a proper manner; the study of the real problem of oscillatory systems with internal friction of a hysteretic type still remains in its most primitive stage because of the nonlinearity of the equations obtained.

We should welcome, therefore, the appearance of G. S. Pisarenko's work devoted precisely to the study of this important and pertinent problem.

The present treatise consists of a theoretical investigation and an exposition of the experimental part.

In the first part the author, basing himself on contemporary conceptions concerning the character of the processes of dissipation of energy in material and, in particular, using the hypothesis of N. N. Davidenkov, sets forth a theory of analysis of oscillations of elastic systems taking into account the dissipation of energy in the material in a manner applicable to a number of engineering problems.

In view of the nonlinearity of the differential equations obtained in this connections, the author has used the ideas of the theory of asymptotic expansion in nonlinear mechanics, and has worked out an original method of calculation of resonance curves, which has proved to be very effective.

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With the aid of this method a series of important and practical problems connected, in particular, with the vibration of turbine blades was solved by the author. Cases of oscillation of blades of constant and of variable cross-section with due allowance for centrifugal forces are thereby considered, as are also cases of oscillation of short blades.

In general, it should be remarked that some typical problems are completely solved in this bood and are carried through to the numerical computations. These problems demonstrate the effectiveness of the method proposed by the author and show the sufficiency of the first approximation.

This circumstance permits us to recommend G. S. Pisarenko's method for utilization in the appropriate design offices.

In connection with the fact that the proposed method envisages the utilization of experimental constants, characterizing the damping properties of materials, considerable attention is given in the work to their determination.

In the experimental part a series of original apparatuses with a very ingenious principle of operation is proposed by the author, which permits the investigation of the damping properties of the materials themselves in a "pure state" under different temperature conditions.

Thus, G. S. Pisarenko's book is a valuable contribution to the little studied but important field of investigation of vibrations (considering dissipation of energy) which are characterized by nonlinear differential equations.

Academician N. N. Bogolyubov

INTRODUCTION

The present monograph is devoted to a relatively little studied but important problem - the investigation of oscillations of elastic systems allowing for dissipation of energy in material.

The development of high-speed machines in the Soviet Union, and, in particular, of turbine and motor construction, demands higher standards in the calculation of the dynamic strength of their components. In this connection the problem of oscillations acquire particular urgency.

At the present time the theory of oscillations and its application to different branches of technology have undergone a very great development, thanks mainly to the work of scientists of our own country. However, several problems of the theory of oscillation which are important for machine construction, in particular, questions connected with the consideration of the dissipation of energy during oscillation, despite their tremendous practical significance for dynamical calculations, have been studied relatively little up to now.

As is well known, during oscillation of actual structures, which are non-conservative oscillatory systems, energy supplied from outside the system is dissipated. The causes of the dissipation of energy are usually divided into external causes and internal causes. Among the external factors are friction of the oscillating system in the medium in which the oscillation occurs and the friction in the connections of the separate elements of the oscillating system. Among internal factors is the incomplete elasticity of the material.

The influence of the external and internal factors on the dissipation of energy varies within wide limits depending on the form of the oscillations, the design of the oscillating system, the material of which this system is made, and also the medium in which the oscillations take place.

Extensive investigations, conducted by a number of authors in a study of damping in material in torsional oscillations, showed that the loss of energy from air resistance is insignificant by comparison with the loss in the material itself, which constitutes not less than 2/3 of the entire loss during oscillation.

Therefore, for rational design of structures, knowledge of the ability of the material and the separate elements of construction made from this material to damp the oscillation is most essential. Sometimes special elements, so-called internal friction dampers, are included in elastic systems in which harmful oscillations (such as torsional oscillation of airplane engine crankshafts) unavoidably occur during use. These dampers, which are not load-carrying members, are intended for the absorption of the energy of oscillation of the elastic system in which they are included; they are, therefore, made of material having great internal damping, for example, rubber.

In several cases of combined oscillation of a group of machine components (for example, shaft-airplane propeller, shaft-disk-blades of a steam turbine, and so forth) by proper adaptation of the separate elements of the system, it is possible to achieve such a degree of interaction that the oscillation of some elements of the system will be damped by the oscillation of the others.

Many investigators have concerned themselves with the study of oscillation of elastic systems taking account of dissipation of energy in material, but the methods of theoretical calculation of internal damping used in many cases led to results contradicting experimental data.

The goal of the present work is — on the basis of very general assumptions which are confirmed by experiment — to provide a new method of theoretical solution of the problem of oscillation of elastic systems with allowance for dissipation of energy in the material.

In the first part of the work a theory of analysis of elastic systems with allowance for hysteretic losses in material is set forth. In the book there is a generalization of the author's previously published works, [8-14] in which the idea of a proposed new method of calculation is presented, based on nonlinear treatment of problems of oscillation which are accompanied by hysteretic losses. In this book, the relation proposed by N. N. Davidenkov is taken as the fundamental relation between stresses and deformations which characterizes the departure from Hooke's Law. This relation reflects the nature of the formation of the hysteresis loop better than others do, as is well known [15] In the monograph the asymptotic methods of nonlinear mechanics proposed by academicians N. M. Krylov and N. N. Bogolyubov, [2,3,4] , which are very effectively adaptable to the solution of a number of important problems in physics and technology, have undergone further development.

In spite of the somewhat complicated nature of the proposed method of calculation, which is based upon direct integration of a differential equation of oscillation of the system with the nonlinear law of internal dissipation of energy in the material, one must admit that in some important problems this method is, on theoretical grounds, the only correct one.

The proposed method of analysis is at the same time sufficiently universal to permit the solution of a problem of oscillation and, in particular, for rods under complicated states of stress, also with any law of nonlinear dependency between stress and strain.

The book shows in a series of concrete problems the effectiveness of the proposed asymptotic methods of non-linear mechanics in the solution of problems of oscillation with allowance for hysteresis, even in cases where the basic nonlinear relationships are complicated. Here it should be noted that in the investigation of certain kinds of oscillation of rods allowing for dissipation of energy in material, other methods based on additional assumptions were applied by a number of authors [7,8,16,17].

Since we attribute great significance to experimental investigations of dissipation of energy in material, without which it is impossible to make theoretical calculations of resonance curves, in the second part of the book we cite material on the experimental study of this problem. The method of investigation is presented and a description of the new experimental apparatuses devised by the author at the Institute of Structural Mechanics of the Academy of Science of the Ukrainian SSR, in the Special Alloy Laboratory of the Academy of Sciences of Ukrainian SSR, and in the Kiev Polytechnical Institute is given. In addition, facts obtained by the author in connection with the study of the influence of various factors on the magnitude of dissipation of energy in material are presented in this book.

The materials included in this treatise grow out of the general conclusion of the investigations conducted by the author in the past few years in the field of vibrations. A number of questions was considered under the influence of Academician N. N. Bogolyubov and Active Member of the Academy of Sciences of the Ukrainian SSR, N. N. Davidenkov, with whom over a period of ten years the author worked at the Institute of Structural Mechanics of the Academy of Science of the Ukrainian SSR. The author considers it his duty to express his profound gratitude to Academician N. N. Bogolyubov and to Active Member of the Academy of Sciences of the Ukrainian SSR, N. N. Davidenkov for their valuable advice which aided in the writing of the present monograph. I also wish to express my profound gratitude to Active Member of the Academy of Sciences of the Ukrainian SSR, G. N. Savin for the important comments made by him, after he had familiarized himself with the manuscript.

In conclusion I consider it my duty to express my great gratitude to the Director of Technical Sciences, D. V. Weinberg, for his assistance in the editing of the book.

Chapter I

Vibrations of a System with One Degree of Freedom

1. Statement of the problem

Under alternating loads, which are accompanied by oscillations, the majority of materials used in machine construction deformation do not follow Hooke's Law when they deform. Energy expended in the deformation of material during each increase of load is not completely released upon removal of the load. Because of this circumstance, if no energy is added from outside the system, the oscillation of such a system decreases with each cycle of oscillation, and when all the energy is spent on the internal processes which take place in the material of the system the oscillations will cease completely (be damped). In the stress-strain diagram for materials possessing the property of internal absorption of energy, the process of dissipation of energy in one cycle of oscillation is pictured in the form of a hysteresis loop. The area of a hysteresis loop determines the quantity of energy dissipated per unit volume of the material in one cycle of oscillation. This amount of energy, which can be expressed as a function of stress, characterizes the "dissipation of energy in material during vibration" or the "internal damping of the oscillations". The dissipation of energy in material per unit volume should be considered as a property of the material, independent of the form and dimensions of the specimen.

In connection with the study of oscillation of elastic systems, questions concerning dissipation of energy in material have for some time now attracted the attention of many scientists — physicists and mechanicians. We shall not occupy ourselves at the present time with an analysis of work which has been done, inasmuch as such an analysis has been given by Professor N. N. Davidenkov [1]. We shall point out only that all existing methods of investigation of damping in material follow two fundamentally different dirrections. One of these directions, originating, apparently, in works of Voigt, is based on the hypothesis of "viscous friction". According to this hypothesis, damping in the material during oscillation is proportional to the rate of deformation or, what amounts to the same thing — to the frequency of oscillation.

The second direction, represented by a considerably smaller number of works, is based on a hypothesis according to which material damping is proportional to the amplitude of oscillation. Experiments must serve as a criterion of the correctness of these hypotheses.

Of all the experimental work devoted to the study of energy dissipation in material, it is difficult to name even one, which was carried out in a thorough manner, and which would support the hypothesis of viscous friction. As far as the second hypothesis is concerned, it is verified by a great amount of painstakingly performed experimental work, which deals both with torsional as well as flexural vibrations.

We should note the fact that authors of many works dating from an even later period adhere to the hypothesis of "viscous friction".* The fact of the matter is that these works are primarily theoretical, and for theoretical analysis of oscillation of elastic systems taking account of energy dissipation in material, the hypothesis of frequency dependent damping of vibrations greatly facilitates the mathematical calculations - the differential equations of oscillation turn out to be linear, which considerably simplifies their solution.

When choosing the hypothesis, which one must use as a basis of investigation, one should start from the analysis of factors influencing the size of the hysteresis loop, as this quantity characterizes the damping in material.

From the experiments of Academician A. F. Yoffe with monocrystals of quartz, it is well known that perfect monocrystals possess complete elasticity. Apparently, hysteresis in metals is due to their polycrystalline structure.** The state of stress of a material consisting of many interconnected but differently oriented grains is heterogeneous, due to anisotropy of the elastic properties. The heterogeneity of the stress is so considerable that on the boundaries of the individual grains, overstressing occurs causing local plastic deformations which increase with the size of the load. As a result of these deformations, the metal changes somewhat its structure and passes into a state of stress which is more homogeneous than the state of

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*We exclude the work of Forster in which the study of damping was carried out by an electro-acoustical method for very small amplitudes without eliminating the loss due to air resistance. (See pp. 48-51, WADC Tech. Report 56-180 by L. J. Demer for references.)

**The physical side of this question has been thoroughly dealt with by N. N. Davidenkov [1].

stress without plastic deformations; i.e., equalization of stress on the grain boundaries takes place.

As a result of this anisotropy of material, when the load is removed, local overstresses occur again, and, as a consequence, local plastic deformations, but now in the opposite direction. On a stress-strain diagram the process of unloading appears as a curve, which is different from the curve of the process of loading. During a steady state of oscillation of the load, the usual closed hysteresis loop is formed.

Without going into the details of the complicated processes which originate in the polycrystalline anisotropic medium of the material of the oscillating system, we must consider it as proven that dissipation of energy in material is caused by isolated plastic deformations and that the magnitude of this dissipation increases with increase of load or, what is the same thing, with increase in the amplitude of oscillation.

This conception permits us to begin from a functional relationship between damping and amplitude of oscillation established directly by experiments, when solving a problem of oscillation of elastic system taking account of energy dissipation in the material.*

In accordance with the hypothesis which has been accepted concerning the relation between energy dissipation in the material and the magnitude of stress, the hysteresis loop must be considered in the derivation of the equation of vibration. As the experimental data show, the hysteresis loop has a form which does not lend itself to description by the equation of one smooth curve which describes the entire contour. The hysteresis loop is limited by two smooth curves,

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*A bibliography of work devoted to this problem can be found in a survey article by N. N. Davidenkov, published in Zhurn. Tekh. Fiz. vol. VIII, no. 6, 1938.

one of which corresponds to loading and the other to the unloading of the sample. In the places where the branches meet a sharp break in the curve is observed. Therefore, the contour of the hysteresis loop must be represented by two equations, one of which refers to the ascending branch and the other to the descending one. Of course, some boundary conditions must be satisfied in the places where the two branches meet. Therefore, the process of oscillation of a system possessing the ability to dissipate energy in the material, is described not by one, but by two differential equations, of which one refers to the ascending movement, the other to the descending one. The fact that for the majority of materials used in machine construction, the branches of the hysteresis loop diverge very slightly from the straight line which characterizes Hooke's Law has great importance for what follows.

This circumstance has prompted us to apply in the solution of the problem, which is the problem in the theory of pseudo-harmonic oscillations, an approximate method which would permit us to obtain reliable results on the basis of a physically justified notion of the nature of damping in the material. The method which it is natural to apply to our case of a "slightly nonlinear" problem is the asymtotic method devised by academicians N. M. Krylov and N. N. Bogolyubov [4] which has proved to be effective in the solution of important problems of technology and physics. Thus, for the solution of nonlinear problems sufficiently close to the linear, we shall use the methods of nonlinear mechanics, based on an application of expansion by powers of a small parameter.

The application of these expansions leads to approximate solutions which does not contain secular terms and which satisfies the given differential equations, uniformly in time, up to a predetermined power of the small parameter.

In our theoretical investigations we start from modern physical conceptions of the mechanism of the dissipation of energy in material during oscillations, and we depend on data found by experiments. Such an approach to the problem, naturally, causes the final formulas to contain physical constants, characterizing the damping properties of material. These constants should be obtained from experiments for each specific material. Therefore, in this book problems on experimental investigation of dissipation of energy in material are also considered.

We shall use the asymptotic method as a means of constructing, in a very effective manner, resonance curves for vibrations of the elastic systems.

In this treatise the general questions of the investigation of the convergence of the series used, and so forth, are not treated, all the more because those questions do not always lend themselves to investigation in a general way. As for general questions of the theory of nonlinear mechanics applied by us, based on asymptotic expansions by powers of a small parameter, they are dealt with in the numerous works of the authors of this theory, Academicians N. M. Krylov and N. N. Bogolyubov.

2. Derivation of the basic equations

It is expedient to begin the discussion of the theory of vibration of elastic systems taking account of energy dissipation in the material with the examination of the simplest model in the form of a system with one degree of freedom.

A weight is suspended from a spring (Figure 1). If only vertical vibrations of the weight Q are possible and if the mass of the spring is small in comparison with that of Q,

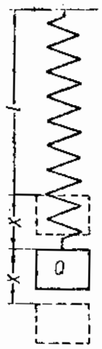

Fig. 1

then it can be considered that such a system has one degree of freedom.

We shall introduce the following notation:

x - vertical displacement of the mass, considered positive in the downward direction.

c - stiffness of the spring, i.e. the load which is necessary to stretch the spring a unit length.

$\Phi(x)$ - a functional, characterizing the energy dissipation in the material of the spring.

ε - a small parameter

t - time

g - acceleration of gravity

ν and n - parameters, depending on the damping properties of the material.

The differential equation of the free vibration of this system can be expressed as

$$\frac{Q}{g}\frac{d^2x}{dt^2}+cx+c\varepsilon\overset{\rightleftarrows}{\Phi}(x)=0. \qquad (2.1)$$

Letting

$$\frac{cg}{Q}=p^2,$$

we obtain the equation

$$\frac{d^2x}{dt^2}+p^2x+p^2\varepsilon\overset{\rightleftarrows}{\Phi}(x)=0. \qquad (2.2)$$

In the equation (2.2) the term $p^2\varepsilon\overset{\rightleftarrows}{\Phi}(x)$ takes into account the damping of vibrations in the material of the suspension, which can be characterized as some sort of retarding force. The value of this term during the upward movement of the load differs from its value during downward movement, a fact indicated by the two arrows in different directions over Φ

Thus, the process of vibration of this system, in which there is a loss of energy of vibration by dissipation in the material of the spring, is expressed by two differential equations. The equation (2.2) can also take the form

$$\frac{d^2x}{dt^2} + p^2\left[x + \epsilon \vec{\Phi}(x)\right] = 0. \tag{2.3}$$

Instead of multiplying x by p^2 when there is no damping, the term $x + \epsilon\Phi(x)$, is used which gives a nonlinear equation (2.3). Here the presence of the small parameter ϵ indicates that the nonlinearity of equation (2.3) is slight, because it is caused by the deviation of the "stress-strain" curve from the linear law of Hooke, and this deviation, as is known, is small. In the case of the ideal elastic system (without damping) the small parameter reduces to zero, and then the term of the equation (2.3) containing the parameter ϵ disappears, and the nonlinear differential equation reduces to the usual linear differential equation of free harmonic vibrations without damping.

$$\frac{d^2x}{dt^2} + p^2 x = 0. \tag{2.4}$$

Vibrations, characterized by equation (2.3), will be transient, damped vibrations. The rapidity of damping of the vibrations will depend on the magnitude of the term $p^2 \epsilon \Phi(x)$.
The action on the system of an external periodic exciting force is required to maintain such vibrations. Since the dissipation of energy in the material (hysteresis losses) is slight, to maintain a steady-state vibration a small external exciting force is obviously also needed. It is expedient to indicate the latter circumstance by introducing the factor ϵ in the periodic external force. Thus, if the external exciting force is designated by $\epsilon P \sin \omega t$, where ϵP is the amplitude, and ω is the frequency of the

external periodic load, then the differential equation of the forced vibrations of the weight, Q, on the spring can be represented by

$$\frac{d^2x}{dt^2} + p^2 [x + \varepsilon \vec{\Phi}(x)] = \varepsilon q \sin \omega t, \tag{2.5}$$

where $q = \frac{gP}{Q}$.

Thus, we obtain the basic equation of the problem of vibrations of a system with one degree of freedom accounting for energy dissipation in the material (here in the spring).

The equation (2.5) contains a small parameter, the magnitude of which is determined by the ratio of the magnitude of the energy which is found from the area of the hysteresis loop, to the potential energy stored per unit of volume of the material of the elastic system at the extreme position of the weight. It is then natural to seek a solution of this equation in the form of expansion in a series with respect to the small parameter:

$$x = x_0 \cos(\omega t + \psi) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \ldots, \tag{2.6}$$

where ψ is the phase shift.

We note that the terms of the series $u_1(t)$, $u_2(t)$, $u_3(t)$ ... are not to contain the fundamental harmonic. Keeping in mind the series (2.6), it is possible to represent the functional $\Phi(x)$ in the form of a Taylor series

$$\begin{aligned} \Phi(x) &= \Phi[x_0 \cos(\omega t + \psi) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \ldots] = \\ &= \Phi[x_0 \cos(\omega t + \psi)] + \varepsilon \Phi'_x[x_0 \cos(\omega t + \psi)]\, u_1(t) + \\ &+ \varepsilon^2 \Phi'_x[x_0 \cos(\omega t + \psi)]\, u_2(t) + \frac{\varepsilon^2}{2!} \Phi''_x[x_0 \cos(\omega t + \psi)]\, u_1^2(t) + \ldots \end{aligned} \tag{2.7}$$

Substituting the series (2.6) into the differential equation (2.5) and using the notation

$$u_0(t) = x_0 \cos(\omega t + \psi), \tag{2.8}$$

we obtain

$$\frac{d^2u_0(t)}{dt^2}+\varepsilon\frac{d^2u_1(t)}{dt^2}+\varepsilon^2\frac{d^2u_2(t)}{dt^2}+\ldots+p^2u_0(t)+p^2\varepsilon u_1(t)+ \\ +p^2\varepsilon^2u_2(t)+\ldots+\varepsilon p^2\Phi[u_0(t)+\varepsilon u_1(t)+\varepsilon^2u_2(t)+\ldots]=\varepsilon q\sin\omega t. \quad (2.9)$$

We separate from equation (2.9) the terms containing the fundamental harmonics

$$\frac{d^2u_0(t)}{dt^2}+p^2u_0(t)+p^2\varepsilon\Phi[u_0(t)]_{\text{гг}}=\varepsilon q\sin\omega t. \quad (2.10)^*$$

Because the term $p^2\varepsilon\Phi[u_0(t)]_{\text{гг}}$ contains only the fundamental harmonic, it is found that the magnitude of this term in the zeroth approximation is equal to the magnitude of the right hand side, i.e.

$$p^2\varepsilon\Phi[u_0(t)]_{\text{гг}}\approx\varepsilon q\sin\omega t. \quad (2.11)$$

The well-known equation for determination of the natural frequency is obtained from equation (2.10)

$$\frac{d^2u_0(t)}{dt^2}+p^2u_0(t)=0. \quad (2.12)$$

This equation in the zeroth approximation could also have been obtained from (2.10) by setting $\varepsilon=0$.

Introducing the substitution (2.8), we get

$$-\omega^2x_0\cos(\omega t+\psi)+p^2x_0\cos(\omega t+\psi)=0,$$

whence

$$\omega^2=p^2=\frac{cg}{Q}=\frac{g}{\delta_{\text{ст}}}, \quad (2.13)$$

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*The Russian subscripts гг denote "principal harmonic"; б/гг will mean "without principal harmonic". (Trans.)

where $\frac{Q}{c} = \delta_{CT}$, the static extension of the spring under the influence of the weight, Q.

Returning to equation (2.9), we equate to zero the expressions which multiply the different powers of the small parameter. From the remaining terms of the equation (2.9), after separation of the main harmonics and considering the expansion (2.7), these additional equations are obtained:

$$\frac{d^2u_1(t)}{dt^2} + p^2u_1(t) + p^2\Phi[x_0 \cos(\omega t+\psi)]_{6/rr} = 0, \tag{2.14}$$

$$\frac{d^2u_2(t)}{dt^2} + p^2u_2(t) + p^2\Phi'_x[x_0 \cos(\omega t+\psi)]_{6/rr}u_1(t) = 0, \tag{2.15}$$

$$\frac{d^2u_3(t)}{dt^2} + p^2u_3(t) + p^2\Phi'_x[x_0 \cos(\omega t+\psi)]_{6/rr}u_2(t) + \\ + \frac{p^2}{2}\Phi''_x[x_0 \cos(\omega t+\psi)]_{6/rr}u_1^2(t) = 0. \tag{2.16}$$

We examine the equations of harmonic balance to determine the actual frequency of vibrations of the weight, Q, and the magnitude of the phase shift.

$$\oint\left\{\frac{d^2x}{dt^2} + p^2[x+\varepsilon\Phi(x, \nu, n)] - \varepsilon q \sin \omega t\right\}\cos \omega t\, dt = 0, \tag{2.17}$$

$$\oint\left\{\frac{d^2x}{dt^2} + p^2[x+\varepsilon\Phi(x, \nu, n)] - \varepsilon q \sin \omega t\right\}\sin \omega t\, dt = 0. \tag{2.18}$$

What we have done by writing these equations is to separate out the principal harmonics in sines and cosines, occurring in the original differential equation (2.5). After substituting the expression for χ from (2.6) into the equations (2.17) and (2.18), we get:

$$\oint\left\{\frac{d^2}{dt^2}\left[x_0\cos(\omega t+\psi)+\varepsilon u_1(t)+\varepsilon^2 u_2(t)+\ldots\right]+\right.$$
$$+p^2\left[x_0\cos(\omega t+\psi)+\varepsilon u_1(t)+\varepsilon^2 u_2(t)+\ldots\right]+$$
$$+p^2\varepsilon\Phi\left[x_0\cos(\omega t+\psi)+\varepsilon u_1(t)+\varepsilon^2 u_2(t)+\ldots\right]-$$
$$\left.-\varepsilon q\sin\omega t\right\}\cos\omega t\,dt=0, \tag{2.19}$$

$$\oint\left\{\frac{d^2}{dt^2}\left[x_0\cos(\omega t+\psi)+\varepsilon u_1(t)+\varepsilon^2 u_2(t)+\ldots\right]+\right.$$
$$+p^2\left[x_0\cos(\omega t+\psi)+\varepsilon u_1(t)+\varepsilon^2 u_2(t)+\ldots\right]+$$
$$+p^2\varepsilon\Phi\left[x_0\cos(\omega t+\psi)+\varepsilon u_1(t)+\varepsilon u_2(t)+\ldots\right]-$$
$$\left.-\varepsilon q\sin\omega t\right\}\sin\omega t\,dt=0. \tag{2.20}$$

Carrying out the integrations in equations (2.19) and (2.20), and retaining the terms containing the small parameter ε to various powers as a factor, equations are obtained which determine the frequency ω and phase shift ψ to different degrees of exactness. Thus, for example, retaining the terms containing the small parameter of zeroth and first powers, formulas are obtained for determining ω and ψ in the first approximation:

$$\pi(p^2-\omega^2)x_0\cos\psi+p^2\oint\varepsilon\Phi\left[x_0\cos(\omega t+\psi)\right]\cos\omega t\,dt=0, \tag{2.21}$$

$$-\pi(p^2-\omega^2)x_0\sin\psi+p^2\oint\varepsilon\Phi\left[x_0\cos(\omega t+\psi)\right]\sin\omega t\,dt-$$
$$-\varepsilon q\pi=0. \tag{2.22}$$

The symbol $\oint$, as it is usually meant, indicates that the integration must be carried out on an entire closed cycle.

Formulas to determine ω and ψ in the second approximation can be obtained, if in the integration of equations (2.19) and (2.20), we retain the terms containing the small parameter in the zeroth, first and second powers, namely:

$$\pi(p^2-\omega^2)x_0\cos\psi+p^2\oint\varepsilon\Phi[x_0\cos(\omega t+\psi)+ + \varepsilon u_1(t)]\cos\omega t\,dt=0, \qquad (2.23)$$

$$-\pi(p^2-\omega^2)x_0\sin\psi+p^2\oint\varepsilon\Phi[x_0\cos(\omega t+\psi)+ + \varepsilon u_1(t)]\sin\omega t\,dt-\varepsilon q\pi=0. \qquad (2.24)$$

To determine the magnitudes of ω and ψ in the third approximation, it is necessary, when integrating equations (2.19) and (2.20), to retain all terms containing as a factor the small parameter from the zeroth to the third power inclusively. As a result the equations obtained are:

$$\pi(p^2-\omega^2)x_0\cos\psi+p^2\oint\varepsilon\Phi[x_0\cos(\omega t+\psi)+\varepsilon u_1(t)+ + \varepsilon^2u_2(t)]\cos\omega t\,dt=0, \qquad (2.25)$$

$$-\pi(p^2-\omega^2)x_0\sin\psi+p^2\oint\varepsilon\Phi[x_0\cos(\omega t+\psi)+\varepsilon u_1(t)+ + \varepsilon^2u_2(t)]\sin\omega t\,dt-\varepsilon q\pi=0. \qquad (2.26)$$

Proceeding in a similar way it is possible to obtain equations from the successive approximations. Before computing the final formulas for determining the displacement x, frequency of vibration ω and the phase shift ψ with accuracy up to the various powers of the small parameter, the functional $\Phi(x)$ must be found. To do this, the hysteresis loop of the material of the bar (spring) is examined. A symmetrical hysteresis loop occurs during vertical vibration of the load suspended from the spring, plotted in coordinates: unit strain ξ — normal (or tangential) stress σ (Figure 2).

The presence of a hysteresis loop demonstrates the variability of the true modulus of elasticity of the material, which is geometrically expressed by the tangent of the angle formed by the tangent to the curve depicting the hysteresis loop and the strain axis, ξ . It is assumed that the true modulus of elasticity of the material of the bar for the ascending and descending branches of the hysteresis loop during the symmetrical cycle (Figure 2) is represented in accordance with the expressions:

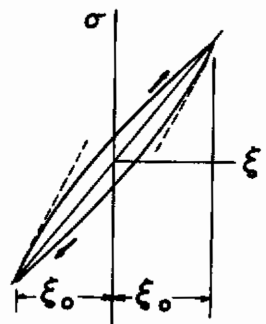

Fig. 2

$$\frac{d\vec{\sigma}}{d\xi} = k\left[1 - f_1(\xi)\right],$$

$$\frac{d\overleftarrow{\sigma}}{d\xi} = k\left[1 + f_2(\xi)\right], \qquad (2.27)$$

where k is the average modulus of the elasticity of the material.

The expressions (2.27) should satisfy the following conditions which follow from the symmetry of the hysteresis loop:

$$\left[\frac{d\vec{\sigma}}{d\xi}\right]_{\xi=-\xi_0} = \left[\frac{d\overleftarrow{\sigma}}{d\xi}\right]_{\xi=\xi_0}; \quad \left[\frac{d\vec{\sigma}}{d\xi}\right]_{\xi=\xi_0} = \left[\frac{d\overleftarrow{\sigma}}{d\xi}\right]_{\xi=-\xi_0}$$

Integrating the expressions (2.27) and determining the constants of integration, by using the condition that when $\xi = \xi_0$ $\vec{\sigma} = \overleftarrow{\sigma}$, we obtain the equations of both branches of the hysteresis loop, which express the dependence between stresses and strains.

$$\vec{\sigma} = k\left[\xi - F_1(\xi)\right],$$

$$\overleftarrow{\sigma} = k\left[\xi + F_2(\xi)\right]. \qquad (2.28)$$

From a comparison of equation (2.28) with differential equation (2.1) it is easy to notice that the product of the functional by the small parameter $\varepsilon\Phi(x)$ is exactly a quantity proportional to the second term in the square brackets of formula (2.28), namely:

$$\varepsilon\vec{\Phi}(\xi) = -F_1(\xi),$$
$$\varepsilon\overleftarrow{\Phi}(\xi) = F_2(\xi),$$

(2.29)

where $\xi = \frac{x}{l}$, l — is some length.

Formulas (2.29) confirm the supposition of two different values of the nonlinear term $\varepsilon\Phi(x)$ for the ascending and descending motion. We shall not break up the differential equation (2.5) into two equations to conform to the two different branches of the hysteresis loop. When using the single equation, which includes the functional $\varepsilon\Phi(x)$, it will be remembered that this term has two values in one cycle of vibration. Thus, when integrating expressions containing $\varepsilon\Phi(x)$ over the entire cycle, in particular, when solving equations (2.28) and (2.29) and similar ones, it must be kept in mind that

$$\oint \varepsilon\Phi(x) = \int_0^{\pi} \varepsilon\overleftarrow{\Phi}(x) + \int_{\pi}^{2\pi} \varepsilon\vec{\Phi}(x).$$

(2.30)

An important argument in favor of the method described is the fact that when deriving all the formulas connected with the study of the vibrations of a system possessing hysteresis losses, one has only to operate with integration of the functional $\varepsilon\Phi(x)$ And so on the basis of (2.6) and (2.29) the functional $\varepsilon\Phi(x)$ for the ascending and descending branches of the hysteresis loop in the first approximation is expressed according to the formulas

$$\varepsilon\vec{\Phi}(x) = \varepsilon\vec{\Phi}(x_0 \cos\varphi),$$
$$\varepsilon\overleftarrow{\Phi}(x) = \varepsilon\overleftarrow{\Phi}(x_0 \cos\varphi),$$

(2.31)

where

$$\varphi = \omega t + \psi.$$

(2.32)

For the second approximation the expressions for $\varepsilon\Phi(x)$ will have the form

$$\varepsilon\vec{\Phi}(x) = \varepsilon\vec{\Phi}[x_0 \cos\varphi + \varepsilon u_1(t)],$$
$$\varepsilon\overleftarrow{\Phi}(x) = \varepsilon\overleftarrow{\Phi}[x_0 \cos\varphi + \varepsilon u_1(t)].$$

(2.33)

In the third approximation $\varepsilon\Phi(x)$ is expressed by the formulas:

$$\varepsilon\vec{\Phi}(x) = \varepsilon\vec{\Phi}[x_0 \cos\varphi + \varepsilon u_1(t) + \varepsilon^2 u_2(t)],$$
$$\varepsilon\overleftarrow{\Phi}(x) = \varepsilon\overleftarrow{\Phi}[x_0 \cos\varphi + \varepsilon u_1(t) + \varepsilon^2 u_2(t)].$$

(2.34)

We remark that in all approximations the limits of the integration remain the same due to the absence of discontinuities in the configuration of the hysteresis loop during a circuit of the latter for one cycle of vibration. For, if it is kept in mind that

$$\int_{\pi}^{\pi+\epsilon a} \Phi(\varphi)\, d\varphi \approx \left[\Phi(\varphi)\right]_{\varphi=\pi} \epsilon a,$$

then

$$\int_{0+\varepsilon\alpha}^{\pi+\varepsilon\beta} \overleftarrow{\Phi}(x)\,dx + \int_{\pi+\varepsilon\beta}^{2\pi+\varepsilon\alpha} \vec{\Phi}(x)\,dx = \int_{0}^{\pi} \overleftarrow{\Phi}(x)\,dx + \int_{\pi}^{2\pi} \vec{\Phi}(x)\,dx +$$
$$+ \varepsilon\beta[\overleftarrow{\Phi}(x)]_{\pi} - \varepsilon\beta[\vec{\Phi}(x)]_{\pi} - \varepsilon\alpha[\overleftarrow{\Phi}(x)]_{0} +$$
$$+ \varepsilon\alpha[\vec{\Phi}(x)]_{2\pi} = \int_{0}^{\pi} \overleftarrow{\Phi}(x)\,dx + \int_{\pi}^{2\pi} \vec{\Phi}(x)\,dx.$$

3. Solution of the problem in the first approximation

It is necessary to calculate the quantity $u_1(t)$ in order to determine the displacement x in the first approximation. The differential equation (2.14) is examined in order to do this:

$$\frac{d^2 u_1(t)}{dt^2} + p^2 u_1(t) = -p^2 \left\{ \Phi\left[x_0 \cos(\omega t + \psi)\right]\right\}_{\sigma/\tau\tau} \cdot \tag{3.1}$$

Since, by convention, the function $u_1(t)$ does not contain the fundamental (first) harmonic, then this harmonic is also absent from the expression in the braces of equation (3.1), which is indicated by the subscript beneath the braces.

Let us consider the expansion

$$\Phi(x_0 \cos\phi) = A(x_0) + \sum_{k=2}^{\infty}\left\{A_k(x_0)\cos k\phi + B_k(x_0)\sin k\phi\right\} ,$$

where, as we know

$$A(x_0) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(x_0\cos\varphi)\, d\varphi,$$

$$A_k(x_0) = \frac{1}{\pi}\int_0^{2\pi} \Phi(x_0\cos\varphi)\cos k\varphi\, d\varphi,$$

$$B_k(x_0) = \frac{1}{\pi}\int_0^{2\pi} \Phi(x_0\cos\varphi)\sin k\varphi\, d\varphi. \tag{3.2}$$

On the basis of (2.32) we can write

$$\Phi\left[x_0\cos(\omega t + \psi)\right] = A(x_0) + \sum_{k=2}^{\infty}\left\{A_k(x_0)\cos k(\omega t + \psi) + B_k(x_0)\sin k(\omega t + \psi)\right\} \tag{3.3}$$

Substituting (3.3) in (3.1) we obtain

$$\frac{d^2u_1(t)}{dt^2} + p^2u_1(t) = -p^2A(x_0) -$$

$$-p^2\sum_{k=2}^{\infty}\{A_k(x_0)\cos k(\omega t+\psi) + B_k(x_0)\sin k(\omega t+\psi)\}. \tag{3.4}$$

From equation (3.4) we find

$$u_1(t) = -A(x_0) +$$

$$+p^2\sum_{k=2}^{\infty}\frac{A_k(x_0)\cos k(\omega t+\psi) + B_k(x_0)\sin k(\omega t+\psi)}{(k\omega)^2-p^2}. \tag{3.5}$$

Denoting the right hand side of the latter equality by $v(\omega t+\psi)$, we represent the expression for the displacement x in the first approximation in the following form, in accordance with (2.6):

$$x = x_0\cos(\omega t+\psi) + \varepsilon v(\omega t+\psi). \tag{3.6}$$

Formulas for determining the frequency ω and phase shift ψ in the first approximation are obtained by examining equations (2.21) and (2.22), which on the basis of (3.2) can be represented as:

$$(p^2-\omega^2)x_0\cos\psi + p^2\varepsilon\{A_1(x_0)\cos\psi + B_1(x_0)\sin\psi\} = 0, \tag{3.7}$$

$$-(p^2-\omega^2)x_0\sin\psi + p^2\varepsilon\{B_1(x_0)\cos\psi - A_1(x_0)\sin\psi\} - \varepsilon q = 0. \tag{3.8}$$

We rewrite (3.7) and (3.8) in the following form:

$$[(p^2-\omega^2)x_0 + p^2\varepsilon A_1(x_0)]\cos\psi + p^2\varepsilon B_1(x_0)\sin\psi = 0, \tag{3.9}$$

$$-[(p^2-\omega^2)x_0 + p^2\varepsilon A_1(x_0)]\sin\psi + p^2\varepsilon B_1(x_0)\cos\psi = \varepsilon q. \tag{3.10}$$

The following formulas are obtained from this system of equations for determining in the first approximation the square of the natural frequency and tangent of the phase shift angle:

$$\omega_I^2 = p^2 + \frac{p^2 \epsilon A_I(x_0) \mp \sqrt{\epsilon^2 q^2 - \epsilon^2 p^4 B_I^2(x_0)}}{x_0}, \tag{3.11}$$

$$\operatorname{tg} \psi_I = \mp \frac{\sqrt{\epsilon^2 q^2 - \epsilon^2 p^4 B_I^2(x_0)}}{p^2 \epsilon B_I(x_0)}. \tag{3.12}$$

4. Solution of the problem in the second approximation

To determine the correction to the displacement of the weight $u_2(t)$ in the second approximation, we use the equation (2.15) taking account of the expansion (2.7)

$$\frac{d^2 u_2(t)}{dt^2} + p_c^2 u_2(t) = -p_c^2 u_1(t) \Phi'_x \left[x_0 \cos(\omega t + \psi)\right] \sigma/_{rr} \tag{4.1}$$

Using the notation

$$F(x, \phi) = u_1(t)\, \Phi'_x \left[x_0 \cos(\omega t + \psi)\right],$$

we consider the expansion

$$F(x, \phi) = A^I(x_0) + \sum_{n=1}^{\infty} \left\{ A_k^I(x_0) \cos k\phi + B_k^I(x_0) \sin k\phi \right\}, \tag{4.2}$$

where

$$A^I(x_0) = \frac{1}{2\pi} \oint F(x, \phi)\, d\phi,$$

$$A_k^I(x_0) = \frac{1}{\pi} \oint F(x, \phi) \cos k\phi \, d\phi,$$

$$B_k^I(x_0) = \frac{1}{\pi} \oint F(x, \phi) \sin k\phi \, d\phi. \tag{4.3}$$

On the basis (4.2), the differential equation (4.1) takes the form:

$$\frac{d^2 u_2(t)}{dt^2} + p_c^2 u_2(t) =$$

$$= -A^I(x_o)p_c^2 - p_c^2 \sum_{k=2}^{\infty} \left[A_k^I(x_o)\cos k\phi + B_k^I(x_o)\sin k\phi\right]. \quad (4.4)$$

Solving this equation with respect to $u_2(t)$ and substituting the value of ϕ according to (2.32), we get

$$u_2(t) = -A^I(x_o) + p_c^2 \sum_{k=2}^{\infty} \frac{1}{(k\omega)^2 - p_c^2}\left[A_k^I(x_o)\cos k(\omega t + \psi) + \right.$$

$$\left. + B^I(x_o)\sin k(\omega t + \psi)\right] = w(\omega t + \psi). \quad (4.5)$$

Thus the magnitude of the displacement in the second approximation will be determined in accordance with (2.6), (3.6), and (4.5) by the formula

$$x_{II} = x_o \cos(\omega t + \psi) + \epsilon v(\omega t + \psi) + \epsilon^2 w(\omega t + \psi). \quad (4.6)$$

We now go on to determine the frequency of vibration ω_{II} and phase shift ψ_{II} in the second approximation. To this end we examine equations (2.23) and (2.24), which we rewrite in the form:

$$(p^2 - \omega^2)x_o \cos\psi + \epsilon p^2\left[A_{II}(x_o)\cos\psi + B_{II}(x_o)\sin\psi\right] = 0, \quad (4.7)$$

$$-(p^2 - \omega^2)x_o \sin\psi + \epsilon p^2\left[B_{II}(x_o)\cos\psi - A_{II}(x_o)\sin\psi\right] = \epsilon q. \quad (4.8)$$

where we have taken

$$A_{II}(x_o) = \frac{1}{\pi}\oint \Phi\left[x_o\cos\phi + \epsilon v(\phi)\right]\cos\phi \, d\phi,$$

$$B_{II}(x_o) = \frac{1}{\pi}\oint \Phi\left[x_o\cos\phi + \epsilon v(\phi)\right]\sin\phi \, d\phi. \quad (4.9)$$

Solving equations (4.7) and (4.8) simultaneously for ω and ψ, proceeding here exactly as in the first approximation, we get the formulas:

$$\omega_{II}^2 = p^2 + \frac{\epsilon p^2 A_{II}(x_0) \mp \sqrt{\epsilon^2 p^2 - \epsilon^2 p^4 B_{II}^2(x_0)}}{x_0}, \tag{4.10}$$

$$\operatorname{tg}\psi_{II} = \mp \frac{\sqrt{\epsilon^2 q^2 - \epsilon^2 p^4 B_{II}^2(x_0)}}{\epsilon p^2 B_{II}(x_0)} \tag{4.11}$$

Thus, by their structures, the formulas of the second approximation differ from the formulas of the first approximation only by the values of the coefficients A and B.

On the basis of the expansion (2.7) the formulas (4.9) are rewritten in the form

$$A_{II}(x_0) = \frac{1}{\pi}\oint \Phi(x_0\cos\varphi)\cos\varphi\, d\varphi + \frac{\varepsilon}{\pi}\oint \Phi'_x(x_0\cos\varphi)v(\varphi)\cos\varphi\, d\varphi,$$

$$B_{II}(x_0) = \frac{1}{\pi}\oint \Phi(x_0\cos\varphi)\sin\varphi\, d\varphi + \frac{\varepsilon}{\pi}\oint \Phi'_x(x_0\cos\varphi)v(\varphi)\sin\varphi\, d\varphi. \tag{4.12}$$

Since in accordance with (3.7)

$$\frac{1}{\pi}\oint \Phi(x_0\cos\phi)\cos\phi\, d\phi = A_I(x_0),$$

$$\frac{1}{\pi}\oint \Phi(x_0\cos\phi)\sin\phi\, d\phi = B_I(x_0).$$

only the second integrals of formulas (4.12) remain to be examined. Taking into account (2.29) and (6.2), and also setting $\xi = \xi_0\cos\varphi$ we can write:

$$\frac{\varepsilon}{\pi}\oint \Phi'_x(x_0\cos\varphi)v(\varphi)\cos\varphi\, d\varphi =$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\nu x_0^n}{n}\frac{d}{d\varphi}[(1-\cos\varphi)^n - 2^{n-1}]v(\varphi)\cos\varphi\, d\varphi -$$

$$- \frac{\nu x_0^n}{\pi n}\int_{\pi}^{2\pi} \frac{d}{d\varphi}[(1+\cos\varphi)^n - 2^{n-1}]v(\varphi)\cos\varphi\, d\varphi =$$

(4.13) cont.

$$= \frac{\nu x_0^n}{\pi}\left\{\int_0^\pi (1-\cos\varphi)^{n-1}\sin\varphi\, v(\varphi)\cos\varphi\, d\varphi + \right.$$

$$\left. + \int_\pi^{2\pi}(1+\cos\varphi)^{n-1}\sin\varphi\, v(\varphi)\cos\varphi\, d\varphi\right\}.$$

(4.13)

Changing the limits of integration in the second integral by replacing φ by $\varphi+\pi$ and, therefore, $v(\varphi)$ by $-v(\varphi)$ we get

$$\frac{\epsilon}{\pi}\oint \Phi'_x(x_0\cos\phi)\, v(\phi)\cos\phi\, d\phi =$$

$$= \frac{v x_0^n}{\pi}\left\{\int_0^\pi (1-\cos\phi)^{n-1}\sin\phi\, v(\phi)\cos\phi\, d\phi - \right.$$

$$\left. -\int_0^\pi (1-\cos\phi)^{n-1}\sin\phi\, v(\phi)\cos\phi\, d\phi\right\} = 0$$

. (4.14)

Therefore,

$$A_{II}(x_0) = A_I(x_0) = \frac{2\nu x_0^n}{\varepsilon\pi n}\int_0^\pi (1-\cos\varphi)^n\cos\varphi\, d\varphi.$$

(4.15)

We determine further the coefficient

$$B_{II}(x_0) = \frac{1}{\pi}\oint \Phi(x_0\cos\varphi)\sin\varphi\, d\varphi + \frac{\varepsilon}{\pi}\oint \Phi'_x(x_0\cos\varphi)\, v(\varphi)\sin\varphi\, d\varphi.$$

We now examine separately each of the integrals of the latter equality

$$\frac{1}{\pi}\oint \Phi(x_0\cos\varphi)\sin\varphi\, d\varphi = \frac{2\nu x_0^n}{\varepsilon\pi n}\left[\int_0^\pi (1-\cos\varphi)^n\sin\varphi\, d\varphi - 2^n\right] =$$

$$= \frac{2^{n+1}x_0^n\nu}{\varepsilon\pi n(n+1)}(1-n) = B_I(x_0);$$

$$\frac{\varepsilon}{\pi}\oint \Phi'_x(x_0\cos\varphi)\, v(\varphi)\sin\varphi\, d\varphi = \frac{\nu x_0^n}{\pi}\int_0^\pi (1-\cos\varphi)^{n-1} v(\varphi)\sin^2\varphi\, d\varphi +$$

$$+ \frac{\nu x_0^n}{\pi}\int_\pi^{2\pi}(1+\cos\varphi)\, v(\varphi)\sin^2\varphi\, d\varphi. \qquad (4.16)$$

(4.16)

Substituting the limits of integration in the last integral, we have

$$\frac{\varepsilon}{\pi}\oint \Phi'_x(x_0\cos\varphi)\,v(\varphi)\sin\varphi\,d\varphi =$$

$$= \frac{\nu x_0^n}{\pi}\left\{\int_0^\pi (1-\cos\varphi)^{n-1}v(\varphi)\sin^2\varphi\,d\varphi - \right.$$

$$\left. - \int_0^\pi (1-\cos\varphi)^{n-1}v(\varphi)\sin^2\varphi\,d\varphi\right\} = 0.$$

(4.17)

Thus

$$B_{II}(x_0) = B_I(x_0) = \frac{2^{n+1}\nu x_0(1-n)}{\varepsilon\pi n(n+1)}.$$

(4.18)

Using the formulas (4.10) and (4.11) and substituting in them the values of $A_{II}(x_0)$ and $B_{II}(x_0)$ from (4.15) and (4.18) it is possible to construct a resonance curve for the second approximation, and to determine the phase shift for this problem of vibration in a system with one degree of freedom considering the dissipation of energy in material.

5. Solution of the problem in the third approximation

In the third approximation, the magnitude of the phase shift ψ_{III} and the frequency of vibration ω_{III} can be determined by starting from equation (2.25) and (2.26). We use the notation:

$$A_{III}(x_0) = \frac{1}{\pi}\oint \{\Phi[x_0\cos\varphi + \varepsilon v(\varphi) + \varepsilon^2 w(\varphi)]\}\cos\varphi\,d\varphi,$$

$$B_{III}(x_0) = \frac{1}{\pi}\oint \{\Phi[x_0\cos\varphi + \varepsilon v(\varphi) + \varepsilon^2 w(\varphi)]\}\sin\varphi\,d\varphi.$$

(5.1)

Then equations (2.25) and (2.26) can take the form:

$$(p^2-\omega^2)x_0\cos\psi+\varepsilon p^2[A_{III}(x_0)\cos\psi+B_{III}(x_0)\sin\psi]=0, \tag{5.2}$$

$$-(p^2-\omega^2)x_0\sin\psi+\varepsilon p^2[B_{III}(x_0)\cos\psi-A_{III}(x_0)\sin\psi]=\varepsilon q. \tag{5.3}$$

Solving simultaneously equations (5.2) and (5.3) for ω and ψ by analogy with the equation for the second approximation, we obtain the formulas

$$\omega_{III}^2=p^2+\frac{\varepsilon p^2 A_{III}(x_0)\mp\sqrt{\varepsilon^2q^2-\varepsilon^2p^4B_{III}^2(x_0)}}{x_0}, \tag{5.4}$$

$$\operatorname{tg}\psi_{III}=\mp\frac{\sqrt{\varepsilon^2q^2-\varepsilon^2p_c^4B_{III}^2(x_0)}}{\varepsilon p^2B_{III}(x_0)} \tag{5.5}$$

From a comparison of the formulas for the third approximation (5.4) and (5.5) and the formulas for the second approximation (4.10) and (4.11), it is noted that the only difference is the value of the coefficients $A(x_0)$ and $B(x_0)$ The determination of these coefficients is the basic content of the successive computational operations when solving a problem in different approximations.

The coefficients $A_{III}(x_0)$ and $B_{III}(x_0)$ in formulas (5.2) and (5.3) will be determined for the third approximation, using formulas (5.1) taking account of the expansion (2.7)

$$\begin{aligned} A_{III}(x_0)=\frac{1}{\pi}\Big[&\oint\Phi(x_0,\cos\varphi)\cos\varphi\,d\varphi+\\ &+\varepsilon\oint\Phi'_x(x_0,\cos\varphi)v(\varphi)\cos\varphi\,d\varphi+\\ &+\varepsilon^2\oint\Phi'_x(x_0,\cos\varphi)w(\varphi)\cos\varphi\,d\varphi+\\ &+\frac{\varepsilon^2}{2}\oint\Phi''_x(x_0,\cos\varphi)v^2(\varphi)\cos\varphi\,d\varphi\Big]; \end{aligned} \tag{5.6}$$

$$B_{III}(x_0)=\frac{1}{\pi}\left[\oint \Phi(x_0, \cos\varphi)\sin\varphi\, d\varphi + \right.$$

$$+\varepsilon\oint \Phi'_x(x_0, \cos\varphi)v(\varphi)\sin\varphi\, d\varphi +$$

$$+\varepsilon^2\oint \Phi'_x(x_0, \cos\varphi)w(\varphi)\sin\varphi\, d\varphi +$$

$$\left. +\frac{\varepsilon^2}{2}\oint \Phi''_x(x_0, \cos\varphi)v^2(\varphi)\sin\varphi\, d\varphi\right].$$

(5.7)

The first two integrals on the right side of the equations (5.6) and (5.7) are equal, respectively, to the coefficients of the second approximation $A_{II}(x_0)$ in (4.15) and $B_{II}(x_0)$ in (4.18).

Therefore, for a complete determination of $A_{III}(x_0)$ and $B_{III}(x_0)$ it is necessary to compute the integrals:

$$I_1=\frac{\varepsilon^2}{\pi}\oint \Phi'_x(x_0\cos\varphi)w(\varphi)\cos\varphi\, d\varphi,$$

$$I_2=\frac{\varepsilon^2}{2\pi}\oint \Phi''_x(x_0\cos\varphi)v^2(\varphi)\cos\varphi\, d\varphi,$$

$$I_3=\frac{\varepsilon^2}{\pi}\oint \Phi'_x(x_0\cos\varphi)w(\varphi)\sin\varphi\, d\varphi,$$

$$I_4=\frac{\varepsilon^2}{2\pi}\oint \Phi''_x(x_0\cos\varphi)v^2(\varphi)\sin\varphi\, d\varphi.$$

In expanded form we have:

$$I_1=\frac{\varepsilon^2}{\pi}\oint \Phi'_x(x_0\cos\varphi)w(\varphi)\cos\varphi\, d\varphi =$$

$$=\frac{\varepsilon\nu x_0^n}{\pi}\left\{\int_0^\pi (1-\cos\varphi)^{n-1}\sin\varphi\cos\varphi\, w(\varphi)d\varphi + \right.$$

$$\left. +\int_\pi^{2\pi}(1+\cos\varphi)^{n-1}\sin\varphi\cos\varphi\, w(\varphi)\,d\varphi\right\} =$$

$$=\frac{\varepsilon\nu x_0^n}{\pi}\left\{\int_0^\pi (1-\cos\varphi)^{n-1}\sin\varphi\cos\varphi\, w(\varphi)\,d\varphi + \right.$$

(5.8)
cont.

$$+\int_0^{\pi}(1-\cos\varphi)^{n-1}\sin\varphi\cos\varphi\, w(\varphi)\,d\varphi\Bigg\}=$$

$$=\frac{2\varepsilon\nu x_0^n}{\pi}\int_0^{\pi}(1-\cos\varphi)^{n-1}\sin\varphi\cos\varphi\, w(\varphi)\,d\varphi. \qquad (5.8)$$

$$I_2=\frac{\varepsilon^2}{2\pi}\oint\Phi_x''(x_0\cos\varphi)\,v^2(\varphi)\,\cos\varphi\,d\varphi=$$

$$=\frac{\varepsilon\nu x_0^n}{2\pi}\Bigg\{\int_0^{\pi}[(n-1)(1-\cos\varphi)^{n-2}\sin^2\varphi+(1-\cos\varphi)^{n-1}\cos\varphi]\times$$

$$\times\cos\varphi v^2(\varphi)\,d\varphi+\int_{\pi}^{2\pi}[(n-1)(1+\cos\varphi)^{n-2}(-\sin^2\varphi)+$$

$$+(1+\cos\varphi)^{n-1}\cos\varphi v^2(\varphi)\,d\varphi\Bigg\}=\frac{\varepsilon\nu x_0^n}{\pi}\int_0^{\pi}[(n-1)(1-\cos\varphi)^{n-2}\sin^2\varphi+$$

$$+(1-\cos\varphi)^{n-1}\cos\varphi]\cos\varphi v^2(\varphi)\,d\varphi. \qquad (5.9)$$

$$I_3=\frac{\varepsilon^2}{\pi}\oint\Phi_x'(x_0\cos\varphi)\,w(\varphi)\,\sin\varphi\,d\varphi=$$

$$=\frac{\varepsilon\nu x_0^n}{\pi}\Bigg\{\int_0^{\pi}(1-\cos\varphi)^{n-1}\sin^2\varphi\, w(\varphi)\,d\varphi+$$

$$+\int_{\pi}^{2\pi}(1+\cos\varphi)^{n-1}\sin^2\varphi\, w(\varphi)\,d\varphi\Bigg\}=$$

$$=\frac{\varepsilon\nu x_0^n}{\pi}\Bigg\{\int_0^{\pi}(1-\cos\varphi)^{n-1}\sin^2\varphi\, w(\varphi)\,d\varphi+\int_0^{\pi}(1-\cos\varphi)^{n-1}\sin^2\varphi\, w(\varphi)\,d\varphi\Bigg\}=$$

$$=\frac{2\varepsilon\nu x_0^n}{\pi}\int_0^{\pi}(1-\cos\varphi)^{n-1}\sin^2\varphi\, w(\varphi)\,d\varphi; \qquad (5.10)$$

$$I_4=\frac{\epsilon^2}{2\pi}\oint\Phi_x''(x_0\cos\phi)\,\upsilon^2(\phi)\sin\phi\,d\phi=$$

$$=\frac{\epsilon\upsilon x_0^n}{2\pi}\Bigg\{\int_0^{\pi}\Big[(n-1)(1-\cos\phi)^{n-2}\sin^2\phi+$$

$$+(1-\cos\phi)^{n-1}\cos\phi\Big]\sin\phi\,\upsilon^2(\phi)\,d\phi+\int_{\pi}^{2\pi}\Big[(n-1)(1+\cos\phi)^{n-2}(-\sin^2\phi)+$$

$$+(1+\cos\phi)^{n-1}\cos\phi\Big]\sin\phi\,\upsilon^2(\phi)\,d\phi\Bigg\}=\frac{\epsilon\upsilon x_0}{\pi}\int\Big[(n-1)(1-\cos\phi)^{n-2}\sin^2\phi+$$

$$+(1-\cos\phi)^{n-1}\cos\phi\Big]\sin\phi\,\upsilon^2(\phi)\,d\phi\,. \qquad (5.11)$$

Substituting the expressions obtained for the integrals into the formulas (5.6) and (5.7), we obtain

$$A_{III}(x_0) = A_{II}(x_0) + \frac{\epsilon \upsilon x_0^n}{\pi} \left\{ \int_0^{\pi} \left[(n-1)(1-\cos\phi)^{n-2} \sin^2\phi + \right.\right.$$

$$\left. + (1-\cos\phi)^{n-1} \cos\phi \right] \cos\phi \, \upsilon^2(\phi) \, d\phi +$$

$$\left. + 2 \int_0^{\pi} (1-\cos\phi)^{n-1} \sin\phi \cos\phi \, w(\phi) \, d\phi \right\} ; \qquad (5.12)$$

$$B_{III}(x_0) = B_{II}(x_0) + \frac{\varepsilon \nu x_0^n}{\pi} \left\{ \int_0^{\pi} [(n-1)(1-\cos\varphi)^{n-2} \sin^2\varphi + \right.$$

$$+ (1-\cos\varphi)^{n-1} \cos\varphi] \sin\varphi \, v^2(\varphi) \, d\varphi +$$

$$\left. + 2 \int_0^{\pi} (1-\cos\varphi)^{n-1} \sin^2\varphi \, w(\varphi) \, d\varphi \right\}.$$

(5.13)

Substituting the values of the coefficients $A_{III}(\chi_0)$ and $B_{III}(\chi_0)$ from (5.12) and (5.13) into formulas (5.4) and (5.5), it is possible to determine the frequency ω_{III} and tangent of the phase shift angle $\tan\psi_{III}$ in the third approximation.

For a complete solution in the third approximation, it is still necessary to determine the term $\varepsilon^3 u_3(t)$ which enters according to (2.6) in the formula for the elongation per unit length χ (taking $l = 1$) in the third approximation. However, this will not be done, since in the second approximation the magnitude of the relative displacement is determined by formula (4.6) with sufficient accuracy. Moreover, it must be remembered that the degree of accuracy of determining the magnitude of frequency of vibration should always be greater than the degree of accuracy for determining the magnitude of displacement in vibration.

6. Equations of the curves which form the hysteresis loop

For establishing the nonlinear relation (2.28) between the stress and strain, leading to the forming of the hysteresis loop, we begin with the expression (2.27) of the true modulus of elasticity.

Remembering that the curves forming the hysteresis loop always have a small curvature, it is possible to represent the tangent of the angle between the tangent to the contour of the loop and the strain axis sufficiently correctly by a power relationship. Examining the hysteresis loop in the coordinates: normal stress σ — elongation per unit length ξ, and using the suggestion of N. N. Davidenkov, we can write the expression for the true modulus of elasticity (2.27) in a symmetrical cycle for the ascending and descending motions as:

$$\frac{d\vec{\sigma}}{d\xi} = E[1-\nu(\xi_0+\xi)^k],$$

$$\frac{d\overleftarrow{\sigma}}{d\xi} = E[1+\nu(\xi_0-\xi)^k]$$

(6.1)

Integrating these equations taking account of the boundary conditions of the branches of the hysteresis loop, we obtain the following final equations of the contour curves of the hysteresis loop:

$$\vec{\sigma}=E\left\{\xi-\frac{\nu}{n}\left[(\xi_0+\xi)^n-2^{n-1}\xi_0^n\right]\right\},$$

$$\overleftarrow{\sigma}=E\left\{\xi+\frac{\nu}{n}\left[(\xi_0-\xi)^n-2^{n-1}\xi_0^n\right]\right\},$$

(6.2)

where E is the modulus of elasticity for extension

ξ_0 is amplitude of the elongation per unit length

ξ is the elongation per unit length at an arbitrary instant of time.

ν and $n=k+1$ are geometrical parameters of the hysteresis loop.

For the same amplitudes of strains, the various materials have different damping, i.e., have a different value of area of the hysteresis loop, which, in turn, depends on the parameters of ν and n. These parameters for materials with different damping will be different and must be determined by experiment.

Here it should be noted that from the point of view of the accepted hypothesis according to which the dissipation of energy in the material is determined by the value of the area of the hysteresis loop, the character of the function (6.1) has no vital importance. For as far as determining the parameters ν and n needed for calculation, we begin with the magnitude of the area of the hysteresis loop at a given amplitude of strain. Then the form of the hysteresis loop, which is determined by the equation (6.2), is of little interest. The second assumption which is implied by the function (6.2), stating that the form of the hysteresis loop is independent of the magnitude of the amplitude of strain, is likewise not essential. It is obvious that the dissipation in the material can be expressed also as a function of the shearing stresses.

7. Construction of the resonance curve

Formula (3.11) is used in this case to construct the resonance curve

$$\omega_1^2 = p^2 + \frac{\varepsilon p^2 A_1(x_0) \mp \sqrt{\varepsilon^2 q^2 - \varepsilon^2 p^4 B_1^2(x_0)}}{x_0}, \tag{7.1}$$

where

$$A_1(x_0) = \frac{2\nu x_0^n}{\varepsilon \pi n}\left[\int_0^{\pi} (1-\cos\varphi)^n \cos\varphi \, d\varphi\right],$$

$$B_1(x_0) = \frac{2^{n+1}\nu x_0^n (1-n)}{\varepsilon \pi n (n+1)}. \tag{7.2}$$

The parameters in the expressions for $A_I(x_o)$ and $B_I(x_o)$ were determined by a bending test of a specimen of St. 20. The values of these parameters are:

$$n=2; \qquad \nu=18,6 \cdot$$

Using these values, we find

$$\int_0^{\pi}(1-\cos\phi)^n \cos\phi\, d\phi =$$

$$=\int_0^{\pi}(1-\cos\phi)^2 \cos\phi\, d\phi = -\pi = -3,141593;$$

$$A_I(x_o) = -18,6\,\frac{x_o^2}{\epsilon},$$

$$B_I(x_o) = -7,894088\,\frac{x_o^2}{\epsilon}. \tag{7.3}$$

Substituting the quantities from (7.3) into formula (3.11), we have

$$\left(\frac{\omega}{p}\right)^2 = 1 + \frac{-18,6\,x_o \mp \sqrt{q_1^2 - 62,316625\,x_o^2}}{x_o}, \tag{7.4}$$

where

$$q_1^2 = \frac{\epsilon^2 q^2}{p^4}.$$

We select the value $q_1^2 = 62,316625 \cdot 10^{-12}$ so that all resonance the maximum relative deformation x_o in the rod l (spring) would not exceed $x_o = 10^{-3}$.

After substituting the quantity q_1 in formula (7.1) and computing the value of ω/p as a function of the magnitude of the strain x_o we obtain the results shown in Table 1.

Table 1

$x_0 \cdot 10^3$	$\left(\frac{\omega}{p}\right)^2_{л}$	$\left(\frac{\omega}{p}\right)_{л}$	$\left(\frac{\omega}{p}\right)^2_{п}$	$\left(\frac{\omega}{p}\right)_{п}$
0,1	0,919203	0,95875	1,077077	1,03782
0,2	0,956841	0,98818	1,035719	1,01770
0,3	0,968213	0,98398	1,020627	1,01023
0,5	0,975412	0,98763	1,005988	1,00299
0,7	0,977149	0,98851	0,996811	0,99840
0,9	0,978116	0,98900	0,988404	0,99419
0,95	0,978751	0,98932	0,985909	0,99293
1,0	0,981400	0,99066	0,981400	0,99066

In Table 1 the quantities $\left(\frac{\omega}{p}\right)_n$ refer to the left branch, and $\left(\frac{\omega}{p}\right)_n$ to the right branch of the resonance curve.

The resonance curve shown on Figure 3 was constructed according to the data of Table 1, where $\chi = \chi_0 \cdot 10^3$

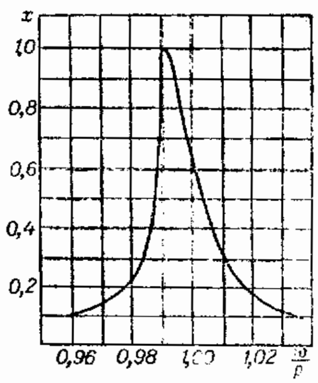

Fig. 3

To prove convergence of the assumed expansion and to establish the degree of approximation of the solution of practical problems by the proposed method, the solutions for displacement, frequency of vibration and phase shift must be found for the succeeding approximations.

As to the solving of the problem for the frequency and magnitude of the phase shift in the second approximation, the theoretical section shows that due to equality of the coefficients $A_I(\chi_0) = A_{II}(\chi_0)$ and $B_I(\chi_0) = B_{II}(\chi_0)$ (formulas (4.15) and (4.18)) the second approximation completely coincides with the first approximation, and, therefore, it is necessary to examine the next, i.e., third, approximation.

The formulas in the third approximation have the form:

$$\left(\frac{\omega}{p}\right)^2 = 1 + \frac{\varepsilon A_{III}(x_0) \mp \sqrt{\varepsilon^2 q^2 \cdot p^{-4} - \varepsilon^2 B_{III}^2(x_0)}}{x_0}. \tag{7.5}$$

Before using the above formula it is necessary to determine the value of εu_1 and $\varepsilon^2 u_2$ the additional terms of the series which approximate the displacement.

Examining the resonance condition (when $\omega = p$), we obtain

$$\varepsilon u_1 = \frac{2\nu x_0^n}{\pi n} \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2-1} \left\{ \cos(2i+1)\varphi \int_0^{\pi} (1-\cos\varphi)^n \cos(2i+1)\varphi\, d\varphi + \right.$$
$$\left. + \sin(2i+1)\varphi \left[\int_0^{\pi} (1-\cos\varphi)\sin(2i+1)\varphi\, d\varphi - \frac{2^n}{2i+1} \right] \right\}. \tag{7.6}$$

The magnitudes of the terms of the above expression for different values of i, are given in Table 2.

Table 2

i	$\int_0^{\pi}(1-\cos\varphi)^n \cos(2i+1)\varphi\, d\varphi$	$\int_0^{\pi}(1-\cos\varphi)^n \sin(2i+1)\varphi\, d\varphi$	$(2i+1)^2-1$	$\frac{2^n}{2i+1}$
1	0	1,6000	8	1,3333
2	0	0,8381	24	0,8000
3	0	0,5841	48	0,5714

Substituting into formula (7.6) the values of the integrals from Table 2, and also the values ν and n we find

$$\varepsilon u_1 = 5{,}920566^2\ (0{,}0333333 \sin 3\varphi + 0{,}0015873 \sin 5\varphi + $$
$$+ 0{,}00026454 \sin 7\varphi).$$

Examining the extreme position of the vibrating mass, $\varphi = 0$, we have

$$\varepsilon u_1 = 0. \tag{7.7}$$

To determine further the magnitude of $\varepsilon^2 u_2$ the formula (4.5) is examined, assuming that $\omega = p$ (in the case of resonance). We will have with these values

$$\varepsilon^2 u_2 = \frac{2\nu x_0^n}{\pi} \sum_{i=2,4,6\ldots}^{\infty} \frac{1}{i^2-1} \left[\cos i\phi \int_0^{\pi} (1-\cos\phi)^{n-1} \sin\phi \cos i\phi\, u_1\, d\phi + \right.$$

$$\left. + \sin i\phi \int_0^{\pi} (1-\cos\phi)^{n-1} \sin\phi \sin i\phi\, u_1\, d\phi \right] = \tag{7.8}$$

$$= 70{,}106204\, x_0^4 \left[\frac{1}{3}(-0{,}0125624) \sin 2\phi + \frac{1}{15}(-0{,}0123509) \sin 4\phi + \right.$$

$$\left. + \frac{1}{35} \cdot 0{,}0056013 \sin 6\phi \right] = 70{,}106204\, x_0^4 (-0{,}00418747 \sin 2\phi -$$

$$- 0{,}000823393 \sin 4\phi + 0{,}000160037 \sin 6\phi)\ .$$

For the extreme position of the mass m when $\varphi = 0$ we obtain

$$\varepsilon^2 u_2 = 0. \tag{7.9}$$

We now proceed to determine the coefficients $A_{III}(x_0)$ and $B_{III}(x_0)$ in formula (7.5):

$$A_{III}(x_0) = A_{II}(x_0) + \frac{\varepsilon\nu x_0^n}{\pi} \left\{ \int_0^{\pi} [(n-1)(1-\cos\varphi)^{n-2} \sin^2\varphi + \right.$$

$$+ (1-\cos\varphi)^{n-1} \cos\varphi] \cos\varphi\, v^2(\varphi)\, d\varphi +$$

$$\left. + 2\int_0^{\pi} (1-\cos\varphi)^{n-1} \sin\varphi \cos\varphi\, w(\varphi)\, d\varphi \right\} \ldots, \tag{7.10}$$

where

$$A_{II}(x_0) = A_I(x_0) = \frac{2\nu x_0^n}{\varepsilon\pi n} \int_0^{\pi} (1-\cos\varphi)^n \cos\varphi\, d\varphi = -18{,}6 \frac{x_0^3}{\varepsilon}.$$

The expression for $v^2(\varphi)$ has, according to (3.6) and (7.6), the form:

$$v^2(\varphi) = \frac{1}{\varepsilon^2} 35{,}053102\, x_0^4 (111{,}111 \cdot 10^{-5} \sin^2 3\varphi +$$

$$+ 2{,}51952 \cdot 10^{-6} \sin^2 5\varphi + 0{,}69914 \cdot 10^{-7} \sin^2 7\varphi +$$

$$+ 2 \cdot 52{,}9099 \cdot 10^{-6} \sin 3\varphi \sin 5\varphi + 2 \cdot 88{,}1799 \cdot 10^{-6} \sin 3\varphi \sin 7\varphi +$$

$$+ 2 \cdot 0{,}419904 \cdot 10^{-6} \sin 5\varphi \sin 7\varphi)$$

or $$v^2(\phi)=\frac{10^{-8}}{\epsilon^2}35{,}053102x_0^4(55685{,}1+5332{,}98\cos 2\phi+$$

$$+881{,}799\cos 4\phi+55555{,}6\cos 6\phi-5290{,}99\cos 8\phi-$$

$$-1007{,}78\cos 10\phi-41{,}9904\cos 12\phi-3{,}49907\cos 14\phi). \quad (7.11)$$

Analogously, we obtain the value $w(\phi)$ in accordance with equation (4.5):

$$w(\phi)=\frac{1}{\epsilon^2}70{,}106204x_0^4(-0{,}00418747\sin 2\phi-$$

$$-0{,}000823393\sin 4\phi+0{,}000160037\sin 6\phi). \quad (7.12)$$

Substituting in formula (5.12) the values of the individual terms from (7.11) and (7.12), and recalling the value $A_{II}(x_0)$ we finally find the expression for $A_{III}(x_0)$ when $n=2$.

$$A_{III}(x_0)=-18{,}6\frac{x_0^2}{\epsilon}+0{,}181530\frac{x_0^6}{\epsilon}. \quad (7.13)$$

Further,

$$B_{III}(x_0)=B_{II}(x_0)+\frac{\nu x_0^n}{\pi}\left\{\int_0^\pi \epsilon\left[(n-1)(1-\cos\phi)^{n-2}\sin^2\phi+\right.\right.$$

$$\left.+(1-\cos\phi)^{n-1}\cos\phi\right]\sin\phi\, v^2(\phi)\,d\phi+$$

$$\left.+2\epsilon\int_0^\pi(1-\cos\phi)^{n-1}\sin^2\phi\, w(\phi)\,d\phi\right\}, \quad (7.14)$$

where, as is known

$$B_{II}(x_0)=B_I(x_0)=7{,}894088\frac{x_0^2}{\epsilon}.$$

Computing the separate integrals in formula (7.14) and substituting their values in formula (5.13), we obtain

$$B_{III}(x_0)=7{,}894088\frac{x_0^2}{\epsilon}-1{,}727017\frac{x_0^6}{\epsilon}. \quad (7.15)$$

Having the values of the coefficients $A_{III}(x_0)$ and $B_{III}(x_0)$, according to formula (5.4) we obtain the frequency of vibrations in the third approximation. Since the magnitudes of the coefficients $A(x_0)$ and $B(x_0)$ in the third approximation differ little from their values in the second and first approximations (when $n = 2$ and $x_0 = 10^{-3}$, only by $1 \cdot 10^{-12}$ %) therefore, to determine the frequency of vibration and phase shift it is sufficient to limit ourselves to the first approximation.

In conclusion, we will examine what extra precision the second and third approximations provide for the displacement. On the basis of (2.6) and (2.32) we have

$$x = x_0 \cos \varphi + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots \tag{7.16}$$

The maximum displacement in the extreme position (when $\varphi = 0$) is, in the zeroth approximation, equal to the amplitude x_0.

The displacement in the first approximation when $\varphi = 0$ on the basis of (7.7) and (7.16) is expressed as:

$$x_{\rm I} = x_0,$$

from which it follows that when $n = 2$ the first approximation exactly coincides with the zeroth approximation. However, it must be kept in mind that, as our calculations indicate, exact agreement does not always occur, i.e., not always when $\varphi = 0$ does $\varepsilon u_1 = 0$ and $\varepsilon^2 u_2 = 0$. Thus, for example, for the value $n = 4.3$ when $x_0 = 10^{-3}$ the improvement of the values of the elongation per unit length has a magnitude of the order of 10%.

On the basis of this calculation and other examples, we conclude that when solving this "hysteresis" problem by methods of nonlinear mechanics, it is possible to obtain sufficiently exact results, if one solves the problem for displacements in the zeroth approximation and frequency of vibration and magnitude of the phase shift in the first approximation.

The fact that for construction of the resonance curve the first approximation, which is based on the consideration of the area of the hysteresis loop, is sufficient indicates that the dissipation of energy in material of the elastic system which affects the variation of amplitude of vibration for a given frequency of forced vibration depends mainly on the magnitude of the area of the hysteresis loop.

Chapter II

Transverse Vibrations of a Bar with Constant Cross-Section

8. Derivation of the basic differential equations

In this chapter, forced transverse vibrations of an elastic bar will be examined, taking account of damping in the material.

This problem is of great theoretical interest, and its solution has considerable significance for machine design practice.

Let the cantilevered bar of a constant cross-section execute forced transverse vibrations under the influence of periodic rotation of the clamped cross-section.

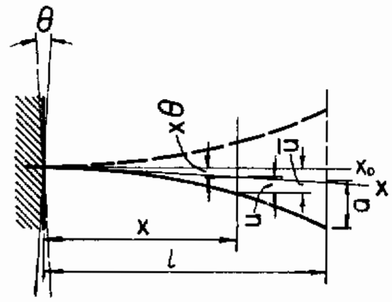

Fig. 4

Turning to the derivation of the differential equation of vibrations of the elastic system, we introduce the following notation (Figure 4):

- x - coordinate axis coinciding with the axis of the bar;
- θ - angle of rotation of the clamped cross-section
- u - deflection of the bar relative its own undeformed axis;
- $\bar{u}$ - total displacement of the cross-section of the bar relative its original axis when $\theta = 0$;
- m - the mass of a unit of length of the bar;
- F - the area of the transverse cross-section of the bar;

ω_c - the natural frequency of vibration of the elastic bar;

ω - the actual frequency of vibrations;

γ - specific gravity of the material of the shaft;

M - the bending moment at the cross-section of the shaft with an abscissa x;

I - moment of inertia of the cross-section of the shaft;

E - modulus of the elasticity of the material;

σ - stress at an arbitrary point of the shaft;

ξ - strain;

ε - a small parameter

We shall assume, that the vibrations take place in one of the principal planes of bending of the shaft and that the dimensions of the transverse cross-section of the shaft are small in comparison to its length. In such a case, it is possible to use the usual differential equation of the deflection curve of a bent bar.

$$M = EI\frac{d^2u}{dx^2}. \tag{8.1}$$

For materials having internal damping, the formula (8.1) is inapplicable.

According to equation (2.28), where the first terms of the right side of both formulas are identical, it is possible, in the general form, to use one formula when considering the nonlinear relationship between normal stress and normal strain.

$$\sigma = \sigma_y + \overleftrightarrow{\sigma}_s(\xi, \nu, \eta), \tag{8.2}$$

where

σ_y is the stress, proportional to the elongation per unit length, with a coefficient of proportionality equal to the average modulus of elasticity of the material;

$\sigma_s(\xi, v, n)$ is that part of stress, which is a power function of the strain and the strain amplitude; here the fixed coefficients entering into this relationship are parameters characterizing the dissipation of energy in material.

The stress σ_y is independent of the direction of strain (loading or unloading), whereas $\sigma_s(\xi, v, n)$ depends on whether we are examining loading or unloading, i.e. whether we take the point on the ascending or descending branch of the hysteresis loop.

Using expression (8.2) for stress, one can represent the differential equation for the flexural vibrations of the vibrating rod in the form:

$$M = EI\frac{\partial^2 u}{\partial x^2} + \varepsilon\Phi\left(\frac{\partial^2 u}{\partial x^2}\right) \tag{8.3}$$

The first term of the right side of the equation (8.3) expressed the magnitude of the moment of the stresses σ_y; this moment, upon (subsequent) unloading, does not form a hysteresis loop. The second term is the magnitude of the bending moment, set up by the stresses $\sigma_s(\xi, v, n)$; this moment completely accounts for the hysteresis loop. In other words, the term $\epsilon\Phi\left(\frac{\partial^2 u}{\partial x^2}\right)$ is the magnitude of the bending moment characterizing the internal losses in the material.

The presence of the small parameter ε in the second term of the right side indicates the small deviation of the magnitude of the total bending moment M at a cross-section x from the magnitude of the elastic moment, equal to $EI\frac{\partial^2 u}{\partial x^2}$ due to the deviation of the stress-strain relationship from Hooke's Law.

Differentiating equations (8.3) twice, we obtain

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right)+\varepsilon\frac{\partial^2}{\partial x^2}\left[\Phi\left(\frac{\partial^2 u}{\partial x^2}\right)\right]=\frac{\partial^2}{\partial x^2}M=q. \tag{8.4}$$

The equation (8.4) is the differential equation of the deflection curve of a bar, subjected to the action of a distributed load with an intensity q used for obtaining the equation of transverse vibrations of this bar. The assumption in this case of an exciting external load in the form of a forced angle of rotation of the fixed cross-section is equivalent to the loading of the vibrating bar with an inertia force distributed along the bar in the following way:

$$q_i=-\frac{\gamma F}{g}\frac{\partial^2\bar{u}}{\partial t^2}=-m\frac{\partial^2 u}{\partial t^2}. \tag{8.5}$$

Keeping in mind that $\bar{u}=x\theta+u$ it is possible to write

$$q_i=m\left(x\frac{\partial^2\theta}{\partial t^2}+\frac{\partial^2 u}{\partial t^2}\right). \tag{8.6}$$

Starting from (8.4) and considering (8.6) we obtain the differential equation of the forced vibrations of the cantilevered bar taking account of damping in the material for forced rotation of the clamped cross-section

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 u}{\partial x^2}\right]+\varepsilon\frac{\partial^2}{\partial x^2}\left[\Phi\left(\frac{\partial^2 u}{\partial x^2}\right)\right]+m\left(x\frac{\partial^2\theta}{\partial t^2}+\frac{\partial^2 u}{\partial t^2}\right)=0. \tag{8.7}$$

In forming the differential equation of the bar, we have not considered the dissipation of energy of vibration due to friction of the vibrating shaft with the external medium. Moreover, we have disregarded the effects of shear deformation and rotatory inertia; i.e., factors, which, in this problem, have secondary importance because the vibrations of a slender bar in the fundamental frequency are under consideration.

By assumption, the shaft has a constant cross-section, consequently, I = constant, m = constant.

Further, we assume, and this is important in our investigation, that the exciting forces, characterized in this case by the change of angle θ, as well as the assumed force of damping of $\varepsilon\frac{\partial^2}{\partial x^2}\left[\Phi\left(\frac{\partial^2 u}{\partial x^2}\right)\right]$, are quite small in comparison to the inertia forces and the elastic forces.

In this connection, it is relevant to mention that both from a theoretical and from a practical point of view it is exactly the cases of slightly damped vibrations caused by small periodic forces which are of greatest interest. The assumption of the smallness of the exciting force, which for this case is equivalent to the smallness of the angle of rotation of the fixed cross-section θ, can be reflected in equation (8.7), by considering the angle of rotation also proportional to the small parameter ε. Taking a harmonic variation of the forced angle of rotation

$$\theta = \varepsilon\beta\cos\omega t = \theta_0\cos\omega t, \tag{8.8}$$

we transform the equation (8.7) for the case of the bar with constant cross-section to the form

$$EI\frac{\partial^4 u}{\partial x^4} + m\frac{\partial^2 u}{\partial t^2} - \varepsilon\beta m x\omega^2\cos\omega t + \varepsilon\frac{\partial^2}{\partial x^2}\left[\Phi\left(\frac{\partial^2 u}{\partial x^2}\right)\right] = 0. \tag{8.9}$$

Introducing the dimensionless coordinates $\zeta = \frac{x}{l}$ and $u^* = \frac{u}{l}$, we rewrite the differential equation (8.9) as

$$EI\frac{\partial^4 u^*}{\partial\zeta^4} + ml^4\frac{\partial^2 u^*}{\partial t^2} - \varepsilon\beta ml^4\zeta\omega^2\cos\omega t + \varepsilon\frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{\partial^2 u^*}{\partial\zeta^2}\right)\right]l = 0. \tag{8.10}$$

We shall assume that the deflection $u^*(\zeta, t)$, the circular frequency of vibration ω and the shift phase ψ can be represented in the form of an expansion by powers of the small parameter ϵ

$$u^*(\zeta,t) = \phi(\zeta)\, a \cos(\omega t + \psi) + \epsilon u_1(\zeta,t) + \epsilon^2 u_2(\zeta,t) + \cdots; \tag{8.11}$$

$$\omega^2 = \omega_c^2 + \epsilon\Delta_1 + \epsilon^2\Delta_2 + \cdots; \tag{8.12}$$

$$\psi = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \cdots \tag{8.13}$$

We introduce the new variable

$$\tau = \omega t + \psi, \tag{8.14}$$

then the differential equation (8.10) takes the form

$$EI\frac{\partial^4 u^*}{\partial\zeta^4} + m l^4 \omega^2 \frac{\partial^2 u^*}{\partial\tau^2} - \epsilon m l^4 \beta\omega^2 \zeta\cos(\tau-\psi) +$$

$$+ \epsilon\frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{\partial^2 u^*}{\partial\zeta^2}\right)\right] l = 0. \tag{8.15}$$

On the basis of (8.13) $\cos(\tau-\psi)$ can be expressed as a series

$$\cos(\tau-\psi) = \cos(\tau-\psi_0)\left[1 - \frac{\epsilon^2(\psi_1 + \epsilon\psi_2 + \cdots)^2}{2!} + \cdots\right] +$$

$$+ \sin(\tau-\psi_0)\left[\epsilon(\psi_1 + \epsilon\psi_2 + \cdots) - \frac{\epsilon^3(\psi_1 + \epsilon\psi_2 + \cdots)^3}{3!} + \cdots\right]. \tag{8.16}$$

Using the change of variable (8.14) and also the expression of $\cos(\tau-\psi)$ in the form of the series (8.16) we substitute the expansions (8.11) and (8.12) in the differential equation (8.10). Further, we group the terms containing as factors the small parameter raised to like powers, and equate to zero the expressions, which multiply the different powers of the small parameter. We obtain

$$EI\frac{d^4\varphi}{d\zeta^4}-ml^4\omega_c^2\varphi=0; \tag{8.17}$$

$$EI\frac{\partial^4 u_1}{\partial\zeta^4}+ml^4\omega_c^2\frac{\partial^2 u_1}{\partial\tau^2}-ml^4\Delta_1 a\varphi\cos\tau-$$

$$-m\beta l^4\zeta\omega_c^2\cos(\tau-\psi_0)+\frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{\partial^2\varphi}{\partial\zeta^2}a\cos\tau\right)\right]l=0; \tag{8.18}$$

$$EI\frac{\partial^4 u_2}{\partial\zeta^4}+ml^4\omega_c^2\frac{\partial^2 u_2}{\partial\tau^2}+ml^4\Delta_1\frac{\partial^2 u_1}{\partial\tau^2}-m\Delta_2 l^4\varphi a\cos\tau-$$

$$-m\beta\zeta l^4\Delta_1\cos(\tau-\psi_0)-m\beta\omega_c^2\zeta l^4\psi_1\sin(\tau-\psi_0)+\frac{\partial^2}{\partial\zeta^2}[\Psi(\zeta,\tau)]l=0. \tag{8.19}$$

The differential equations (8.17), (8.18), and (8.19) form the basic system of equations with the aid of which it is possible to examine, to any degree of approximation, the forced transverse vibrations of a bar taking account of the dissipation of energy in the material.

9. Determination of the deflections and frequencies of vibrations in the zeroth approximation

To solve the present problem in the zeroth approximation, it is sufficient to consider the solution of equation (8.17), which describes the transverse vibrations of a beam with a constant cross-section. The solution of this unperturbed equation, which we will consider as the zeroth approximation for solving our problem can be written as

$$\varphi=C_1(\cos k\zeta+\operatorname{ch} k\zeta)+C_2(\cos k\zeta-\operatorname{ch} k\zeta)+$$
$$+C_3(\sin k\zeta+\operatorname{sh} k\zeta)+C_4(\sin k\zeta-\operatorname{sh} k\zeta), \tag{9.1}$$

$$k^4=\frac{\omega_c^2 ml^4}{EI}. \tag{9.2}$$

The constants of integration C_1, C_2, C_3, and C_4 must be determined from the end conditions. For a bar with one end fixed and one free, these conditions are:

$$1)\ [\varphi]_{\zeta=0}=0; \qquad 2)\ \left[\frac{d\varphi}{d\zeta}\right]_{\zeta=0}=0;$$
$$3)\ \left[\frac{d^2\varphi}{d\zeta^2}\right]_{\zeta=1}=0; \qquad 4)\ \left[\frac{d^3\varphi}{d\zeta^3}\right]_{\zeta=1}=0. \tag{9.3}$$

On the basis of the first two conditions $C_1 = C_3 = 0$ and the solution of (9.1) has the form of

$$\varphi=C_2(\cos k\zeta-\operatorname{ch} k\zeta)+C_4(\sin k\zeta-\operatorname{sh} k\zeta). \tag{9.4}$$

The constants C_2 and C_4 are determined from the last two equations of (9.3) and from the condition that the maximum deflection at the end of the bar (when $\tau = 0$) is equal to the amplitude of vibration of the end of the shaft a, i.e.

$$[u(\zeta,\tau)]_{\substack{\zeta=1\\ \tau=0}}=a; \tag{9.3'}$$

$$\left[\frac{d^2\varphi}{d\zeta^2}\right]_{\zeta=1}=k^2[(-\cos k-\operatorname{ch} k)C_2+(-\sin k-\operatorname{sh} k)C_4]=0;$$
$$\left[\frac{d^3\varphi}{d\zeta^3}\right]_{\zeta=1}=k^3[(\sin k-\operatorname{sh} k)C_2+(-\cos k-\operatorname{ch} k)C_4]=0. \tag{9.5}$$

The ratio of the constants of integration is, on the basis of the last equations,

$$\frac{C_2}{C_4}=\frac{\cos k+\operatorname{ch} k}{\sin k-\operatorname{sh} k}=-\frac{\sin k+\operatorname{sh} k}{\cos k+\operatorname{ch} k}.$$

Introducing certain constant multipliers A or A_1, we write

$$C_2=A(\cos k+\operatorname{ch} k)=-A_1(\sin k+\operatorname{sh} k),$$
$$C_4=A(\sin k-\operatorname{sh} k)=A_1(\cos k+\operatorname{ch} k).$$

Thus, the integral can now be expressed in the form

$$\varphi = A[(\cos k + \operatorname{ch} k)(\cos k - \operatorname{ch} k) + \\ + (\sin k - \operatorname{sh} k)(\sin k - \operatorname{sh} k)].$$

(9.6)

The constant A, is determined from the condition (9.3')

$$[u(\zeta, t)]_{\substack{\zeta=1 \\ t=0}} = A[\cos^2 k - \operatorname{ch}^2 k + \sin^2 k - 2\operatorname{sh} k \sin k + \operatorname{sh}^2 k]\, a = a,$$

from which

$$A = \frac{1}{2 \sin k \operatorname{sh} k}. \tag{9.7}$$

Consequently

$$C_3 = \frac{\cos k + \operatorname{ch} k}{2 \sin k \operatorname{sh} k},$$

$$C_4 = \frac{\sin k - \operatorname{sh} k}{2 \sin k \operatorname{sh} k}.$$

The solution (9.1) of the differential equation (8.17) will be

$$\varphi = \frac{1}{2 \sin k \operatorname{sh} k}[(\cos k + \operatorname{ch} k)(\operatorname{ch} k\zeta - \cos k\zeta) + \\ + (\sin k - \operatorname{sh} k)(\operatorname{sh} k\zeta - \sin k\zeta)].$$

(9.8)

On the basis of (8.11) and (9.8) the deflection function of the bar in the zeroth approximation is

$$u^*(\zeta, t) = \frac{a \cos \tau}{2 \sin k \operatorname{sh} k}[(\cos k + \operatorname{ch} k)(\operatorname{ch} k\zeta - \cos k\zeta) + \\ + (\sin k - \operatorname{sh} k)(\operatorname{sh} k\zeta - \sin k\zeta)].$$

(9.9)

The natural frequency of vibrations ω_c is determined by the equation:

$$\cos k \operatorname{ch} k = -1 \tag{9.10}$$

The successive roots of this equation, as is known, have the values

k_1	k_2	k_3	k_4	k_5	k_6
1,875	4,694	7,855	10,996	14,137	17,279

$$k_i = \frac{1}{2}(2i-1)\pi, \quad \text{where} \quad i > 4.$$

The frequency of vibrations of any mode is, on the basis of (9.2),

$$[\bar{\omega}_c]_i = \frac{[\omega_c]_i}{2\pi} = \frac{k_i^2}{2\pi l^2}\sqrt{\frac{EI}{m}}. \qquad (9.11)$$

In particular, for the fundamental frequency of vibrations we obtain

$$[\bar{\omega}_c]_1 = \frac{1}{2\pi}\left(\frac{1,875}{l}\right)^2\sqrt{\frac{EI}{m}} = \frac{3,515}{2\pi l^2}\sqrt{\frac{EI}{m}}.$$

10. Determination of the frequency and phase shift in the first approximation

We now turn to equation (8.18) from which we shall find the quantities Δ_1 and $\sin\psi_0$, which are necessary for the solution of the problem in the first approximation, in accordance with the expansions (8.12) and (8.13).

The balance of energy of vibration of the shaft for one cycle will be examined. To do this, we multiply the equation (8.18) first by $\varphi \sin\tau \, d\zeta \, d\tau$, and a second time by $\varphi \cos\tau \, d\zeta \, d\tau$. Further, we equate to zero the integrals of the products just formed taken over the entire length of the bar for one cycle.

$$\oint\int_0^1 \left\{ EI \frac{\partial^4 u_1}{\partial\zeta^4} + ml^4\omega_c^2 \frac{\partial^2 u_1}{\partial\tau^2} - a\Delta_1 ml^4\varphi\cos\tau - \right.$$
$$\left. - m\beta\omega_c^2\zeta l^4\cos(\tau-\psi_0) + \frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\right] l \right\} \varphi\sin\tau \, d\zeta \, d\tau = 0; \qquad (10.1)$$

$$\oint\int_0^1\left\{EI\frac{\partial^4 u_1}{\partial\zeta^4}+ml^4\omega_c^2\frac{\partial^2 u_1}{\partial\tau^2}-a\Delta_1 l^4 m\varphi\cos\tau-m\beta\omega_c^2 l^4\zeta\cos(\tau-\psi_0)+\right.$$
$$\left.+\frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\right]l\right\}\varphi\cos\tau\, d\zeta\, d\tau=0. \qquad (10.2)$$

The integral of the first two terms both in equation (10.1), and in equation (10.2) is equal to zero, i.e.

$$\int_0^{2\pi}\int_0^1\left[EI\frac{\partial^4 u_1}{\partial\zeta^4}+ml^4\omega_c^2\frac{\partial^2 u_1}{\partial\tau^2}\right]\varphi\sin\tau\, d\zeta\, d\tau=0. \qquad (10.3)$$

The correctness of equation (10.3) can easily be proven. For, integrating the first term by parts with respect to ζ, and the second with respect to τ and considering the conditions at the ends of the bar (9.3), we obtain the relations

$$\int_0^1\frac{\partial^4 u_1}{\partial\zeta^4}\varphi\, d\zeta=\int_0^1\frac{d^4\varphi}{d\zeta^4}u_1\, d\zeta,$$
$$\int_0^{2\pi}\frac{\partial^2 u_1}{\partial\tau^2}\sin\tau\, d\tau=-\int_0^{2\pi}u_1\sin\tau\, d\tau.$$

Then equation (10.3) can be rewritten as follows:

$$\int_0^{2\pi}\int_0^1\left[EI\frac{d^4\varphi}{d\zeta^4}u_1+ml^4\omega_c^2\frac{\partial^2 u_1}{\partial\tau^2}\varphi\right]\sin\tau\, d\zeta\, d\tau=$$
$$=\int_0^{2\pi}\int_0^1\left[EI\frac{d^4\varphi}{d\zeta^4}-ml^4\omega_c^2\varphi\right]u_1\sin\tau\, d\zeta\, d\tau. \qquad (10.4)$$

But the right part of the last equation is equal to zero because of (8.17) and, consequently, the left side is also equal to zero, which proves the correctness of (10.3). Thus, the differential equation (10.1) splits into two equations: equation (10.3) and

$$\oint\int_0^1\left\{-a\Delta_1 l^4 m\varphi\cos\tau - m\beta l^4\omega_c^2\zeta\cos(\tau-\psi_0)+ + l\frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\right]\right\}\varphi\sin\tau\, d\zeta\, d\tau=0. \tag{10.5}$$

Analogously to equation (10.1), equation (10.2) splits into the following two equations:

$$\int_0^{2\pi}\left\{EI\frac{\partial^4 u_1}{\partial\zeta^4}+ml^4\omega_c^2\frac{\partial^2 u_1}{\partial\tau^2}\right\}\varphi\cos\tau\, d\zeta\, d\tau=0; \tag{10.6}$$

$$\oint\int_0^1\left\{-a\Delta_1 l^4 m\varphi\cos\tau - ml^4\beta\omega_c^2\zeta\cos(\tau-\psi_0)+ + l\frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\right]\right\}\varphi\cos\tau\, d\zeta\, d\tau=0. \tag{10.7}$$

Solving equation (10.7) for Δ_1, we obtain

$$\Delta_1=\frac{1}{am\pi l^4\int_0^1\varphi^2 d\zeta}\left\{\oint\int_0^1\frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\right]l\varphi\cos\tau\, d\zeta\, d\tau - - m\omega_c^2 l^4\beta\pi\cos\psi_0\int_0^1\zeta\varphi\, d\zeta\right\}. \tag{10.8}$$

The first integral in the braces of the above equation can, on the basis of (8.3) and (8.17), be transformed into the following form:

$$l\oint\int_0^1\frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{d^2\phi}{d\zeta^2}a\cos\tau\right)\right]\phi\cos\tau\, d\zeta\, d\tau = = l\oint\int_0^1\frac{\partial^2}{\partial\zeta^2}\left[M\left(\frac{d^2\phi}{d\zeta^2}a\cos\tau\right)\right]\phi\cos\tau\, d\zeta\, d\tau - -\oint\int_0^1\frac{\partial^2}{\partial\zeta^2}\left[EI\frac{d^2\phi}{d\zeta^2}a\cos\tau\right]\phi\cos\tau\, d\zeta\, d\tau. \tag{10.9}$$

But because

$$\oint\int_0^1 \frac{\partial^2}{\partial\zeta^2}\left[EI\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right]\varphi\cos\tau\, d\zeta\, d\tau =$$

$$=\int_0^{2\pi}\int_0^1 EI\frac{d^4\varphi}{d\zeta^4}\varphi a\cos^2\tau\, d\zeta\, d\tau =$$

$$= m\omega_c^2 l^4\int_0^{2\pi}\int_0^1 a\varphi^2\cos^2\tau\, d\zeta\, d\tau = aml^4\omega_c^2\pi\int_0^1\varphi^2\, d\zeta,$$

then

$$l\varepsilon\oint\int_0^1\frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\right]\varphi\cos\tau\, d\zeta\, d\tau =$$

$$= l\oint\int_0^1\frac{\partial^2}{\partial\zeta^2}\left[M\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\right]\varphi\cos\tau\, d\zeta\, d\tau - aml^4\omega_c^2\pi\int_0^1\varphi^2\, d\zeta.$$

Then from equation (10.8) we find

$$\varepsilon\Delta_1 = \left(aml^4\pi\int_0^1\varphi^2 d\zeta\right)^{-1}\left\{l\oint\int_0^1\frac{\partial^2}{\partial\zeta^2}\left[M\left(\frac{d\varphi}{d\zeta}a\cos\varphi\right)\right]\varphi\cos\tau\, d\zeta\, d\tau - \right.$$

$$\left. - m\theta_0 l^4\omega_c^2\int_0^1\zeta\varphi\, d\zeta\right\} - \omega_c^2, \tag{10.10}$$

where

$$\theta_0 = \varepsilon\beta.$$

On the basis of (8.12) and (10.10) the square of the frequency in the first approximation is equal to

$$\omega^2 = \omega_c^2 + \varepsilon\Delta_1 =$$

$$= \left(aml^4\pi\int_0^1\varphi^2\, d\zeta\right)^{-1}\left\{l\oint\int_0^1\frac{\partial^2}{\partial\zeta^2}\left[M\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\right]\varphi\cos\tau\, d\zeta\, d\tau - \right.$$

$$\left. - m\theta_0 l^4\omega_c^2\pi\cos\psi_0\int_0^1\zeta\varphi\, d\zeta\right\}. \tag{10.11}$$

If equation (10.5) is used, we will obtain the formula for determining the phase shift in the first approximation. Integrating (10.5) with respect to τ and multiplying the equation by the small parameter ε, we obtain

$$ml^4\pi\theta_0\omega_c^2\int_0^1 \zeta\varphi \sin\psi_0\, d\zeta =$$
$$= l\varepsilon \oint\int_0^1 \frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\right]\varphi\sin\tau\, d\zeta\, d\tau. \qquad (10.12)$$

Since

$$\int_0^{2\pi}\int_0^1\left[\frac{\partial^2}{\partial\zeta^2}\left(EI\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\right]\varphi\sin\tau\, d\zeta\, d\tau = 0,$$

equation (10.9) can be transformed into

$$\omega_c^2\theta_0\int_0^1\zeta\varphi\sin\psi_0\, d\zeta = \frac{1}{\pi ml^4}\oint\int_0^1\left\{\frac{\partial^2}{\partial\zeta^2}\left(EI\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)+\right.$$
$$\left.+l\frac{\partial^2}{\partial\zeta^2}\left[\varepsilon\Phi\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\right]\right\}\varphi\sin\tau\, d\zeta\, d\tau =$$
$$= \frac{1}{\pi ml^3}\oint\int_0^1\frac{\partial^2}{\partial\zeta^2}\left[M\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\right]\varphi\sin\tau\, d\zeta\, d\tau.$$

Integrating the right side of the last equation by parts, we have

$$\theta_0\omega_c^2\sin\psi_0\int_0^1\zeta\varphi\, d\zeta = \frac{1}{\pi ml^3}\oint\int_0^1\left[M\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\frac{d^2\varphi}{d\zeta^2}d(\cos\tau)\right]d\zeta,$$

whence

$$\sin\psi_0 = \left[\theta_0\pi ml^3\omega_c^2\int_0^1\zeta\varphi\, d\zeta\right]^{-1}\oint\int_0^1 M\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\frac{d^2\varphi}{d\zeta^2}\sin\tau\, d\zeta\, d\tau. \qquad (10.13)$$

In formulas (10.11) and (10.13) the magnitudes of frequency of vibrations ω and the sine of the phase shift $\sin\psi_0$ are expressed in the first approximation by an integral of the bending moment at cross-sections of the bar along its entire length for one cycle of vibration. When integrating over the cycle from 0 to π, the expression of the moment corresponding to the descending branch of the hysteresis loop $\overleftarrow{M}$ is taken, and integrating from π to 2π the moment corresponding to ascending branch $\vec{M}$ is taken.

According to equations (6.2) and (8.3) the bending moment at a cross-section for the ascending and descending branches of the hysteresis loop can be represented by the expressions:

$$\vec{M} = \frac{1}{l} EI \frac{d^2\varphi}{d\zeta^2} a \cos\tau - \int_F E\frac{\nu}{n}\left[(\xi_0+\xi)^n - 2^{n-1}\xi_0^n\right] z\,dF;$$

$$\tilde{M} = \frac{1}{l} EI \frac{d^2\varphi}{d\zeta^2} a \cos\tau + \int_F E\frac{\nu}{n}\left[(\xi_0-\xi)^n - 2^{n-1}\xi_0^n\right] z\,dF. \tag{10.14}$$

For a bar of rectangular cross-section with width, b, and height, h, the maximum amplitude of the strain will be

$$(\xi_0)_{max.} = \frac{1}{l}\left(\frac{\partial^2 u}{\partial\zeta^2}\right)_{max.}\frac{h}{2} = a\,\frac{h}{2l}\,\frac{d^2\phi}{d\zeta^2}$$

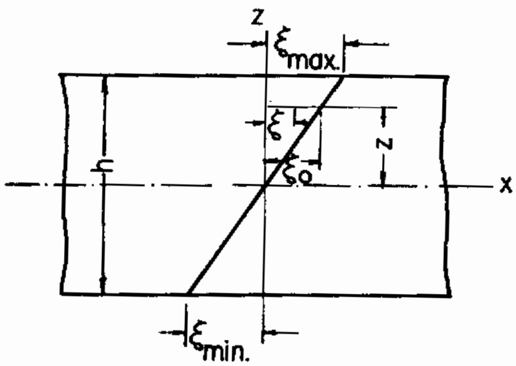

Fig. 5

Assuming a linear distribution of strain through the height of the cross-section of the bar in bending, we write the expression for ξ_0 and ξ (Figure 5) at any point at a distance z, from the neutral axis

$$\xi_0 = (\xi_0)_{max.}\frac{2z}{h} = \frac{az}{l}\,\frac{d^2\phi}{d\zeta^2};$$

$$\xi = \frac{az}{l}\,\frac{d^2\phi}{d\zeta^2}\cos\tau \tag{10.15}$$

If we substitute in (10.14) the values from (10.15), we obtain the following expressions for the bending moments:

$$\vec{M} = \frac{1}{l} EI \frac{d^2\varphi}{d\zeta^2} a \cos\tau - 2\int_0^{\frac{h}{2}} E\frac{\nu}{n}\left[\frac{a}{l}\frac{d^2\varphi}{d\zeta^2}z + \right.$$

$$\left. + \frac{a}{l}\frac{d^2\varphi}{d\zeta^2}z\cos\tau\right]^n - 2^{n-1}\left[\frac{a}{l}\frac{d^2\varphi}{d\zeta^2}z\right]^n\Bigg\} bz\,dz;$$

$$\overleftarrow{M} = \frac{1}{l} EI \frac{d^2\varphi}{d\zeta^2} a \cos\tau + 2\int_0^{\frac{h}{2}} E\frac{\nu}{n}\left[\frac{a}{l}\frac{d^2\varphi}{d\zeta^2}z - \right.$$

$$\left. - \frac{a}{l}\frac{d^2\varphi}{d\zeta^2}z\cos\tau\right]^n - 2^{n-1}\left[\frac{a}{l}\frac{d^2\varphi}{d\zeta^2}z\right]^n\Bigg\} bz\,dz. \qquad (10.16)$$

After integrating, we obtain

$$\vec{M} = \frac{1}{l} EI \frac{d^2\varphi}{d\zeta^2} a\cos\tau - \frac{bh^{n+2}a^n}{l^n(n+2)2^{n+2}}\frac{E\nu}{n}\left(\frac{d^2\varphi}{d\zeta^2}\right)^n[2(1+\cos\tau)^n - 2^n];$$

$$\overleftarrow{M} = \frac{1}{l} EI \frac{d^2\varphi}{d\zeta^2} a\cos\tau + \frac{bh^{n+2}a^nE\nu}{l^n(n+2)2^{n+2}n}\left(\frac{d^2\varphi}{d\zeta^2}\right)^n[2(1-\cos\tau)^n - 2^n].$$

(10.17)

Remembering that the moment of inertia of a rectangular cross-section is

$$I = \frac{bh^3}{12},$$

we rewrite (10.17)

$$\vec{M} = \frac{1}{l} EI\left\{\frac{d^2\varphi}{d\zeta^2} a\cos\tau - \frac{3a^nh^{n-1}\nu}{2^nn(n+2)l^{n-1}}[2(1+\cos\tau)^n - 2^n]\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right\};$$

$$\overleftarrow{M} = \frac{1}{l} EI\left\{\frac{d^2\varphi}{d\zeta^2} a\cos\tau + \frac{3a^nh^{n-1}\nu}{2^nn(n+2)l^{n-1}}[2(1-\cos\tau)^n - 2^n]\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right\}.$$

(10.18)

Since we have at our disposal the expressions for the bending moments in the cross-sections of the vibrating bar for the upward and downward motions, we can calculate the values of the integrals from the expressions containing the moments in formulas (10.11) and (10.13) in order to determine ω and $\sin\psi$.

On the basis of (10.18) we have

$$\oint\int_0^1 \frac{\partial^2}{\partial\zeta^2}\left[M\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\right]\varphi\cos\tau\,d\zeta\,d\tau=$$

$$=\oint\int_0^1 M\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\frac{d^2\varphi}{d\zeta^2}\cos\tau\,d\zeta\,d\tau=$$

$$=\int_\pi^{2\pi}\int_0^1 \vec{M}\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\frac{d^2\varphi}{d\zeta^2}\cos\tau\,d\zeta\,d\tau+$$

$$+\int_0^\pi\int_0^1 \overleftarrow{M}\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\frac{d^2\varphi}{d\zeta^2}\cos\tau\,d\zeta\,d\tau=$$

$$=EI\left\{\frac{\pi a k^4}{4l}+\frac{12h^{n-1}\nu a^n}{l^n n(n+2)2^n}\int_0^\pi\int_0^1\left(\frac{d^2\varphi}{d\zeta^2}\right)^{n+1}(1-\cos\tau)^n\cos\tau\,d\zeta\,d\tau\right\};\tag{10.19}$$

$$\oint\int_0^1 \frac{\partial^2}{\partial\zeta^2}\left[M\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\right]\varphi\sin\tau\,d\zeta\,d\tau=$$

$$=\oint\int_0^1 M\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\frac{d^2\varphi}{d\zeta^2}\sin\tau\,d\zeta\,d\tau=$$

$$=\int_\pi^{2\pi}\int_0^1 \vec{M}\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\frac{d^2\varphi}{d\zeta^2}\sin\tau\,d\zeta\,d\tau+$$

$$+\int_0^\pi\int_0^1 \overleftarrow{M}\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\frac{d^2\varphi}{d\zeta^2}\sin\tau\,d\zeta\,d\tau=$$

$$=-\frac{12(n-1)EIa^n h^{n-1}\nu}{n(n+1)(n+2)l^n}\int_0^1\left[\frac{d^2\varphi}{d\zeta^2}\right]^{n+1}d\zeta.\tag{10.20}$$

Substituting further the values of the integrals (10.19) and (10.20), and also those of the $\int_0^1\zeta\phi\,d\zeta$, $\int_0^1\phi^2 d\zeta$ i $\int_0^1\left(\frac{d^2\phi}{d\zeta^2}\right)d\zeta$ integrals in formulas (10.11) and (10.13), with consideration of (9.2) we obtain

$$\omega^2 = \omega_c^2 + \frac{48h^{n-1}a^{n-1}\nu\omega_c^2}{2^n l^{n-1} n(n+2)\pi k^4}\int_0^\pi\int_0^1\left(\frac{d^2\varphi}{d\zeta^2}\right)^{n+1}(1-\cos\tau)^n\cos\tau\, d\zeta\, d\tau -$$
$$-\frac{4\theta_0\omega_c^2(\cos k+\operatorname{ch} k)}{k^3 a\sin k\operatorname{sh} k}\cos\psi_0;$$
$$\sin\psi_0 = \frac{-12(n-1)a^n h^{n-1}\nu\sin k\operatorname{sh} k}{n(n+1)(n+2)\theta_0 k^3 l^{n-1}\pi(\cos k-\operatorname{ch} k)}\int_0^1\left(\frac{d^2\varphi}{d\zeta^2}\right)^{n+1}d\zeta. \tag{10.21}$$

We determine the integral from the last equation and substitute its value in the expression for the frequency; further, we divide the right and left parts of this equation by ω_c^2 . As a result we obtain

$$\left(\frac{\omega}{\omega_c}\right)^2 = 1-\frac{4\theta_0(\cos k+\operatorname{ch} k)}{ak^3\sin k\operatorname{sh} k}\left[\frac{(n+1)\sin\psi_0}{2^n(n-1)}\int_0^\pi(1-\cos\tau)^n\cos\tau\, d\tau+\cos\psi_0\right]. \tag{10.22}$$

Formulas (10.21) and (10.22) enable us to calculate in the first approximation the frequency of vibrations and the magnitude of the phase shift as functions of the amplitude of vibration, and also to construct the resonance curve. The constants ν and n in the formulas, must be found by experiments.

11. Determination of the deflections of the bar in the first approximation

We must begin the derivation of formulas for the determination of deflection in the first approximation with the determination of the second term in the series expansion (8.11): $u(\zeta, t)$, or $u_1(\zeta, \tau)$. For this, we will use equation (8.18) which we shall rewrite in the form

$$EI\frac{\partial^4 u_1}{\partial\zeta^4} + m\omega_c^2 l^4\frac{\partial^2 u_1}{\partial\tau^2} = F(\zeta, \tau), \tag{11.1}$$

where

$$F(\zeta,\tau)=ml^4\Delta_1 a\varphi\cos\tau+ml^4\beta\zeta\omega_c^2\cos(\tau-\psi_0)-$$
$$-\frac{\partial^2}{\partial\zeta^2}\left[\Phi\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right]l. \qquad (11.2)$$

In accordance with equations (10.5), (10.6) the function $F(\zeta,\tau)$ is subjected to the condition

$$\oint\int_0^1 F(\zeta,\tau)\,\varphi\sin\tau\,d\zeta\,d\tau=0,$$
$$\oint\int_0^1 F(\zeta,\tau)\,\varphi\cos\tau\,d\zeta\,d\tau=0.$$

For the solution of the differential equations (11.1), we substitute $u_1(\zeta,\tau)$ and $F(\zeta,\tau)$ in the form of the series

$$u_1(\zeta,\tau)=u_0(\zeta)+\sum_{j=1}^{\infty}\{u_j^s(\zeta)\sin j\tau+u_j^c(\zeta)\cos j\tau\}; \qquad (11.3)$$

$$F(\zeta,\tau)=F_0(\zeta)+\sum_{j=1}^{\infty}\{F_j^s(\zeta)\sin j\tau+F_j^c(\zeta)\cos j\tau\}, \qquad (11.4)$$

where

$$F_0(\zeta)=\frac{1}{\pi}\oint F(\zeta,\tau)\,d\tau=\frac{1}{\pi}\left\{\int_0^{2\pi}[\Delta_1 l^4 m\varphi a\cos\tau+ml^4\beta\omega_c^2\zeta\cos(\tau-\psi_0)]-\right.$$
$$\left.-\oint\frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\right]l\right\}d\tau. \qquad (11.5)$$

We shall show that the right side of the last equation goes to zero. As far as the first integral is concerned (the expression in square brackets), that it vanishes is evident, however, that the last integral is zero must be proved. From (8.3), (10.9), and (11.2) it follows that

$$\varepsilon \oint \frac{\partial^2}{\partial \zeta^2} \left[\Phi \left(\frac{d^2\varphi}{d\zeta^2} a \cos \tau \right) \right] d\tau = \oint \frac{\partial^2}{\partial \zeta^2} \left[M \left(\frac{d^2\varphi}{d\zeta^2} a \cos \tau \right) \right] d\tau -$$

$$- \oint \frac{\partial^2}{\partial \zeta^2} \left[\frac{1}{l} EI \left(\frac{d^2\varphi}{d\zeta^2} a \cos \tau \right) \right] d\tau =$$

$$= - \int_0^{2\pi} \frac{1}{l} EI \frac{d^4\varphi}{d\zeta^4} a \cos \tau \, d\tau + \int_0^{2\pi} \frac{1}{l} EI \frac{d^4\varphi}{d\zeta^4} a \cos \tau \, d\tau +$$

$$+ EI \left\{ \int_0^{\pi} \frac{3a^n h^{n-1} \nu}{2^n (n+2) n l^n} [2(1-\cos \tau)^n - 2^n] \frac{d^2}{d\zeta^2} \left[\frac{d^2\varphi}{d\zeta^2} \right]^n d\tau - \right.$$

$$\left. - \int_0^{2\pi} \frac{3a^n h^{n-1} \nu}{2^n (n+2) n l^n} [2(1+\cos \tau)^n - 2^n] \frac{d^2}{d\zeta^2} \left[\left(\frac{d^2\varphi}{d\zeta^2} \right)^n \right] d\tau \right\} = 0.$$

Since $\varepsilon \neq 0$,

$$\oint \frac{\partial^2}{\partial \zeta^2} \left[\Phi \left(\frac{d^2\varphi}{d\zeta^2} a \cos \tau \right) \right] d\varphi = 0$$

and therefore

$$F_0(\zeta) = 0. \tag{11.6}$$

To determine $u_1(\zeta, \tau)$ from equation (11.1) it is necessary to solve a system of differential equations which is formed like (11.1) using of the corresponding terms of expansions (11.3) and (11.4) and add the resulting values for $u_j(\zeta)$. Substituting the zeroth terms of expansions (11.3) and (11.4) into equation (11.1), we find

$$EI \frac{d^4 u_0(\zeta)}{d\zeta^4} = F_0(\zeta).$$

Keeping in mind (11.6), we obtain

$$EI \frac{d^4 u_0(\zeta)}{d\zeta^4} = 0. \tag{11.7}$$

After integrating the last equation and determining the constants of integration from the boundary conditions (9.3), we find that

$$u_0(\zeta)=0. \tag{11.8}$$

Substituting the jth term of the expansions of the series (11.3) and (11.4) into equation (11.1), we obtain

$$EI\frac{d^4u_j(\zeta)}{d\zeta^4}-mj^2l^4\omega_c^2u_j(\zeta)=F_j(\zeta). \tag{11.9}$$

For generality, we drop the scripts s and c on $u_j(\zeta)$ and $F_j(\zeta)$.

We shall substitute the function $u_j(\zeta)$ in the form of an expansion in the fundamental functions

$$u_j(\zeta)=\sum_1^\infty c_k^{(j)}\varphi_k(\zeta), \tag{11.10}$$

where the function $\varphi_k(\zeta)$, which satisfies the boundary conditions of the problem (9.3) is an integral of the differential equation

$$\frac{d^2}{d\zeta^2}\left[EI\frac{d^2\varphi_k(\zeta)}{d\zeta^2}\right]-\lambda_k\varphi_k(\zeta)=0, \tag{11.11}$$

and λ_k are the characteristic values of the parameter.

It is easily seen that the fundamental function $\varphi_k(\zeta)$ is nothing else but the function $\varphi(\zeta)$, determined by formula (9.8) and representing the integral of the basic equation (8.17), but differing from it by a constant multiplier.

Substituting the series (11.10) into equation (11.9) and keeping in mind (11.11), we obtain

$$EI\frac{d^4u_j(\zeta)}{d\zeta^4}=\sum_{k=1}^\infty EI\,c_k^{(j)}\frac{d^4\varphi_k(\zeta)}{d\zeta^4}=\sum_1^\infty c_k^{(j)}\lambda_k\varphi_k(\zeta)$$

$$\sum_{k=1}^\infty c_k^{(j)}\lambda_k\varphi_k(\zeta)-mj^2\omega_c^2l^4\sum_{k=1}^\infty c_k^{(j)}\varphi_k(\zeta)=F_j(\zeta). \tag{11.12}$$

We expand the function $F_j(\zeta)$ in the fundamental functions

$$F_j(\zeta)=\sum_{k=1}^{\infty} a_k^{(j)}\varphi_k(\zeta), \tag{11.13}$$

where $a_k^{(j)}$ is a Fourier series coefficient which is determined by the formula

$$a_k^{(j)}=2\int_0^1 F_j(\zeta)\,\varphi_k(\zeta)\,d\zeta. \tag{11.14}$$

Substituting the series (11.13) into equation (11.12) we find

$$\sum_{k=1}^{\infty} c_k^{(j)}\varphi_k(\zeta)\,(\lambda_k-ml^4j^2\omega_c^2)=\sum_{k=1}^{\infty} a_k^{(j)}\varphi_k(\zeta)$$

or

$$\sum_{k=1}^{\infty} c_k^{(j)}\varphi_k(\zeta)\,(\lambda_k-ml^4j^2\omega_c^2)-a_k^{(j)}\varphi_k(\zeta)=0.$$

Equating to zero the expressions which multiply the same functions $\varphi_k(\zeta)$, we obtain

$$c_k^{(j)}(\lambda_k-ml^4j^2\omega_c^2)-a_k^{(j)}=0,$$

whence

$$c_k^{(j)}=\frac{a_k^{(j)}}{\lambda_k-ml^4j^2\omega_c^2}. \tag{11.15}$$

Replacing $a_k^{(j)}$ by its value from (11.14) we obtain

$$c_k^{(j)}=\frac{2\int_0^1 F_j(\zeta)\varphi_k(\zeta)\,d\zeta}{\lambda_k-mj^2l^4\omega_c^2}.$$

By virtue of (11.10) it is possible to represent $u_j(\zeta)$ in the following form:

$$u_j=\sum_{k=1}^{\infty}\frac{2\int_0^1 F_j(\zeta)\,\varphi_k(\zeta)\,d\zeta}{(\lambda_k-mj^2\omega_c^2l^4)}\,\varphi_k(\zeta). \tag{11.16}$$

Then taking into account (11.8) and (11.16), we have

$$u_1(\zeta,\tau)=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left\{\frac{2\int_0^1 F_j(\zeta)\varphi_k(\zeta)\,d\zeta}{(\lambda_k-mj^2l^4\omega_c^2)}\varphi_k(\zeta)\sin j\tau+\right.$$
$$\left.+\frac{2\int_0^1 F_j(\zeta)\varphi_k(\zeta)\,d\zeta\,\varphi_k(\zeta)\cos j\tau}{(\lambda_k-l^4mj^2\omega_c^2)}\right\} \tag{11.17}$$

In order to make use of equation (11.17) it is necessary to determine the functions $F_j^s(\zeta)$ and $F_j^c(\zeta)$, which are the coefficients of the Fourier series expansions of the function $F(\zeta,\tau)$, determined by formula (11.2). The coefficients of the sine expansion will be

$$F_1^s(\zeta)=\frac{1}{\pi}\oint F(\zeta,\tau)\sin\tau\,d\tau=m\beta l^4\omega_c^2\zeta\sin\psi_0-$$
$$-\frac{12EIa^n(n-1)h^{n-1}\nu}{l^{n-1}\varepsilon\pi n(n+1)(n+2)}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]; \tag{11.18}$$

$$F_j^s(\zeta)=\frac{1}{\pi}\oint F(\zeta,\tau)\sin j\tau\,d\tau=\frac{1}{\pi}\int_0^{2\pi}a\Delta_1 l^4m\varphi(\zeta)\cos\tau\sin j\tau\,d\tau+$$
$$+\frac{1}{\pi}\int_0^{2\pi}l^4m\beta\omega_c^2\zeta\cos(\tau-\psi_0)\sin j\tau\,d\tau-$$
$$-\frac{3EIa^nh^{n-1}\nu}{\varepsilon l^{n-1}2^n(n+2)n\pi}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\left\{\int_0^{\pi}[2(1-\cos\tau)^n-2^n]\sin j\tau\,d\tau-\right.$$
$$\left.-\int_0^{2\pi}[2(1+\cos\tau)^n-2^n]\sin j\tau\,d\tau\right\}. \tag{11.19}$$

Having performed the integration on the right side of equation (11.19) we find that for even j

$$F_{j=2i}^s(\zeta)=0 \tag{11.20}$$

And for odd j

$$F^s_{j=2i+1}(\zeta) = \frac{12EIa^n h^{n-1}\nu}{\varepsilon l^{n-1} n(n+2)(2i+1)\pi} \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right] \times$$
$$\times \left\{1 - \frac{2i+1}{2^n}\int_0^{\pi}(1-\cos\tau)^n \sin(2i+1)\tau\, d\tau\right\},$$

(11.21)

where $i = 1, 2, 3, 4, \ldots$

We now give the coefficients of the development of function $F(\zeta, \tau)$ into a cosine Fourier series

$$F_1^c = \frac{1}{\pi}\oint F(\zeta, \tau)\cos\tau\, d\tau = \Delta_1 m l^4 \varphi(\zeta) a +$$
$$+ m l^4 \beta \omega_c^2 \zeta \cos\psi_0 - \frac{12EIa^n h^{n-1}\nu}{l^{n-1} 2^n \varepsilon n(n+2)\pi} \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right] \times$$
$$\times \int_0^{\pi}(1-\cos\tau)^n \cos\tau\, d\tau;$$

$$F_j^c(\zeta) = \frac{1}{\pi}\oint F(\zeta, \tau)\cos j\tau\, d\tau = \frac{1}{\pi}\int_0^{2\pi}\Delta_1 l^4 m \varphi(\zeta) a \cos\tau \cos j\tau\, d\tau + \qquad (11.22)$$
$$+ \frac{1}{\pi}\int_0^{2\pi} m l^4 \beta \omega_c^2 \zeta \cos(\tau - \psi_0)\cos j\tau\, d\tau - \frac{3EIa^n h^{n-1}\nu}{2^n \varepsilon l^{n-1} n(n+2)\pi} \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{a\zeta^2}\right)^n\right] \times$$
$$\times \left\{\int_0^{\pi}[2(1-\cos\tau)^n - 2^n]\cos j\tau\, d\tau - \int_{\pi}^{2\pi}[2(1+\cos\tau)^n - 2^n]\cos j\tau\, d\tau\right\}. \;(11.23)$$

(11.23)

Integrating the right side of equation (11.23), we find that for even j

$$F^c_{j=2i}(\zeta) = 0 \qquad (11.24)$$

and for odd j

$$F_{j=2i+1}(\zeta) = -\frac{12EIa^n h^{n-1}\nu}{2^n \varepsilon n(n+2) l^{n-1}\pi} \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right] \times$$
$$\times \int_0^\pi (1-\cos\tau)^n \cos(2i+1)\tau\, d\tau. \qquad (11.25)$$

Thus, the expansion of the function $F(\zeta, \tau)$ into a Fourier series will take the final form of

$$F(\zeta,\tau) = \left\{\sin\psi_0 m l^4 \beta\omega_c^2\zeta - \frac{12EIa^n(n-1)h^{n-1}\nu}{\varepsilon l^{n-1} n(n+1)(n+2)\pi}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\right\}\sin\tau +$$
$$+\left\{\Delta_1 m l^4 a\varphi(\zeta) + m l^4 \beta\omega_c^2 \zeta\cos\psi_0 - \frac{12EIa^n h^{n-1}}{2^n \varepsilon n(n+2) l^{n-1}\pi}\times\right.$$
$$\left.\times\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\int_0^\pi (1-\cos\tau)^n\cos\tau\, d\tau\right\}\cos\tau +$$
$$+\sum_{i=1}^{\infty}\left\{\frac{12EIa^n h^{n-1}}{\varepsilon l^{n-1} n(n+2)(2i+1)\pi}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\times\right.$$
$$\left.\times\left[1-\frac{2i+1}{2n}\int_0^\pi (1-\cos\tau)^n \sin(2i+1)\tau\, d\tau\right]\right\}\sin(2i+1)\tau -$$
$$-\sum_{i=1}^{\infty}\left\{\frac{12EIa^n h^{n-1}\nu}{\varepsilon l^{n-1} n(n+2)\pi}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^2\right]\times\right.$$
$$\left.\times\int_0^\pi (1-\cos\tau)^n\cos(2i+1)\tau\, d\tau\right\}\cos(2i+1)\tau. \qquad (11.26)$$

The coefficients of the expansion of $F(\zeta)$ into a Fourier series in the fundamental functions in accordance with formula (11.14) are determined in the following form based on (11.18) and (11.25):

$$a_{j,s}^{(1)} = 2\int_0^1 F_1^s(\zeta)\varphi_j(\zeta)\, d\zeta = 2\int_0^1 \sin\psi_0 m l^4 \beta\omega_c^2 \zeta\varphi_j(\zeta)\, d\zeta -$$
$$-\frac{48EIa^n(n-1)h^{n-1}\nu}{\varepsilon l^{n-1} n(n+1)(n+2)\pi}\int_0^1 \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi_j(\zeta)\, d\zeta. \qquad (11.27)$$

Keeping in mind that according to (10.5)

$$ml^4\pi\beta\omega_c^2\int_0^1 \zeta\varphi(\zeta)\sin\psi_0 d\zeta =$$

$$= \int_0^1 \oint \frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}a\cos\tau\right)l\right]\varphi(\zeta)\sin\tau\, d\tau\, d\zeta,$$

we transform the expression (11.27) into the form

$$a_{1,s}^{(1)} = \frac{2}{\pi}\oint\int_0^1 \frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}a\cos\tau\right)l\right]\varphi(\zeta)\sin\tau\, d\zeta\, d\tau -$$

$$-\frac{48EIa^n(n-1)h^{n-1}\nu}{\varepsilon l^{n-1}n(n+1)(n+2)\pi}\int_0^1 \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi(\zeta)\, d\zeta =$$

$$= \frac{2}{\pi}\left\{\int_0^\pi\int_0^1 \frac{3EIh^{n-1}a^n\nu}{2^n\varepsilon l^{n-1}n(n+2)}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right][2(1-\cos\tau)^n-2^n]\varphi(\zeta)\sin\tau\, d\zeta\, d\tau -\right.$$

$$\left.-\int_\pi^{2\pi}\frac{3EIa^nh^{n-1}}{2^n\varepsilon l^{n-1}n(n+2)}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right][2(1+\cos\tau)^n-2^n]\varphi(\zeta)\sin\tau\, d\zeta\, d\tau\right\} -$$

$$-\frac{48EIa^n(n-1)h^{n-1}\nu}{\varepsilon l^{n-1}n(n+1)(n+2)}\int_0^1 \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi(\zeta)\, d\zeta = 0; \tag{11.28}$$

$$a_{1,c}^{(1)} = 2\int_0^1 F_1^c(\zeta)\varphi(\zeta)\, d\zeta = 2\int_0^1 \Delta_1 l^4 m\varphi(\zeta)a\, d\zeta +$$

$$+2ml^4\beta\omega_c^2\cos\psi_0\int_0^1 \zeta\varphi(\zeta)\, d\zeta - \frac{48EIa^nh^{n-1}}{\varepsilon l^{n-1}2^n n(n+2)\pi}\times$$

$$\times\int_0^1 \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi(\zeta)\, d\zeta\int_0^\pi (1-\cos\tau)\cos\tau\, d\tau. \tag{11.29}$$

On the basis of (10.8) and (10.20) we have

$$\Delta_1 aml^4\int_0^1 \varphi^2(\zeta)\, d\zeta + ml^4\omega_c^2\beta\cos\psi_0\int_0^1 \zeta\varphi(\zeta)\, d\zeta =$$

$$= \frac{l}{\pi}\oint\int_0^1 \frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}a\cos\tau\right)\right]\varphi(\zeta)\cos\tau\, d\zeta\, d\tau =$$

$$= \frac{12EIa^nh^{n-1}\nu}{2^n\varepsilon l^{n-1}n(n+2)\pi}\int_0^1 \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi(\zeta)\, d\zeta\int_0^\pi (1-\cos\tau)^n\cos\tau\, d\tau.$$

Substituting the last equation into (11.29) we finally obtain

$$a^{(1)}_{1,c}=2\int_0^1 F_1^c(\zeta)\varphi_1(\zeta)\,d\zeta=0. \tag{11.30}$$

For values of $k>1$ the coefficients of the first harmonic will be

$$\begin{aligned}a^{(1)}_{k,s}&=2\int_0^1 F_1^s(\zeta)\varphi_k(\zeta)\,d\zeta=2\int_0^1 \sin\psi_0\, ml^4\beta\omega_c^2\zeta\varphi_k(\zeta)\,d\zeta-\\&-\frac{48EIa^n(n-1)h^{n-1}\nu}{\varepsilon l^{n-1}n(n+1)(n+2)\pi}\int_0^1\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi_k(\zeta)\,d\zeta;\end{aligned} \tag{11.31}$$

$$\begin{aligned}a^{(1)}_{k,c}&=2\int_0^1 F_1^c(\zeta)\varphi_k(\zeta)\,d\zeta=2\int_0^1 \Delta_1 ml^4\varphi(\zeta)\varphi_k(\zeta)\,a\,d\zeta+\\&+2ml^4\beta\omega_c^2\cos\psi_0\int_0^1\zeta\varphi_k(\zeta)\,d\zeta-\frac{48EIa^nh^{n-1}\nu}{\varepsilon l^{n-1}2^nn(n+2)\pi}\times\\&\times\int_0^1\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi_k(\zeta)\,d\zeta\int_0^\pi(1-\cos\tau)\cos\tau\,d\tau.\end{aligned}$$

By virtue of the orthogonality of the functions $\varphi_k(\zeta)$ the integral of the first term of the right side of the last equation goes to zero, and as a result we obtain

$$\begin{aligned}a^{(1)}_{k,c}&=2ml^4\beta\omega_c^2\cos\psi_0\int_0\zeta\varphi_k(\zeta)\,d\zeta-\frac{48EIa^nh^{n-1}\nu}{\varepsilon l^{n-1}n(n+2)\pi}\times\\&\times\int_0^1\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi_k(\zeta)\,d\zeta\int_0^\pi(1-\cos\tau)^n\cos\tau\,d\tau.\end{aligned} \tag{11.32}$$

On the basis of (11.20) and (11.24) the coefficients $a_k^{(j)}$ for even j, will be

$$a^{(2i)}_{k\,s}=2\int_0^1 F_{j=2i}(\zeta)\varphi_k(\zeta)\,d\zeta=0; \tag{11.33}$$

$$a^{(2i)}_{k,c}=2\int_0^1 F_{j=2i}(\zeta)\varphi_k(\zeta)\,d\zeta=0. \tag{11.34}$$

In accordance with (11.21), (11.30), and (11.14), the coefficients $a_k^{(j)}$ for odd j will be

$$a_{k,s}^{(2i+1)} = 2\int_0^1 F_{2i+1}^s(\zeta)\varphi_k(\zeta)\,d\zeta = \frac{48EIa^n h^{n-1}\nu}{\varepsilon l^{n-1} n(n+2)(2i+1)\pi}\left[1-\frac{2i+1}{2^n}\times\right.$$

$$\left.\times\int_0^\pi (1-\cos\tau)^n \sin(2i+1)\tau\,d\tau\right]\int_0^1 \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi_k(\zeta)\,d\zeta;$$

$$a_{k,c}^{(2i+1)} = 2\int_0^1 F_{2i+1}^c(\zeta)\varphi_k(\zeta)\,d\zeta = \frac{48EIa^n h^{n-1}\nu}{\varepsilon l^{n-1} 2^n n(n+2)\pi}\times$$

$$\times\int_0^\pi (1-\cos\tau)^n \cos(2i+1)\tau\,d\tau\int_0^1 \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi_k(\zeta)\,d\zeta.$$

Using the notation

$$B_{2i+1}^s = \frac{48EIa^n h^{n-1}\nu}{\varepsilon l^{n-1} n(n+2)(2i+1)\pi}\left[1-\frac{2i+1}{2^n}\int_0^\pi (1-\cos\tau)^n \sin(2i+1)\tau\,d\tau\right];$$

$$B_{2i+1}^c = -\frac{48EIa^n h^{n-1}\nu}{\varepsilon l^{n-1} 2^n n(n+2)\pi}\int_0^\pi (1-\cos\tau)^n \cos\tau(2i+1)\tau\,d\tau;$$

(11.35)

we rewrite expressions for $a_k^{(j)}$ in an abbreviated form

$$a_{k,s}^{(2i+1)} = B_{2i+1}^s\int_0^1 \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi_k(\zeta)\,d\zeta; \qquad (11.36)$$

$$a_{k,c}^{(2i+1)} = B_{2i+1}^c\int_0^1 \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi_k(\zeta)\,d\zeta.$$

(11.37)

Now the coefficients of the expansion of the function $u_j(\zeta)$ in the fundamental functions, according to (11.10) are determined by formula (11.13) on the basis of (11.28)

$$\left[c_{k,s}^{(j)}\right]_{\substack{j=1\\k=1}} = \frac{a_{1,s}^{(1)}}{\lambda - ml^4\omega_c^2} = 0.$$

(11.38)

Similarly, on the basis of (11.30), (11.31), and (11.32), we have

$$\left[c_{k,s}^{(j)}\right]_{\substack{j=1\\k>1}} = \frac{a_{k,s}^{(1)}}{\lambda_k - ml^4\omega_c^2};$$

(11.39)

$$\left[c_{k,c}^{(j)}\right]_{\substack{j=1\\k>1}} = \frac{a_{k,c}^{(1)}}{\lambda_k - m\omega_c^2 l^4}. \tag{11.40}$$

Keeping formulas (11.33) and (11.34) in mind we obtain, for even j

$$c_{k,s}^{(2i)} = \frac{a_{k,s}^{(2i)}}{\lambda_k - ml^4 (2i)^2 \omega_c^2} = 0, \tag{11.41}$$

$$c_{k,c}^{(2i)} = \frac{a_{k,c}^{(2i)}}{\lambda_k - m(2i)^2 \omega_c^2 l^4} = 0. \tag{11.42}$$

If j is odd, on the basis of formulas (11.36) and (11.37), the coefficients $c_{k,s}^{(j)}$ and $c_{k,c}^{(j)}$ will be expressed by the formulas:

$$c_{k,s}^{(2i+1)} = \frac{B_{2i+1}^{s} \int\limits_0^1 \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi_k(\zeta)\,d\zeta}{\lambda_k - ml^4(2i+1)^2\omega_c^2}, \tag{11.43}$$

$$c_{k,c}^{(2i+1)} = \frac{B_{2i+1}^{c} \int\limits_0^1 \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi_k(\zeta)\,d\zeta}{\lambda_k - ml^4(2i+1)^2\omega_c^2}. \tag{11.44}$$

The expansion of the function $u_1(\zeta, \tau)$ into the series (11.17) can finally be presented in the form

$$u_1(\zeta,\tau) = \sum_{k=2}^{\infty}\left[\frac{\varphi_k(\zeta)}{\lambda_k - ml^4\omega_c^2}\left(a_{k,s}^{(1)}\sin\tau + a_{k,c}^{(1)}\cos\tau\right)\right] +$$
$$+ \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\left\{\frac{\int\limits_0^1 \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi_k(\zeta)\,d\zeta}{\lambda_k - ml^4(2i+1)^2\omega_c^2}\,\varphi_k(\zeta)\times\right.$$
$$\left.\times\left[B_{2i+1}^{s}\sin(2i+1)\tau + B_{2i+1}^{c}\cos(2i+1)\tau\right]\right\}. \tag{11.45}$$

Knowing the function $u_1(\zeta, \tau)$, the solution of the problem can be found in the first approximation. On the basis of (8.11) the value of deflection in the first approximation will be

$$u(\zeta, \tau) = \varphi(\zeta) a \cos \tau + \varepsilon u_1(\zeta, \tau).$$

Substituting in the last equation, the expressions for $u(\zeta, \tau)$ from (11.45) we obtain an expression for determining the deflection of the bar in the first approximation:

$$u(\zeta, \tau) = a\varphi(\zeta) \cos \tau + \varepsilon \sum_{k=2}^{\infty} \left[\frac{\varphi_k(\zeta)}{\lambda_k - ml^4\omega_c^2} (a_{k,s}^{(1)} \sin \tau + a_{k,c}^{(1)} \cos \tau) \right] +$$
$$+ \varepsilon \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \left\{ \frac{\int_0^1 \frac{d^2}{d\zeta^2} \left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2} \right)^n \right] \varphi_k(\zeta)\, d\zeta}{\lambda_k - ml^4(2i+1)^2\omega_c^2} \varphi_k(\zeta) \times \right.$$
$$\left. \times \left[B_{2i+1}^s \sin(2i+1)\tau + B_{2i+1}^c \cos(2i+1)\tau \right] \right\}. \qquad (11.46)$$

12. Determination of the frequency of vibrations and the phase shift in the second approximation

To determine the frequency of vibrations and the phase shift in the second approximation, it is necessary to find the additional terms $\varepsilon^2 \Delta_2$ and $\varepsilon^2 \psi_1$, entering into the series (8.12) and (8.13). For this purpose, we will analyze equation (8.19), which was obtained as a result of equating to zero, the expression which multiplies ε^2 in the equation (8.17). We shall multiply equation (8.19) first by $\varphi(\zeta) \cos \tau\, d\tau$, and a second time by $\varphi(\zeta) \sin \tau\, d\tau$, then we integrate the equations obtained along the length of the bar for one cycle of vibrations and equate the result zero. As a result, we will obtain a system of equations:

$$\oint\int_0^1 \left\{ EI\frac{\partial^4 u_2}{\partial \zeta^4} + ml^4\omega_c^2\frac{\partial^2 u_2}{\partial \tau^2} + ml^4\Delta_1\frac{\partial^2 u_1}{\partial \tau^2} - ml^4\Delta_2\varphi(\zeta)\,a\cos\tau - \right.$$
$$- ml^4\zeta\beta\Delta_1\cos(\tau-\psi_0) - ml^4\omega_c^2\zeta\beta\psi_1\sin(\tau-\psi) +$$
$$\left. + \frac{\partial^2}{\partial \zeta^2}[\Psi(\zeta,\ \tau)] \right\}\varphi(\zeta)\cos\tau\, d\zeta\, d\tau = 0; \qquad (12.1)$$

$$\oint\int_0^1 \left\{ EI\frac{\partial^4 u_2}{\partial \zeta^4} + ml^4\omega_c^2\frac{\partial^2 u_2}{\partial \tau^2} + ml^4\Delta_1\frac{\partial^2 u_1}{\partial \tau^2} - ml^4\Delta_2\varphi(\zeta)\,a\cos\tau - \right.$$
$$- ml^4\beta\zeta\Delta_1\cos(\tau-\psi_0) - ml^4\beta\zeta\omega^2\psi_1\sin(\tau-\psi_0) +$$
$$\left. + \frac{\partial^2}{\partial \zeta^2}[\Psi(\zeta,\ \tau)] \right\}\varphi(\zeta)\sin\tau\, d\zeta\, d\tau = 0.$$

$$(12.2)$$

Each of these equations can be split into two equations

$$\oint\int_0^1 \left[EI\frac{\partial^4 u_2}{\partial \zeta^4} + ml^4\omega_c^2\frac{\partial^2 u_2}{\partial \tau^2} \right]\varphi(\zeta)\cos\tau\, d\zeta\, d\tau = 0, \qquad (12.3)$$

$$\oint\int_0^1 \left\{ ml^4\Delta_1\frac{\partial^2 u_1}{\partial \tau^2} - ml^4\Delta_2\varphi(\zeta)\,a\cos\tau - ml^4\beta\zeta\Delta_1\cos(\tau-\psi_0) - \right.$$
$$\left. - ml^4\beta\omega_c^2\zeta\psi_1\sin(\tau-\psi_0) + l\frac{\partial^2}{\partial \zeta^2}[\Psi(\zeta,\ \tau)] \right\}\varphi(\zeta)\cos\tau\, d\zeta\, d\tau = 0;$$

$$(12.4)$$

$$\oint\int_0^1 \left[EI\frac{\partial^4 u_2}{\partial \zeta^4} + ml^4\omega_c^2\frac{\partial^2 u_2}{\partial \tau^2} \right]\varphi(\zeta)\sin\tau\, d\zeta\, d\tau = 0, \qquad (12.5)$$

$$\oint\int_0^1 \left\{ ml^4\Delta_1\frac{\partial^2 u_1}{\partial \tau^2} - ml^4\Delta_2\varphi(\zeta)\,a\cos\tau - ml^4\beta\zeta\Delta_1\cos(\tau-\psi_0) - \right.$$
$$\left. - ml^4\beta\omega_c^2\zeta\psi_1\sin(\tau-\psi_0) + l\frac{\partial^2}{\partial \zeta^2}[\Psi(\zeta,\ \tau)] \right\}\varphi(\zeta)\sin\tau\, d\zeta\, d\tau = 0.$$

$$(12.6)$$

The correctness of such a splitting of equations (12.1) and (12.2) can be shown in the same way it was done in connection with equation (10.3).

Turning to the solution of equations (12.4) and (12.6), it is necessary to clarify what the function $\Psi(\zeta,\tau)$, which is contained in these equations, represents. From equations (8.17) and (9.1), where the function $\Psi(\zeta,\tau)$ was first introduced, it is easy to establish a dimensional analysis that $\Psi(\zeta,\tau)$ represents the value of bending moment of the forces of damping, which occur in a section of the bar. This moment has a value of the second order of smallness, which is indicated by the factor of the small parameter to the second power, ε^2 . Having this in mind, it is possible to represent in the second approximation the bending moments for the upward and downward movements, in the following way:

$$\vec{M}_{II} = \frac{1}{l} EI\left[\frac{d^2\varphi(\zeta)}{d\zeta^2} a\cos\tau + \varepsilon\frac{d^2u_1}{d\zeta^2}\right] + \\ + \varepsilon\vec{\Phi}\left[\frac{d^2\varphi(\zeta)}{d\zeta^2} a\cos\tau\right] + \varepsilon^2\vec{\Psi}(\zeta,\tau);$$

$$\overleftarrow{M}_{II} = \frac{1}{l} EI\left[\frac{d^2\varphi(\zeta)}{d\zeta^2} a\cos\tau + \varepsilon\frac{\partial^2u_1}{\partial\zeta^2}\right] + \\ + \varepsilon\overleftarrow{\Phi}\left[\frac{d^2\varphi(\zeta)}{d\zeta^2} a\cos\tau\right] + \varepsilon^2\overleftarrow{\Psi}(\zeta,\tau).$$

(12.7)

Similarly, we represent expressions for the bending moments (10.14) in the first approximation:

$$\vec{M}_{I} = \frac{1}{l} EI\frac{d^2\varphi(\zeta)}{d\zeta^2} a\cos\tau + \varepsilon\vec{\Phi}\left[\frac{d^2\varphi(\zeta)}{d\zeta^2} a\cos\tau\right];$$

$$\overleftarrow{M}_{I} = \frac{1}{l} EI\frac{d^2\varphi(\zeta)}{d\zeta^2} a\cos\tau + \varepsilon\overleftarrow{\Phi}\left[\frac{d^2\varphi(\zeta)}{d\zeta^2} a\cos\tau\right].$$

(12.8)

From equations (12.7) and (12.8) we find

$$\vec{\Psi}(\zeta,\tau) = \frac{1}{\varepsilon^2}\left[\vec{M}_{II} - \vec{M}_{I} - \frac{1}{l}EI\varepsilon\frac{\partial^2u_1}{\partial\zeta^2}\right];$$

$$\overleftarrow{\Psi}(\zeta,\tau) = \frac{1}{\varepsilon^2}\left[\overleftarrow{M}_{II} - \overleftarrow{M}_{I} - \frac{1}{l}EI\varepsilon\frac{\partial^2u_1}{\partial\zeta^2}\right].$$

(12.9)

On the right side of the last formulas, all the quantities with the exception of the magnitude of the bending moment in the second approximation M_{II}, are known. To determine M_{II}, we will express the strains ξ by means of the derivatives of the expression $u(\zeta, \tau)$, determined by the function (11.46) in the same way as we did in the calculation of bending moments in the first approximation

$$\xi=\left\{a\frac{d^2\varphi(\zeta)}{d\zeta^2}\cos\tau+\varepsilon\sum_{k=2}^{\infty}\frac{\frac{d^2\varphi(\zeta)}{d\zeta^2}}{\lambda_k-ml^4\omega_c^2}(a_{k,s}^{(1)}\sin\tau+a_{k,c}^{(1)}\cos\tau)+\right.$$
$$+\varepsilon\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{\int_0^1\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi_k(\zeta)\,d\zeta}{\lambda_k-ml^4(2i+1)^2\omega_c^2}\frac{d^2\varphi_k(\zeta)}{d\zeta^2}\times$$
$$\left.\times[B_{2i+1}^s\sin(2i+1)\tau+B_{2i+1}^c\cos(2i+1)\tau]\right\}z; \tag{12.10}$$

$$\xi_0=[\xi]_{\tau=0}=\left\{a\frac{d^2\varphi(\zeta)}{d\zeta^2}+\varepsilon\sum_{k=2}^{\infty}\frac{a_{k,c}^{(1)}}{\lambda_k-ml^4\omega_c^2}\frac{d^2\varphi(\zeta)}{d\zeta^2}+\right.$$
$$\left.+\varepsilon\sum_{i=1}^{\infty}\sum_{i=k}^{\infty}\frac{\int_0^1\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi_k(\zeta)\,d\zeta}{\lambda_k-m(2i+1)^2l^4\omega_c^2}B_{2i+1}^c\frac{d^2\varphi_k(\zeta)}{d\zeta^2}\right\}z. \tag{12.11}$$

On the basis of (12.10), (12.11), and (10.14) expressions for the bending moments in a section of the bar, can be presented in the following form:

$$\vec{M}_{II}=\int_{-\frac{h}{2}}^{+\frac{h}{2}}E\left\{a\frac{d^2\varphi(\zeta)}{d\zeta^2}\cos\tau+\varepsilon\sum_{k=2}^{\infty}\frac{\frac{d^2\varphi(\zeta)}{d\zeta^2}}{\lambda_k-ml^4\omega_c^2}(a_{k,s}^{(1)}\sin\tau+a_{k,c}^{(1)}\cos\tau)+\right.$$
$$+\varepsilon\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{\int_0^1\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi_k(\zeta)\,d\zeta}{\lambda_k-ml^4(2i+1)^2\omega_c^2}\frac{d^2\varphi_k(\zeta)}{d\zeta^2}[B_{2i+1}^s\sin(2i+1)\tau+ \tag{12.12}$$

$$+ B^{c}_{2i+1}\cos(2i+1)\Big] z - \frac{\nu}{n}\Big\{ a\frac{d^2\phi(\zeta)}{d\zeta^2} + \epsilon\sum_{k=2}^{\infty}\frac{a^{(1)}_{k,c}\frac{d^2\phi_k}{d\zeta^2}}{\lambda - ml^4\omega_c^2} +$$

$$+ \epsilon\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{\int_0^1 \frac{d^2}{d\zeta^2}\Big[\Big(\frac{d^2\phi(\zeta)}{d\zeta^2}\Big)^n\Big]\phi_k(\zeta)d\zeta}{\lambda_k - m(2i+1)^2 l^4\omega_c^2} B^{c}_{2i+1}\frac{d^2\phi_k(\zeta)}{d\zeta^2} + a\frac{d^2\phi_k(\zeta)}{d\zeta^2}\cos\tau +$$

$$+ \epsilon\sum_{k=2}^{\infty}\frac{\frac{d^2\phi_k(\zeta)}{d\zeta^2}}{\lambda_k - ml^4\omega_c^2}\Big(a^{(1)}_{k,s}\sin\tau + a^{(1)}_{k,c}\cos\tau\Big) +$$

$$+ \epsilon\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{\int_0^1 \frac{d^2}{d\zeta^2}\Big[\Big(\frac{d^2\phi(\zeta)}{d\zeta^2}\Big)^n\Big]\phi_k(\zeta)d\zeta}{\lambda_k - ml^4(2i+1)^2\omega_c^2}\frac{d^2\phi_k(\zeta)}{d\zeta^2}\Big[B^{s}_{2i+1}\sin(2i+1)\tau +$$

$$+ B^{c}_{2i+1}\cos(2i+1)\tau\Big]\Big\}^n z^n + 2^{n-1}\frac{\nu}{n}\Big[a\frac{d^2\phi(\zeta)}{d\zeta^2} + \epsilon\sum_{k=2}^{\infty}\frac{\frac{d^2\phi(\zeta)}{d\zeta^2}a^{(1)}_{k,c}}{\lambda_k - ml^4\omega_c^2} +$$

$$+ \epsilon\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{\int_0^1 \frac{d^2}{d\zeta^2}\Big[\Big(\frac{d^2\phi(\zeta)}{d\zeta^2}\Big)^n\Big]\phi_k(\zeta)d\zeta}{\lambda_k - m(2i+1)^2 l^4\omega_c^2} B^{c}_{2i+1}\frac{d^2\phi_k(\zeta)}{d\zeta^2}\Big]^n z^n\Big\} bz\,dz; \qquad (12.12)$$

$$\overleftarrow{M}_{II} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} E\Big\{ a\frac{d^2\phi(\zeta)}{d\zeta^2}\cos\tau + \epsilon\sum_{k=2}^{\infty}\frac{\frac{d^2\phi(\zeta)}{d\zeta^2}}{\lambda_k - ml^4\omega_c^2}\Big(a^{(1)}_{k,s}\sin\tau + a^{(1)}_{k,c}\cos\tau\Big) +$$

$$+ \epsilon\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{\int_0^1 \frac{d^2}{d\zeta^2}\Big[\Big(\frac{d^2\phi(\zeta)}{d\zeta^2}\Big)^n\Big]\phi_k(\zeta)d\zeta}{\lambda_k - m(2i+1)^2 l^4\omega_c^2}\frac{d^2\phi(\zeta)}{d\zeta^2}\Big[B^{s}_{2i+1}\sin(2i+1)\tau +$$

(12.13) cont.

$$+ B^{c}_{2i+1}\cos(2i+1)\tau]z + \frac{\nu}{n}\left\{a\frac{d^2\varphi(\zeta)}{d\zeta^2} + \varepsilon\sum_{k=2}^{\infty}\frac{a^{(1)}_{k,c}\frac{d^2\varphi_k(\zeta)}{d\zeta^2}}{\lambda_k - ml^4\omega_c^2} + \right.$$

$$+ \varepsilon\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{\int_0^1 \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]}{\lambda_k - m(2i+1)^2 l^4\omega_c^2} B^{c}_{2i+1}\frac{d^2\varphi_k(\zeta)}{d\zeta^2} - a\frac{d^2\varphi(\zeta)}{d\zeta^2}\cos\tau -$$

$$- \varepsilon\sum_{k=2}^{\infty}\frac{\frac{d^2\varphi(\zeta)}{d\zeta^2}}{\lambda_k - ml^4\omega_c^2}(a^{(1)}_{k,s}\sin\tau + a^{(1)}_{k,c}\cos\tau) -$$

$$- \varepsilon\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{\int_0^1 \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi_k(\zeta)\,d\zeta}{\lambda_k - m(2i+1)^2 l^4\omega_c^2}\ \frac{d^2\varphi(\zeta)}{d\zeta^2}\,[B^{s}_{2i+1}\sin(2i+1)\tau +$$

$$+ B^{c}_{2i+1}\cos(2i+1)\tau]\Big\}^n z^n - 2^{n-1}\frac{\nu}{n}\left[a\frac{d^2\varphi(\zeta)}{d\zeta^2} + \varepsilon\sum_{k=2}^{\infty}\frac{\frac{d^2\varphi(\zeta)}{d\zeta^2}a^{(1)}_{k,c}}{\lambda_k - ml^4\omega_c^2} + \right.$$

$$\left. + \varepsilon\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{\int_0^1 \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\right]\varphi_k(\zeta)\,d\zeta}{\lambda_k - ml^4(2i+1)^2\omega_c^2}\ B^{c}_{2i+1}\frac{d^2\varphi_k(\zeta)}{d\zeta^2}\right]^n z^n\Big\}\, bz\,dz. \qquad (12.13)$$

In this way, using formulas (12.9) and substituting in them the value of the moments from (10.4), (12.12) and (12.13) it is possible to determine the values of functional $\Psi(\zeta,\tau)$ for the upward and downward motions.

Now we can go on to the determination of $\varepsilon^2\Delta_2$ and $\varepsilon\psi_1$ from equations (12.4) and (12.6). Solving equation (12.4) for Δ_2 and multiplying both sides of the equation by the square of the small parameter, we obtain

$$\varepsilon^2\Delta_2 = \left[a\int_0^{2\pi} m\int_0^1 \varphi^2(\zeta)\cos^2\tau\,d\zeta\,d\tau\right]^{-1}\left\{\int_0^{2\pi}\int_0^1 m(\varepsilon\Delta_1)\varepsilon\frac{\partial^2 u_1}{\partial\tau^2}\varphi(\zeta)\cos\tau\,d\zeta\,d\tau - \right.$$

$$- \int_0^{2\pi}\int_0^1 m\theta_0\zeta(\varepsilon\Delta_1)\cos(\tau-\psi_0)\varphi(\zeta)\cos\tau\,d\zeta\,d\tau -$$

$$-\int_0^{2\pi}\int_0^{1} m\theta_0\zeta\omega_c^2(\varepsilon\psi_1)\sin(\tau-\psi_0)\varphi(\zeta)\cos\tau\, d\zeta\, d\tau +$$

$$+\int_0^{\pi}\frac{\varepsilon^2}{l^3}\int_0^1 \overleftarrow{\Psi}(\zeta,\tau)\frac{d^2\varphi(\zeta)}{d\zeta^2}\cos\tau\, d\zeta\, d\tau+\int_{\pi}^{2\pi}\frac{\varepsilon^2}{l^3}\int_0^1\vec{\Psi}(\zeta,\tau)\frac{d^2\varphi(\zeta)}{d\zeta^2}\cos\tau\, d\zeta\, d\tau\Big\}.$$

After some transformations, this formula can be represented in the form:

$$\varepsilon^2\Delta_2=\frac{4}{am\pi}\Big\{\int_0^{2\pi}\int_0^1 m(\varepsilon\Delta_1)\varepsilon\frac{\partial^2 u_1}{\partial\tau^2}\frac{d^2\varphi(\zeta)}{d\zeta^2}\cos\tau\, d\zeta\, d\tau -$$

$$-\int_0^1 m\theta_0\pi\cos\psi_0(\varepsilon\Delta_1)\zeta\varphi(\zeta)\,d\zeta+\int_0^1 m\theta_0\omega_c^2\zeta\sin\psi_0(\varepsilon\psi_1)\varphi(\zeta)\,d\zeta+$$

$$+\Big[\varepsilon^2\int_0^{\pi}\overleftarrow{\Psi}(\zeta,\tau)\frac{d^2\varphi}{d\zeta^2}\cos\tau\, d\zeta\, d\tau+\varepsilon^2\int_{\pi}^{2\pi}\vec{\Psi}(\zeta,\tau)\frac{d^2\varphi(\zeta)}{d\zeta^2}\cos\tau\, d\zeta\, d\tau\Big]\frac{1}{l^3}\Big\}. \qquad (12.14)$$

In the last formula only the quantity $\varepsilon\psi_1$, is unknown; it can be determined from equation (12.6) and is expressed by the formula:

$$\varepsilon\psi_1=\Big[\int_0^{2\pi}\int_0^1 m\theta_0\omega_c^2\zeta\sin(\tau-\psi_0)\varphi(\zeta)\sin\tau\, d\zeta\, d\tau\Big]^{-1}\times$$

$$\times\Big\{\int_0^{2\pi}\int_0^1 m(\varepsilon\Delta_1)\varepsilon\frac{\partial^2 u_1}{\partial\zeta^2}\varphi(\zeta)\sin\tau\, d\zeta\, d\tau -$$

$$-\int_0^{2\pi}\int_0^1 m(\varepsilon^2\Delta_2)\varphi^2(\zeta)a\cos\tau\sin\tau\, d\zeta\, d\tau -$$

$$-\int_0^{2\pi}\int_0^1 m\theta_0\zeta(\varepsilon\Delta_1)\cos(\tau-\psi_0)\varphi(\zeta)\sin\tau\, d\zeta\, d\tau +$$

$$+\Big[\int_0^{\pi}\varepsilon^2\int_0^1\overleftarrow{\Psi}(\zeta,\tau)\frac{d^2\varphi(\zeta)}{d\zeta^2}\sin\tau\, d\zeta\, d\tau+\int_0^{2\pi}\varepsilon^2\int_0^1\vec{\Psi}(\zeta,\tau)\frac{d^2\varphi(\zeta)}{d\zeta^2}\sin\tau\, d\zeta\, d\tau\Big]\frac{1}{l^3}\Big\}.$$

After simplification of the right side of the last expression, we obtain

$$\varepsilon\psi_1 = \left[\int_0^1 m\theta_0\omega_c^2\pi\zeta\cos\psi_0\varphi(\zeta)\,d\zeta\right]^{-1}\times$$
$$\times\left\{\int_0^{2\pi}\int_0^1 m(\varepsilon\Delta_1)\,\varepsilon\frac{\partial^2 u_1}{d\zeta^2}\varphi(\zeta)\sin\tau\,d\zeta\,d\tau -\right.$$
$$-\int_0^1 m\theta_0\sin\psi_0(\varepsilon\Delta_1)\pi\zeta\varphi(\zeta)\,d\zeta + \left[\int_0^{\pi}\int_0^1 \vec{\Psi}(\zeta,\tau)\frac{d^2\varphi(\zeta)}{d\zeta^2}\sin\tau\,d\zeta\,d\tau +\right.$$
$$\left.\left. + \int_0^{2\pi}\varepsilon^2\int_0^1 \vec{\Psi}(\zeta,\tau)\frac{d^2\varphi(\zeta)}{d\zeta^2}\sin\tau\,d\zeta\,d\tau\right]\frac{1}{l^3}\right\}. \tag{12.15}$$

In this way, having determined $\varepsilon\psi_1$ by formula (12.15) and having substituted its value in (12.14), we obtain the quantity $\varepsilon^2\Delta_2$. Then, the square of frequency in the second approximation will equal

$$\omega^2 = \omega_c^2 + \varepsilon\Delta_1 + \varepsilon^2\Delta_2 \tag{12.16}$$

and the phase shift,

$$\psi_{II} = \psi_0 + \varepsilon\psi_1. \tag{12.17}$$

The expressions which have been obtained for the bending moments (12.12), (12.13), and, consequently, also formulas (12.14) and (12.15) for the determination of $\varepsilon^2\Delta_2$ and $\varepsilon\psi_1$ are quite complicated. However, when the exponent n is an even integer, these formulas are considerably simplified.

From formulas (12.12) and (12.13) it follows that, for even n,

$$\vec{M}_{II} = \overleftarrow{M}_{II} = \frac{EI}{l}\frac{\partial^2 u(\zeta,\tau)}{\partial\zeta^2} = \frac{EI}{l}\left[\frac{d^2\varphi(\zeta)}{d\zeta^2}a\cos\tau + \varepsilon\frac{\partial^2 u_1}{\partial\zeta^2}\right]. \tag{12.18}$$

Comparing (10.18) with (12.7) we conclude that in this case

$$\varepsilon^2 \vec{\Psi}(\zeta, \tau) = -\varepsilon \vec{\Phi}\left(\frac{d^2\varphi(\zeta)}{d\zeta^2} a \cos\tau\right) =$$
$$= \frac{3EIa^n h^{n-1}\nu}{2^n l^n n(n+2)}\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n [2(1+\cos\tau)^n - 2^n];$$

(12.19)

$$\varepsilon \overleftarrow{\Psi}(\zeta, \tau) = -\varepsilon \overleftarrow{\Phi}\left(\frac{d^2\varphi(\zeta)}{d\zeta^2} a \cos\tau\right) =$$
$$= -\frac{3EIa^n h^{n-1}\nu}{2^n l^n n(n+2)}\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n [2(1-\cos\tau)^n - 2^n].$$

(12.20)

Equation (12.14) can be represented in the form:

$$\Delta_2 = \frac{4}{am\pi}\oint\int_0^1 \left\{ml^3\Delta_1 \frac{\partial^2 u_1}{\partial\zeta^2} - ml^3\Delta_1\zeta\beta\cos(\tau-\psi_0) - \right.$$
$$\left. - ml^3\zeta\omega_\zeta^2\psi_1 \sin(\tau-\psi_0) + \frac{\partial^2}{\partial\zeta^2}[\Psi(\zeta, \tau)]\right\}\varphi(\zeta)\cos\tau\, d\tau\, d\zeta.$$

(12.21)

Here

$$\oint\int_0^1 \frac{\partial^2}{\partial\zeta^2}[\Psi(\zeta, \tau)]\varphi(\zeta)\, d\zeta\, d\tau = \int_\pi^{2\pi}\frac{\partial^2}{\partial\zeta^2}[\vec{\Psi}(\zeta, \tau)]\varphi(\zeta)\cos\tau\, d\zeta\, d\tau +$$
$$+ \int_0^\pi\int_0^1 \frac{\partial^2}{\partial\zeta^2}[\overleftarrow{\Psi}(\zeta, \tau)]\varphi(\zeta)\cos\tau\, d\zeta\, d\tau.$$

Integrating by parts twice and taking account of the boundary conditions, we obtain

$$\varepsilon^2\oint\int_0^1 \frac{\partial^2}{\partial\zeta^2}[\Psi(\zeta, \tau)]\varphi(\zeta)\cos\tau\, d\zeta\, d\tau =$$
$$= \int_\pi^{2\pi}\frac{3EIa^n h^{n-1}\nu}{2^n l^n n(n+2)}\left[\frac{d^2\varphi(\zeta)}{d\zeta^2}\right]^{n+1}[2(1+\cos\tau)^n - 2^n]\cos\tau\, d\zeta\, d\tau -$$
$$- \int_0^\pi\int_0^1 \frac{3EIa^n h^{n-1}\nu}{2^n l^n n(n+2)}\left[\frac{d^2\varphi(\zeta)}{d\zeta^2}\right]^{n+1}[2(1-\cos\tau)^n - 2^n]\cos\tau\, d\zeta\, d\tau.$$

(12.22)

After making the change of variable $\tau = \tau_1 + \pi$ in the first integral of the right side of equation (12.22), we have

$$\varepsilon^2 \oint\int_0^1 \frac{\partial^2}{\partial\zeta^2}[\Psi(\zeta,\tau)]\,\varphi(\zeta)\cos\tau\,d\zeta\,d\tau =$$

$$= -\frac{12EIa^n h^{n-1}\nu}{2^n l^n n(n+2)}\int_0^{\pi}(1-\cos\tau)^n\cos\tau\,d\tau\int_0^1\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^{n+1}d\zeta. \tag{12.23}$$

Substituting the values of the individual integrals into equation (12.21) we obtain

$$\varepsilon^2\Delta_2 = \frac{4}{\pi m a}\left\{\frac{-\pi m\theta_0(\varepsilon\Delta_1)\cos\psi_0}{k^2} + \frac{\pi m\theta_0\omega_c^2(\varepsilon\psi_0)\sin\psi_0}{k^2} - \right.$$
$$\left. -\frac{12EIa^n h^{n-1}\nu}{2^n n l^n(n+2)}\int_0^{\pi}(1-\cos\tau)^n\cos\tau\,d\tau\int_0^1\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^{n+1}d\zeta\right\}. \tag{12.24}$$

From (12.15) we find

$$\varepsilon\psi_1 = \frac{k^2}{m\theta\omega_c^2\pi\cos\psi_0}\left\{-\frac{m\theta_0(\varepsilon\Delta_1)\pi}{k^2}\sin\psi_0 + \right.$$
$$\left. +\varepsilon^2\oint\int_0^1\frac{\partial^2}{\partial\zeta^2}[\Psi(\zeta,\tau)]\,\varphi(\zeta)\sin\tau\,d\zeta\,d\tau\right\}. \tag{12.25}$$

We compute the integral entering into the last equation

$$\varepsilon^2\oint\int_0^1\frac{\partial^2}{\partial\zeta^2}[\Psi(\zeta,\tau)]\,\varphi(\zeta)\sin\tau\,d\zeta\,d\tau =$$

$$= \frac{3EIa^n h^{n-1}\nu}{2^n n(n+2)l^n}\left\{\int_{\pi}^{2\pi}\int_0^1\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^{n+1}[2(1+\cos\tau)^n - 2^n]\sin\tau\,d\zeta\,d\tau - \right.$$

$$\left. -\int_0^{\pi}\int_0^1\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^{n+1}[2(1-\cos\tau)^n - 2^n]\sin\tau\,d\zeta\,d\tau\right\} =$$

$$= \frac{12EIa^n h^{n-1}\nu}{2^n n(n+2)l^n}\left[2^n - \frac{2^{n+1}}{n+1}\right]\int_0^1\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^{n+1}d\zeta =$$

$$= \frac{12EIa^n h^{n-1}\nu(n-1)}{n(n+1)(n+2)l^n}\int_0^1\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^{n+1}d\zeta.$$

Then

$$\varepsilon\psi_1 = \frac{k^2}{m\theta_0\omega_c^2\pi\cos\psi_0}\left\{-\frac{m\theta_0(\varepsilon\Delta_1)\pi}{k^2}\sin\psi_0 + \right.$$

$$\left. + \frac{12EIa^n h^{n-1}\nu(n-1)}{n(n+1)(n+2)l^n}\int_0^1\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^{n+1}d\zeta\right\}.$$

(12.26)

From equation (12.26) it follows that

$$\frac{\pi(\varepsilon\psi_1)\omega_c^2 m\theta_0}{k^2}\sin\psi_0 =$$

$$= \operatorname{tg}\psi_0\left\{\frac{12EIa^n h^{n-1}(n-1)\nu}{n(n+1)(n+2)l^{n+3}}\int_0^1\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^{n+1}d\zeta - \frac{m\theta_0(\varepsilon\Delta_1)\pi}{k^2}\sin\psi_0\right\}.$$

(12.27)

Substituting the last equation into (12.21) we obtain

$$\varepsilon^2\Delta_2 = \frac{4}{ma\pi}\left\{-\frac{\pi m\theta_0\cos\psi_0}{k^2}(\varepsilon\Delta_1) - \frac{\pi m\theta_0\sin^2\psi_0}{k^2\cos\psi_0}(\varepsilon\Delta_1) - \right.$$

$$- \frac{12EIa^n h^{n-1}\nu}{2^n n(n+2)l^{n+3}}\int_0^1(1-\cos\tau)^n\cos\tau\,d\tau\int_0^1\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^{n+1}d\zeta +$$

$$+ \frac{12EIa^n h^{n-1}\nu(n-1)}{n(n+1)(n+2)l^{n+3}}\int_0^1\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^{n+1}d\zeta = \frac{4}{ma\pi}\left\{-\frac{\pi m\theta_0}{k^2\cos\psi_0}(\varepsilon\Delta_1) + \right.$$

$$+ \frac{12EIa^n h^{n+1}\nu(n-1)\operatorname{tg}\psi_0}{n(n+1)(n+2)l^{n+3}}\int_0^1\left(\frac{d^2\varphi}{d\zeta^2}\right)^{n+1}d\zeta - \frac{12EIa^n h^{n-1}\nu}{2^n n(n+2)l^{n+3}}\times$$

$$\left.\times\int_0^1\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^{n+1}d\zeta\int_0^\pi(1-\cos\tau)^n\cos\tau\,d\tau\right\}.$$

(12.28)

After substituting in the last equation the value of $(\varepsilon\Delta_1)$ we have

$$\varepsilon^2\Delta_2 = \frac{4EI\theta_0}{\pi mal^4}\left\{\frac{\pi(n+1)\sin\psi_0}{(n-1)2^n}\int_0^\pi(1-\cos\tau)\cos\tau\,d\tau\left(k^2 + \frac{4\theta_0}{a\cos\psi_0}\right) + \right.$$

$$\left. + \frac{4\pi\theta_0}{a} - \frac{k^2\sin^2\psi\pi}{\cos\psi_0}\right\}.$$

(12.29)

On the basis of (12.16) we find

$$\frac{\omega^2}{\omega_C^2} = 1 + \frac{\epsilon \Delta_1}{\omega_C^2} + \frac{\epsilon^2 \Delta_2}{\omega_C^2} \qquad (12.30)$$

After substitution of the values of $\epsilon \Delta_1$ and $\epsilon^2 \Delta_2$ into (12.30) we will obtain a formula for the calculation of frequency of vibrations in the second approximation

$$\frac{\omega^2}{\omega^2} = 1 - \frac{4\theta_0}{2k^2}\cos\psi_0 - \frac{4\theta_0 \sin^2\psi_0}{\pi a k^2 (n+1)\cos\psi_0} +$$

$$+ \frac{16\theta_0^2}{a^2 k^4}\left[\frac{(n-1)}{2^n (n-1)} \operatorname{tg}\psi_0 \int_0^{\pi} (1-\cos\tau)^n \cos\tau \, d\tau + 1\right] \cdot \qquad (12.31)$$

The magnitude of the phase shift in the second approximation, is determined by formula (12.12) on the basis of (12.21) and (12.26).

By analyzing the physical nature of the problem, it is easy to convince oneself that the calculation of the frequency of vibrations and the phase shift in the first approximation is dependent on the consideration of the area of the hysteresis loop; however, the calculation of these quantities in the second approximation is based on the consideration of the form of the hysteresis loop. It is evident that the form of the hysteresis loop will not effect materially the amount of damping, even when the form of the loop differs from the actual one. The only important condition is that the area of the assumed loop be equal to the area of the true loop. When solving real problems of vibrations in the material by the proposed method, which is based on the theory of nonlinear mechanics, there is no need to resort to second approximation because, to solve technical problems, the first approximation, as will be shown later, gives a precision which is entirely adequate.

13. Construction of the resonance curve

In conclusion of the present chapter, we will give a sample calculation of forced transverse vibrations of a rod of constant cross-section and will show the application of the formulas obtained for determination of frequency in the first approximation,

$$\left(\frac{\omega_1}{\omega_c}\right) = 1 - \frac{4\theta_0(\cos k + \operatorname{ch} k)}{ak^2 \sin k \operatorname{sh} k}\left[\frac{(n+1)\sin\psi_0}{2^n(n-1)} \times \right.$$

$$\left. \times \int_0^{\pi} (1-\cos\tau)^n \cos\tau\, d\tau + \cos\psi_0\right]. \tag{13.1}$$

We take the following basic data:

1) The material of the rod is St. 20, for which the following parameters were found by experiment:

$$n = 2, \quad \nu = 18{,}6.$$

2) We take the amplitude of the forced angle of rotation of the fixed section of the rod, as

$$\theta_0 = 0{,}0001.$$

3) Dimensions of the rod:

$$l = 40{,}5 \text{ см},$$
$$h = 1{,}5 \text{ см},$$
$$b = 3{,}0 \text{ см}.$$

Applying these data, we find

$$\int_0^{\pi} (1-\cos\tau)^n \cos\tau\, d\tau = \int_0^{\pi} (1-\cos\tau)^2 \cos\tau\, d\tau = -3{,}141593;$$

$$k = 1{,}875. \tag{13.2}$$

After substitution of the known quantities into formula (13.1) we obtain

$$\left(\frac{\omega_1}{\omega_c}\right)^2 = 1 + \frac{1{,}13778\cdot 10^{-4}}{a}(2{,}356195\sin\psi_0 - \cos\psi_0). \tag{13.3}$$

In the construction of the resonance curve $a = f\left(\frac{\omega_I}{\omega_c}\right)$ in addition to formula (13.3), we use formula (10.21) to determine the value of the phase shift

$$\sin\psi_0 = -\frac{12(n-1)a^n h^{n-1}\nu\sin k\,\mathrm{sh}\,k}{n(n+1)(n+2)\theta_0 k^2 l^{n-1}\pi(\cos k + \mathrm{ch}\,k)}\int_0^1\left(\frac{d^2\varphi}{d\zeta^2}\right)^{n+1}d\zeta. \tag{13.4}$$

Further, we calculate the value of integral entering in the right side of the last expression

$$\int_0^1\left(\frac{d^2\varphi}{d\zeta^2}\right)^{n+1}d\zeta = \int_0^1\left\{\frac{k^2}{2\sin k\,\mathrm{sh}\,k}[(\cos k+\mathrm{ch}\,k)(\mathrm{ch}\,k\zeta+\cos k\zeta)+\right.$$

$$\left.+(\sin k-\mathrm{sh}\,k)(\mathrm{sh}\,k\zeta+\sin k\zeta)]\right\}^3 d\zeta = 47{,}3738$$

Substituting into formula (10.2) the known quantities, we obtain

$$\sin\psi_0 = -1{,}15511\cdot 10^4 a^n \tag{13.5}$$

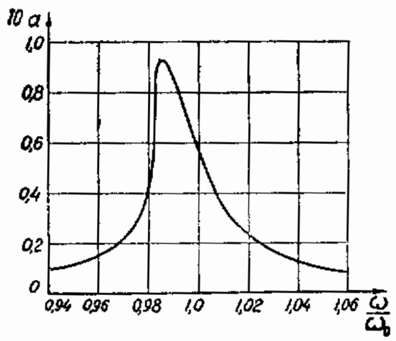

FIG. 6

Using formulas (13.3) and (13.5) and taking $n=2$, we compile a table of values of amplitude of vibrations as a function of the ratio of frequencies $\frac{\omega}{\omega_c}$, which is required for the construction of the resonance curve $a=f\left(\frac{\omega}{\omega_c}\right)$.

The results of calculations are given in Table 2; here the indices π and II on the ratio of frequencies, as in the preceeding chapter, indicate, respectively, the left and right branches of the resonance curve.

The resonance curve of Fig. 6 is plotted in accordance with the data of Table 3.

Then we determine the correction to the value of the square frequency, by solving the problem in the second approximation.

Table 3

a	$\left(\frac{\omega}{\omega_c}\right)^2_л$	$\left(\frac{\omega}{\omega_c}\right)_л$	$\left(\frac{\omega}{\omega_c}\right)^2_п$	$\left(\frac{\omega}{\omega_c}\right)_п$
0,001	0,883133	0,9397	1,110673	1,0539
0,002	0,936978	0,9680	1,050635	1,0250
0,003	0,952990	0,9762	1,028430	1,0141
0,004	0,959660	0,9796	1,015567	1,0077
0,005	0,962729	0,9812	1,006303	1,0032
0,006	0,964174	0,9819	0,9986658	0,9993
0,007	0,964924	0,9823	0,991723	0,9959
0,008	0,965650	0,9827	0,984805	0,9924
0,009	0,967668	0,9837	0,976592	0,9882
0,010	—	—	—	—

Let us examine the ratio of this correction to the square of the natural frequency, which is determined by formula

$$\frac{\varepsilon^2 \Delta_2}{\omega_c^2} = \frac{4\theta_0}{k^4 a}\left\{\frac{(n+1)\sin\psi_0}{2^n(n-1)}\int_0^{\pi}(1-\cos\tau)^n\cos\tau\, d\tau\left[k^2+\frac{4\theta_0}{a\cos\psi_0}\right]+\right.$$
$$\left.+\frac{4\theta_0}{a}-\frac{k^2\sin^2\psi_0}{\cos\psi_0}\right\}. \qquad (13.6)$$

Substituting in this formula the values taken previously

$$\theta_0 = 10^{-4};\ n=2;\ l=40{,}5\ см;\ \ k=1{,}875,$$

and having also the computed value

$$\sin\psi_0 = -1{,}15511\cdot 10^4 a^n,$$

we find the various values of $\frac{\varepsilon \Delta_2}{\omega_c^2}$ and the corresponding ratios of the square of the frequency in the second approximation to the magnitude of the natural frequency as a function of the amplitude (Table 3a).

Table 3a

a	$\left(\frac{\varepsilon^2 \Delta_2}{\omega_c^2}\right)_л$	$\left(\frac{\omega_{II}}{\omega_c}\right)^2_л$	$\left(\frac{\varepsilon^2 \Delta_2}{\omega_c^2}\right)_п$	$\left(\frac{\omega_{II}}{\omega_c}\right)^2_п$
0,001	0,016379	0,899512	0,015705	1,126378
0,002	0,009661	0,946639	0,009198	1,059833
0,003	0,010670	0,963660	0,010786	1,039216
0,004	0,012566	0,972226	0,013826	1,029393
0,005	0,014387	0,977116	0,017149	1,023452
0,006	0,015721	0,979895	0,0166561	1,015227
0,007	0,016051	0,980957	0,027828	1,019551
0,008	0,013957	0,979607	0,035994	1,020799
0,009	0,002330	0,965338	0,058389	1,034981
0,010	—	—	—	—

Table 4

a	$\left(\frac{\omega_{II}}{\omega_c}\right)_л$	$\left(\frac{\omega_{II}-\omega_I}{\omega_I}\right)_л \cdot 100\%$	$\left(\frac{\omega_{II}}{\omega_c}\right)_п$	$\left(\frac{\omega_{II}-\omega_I}{\omega_I}\right)_п \cdot 100\%$
0,001	0,9485	0,94	1,0613	0,70
0,002	0,9729	0,51	1,0295	0,44
0,003	0,9816	0,55	1,0194	0,52
0,004	0,9860	0,65	1,0146	0,68
0,005	0,9885	0,74	1,0117	0,85
0,006	0,9899	0,81	1,0076	0,83
0,007	0,9905	0,83	1,0097	1,38
0,008	0,9897	0,71	1,0104	1,81
0,009	0,9825	0,58	1,0173	2,94
0,010	—	—	—	—

Values of $\frac{\omega_{II}}{\omega_c}$, obtained for different amplitudes and also the values of corrections, found in the determination of freouency in the second approximation, are given in Table 4. For comparison with results of calculations in the first approximation, the values of

$\frac{\omega_{II} - \omega_I}{\omega_I} \cdot 100\,\%$ are also given. The table shows that the greatest increase in accuracy of determination of the magnitude of the actual frequency of vibrations (at a certain amplitude) in the second approximation, consists of about 3% of the frequency obtained in the first approximation. This confirms the supposition that the dissipation of energy in the material is determined by the area of the hysteresis loop and is accounted for by the solution in the first approximation. The shape of the hysteresis loop which is considered in the second approximation, affects but slightly. the magnitude of the dissipation. Therefore, when constructing the resonance curve to a sufficiently high degree of accuracv, one can confine oneself to the first approximation.

Chapter III

Transverse Vibrations of a Bar with Variable Cross-Section

14. Derivation of the basic equations

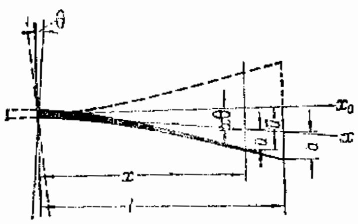

Fig. 7

In the present chapter, the previously developed theory of analysis of vibrations of a rod of constant cross-section, taking account of damping in the material is applied to a rod of variable cross-section. All the basic premises regarding the character of excitation of vibrations and conditions at the ends of the rod remain the same as in the case of vibrations of rod of constant cross-section. Because the problem was worked out in general form from the start in the last chapter, differential equation (8.7) which was obtained is equally valid in the case of variable cross-section (Fig.7). Keeping in mind that for a rod of variable cross-section

$$I = I(x),\ m = \frac{\gamma F(x)}{g},$$

we rewrite equation (8.7) in the following form:

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 u}{\partial x^2}\right] + \varepsilon\frac{\partial^2}{\partial x^2}\left[\Phi\left(\frac{\partial^2 u}{\partial x^2}\right)\right] + \frac{\gamma}{g}F(x)\left(x\frac{\partial^2\theta}{\partial t^2} + \frac{\partial^2 u}{\partial t^2}\right) = 0. \tag{14.1}$$

Taking, as in the case of vibrations of a bar of constant cross-section, a harmonic variation of the forced motion of the angle of rotation $\theta = \varepsilon\beta\cos\omega t = \theta_0 \cos\omega t$ and designating the density of the material by ρ, we rewrite equation (14.1) in dimensionless coordinates:

$$\frac{\partial^2}{\partial\zeta^2}\left[l^4 EI(\zeta)\frac{\partial^2 u^*}{\partial\zeta^2}\right] + \varrho l^6 F(\zeta)\frac{\partial^2 u^*}{\partial t^2} - \\ - \theta_0 \varrho l^6 F(\zeta)\,\omega^2\zeta\cos\omega t + \varepsilon\frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{\partial^2 u^*}{\partial\zeta^2}\right)\right] = 0. \tag{14.2}$$

Since this differential equation includes a small parameter, it is natural to look for the magnitude of deflection $u^*(\zeta, t)$, the frequency of vibrations ω and the phase shift ψ in the form of an expansion in powers of this small parameter:

$$u^*(\zeta, t) = \varphi(\zeta)\,a\cos(\omega t + \psi) + \varepsilon u_1(\zeta, t) + \varepsilon^2 u_2(\zeta, t) + \dots; \tag{14.3}$$

$$\omega^2 = \omega_c^2 + \varepsilon\Delta_1 + \varepsilon^2\Delta_2 + \dots; \tag{14.4}$$

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \dots. \tag{14.5}$$

Replacing $(\omega t + \psi)$ by a new variable τ and expanding $\cos(\tau - \psi)$ in a series, then, substituting the expressions (14.3), (14.4), and (14.5) into (14.2), we group the terms of the expression obtained which contain the same power of the small parameter and equate them to zero. As a result, we will have the following basic system of differential equations:

$$\frac{\partial^2}{\partial\zeta^2}\left[l^4 EI(\zeta)\frac{\partial^2\varphi(\zeta)}{\partial\zeta^2}\right]a\cos\tau - \varrho l^6 F(\zeta)\,\varphi(\zeta)\,\omega_c^2\,a\cos\tau = 0; \tag{14.6}$$

$$\frac{\partial^2}{\partial\zeta^2}\left[l^4 EI(\zeta)\frac{\partial^2 u_1}{\partial\zeta^2}\right]+\varrho l^6 F(\zeta)\omega_c^2\frac{\partial^2 u_1}{\partial\tau^2}-\varrho l^6 F(\zeta)\Delta_1 a\varphi(\zeta)\cos\tau -$$

$$-\varrho F(\zeta) l^6\zeta\omega_c^2\beta\cos(\tau-\psi_0)+\frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)a\cos\tau\right]l=0; \qquad (14.7)$$

$$\frac{\partial^2}{\partial\zeta^2}\left[l^4 EI(\zeta)\frac{\partial^2 u_2}{\partial\zeta^2}\right]+\varrho l^6 F(\zeta)\omega_c^2\frac{\partial^2 u_2}{\partial\tau^2}+\varrho l^6 F(\zeta)\Delta_1\frac{\partial^2 u_1}{\partial\tau^2} -$$

$$-\varrho l^6 F(\zeta)\Delta_2\varphi(\zeta)a\cos\tau-\varrho l^6 F(\zeta)\beta\Delta_1\zeta\cos(\tau-\psi_0) -$$

$$-\varrho l^6 F(\zeta)\beta\omega_c^2\zeta\psi_1\sin(\tau-\psi_0)+l\frac{\partial^2}{\partial\zeta^2}[\Psi(\zeta,\tau)]=0; \qquad (14.8)$$

15. Solution of the problem in the zeroth approximation (free vibrations)

Let us rewrite (14.6) after cancelling $a\cos\tau$

$$\frac{\partial^2}{\partial\zeta^2}\left[l^4 EI(\zeta)\frac{d^2\varphi(\zeta)}{d\zeta^2}\right]-\varrho l^6\omega_c^2 F(\zeta)\varphi(\zeta)=0. \qquad (15.1')$$

For the solution of this differential equation, which represents the equation of free vibrations of the rod of variable cross-section, we shall use an approximate method, based on the theory of perturbations. If for the given region and the given boundary conditions, the eigenvalues λ_n are known; and the corresponding normalized and orthogonal eigenfunctions u_n of the linear self-adjoint* differential equation

$$L\left[u_n\right] + \lambda_n u_n = 0,$$

then, using a method based on the theory of perturbation, it is possible to calculate the eigenvalues and the fundamental functions of the eigenvalue problem of a "nearby" or "perturbed" differential equation

$$L\left[\bar{u}_n\right] - \epsilon r \bar{u}_n + \bar{\lambda}_n \bar{u}_n = 0.$$

Here, the boundary conditions and region remain the same. The following notation is used above: r is a given function which is continuous in the basic region; ϵ is a parameter; u_n and λ_n are respectively the fundamental functions and eigenvalues of the new problem. It is presumed that both the new eigenvalues, and functions admit development by powers of the perturbation parameter. In many cases, where the problem does not include such a parameter, one can be introduced, in which case it is necessary to transform the differential equations in an appropriate way.

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*If the differential expression $M[v] = (av')' + (bv)' - cv - dv$ is uniquely determined by the differential expression $L[u] = (au')' - bu' + (cu)' - du$ and vice versa, with the aid of the requirement, that the integrals on the left side of the the formula $\int_{x_0}^{x_1} \left\{ vL[u] - uM[v] \right\} dx = \left[a(u'v - v'u) + (c-b)uv \right] \Big|_{x_0}^{x_1}$ can be expressed by the values of the function and its derivatives on the boundary alone, then two expressions are called mutually adjoint. If $L[u] = M[u]$ identically, the differential expression $L[u]$ is called self-adjoint. (See R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. 1, Chapter V.)

The boundary conditions of our problem are:

$$[\varphi(\zeta)]_{\zeta=0}=0;\ \left[\frac{d\varphi(\zeta)}{d\zeta}\right]_{\zeta=0}=0;\ \left[\frac{d^2\varphi(\zeta)}{d\zeta^2}\right]_{\zeta=1}=0;\ \left[\frac{d^3\varphi(\zeta)}{d\zeta^3}\right]_{\zeta=1}=0. \tag{15.1}$$

Let us suppose that the deflection function $\varphi(\zeta)$ and the eigenvalues $\lambda=\rho\omega_c^2$ can be expanded in series of some parameter in the following way:

$$\varphi(\zeta)=\varphi_n+\varepsilon v_n+\varepsilon^2 w_n+\ldots; \tag{15.2}$$

$$\lambda=\lambda_n+\varepsilon\mu_n+\varepsilon^2\varkappa+\ldots, \tag{15.3}$$

where $\varphi_n, v_n, w_n \ldots$ are functions of ζ .

We shall also suppose that the variable moment of inertia of the rod and the variable cross sectional area are expressed, respectively, as:

$$l^4 I(\zeta)=I_0+\varepsilon l^4 I_1(\zeta); \tag{15.4}$$

$$l^2 F(\zeta)=F_0+\varepsilon l^2 F(\zeta). \tag{15.5}$$

Here I_0 and F_0 denote, respectively, the averaged values of the moment of inertia and the area of cross-section of the rod, which can be determined for example, by the formulas

$$I_0=\frac{1}{l}\int_0^l I(x)\,dx, \tag{15.6}$$

$$F_0=\frac{1}{l}\int_0^l F(x)\,dx. \tag{15.7}$$

Substituting expressions (15.2) — (15.7) into the differential equation (14.6) we obtain the perturbed differential equation

$$\frac{d^2}{d\zeta^2}\left\{[I_0+\varepsilon l^4 I_1(\zeta)]\left[\left(\frac{d^2\varphi_n}{d\zeta^2}+\varepsilon\frac{d^2v_n}{d\zeta^2}+\varepsilon^2\frac{d^2w_n}{d\zeta^2}+\dots\right)\right]E\right\}-$$
$$-l^4(\lambda_n+\varepsilon\mu_n+\varepsilon^2\varkappa_n+\dots)[F_0+\varepsilon l^4F_1(\zeta)](\varphi_n+\varepsilon v_n+\varepsilon^2 w_n+\dots)=0 \qquad (15.8)$$

Grouping in equation (15.8) the terms containing identical powers of the small parameter and equating to zero the expressions which multiply the powers of the parameter, we obtain a system of differential equations:

$$\frac{d^2}{d\zeta^2}\left[EI_0\frac{d^2\varphi_n}{d\zeta^2}\right]-\lambda_n l^4F_0\varphi_n=0; \qquad (15.9)$$

$$E\left\{I_0\frac{d^4v_n}{d\zeta^4}+\frac{d^2}{d\zeta^2}\left[l^4I_1(\zeta)\frac{d^2\varphi_n}{d\zeta^2}\right]\right\}-\lambda_n l^4F_0v_n-$$
$$-F_0l^4\mu_n\varphi_n-\lambda_n l^4F_1(\zeta)\varphi_n=0; \qquad (15.10)$$

$$E\left\{I_0\frac{d^4w_n}{d\zeta^4}+\frac{d^2}{d\zeta^2}\left[l^4I_1(\zeta)\frac{d^2v_n}{d\zeta^2}\right]\right\}-\lambda_n l^4F_0w_n-\mu l^4F_0v_n-$$
$$-\varkappa_n l^4F_0\varphi_n-\lambda_n l^4F_1(\zeta)v_n-\mu_n l^4F_1(\zeta)\varphi_n=0; \qquad (15.11)$$

Evidently, the basic, unperturbed differential equation for the problem in question of vibrations of a rod of variable cross-section, will be equation (15.9) which corresponds to the equation of free vibrations of a rod of constant cross-section, the solution of which is known. Introducing the notation

$$k_n^4=\frac{\lambda_n l^4F_0}{EI_0}, \qquad (15.12)$$

The equation (15.9) can be represented as

$$\frac{d^4\varphi_n}{d\zeta^4}-k_n^4\varphi_n=0. \qquad (15.13)$$

The solution of the last equation under the boundary conditions (15.1) can be written as follows in accordance with (9.1)

$$\varphi_n = -C_n\{(\cos k_n + \operatorname{ch} k_n)(\operatorname{ch} k_n\zeta - \cos k_n\zeta) + (\sin k_n - \operatorname{sh} k_n)(\operatorname{sh} k_n\zeta - \sin k_n\zeta)\}. \tag{15.14}$$

The constant C_n, entering in equation (15.14) is chosen from the normalizing condition of the function

$$\int_0^1 \varphi_n \, d\zeta = 1. \tag{15.15}$$

Omitting the intermediate computations, connected with the determination of constant C_n from condition (15.15), we write the final solution of the differential equation (15.13), after determining C_n, in the form:

$$\varphi_n = \operatorname{ch} k_n\zeta - \cos k_n\zeta + \frac{\operatorname{sh} k_n - \sin k_n}{\cos k_n + \operatorname{ch} k_n}(\sin k_n\zeta - \operatorname{ch} k_n\zeta). \tag{15.16}$$

We find the characteristic numbers (i.e. frequencies of vibrations) from the frequency equation, which, as is known, has the form

$$\cos k_n \operatorname{ch} k_n = -1, \tag{15.17}$$

where n = 1, 2, 3, 4, . . .

The roots of equation (15.17) (see table on page 53) have values $k_1 = \pm 1{,}875$; $k_2 = \pm 4{,}694$; $k_3 = \pm 7{,}855$ etc.
The natural frequencies of vibrations are determined by the formula

$$\lambda_n = \left(\frac{k_n}{l}\right)^4 \frac{EI_0}{F_0}. \tag{15.18}$$

To obtain the first approximation, it is necessary to examine equation (15.10). Using the notation

$$m_n = \frac{\mu_n F_0 l^4}{EI_0}, \tag{15.19}$$

equation (15.10) can be transformed into:

$$\frac{d^4 v_n}{d\zeta^4} - k_n^4 v_n = m_n \varphi_n + k_n^4 \frac{l^2 F_1(\zeta)}{F_0} \varphi_n - \frac{d^2}{d\zeta^2}\left[\frac{l^4 I_1(\zeta)}{I_0} \frac{d^2\varphi_n}{d\zeta^2}\right]. \tag{15.20}$$

To determine v_n and m_n, we will introduce, as an unknown quantity, the coefficient of the expansion (Fourier coefficient)

$$a_{nj} = \int_0^1 v_n \varphi_j \, d\zeta \tag{15.21}$$

of the function v_n in the fundamental functions φ_j. We multiply equation (15.13) (changing the index n to i) by v_n and equation (15.20) by φ_i and subtract the first from the second equation. Then we have,

$$\frac{d^4 v_n}{d\zeta^4}\varphi_i - \frac{d^4\varphi_i}{d\zeta^4} v_n + v_n\varphi_i (k_i^4 - k_n^4) =$$
$$= m_n \varphi_n \varphi_i + k_n^4 \frac{l^2 F_1(\zeta)}{F_0}\varphi_i\varphi_n - \frac{d^2}{d\zeta^2}\left[\frac{l^4 I_1(\zeta)}{I_0}\frac{d^2\varphi_n}{d\zeta^2}\right]\varphi_i.$$

The equation obtained is integrated along the length of the rod

$$\int_0^1 \left[\frac{d^4 v_n}{d\zeta^4}\varphi_i - \frac{d^4\varphi_i}{d\zeta^4} v_n\right] d\zeta + (k_i^4 - k_n^4)\int_0^1 v_n\varphi_i \, d\zeta = m_n \int_0^1 \varphi_n\varphi_i \, d\zeta +$$
$$+ k_n^4 \int_0^1 \frac{l^2 F_1(\zeta)}{F_0}\varphi_n\varphi_i \, d\zeta - \int_0^1 \frac{d^2}{d\zeta^2}\left[\frac{l^4 I_1(\zeta)}{I_0}\frac{d^2\varphi_n}{d\zeta^2}\right]\varphi_i \, d\zeta. \tag{15.22}$$

Integrating by parts, it is easy to prove, that

$$\int_0^1 \frac{d^4 v_n}{d\zeta^4}\varphi_i d\zeta = \int_0^1 \frac{d^4\varphi_n}{d\zeta^4} v_n d\zeta.$$

Therefore, the first integral in equation (15.22) is identically equal to zero. From examination of the remaining integrals of this equation, it follows that $k_i = k_n$ for $i = n$. Then from equation (15.22) we find

$$m_n = \int_0^1 \frac{d^2}{d\zeta^2}\left[\frac{l^4 I_1(\zeta)}{I_0}\frac{d^2\varphi_n}{d\zeta^2}\right]\varphi_n d\zeta - k_n^4 \int_0^1 \frac{l^2 F_1(\zeta)}{F_0}\varphi_n^2 d\zeta. \qquad (15.23)$$

We transform the first integral of equation (15.23) taking account of the boundary conditions of (15.1)

$$\int_0^1 \frac{d^2}{d\zeta^2}\left[\frac{l^4 I_1(\zeta)}{I_0}\frac{d^2\varphi_n}{d\zeta^2}\right]\varphi_n d\zeta = \frac{d}{d\zeta}\left[\frac{l^4 I_1(\zeta)}{I_0}\frac{d^2\varphi_n}{d\zeta^2}\right]\varphi_n\Big|_0^1 -$$

$$-\int_0^1 \frac{d\varphi_n}{d\zeta}\frac{d}{d\zeta}\left[\frac{l^4 I_1(\zeta)}{I_0}\frac{d^2\varphi_n}{d\zeta^2}\right] = -\frac{d\varphi_n}{d\zeta^2}\frac{l^4 I_1(\zeta)}{I_0}\frac{d^2\varphi_n}{d\zeta^2}\Big|_0^1 +$$

$$+\int_0^1 \frac{l^4 I_1(\zeta)}{I_0}\left[\frac{d^2\varphi_n}{d\zeta^2}\right]^2 d\zeta = \int_0^1 \frac{l^4 I(\zeta)}{I_0}\left[\frac{d^2\varphi_n}{d\zeta^2}\right]^2 d\zeta.$$

Then the expression for m_n will take the form

$$m_n = \int_0^1 \frac{l^4 I_1(\zeta)}{I_0}\left[\frac{d^2\varphi_n}{d\zeta^2}\right]^2 d\zeta - k_n^4 \int_0^1 \frac{l^2 F_1(\zeta)}{F_0}\varphi_n^2 d\zeta. \qquad (15.24)$$

For $i \neq n$ we obtain from equation (15.22), having in mind (15.21):

$$(k_i^4 - k_n^4) a_{ni} = k_n^4 \int_0^1 \frac{l^2 F_1(\zeta)}{F_0}\varphi_n\varphi_i d\zeta - \int_0^1 \frac{d^2}{d\zeta^2}\left[\frac{l^4 I_1(\zeta)}{I_0}\frac{d^2\varphi_n}{d\zeta^2}\right]\varphi_i d\zeta.$$

After this, the Fourier coefficient equals

$$a_{ni} = \frac{1}{k_i^4 - k_n^4}\left\{k_n^4 \int_0^1 \frac{l^2 F_1(\zeta)}{F_0}\varphi_n\varphi_i d\zeta - \int_0^1 \frac{d^2}{d\zeta^2}\left[\frac{l^4 I_1(\zeta)}{I_0}\frac{d^2\varphi_n}{d\zeta^2}\right]\varphi_i d\zeta\right\}. \qquad (15.25)$$

Now, the unknown function $v_n(\zeta)$ can be represented in an infinite series

$$v_n = \sum_{j=1}^{\infty}{}' \frac{\varphi_j}{k_j - k_n^4} \left\{ k_n^4 \int_0^1 \frac{l^2 F_1(\zeta)}{F_0} \varphi_n \varphi_j \, d\zeta - \right.$$
$$\left. - \int_0^1 \frac{d^2}{d\zeta^2} \left[\frac{l^4 I_1(\zeta)}{I_0} \frac{d^2 \varphi_n}{d\zeta^2} \right] \varphi_j \, d\zeta \right\}. \quad (15.26)$$

The prime on the sum sign indicates that terms with index $j = n$ should be dropped.

From equation (15.19) we find

$$\mu_n = \frac{EI_0 m_n}{l^4 F_0}. \quad (15.27)$$

In the same way it is possible to obtain the second approximation. To do this we examine equation (15.11) which, in accordance with (15.8) can be rewritten as

$$\frac{d^4 w_n}{d\zeta^4} - k_n^4 w_n = -\frac{d^2}{d\zeta^2} \left[\frac{l^4 I_1(\zeta)}{I_0} \frac{d^2 v_n}{d\zeta^2} \right] +$$
$$+ m_n v_n + p_n \varphi_n + k_n^4 \frac{l^2 F_1(\zeta)}{F_0} v_n + m_n \frac{l^2 F_1(\zeta)}{F_0} \varphi_n, \quad (15.28)$$

where

$$p_n = \frac{\varkappa_n F_0}{EI_0}. \quad (15.29)$$

We represent the function w_n in the form of a series expansion in the fundamental functions.

$$w_n = \sum_{t=1}^{\infty} b_{nt} \varphi_t, \quad (15.29')$$

$$b_{nt} = \int_0^1 w_n \varphi_t \, d\zeta. \quad (15.30)$$

We multiply the basic equation (15.13) by w_n, first replacing the index n by i in it, and equation (15.28) by φ_i, and, subtracting the first equation from the second we obtain

$$\frac{d^4 w_n}{d\zeta^4}\phi_i - k_n^4 w_n \phi_i - \frac{d^4\phi_i}{d\zeta^4} w_n + k_i^4 \phi_i w_n =$$

$$= -\frac{d^2}{d\zeta^2}\left[\frac{l^4 I_1(\zeta)}{I_0}\frac{d^2 v_n}{d\zeta^2}\right]\phi_i + m_n \phi_n v_n + p_n \phi_n \phi_i +$$

$$+ k_n^4 \frac{l^2 F_1(\zeta)}{F_0} v_n \phi_i + m_n \frac{l^2 F_1(\zeta)}{F_0}\phi_n \phi_i . \quad (15.31)$$

We integrate equation (15.31) along the length of the rod (from 0 to 1), then for $i = n$, by virtue of (15.15), we obtain

$$\int_0^1 p_n \varphi_n^2 \, d\zeta = p_n.$$

By integrating by parts taking account of the boundary conditions, we find that

$$\int_0^1 \left[\frac{d^4 w_n}{d\zeta^4}\varphi_n - \frac{d^4\varphi_n}{d\zeta^4} w_n\right] d\zeta = 0;$$

$$\int_0^1 (k_i^4 - k_n^4) w_n \varphi_i \, d\zeta = 0.$$

Because

$$k_i = k_n,$$

$$p_n = \int_0^1 \frac{d^2}{d\zeta^2}\left[\frac{l^4 I_1(\zeta)}{I_0}\frac{d^2 v_n}{d\zeta^2}\right]\varphi_n \, d\zeta - m_n \int_0^1 \varphi_n v_n \, d\zeta -$$

$$- k_n^4 \int_0^1 \frac{l^2 F_1(\zeta)}{F_0} v_n \varphi_n \, d\zeta - m_n \int_0^1 \frac{l^2 F_1(\zeta)}{F_0}\varphi_n^2 \, d\zeta.$$

Integrating the first integral by parts, taking account of the boundary conditions, we find, finally

$$p_n = \int_0^1 \frac{l^4 I_1(\zeta)}{I_0} \frac{d^2 v_n}{d\zeta^2} \frac{d^2 \varphi_n}{d\zeta^2} d\zeta - m \int_0^1 v_n \varphi_n d\zeta -$$

$$- k_n^4 \int_0^1 \frac{l^2 F_1(\zeta)}{F_0} v_n \varphi_n d\zeta - m_n \int_0^1 \frac{l^2 F_1(\zeta)}{F_0} \varphi_n^2 d\zeta. \tag{15.32}$$

Knowing p_n, we will determine x_n. Integrating equation (15.31) along the length of the rod for the case, when $i \neq n$ in accordance with (15.16), we obtain

$$(k_i^4 - k_n^4) \int_0^1 w_n \varphi_i d\zeta = - \int_0^1 \frac{l^4 I_1(\zeta)}{I_0} \frac{d^2 v_n}{d\zeta^2} \frac{d^2 \varphi_i}{d\zeta^2} d\zeta +$$

$$+ k_n^4 \int_0^1 \frac{l^2 F_1(\zeta)}{F_0} v_n \varphi_i d\zeta + m_n \int_0^1 \frac{l^2 F_1(\zeta)}{F_0} \varphi_n \varphi_i d\zeta. \tag{15.33}$$

From (15.30) and (15.33) it follows that the Fourier coefficient b_{ni} can be calculated from the formula

$$b_{ni} = \frac{1}{k_n^4 - k_i^4} \left\{ \int_0^1 \frac{l^4 I_1(\zeta)}{I_0} \frac{d^2 v_n}{d\zeta^2} \frac{d^2 \varphi_i}{d\zeta^2} d\zeta - \right.$$

$$\left. - k_n^4 \int_0^1 \frac{l^2 F_1(\zeta)}{F_0} \varphi_i v_n d\zeta - m_n \int_0^1 \frac{l^2 F_1(\zeta)}{F_0} \varphi_n \varphi_i d\zeta \right\}. \tag{15.34}$$

The functions w_n can be represented as an infinite series, in accordance with (15.29'):

$$w_n = \sum_{j=1}^{\infty}{}' \frac{\varphi_j}{k_n^4 - k_j^4} \left\{ \int_0^1 \frac{l^4 I_1(\zeta)}{I_0} \frac{d^2 v_n}{d\zeta^2} \frac{d^2 \varphi_j}{d\zeta^2} d\zeta - \right.$$

$$\left. - k_n^4 \int_0^1 \frac{l^2 F_1(\zeta)}{F_0} v_n \varphi_j d\zeta - m_n \int_0^1 \frac{l^2 F_1(\zeta)}{F_0} \varphi_n \varphi_j d\zeta \right\}. \tag{15.35}$$

Here, as in equation (15.25), in performing the summation, the terms with index $j = n$ should be dropped. In a similar way, we might find also the successive approximations. However, for the solution of the basic problem, i.e. the solution of equation (14.8) we will limit ourselves to the second approximation.

The solution of the basic equation (14.6) in the zeroth approximation, on the basis of (15.1'), (15.3), (15.8), (15.17), (15.18), (15.19), (15.24), (15.26), (15.27), (15.29), (15.32), and (15.35), taking the parameter ε equal to one, can be written out as:

$$\varphi(\zeta) = \varphi_n + v_n + w_n,$$
$$\lambda = \lambda_n + \mu_n + \varkappa_n \tag{15.36}$$

Substituting for the right side in these formulas the appropriate values from formulas (15.19), (15.26), (15.35), (15.12), (15.17), and (15.29) we obtain the deflection function:

$$\begin{aligned}\varphi(\zeta) = \frac{1}{\cos k_n - \operatorname{ch} k_n} [(\cos k_n + \operatorname{ch} k_n)(\cos k_n\zeta - \operatorname{ch} k_n\zeta) + \\ + (\sin k_n - \operatorname{sh} k_n)(\sin k_n\zeta - \operatorname{ch} k_n\zeta)] - \\ - \sum_{j=1}^{\infty}{}' \frac{\varphi_j}{k_n^4 - k_j^4} \int_0^1 k_n^4 \frac{l^3 F_1(\zeta)}{F_0} \varphi_n \varphi_j \, d\zeta - \int_0^1 \frac{d^2}{d\zeta^2}\left[\frac{l^4 I_1(\zeta)}{I_0} \frac{d^2\varphi_n}{d\zeta^2}\right]\varphi_j \, d\zeta - \\ - \sum_{j=1}^{\infty}{}' \frac{\varphi_j}{k_n^4 - k_j^4} \left\{ \int_0^1 \frac{l^4 I_1(\zeta)}{I_0} \frac{d^2 v_n}{d\zeta^2} \frac{d^2\varphi_j}{d\zeta^2} \, d\zeta - \right. \\ \left. - k_n \int_0^1 \frac{l^3 F_1(\zeta)}{F_0} v_n \varphi_j \, d\zeta - m_n \int_0^1 \frac{l^3 F_1(\zeta)}{F_0} \varphi_n \varphi_j \, d\zeta \right\}.\end{aligned} \tag{15.37}$$

The square of frequency of vibrations is:

$$\omega_c^2 = \frac{k_n^4 EI_0}{l^4 \varrho F_0} + \frac{EI_0}{l^4 \varrho F_0}\left\{\int_0^1 \frac{d^2}{d\zeta^2}\left[\frac{l^4 I_1(\zeta)}{d\zeta^2}\frac{d^2\varphi_n}{d\zeta^2}\right]\varphi_n\,d\zeta - \right.$$

$$\left. - k_n^4 \int_0^1 \frac{l^2 F_1(\zeta)}{F_0}\varphi_n^2\,d\zeta\right\} + \frac{EI_0}{l^4 \varrho F_0}\left\{-\int_0^1 \frac{l^4 I_1(\zeta)}{I_0}\frac{d^2 v_n}{d\zeta^2}\frac{d^2\varphi_n}{d\zeta^2}d\zeta + \right.$$

$$\left. + m_n \int_0^1 v_n\varphi_n\,d\zeta + k_n^4 \int_0^1 \frac{l^2 F_1(\zeta)}{F_0} v_n\varphi_n\,d\zeta - m_n \int_0^1 \frac{l^2 F_1(\zeta)}{F_0}\varphi_n^2\,d\zeta\right\}. \quad (15.38)$$

The magnitude of the deflection of the rod of variable cross-section, determined in the zeroth approximation, in accordance with (14.3), is equal to

$$u^*(\zeta, t) = \varphi(\zeta)\,a\cos(\omega t + \psi), \quad (15.39)$$

where $\varphi(\zeta)$ is determined in accordance with (15.37).

16. Determination of the frequency of vibrations and phase shift in the first approximation.

To solve the problem in the first approximation the quantities Δ_1 and ψ_0, which enter into the expansions (14.4) and (14.5), must be determined.

Let us examine equation (14.7) which was obtained by equating to zero the expression which multiplies the small parameter raised to the first degree in equation (14.2). In conformity with (15.4) and (15.5) to maintain the same degree of precision in powers of the small parameter, the variable $I(\zeta)$ and $F(\zeta)$ in equation (14.7) should be replaced by the constants I_0 and F_0 taken from the expansions of (15.4) and (15.5). The quantities I_0 and F_0 are the first terms of the expressions for $I(\zeta)$ and $F(\zeta)$ and represent the average values of the moment of inertia and the cross-sectional area of the bar.

In this way equation (14.7) can be rewritten as:

$$EI_0 \frac{d^4 u_1}{\partial \zeta^4} + \left[\varrho \omega_c^2 F_0 \frac{\partial^2 u_1}{d\tau^2} - \varrho \Delta_1 F_0 a \varphi \cos \tau - \varrho \omega_c^2 F_0 \beta \zeta \cos(\tau - \psi_0) \right] l^4 + $$
$$+ l \frac{\partial^2}{\partial \zeta^2} \left[\Phi \left(\frac{d^2 \varphi}{d\zeta^2} a \cos \tau \right) \right] = 0. \quad (16.1)$$

To determine Δ_1 and ψ_0 we examine the equations of harmonic balance. For this purpose we multiply equation (16.1) first by $\varphi \sin \tau d\zeta d\tau$, and then by $\varphi \cos \tau d\zeta d\tau$ and to equate to zero the integrals of the resulting expressions along the whole length of the bar for a single cycle. Then

$$1) \quad \oint \int_0^1 \left\{ EI_0 \frac{\partial^4 u_1}{\partial \zeta^4} + \left[\varrho \omega_c^2 F_0 \frac{\partial^2 u_1}{\partial \tau^2} - \varrho \Delta_1 F_0 a \varphi \cos \tau - \right. \right.$$
$$\left. \left. - \varrho \omega_c^2 F_0 \beta \zeta \cos(\tau - \psi_0) \right] l^4 + l \frac{\partial^2}{\partial \zeta^2} \left[\Phi \left(\frac{d^2 \varphi}{d\zeta^2} a \cos \tau \right) \right] \right\} \varphi \sin \tau \, d\zeta \, d\tau = 0; \quad (16.2)$$

$$2) \quad \oint \int_0^1 EI_0 \frac{\partial^4 u_1}{\partial \zeta^4} + \left[\varrho \omega_c^2 F_0 \frac{\partial^2 u_1}{\partial \tau^2} - \varrho \Delta_1 F_0 a \varphi \cos \tau - \right.$$
$$\left. - \varrho \omega_c^2 F_0 \beta \zeta \cos(\tau - \psi_0) \right] l^4 + \frac{\partial^2}{\partial \zeta^2} \left[\Phi \left(\frac{d^2 \varphi}{d\zeta^2} a \cos \tau \right) \right] \Big\} \varphi \cos \tau \, d\zeta \, d\tau = 0. \quad (16.3)$$

Integrating these equations, considering the conditions (15.1) at the ends of the bar and considering also that the function u_1 does not contain the principal harmonic, we solve the equations for Δ_1 and $\sin \psi_0$. We obtain

$$\Delta_1 = \left[\varrho F_0 a \pi \int_0^1 \varphi^2 d\zeta \right]^{-1} \left\{ \frac{1}{l^3} \oint \int_0^1 \left[\Phi \left(\frac{d^2 \varphi}{d\zeta^2} a \cos \tau \right) \right] \frac{d^2 \varphi}{d\zeta^2} \cos \tau \, d\zeta \, d\tau - \right.$$
$$\left. - \varrho \omega_c^2 F_0 \beta \pi \cos \psi_0 \int_0^1 \zeta \varphi(\zeta) \, d\zeta \right\}; \quad (16.4)$$

$$\sin\psi_0 = \left[\varrho\omega_c^2 F_0 \beta\pi \int_0^1 \zeta\varphi\, d\zeta\right]^{-1} \times$$

$$\times \frac{1}{l^3} \oint \int_0^l \left[\Phi\left(\frac{d^2\varphi}{d\zeta^2} a\cos\tau\right)\right] \frac{d^2\varphi}{d\zeta^2} \sin\tau\, d\zeta\, d\tau. \qquad (16.5)$$

Keeping in mind, further, that

$$1)\ \varepsilon \oint \int_0^1 \left[\Phi\left(\frac{d^2\varphi}{d\zeta^2} a\cos\tau\right)\right] \frac{d^2\varphi}{d\zeta^2} \sin\tau\, d\zeta\, d\tau =$$

$$= \oint \int_0^1 M\left[\frac{d^2\varphi}{d\zeta^2} a\cos\tau\right] \frac{d^2\varphi}{d\zeta^2} \sin\tau\, d\zeta\, d\tau;$$

$$2)\ \varepsilon \oint \int_0^1 \left[\Phi\left(\frac{d^2\varphi}{d\zeta^2} a\cos\tau\right)\right] \frac{d^2\varphi}{d\zeta^2} \cos\tau\, d\zeta\, d\tau =$$

$$= \oint \int_0^1 M\left[\frac{d^2\varphi}{d\zeta^2} a\cos\tau\right] \frac{d^2\varphi}{d\zeta^2} \cos\tau\, d\zeta\, d\tau - \frac{1}{l} EI_0 \pi a \int_0^1 \left(\frac{d^2\varphi}{d\zeta^2}\right)^2 d\zeta,$$

we express formulas (16.4) and (16.5) as follows:

$$\varepsilon\Delta_1 = \left[\varrho F_0 a\pi \int_0^1 \varphi^2\, d\zeta\right]^{-1} \left\{ \frac{1}{l^3} \oint \int_0^1 M\left[\frac{d^2\varphi}{d\zeta^2} a\cos\tau\right] \frac{d^2\varphi}{d\zeta^2} \cos\tau\, d\zeta\, d\tau -\right.$$

$$\left. - \frac{1}{l^4} EI_0 \pi a \int_0^1 \left(\frac{d^2\varphi}{d\zeta^2}\right)^2 d\zeta - \varrho\omega_c^2 F_0 \theta_0 \pi \cos\psi_0 \int_0^1 \zeta\varphi\, d\zeta \right\}; \qquad (16.6)$$

$$\sin\psi_0 = \left[\varrho\omega_c^2 F_0 \theta_0 \pi \int_0^1 \zeta\varphi\, d\zeta\right]^{-1} \times$$

$$\times \frac{1}{l^3} \oint \int_0^1 M\left[\frac{d^2\varphi}{d\zeta^2} a\cos\tau\right] \frac{d^2\varphi}{d\zeta^2} \sin\tau\, d\zeta\, d\tau. \qquad (16.7)$$

The square of frequency of vibrations in the first approximation, on the basis of (14.4), equals

$$\omega^2=\omega_c^2+\varepsilon\Delta_1=\omega_c^2+\left[\varrho F_0 a\pi\int_0^1\varphi^2\,d\zeta\right]^{-1}\left\{\oint\int_0^1 M\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\frac{1}{l^2}\times\right.$$

$$\left.\times\frac{d^2\varphi}{d\zeta^2}\cos\tau\,d\zeta\,d\tau-\frac{1}{l^4}EI_0\pi a\int_0^1\left(\frac{d^2\varphi}{d\zeta^2}\right)^2 d\zeta-\varrho\omega_c^2F_0\theta_0\pi\cos\psi_0\int_0^1\zeta\varphi\,d\zeta.\right. \tag{16.8}$$

In this way the final formulas for the determination of the magnitude of the phase shift and the frequency in the first approximation are obtained. These formulas contain the bending moment, the calculation of which is given below.

In accordance with equation (8.4)', for the solution of the problem in the first approximation:

$$M_0=EI_0\frac{d^2\varphi}{d\zeta^2}a\cos\tau;\quad M_s=\varepsilon\Phi\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right);$$

$$M=M\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right). \tag{16.9}$$

The elastic bending moment M_0, will evidently have the same value for both branches of the hysteresis loop. As for M_s , which is due to the force of internal friction, and, therefore, for the total bending moment also, their values will be different for each of the branches of the hysteresis loop. The symbols for of the respective moments will be marked by superior arrows: directed to the right for the ascending branch of the hysteresis loop, and to the left for the descending branch.

In this way we can write down two expressions for the bending moments of the two branches of the hysteresis loop:

$$\vec{M}=M_0+\vec{M}_s;\quad \overleftarrow{M}=M_0+\overleftarrow{M}_s. \tag{16.10}$$

Because of equation (10.14), for a bar of the variable section we have

$$\vec{M}_e = -\iint_F E\frac{\nu}{n}\left[(\xi_0+\xi)^n - 2^{n-1}\xi_0^n\right] z\, dz\, dy,$$

$$\overleftarrow{M}_e = \iint_F E\frac{\nu}{n}\left[(\xi_0-\xi)^n - 2^{n-1}\xi_0^n\right] z\, dz\, dy,$$

$$M_0 = \iint_F E\xi z\, dz\, dy,$$

(16.11)

Where z and y are the coordinates of an element of area of the variable section of the bar, the longitudinal axis of which coincides with the x-axis (Fig. 8).

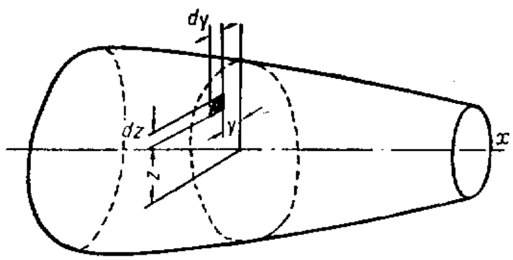

Fig. 8

The strains ξ_0 and ξ, which enter into the equation (16.11) in the present case have the values:

$$(\xi_0)_{max} = \frac{1}{l^n}\left(\frac{\partial^2 u}{\partial \zeta^2}\right)_{max} z_{max} = \frac{1}{l^n}\, a\, \frac{d^2\phi}{d\zeta^2} z_{max} = a\frac{1}{l^n}\frac{d^2\phi}{d\zeta^2}\frac{h_x}{2},$$

$$\xi_0 = (\xi_0)_{max}\frac{2z}{h_x} = \frac{1}{l^n}\, a\, \frac{d^2\phi}{d\zeta^2} z,$$

$$\xi = \frac{1}{l^n}\, a\, \frac{d^2\phi}{d\zeta^2} z \cos\tau .$$

(16.12)

Substituting the equations (16.12) into equations (16.11) we obtain

$$\vec{M}_s = -\iint_F \frac{1}{l^n} E \frac{\nu}{n}\left[\left(a\frac{d^2\varphi}{d\zeta^2}z + a\frac{d^2\varphi}{d\zeta^2}z\cos\tau\right)^n - 2^{n-1}a^n\left(\frac{d^2\varphi}{d\zeta^2}\right)^n z^n\right] z\,dz\,dy =$$

$$= -\iint_F E\frac{\nu}{n}\frac{1}{l^n} z^{n+1}a^n\left[\frac{d^2\varphi}{d\zeta^2}\right]^n\left[(1+\cos\tau)^n - 2^{n-1}\right]dz\,dy$$

or

$$\vec{M}_s = -\frac{1}{l^n}E\frac{\nu}{n}a^n\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\left[(1+\cos\tau)^n - 2^{n-1}\right]\iint_F z^{n+1}\,dz\,dy. \tag{16.13}$$

Similarly we obtain

$$\overleftarrow{M}_s = E\frac{1}{l^n}\frac{\nu}{n}a^n\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\left[(1-\cos\tau)^n - 2^{n-1}\right]\iint_F z^{n+1}\,dz\,dy. \tag{16.14}$$

Keeping equations (16.9), (16.10), (16.13), and (16.14) in mind, we now find the integrals of the expressions containing the bending moments in formulas (16.4) and (16.5),

$$1)\quad \oint\int_0^1 M\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\frac{d^2\varphi}{d\zeta^2}\sin\tau\,d\zeta\,d\tau =$$

$$= E\frac{\nu}{n}a^n\frac{2^{n+1}(n-1)}{l^n(n+1)}\int_0^1\left\{\left(\frac{d^2\varphi}{d\zeta^2}\right)^{n+1}\iint_F z^{n+1}\,dz\,dy\right\}d\zeta; \tag{16.15}$$

$$2)\quad \oint\int_0^1 M\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\frac{d^2\varphi}{d\zeta^2}\cos\tau\,d\zeta\,d\tau = \frac{1}{l}Ea\pi\int_0^1 I_0\left(\frac{d^2\varphi}{d\zeta^2}\right)^2 d\zeta +$$

$$+ 2E\frac{\nu}{n}\frac{a^n}{l^n}\int_0^\pi(1-\cos\tau)^n\cos\tau\,d\tau\int_0^1\left\{\left(\frac{d^2\varphi}{d\zeta^2}\right)^{n+1}\iint_F z^{n+1}\,dz\,dy\right\}d\zeta. \tag{16.16}$$

Substituting (16.15) in (16.5), and (16.16) into (16.6), we obtain the formulas for computation:

$$\sin\psi_0 = \frac{\nu a^n 2^{n+1}(n-1)E}{l^{n+3}\,n(n+1)\,\varepsilon\rho\omega_c^2 F_0\beta\pi\int_0^1\zeta\varphi\,d\zeta}\int_0^1\left\{\left(\frac{d^2\varphi}{d\zeta^2}\right)^{n+1}\iint_F z^{n+1}\,dz\,dy\right\}d\zeta; \tag{16.17}$$

$$\omega^2=\omega_c^2+\left[\varrho F_0 a\pi\int\limits_0^1\varphi^2 d\zeta\right]^{-1}\left\{\frac{2Ea^n}{l^{n+3}}\frac{\nu}{n}\int\limits_0^\pi(1-\cos\tau)^n\cos\tau\,d\tau\times\right.$$
$$\left.\times\int\limits_0^1\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^{n+1}\iint\limits_F z^{n+1}dz\,dy\right]d\zeta-\varrho\omega_c^2F_0\theta_0\pi\cos\psi_0\int\limits_0^1\zeta\psi\,d\zeta\right\}. \qquad (16.18)$$

On the basis of (16.17) the value of the integral entering into equation (16.18) can be written

$$\int\limits_0^1\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^{n+1}\iint\limits_F z^{n+1}dz\,dy\right]d\zeta=\frac{\varrho\omega_c^2F_0\theta_0\pi(n+1)\,n\sin\psi_0}{E(n-1)\,\nu a^n 2^{n+2}\,l^{-(n+3)}}\int\limits_0^1\zeta\varphi\,d\zeta.$$

Substituting the last expression into equation (16.18) and performing the necessary reductions, we obtain

$$\omega^2=\omega_c^2+\omega_c^2\theta_0\int\limits_0^1\zeta\varphi\,d\zeta\left[\sin\psi_0\frac{n+1}{2^{n+2}(n-1)}\int\limits_0^\pi(1-\cos\tau)^n\cos\tau\,d\tau-\right.$$
$$\left.-\cos\psi_0\right]\frac{1}{a\int\limits_0^1\varphi^2\,d\zeta}. \qquad (16.19)$$

After dividing both sides of equation (16.19) by ω_c^2 the formula for the determination of the frequency of vibrations in the first approximation will finally take the form

$$\frac{\omega^2}{\omega_c^2}=1+\frac{\theta_0\int\limits_0^1\zeta\varphi\,d\zeta}{a\int\limits_0^1\varphi^2\,d\zeta}\left[\frac{n+1}{2^{n+1}}\int\limits_0^\pi(1-\cos\tau)^n\cos\tau\,d\tau\sin\psi_0-\cos\psi_0\right]. \qquad (16.20)$$

We note that the structure of the formula for the frequency of vibrations of a bar of variable cross-section has remained the same as for the bar of constant cross-section, the only difference being in the different expressions for the function φ .

17. Determination of the deflection in the first approximation

In order to determine the value of the deflection in the first approximation in accordance with the series (14.3), it is necessary, first of all, to calculate $u_1(\zeta,\tau)$. For this purpose let us examine equation (16.1), which, with the help of the notation

$$L(\zeta,\tau)=[\varrho\Delta_1 F_0 a\varphi\cos\tau+\varrho\omega_c^2 F_0\beta\zeta\cos(\tau-\psi)]\,l^4-\\-\frac{\partial^2}{\partial\zeta^2}\left[\Phi\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\right]l \tag{17.1}$$

can be expressed as:

$$EI_0\frac{\partial^4 u_1}{\partial\zeta^4}+l^4\varrho\omega_c^2 F_0\frac{\partial^2 u_1}{\partial\tau^2}=L(\zeta,\tau). \tag{17.2}$$

In accordance with (16.2) and (16.3) the function $L(\zeta,\tau)$ satisfies the following conditions:

$$\oint\int_0^1 L(\zeta,\tau)\varphi\sin\tau\,d\zeta\,d\tau=0,$$
$$\oint\int_0^1 L(\zeta,\tau)\varphi\cos\tau\,d\zeta\,d\tau=0. \tag{17.3}$$

We represent $u_1(\zeta,\tau)$ and $L(\zeta,\tau)$ as Fourier series expansions

$$u_1(\zeta,\tau)=u_0(\zeta)+\sum_{k=2}^{\infty}u_k^s(\zeta)\sin k\tau+\sum_{k=2}^{\infty}u_k^c(\zeta)\cos k\tau, \tag{17.4}$$

$$L(\zeta, \tau) = L_0(\zeta) + \sum_{k=2}^{\infty} L_k^s(\zeta) \sin k\tau + \sum_{k=2}^{\infty} L_k^c(\zeta) \cos k\tau. \tag{17.5}$$

Comparing the corresponding terms of series (17.4) and (17.5) in equation (17.2) we obtain

$$EI_0 \frac{d^4 u_0}{d\zeta^4} = L_0(\zeta), \tag{17.6}$$

where

$$L_0(\zeta) = \frac{1}{\pi} \oint L(\zeta, \tau)\, d\tau = \frac{l^4}{\pi} \varrho \omega_c^2 F_0 \beta \zeta \int_0^{2\pi} \cos(\tau - \psi_0)\, d\tau +$$
$$+ \frac{l^4}{\pi} \varrho \Delta_1 F_0 a\varphi \int_0^{2\pi} \cos \tau\, d\tau - \frac{l}{\pi} \oint \frac{\partial^2}{\partial \zeta} \left[\Phi \left(\frac{d^2\varphi}{d\zeta^2} a \cos \tau \right) \right] =$$
$$= \frac{1}{\varepsilon} \frac{\partial^2}{\partial \zeta^2} \left\{ -l \oint M \left(\frac{d^2\varphi}{d\zeta^2} a \cos \tau \right) d\tau - \oint EI_0 a \cos \tau \frac{d^2\varphi}{d\zeta^2} d\tau \right\} =$$
$$= \frac{l^{1-n}}{\varepsilon} \frac{\partial^2}{\partial \zeta^2} \left\{ - \int_0^{2\pi} E \frac{\nu}{n} a^n \left(\frac{d^2\varphi}{d\zeta^2} \right)^n [(1 + \cos \tau)^n - 2^{n-1}] \left[\iint_F z^{n+1}\, dz\, dy \right] d\tau + \right.$$
$$\left. + \int_0^{\pi} E \frac{\nu}{n} a^n \left(\frac{d^2\varphi}{d\zeta^2} \right)^n [(1 - \cos \tau)^n - 2^{n-1}] \left[\iint_F z^{n+1}\, dz\, dy \right] d\tau \right\} = 0. \tag{17.7}$$

Thus $L_0(\zeta) = 0$. Therefore, from (17.6) we have

$$EI_0 \frac{d^4 u_0}{d\zeta^4} = 0. \tag{17.8}$$

Integrating the last equation considering the boundary conditions, we find

$$u_0 = 0. \tag{17.9}$$

Substituting the kth terms of the series into equation (17.2) we obtain

$$EI_0 \frac{d^4 u_k(\zeta)}{d\zeta^4} - l^4 \varrho \omega_c^2 F_0 k^2 u_k(\zeta) = L_k(\zeta). \tag{17.10}$$

We expand $u_k(\zeta)$ in the fundamental functions:

$$u_k(\zeta) = \sum_{n=1}^{\infty} c_n^{(k)} \phi_n(\zeta) \tag{17.11}$$

where $\varphi_n(\zeta)$ — a function which satisfies the boundary conditions of the problem is, as is known, a solution of the differential equation (17.8).

Substituting (17.11) into (17.8) and having in mind equation (15.9) we obtain

$$EI \frac{d^4 u_k}{d\zeta^4} = \sum_{n=1}^{\infty} EI\, c_n^{(k)} \frac{d^4 \phi_n}{d\zeta^4} = \sum_{n=1}^{\infty} c_n^{(k)} \lambda_n' \phi_n ,$$

$$\sum_{n=1}^{\infty} c_n^{(k)} \lambda_n' \phi_n - l^4 \rho F_0 \omega_c^2 k^2 \sum_{n=1}^{\infty} c_n^{(k)} \phi_n = L_k(\zeta), \tag{17.12}$$

where

$$\lambda_n F_0 l^4 = \lambda_n' .$$

Now let us expand $L_k(\zeta)$ in the same fundamental functions

$$L_k(\zeta) = \sum_{n=1}^{\infty} a_n^{(k)} \phi_n , \tag{17.13}$$

where the Fourier coefficient is:

$$a_n^{(k)} = 2\int_0^1 L_k(\zeta) \phi_n \, d\zeta . \tag{17.14}$$

Substituting (17.13) into (17.12), we find

$$\sum_{n=1}^{\infty} c_n^{(k)} \lambda_n' \phi_n - l^4 \rho F_0 \omega_c^2 k^2 \sum_{n=1}^{\infty} c_n^{(k)} \phi_n = \sum_{n=1}^{\infty} a_n^{(k)} \phi_n$$

or

$$\sum_{n=1}^{\infty} \left\{ c_n^{(k)} \left[\lambda_n' - l^4 \rho F_0 \omega_c^2 k^2 \right] - a_n^{(k)} \right\} \phi_n = 0 .$$

Equating to zero the coefficients of the same φ_n , we have

$$c_n^{(k)}[\lambda_n' - l^4 \varrho F_0 \omega_c^2 k^2] - a_n^{(k)} = 0;$$

$$c_n^{(k)} = \frac{a_n^{(k)}}{\lambda_n' - \varrho \omega_c^2 F_0 k^2 l^4}. \tag{17.15}$$

Substituting the value of $a_n^{(k)}$ from (17.14) we rewrite formula (17.14) as:

$$c_n^{(k)} = \frac{2\int_0^1 L_k(\zeta)\varphi_n d\zeta}{\lambda_n' - \varrho F_0 \omega_c^2 k^2 l^4}. \tag{17.16}$$

Then in accordance with (17.14) $u_k(\zeta)$ will be expressed as follows:

$$u_k(\zeta) = \sum_{n=1}^{\infty} \frac{2\int_0^l L_k(\zeta)\varphi_n d\zeta}{\lambda_n' - \varrho F_0 \omega_c^2 k^2 l^4}\varphi_n, \tag{17.17}$$

and because of (17.4) and (17.9) $u_1(\zeta, \tau)$ will be represented by the double series:

$$u_1(\zeta, \tau) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left\{ \frac{2\int_0^1 L_k^s(\zeta)\varphi_n d\zeta}{\lambda_n - \varrho F_0 \omega_c^2 k^2 l^4}\varphi_n \sin k\tau + \right.$$

$$\left. + \frac{2\int_0^1 L_k^c(\zeta)\varphi_n d\zeta}{\lambda_n - \varrho F_0 \omega_c^2 k^2 l^4}\varphi_n \cos k\tau \right\}. \tag{17.18}$$

The functions $L_k^s(\zeta)$ and $L_k^c(\zeta)$, entering into the equation (17.18), represent the coefficients of the Fourier series expansion of the function $L(\zeta, \tau)$. The coefficient of $\sin \tau$ is determined by the formula:

$$L_1^s(\zeta) = \frac{1}{\pi}\oint L(\zeta, \tau)\sin\tau\, d\tau.$$

Taking into account (17.1) and also keeping in mind that in accordance with (16.9), (16.13), and (16.14),

$$\vec{\Phi}\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)=\frac{-E\nu a^n}{\varepsilon n l^n}\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\left[(1+\cos\tau)^n-2^{n-1}\right]\iint_F z^{n+1}\,dz\,dy, \tag{17.19}$$

$$\overleftarrow{\Phi}\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)=\frac{E\nu a^n}{\varepsilon n l^n}\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\left[(1-\cos\tau)^n-2^{n-1}\right]\iint_F z^{n+1}\,dz\,dy, \tag{17.20}$$

we represent $L_1^s(\zeta)$ in the form

$$L_1^s(\zeta)=\frac{1}{\pi}\left\{\int_0^{2\pi} l^4\varrho F_0\Delta_1 a\varphi\cos\tau\sin\tau\,d\tau+\int_0^{2\pi} l^4\varrho F_0\zeta\omega_c^2\beta\cos(\tau-\psi_0)\sin\tau\,d\tau+\right.$$
$$+\frac{1}{l^{n-1}}\frac{E\nu a^n}{\varepsilon n}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\int_0^{2\pi}\left[(1+\cos\tau)^n-2^{n-1}\right]\iint_F z^{n+1}\,dz\,dy\sin\tau\,d\tau-$$
$$\left.-\frac{E\nu a^n}{\varepsilon n}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta}\right)^n\cdot\int_0^{\pi}\left[(1-\cos\tau)^n-2^{n-1}\right]\iint_F z^{n+1}\,dz\,dy\sin\tau\,d\tau\right]\right\}.$$

After integrating we obtain

$$L_1^s(\zeta)=l^4\varrho F_0\omega_c^2\beta\zeta\sin\psi_0+\frac{(n-1)a^nE\nu}{\varepsilon n(n+1)\pi l^{n-1}}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\iint_F z^{n+1}\,dz\,dy. \tag{17.21}$$

The coefficients of $\sin k\tau$ will be determined as follows:

$$L_k^{(s)}(\zeta)=\frac{1}{\pi}\oint L(\zeta,\ \tau)\sin k\tau\,d\tau=\frac{l^4}{\pi}\left\{\int_0^{2\pi}\varrho F_0\Delta_1 a\varphi\cos\tau\sin k\tau\,d\tau+\right.$$
$$\left.+\int_0^{2\pi}\varrho F_0\omega_c^2\beta\zeta\cos(\tau-\psi_0)\sin k\tau\,d\tau\right\}+\frac{E\nu a^n}{\varepsilon n\pi l^{n-1}}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\left\{\int_0^{2\pi}\left[(1+\cos\tau)^n-\right.\right.$$
$$\left.\left.-2^{n-1}\right]\sin k\tau\,d\tau-\int_0^{\pi}\left[(1-\cos\tau)^n-2^{n-1}\right]\sin k\tau\,d\tau\right\}\iint_F z^{n+1}\,dz\,dy.$$

Integrating the right side of the last ecuation we find that for $k=2i$

(17.22)

and for

$$k = 2i + 1$$

$$L_{2i+1}^{(s)}(\zeta) = \frac{E\nu a^n 2^{n+2} l^{1-n}}{\varepsilon\pi n(2i+1)} \frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right] \iint z^{n+1}\,dz\,dy \times$$
$$\times \left\{1 - \frac{2i+1}{2^n}\int_0^{\pi} (1-\cos\tau)^n \sin(2i+1)\tau\,d\tau\right\}, \tag{17.23}$$

where i = 1, 2, 3, 4 . . .

We now determine the coefficients of the cosines. The coefficient of $\cos\tau$ is determined from the formula:

$$L_1^{(c)}(\zeta) = \frac{1}{\pi}\oint L(\zeta,\tau)\cos\tau\,d\tau = \frac{l^4}{\pi}\left\{\int_0^{2\pi} \varrho F_0 \Delta_1 a\varphi \cos^2\tau\,d\tau + \right.$$
$$\left. + \int_0^{2\pi} \varrho F_0 \zeta\omega_c^2\beta\cos(\tau-\psi_0)\cos\tau\,d\tau\right\} + \frac{E\nu a^n}{\varepsilon n\pi}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right] \times$$
$$\times l^{1-n}\iint_F z^{n+1}\,dz\,dy\left\{\int_0^{2\pi}\left[(1+\cos\tau)^n - 2^{n-1}\right]\cos\tau\,d\tau - \right.$$
$$\left. - \int_0^{\pi}\left[(1-\cos\tau)^n - 2^{n-1}\right]\cos\tau\,d\tau\right\}.$$

Integrating we find

$$L_1^{(c)}(\zeta) = l^4(\varrho F_0\Delta_1 a\varphi + \varrho F_0\zeta\omega_c^2\beta\cos\psi_0) - \frac{2E\nu a^n}{\varepsilon n\pi}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right] \times$$
$$\times l^{1-n}\iint_F z^{n+1}\,dz\,dy \int_0^{\pi}(1-\cos\tau)^n\cos\tau\,d\tau. \tag{17.24}$$

The coefficient of $\cos k\tau$ is found by the formula

$$L_k^{(c)}(\zeta) = \frac{1}{\pi}\oint L(\zeta, \tau)\cos k\tau\, d\tau = \frac{l^4}{\pi}\left\{ \int_0^{2\pi} \varrho F_0 \Delta_1 a\varphi \cos\tau \cos k\tau\, d\tau + \right.$$

$$\left. + \int_0^{2\pi} \varrho F_0 \zeta \omega_c^2 \beta \cos(\tau - \psi_0)\cos k\tau\, d\tau \right\} +$$

$$+ l^{1-n}\frac{E\nu a^n}{\varepsilon\pi n}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\left\{ \int_\pi^{2\pi} [(1+\cos\tau)^n - 2^{n-1}]\cos k\tau\, d\tau - \right.$$

$$\left. - \int_0^{\pi} [(1-\cos\tau)^n - 2^{n-1}]\cos k\tau\, d\tau \right\} \iint_F z^{n+1}\, dz\, dy.$$

(17.25)

By integrating we find that for $k = 2i$

$$L_k^{(c)}(\zeta) = 0$$

and for $k = 2i+1$

$$L_{2i+1}^{(c)}(\zeta) = \left\{ -\frac{2E\nu a^n}{l^{n-1}\varepsilon n\pi}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right] \times \right.$$

$$\left. \times \int_0^{\pi} (1-\cos\tau)^n \cos(2i+1)\tau\, d\tau \right\} \iint_F z^{n+1}\, dz\, dy.$$

(17.26)

On the basis of the formulas obtained, the series expansion of the function $L(\zeta, \tau)$ is finally represented as the series

$$L(\zeta, \tau) = \left\{ l^4 \varrho F_0 \omega_c^2 \beta\zeta \sin\psi_0 + \frac{(n-1)Ea^n\nu}{\varepsilon n(n+1)\pi}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\overline{I} \right\}\sin\tau +$$

$$+ \left\{ l^4(\varrho F_0 \Delta_1 a\varphi + \varrho F_0 \zeta\omega_c^2\beta\cos\psi_0) - \frac{2E\nu a^n}{\varepsilon\pi n}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\overline{I}\int_0^{\pi}(1-\cos\tau)^n \times \right.$$

$$\left. \times \cos\tau\, d\tau \right\}\cos\tau + \sum_{i=1}^{\infty}\left\{ \frac{E\nu a^n 2^{n+2}}{\varepsilon n(2i+1)}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\overline{I}\left[1 - \frac{2i+1}{\pi 2^n} \times \right.\right.$$

$$\times \int_0^\pi (1-\cos\tau)^n \sin(2i+1)\,\tau\, d\tau\Big]\Big\}\sin(2i+1)\,\tau - \sum_{i=1}^{\infty}\Big\{\frac{2E\nu a^n}{\varepsilon n\pi}\frac{d^2}{d\zeta^2}\times$$
$$\times\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\bar{I}\int_0^\pi (1-\cos\tau)^n\cos(2i+1)\,\tau\, d\tau\Big\}\cos(2i+1)\,\tau, \qquad (17.27)$$

where

$$\bar{I} = l^{1-n}\iint_F z^{n+1}\,dz\,dy.$$

The coefficients $a_n^{(k)}$ of the expansion of the function $L(\zeta)$ in the fundamental functions $\varphi_n(\zeta)$ are determined from the following expression, in accordance with formula (17.20):

$$a_{1,s}^{(1)} = 2\int_0^1 L_1^s(\zeta)\varphi_1(\zeta)\,d\zeta = 2\int_0^1 l^4\varrho F_0\omega_{c}^2\beta\zeta\sin\psi_0\varphi_1\,d\zeta +$$
$$+2\int_0^1 \frac{(n-1)Ea^n\nu}{\varepsilon n(n+1)\pi}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\bar{I}\varphi_1\,d\zeta.$$

From (16.8) it follows that

$$l^4\varrho F_0\omega_{c}^2\beta\sin\psi_0\int_0^1 \zeta\varphi(\zeta) = l\oint\int_0^1 \Phi\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\frac{d^2\varphi}{d\zeta^2}\sin\tau\,d\zeta\,d\tau =$$
$$= \frac{E\nu a^{n+2}}{n(n+1)}\int_0^1\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^{n+1}\bar{I}\right]d\zeta.$$

Then

$$a_{1,s}^{(1)} = 2\int_0^1 \frac{(n-1)Ea^n\nu}{\varepsilon n(n+1)\pi}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\bar{I}\varphi_1\,d\zeta -$$
$$-\int_0^1 \frac{(n-1)Ea^n\nu}{\varepsilon n(n+1)\pi}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\bar{I}\varphi_1\,d\zeta = 0; \qquad (17.28)$$

$$a^{(1)}_{1,c}=2\int_0^1 L_1^c(\zeta)\,\varphi\,d\zeta=2\int_0^1\Big\{l^4(\varrho F_0\Delta_1 a\varphi+\varrho F_0\zeta\omega_c^2\beta\cos\psi_0)-$$

$$-\frac{2E\nu a^n}{\varepsilon n\pi}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\bar{I}\int_0^\pi(1-\cos\tau)^n\cos\tau\,d\tau\Big\}\varphi_1\,d\zeta.$$

On the basis of (16.9) it follows that

$$l^4(\varrho F_0\beta\omega_c^2\pi\cos\psi_0\int_0^1\zeta\varphi\,d\zeta+\varrho F_0 a\Delta_1\pi\int_0^1\varphi^2\,d\zeta)=$$

$$=l\oint\int_0^1\Phi\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right)\frac{d^2\varphi}{d\zeta^2}\cos\tau\,d\zeta\,d\tau=$$

$$=\frac{2E\nu a^n}{\varepsilon n}\int_0^1\int_0^\pi(1-\cos\tau)^n\left(\frac{d^2\varphi}{d\zeta^2}\right)^{n+1}\bar{I}\cos\tau\,d\zeta\,d\tau.$$

Substituting the last expression into the preceding formula, we obtain

$$a^{(1)}_{1,c}=2\int_0^1 L_1^{(c)}(\zeta)\varphi_1(\zeta)\,d\zeta=2\int_0^1\frac{2Ea^n}{\varepsilon n\pi}\int_0^\pi(1-\cos\tau)^n\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\bar{I}\,d\zeta\,d\tau-$$

$$-2\int_0^1\frac{2Ea^n}{\varepsilon n\pi}\int_0^\pi(1-\cos\tau)^n\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\varphi_1\bar{I}\,d\zeta=0.$$

(17.29)

For the values of $n > 1$ the coefficient of the first harmonic is

$$a^{(1)}_{n,s}=2\int_0^1 L_1^s(\zeta)\varphi_n\,d\zeta=2\int_0^1 l^4\varrho F_0\omega_c^2\beta\zeta\sin\psi_0\varphi_n\,d\zeta+$$

$$+2\int_0^1\frac{(n-1)Ea^n\nu}{\varepsilon n(n+1)\pi}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\bar{I}\varphi_n\,d\zeta;$$

$$a^{(1)}_{n,c}=2\int_0^1 L_1^c(\zeta)\varphi_n\,d\zeta=2\int_0^1\Big\{l^4(\varrho F_0\Delta_1 a\varphi+\varrho F_0\zeta\omega_c^2\beta\cos\psi_0)-$$

$$-\frac{2E\nu a^n}{\varepsilon n\pi}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\bar{I}\int_0^\pi(1-\cos\tau)^n\cos\tau\,d\tau\Big\}\varphi\,d\zeta.$$

(17.30)

By virtue of the orthogonality of the functions φ_n, the first integral of the last expression becomes zero and we obtain

$$a_{n,c}^{(1)}=2\int_0^1\left\{\mu\rho F_0\omega_c^2\beta\zeta\cos\psi_0-\frac{2Era^n}{\varepsilon n\pi}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\bar{I}\times\right.$$
$$\left.\times\int_0^\pi(1-\cos\tau)^n\cos\tau\,d\tau\right\}\varphi_n\,d\zeta. \qquad (17.31)$$

In accordance with equations (17.22) and (17.27), we have

$$a_{n,s}^{(2i)}=2\int_0^1 L_{2i}^{(s)}(\zeta)\varphi_n\,d\zeta=0,$$
$$a_{n,c}^{(2i)}=2\int_0^1 L_{2i}^{(c)}(\zeta)\varphi_n\,d\zeta=0, \qquad (17.32)$$

and on the basis of (17.23) and (17.26)

$$a_{n,s}^{(2i+1)}=2\int_0^1\frac{Era^n2^{n+2}}{\varepsilon n(2i+1)}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\bar{I}\left[1-\frac{2i+1}{2^n}\times\right.$$
$$\left.\times\int_0^\pi(1-\cos\tau)^n\sin(2i+1)\tau\,d\tau\right]\varphi_n\,d\zeta;$$
$$a_{n,c}^{(2i+1)}=2\int_0^1-\frac{2Era^n}{\varepsilon n\pi}\frac{d^2}{d\zeta^2}\left[\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\right]\int_0^\pi(1-\cos\tau)^n\times$$
$$\times\cos(2i+1)\tau\,d\tau\,\bar{I}\Big\}\varphi_n\,d\zeta. \qquad (17.33)$$

The coefficients of the expansion of the function $u_n(\zeta)$ in fundamental functions (in accordance with formula (17.11)) equals:

$$\left[c_{n,s}^{(k)}\right]_{\substack{k=1\\n=1}} = \frac{a_{1,s}^{(1)}}{\lambda_1' - l^4 \varrho F_0 \omega_c^2} = 0,$$

$$\left[c_{n,c}^{(k)}\right]_{\substack{k=1\\n=1}} = \frac{a_{1,c}^{(1)}}{\lambda_1' - l^4 \varrho F_0 \omega_c^2} = 0,$$

$$\left[c_{n,s}^{(k)}\right]_{\substack{k=1\\n>1}} = \frac{a_{1,s}^{(1)}}{\lambda_n' - l^4 \varrho F_0 \omega_c^2},$$

$$\left[c_{n,c}^{(k)}\right]_{\substack{k=1\\n>1}} = \frac{a_{1,c}^{(1)}}{\lambda_n' - l^4 \varrho F_0 \omega_c^2}. \tag{17.34}$$

From (17.32) and (17.33) we have

$$\left[c_{n,s}^{(k)}\right]_{k=2i} = \frac{a_{k,s}^{(2i)}}{\lambda_n' - l^4 \varrho F_0 (2i)^2 \omega_c^2} = 0;$$

$$\left[c_{n,c}^{(k)}\right]_{k=2i} = \frac{a_{k,c}^{(2i)}}{\lambda_n' - l^4 \varrho F_0 (2i)^2 \omega_c^2} = 0;$$

$$\left[c_{n,s}^{(k)}\right]_{k=2i+1} = \frac{a_{n,s}^{(2i+1)}}{\lambda_n' - l^4 \varrho F_0 (2i+1)^2 \omega_c^2};$$

$$\left[c_{n,c}^{(k)}\right]_{k=2i+1} = \frac{a_{n,c}^{(2i+1)}}{\lambda_n' - l^4 \varrho F_0 (2i+1)^2 \omega_c^2}. \tag{17.35}$$

From (17.9), (17.34), and (17.5) it follows that

$$u_0(\zeta) = 0; \quad u_{1,1}^{(s)}(\zeta) = 0; \quad u_{1,1}^{(c)}(\zeta) = 0,$$
$$u_{n,2i}^{(s)}(\zeta) = 0; \quad u_{n,2i}^{(c)}(\zeta) = 0;$$
$$u_{1,n}^{(s)}(\zeta) = c_{n,s}^{(1)} \varphi_i; \quad u_{1,n}^{(c)}(\zeta) = c_{n,c}^{(1)} \varphi_n;$$
$$u_{(2i+1),n}^{(s)}(\zeta) = c_{n,s}^{(2i+1)} \varphi_n; \quad u_{(2i+1),n}^{(c)}(\zeta) = c_{n,c}^{(2i+1)} \varphi_n. \tag{17.36}$$

From these equations it is possible to form the expression for $u(\zeta, \tau)$, namely:

$$u_1(\zeta, \tau) = \sum_{n=2}^{\infty} \left[\frac{\varphi_n(\zeta)}{\lambda_n' - l^4 \varrho F_0 \omega_c^2} (a_{n,s}^{(1)} \sin \tau + a_{n,c}^{(1)} \cos \tau) \right] + \\ - \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{c_{n,s}^{(2i-1)} \varphi_n}{\lambda_n' - l^4 \varrho F_0 \omega_c^2 (2i-1)^2} \varphi_n(\zeta) \sin (2i-1) \tau + \right. \\ \left. + \frac{c_{n,c}^{(2i+1)} \varphi_n(\zeta)}{\lambda_n' - l^4 \varrho F_0 \omega_c^2 (2i+1)^2} \varphi_n(\zeta) \cos (2i+1) \tau \right].$$

(17.37)

Based on (14.3) the value of the deflections in the first approximation can be represented by

$$u(\zeta, \tau) = a \varphi(\zeta) \cos \tau + \epsilon u_1(\zeta, \tau).$$

(17.38)

In using the formula (17.38) the expression for $\phi(\zeta)$ must be taken from formula (15.37) and $u_1(\zeta, \tau)$ from formula (17.37). The small parameter, ϵ, which multiplies u_1 will cancel because it enters in the denominators of all the terms of the expression for $u_1(\zeta, \tau)$.

As a result of the investigation of the forced transverse vibrations of a cantilevered bar of variable cross-section considering energy dissipation in the material by using the methods of the theory of perturbations, we have approximate formulas which permits us to determine the magnitude of the deflection of the bar, the frequency of vibrations and the magnitude of the phase shift of the vibrations. Based on formula (17.7) and (17.10) it is possible to construct a resonance curve for a bar of variable section made from any material. The value of the constants η and ν entering into the formulas must be determined by experiment.

Knowing the function $u_1(\zeta, \tau)$, the problem can be solved in the second approximation. For that it would be necessary to consider equation (14.8) obtained by setting equal to zero the coefficient of the small parameter to the second degree in equation (14.2) after substituting (14.3) and (14.5) into it. Here, the terms containing I_0 left from equation (14.7) after separating out the terms containing $\varepsilon I_1(x)$ for solving the problem at the first approximation must be considered.

In the present instance we shall confine ourselves to solving the problem in the first approximation, keeping in mind that the precision obtained is entirely sufficient, as was shown in the previous chapters.

18. The construction of the resonance curve

Let us construct a resonance curve for a bar with the following data (Fig. 9, 10):

$$l = 40 \text{ см},$$
$$F(x) = 3\left(1{,}5 - \frac{x}{40}\right),$$
$$I(x) = \frac{(60-x)^3}{4^4 \cdot 10^3},$$
$$E = 2{,}08 \cdot 10^6 \text{ кг/см}^2.$$

In the sample calculation we shall use the coordinates x, u. First of all we shall determine the frequency of the free vibrations of the bar of variable cross-section in the zeroth approximation. According to the formulas (14.13) and (14.14) the averaged values of the moment of inertia and the area of the cross-section will have the following values:

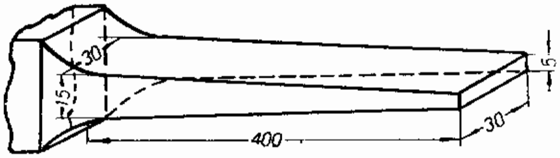

Fig. 9

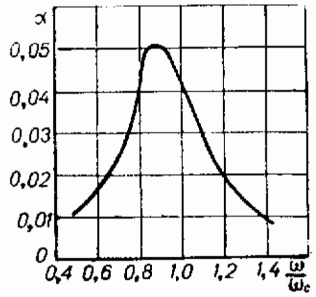

Fig. 10

$$I_0 = \frac{1}{40}\int_0^{40} \frac{1}{4}\left(1{,}5 - \frac{x}{40}\right)^3 dx = \frac{5}{16}\, см^4,$$

$$F_0 = \frac{1}{40}\int_0^{40} \left(1{,}5 - \frac{x}{40}\right) dx = 3\, см^2.$$

The roots of equation (15.17), using a coordinate x instead of ζ, equals

$$k_1 l = 1{,}8751;\ k_2 l = 4{,}6941;\ k_3 l = 7{,}8548;\ k_4 l = 10{,}9955.$$

Then for $l = 40$ cm we have

$$k_1 = 0{,}0468775;\ k_2 = 0{,}1173525;\ k_3 = 0{,}19637;\ k_4 = 0{,}274888$$

and correspondingly

$$k_1^4 = 4{,}82901 \cdot 10^{-6};\quad k_2^4 = 189{,}66 \cdot 10^{-6};\quad k_3^4 = 1486{,}97 \cdot 10^{-6};\quad k_4^4 = 5709{,}83 \cdot 10^{-6}.$$

In accordance with (15.17)

$$k_n = \frac{EI_0 k_n^4}{F_0}$$

and, therefore, from

$$\lambda_n = \frac{EI_0 k_n^4}{F_0}$$

we obtain

$$\lambda_1 = 1{,}0465;\ \lambda_2 = 41{,}093;\ \lambda_3 = 322{,}184;\ \lambda_4 = 1237{,}166.$$

The circular frequency will be determined by the expression

$$\omega_n^2 = \lambda_n \cdot \frac{1}{\varrho},$$

where

$$\rho = \frac{\gamma}{g} = \frac{7{,}85 \cdot 10^{-3}}{981} = 8 \cdot 10^{-6}\ \text{Kg sec}^2/\text{CM}^4.$$

In the zeroth approximation the square of the circular frequency equals

$$\omega_1^2 = \frac{1{,}0465 \cdot 10^6}{8} = 130\,800.$$

We now determine the natural frequency of vibrations of the bar of variable cross-section in the first approximation. To do this it is necessary to find the second term of the series (14.12); that is, the first correction term, $\varepsilon\mu$. The determination of this quantity, in accordance with formula (15.19), is connected with the preliminary calculation of the value of m_n , determined by the formula

$$m_n = \int_0^l \frac{I_1(x)}{I_0} \left[\frac{d^2\varphi}{dx^2}\right]^2 dx - k_n^4 \int_0^l \frac{F(x)}{F_0} \varphi_n^2(x)\, dx. \tag{18.1}$$

In accordance with (14.13) and (14.14), we make the following substitution:

$$\begin{aligned} \varepsilon I_1(x) &= I(x) - I_0, \\ \varepsilon F_1(x) &= F(x) - F_0. \end{aligned} \tag{18.2}$$

We transform (18.1) to

$$\varepsilon m_n = \int_0^l \frac{I(x)}{I_0}\left[\frac{d^2\varphi}{dx^2}\right]^2 dx - k_n^4 \int_0^l \frac{F(x)}{F_0}\varphi_n^2\, dx. \tag{18.3}$$

Knowing the value of εm_n, on the basis of (15.19), we obtain

$$\varepsilon\mu = \frac{EI_0 \varepsilon m_n}{F_0},$$

where n = 1, 2, 3 . . .

Keeping in mind the expression for $\varphi_n(x)$, the deflection function of the bar (normalized in our case),

$$\varphi_n(x) = -\frac{1}{\sqrt{l}}(\cos k_n x - \operatorname{ch} k_n x) - \frac{1}{\sqrt{l}}\frac{\sin k_n l - \operatorname{sh} k_n l}{\cos k_n l + \operatorname{ch} k_n l}(\sin k_n x - \operatorname{sh} k_n x)$$

and carrying out the.calculations we obtain the final results for $\varepsilon\mu$, given in Table 5.

We now turn to the determination of the correction $\varepsilon^2 k$ required for the calculation in the second approximation of the eigenvalue λ in accordance with (15.3).

For this, it is essential to determine beforehand the first correction term $\varepsilon v(x)$ for the deflection function of the bar obtained in the first approximation.

Table 5

n	$k_n^4 \cdot 10^6$	$\int_0^l \frac{I(x)}{I_0}\left(\frac{d^2\varphi}{dx^2}\right)^2 dx$	$\int_0^l \frac{F(x)}{F_0}\varphi_n^2\, dx$	λ_n	$\varepsilon m_n \cdot 10^6$	$\varepsilon\mu_n$
1	4,83	8,866	0,693345	1,05	5,6174	1,21710
2	189,66	229,933	0,905815	41,09	58,1351	12,59618
3	1486,97	1596,574	0,968417	322,84	156,5786	33,92507
4	5709,83	5933,878	0,982220	1237,17	325,5690	70,53906

According to formula (15.26) we have

$$v_n(x)=\sum_{j=1}^{\infty}{}' \frac{\varphi_j(x)}{k_n^4-k_j^4}\left\{k_n^4\int_0^l \frac{F_1(x)}{F_0}\varphi_n(x)\varphi_j(x)\,dx - \right.$$
$$\left. -\int_0^l \frac{d^2}{dx^2}\left[\frac{I_1(x)}{I_0}\frac{d^2\varphi_n}{dx^2}\right]\varphi_j(x)\,dx\right\}.$$

Substituting in the last formula in place of $I_1(x)$ and $F_1(x)$ their values in accordance with formulas (18.2) and effecting the necessary transformations, we obtain

$$\varepsilon v(x)=\sum_{j=1}^{\infty}{}' \frac{\varphi_j(x)}{k_n^4-k_j^4}\left\{k_n^4\int_0^l \frac{F(x)}{F_0}\varphi_n(x)\varphi_j(x)\,dx - \right.$$
$$\left. -\int_0^l \frac{I(x)}{I_0}\frac{d^2\varphi_n}{dx}\frac{d^2\varphi_j}{dx^2}\,dx\right\}. \tag{18.4}$$

Calculating the various quantities entering into the right side of the equation we obtain for different n and j the values of the integrals given in Table 6.

Table 6

n	j	$k_n\cdot 10^6$	$\frac{1}{k_n^4-k_j^4}$	$\int_0^l \frac{F(x)}{F_0}\varphi_n\varphi_j\,dx$	$\int_0^l \frac{I(x)}{I_0}\frac{d^2\varphi_n}{dx^2}\frac{d^2\varphi_j}{dx^2}\,dx$
2	1	189,66	5410,35	15,35298	1,66941
3	2	1486,97	770,826	18,82911	29,44116
4	3	5709,83	236,806	22,27678	155,63820
1	4	4,83	− 175,285	0,79574	0,93414
2	4	—	− 181,154	− 1,78092	17,28453
3	1	—	674,700	− 2,09226	1,31172

The values of $\varepsilon v(x)$, for different n will equal (values n are shown as indices on v)

$$\varepsilon v_1(x) = 0{,}0863097\varphi_2(x) + 0{,}0089183\varphi_3(x) + 0{,}00163067\varphi_4(x);$$
$$\varepsilon v_2(x) = 0{,}0672201\varphi_1(x) + 0{,}199413\varphi_3(x) + 0{,}0319235\varphi_4(x);$$
$$\varepsilon v_3(x) = 0{,}01214056\varphi_1(x) - 0{,}0111217\varphi_2(x) + 0{,}290119\varphi_4(x);$$
$$\varepsilon v_4(x) = 0{,}00632876\varphi_1(x) - 0{,}0497327\varphi_2(x) - 0{,}0673514\varphi_3(x).$$

Having the expressions for εv_n at our disposal, we can go on to determining the corrections $\varepsilon \varkappa$ for the second approximation. Starting from formula (15.32) and having in mind formula (18.2) we will form expressions to determine p_n.

$$\varepsilon^2 p = -\int_0^l \frac{I(x)}{I_0}\frac{d^2 v_n}{dx^2}\frac{d^2\varphi_n}{dx^2}dx + k_n^4\int_0^l \frac{F(x)}{F_0}v_n\varphi_n\,dx - m\int_0^l \frac{F(x)}{F_0}\varphi_n^2\,dx + m_n. \tag{18.5}$$

The results of calculations of the quantities entering in the last formula are listed in Table 7.

Table 7

n	$\int_0^l \frac{I(x)}{I_0}\varepsilon\frac{d^2\varphi_n}{dx^2}\frac{d^2 v_n}{dx^2}dx$	$\int_0^l \frac{F(x)}{F_0}\varepsilon\varphi_n v_n\,dx$	$1-\int_0^l \frac{F(x)}{F_0}\varphi_n^2\,dx$	$\varepsilon^2 p_n \cdot 10^6$
1	−1,5731	0,13078	0,30655	0,2121
2	−65,4262	0,48266	0,94185	−50,7966
3	3,66334	0,54582	0,03158	−235,7257
4	112,8811	−0,11813	0,01778	−51,2200

Using the value of $\varepsilon^2 p_n$, on the basis of formula (15.29), we shall find the quantity $\varepsilon^2\varkappa$. As we are interested only in the first frequency, we give in Table 8 the first eigenvalue, λ, calculated in the second approximation.

Table 8

n	λ_n	$\varepsilon\mu_n$	$\varepsilon^2 x$	λ
1	1,0465	1,2171	0,0460	2,2996

On the basis of Table 8 the square of the frequency of the free vibrations of the bar of variable cross-section is:

$$\omega^2 = \frac{\lambda}{\varrho} = \frac{2{,}299558}{8\cdot 10^{-6}} = 287445. \tag{18.6}$$

Therefore,

$$\omega = 536.$$

The frequency is equal to

$$\bar{\omega} = \frac{\omega}{2\pi} = 85{,}5 \text{ гц}^{*}.$$

For constructing the resonance curve it is necessary also to determine in the second approximation the values of the deflection function of the rod in question. For this purpose we calculate the third term of the expansion (14.11) of the deflection function w_n, which is expressed by the formula:

$$w_n = \sum{}' \frac{\varphi_j}{k_n^4 - k_j^4} \left\{ \int_0^l \frac{I(x)}{I_0} \frac{d^2 v_n}{dx^2} \frac{d^2 \varphi_j}{dx^2} dx - \right.$$
$$\left. - k_n^4 \int_0^l \frac{F(x)}{F_0} v_n \varphi_j dx - \varepsilon m \int_0^l \frac{F(x)}{F_0} \varphi_n \varphi_j dx \right\}.$$

---

*The value of the frequency determined experimentally amounted to 88 cycles/sec.

Using this formula we present the values of $\varepsilon^2 w_n$, calculated for various n, (the values of n are indicated as indices on w):

$$\varepsilon^2 w_1(x) = -0{,}116349\varphi_2(x) - 0{,}284610\varphi_3(x) - 0{,}00674764\varphi_4(x);$$

$$\varepsilon^2 w_2(x) = -0{,}0729895\varphi_1(x) - 0{,}0247867\varphi_3(x) - 0{,}0882455\varphi_4(x);$$

$$\varepsilon^2 w_3(x) = -0{,}00990945\varphi_1(x) - 0{,}00599775\varphi_2(x) - 0{,}272287\varphi_4(x);$$

$$\varepsilon^2 w_4(x) = 0{,}00158542\varphi_1(x) + 0{,}0536325\varphi_2(x) + 0{,}055768\varphi_3(x).$$

Knowing the quantities $\varphi_n(x)$, $\varepsilon v(x)$ and $\varepsilon^2 w(x)$, we represent the deflection function of the bar of variable cross-section in the second approximation (without taking account of the dissipation of energy in the material) by the series:

$$\varphi(x) = \varphi_n(x) + \varepsilon v_n(x) + \varepsilon^2 w_n(x). \tag{18.7}$$

Knowing the magnitude of the free frequency of vibration from (18.6) and the deflection function $\varphi(x)$, we make a resonance curve starting from formula (16.20) which can be represented in the form:

$$\left(\frac{\omega}{\omega_c}\right)^2 = 1 + \frac{\theta_0 \int\limits_0^l x\varphi(x)\,dx}{a \int\limits_0^l \varphi_2(x)\,dx} \left[\frac{(n+1)\sin\psi_0}{2^{n+1}(n-1)} \times \right.$$

$$\left. \times \int\limits_0^\pi (1-\cos\tau)^n \cos\tau\, d\tau - \cos\psi_0 \right], \tag{18.8}$$

where

$$\varphi(x) = \varphi_n + \varepsilon v_n(x) + \varepsilon^2 w_n(x) = \varphi_1(x) + \varepsilon v_1(x) + \varepsilon^2 w_1(x).$$

Keeping in mind the equations

$$\varepsilon v_1(x) = -0{,}0863097\varphi_2(x) - 0{,}00891834\varphi_2(x) - 0{,}00163067\varphi_4(x);$$

$$\varepsilon^2 w_1(x) = -0{,}116349\varphi_2(x) - 0{,}28461\varphi_3(x) - 0{,}00674764\varphi_4(x),$$

for the deflection of a cross-section of the bar $\varphi(x)$ we obtain the following expression:

$$\varphi(x) = \varphi_1(x) - 0{,}20659\varphi_2(x) - 0{,}0373793\varphi_3(x) - 0{,}00837831\varphi_4(x).$$

Taking as in the case of vibrations of a bar of constant cross-section

$$\nu = 18{,}6;\ n = 2 \quad \text{and} \quad \theta_0 = 0{,}000216$$

and the length of the bar $l = 40$ cm, we find the values of integrals entering into formula (18.8). We obtain on calculation:

$$-\int_0^\pi (1-\cos\tau)^2 \cos\tau \, d\tau = \pi;$$

$$\frac{\int_0^l x\varphi(x)\,dx}{\int_0^l \varphi^2(x)\,dx} = 124{,}3063.$$

Substituting the values of the integrals obtained and the magnitude of the amplitude of the angle of rotation of the fixed section θ_0 into the equation (18.8) we shall finally have

$$\left(\frac{\omega}{\omega_c}\right)^2 = 1 + \frac{0{,}0268502}{a\cdot 2{,}694379}(1{,}178097\sin\psi_0 - \cos\psi_0). \tag{18.9}$$

For the construction of the resonance curve it is necessary to consider, in addition to formula (18.9), the formula for the determination of the sine of the phase shift angle

$$\sin\psi = \frac{\nu a^n 2^{n+2} E}{n(n+1)J}\int_0^l \left\{\left(\frac{d^2\varphi(x)}{dx^2}\right)^{n+1} \iint_F z^{n+1}\,dz\,dy\right\} dx, \tag{18.10}$$

where

$$J = \varepsilon\varrho\omega_c^2 F_0 \beta\pi \int_0^1 \zeta\varphi\, d\zeta.$$

For the dimensions of the bar of variable cross-section which have been chosen,

$$h_x = \frac{60-x}{40}, \quad b = 3 \text{ см} \quad \text{и} \quad F_0 = 4{,}5 \text{ см}^2$$

the expression of the double integral entering in formula (18.10) becomes

$$\iint_F z^{n+1}\,dz\,dy = \frac{2b}{n+2}\left(\frac{60-x}{80}\right)^{n+2} = \frac{6(60-x)^4}{4\cdot 80^4}.$$

The value of the outer integral in equation (18.10) is calculated approximately by using Simpson's formula. As a result of these calculations we find

$$\int^l \left(\frac{d^2\varphi}{dx^2}\right)^{n+1}(60-x)^{n+2}\,dx = \int_0^{40}\left(\frac{d^2\varphi}{dx^2}\right)^3 (60-x)^4\,dx = -2\cdot 2{,}533964.$$

Substituting now all the known quantities into formula (18.10) and keeping in mind that $J_3 = \varepsilon\varrho\omega_c^2 F_0\beta\pi\int_0^l xy\,dx,$

we shall obtain the expression for $\sin\psi_0$ as a function of amplitude a.

$$\sin\psi_0 = -52{,}451025\, a^n = -52{,}451025\, a^2. \tag{18.11}$$

The function $\varphi(x)$, contained in the expression for $\sin\psi_0$, consists of the normalized functions and does not satisfy the condition at the end of the bar with regard to deflection. Therefore, when determining the true magnitudes of the amplitudes of vibrations a as functions of the frequency of excitation, it is necessary to introduce in the expression for $\sin\psi_0$ a coefficient found from equating the magnitude of maximum deflection at the end of the bar to the magnitude of the amplitude a, that is:

$$[u(x,\tau)]_{\substack{x=l\\ \tau=0}} = a. \tag{18.12}$$

As is known,

$$u(x,\tau) = \varphi(x)\, a \cos\tau$$

and for the function $\varphi(x)$, consisting of a certain series of the normalized functions φ_n, we have

$$u(x,\tau)_{\substack{x=40\\ \tau=0}} = 0{,}371143\, a.$$

Therefore, the constant coefficient by which the deflection function should be multiplied to satisfy conditions (18.12) at the end of the bar is

$$\alpha = \frac{1}{0{,}371143} = 2{,}694379.$$

Multiplying the coefficient a^n by $\alpha^n = 7{,}2597$ (for $n = 2$) we will obtain an expression for $\sin\psi$ corresponding in structure to formula (18.10), namely:

$$\sin\psi_0 = -380{,}777552\ a^2.$$

To construct the resonance curve we use formulas (18.9) and (18.13). As a result of calculations we find the values of the relative amplitudes of vibration a of the bar of variable cross-section as a function of the ratio of the frequency of the external disturbing force to the frequency of free vibrations, $\frac{\omega}{\omega_c}$. These values are given in Table 9.

Table 9

a	$\left(\frac{\omega}{\omega_c}\right)^2_л$	$\left(\frac{\omega}{\omega_c}\right)_л$	$\left(\frac{\omega}{\omega_c}\right)^2_п$	$\left(\frac{\omega}{\omega_c}\right)_п$
0,01	—	—	1,951105	1,397
0,02	0,418608	0,647	1,402578	1,184
0,03	0,553829	0,744	1,177951	1,085
0,04	0,623629	0,789	1,018742	1,009
0,045	0,660041	0,812	0,937626	0,968
0,048	0,685791	0,828	0,885055	0,941
0,05	0,714774	0,845	0,838192	0,915
0,05125	0,770926	0,878	0,770926	0,878

The curve of the resonance is plotted in Fig. 10 in accordance with the tabular data. The character of the curve indicates the substantial nonlinearity of the present vibrating system in the analysis of which energy dissipation was taken into account.

Chapter IV

Transverse Vibrations of Turbine Blades of Constant Cross-Section in a Field of Centrifugal Forces

19. Derivation of the equation of vibrations

We shall examine the forced transverse vibrations of a prismatic bar of length l attached by one end to the periphery of an absolutely rigid rotating disc of radius r_o We shall assume that the disc rotates at a constant number of revolutions n per minute. Further, we shall assume that the bar under consideration carries at its free end a concentrated mass m and is subjected to the action of a bending moment M_σ proportional to the angle of rotation of the end of the bar.

Let the external exciting force have the form of a uniformly distributed transverse load, varying sinusoidally,

$$q = q_0 \cos \omega t,$$

where q_0 is the amplitude of the load, ω is the frequency and t is time.

The vibrating system in question is shown schematically in Fig. 11.

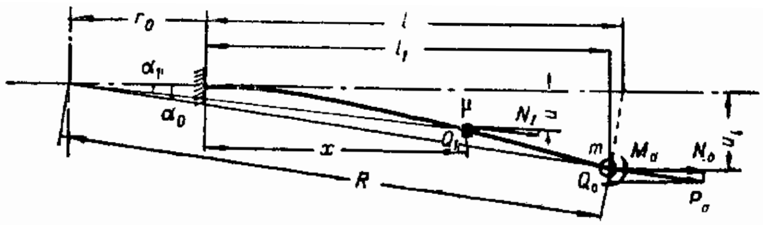

Fig. 11

The following forces are denoted by vectors: P_0 is the centrifugal force due to the concentrated mass m, P is that due to the mass μ of a unit length of the rod; N_0, Q_0, and N_1, Q_1 are respectively, the horizontal (parallel to the x axis) and vertical forces, which make up the centrifugal forces P_0 and P.

Let us examine the expressions for these components. The force directed parallel to the x axis, caused by the action on the bar of the centrifugal force P_0, which is due to the concentrated mass m,

$$N_0 = P_0 \cos \alpha_0 = P_0 \frac{r_0 + l_1}{R},$$

where R is the distance between the axis of rotation O and mass m.

Due to the smallness of the deformations it can be presumed that

$$R \cong r_0 + l \approx r_0 + l_1.$$

Then with sufficient accuracy it can be supposed that

$$N_0 \approx P_0.$$

Similarly we find the expression for the vertical component of the force

$$Q_0 = P_0 \sin \alpha_0 = P_0 \frac{u_{x=l}}{r_0 + l}.$$

Keeping in mind that

$$P_0 = m\bar{\omega}^2 R \cong m\bar{\omega}^2 (r_0 + l),$$

where $\bar{\omega} = \frac{\pi n}{30}$ is the angular velocity of the disc, we obtain

$$N_0 = m\bar{\omega}^2 (r_0 + l); \quad Q_0 = m\bar{\omega}^2 u_x = l. \tag{19.1}$$

The centrifugal force on the mass of the part of the bar of length $l - x$, acting at a section of the bar at a distance x (from the root) will be

$$P = \bar{\omega}^2 \int_x^l \mu (r_0 + \zeta)\, d\zeta = \bar{\omega}^2 \mu \left(r_0 l + \frac{l^2}{2} - r_0 x - \frac{x^2}{2} \right).$$

We now introduce the notations:

$$P_1 = \mu\bar{\omega}^2 \left(r_0 l + \frac{l^2}{2} \right); \quad P_x = \mu\bar{\omega}^2 \left(r_0 x + \frac{x^2}{2} \right).$$

The force acting on the cross-section in the direction parallel to the x-axis caused by the centrifugal forces on the mass of the bar itself is

$$N_1 = P \cos \alpha_1 \approx P = P_1 - P_x,$$

and the component of centrifugal force directed perpendicularly to the x-axis equals

$$Q_1 = P \sin \alpha_1 = \frac{Pu}{r_0 + x} = \frac{P_1 - P_x}{r_0 + x} u.$$

Thus, the force acting on any section of the bar at a distance of x from the origin of coordinates in the direction parallel to the coordinate axis of x, will be equal to:

$$N_x = N_0 + P = N_0 + P_1 - P_x.$$

We shall call this force the horizontal force.

The expression for the vertical force, acting on the same cross-section of the bar, will take the form

$$Q_x = Q_{x=l} + Q_1 = Q_{x=l} + \frac{P_1 - P_x}{r_0 + x} u.$$

In forming of differential equation of vibrations of the bar in question we shall assume that the vibrations take place in one of the principal planes of flexure and that the dimensions of transverse cross-section of the bar are small in comparison to its length. Therefore, the influence of the shear forces can be disregarded.

Neither will we account for the vertical forces Q_x because of their smallness. Further, we will presume that the following nonlinear relation between the stresses and deformations for loading and unloading of material holds:

$$\begin{aligned} \vec{\sigma} &= E[\xi + f(\vec{\xi})], \\ \overleftarrow{\sigma} &= E[\xi + f(\overleftarrow{\xi})], \end{aligned} \tag{19.2}$$

where $E\vec{f}(\xi)$ and $E\overleftarrow{f}(\xi)$ are certain stress increments which characterize the deviation of the curve of the ascending and descending branches of the hysteresis loop from the linear law expressed by the term $E\xi$.

In conformity with formulas (19.2), the bending moment acting on a cross-section of the bar at any instant of vibration can be expressed by the second derivative of deflection in the following way:

$$-M = M\left(\frac{d^2u}{dx^2}\right) = EI\frac{d^2u}{dx^2} + \varepsilon\vec{\Phi}\left(\frac{d^2u}{dx^2}\right), \tag{19.3}$$

where $\varepsilon\Phi\left(\frac{d^2u}{dx^2}\right)$ is a functional which characterizes the incomplete elasticity of the material. The values of the latter for the ascending and descending motions corresponding to the different branches of the hysteresis loop will be different.

The equation of equilibrium of the moments of the forces acting on the element of the bar will have the form

$$\frac{dM}{dx}=Q-N_x\frac{du}{dx}.$$

Differentiating the last equation with respect to x, we obtain

$$\frac{d^2M}{dx^2}=\frac{dQ}{dx}-\frac{dN}{dx}\frac{du}{dx}-N_x\frac{d^2u}{dx^2}. \tag{19.4}$$

Considering the known relation

$$q_1=-\frac{dQ}{dx}, \tag{19.5}$$

and also keeping in mind that in our case $u=f(x,t)$, on the basis of equations (19.3), (19.4), and (19.5), one can write

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 u}{\partial x^2}\right]+\frac{\partial^2}{\partial x^2}\left[\varepsilon\vec{\Phi}\left(\frac{\partial^2 u}{\partial x^2}\right)\right]-\frac{dN_x}{dx}\frac{\partial u}{\partial x}-N_x\frac{\partial^2 u}{\partial x^2}=q_1. \tag{19.6}$$

This differential equation of the bar subjected to the action of uniformly distributed load of intensity q_1 is used to obtain the equation of transverse vibrations of the bar.

Considering that the intensities of the forces of inertia and of the external exciting forces are expressed, respectively, by the formulas

$$q_i=-\mu\frac{\partial^2 u}{\partial t^2}, \quad q_r=\varepsilon q\cos\omega t,$$

Disregarding the rotatory inertia and also not considering the transverse force Q_D because of its smallness, we obtain from (19.6) the differential equation of transverse vibrations of the bar in question:

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 u}{\partial x^2}\right]+\frac{\partial^2}{\partial x^2}\left[\varepsilon\overleftrightarrow{\Phi}\left(\frac{\partial^2 u}{\partial x^2}\right)\right]-\frac{dN_x}{dx}\frac{\partial u}{\partial x}-$$
$$-N_x\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial t^2}-\varepsilon q\cos\omega t=0. \tag{19.7}$$

In the light of the fact that the dissipation of energy in the material is relatively small we shall assume that the external periodic force necessary to maintain the vibrations is also small. This latter circumstance is taken into account in equation (19.7) by the introduction of a small parameter ε as a factor, both in the term which accounts for the losses in the material and in the term characterizing the magnitude of the external exciting force.

It should also be remembered that the presence in equation (19.7) of the term $\frac{\partial^2}{\partial x^2}\left[\varepsilon\overleftrightarrow{\Phi}\left(\frac{d^2 u}{dx^2}\right)\right]$, which characterizes the dissipation of energy in the material and which has a different expression for the ascending and descending motions (for loading and unloading) indicates that the vibrations of the bar are expressed not by one but by two differential equations.

For convenience in handling equation (19.7) we introduce a dimensionless coordinate $\zeta=\frac{x}{l}$ and a dimensionless deflection $u^*=\frac{u}{l}$, where l is the length of the bar in question. Then equation (19.7) can be rewritten, as follows, in dimensionless quantities:

$$\frac{\partial^4 u^*}{\partial \zeta^4} + \frac{l \epsilon}{EI} \frac{\partial^2}{\partial \zeta^2}\left[\overline{\overline{\Phi}}\left(\frac{\partial^2 u^*}{\partial \zeta^2}\right)\right] + \frac{l^3}{EI}(a r_0 + a l \zeta)\frac{\partial u^*}{\partial \zeta} -$$

$$-\frac{l}{EI}\left(P_0 - a r_0 l \zeta - \frac{a l^2}{2}\zeta^2\right)\frac{\partial^2 u}{\partial \zeta^2} + \frac{\mu l^4}{EI}\frac{\partial^2 u^*}{\partial t^2} - \frac{\epsilon q l^3}{EI}\cos\omega t = 0, \quad (19.8)$$

where the following notations are introduced:

$$a = \mu \bar{\omega}^2$$

$$P_0 = N_0 + P_1 = m\bar{\omega}^2(r_0 + l) + \mu\bar{\omega}^2\left(r_0 l + \frac{l^2}{2}\right) \cdot \quad (19.9)$$

20. Methods of approximate solution of the equation of vibrations in this problem

To solve the differential equation (19.8) the non-linearity of which is caused by the imperfect elasticity of material, we shall use, as in previous chapters, the method of nonlinear mechanics based on the expansion by powers of a small parameter. We will represent the deflection $u^*(\zeta, t)$, the frequency of vibrations ω, and the magnitude of the phase shift between the stress and strain ψ in the form of the following expansions in series of powers of the small parameter ϵ:

$$u^*(\zeta, t) = \phi(\zeta) a \cos(\omega t + \psi) + \epsilon u_1(\zeta, t) + \epsilon^2 u_2(\zeta, t) + \cdots; \quad (20.1)$$

$$\omega^2 = \omega_c^2 + \epsilon \Delta_1 + \epsilon \Delta_2 + \cdots; \quad (20.2)$$

$$\psi = \psi_0 + \epsilon \psi_1 + \epsilon^2 \psi_2 + \cdots, \quad (20.3)$$

where a is the dimensionless amplitude of the deflection of the free end of the bar, ω_c is the frequency of free vibrations of the bar.

We now introduce the new variable

$$\tau = \omega t + \psi \tag{20.4}$$

and transform $\cos \omega t = \cos(\tau - \psi)$, using the expression for ψ as the series (20.3),

$$\cos(\tau - \psi) = \cos(\tau - \psi_0 - \epsilon \psi_1 - \epsilon^2 \psi_2 - \cdots) =$$

$$= \cos(\tau - \psi_0) \cos \epsilon (\psi_1 + \epsilon \psi_2 + \cdots) + \sin(\tau - \psi_0) \sin \epsilon (\psi_1 + \epsilon \psi_2 + \cdots). \tag{20.5}$$

Further, $\cos \epsilon (\psi_1 + \epsilon \psi_2 + \cdots)$ and $\sin \epsilon (\psi_1 + \epsilon \psi_2 + \cdots)$ are expressed as the series

$$\cos \varepsilon (\psi_1 + \varepsilon \psi_2 + \ldots) = 1 - \frac{\varepsilon^3 (\psi_1 + \varepsilon \psi_2 + \ldots)^2}{2!};$$

$$\sin \varepsilon (\psi_1 - \varepsilon \psi_2 + \ldots) = \varepsilon (\psi_1 + \varepsilon \psi_2 + \ldots) - \frac{\varepsilon^3 (\psi_1 + \varepsilon \psi_2 + \ldots)}{3!}.$$

Substituting the values of the latter into formula (20.5) and dropping terms containing the small parameter, ε, to higher than the second power, we obtain

$$\cos(\tau - \psi) = \cos(\tau - \psi_0) + \varepsilon \psi_1 (\tau - \psi_0) + \\ + \varepsilon^2 \left[\psi_2 \sin(\tau - \psi_0) - \frac{\psi_1^2}{2} \cos(\tau - \psi_0) \right]. \tag{20.6}$$

After having substituted the series (20.1) — (20.3) into the differential equation (19.8) and taking account of the change of variable (20.4) and of equation (20.6), we obtain

$$\frac{\partial^4 \phi}{\partial \zeta^4} a \cos \tau + \epsilon \frac{\partial^4 u_1}{\partial \zeta^4} + \epsilon^2 \frac{\partial^4 u_2}{\partial \zeta^4} + \cdots + \frac{l \epsilon}{EI} \frac{\partial^2}{\partial \zeta^2} \Big[\bar{\bar{\Phi}} \Big(\frac{\partial^2 \phi}{\partial \zeta^2} a \cos \tau + \\ + \epsilon \frac{\partial^2 u_1}{\partial \zeta^2} + \epsilon^2 \frac{\partial^2 u_2}{\partial \zeta^2} + \cdots \Big) \Big] + \frac{l^3}{EI} (\alpha r_0 + \alpha l \zeta) \Big[\frac{\partial \phi}{\partial \zeta} a \cos \tau + \tag{20.7}$$

cont.

$$+\varepsilon\frac{\partial u_1}{\partial\zeta}+\varepsilon^2\frac{\partial u_2}{\partial\zeta}+\dots\Big]-\frac{l^2}{EI}\Big(P_0-\alpha r_0 l\zeta-\frac{\alpha l^2}{2}\zeta^2\Big)\times$$

$$\times\Big[\frac{d^4\varphi}{d\zeta^2}a\cos\tau+\varepsilon\frac{\partial^2 u_1}{\partial\zeta^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial\zeta^2}+\dots\Big]+$$

$$+\frac{\mu l^4}{EI}(\omega_c^2+\varepsilon\Delta_1+\varepsilon^2\Delta_2+\dots)\Big(-\varphi a\cos\tau+\varepsilon\frac{\partial^2 u_1}{\partial\tau^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial\tau^2}+\dots\Big)-$$

$$-\frac{\varepsilon l^3 q}{EI}\Big\{\cos(\tau-\psi)+\varepsilon\psi_1\sin(\tau-\psi)+$$

$$+\varepsilon^2\Big[\psi_2\sin(\tau-\psi)-\frac{\psi_1^2}{2}\cos(\tau-\psi_0)\Big]\Big\}=0. \qquad (20.7)$$

We group the terms of the last equation in such a way that each of its terms contains as a factor the small parameter to the zeroth, first, second etc., power. After that, inasmuch as $\varepsilon\neq 0$, we can equate to zero the factors of the various powers of the small parameter. After carrying this out, instead of equation (20.7) we will obtain the following system of differential equations:

$$\frac{d^4\varphi}{d\zeta^4}+\frac{l^3}{EI}(\alpha r_0+\alpha l\zeta)\frac{d\varphi}{d\zeta}-\frac{l^2}{EI}\Big(P_0-\alpha r_0 l\zeta-\frac{\alpha l^2}{2}\zeta^2\Big)\frac{d^2\varphi}{d\zeta^2}-\frac{\mu l^4}{EI}\omega_c^2\varphi=0; \qquad (20.8)$$

$$\frac{\partial^4 u_1}{\partial\zeta^4}+\frac{l^3}{EI}(\alpha r_0+\alpha l\zeta)\frac{\partial u_1}{\partial\zeta}-\frac{l^2}{EI}\Big(P_0-\alpha r_0 l\zeta-\frac{\alpha l^2}{2}\zeta^2\Big)\frac{\partial^2 u_1}{\partial\zeta^2}-$$

$$-\frac{\mu l^4}{EI}\Delta_1\varphi a\cos\tau+\frac{\mu l^4}{EI}\omega_c^2\frac{\partial^2 u_1}{\partial\tau^2}+\frac{l}{EI}\frac{\partial^2}{\partial\zeta^2}\Big[\vec{\Phi}\Big(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\Big)\Big]-$$

$$-\frac{ql^3}{EI}\cos(\tau-\psi_0)=0; \qquad (20.9)$$

$$\frac{\partial^4 u_2}{\partial\zeta^2}+\frac{l^3}{EI}(\alpha r_0+\alpha l\zeta)\frac{\partial u_2}{\partial\zeta}-\frac{l^2}{EI}\Big(P_0-\alpha r_0 l\zeta-\frac{\alpha l^2}{2}\zeta^2\Big)\frac{\partial^2 u_2}{\partial\zeta^2}-$$

$$-\frac{\mu l^4}{EI}\Delta_2\varphi a\cos\tau+\frac{\mu l^4}{EI}\omega_c^2\frac{\partial^2 u_2}{\partial\tau^2}+\frac{l}{EI}\frac{\partial^2}{\partial\zeta^2}[\Psi(\zeta,\tau)]-$$

$$-\psi_1\sin(\tau-\psi_0)=0; \qquad (20.10)$$

Here $\Psi(\zeta, \tau)$ is a functional which defines more accurately the magnitude of energy dissipation in the material to the second approximation. The equations obtained (20.8) — (20.10), are the basic ones for the investigation, in the various approximations, of the influence of energy dissipation in the material on the vibrations of the bar.

21. Solution of the problem in the zeroth approximation

To determine the deflection function and frequency of vibrations in the zeroth approximation, i.e. without accounting for the energy dissipation in the material, it is necessary to solve equation (20.8).

We rewrite this equation in the form

$$\frac{d^4\varphi}{d\zeta^4} - (d + c\zeta + f\zeta^2)\frac{d^2\varphi}{d\zeta^2} - (c + g\zeta)\frac{d\varphi}{d\zeta} - r\varphi = 0, \tag{21.1}$$

where

$$d = \frac{l^2 P_0}{EI}; \quad c = -\frac{l^3 r_0 \alpha}{EI}; \quad f = -\frac{l^4 \alpha}{2EI};$$
$$g = -\frac{l^4 \alpha}{EI}; \quad r = \frac{l^4 \mu}{EI}\omega_c^2. \tag{21.2}$$

The solution of the homogeneous differential equation of the fourth order (21.1) with variable coefficients can be found in the form of a series:

$$\varphi = A_0 + A_1\zeta + A_2\zeta^2 + A_3\zeta^3 + \dots + A_n\zeta^n + \dots, \tag{21.3}$$

where $A_0, A_1, A_2 \dots, A_n \cdots$ are certain constants.

Substituting (21.3) into the differential equation (21.1) and grouping the terms containing the same powers of ζ, it is possible then to equate to zero each of the expressions multiplying the various powers of ζ . Having carried this out, we obtain a system of equations

$$
\begin{aligned}
&4\cdot 3\cdot 2\cdot A_4 - 2dA_2 - cA_1 - rA_0 = 0,\\
&5\cdot 4\cdot 3\cdot 2\cdot A_5 - 3\cdot 2\cdot dA_3 - 2cA_2 - 2cA_2 - gA_1 - rA_1 = 0,\\
&6\cdot 5\cdot 4\cdot 3\cdot A_6 - 4\cdot 3dA_4 - 3\cdot 2cA_3 - 2fA_2 - 3cA_3 - 2gA_2 - rA_2 = 0,\\
&7\cdot 6\cdot 5\cdot 4\cdot A_7 - 5\cdot 4dA_5 - 4\cdot 3cA_4 - 3\cdot 2fA_3 - 4cA_4 - 3gA_3 - rA_3 = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(n+4)(n+3)(n+2)(n+1)A_{n+4} - (n+2)(n+1)dA_{n+1} -\\
&\quad - n(n+1)cA_{n+1} - n(n-1)fA_n - (n+1)cA_{n+1} -\\
&\quad - ngA_n - 2A_n = 0 .
\end{aligned}
\tag{21.4}
$$

From this system of equations (21.4) we obtain

$$
\begin{aligned}
A_4 &= \frac{2!}{4!}dA_2 + \frac{c}{4!}A_1 + \frac{r}{4!}A_0,\\
A_5 &= \frac{3!}{5!}dA_3 + \frac{2\cdot 2!}{5!}cA_2 + \frac{(g+r)}{5!}A_1,\\
A_6 &= \frac{4!}{6!}dA_4 + \frac{3\cdot 3!}{6!}cA_3 + \frac{2!(2f+2g+r)}{6!}A_2,\\
A_7 &= \frac{5!}{7!}dA_5 + \frac{4\cdot 4!}{7!}cA_4 + \frac{3!(3\cdot 2f+3g+r)}{7!}A_3,\\
A_8 &= \frac{6!}{8!}dA_6 + \frac{5\cdot 5!}{8!}cA_5 + \frac{4!(4\cdot 3f+4g+r)}{8!}A_4,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
A_n &= \frac{(n-2)!}{n!}dA_{n-2} + \frac{(n+3)(n-3)!}{n!}cA_{n-3} +\\
&+ \frac{(n-4)![(n-4)(n-5)f+(n-4)g+r]}{n!}A_{n-4}.
\end{aligned}
\tag{21.5}
$$

From equations (21.5) it follows that all the constants beginning with A_4 and above can be expressed in terms of A_o, A_1, A_2, and A_3.

In this way the solution of equation (21.1) represented by the series (21.3) can be reduced to a solution containing a number of constants corresponding to the order of the differential equation.

We now prove this. Substituting in the expressions for A_6, A_7, A_8 and so forth, in place of A_4 and A_5 their expressions in A_o, A_1, A_2, and A_3, we find

$$A_6 = \frac{3 \cdot 3!}{6!} cA_3 + \frac{2!}{6!}(d^2 + 2 \cdot 3f + r)A_2 + \frac{1}{6!} cdA_1 + \frac{3d}{6!} A_0 ;$$

$$A_7 = \frac{3!}{7!}\left[d^2 + (3 \cdot 4f + r)\right]A_3 + \frac{2!}{7!}(2cd + 4cd)A_2 + \\ + \frac{1}{7!}\left[(2f + r) + 4c^2\right]A_1 + \frac{4rc}{7!} A_0 ;$$

$$A_8 = \frac{3!}{8!}(3cd + 5cd)A_3 + \frac{2!}{8!}\left[d^2 + (2 \cdot 3f + r)d + 5 \cdot 2c^2 + \\ + (4 \cdot 5f + r)d\right]A_2 + \frac{1}{8!}\left[cd^2 + 5c(2f + r) + c(4 \cdot 5f + r)\right]A_1 + \\ + \frac{1}{8!}\left[rd^2 + (4 \cdot 5f + r)r\right]A_0 ;$$

Further, substituting A_4, A_5, A_6, A_7, A_8 . . . , expressed in terms of A_o, A_1, A_2, and A_3 in equation (21.3) and designating the expressions multiplying A_o, A_1, A_2, and A_3, respectively by $\phi_0(\zeta)$, $\phi_1(\zeta)$, $\phi_2(\zeta)$, and $\phi_3(\zeta)$, in place of series (21.3), the solution of (21.1) can be represented as

$$\phi(\zeta) = A_0\phi_0(\zeta) + A_1\phi_1(\zeta) + A_2\phi_2(\zeta) + A_3\phi_3(\zeta) .$$

In this way the solution of differential equation (21.1) represented by the series (21.3) constitutes the general solution and contains four constants of integrations: A_0, A_1, A_2 and A_3.

The boundary conditions of this problem from which the constants of integrations must be determined are:

$$[u^*(\zeta, \tau)]_{\substack{\tau=0\\ \zeta=0}} = 0; \quad \left[\frac{\partial u^*(\zeta, \tau)}{\partial \zeta}\right]_{\substack{\tau=0\\ \zeta=0}} = 0;$$

$$\left[\frac{\partial^2 u^*(\zeta, \tau)}{\partial \zeta^2}\right]_{\substack{\tau=0\\ \zeta=1}} = \frac{lM_6}{EI}; \quad \left[\frac{\partial^3 u^*(\zeta, \tau)}{\partial \zeta^3}\right]_{\substack{\tau=0\\ \zeta=1}} = \frac{l^2 (Q)_{\zeta=1}}{EI},$$

(21.6)

where $(Q)_{\zeta=1}$ is the shear force at the end of the bar, the magnitude of which, in the present case is

$$(Q)_{\zeta=1} = N_0 \sin\left[\left(\frac{\partial u^*(\zeta, \tau)}{\partial \zeta}\right)_{\substack{\tau=0\\ \zeta=1}}\right] \approx N_0 \left[\frac{\partial u^*(\zeta, \tau)}{\partial \zeta}\right]_{\substack{\tau=0\\ \zeta=1}};$$

(21.7)

M_σ is the elastic bending moment acting on the end of the bar; its value depends on the deflection and can be expressed by the formula

$$M_6 = K\left[\frac{\partial u^*(\zeta, \tau)}{\partial \zeta}\right]_{\substack{\tau=0\\ \zeta=1}},$$

where K is a constant coefficient.

In conformity with equation (20.1) the conditions (21.6) at the ends which apply to the solution (21.3) should be rewritten as:

$$[\varphi(\zeta)]_{\zeta=0}=0; \quad \left[\frac{d\varphi(\zeta)}{d\zeta}\right]_{\zeta=0}=0;$$
$$\left[\frac{d^2\varphi(\zeta)}{d\zeta^2}\right]_{\zeta=1}=P\left[\frac{d\varphi(\zeta)}{d\zeta}\right]_{\zeta=1}; \quad \left[\frac{d^3\varphi(\zeta)}{d\zeta^3}\right]_{\zeta=1}=d_0\left[\frac{d\varphi(\zeta)}{d\zeta}\right]_{\zeta=1}. \tag{21.8}$$

Here

$$P=\frac{Kl}{EI}; \quad d_0=\frac{N_0 l^2}{EI}.$$

In accordance with the first two conditions (21.8), from the solution (21.3), we have

$$[\varphi(\zeta)]_{\zeta=0}=A_0=0; \quad \left[\frac{d\varphi(\zeta)}{d\zeta}\right]_{\zeta=0}=A_1=0.$$

Then, on the basis of formulas (21.5), the solution (21.3) can be presented in a general form as follows:

$$\phi(\zeta)=A_2\zeta^2+A_3\zeta^3+\sum_{n=4}^{\infty}\frac{\zeta^n}{n!}\Big\{(n-2)!\; dA_{n-2}+(n-3)(n-3)!\; cA_{n-3}+$$
$$+(n-4)!\Big[(n-4)(n-5)f+(n-4)g+r\Big]A_{n-4}\Big\}. \tag{21.9}$$

The unknown constants in the last expression are just A_2 and A_3, by which, according to equations (21.5) all the constants A for any value n, will be expressed. Besides the constants of integration A_2 and A_3, the solution (21.9) contains a third, as yet unknown, constant quantity $r=\frac{l^4\mu}{EI}\omega_c^2$, containing the frequency of the free vibrations of the bar ω_c in the lowest mode, which we are looking for.

To determine constants A_2, A_3 and r, we use the remaining two boundary conditions (21.8) as well as the conditions that the value of the function at the end of the bar equals unity i.e. these three conditions:

$$\left[\frac{d^2\varphi(\zeta)}{d\zeta^2}\right]_{\zeta=1} = P\left[\frac{d\varphi(\zeta)}{\partial\zeta}\right]_{\zeta=1}; \quad \left[\frac{d^3\varphi(\zeta)}{d\zeta^3}\right]_{\zeta=1} = d_0\left[\frac{d\varphi(\zeta)}{d\zeta}\right]_{\zeta=1};$$
$$[\varphi(\zeta)]_{\zeta=1} = 1. \tag{21.10}$$

If we now represent equation (21.9) in the most general form

$$\varphi(\zeta) = \sum_{n=2}^{\infty} A_n \zeta^n, \tag{21.11}$$

the conditions (21.10) for determination of the constants A_2, A_3 and r can be expressed as follows:

$$1) \quad \sum_{n=2}^{\infty} n(n-1)A_n = P\sum_{n=2}^{\infty} nA_n,$$
$$2) \quad \sum_{n=2}^{\infty} n(n-1)(n-2)A_n = d_0\sum_{n=2}^{\infty} nA_n,$$
$$3) \quad \sum_{n=2}^{\infty} A_n = 1. \tag{21.12}$$

Using these equations and also formulas (21.4) and (21.5) it is possible to determine the constants of integration A_2 and A_3 as well as the magnitude of r, and from it the lowest frequency of free vibrations.*

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*Here the value of r is found from the partial determinant formed from the first two conditions, (21.10) or (21.12).

$$\omega_c = \sqrt{\frac{rEI}{\mu l^4}}. \tag{21.13}$$

The series (21.9) or (21.11) are convergent in the interval from 0 to 1 which can be easily demonstrated on the basis of the d'Alembert test. At the end of the bar ($\zeta = 1$) the sum of the series equals 1, which corresponds to the boundary conditions of the present problem. In this way, having determined the required number of coefficients A_n (usually their number does not exceed 10 to 12) and substituting their values into formula (21.9), we will obtain the final expression for the deflection which is then the solution of the differential equation (19.8) in the zeroth approximation.

22. Determination of the frequency of vibrations and the phase shift in the first approximation

In accordance with equations (20.2) and (20.3), the given problem in the first approximation is associated with the solution of equation (20.9), from which we find Δ_1 and $\sin\psi_0$. These values can be found from equation (20.9) if we examine the balance of energy of the vibrating bar (potential and kinetic) for one cycle of vibration.

Multiplying equation (20.9) first by $\phi \sin\tau \, d\zeta \, d\tau$, and a second time by $\phi \cos\tau \, d\zeta \, d\tau$ and equating to zero the integrals in both cases taken along the whole length of the bar for one cycle of vibration, we obtain

$$\oint\int_0^1 \left\{ \frac{\partial^4 u_1}{\partial\zeta^4} + \frac{l^3}{EI}(\alpha r_0 + \alpha l\zeta)\frac{\partial u_1}{\partial\zeta} - \frac{l^2}{EI}\left(P_0 - \alpha r_0 l\zeta - \frac{\alpha l^2}{2}\zeta\right)\frac{\partial^2 u_1}{\partial r^2} - \right.$$
$$- \frac{\mu l^4}{EI}\Delta_1 \varphi a \cos\tau + \frac{\mu l^4}{EI}\omega_c^2 \frac{\partial^2 u_1}{\partial\tau^2} + \frac{l}{EI}\frac{\partial^2}{\partial\zeta^2}\left[\vec{\Phi}\left(\frac{d^2\varphi}{d\zeta^2}a.\cos\tau\right)\right] -$$
$$\left. - \frac{l^3 q}{EI}\cos(\tau - \psi_0)\right\}\varphi \sin\tau \, d\zeta \, d\tau = 0, \tag{22.1}$$

$$\oint\int_0^1\Big\{\frac{\partial^4 u_1}{\partial\zeta^4}+\frac{l^3}{EI}(\alpha r_0+\alpha l\zeta)\frac{\partial u_1}{\partial\zeta}-\frac{l^2}{EI}\Big(P_0-\alpha r_0 l\zeta-\frac{\alpha l^2}{2}\zeta^2\Big)\frac{\partial^2 u_1}{\partial\zeta^2}-$$
$$-\frac{\mu l^4}{EI}\Delta_1\varphi a\cos\tau+\frac{\mu l^4}{EI}\omega_c^2\frac{\partial^2 u_1}{\partial\tau^2}+\frac{l}{EI}\frac{\partial^2}{\partial\zeta^2}\Big[\vec{\Phi}\Big(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\Big)\Big]-$$
$$-\frac{l^3 q}{EI}\cos(\tau-\psi_0)\Big\}\varphi\cos\tau\, d\zeta\, d\tau=0. \qquad (22.2)$$

Integration of these equations by parts with respect to ζ and τ taking account of the conditions at the ends of the bar (21.10) also considering that $u_1(\zeta,\tau)$ does not contain the principal harmonic, one can show:

$$1)\quad \oint\int_0^1\Big[EI\frac{\partial^4 u_1}{\partial\zeta^4}-l^2\Big(P_0-\alpha r_0\zeta l-\frac{\alpha l^2}{2}\zeta^2\Big)\frac{\partial^2 u_1}{\partial\zeta^2}+l^3(\alpha r_0+\alpha l\zeta)\frac{\partial u_1}{\partial\zeta}+$$
$$+\mu l^4\omega_c^2\frac{\partial^2 u_1}{\partial\tau^2}\Big]\varphi\sin\tau\, d\zeta\, d\tau=0; \qquad (22.3)$$

$$2)\quad \oint\int_0^1\Big[EI\frac{\partial^4 u_1}{\partial\zeta^4}-l^2\Big(P_0-\alpha r_0 l\zeta-\frac{\alpha l^2}{2}\zeta\Big)\frac{\partial^2 u}{\partial\zeta^2}+l^3(\alpha r_0+\alpha l\zeta)\frac{\partial u_1}{\partial\zeta}+$$
$$+\mu l^4\omega_c^2\frac{\partial^2 u_1}{\partial\tau^2}\Big]\varphi\cos\tau\, d\zeta\, d\tau=0. \qquad (22.4)$$

In this way the two equations of balance (22.1) and (22.2) can be replaced by four; viz., by equations (22.3) and (22.4) and in addition, by the following equations:

$$\oint\int_0^1\Big\{-\alpha\mu l^3\varphi\Delta_1\cos\tau-l^2 q\cos(\tau-\psi_0)+$$
$$+\frac{\partial^2}{\partial\zeta^2}\Big[\vec{\Phi}\Big(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\Big)\Big]\Big\}\varphi\sin\tau\, d\zeta\, d\tau=0; \qquad (22.5)$$

$$\oint\int_0^1\Big\{-a\mu l^3\varphi\Delta_1\cos\tau-l^2 q\cos(\tau-\psi_0)+$$
$$+\frac{\partial^2}{\partial\zeta^2}\Big[\vec{\Phi}\Big(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\Big)\Big]\Big\}\varphi\cos\tau\, d\zeta\, d\tau=0. \qquad (22.6)$$

Solving the last equation for Δ_1 , we find

$$\Delta_1 = \left[a\mu l^3 \pi \int_0^1 \varphi^2 d\zeta \right]^{-1} \left\{ \Phi \int_0^1 \frac{\partial^2}{\partial \zeta^2} \left[\vec{\bar{\Phi}} \left(\frac{d^2\varphi}{d\zeta^2} a \cos \tau \right) \right] \varphi \cos \tau \, d\zeta - \right.$$
$$\left. - q l^2 \pi \cos \psi_0 \int_0^1 \varphi \, d\zeta \right\}.$$

The first integral in the brackets of the last equation, can be expressed in terms of the bending moment acting on the cross-section on the basis of (19.3) as follows:

$$\varepsilon \oint \int_0^1 \frac{\partial^2}{\partial \zeta^2} \left[\vec{\bar{\Phi}} \left(\frac{d^2\varphi}{d\zeta^2} a \cos \tau \right) \right] \varphi \cos \tau \, d\zeta \, d\tau =$$
$$= \oint \int_0^1 \frac{\partial^2}{\partial \zeta^2} \left[M \left(\frac{d^2\varphi}{d\zeta^2} a \cos \tau \right) \right] \varphi \cos \tau \, d\zeta \, d\tau -$$
$$- \oint \int_0^1 \frac{1}{l} \frac{\partial^2}{\partial \zeta^2} \left[EI \frac{d^2\varphi}{d\zeta^2} a \cos \tau \right] \varphi \cos \tau \, d\zeta \, d\tau.$$

Then the square of frequency in the first approximation can be expressed as:

$$\omega^2 = \omega_c^2 + \varepsilon \Delta_1 = \omega_c^2 + \left[a\mu l^3 \pi \int_0^1 \varphi^2 d\zeta \right]^{-1} \left\{ \oint \int_0^1 \frac{\partial^2}{\partial \zeta^2} \left[M \left(\frac{d^2\varphi}{d\zeta^2} a \cos \tau \right) \right] \times \right.$$
$$\left. \times \varphi \cos \tau \, d\zeta \, d\tau - \frac{1}{l} a\pi EI \int_0^1 \frac{d^4\varphi}{d\zeta^4} \varphi \, d\zeta - q l^2 \pi \cos \psi_0 \int_0^1 \varphi \, d\zeta \right\}. \tag{22.7}$$

The sine of the phase shift angle in the first approximation, as in the previous case, is found from equation (22.3)

$$\sin \psi_0 = \left[q l^2 \pi \int_0 \varphi \, d\zeta \right]^{-1} \oint \int_0^1 \frac{\partial^2}{\partial \zeta^2} \left[M \left(\frac{\partial^2 \varphi}{\partial \zeta^2} a \cos \tau \right) \right] \varphi \sin \tau \, d\zeta \, d\tau. \tag{22.8}$$

Using formulas (22.7) and (22.8) we can construct a resonance curve of vibrations of the bar in question accounting for the energy of dissipation in the material. In order to use these formulas it is necessary to find an expression for the bending moment $M\left(\frac{d^2\varphi}{d\zeta^2} a \cos \tau\right)$. Taking the expressions for the stresses (19.2) in the form (6.2) it is possible to transform formulas (22.7) and (22.8) similarly to the way it was accomplished in chapter 2 and finally to embody the results in the form:

$$\omega^2=\omega_c^2+\left[a\mu l^3\pi\int_0^1\varphi^2\,d\zeta\right]^{-1}\left\{2E\frac{\nu}{n}\frac{1}{l^n}\int_0^1(1-\cos\tau)^n\times\right.$$
$$\left.\times\cos\tau\,d\tau\int_0^1\left[\frac{d^2}{d\zeta^2}\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\varphi\iint_F z^{n+1}\,dz\,dy\right]d\zeta-ql^3\pi\cos\psi_0\int_0^1\varphi\,d\zeta\right\}; \quad (22.9)$$

$$\sin\psi_0=\left[ql^3\pi\int_0^1\varphi\,d\zeta\right]^{-1}E\frac{\nu}{n}a^n\frac{2^{n+1}(n-1)}{(n+1)l^n}\times$$
$$\times\int_0^1\left\{\frac{d^2}{d\zeta^2}\left(\frac{d^2\varphi}{d\zeta^2}\right)^n\varphi\iint_F z^{n+1}\,dz\,dy\right\}d\zeta. \quad (22.10)$$

Using these formulas, it is possible to construct a resonance curve of the vibrations of a blade of constant cross-section with one restraining shroud.

On the basis of equation (20.1) the magnitude of the deflection of the blade can be found by the formula

$$u^*(\zeta,t)=\varphi(\zeta)\,a\cos(\omega t+\psi_0).$$

In this case also we shall confine ourselves to the solution of the problem in the first approximation.

As far as the precision of the first approximation for the solution of technical problems is concerned, as was shown in the previous chapters devoted to questions of vibrations accounting for energy dissipation in the material, it is entirely sufficient. The extra precision in the second approximation amounts to less than 3% for the frequency and less than 0.2% for the displacement.

23. Sample calculation

To illustrate the application of the formulas obtained in the previous paragraphs we shall construct the resonance curve for an actual blade of a steam turbine; a sketch of the blade is shown in Figure 12. The basic data for the

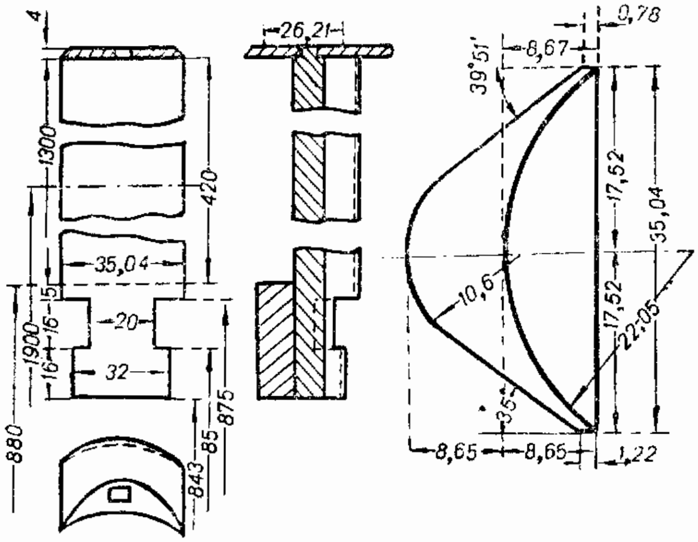

Fig. 12

calculations are as follows.* The length of the blade l = 42 cm, the outer radius of the disc r = 88 cm; rpm of the turbine 1400, area for the cross-section of the blade

*Data for sample calculation are taken from the book of Prof. M. I. Yanovskiy, Design and Calculation of the Strength of Steam Turbine Parts, 1947, p. 81.

F = 1.861 cm^2, the moment of inertia of the blade in the plane of minimum rigidity, I = 0.2268 cm^4, the mass μ of one segment of the shroud of length t = 2.621 cm amounts to 0.29363.10^{-3} kg sec^2/cm; the parameters of the hysteresis loop for the chosen steel, based on experimental data are taken as n = 2 and ν = 3.1; the modulus of elasticity in tension E = 2.2 X 10^6 kg/cm^2.

On the basis of a calculation the natural frequency of vibrations of the blade (without considering the centrifugal forces of the mass on the blade and on a shroud) was found as ω_{i0} =3648 sec^{-1}. The natural frequency of vibration of the blade calculated taking account of centrifugal force on mass of the blade and the shroud according to formula (21.13) equaled ω_c = 4585 sec^{-1}.

On the basis of formula (21.9)—(21.12) the function (21.9) for the example in question can be expressed as:

$$\begin{aligned}\varphi(\zeta) = {} & 1{,}793227\zeta^2 - 1{,}2607475\zeta^3 + 0{,}817416\zeta^4 - 0{,}594793\zeta^5 + 0{,}348279\zeta^6 - \\ & - 0{,}153908\zeta^7 + 0{,}0708347\zeta^8 - 0{,}0269864\zeta^9 + 0{,}00902047\zeta^{10} - \\ & - 0{,}00302912\zeta^{11} + 0{,}000815697\zeta^{12} - 0{,}000238485\zeta^{13} + \\ & + 0{,}000114719\zeta^{14} - 0{,}0000103836\zeta^{15} + \ldots\end{aligned} \qquad (23.1)$$

The following values for the integrals entering into the final formula are found using the assumed data:

$$\int_0^1 \varphi(\zeta)\,d\zeta = 0{,}383214,$$

$$\int_0^1 \varphi^2(\zeta)\,d\zeta = 0{,}236730,$$

$$\int_0^\pi (1-\cos\tau)^n \cos\tau\,d\tau = -\pi,$$

$$\iint_F z^{n+1}\,dz\,dy = 0{,}35652. \qquad (23.2)$$

Formulas (22.9) and (22.10) for the construction of the resonance curve will take the following form after substitution in the known values:

$$\left(\frac{\omega}{\omega_c}\right)^2 = 1 - [2{,}356204 \sin\psi_0 + \cos\psi_0]\,\frac{12{,}3115q}{a\cdot 10^3}, \qquad (23.3)$$

$$\sin\psi_0 = 0{,}272618\cdot 10^3\frac{a^2}{q}, \qquad (23.4)$$

where, as is known, ψ_0 is the phase shift, $a = \frac{A}{l}$ is the dimensionless magnitude of the amplitude of vibrations, q is the amplitude of the external exciting force, ω and ω_c are, respectively, the forced and natural frequencies of vibrations.

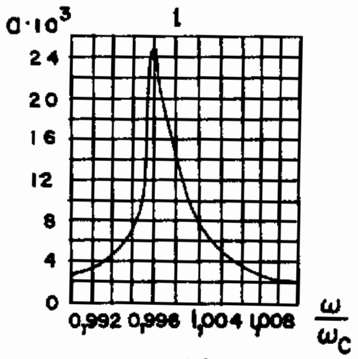

Fig. 13

Using formulas (23.3) and (23.4) and knowing the magnitude of the disturbing force q, it is possible to construct a resonance curve for transverse vibrations of the blade. For a value of alternating exciting force of amplitude per unit length of blade q = 0.004 kg/cm, formulas (23.3) and (23.4) can be represented as

$$\left(\frac{\omega}{\omega_c}\right)^2 = 1 - \frac{0{,}04924}{a\cdot 10^3}(2{,}3562\sin\psi_0 + \cos\psi_0),$$

$$\sin\psi_0 = 1{,}62275\cdot 10^3\, a^2.$$

Using these formulas we obtain the results shown in Table 10.

Table 10

$a \cdot 10^3$	$\sin \psi_0$	$\cos \psi_0$	$\left(\frac{\omega}{\omega_c}\right)^2_л$	$\left(\frac{\omega}{\omega_c}\right)^2_п$	$\left(\frac{\omega}{\omega_c}\right)_л$	$\left(\frac{\omega}{\omega_c}\right)_п$
2	0,0065	0,99998	0,97500	1,02424	0,9874	1,012
4	0,0260	0,99966	0,98694	1,01155	0,9934	1,0057
8	0,1039	0,99459	0,99237	1,00462	0,9962	1,0024
12	0,2337	0,97237	0,99375	1,00173	0,9969	1,0008
16	0,4154	0,90966	0,99419	0,99979	0,9971	0,9999
20	0,6491	0,76072	0,99437	0,99811	0,9971	0,9990
24,824	1	0	0,99533	0,99533	0,9977	0,9977

The resonance curve constructed on the basis of data in Table 10 is given in Figure 13. At maximum amplitude of vibrations, the magnitude of stress (at the root of the blade) for the given amplitude of load q = 0.004 kg/cm amounts to about 3400 kg/cm^2. The resonance curve permits us to conclude that the displacement of the maximum to the left is negligible, though the amplitude of the vibrations at $\frac{\omega}{\omega_c} = 1$ is 40% less than the maximum amplitude at $\frac{\omega}{\omega_c} = 0.9077$. The shape of the resonance curve is characteristic of slightly nonlinear vibrations of systems, the characteristic example of which is the present problem of vibrations accounting for dissipation of energy in the material.

Chapter V

Transverse Vibrations of a Turbine Blade in the Case of Slowly Changing Frequency of Excitation

24. Basic differential equations and methods of solution

The problem of blade vibrations for a non-steady-state regime is of great interest in connection with the study of the behavior of a blade for slow transition* from subresonant frequencies of vibration to super-resonant frequencies.

The theory of analysis of vibrations of elastic systems with slowly changing parameters in a manner applicable to some problems was worked out by Yu. A. Mitropolsky (5). We shall not dwell upon the general results, instead we shall examine the theory of construction of a resonance curve for one particular, practically important, case of blade vibrations, in the case of a change in time of only one

*By "slow" we mean a rate of change of frequency such that the frequency changes 1 to 2% of its magnitude per second.

parameter, for example, the external frequency of excitation. We shall assume that the amplitude of the exciting force, as well as the other parameters (mass, stiffness) remain constant.

In this case of non-steady-state vibration, the differential equation can be obtained from the previously derived equation (19.8) by replacement of the term which characterizes the external excitation, namely:

$$\frac{\partial^4 u^*}{\partial \zeta^4} - \frac{l^2}{EI}\left(P_0 - \alpha r_0 l\zeta - \frac{\alpha l^2}{2}\zeta^2\right)\frac{\partial^2 u^*}{\partial \zeta^2} + \frac{\alpha l^3}{EI}(r_0 + l\zeta)\frac{\partial u^*}{\partial \zeta} + \frac{\mu l^4}{EI}\frac{\partial^2 u^*}{\partial t^2} + \\ + \frac{l^6}{EI}\frac{\partial^2}{\partial \zeta^2}\left[\vec{\Phi}\left(\frac{\partial^2 u^*}{\partial \zeta^2}\right)\right] - \frac{\varepsilon q_0 l^3}{EI}\sin\int_{t_0}^{t} p(\varepsilon t)\,dt = 0 \quad (24.1)$$

We seek the solution of (24.1), following the methods of nonlinear mechanics, in the form of the following asymptotic series:

$$u^*(\zeta, t) = \varphi(\zeta)\,a\cos(\theta+\psi) + \varepsilon u_1(\zeta, \varepsilon t, a, \theta, \theta+\psi) + \\ + \varepsilon^2 u_2(\zeta, \varepsilon t, a, \theta, \theta+\psi) + \ldots, \quad (24.2)$$

in which $\varphi(\zeta)$ is the solution of the "undisturbed" homogeneous equation

$$\frac{\partial^4 u^*}{\partial \zeta^4} - \frac{l^2}{EI}\left(P_0 - \alpha l r_0 \zeta - \frac{\alpha l^2}{2}\zeta^2\right)\frac{\partial^2 u^*}{\partial \zeta^2} + \frac{\alpha l^3}{EI}(r_0 + l\zeta)\frac{\partial u^*}{\partial} + \frac{\mu l^4}{EI}\frac{\partial^2 u^*}{\partial t^2} = 0;$$

and u_1 and u_2 represent periodic functions of the angles $\theta = \int_{t_0}^{t} p(\varepsilon t)\,dt$ and $\theta + \psi$ with period 2π. The magnitudes of amplitude, a, and the phase shift ψ are found from the following system of differential equations:

$$\frac{da}{dt} = \varepsilon A_1(\varepsilon t, a, \psi) + \varepsilon^2 A_2(\varepsilon t, a, \psi) + \ldots \quad (24.3)$$

$$\frac{d\psi}{dt} = p_c - p(\varepsilon t) + \varepsilon B_1(\varepsilon t, a, \psi) + \varepsilon^2 B_2(\varepsilon t, a, \psi) + \cdots, \tag{24.4}$$

where p_c is the natural frequency of vibrations of the blade found from (21.13).

The further solution of this problem is reduced to the determination of functions u_1, u_2 and also of A_1, A_2 so that the series (24.2) satisfies equation (24.1) for values of a and ψ determined from (24.3) and (24.4). In principle, the finding of $u_1, u_2; A_1, A_2 \cdots$ and $B_1, B_2 \cdots$ does not present theoretical difficulties, but in practice, in view of the rapid complication of the formulas, only a few terms of the series (24.2) can be determined.

As was shown in the works of Yu. I. Mitropolsky, the results of the solution of the problems with slow changing parameters in the first approximation give, for all practical purposes, quite satisfactory results. The increase in accuracy of the natural frequency in the second approximation amounts to only hundredths of a percent (0.02%) of the natural frequency of the undisturbed system. In the light of this, in solving a rather complicated problem it is advisable to limit oneself to the examination of the first approximation for which only the functions $A_1(\varepsilon t, a, \psi)$ and $B_1(\varepsilon t, a, \psi)$ will be required, which is easiest to accomplish, proceeding from the examination of equations of harmonic balance.

$$\int_0^1 \int_0^{2\pi} \left\{ \frac{\partial^4 u^*}{\partial \zeta^4} - \frac{l^2}{EI}\left(P_0 - \alpha r_0 l\zeta - \frac{\alpha l^2}{2}\zeta^2\right)\frac{\partial^2 u^*}{\partial \zeta^2} + \right.$$
$$+ \frac{\alpha l^3}{EI}(r_0 + l\zeta)\frac{\partial u^*}{\partial \zeta} + \frac{\mu l^4}{EI}\frac{\partial^2 u^*}{\partial t^2} + \frac{l\varepsilon}{EI}\frac{\partial^2}{\partial \zeta^2}\left[\vec{\Phi}\left(\frac{\partial^2 u^*}{\partial \zeta^2}\right)\right] -$$
$$\left. - \frac{\varepsilon q_0 l^3}{EI}\sin\theta \right\} \varphi(\zeta)\cos(\theta + \psi)\, d(\theta + \psi)\, d\zeta = 0; \tag{24.5}$$

$$\int_0^1 \int_0^{2\pi} \left\{ \frac{\partial^4 u^*}{\partial \zeta^4} - \frac{l^2}{EI}\left(P_0 - \alpha r_0 l\zeta - \frac{\alpha l^2}{2}\zeta\right)\frac{\partial^2 u^*}{\partial \zeta^2} + \frac{\alpha l^3}{EI}(r_0 + l\zeta)\frac{\partial u^*}{\partial \zeta} + \right.$$
$$+ \frac{\mu l^4}{EI}\frac{\partial^2 u^*}{\partial t^2} + \frac{\varepsilon l}{EI}\frac{\partial^2}{\partial \zeta^2}\left[\vec{\Phi}\left(\frac{\partial^2 u^*}{\partial \zeta^2}\right)\right] -$$
$$\left. - \frac{\varepsilon q_0 l^3}{EI}\sin\theta \right\} \varphi(\zeta)\sin(\theta+\psi)\, d(\theta+\psi)\, d\zeta = 0. \quad (24.6)$$

Substituting in the last equation, for $u^*(\zeta, t)$ the values, represented as the series (24.2) we obtain

$$\int_0^1 \int_0^{2\pi} \left\{ \frac{\partial^4 \varphi(\zeta)}{\partial \zeta^4} a\cos(\theta+\psi) + \varepsilon\frac{\partial^4 u_1}{\partial \zeta^4} + \varepsilon^2 \frac{\partial^4 u_2}{\partial \zeta^4} + \ldots + \right.$$
$$+ \left(P_0 - \alpha r_0 l\zeta - \frac{\alpha l^2}{2}\zeta^2\right)\left[\frac{d^2\varphi(\zeta)}{\partial \zeta^2} a\cos(\theta+\psi) + \varepsilon\frac{\partial^2 u_1}{\partial \zeta^2} + \varepsilon^2\frac{\partial^2 u_2}{\partial \zeta^2} + \ldots\right] +$$
$$+ \frac{\alpha l^3}{EI}(r_0 + l\zeta)\left[\frac{d\varphi(\zeta)}{d\zeta} a\cos(\theta+\psi) + \varepsilon\frac{\partial u_1}{\partial \zeta} + \varepsilon^2\frac{\partial u_2}{\partial \zeta} + \ldots\right] +$$
$$+ \frac{\mu l^4}{EI}\left\{ \varphi(\zeta) a\left[-(p_c + \varepsilon B_1 + \varepsilon^2 B_2 + \ldots)^2 \cos(\theta+\psi) - \right.\right.$$
$$-2(\varepsilon A_1 + \varepsilon^2 A_2 + \ldots)(p_c + \varepsilon B_1 + \varepsilon^2 B_2 + \ldots)\sin(\theta+\psi) -$$
$$\left. - (\varepsilon \dot{B}_1 + \varepsilon^2 \dot{B}_2 + \ldots) a\sin(\theta+\psi) + (\varepsilon \dot{A}_1 + \varepsilon^2 \dot{A}_2 + \ldots)\cos(\theta+\psi)\right] + \varepsilon\frac{\partial^2 u_1}{\partial t^2} +$$
$$\left. + \varepsilon^2\frac{\partial^2 u_2}{\partial t^2} + \ldots \right\} + \frac{\varepsilon l}{EI}\frac{\partial^2}{\partial \zeta^2}\vec{\Phi}\left[\frac{d^2\varphi(\zeta)}{\partial \zeta^2} a\cos(\theta+\psi) + \right.$$
$$\left.\left. + \varepsilon\frac{\partial^2 u_1}{\partial \zeta^2} + \varepsilon^2\frac{\partial^2 u_2}{\partial \zeta^2} + \ldots\right] + \frac{\varepsilon q_0 l^3}{EI}\sin\theta \right\} \varphi\cos(\theta+\psi) \times$$
$$\times d(\theta+\psi)\, d\zeta = 0; \quad (24.7)$$

$$\int_0^1 \int_0^{2\pi} \left\{ \frac{d^4\varphi(\zeta)}{\partial \zeta^4} a\cos(\theta+\psi) + \varepsilon\frac{\partial^4 u_1}{\partial \zeta^4} + \varepsilon^2\frac{\partial^4 u_2}{\partial \zeta^4} + \ldots + \right.$$
$$+ \left(P_0 + \alpha r_0 l\zeta - \frac{\alpha l^2}{2}\zeta^2\right)\left[a\frac{d^2\varphi(\zeta)}{\partial \zeta^2} a\cos(\theta+\psi) + \varepsilon\frac{\partial^2 u_1}{\partial \zeta^2} + \right.$$
$$+ \varepsilon^2\frac{\partial^2 u_2}{\partial \zeta^2} + \ldots + \frac{\alpha l^3}{EI}(r_0 + l\zeta)\left[\frac{d\varphi(\zeta)}{d\zeta} a\cos(\theta+\psi) + \right.$$
$$+ \varepsilon\frac{\partial u_1}{\partial \zeta} + \varepsilon^2\frac{\partial u_2}{\partial \zeta} + \ldots\right] + \frac{\mu l^4}{EI}\left\{ \varphi(\zeta) a\left[-(p_c + \varepsilon B_1 + \right.\right. \quad (24.8)$$

(24.8) cont.

$$+\varepsilon^2 B_2+\ldots)^2 \cos(\theta+\psi)-2(\varepsilon A_1+\varepsilon^2 A_2+\ldots)(p_c+\varepsilon B_1+\varepsilon^2 B_2+\ldots)\sin(\theta+\psi)-$$

$$-(\varepsilon \dot{B}_1+\varepsilon^2 \dot{B}_2+\ldots)\,a\sin(\theta+\psi)+\varepsilon_2 A_1+\varepsilon^2 A_2+\ldots)\cos(\theta+\psi)]+$$

$$+\varepsilon\frac{\partial^2 u_1}{\partial t^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial t^2}+\ldots\Big\}+\frac{\varepsilon l}{EI}\frac{\partial^2}{\partial\zeta^2}\vec{\Phi}\Big[\frac{d^2\varphi(\zeta)}{d\zeta^2}a\cos(\theta+\psi)+\varepsilon\frac{\partial^2 u_1}{\partial\zeta^2}+$$

$$+\varepsilon\frac{\partial^2 u_2}{\partial\zeta^2}+\ldots\Big]-\frac{\varepsilon q_0 l^3}{EI}\sin\theta\Big\}\varphi\sin(\theta+\psi)\,d(\theta+\psi)\,d\zeta=0. \qquad (24.8)$$

Here

$$\dot{A}_1=\frac{\partial A_1}{\partial a}\frac{\partial a}{\partial t}+\frac{\partial A_1}{\partial\psi}\frac{\partial\psi}{\partial t}+\varepsilon\frac{\partial A_1}{\partial t},$$

$$\dot{B}_1=\frac{\partial B_1}{\partial a}\frac{\partial a}{\partial t}+\frac{\partial B_1}{\partial\psi}\frac{\partial\psi}{\partial t}+\varepsilon\frac{\partial B_1}{\partial t}.$$

Keeping in mind equations (24.3) and (24.4), the expressions for $\dot{A}_1$ and $\dot{B}_1$ can be presented in the first approximation, that is, up to terms in the small parameter raised to the first power, in the following way:

$$\dot{A}_1=[p_c-p(\varepsilon t)]\frac{\partial A_1}{\partial\psi}; \qquad (24.9)$$

$$\dot{B}_1=[p_c-p(\varepsilon t)]\frac{\partial B_1}{\partial\psi}. \qquad (24.10)$$

The equations of the first approximation for the determination of the frequency of vibrations θ and the phase shift angle ψ are obtained, if in the equations (24.7) and (24.8) we equate to zero the expressions which multiply the small parameter to the first power, and consider also (24.9) and (24.10).

$$\int_0^1 \int_0^{2\pi} \left\{ -\frac{2\mu l^4}{EI} \varphi(\zeta) \left\{ a p_c B_1 \cos(\theta+\psi) + p_c A_1 \sin(\theta+\psi) + \right.\right.$$
$$+ \frac{1}{2} \frac{\partial B_1}{\partial \psi} [p_c - p(\varepsilon t)] a \sin(\theta+\psi) - \frac{1}{2} \frac{\partial A_1}{\partial \psi} [p_c - p(\varepsilon t)] \cos(\theta+\psi) \Big\} +$$
$$+ \frac{l}{EI} \frac{\partial^2}{\partial \zeta^2} \left[\vec{\Phi} \left(\frac{d^2 \varphi(\zeta)}{\partial \zeta^2} a \cos(\theta+\psi) \right) \right] + \frac{\partial^4 u_1}{\partial \zeta^4} -$$
$$- \frac{l^2}{EI} \left(P_0 - \alpha r_0 l \zeta - \frac{\alpha l^2}{2} \zeta^2 \right) \frac{\partial^2 u_1}{\partial \zeta^2} + \frac{\mu l^4}{EI} \frac{\partial^2 u_1}{\partial t^2} +$$
$$\left. + \frac{\alpha l^3}{EI} (r_0 + l\zeta) \frac{\partial u_1}{\partial \zeta} - q_0 l^3 \frac{\sin\theta}{EI} \right\} \varphi(\zeta) \cos(\theta+\psi)\, d(\theta+\psi)\, d\zeta = 0;$$

$$\int_0^1 \int_0^{2\pi} \left\{ -\frac{2\mu l^4}{EI} \varphi(\zeta) \left\{ a p_c B_1 \cos(\theta+\psi) + p_c A_1 \sin(\theta+\psi) + \right.\right.$$
$$+ \frac{1}{2} \frac{\partial B_1}{\partial \psi} [p_c - p(\varepsilon t)] a \sin(\theta+\psi) - \frac{1}{2} \frac{\partial A_1}{\partial \psi} [p_c - p(\varepsilon t)] \cos(\theta+\psi) \Big\} +$$
$$+ \frac{l}{EI} \frac{\partial^2}{\partial \zeta} \left[\vec{\Phi} \left(\frac{d^2 \varphi(\zeta)}{\partial \zeta^2} a \cos(\theta+\psi) \right) \right] + \frac{\partial^4 u_1}{\partial \zeta^4} -$$
$$- \frac{l^2}{EI} \left(P_0 - \alpha r_0 l \zeta - \frac{\alpha l^2}{2} \zeta \right) \frac{\partial^2 u_1}{\partial \zeta^2} + \frac{\mu l^4}{EI} \frac{\partial^2 u_1}{\partial t^2} +$$
$$\left. + \frac{\alpha l^3}{EI} (r_0 + l\zeta) \frac{\partial u_1}{\partial \zeta} - q_0 l^3 \frac{\sin\theta}{EI} \right\} \varphi(\zeta) \sin(\theta+\psi)\, d(\theta+\psi)\, d\zeta = 0.$$

Carrying out the integration in the last equations with respect to $(\theta + \psi)$ and substituting for $\sin\theta$ in the last term in the curly brackets $\sin(\theta+\psi)\cos\psi - \cos(\theta+\psi)\sin\psi$ and taking account of the fact that u_1 does not contain the fundamental harmonic, we obtain

$$\int_0^1 \left\{ -2\mu l^3 \varphi^2(\zeta) \left[a p_c \pi B_1 - \frac{1}{2} (p_c - p(\varepsilon t)) \pi \frac{\partial A_1}{\partial \psi} \right] + \right.$$
$$+ \oint \left\{ \frac{\partial^2}{\partial \zeta} \vec{\Phi} \left[\frac{d^2 \varphi(\zeta)}{\partial \zeta^2} a \cos(\theta+\psi) \right] \cos(\theta+\psi)\, d(\theta+\psi) + \right.$$
$$\left.\left. + q_0 l^2 \pi \sin\psi \right\} \varphi(\zeta) \right\} d\zeta = 0; \qquad (24.11)$$

$$\int_0^1 \left\{ -2\mu l^3 \varphi^2(\zeta) \left[p_c \pi A_1 + \frac{1}{2} (p_c - p(\varepsilon t)\, \pi a \frac{\partial B_1}{\partial \psi} \right] + \right.$$
$$+ \oint \left\{ \frac{\partial^2}{\partial \zeta^2} \vec{\bar{\Phi}} \left[\frac{d^2 \varphi(\zeta)}{d\zeta^2} a \cos(\theta+\psi) \right] \sin(\theta+\psi)\, d(\theta+\psi) - \right.$$
$$\left. \left. - q_0 l^2 \pi \cos\psi \right\} \varphi(\zeta) \right\} d\zeta = 0. \tag{24.12}$$

We denote

$$\alpha_1 = -\int_0^1 2\mu l^3 \phi^2(\zeta)\, d\zeta; \tag{24.13}$$

$$\beta = \int_0^1 \phi(\zeta)\, d\zeta; \tag{24.14}$$

$$\Phi_c(a) = \oint \int_0^1 \frac{\partial^2}{\partial \zeta^2} \vec{\bar{\Phi}} \left[\frac{d^2\phi(\zeta)}{d\zeta^2} a \cos(\theta+\psi) \right] \phi(\zeta) \cos(\theta+\psi)\, d(\theta+\psi)\, d\zeta; \tag{24.15}$$

$$\Phi_s(a) = \oint \int_0^1 \frac{\partial^2}{\partial \zeta^2} \vec{\bar{\Phi}} \left[\frac{d^2\phi(\zeta)}{\partial \zeta^2} a \cos(\theta+\psi) \right] \phi(\zeta) \sin(\theta+\psi)\, d(\theta+\psi)\, \partial\zeta \tag{24.16}$$

Equations (24.11) and (24.12) can then be written in an abbreviated form:

$$\alpha_1 p_c \pi a B_1 - \frac{1}{2} \alpha_1 \pi [p_c - p(\varepsilon t)] \frac{\partial A_1}{\partial \psi} + \Phi_c(a) + \beta q_0 l^2 \pi \sin\psi = 0, \tag{24.17}$$

$$\alpha_1 p_c \pi A_1 + \frac{1}{2} \alpha_1 \pi a [p_c - p(\varepsilon t)] \frac{\partial B_1}{\partial \psi} + \Phi_s(a) - \beta q_0 l^2 \pi \cos\psi = 0. \tag{24.18}$$

The system obtained of two differential equations should be solved simultaneously for A_1 and B_1 . From equation (24.18) we have

$$\frac{dA_1}{\partial \psi} = -\frac{a}{2p_c} [p_c - p(\varepsilon t)] \frac{\partial^2 B_1}{\partial \psi^2} - \frac{\beta q_0 l^2}{p_c \alpha_1} \sin\psi.$$

Substituting this expression for the derivative into equation (24.17) we obtain

$$\alpha_1 p_c \pi a B_1 + \frac{\alpha_1 \pi a}{2p_c} [p_c - p(\varepsilon t)]^2 \frac{\partial^2 B_1}{\partial \psi^2} +$$
$$+ \frac{\beta q l^2 \pi}{2p_c} (2p_c - p(\varepsilon t) \sin\psi + \Phi_c(a) = 0. \tag{24.19}$$

Dividing the entire equation by coefficient of the second derivative of B_1, we obtain

$$\frac{\partial^2 B_1}{\partial\psi^2} + \frac{4p_c^2}{[p_c - p(\epsilon t)]^2} B_1 = -\frac{4p_c \Phi_c(a)}{\alpha_1 \pi a [p_c - p(\epsilon t)]^2} - \frac{2q_0 l^2 \beta \sin\psi}{\alpha_1 a [p_c - p(\epsilon t)]^2} [3p_c - p(\epsilon t)]. \tag{24.20}$$

The solution of this differential equation is*

$$B_1 = C \sin\psi - \frac{\Phi_c(a)}{\pi \alpha_1 a p_c}, \tag{24.21}$$

where C is some constant, which we find if, after substitution of equation (24.21) in equation (24.20), we equate in its right and left sides, the coefficients of $\sin\psi$, that is:

$$C\left[\frac{4p_c^2}{[p_c - p(\epsilon t)]^2} - 1\right] = -\frac{2q_0 l^2 \beta [3p_c - p(\epsilon t)]}{[p_c - p(\epsilon t)]^2 \alpha_1 a}.$$

Whence

$$C = -\frac{2q_0 l^2 \beta}{\alpha_1 a [p_c + p(\epsilon t)]}.$$

In this way we finally obtain

$$B_1 = -\frac{2q_0 l^2 \beta}{\alpha_1 a (p_c + p(\epsilon t))} \sin\psi - \frac{\Phi_c(a)}{\alpha_1 \pi a p_c}. \tag{24.22}$$

Taking the derivative of B_1 with respect to ψ, according

---

*Due to the fact that the complete solution must not contain secular terms, the complementary solution is discarded.

to formula (24.22) and substituting the expression for it into equation (24.18) we obtain an equation from which we determine the function A_1 .

$$\frac{\partial B_1}{\partial \psi} = -\frac{2q_0 l^2 \beta}{\alpha_1 a [p_c + p(\varepsilon t)]} \cos \psi;$$

$$\alpha_1 p_c \pi A_1 - \frac{\pi q_0 l^2 \beta [p_c - p(\varepsilon t)]}{[p_c + p(\varepsilon t)]} \cos \psi + \Phi_s(a) - q_0 \beta l^2 \pi \cos \psi = 0,$$

from which

$$A_1 = \frac{2q_0 l^2 \beta \cos \psi}{\alpha_1 [p_c + p(\varepsilon t)]} - \frac{\Phi_s(a)}{\alpha_1 \pi p_c}. \qquad (24.23)$$

On the basis of (24.3) and (24.23) in the first approximation we can write

$$\frac{da}{dt} = \frac{2\varepsilon q_0 l^2 \beta \cos \psi}{\alpha_1 [p_c + p(\varepsilon t)]} - \frac{\varepsilon \Phi_s(a)}{\alpha_1 \pi p_c}. \qquad (24.23')$$

Using (24.4) and (24.22) we find in the first approximation

$$\frac{d\psi}{dt} = p_c - p(\varepsilon t) - \frac{2q_0 l^2 \beta \sin \psi}{\alpha_1 a (p_c - p(\varepsilon t))} - \frac{\Phi_c(a)}{\alpha_1 \pi a p_c}, \qquad (24.24)$$

where $\alpha_1, \beta, \Phi_s(a)$ and $\Phi_c(a)$ are defined by formulas (24.13) — (24.16).

25. Construction of the resonance curve for vibrations of a blade

For the construction of the resonance curve it is necessary to solve equations (24.23') and (24.24) simultaneously. This requires the expression of the frequency

$p(\varepsilon t)$ as a function of time.

$$p(\varepsilon t) = p_c' + \varepsilon t, \qquad (25.1)$$

where it is assumed that $p_c' < p_c$.

Examining the length of time of vibration of the system, during which the frequency varies from $0.8 p_c$ to $1.2 p_c$, equation (25.1) can be rewritten thus:

$$p(\varepsilon t) = 0{,}8\, p_c + \varepsilon t.$$

Then evidently it is possible to write:

$$\varepsilon t = 0{,}4\, p_c.$$

Taking the time t = 1 sec. during which for the passage through the resonance the frequency changes from $0.8 p_c$ to $1.2 p_c$, the small parameter ε, can be determined, which represents the speed of passage through the resonance, i.e.

$$\varepsilon = \frac{0{,}4 p_c}{t} = \frac{0{,}4 p_c}{1} = 0{,}4 p_c.$$

In accordance with the data given in section 23 the construction of the resonance curve is carried out for the following values of the quantities entering into the formulas (24.23) and (24.24):

$$q_0 = 4 \cdot 10^{-3} \ \text{кг/см},$$
$$l = 42 \ \text{см},$$
$$p_c = 458{,}5 \ ^1/\text{сек},$$
$$\beta = \int_0^1 \varphi(\zeta)\, d\zeta = 0{,}3832;$$
$$a_1 = -\int_0^1 2\mu l^3 \varphi^3(\zeta)\, d\zeta = -0{,}5224;$$
$$n = 2, \ \nu = 3{,}1,$$
$$\iint_F z^{n+1}\, dz\, dy = 0{,}3565,$$
$$E = 2{,}2 \cdot 10^6 \ \text{кг/см}^2.$$

Moreover, according to (24.15) and (24.16) we have

$$\varepsilon\Phi_s(a) = \oint\int_0^1 \frac{\partial^2}{\partial\zeta^2}\left\{\mp\frac{\nu E}{nl^n}a^n\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\times\right.$$

$$\left.\times[(1\pm\cos(\theta+\psi)^n-2^{n-1}]\right\}\varphi(\zeta)\sin(\theta+\psi)\,d(\theta+\psi)\iint_F z^{n+1}\,dz\,dy,$$

$$\varepsilon\Phi_c(a) = \oint\int_0^1 \frac{\partial^2}{\partial\zeta^2}\left\{\mp\frac{\nu E}{nl^n}a^n\left(\frac{d^2\varphi(\zeta)}{\partial\zeta^2}\right)^n\times\right.$$

$$\left.\times[(1\pm\cos(\theta+\psi)^n-2^{n-1}]\right\}\varphi(\zeta)\cos(\theta+\psi)\,d(\theta+\psi)\iint_F z^{n+1}\,dz\,dy;$$

$$\int_0^1 \frac{\partial^2}{\partial\zeta^2}\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^2\varphi(\zeta)\,d\zeta = 30,$$

$$\varepsilon\Phi_s(a) = -\frac{\nu E\cdot 0{,}35652\cdot 30a^2}{3l^2} = -5{,}5135\cdot 10^4a^2;$$

$$\varepsilon\Phi_c(a) = \frac{\nu E\pi\cdot 0{,}35652\cdot 30a^2}{l^2} = -12{,}99\cdot 10^{-4}a^2.$$

After substitution of the values obtained into formulas (24.23) and (24.24), for the problem in question these formulas will finally take the following form:

$$\frac{da}{dt} = -\frac{10{,}3526\cos\psi}{825{,}3+\varepsilon t} - 0{,}73276\cdot 10^2a^2; \tag{25.2}$$

$$\frac{d\psi}{dt} = 91{,}7 - \varepsilon t + \frac{10{,}3526\sin\psi}{(825{,}3+\varepsilon t)a} - 172{,}65a. \tag{25.3}$$

For the solution of the system of the differential equation (25.2) and (25.3) which we have just obtained, under the initial conditions

$$\left(\frac{da}{dt}\right)_{t=0} = 0,$$

$$\left(\frac{d\psi}{dt}\right)_{t=0} = 0,$$

it is expedient to use a numerical method according to which amplitude a and phase shift ψ can be represented by the following series:

$$a=a_0+\left(\frac{da}{dt}\right)\Delta t+\frac{1}{2}\left(\frac{d^2a}{dt^2}\right)(\Delta t)^2+\frac{1}{6}\left(\frac{d^3a}{dt^3}\right)(\Delta t)^3;$$

$$\psi=\psi_0+\left(\frac{d\psi}{dt}\right)\Delta t+\frac{1}{2}\left(\frac{d^2\psi}{dt^2}\right)(\Delta t)^2+\frac{1}{6}\left(\frac{d^3\psi}{dt^3}\right)(\Delta t)^3.$$

The results of calculations for Δt = 0.1 sec. for the chosen interval of time of passage through resonance t = 1 sec. are shown in Table 11.

Table 11

t	Δt	a	ψ	$\frac{da}{dt}$	$\frac{d\psi}{dt}$	$\frac{d^2a}{dt^2}$	$\frac{d^2\psi}{dt^2}$	$\frac{d^3a}{dt^3}$	$\frac{d^3\psi}{dt^3}$
0		$1{,}231\cdot10^{-4}$	4,647	0	0	$-2{,}453\cdot10^{-7}$	−160,8	2,011	−10,1
0,5	0,05	$1{,}650\cdot10^{-4}$	4,377	—	—	—	—	—	—
0,1	0,05	$4{,}581\cdot10^{-4}$	3,972	$0{,}827\cdot10^{-2}$	63,53	−0,581	−180,5	27,92	3110,3
0,2	0,1	$1{,}173\cdot10^{-4}$	9,941	$0{,}945\cdot10^{-2}$	59,79	−0,356	−684,0	15,55	8186,5
0,3	0,1	$2{,}928\cdot10^{-3}$	13,865	$0{,}258\cdot10^{-2}$	50,02	0,574	−184,5	6,493	−860,0
0,4	0,1	$7{,}138\cdot10^{-3}$	17,799	$0{,}535\cdot10^{-2}$	28,48	0,278	−208,4	−6,870	−9,633
0,5	0,1	$7{,}918\cdot10^{-3}$	19,603	$-1{,}285\cdot10^{-2}$	9,61	$9{,}110\cdot10^{-2}$	−169,9	−0,704	−10,55
0,6	0,1	$6{,}972\cdot10^{-3}$	19,712	$-0{,}723\cdot10^{-2}$	8,33	$-6{,}118\cdot10^{-2}$	−182,4	−1,168	−3,700
0,7	0,1	$5{,}748\cdot10^{-3}$	17,966	$0{,}931\cdot10^{-2}$	29,10	$25{,}33\cdot10^{-2}$	−174,8	7,183	−77,30
0,8	0,1	$7{,}270\cdot10^{-3}$	14,169	$0{,}353\cdot10^{-2}$	44,78	$-47{,}20\cdot10^{-2}$	−180,3	−2,378	81,86
0,9	0,1	$4{,}161\cdot10^{-3}$	8,903	—	—	—	—	—	—

The resonance curve of Fig. 14 was constructed in accordance with data of Table 11. Comparing the resulting

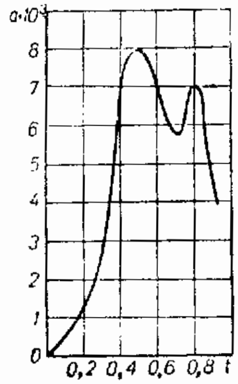

Fig. 14

curve of change in amplitude versus time for the transition through resonance from $0.8 p_c$ to $1.2 p_c$ during 1 sec. with the resonance curve for steady-state vibrations (Fig. 13), it can be concluded that in the former case, for rapid passage through resonance a shift of maximum amplitude to the right can be observed and also, a substantial decrease (about four times) of maximum value of the amplitude takes place in comparison with its resonance value for a steady-state regime of vibrations.

The marked decrease of amplitude in the present case should be explained by the fact that for a rapid transition through resonance the amplitude does not have time to grow to values corresponding to a steady-state regime of vibrations.

Chapter VI

Transverse Vibrations of Short Bars Applied to Design of Turbine Blades

26. Derivation of the basic differential equation

The problem of transverse vibrations of short bars, taking account of energy dissipation in the materials is of great practical interest. We shall solve this problem approximately using the methods of structural mechanics. For greater accuracy the rotatory inertia of the elements of mass of the bar and also the shear deformation will be taken into account.

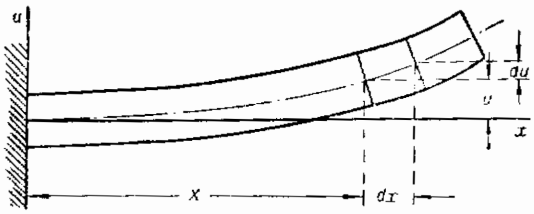

Fig. 15

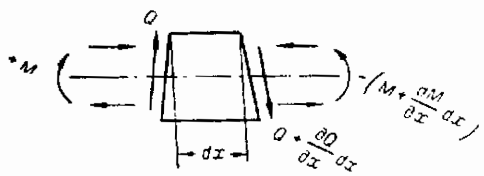

Fig. 16

We now turn to the derivation of the differential equation of transverse vibrations of a short bar. We take the system of coordinates in such a way that at rest, the longitudinal axis of the rod coincides with the x-axis (Fig. 15). The forces acting on an element of the bar which we have isolated are shown in Fig. 16.

First we examine the forces affecting the translational motion. They are the shear forces $+Q-\left(Q+\frac{\partial Q}{\partial x}dx\right)$, and also the inertia forces $-\varrho F\frac{\partial^2 u}{\partial t^2}dx$. Projecting these forces on a vertical axis we obtain

$$-\varrho F\frac{\partial^2 u}{\partial t_2}+\frac{\partial Q}{\partial x}=0, \tag{26.1}$$

where ϱ is the density of the material, F is the cross-sectional area of the rod, u is the translational displacement of the element. Besides the translational motion, the element in question also rotates in the plane under the action of the normal and shear stresses.

In order to form the equation of motion taking account of the inertia of rotation of the element of the rod, it is necessary to express the angle between the axis of an element and the x-axis; this angle depends not only on the rotation of the cross-section of the rod but also on its shear. Denoting by θ the angle of inclination of the tangent to the deflection curve, without taking account of the shear forces and by γ the angle of shear at the neutral axis in the same cross-section we obtain the total angle between the axis of the element and the x-axis:

$$\frac{du}{dx}=\theta+\gamma.$$

Well known relations hold between the bending moment and the shear force, on one hand, and the angles θ and γ on the other:

$$M = -EI_y \frac{d\theta}{dx}, \quad Q = k\gamma FG = k\left(\frac{du}{dx} - \theta\right) FG, \tag{26.2}$$

where k is a coefficient depending on the form of the cross-section, G is the shear modulus of elasticity.

The moment of inertia for rotation of the mass of the element of the bar equals

$$dy \frac{\partial^2\theta}{\partial t^2} = \frac{\partial^2\theta}{\partial t^2} \int_F u^2\, dm = \frac{\partial^2\theta}{\partial t^2} \int_F u^2 \varrho\, dF\, dx = \varrho I_y \frac{\partial^2\theta}{\partial t^2} dx.$$

Thus, the equation of the dynamic equilibrium of the moments of the forces assumes the form

$$\left(Q - \frac{\partial M}{\partial x}\right) dx = \varrho I_y \frac{\partial^2\theta}{\partial t^2} dx. \tag{26.3}$$

Using the relation (26.2) we obtain the differential equations of the rotation and of translatory motion of the element:

$$k\left(\frac{\partial u}{\partial x} - \theta\right) FG + EI_y \frac{\partial^2\theta}{\partial x^2} - \varrho I_y \frac{\partial^2\theta}{\partial t^2} = 0, \tag{26.4}$$

$$-\varrho F \frac{\partial^2 u}{\partial t^2} = -k\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial\theta}{\partial x}\right) FG. \tag{26.5}$$

Eliminating the angle θ, from equations (26.4) and (26.5), we obtain a differential equation of vibrations of a short bar taking account of the rotatory inertia of the mass of the rod and the shearing deformation.

$$EI_y \frac{\partial^4 u}{\partial x^4} + \varrho F \frac{\partial^2 u}{\partial t^2} - \left(\varrho I_y + \varrho \frac{EI_y}{kG}\right) \frac{\partial^4 u}{\partial x^2 \partial t^2} + \frac{\varrho^2 I_y}{kG} \frac{\partial^4 u}{\partial t^4} = 0. \tag{26.6}$$

In the case of forced, steady-state vibrations in the presence of damping forces the differential equation (26.6) must be supplemented by a term characterizing the dissipation of energy in vibrations and also by a term depending on the external exciting force which maintains the vibration at one amplitude in the steady state.

If we adhere to the hypothesis on the dependence of the dissipation of energy in the material in vibrations on the value of normal and shear stresses, the equation of the steady-state forced transverse vibrations of the short rod, accounting for dissipation, can be expressed, as follows, in a general form:

$$EI_y\frac{\partial^4 u}{\partial x^4}+\varrho F\frac{\partial^2 u}{\partial t^2}-\left(\varrho I_y+\frac{EI_y\varrho}{kG}\right)\frac{\partial^4 u}{\partial x^2\partial t^2}+\frac{\varrho^2 I_y}{kG}\frac{\partial^4 u}{\partial t^4}+$$
$$+\varepsilon\frac{\partial^2}{\partial x^2}\overset{\rightleftarrows}{\Phi}(u)+\varepsilon\frac{\partial}{\partial x}\overset{\rightleftarrows}{\Psi}(u)=\varepsilon q\cos\omega t. \tag{26.7}$$

Here $\epsilon\frac{\partial^2}{\partial x^2}\overset{\rightleftarrows}{\Phi}(u)$ and $\epsilon\frac{\partial}{\partial x}\overset{\rightleftarrows}{\Psi}(u)$ account for the dissipation of energy in the material due to the normal and shear stresses respectively. The term $\epsilon q\cos\omega t$ depends on the external periodical disturbing force; ϵ is a small parameter.

Before turning to the solution of the differential equation (26.7) we examine the functionals $\overset{\rightleftarrows}{\Phi}(u)$ and $\overset{\rightleftarrows}{\Psi}(u)$. As before, we proceed from the nonlinear dependence between normal stress and strain during loading and unloading of the material

$$\vec{\sigma}=\sigma_y+\vec{f}(\xi),$$
$$\overleftarrow{\sigma}=\sigma_y+\overleftarrow{f}(\xi), \tag{26.8}$$

where σ_y is the "elastic" stress, the value of which is not altered for loading or unloading; $f(\xi)$ is the stress arising from the losses in the material; its values differ for loading and unloading.

In conformity with equations (26.8) the bending moment in the cross-section of the rod at any time during loading and unloading can be expressed thus:

$$\overrightarrow{M} = M_y + \overrightarrow{M}_s \ ,$$

$$\overleftarrow{M} = M_y + \overleftarrow{M}_s \ , \qquad (26.9)$$

where M_y is the moment of the elastic forces, and M_s is the moment of the dissipative forces, which is just $\varepsilon \Phi(u)$. Considering that

$$M_y = M = EI_y \frac{\partial\theta}{\partial x} = EI_y \left(\frac{\partial^2 u}{\partial x^2} - \frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right),$$

$$\sigma_y = \left(\frac{\partial^2 u}{\partial x^2} - \frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right) Ez,$$

(26.10)

we replace (26.8) by the relation

$$\overrightarrow{\overleftarrow{\sigma}} = E\left\{\xi \mp \frac{\eta}{\varkappa}\left[(\xi_0 \pm \xi)^{\varkappa} - 2^{\varkappa-1}\xi_0^{\varkappa}\right]\right\}, \qquad (26.11)$$

where η and $\varkappa$ are the parameters of the hystersis loop, which must be determined for each material from experiment. Then

$$\xi_0 = \left(\frac{\partial^2 u}{\partial x^2} - \frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right)_{t=0} z,$$

$$\xi = \left(\frac{\partial^2 u}{\partial x^2} - \frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right) z,$$

(26.12)

where z is the coordinate of the point of cross-section. The equations for the bending moment in expanded form can be expressed as:

$$\vec{M}=E\int_F\left(\frac{\partial^2 u}{\partial x^2}-\frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right)z^2\,dF-\frac{\eta F}{\varkappa}\int_F\left\{\left[\left(\frac{\partial^2 u}{\partial x^2}-\frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right)_{t=0}+\right.\right.$$
$$\left.\left.+\left(\frac{\partial^2 u}{\partial x^2}-\frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right)\right]^{\varkappa}z^{\varkappa}-2^{\varkappa-1}\left(\frac{\partial^2 u}{\partial x^2}-\frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right)^{\varkappa}_{t=0}z^{\varkappa}\right\}z\,dF;$$
$$\overleftarrow{M}=E\int_F\left(\frac{\partial^2 u}{\partial x^2}-\frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right)z^2\,dF+\frac{\eta E}{\varkappa}\int_F\left\{\left[\left(\frac{\partial^2 u}{\partial x^2}-\frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right)_{t=0}-\right.\right.$$
$$\left.\left.-\left(\frac{\partial^2 u}{\partial x^2}-\frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right)\right]^{\varkappa}z^{\varkappa}-2^{\varkappa-1}\left(\frac{\partial^2 u}{\partial x^2}-\frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right)^{\varkappa}_{t=0}z^{\varkappa}\right\}z\,dF. \qquad (26.13)$$

It follows from (26.13) that

$$\varepsilon\vec{\Phi}(u)=-\frac{\eta E}{\varkappa}\int_F\left\{\left[\left(\frac{\partial^2 u}{\partial x^2}-\frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right)_{t=0}+\right.\right.$$
$$\left.\left.+\left(\frac{\partial^2 u}{\partial x^2}-\frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right)\right]^{\varkappa}z^{\varkappa}-2^{\varkappa-1}\left(\frac{\partial^2 u}{\partial x^2}-\frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right)^{\varkappa}_{t=0}z^{\varkappa}\right\}z\,dF;$$
$$\varepsilon\overleftarrow{\Phi}(u)=\frac{\eta E}{\varkappa}\int_F\left\{\left[\left(\frac{\partial^2 u}{\partial x^2}-\frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right)_{t=0}-\right.\right.$$
$$\left.\left.-\left(\frac{\partial^2 u}{\partial x^2}-\frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right)\right]^{\varkappa}z^{\varkappa}-2^{\varkappa-1}\left(\frac{\partial^2 u}{\partial x^2}-\frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right)^{\varkappa}_{t=0}z^{\varkappa}\right\}z\,dF.$$

(26.14)

To determine the form of the functional $\Psi(u)$, we write the expression for the shear force which appears at a section of the short bar during its transverse vibration:

$$\vec{Q}=Q_y+\vec{Q}_*,$$
$$\overleftarrow{Q}=Q_y+\overleftarrow{Q}_*, \qquad (26.15)$$

where Q_y is the shear force, which represents a resultant of shear stresses; here

$$Q_y=k\left(\frac{\partial u}{\partial x}-\theta\right)FG.$$

(26.16)

Considering that

$$\frac{\partial\theta}{\partial x}=\frac{\partial^2 u}{\partial x^2}-\frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2},$$

we obtain

$$Q_s = \int_x^l k\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\varrho}{kG}\frac{\partial^2 u}{\partial t^2}\right) FG\,dx = \int_x^l \varrho F \frac{\partial^2 u}{\partial t^2}\,dx.$$

Q_s is the shear force brought about by the losses in material due to the shear stresses, which, by virtue of (26.7), is $\varepsilon\Psi(u)$.

We obtain the expression $\varepsilon\Psi(u)$ for the descending and ascending motions if we assume a certain nonlinear relation between the shear strain γ and the shear stresses τ. Following the accepted hypothesis, we assume

$$\tau = G\left\{\gamma \mp \frac{\nu}{n}\left[(\gamma_0 \pm \gamma)^n - 2^{n-1}\gamma_0^n\right]\right\},$$

where G is the shear modulus, and ν and n are the parameters of the hystersis loop, obtained in the coordinates $\gamma - \tau$ and which are determined from experiment. Further,

$$\tau = \frac{QS_{(z)}}{yI_y} = \frac{S_{(z)}}{yI_y}\int_x^l \varrho F \frac{\partial^2 u}{\partial t^2}\,dx = G\gamma,$$

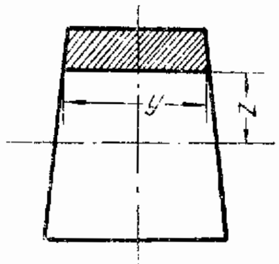

Fig. 17

where Q is the shear force; $S_{(z)}$ is the first moment of part of the cross-section of the bar (Fig. 17), y is the width of rod at a distance z from the y principal axis of inertia of the cross-section. The shear strain equals

$$\gamma_0 = \frac{S_{(z)}\varrho F}{GyI_y}\left[\int_x^l \frac{\partial^2 u}{\partial t^2}\,dx\right]_{t=0},$$

$$\gamma = \frac{S_{(z)}\varrho F}{GyI_y}\int_x^l \frac{\partial^2 u}{\partial t^2}\,dx. \qquad (26.17)$$

Then the expanded expressions (26.15) for the shear force at any time during ascending and descending motions take the following form:

$$\vec{Q} = \int_F \frac{S_{(z)}\varrho F}{yI_y} \int_x^l \frac{\partial^2 u}{\partial t^2} dx\, dF - \frac{\nu G}{n} \int_F \Big\{ \Big[\frac{S_{(z)}\varrho F}{yI_yG} \Big(\int_x^l \frac{\partial^2 u}{\partial t^2} dx \Big)_{t=0} + \frac{S_{(z)}\varrho F}{yI_yG} \int_x^l \frac{\partial^2 u}{\partial t^2} dx \Big]^n - 2^{n-1} \Big[\frac{S_{(z)}\varrho F}{yI_yG} \Big(\int_x^l \frac{\partial^2 u}{\partial t^2} dx \Big)_{t=0} \Big]^n \Big\} dF;$$

$$\overleftarrow{Q} = \int_F \frac{S_{(z)}\varrho F}{yI_y} \int_x^l \frac{\partial^2 u}{\partial t^2} dx\, dF + \frac{\nu G}{n} \int_F \Big\{ \Big[\frac{S_{(z)}\varrho F}{yI_yG} \Big(\int_x^l \frac{\partial^2 u}{\partial t^2} dx \Big)_{t=0} - \frac{S_{(z)}\varrho F}{yI_yG} \int_x^l \frac{\partial^2 u}{\partial t^2} dx \Big]^n - 2^{n-1} \Big[\frac{S_{(z)}\varrho F}{yI_yG} \Big(\int_x^l \frac{\partial^2 u}{\partial t^2} dx \Big)_{t=0} \Big]^n \Big\} dF. \tag{26.18}$$

The first terms of equations (26.18) represent just the shear force Q_y , determined by the integral (26.16), and the expressions containing the factor $\frac{\nu}{n}$, represent the functionals which have been examined:

$${}_\varepsilon\vec{\Psi}(u) = \vec{Q}_s = -\frac{\nu G}{n} \int_F \Big\{ \Big[\frac{S_{(z)}\varrho F}{yI_yG} \Big(\int_x^l \frac{\partial^2 u}{\partial t^2} dx \Big)_{t=0} + \frac{S_{(z)}\varrho F}{yI_yG} \int_x^l \frac{\partial^2 u}{\partial t^2} dx \Big]^n - 2^{n-1} \Big[\frac{S_{(z)}\varrho F}{yI_yG} \Big(\int_x^l \frac{\partial^2 u}{\partial t^2} dx \Big)_{t=0} \Big]^n \Big\} dF;$$

$${}_\varepsilon\overleftarrow{\Psi}(u) = \overleftarrow{Q}_s = \frac{\nu G}{n} \int_F \Big\{ \Big[\frac{S_{(z)}\varrho F}{yI_yG} \Big(\int_x^l \frac{\partial^2 u}{\partial t^2} dx \Big)_{t=0} - \frac{S_{(z)}\varrho F}{yI_yG} \int_x^l \frac{\partial^2 u}{\partial t^2} dx \Big]^n - 2^{n-1} \Big[\frac{S_{(z)}\varrho F}{yI_yG} \Big(\int_x^l \frac{\partial^2 u}{\partial t^2} dx \Big)_{t=0} \Big]^n \Big\} dF. \tag{26.19}$$

For convenience in using the differential equation (26.7) we introduce the dimensionless coordinates $\zeta = \frac{x}{l}$ and the dimensionless deflections $u^* = \frac{u}{l}$, where l is the length of the vibrating rod. Equation (26.7) in dimensionless quantities can be rewritten as:

$$\frac{\partial^4 u^*}{\partial \zeta^4} + \frac{\varrho F l^4}{E I_y} \frac{\partial^2 u^*}{\partial t^2} - \left(\frac{\varrho l^2}{E} + \frac{\varrho l^2}{kG}\right) \frac{\partial^4 u^*}{\partial \zeta^2 \partial t^2} + \frac{\varrho^2 l^4}{kGE} \frac{\partial^4 u^*}{\partial t^4} + \\ + \frac{\varepsilon l}{E I_y} \frac{\partial^2}{\partial \zeta^2} \vec{\Phi}(u^*) + \frac{\varepsilon l^2}{E I_y} \frac{\partial}{\partial \zeta} \vec{\Psi}(u^*) - \frac{\varepsilon q l^3}{E I_y} \cos \omega t = 0 \qquad (26.20)$$

To simplify the writing we introduce the following notations:

$$\alpha_1 = \frac{\varrho F l^4}{E I_y}; \quad \alpha_2 = \varrho l^2 \left(\frac{1}{E} + \frac{1}{kG}\right); \quad \alpha_3 = \frac{\varrho^2 l^4}{kGE}; \quad \alpha_4 = \frac{l^3}{E I_y}. \qquad (26.21)$$

Then

$$\frac{\partial^4 u^*}{\partial \zeta^4} + \alpha_1 \frac{\partial^2 u^*}{\partial t^2} - \alpha_2 \frac{\partial^4 u^*}{\partial \zeta^2 \partial t^2} + \alpha_3 \frac{\partial^4 u^*}{\partial t^4} + \frac{\alpha_4}{l^2} \varepsilon \frac{\partial^2}{\partial \zeta^2} \vec{\Phi}(u^*) + \\ + \frac{\alpha_4}{l} \varepsilon \frac{\partial}{\partial \zeta} \vec{\Psi}(u^*) - \alpha_4 \varepsilon q \cos \omega t = 0. \qquad (26.22)$$

To solve the nonlinear differential equations obtained, (26.22), with a "slight" nonlinearity, we shall apply, as in the previous chapters of this book, the methods of nonlinear mechanics based on developments by powers of a small parameter ε . We represent the value of the dimensionless deflection as a power series in the small parameter ε :

$$u^*(\zeta, t) = \varphi(\zeta) a \cos(\omega t + \psi) + \varepsilon u_1(\zeta, t) + \varepsilon^2 u_2(\zeta, t) + \ldots \qquad (26.23)$$

We also represent in series the square of the frequency of vibrations of the rod, ω^2 , and the magnitude of the phase shift ψ :

$$\omega^2 = \omega_c^2 + \varepsilon \Delta_1 + \varepsilon^2 \Delta_2 + \varepsilon^3 \Delta_3 + \ldots, \qquad (26.24)$$

$$\psi = \psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \varepsilon^3 \psi_3 + \ldots, \qquad (26.25)$$

where a is the amplitude of deflection at the free end of the rod, ω_c is the natural frequency of vibrations of the bar. We introduce the new variable

$$\tau = \omega t + \psi \qquad (26.26)$$

and transform $\cos \omega t = \cos(\tau - \psi)$, using the expression (26.25) for the phase shift ψ :

$$\cos(\tau-\psi)=\cos(\tau-\psi_0-\varepsilon\psi_1-\varepsilon^2\psi_2-\dots)=$$
$$=\cos(\tau-\psi_0)\cos\varepsilon(\psi_1+\varepsilon\psi_2+\dots)+\sin(\tau-\psi_0)\sin\varepsilon(\psi_1+\varepsilon\psi_2+\dots). \qquad (26.27)$$

Further, we also represent $\cos\varepsilon(\psi_1+\varepsilon\psi_2+\dots)$ and $\sin\varepsilon(\psi_1+\varepsilon\psi_2+\dots)$ in the form of series

$$\cos\varepsilon(\psi_1+\varepsilon\psi_2+\dots)=1-\frac{\varepsilon^2(\psi_1+\varepsilon\psi_2+\dots)^2}{2!};$$
$$\sin\varepsilon(\psi_1+\varepsilon\psi_2+\dots)=\varepsilon(\psi_1+\varepsilon\psi_2+\dots)-\frac{\varepsilon^3(\psi_1+\varepsilon\psi_2+\dots)^3}{3!}.$$

Substituting the values of the latter into formula (26.27) and neglecting the terms containing the small parameter to a power higher than the first, we obtain

$$\cos(\tau-\psi)=\cos(\tau-\psi_0)+\varepsilon\psi_1\sin(\tau-\psi_0)+$$
$$+\varepsilon^2\left[\psi_2\sin(\tau-\psi_0)-\frac{\psi_1^2}{2}\cos(\tau-\psi_0)\right]. \qquad (26.28)$$

We introduce the series (26.23)—(26.25) into the differential equation (26.22). Considering, moreover, the change of variable (26.26) we obtain

$$\frac{\partial^4\varphi}{\partial\zeta^4}a\cos\tau+\varepsilon\frac{\partial^4 u_1}{\partial\zeta^4}+\varepsilon^2\frac{\partial^4 u_2}{\partial\zeta^4}+\dots+\alpha_1(\omega_c^2+\varepsilon\Delta_1+\varepsilon^2\Delta_2+\dots)\times$$
$$\times\left[-\varphi(\zeta)a\cos\tau+\varepsilon\frac{\partial^2 u_1}{\partial\tau^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial\tau^2}+\dots\right]-\alpha_2\left[(\omega_c^2+\varepsilon\Delta_1+\varepsilon^2\Delta_2+\dots)\times\right.$$
$$\left.\times\left(-\frac{d^2\varphi}{d\zeta^2}a\cos\tau+\varepsilon\frac{\partial^4 u_1}{\partial\zeta^2\partial\tau^2}+\varepsilon^2\frac{\partial^4 u_2}{\partial\zeta^2\partial\tau^2}+\dots\right)\right]+$$
$$+\alpha_3\left[(\omega_c^2+\varepsilon\Delta_1+\varepsilon^2\Delta_2+\dots)^2\left(\varphi a\cos\tau+\varepsilon\frac{\partial^4 u_1}{\partial\tau^4}+\varepsilon^2\frac{\partial^4 u_2}{\partial\tau^4}+\dots\right)\right]+$$
$$+\frac{\alpha_4}{l^2}\varepsilon\frac{\partial^2}{\partial\zeta^2}[\vec{\Phi}(\varphi a\cos\tau+\varepsilon u_1+\varepsilon^2 u_2+\dots)]+$$
$$+\frac{\alpha_4}{l}\varepsilon\frac{\partial}{\partial\zeta}\left[\vec{\Psi}(\varphi a\cos\tau+\varepsilon u_1+\varepsilon^2 u_2+\dots)\right]-\alpha_4\varepsilon q\left\{\cos(\tau-\psi_0)+\right.$$
$$\left.+\varepsilon\psi_1\sin(\tau-\psi_0)+\varepsilon^2\left[\psi_2\sin(\tau-\psi_0)-\frac{\psi_1^2}{2}\cos(\tau-\psi_0)\right]\right\}=0. \qquad (26.29)$$

We group together the terms of the last equation containing as a factor the small parameter in the zeroth, first, second powers and so on; that, inasmuch as $\epsilon \neq 0$ one can equate to zero, the terms multiplying the various powers of the small parameter.

After having carried this out, instead of equation (26.29) we obtain the following system of differential equations:

$$\frac{d^4\varphi}{d\zeta^4} + \alpha_2\omega_c^2\frac{d^2\varphi}{d\zeta^2} + \alpha_3\omega_c^4\varphi - \alpha_1\omega_c^2\varphi = 0; \qquad (26.30)$$

$$\frac{\partial^4 u_1}{\partial\zeta^4} - \alpha_2\omega_c^2\frac{\partial^4 u_1}{\partial\zeta^2\,\partial\tau^2} + \alpha_1\omega_c^2\frac{\partial^2 u_1}{\partial\tau^2} - \alpha_1\Delta_1\varphi a\cos\tau +$$
$$+\alpha_2\Delta_1\frac{d^2\varphi}{d\zeta^2}a\cos\tau + \alpha_3\omega_c^4\frac{\partial^4 u_1}{\partial\tau^4} + 2\alpha_3\omega_1^2\Delta_1\varphi a\cos\tau +$$
$$+\frac{\alpha_4}{l^2}\frac{\partial^3}{\partial\zeta^2}\vec{\Phi}(\varphi a\cos\tau) + \frac{\alpha_4}{l}\frac{\partial}{\partial\zeta}\vec{\Psi}(\varphi a\cos\tau) - \alpha_4 q\cos(\tau-\psi_0) = 0; \qquad (26.31)$$

$$\frac{\partial^4 u_2}{\partial\zeta^4} - \alpha_2\omega_c^2\frac{\partial^4 u_2}{\partial\zeta^2\,\partial\tau^2} + \alpha_1\omega_c^2\frac{\partial^2 u_2}{\partial\tau^2} - \alpha_1\Delta_2\varphi a\cos\tau + \alpha_1\Delta_1\frac{\partial^2 u_1}{\partial\tau^2} +$$
$$+\alpha_2\Delta_2\frac{\partial^2\varphi}{\partial\zeta^2}a\cos\tau - \alpha_2\Delta_1\frac{\partial^4 u_1}{\partial^2\zeta\,\partial\tau^2} + \alpha_3\Delta_1^2\varphi a\cos\tau + 2\alpha_3\omega_c^2\Delta_1\frac{\partial^4 u_1}{\partial\tau^4} +$$
$$+2\alpha_3\omega_c^2\Delta_2\varphi a\cos\tau + \alpha_3\omega_c^4\frac{\partial^4 u_2}{\partial\tau^4} + \alpha_4\frac{\partial^2}{\partial\zeta^2}\vec{F}(\zeta,\tau) +$$
$$+\alpha_4\frac{\partial}{\partial\zeta}\vec{f}(\zeta,\tau) - \alpha_4 q\psi_1\sin(\tau-\psi_0) = 0.$$

. . . . (26.32)
. . . .

In equation (26.32) $\vec{F}(\zeta,\tau)$ and $\vec{f}(\zeta,\tau)$ are functionals which define more accurately in the second approximation the magnitude of the dissipation of energy in the material under the influence of normal and shear stresses.

Equations (26.30) to (26.32) are the basic ones, with the help of which it is possible to examine the influence of the dissipation of energy on the transverse vibrations of the short rod in question to various degrees of accuracy.

27. Solution of the problem in the zeroth approximation

To determine the deflection function and the frequency of vibrations of the rod in the zeroth approximation it is necessary to solve equation (26.30).

Denoting

$$-\alpha_1\omega_c^2+\alpha_3\omega_c^4=\lambda, \tag{27.1}$$

we write equation (26.30) in the form

$$\frac{d^4\varphi}{d\zeta^4}+\alpha_2\omega_c^2\frac{d^2\varphi}{d\zeta^2}+\lambda\varphi=0. \tag{27.2}$$

The corresponding characteristic equation is:

$$K^4+\alpha_2\omega_c^2K^2+\lambda=0. \tag{27.3}$$

Solving this equation we find

$$K_1^2=-\frac{\alpha_2\omega_c^2}{2}+\sqrt{\frac{\alpha_2^2\omega_c^4}{4}-\lambda},\quad K_2^2=-\frac{\alpha_2\omega_c^2}{2}-\sqrt{\frac{\alpha_2^2\omega_c^4}{4}-\lambda}. \tag{27.4}$$

The general integral of the equation (27.2) is

$$\varphi=\overline{C}_1e^{K_1\zeta}+\overline{C}_2e^{-K_1\zeta}+\overline{C}_3e^{K_2\zeta}+\overline{C}_4e^{-K_2\zeta}.$$

It is convenient to represent the expression for φ in the form of a combination of hyperbolic and trigonometric functions

$$\varphi=C_1\operatorname{sh}K_1\zeta+C_2\operatorname{ch}K_1\zeta+C_3\sin K_2\zeta+C_4\cos K_2\zeta, \tag{27.5}$$

where C_1, C_2, C_3, C_4 are constants of integrations which can be determined from the conditions at the ends of the vibrating bar.

Before turning to the determination of the constants of integration, we shall write expressions for the angle of rotation, bending moment, and the shear force:

$$\frac{\partial}{\partial\zeta}(\varphi a\cos\tau)=\theta'+\gamma'^{*},$$

$$\gamma'=\frac{Q_x}{kFG}=-\frac{\varrho l^2}{kG}\int_{\zeta}^{1}\frac{\partial^2}{\partial t^2}(\varphi a\cos\tau)\,d\zeta,$$

$$M=-\frac{EI}{l}\frac{\partial\theta'}{\partial\zeta}=-\frac{EI}{l}\left[\frac{\partial^2\varphi}{\partial\zeta^2}a\cos\tau+\frac{\partial}{\partial\zeta}\left(\frac{\varrho l^2}{kG}\int_{\zeta}^{1}\frac{\partial^2}{\partial t^2}(\varphi a\cos\tau)\,d\xi\right)\right],$$

$$M=-\frac{EIa}{l}\left[\frac{\partial^2\varphi}{\partial\zeta^2}+\frac{\varrho l^2\omega_c^2}{kG}\varphi\right]\cos\tau. \tag{27.6}$$

The expression for the shear force acting in an arbitrary section is found by using the equation of equilibrium (26.3),

$$Q=\frac{\partial M}{\partial x}+\varrho I\frac{\partial^2\theta'}{\partial t^2},$$

$$\theta'=\frac{\partial\varphi}{\partial\zeta}a\cos\tau+\frac{\varrho l^2}{kG}\int_{\zeta}^{1}\frac{\partial^2}{\partial t^2}(\varphi a\cos\tau)\,d\xi,$$

$$Q=-\frac{EIa}{l^2}\left(\frac{\partial^3\varphi}{\partial\zeta^3}+\frac{\varrho l^2\omega^2}{kG}\frac{\partial\varphi}{\partial\zeta}+\frac{\varrho l^2\omega_2}{E}\frac{\partial\varphi}{\partial\zeta}-\frac{\varrho^2 l^4\omega^4}{kGF}\int_{\zeta}^{1}\varphi\,d\xi\right)\cos\tau. \tag{27.7}$$

After substitution of values of the function and its derivatives into the formulas for θ', M, and Q, we obtain

$$\theta'=\Big\{a(K_1C_1\operatorname{ch}K_1\zeta+K_1C_2\operatorname{sh}K_1\zeta+K_2C_3\cos K_2\zeta-K_2C_4\sin K_2\zeta)+$$
$$+\frac{\varrho l^2\omega_c^2a}{kG}\left[\frac{C_1}{K_1}(\operatorname{ch}K_1-\operatorname{ch}K_1\zeta)+\frac{C_2}{K_1}(\operatorname{sh}K_1-\operatorname{sh}K_1\zeta)-\right.$$
$$\left.-\frac{C_3}{K_2}(\cos K_2-\cos K_2\zeta)+\frac{C_4}{K_2}(\sin K_2-\sin K_2\zeta)\right]\Big\}\cos\tau; \tag{27.8}$$

---

*The primes indicate that the expansion $u(\zeta,\tau)$ is limited to the first term.

$$M=-\frac{EIa}{l}\left[(K_1^2C_1\,\mathrm{sh}\,K_1\zeta+K_1^2C_2\,\mathrm{ch}\,K_1\zeta-K_2^2C_3\sin K_2\zeta-K_2^2C_4\cos K_2\zeta)+\right.$$
$$\left.+\frac{\varrho l^2\omega_c^2}{kG}(C_1\,\mathrm{sh}\,K_1\zeta+C_2\,\mathrm{ch}\,K_1\zeta+C_3\sin K_2\zeta+C_4\cos K_2\zeta)\right]\cos\tau; \qquad (27.9)$$

$$Q=-\frac{EIa}{l^2}\left\{(K_1^3C_1\,\mathrm{ch}\,K_1\zeta+K_1^3C_2\,\mathrm{sh}\,K_1\zeta-K_2^3C_3\cos K_2\zeta+\right.$$
$$+K_2^3C_4\sin K_2\zeta)+\left(\frac{\varrho l^2\omega_c^2}{kG}+\frac{\varrho l^2\omega_c^2}{E}\right)(K_1C_1\,\mathrm{ch}\,K_1\zeta+K_1C_2\,\mathrm{sh}\,K_1\zeta+$$
$$+K_2C_3\cos K_2\zeta-K_2C_4\sin K_2\zeta)-\frac{\varrho^2l^4\omega_c^4}{kGE}\left[\frac{C_1}{K_1}(\mathrm{ch}\,K_1-\mathrm{ch}\,K_1\zeta)+\right.$$
$$+\frac{C_2}{K_1}(\mathrm{sh}\,K_1-\mathrm{ch}\,K_1\zeta)-\frac{C_3}{K_2}(\cos K_2-\cos K_2\zeta)+$$
$$\left.\left.+\frac{C_4}{K_2}(\sin K_2-\sin K_2\zeta)\right]\right\}\cos\tau. \qquad (27.10)$$

To determine the constants of integration C_1, C_2, C_3 and C_4 we use the following conditions at the ends of the rod:

$$(u^*)_{\zeta=0}=0;\quad(\theta)_{\zeta=0}=0;\quad(M)_{\zeta=1}=0;\quad(Q)_{\zeta=1}=0. \qquad (27.11)$$

On the basis of (27.11) and (27.8) — (27.10) we have:

$$C_2+C_4=0,$$
$$C_1\left[K_1+\frac{\alpha}{K_1}(\mathrm{ch}\,K_1-1)\right]+C_2\alpha\frac{\mathrm{sh}\,K_1}{K_1}+$$
$$+C_3\left[K_2-\frac{\alpha}{K_2}(\cos K_2-1)\right]+C_4\frac{\sin K_2}{K_2}\alpha=0,$$
$$C_1(K_1^2+\alpha)\,\mathrm{sh}\,K_1+C_2(K_1^2+\alpha)\,\mathrm{ch}\,K_1-$$
$$-C_3(K_2^2-\alpha)\sin K_2-C_4(K_2^2-\alpha)\cos K_2=0,$$
$$C_1(K_1^3+\beta K_1)\,\mathrm{ch}\,K_1+C_2(K_1^3+\beta K_1)\,\mathrm{sh}\,K_1-$$
$$-C_3(K_2^3-\beta K_2)\cos K_2+C_4(K_2^3-\beta K_2)\sin K_2=0, \qquad (27.12)$$

where

$$\alpha = \frac{\varrho l^3 \omega_c^2}{kG}; \quad \beta = \varrho l^3 \omega_c^2 \left(\frac{1}{kG} + \frac{1}{E}\right).$$

From conditions (27.12) we obtain

$$C_2 = -C_4$$

$$C_1\left[K_1 + \frac{\alpha}{K_1}(\operatorname{ch} K_1 - 1)\right] + C_2\alpha\left[\frac{\operatorname{sh} K_1}{K_1} - \frac{\sin K_2}{K_2}\right] + $$
$$+ C_3\left[K_2 - \frac{\alpha}{K_2}(\cos K_2 - 1)\right] = 0,$$
$$C_1(K_1^2 + \alpha)\operatorname{sh} K_1 + C_2\left[(K_1^2+\alpha)\operatorname{ch} K_1 + (K_2^2 - \alpha)\cos K_2\right] -$$
$$- C_3(K_2^2 - \alpha)\sin K_2 = 0,$$
$$C_1(K_1^3 + \beta K_1)\operatorname{ch} K_1 + C_2\left[(K_1^3 + \beta K_1)\operatorname{sh} K_1 -\right.$$
$$\left. -(K_2^3 - \beta K_2)\sin K_2\right] - C_3(K_2^3 - \beta K_2)\cos K_2 = 0. \qquad (27.13)$$

By equating the determinant of this system to zero, we get the frequency equation:

$$\begin{vmatrix} \left[K_1 + \frac{\alpha}{K_1}(\operatorname{ch} K_1 - 1)\right]; & \alpha\left[\frac{\operatorname{sh} K_1}{K_1} - \frac{\sin K_2}{K_2}\right]; & \left[K_2 - \frac{\alpha}{K_2}(\cos K_2 - 1)\right] \\ \left[(K_1^2+\alpha)\operatorname{sh} K_1\right]; & \left[(K_1^2+\alpha)\operatorname{ch} K_1 + (K_2^2-\alpha)\cos K_2\right]; & -\left[(K_2^2-\alpha)\sin K_2\right] \\ \left[(K_1^3+\beta K_1)\operatorname{ch} K_1\right]; & \left[(K_1^3+\beta K_1)\operatorname{sh} K_1 - (K_2^3-\beta K_2)\sin K_2\right]; & -\left[(K_2^3-\beta K_2)\cos K_2\right] \end{vmatrix} = 0.$$

We write this determinant in expanded form and make the necessary transformations; as a result we obtain the frequency equation:

$$2K_1K_2(K_1^2K_2^2 + \alpha^2) - K_1K_2(K_1^4 + K_2^4 - 2\alpha^2)\operatorname{ch} K_1 \cos K_2 +$$
$$+ [(K_1^2K_2^2 + \alpha^2)(K_1^2 - K_2^2) + \alpha(K_1^2 + K_2^2)^2]\operatorname{sh} K_1 \sin K_2 = 0.$$

(27.14)

Turning to the determination of values of the constants of integration, we shall proceed as follows. From equation (27.13) we find

$$\frac{C_1}{C_3}=\frac{K_1(K_2^2-\alpha)+K_1(K_1^2+\alpha)\operatorname{ch}K_1\cos K_2-K_2(K_2^2-\alpha)\operatorname{sh}K_1\sin K_2}{K_2(K_1^2+\alpha)+K_2(K_2^2-\alpha)\operatorname{ch}K_1\cos K_2+K_1(K_1^2+\alpha)\operatorname{sh}K_1\sin K_2}.$$

Or

$$\begin{aligned}
C_1 &= A[K_1(K_2^2-\alpha)+K_1(K_1^2+\alpha)\operatorname{ch}K_1\cos K_2-\\
&\qquad -K_2(K_2^2-\alpha)\operatorname{sh}K_1\sin K_2],\\
C_2 &= A[K_2(K_2^2-\alpha)\operatorname{ch}K_1\sin K_2-K_1(K_1^2+\alpha)\operatorname{sh}K_1\cos K_2],\\
C_3 &= A[K_2(K_1^2+\alpha)+K_2(K_2^2-\alpha)\operatorname{ch}K_1\cos K_2+\\
&\qquad +K_1(K_1^2+\alpha)\operatorname{sh}K_1\sin K_2],\\
C_4 &= A[K_1(K_1^2+\alpha)\operatorname{sh}K_1\cos K_2-K_2(K_2^2-\alpha)\operatorname{ch}K_1\sin K_2],
\end{aligned} \tag{27.15}$$

where A is a constant, which can be determined from the condition that the maximum deflection at the end of the vibrating rod is equal to the amplitude of vibrations.

$$[u^*(\zeta,\tau)]_{\substack{\zeta=1\\ \tau=0}}=a. \tag{27.16}$$

Examining the zeroth approximation we find on the basis of (26.23), (26.3), (27.15) and (27.16)

$$\begin{aligned}
&[u^*(\zeta,\tau)]_{\substack{\zeta=1\\ \tau=0}}=[\varphi(\zeta)a\cos\tau]_{\substack{\zeta=1\\ \tau=0}}=\\
&=A[(K_1^2+K_2^2)(K_1\operatorname{sh}K_1+K_2\sin K_2)]a=a,
\end{aligned}$$

from which

$$A=\frac{1}{(K_1^2+K_2^2)(K_1\operatorname{sh}K_1+K_2\sin K_2)}$$

Knowing the expressions for the coefficient A, on the basis of (27.15) we can derive the formulas for the constants of integration

$$\begin{aligned}
C_1 &= \frac{K_1(K_2^2-\alpha)+K_1(K_1^2+\alpha)\operatorname{ch}K_1\cos K_2-K_2(K_2^2-\alpha)\operatorname{sh}K_1\sin K_2}{(K_1^2+K_2^2)(K_1\operatorname{sh}K_1+K_2\sin K_2)};\\
C_2 &= \frac{K_2(K_2^2-\alpha)\operatorname{ch}K_1\sin K_2-K_1(K_1^2+\alpha)\operatorname{sh}K_1\cos K_2}{(K_1^2+K_2^2)(K_1\operatorname{sh}K_1+K_2\sin K_2)};
\end{aligned} \tag{27.17}$$

(27.17) cont.

$$C_3 = \frac{K_2(K_1^2+\alpha)+K_2(K_2^2-\alpha)\,\mathrm{ch}\,K_1\cos K_2+K_1(K_1^2+\alpha)\,\mathrm{sh}\,K_1\sin K_2}{(K_1^2+K_2^2)(K_1\,\mathrm{sh}\,K_1+K_2\sin K_2)};$$

$$C_4 = \frac{K_1(K_1^2+\alpha)\,\mathrm{sh}\,K_1\cos K_2-K_2(K_2^2-\alpha)\,\mathrm{ch}\,K_1\sin K_2}{(K_1^2+K_2^2)(K_1\,\mathrm{sh}\,K_1+K_2\sin K_2)} \tag{27.17}$$

Substituting the expressions for the constants of integration into (27.6) we obtain in final form a formula for the determination of the deflection function $\varphi(\zeta)$

$$\varphi(\zeta) = \frac{1}{(K_1^2+K_2^2)(K_1\,\mathrm{sh}\,K_1+K_2\sin K_2)}\times$$
$$\times\{[(K_2^2-\alpha)+K_1(K_1^2+\alpha)\,\mathrm{ch}\,K_1\cos K_2-K_2(K_2^2-\alpha)\,\mathrm{sh}\,K_1\sin K_2]\,\mathrm{sh}\,K_1\zeta+$$
$$+[K_2(K_2^2-\alpha)\,\mathrm{ch}\,K_1\sin K_2-K_1(K_1^2+\alpha)\,\mathrm{sh}\,K_1\cos K_2]\cos K_1\zeta+$$
$$+[K_2(K_1^2+\alpha)+K_2(K_2^2-\alpha)\,\mathrm{ch}\,K_1\cos K_2+$$
$$+K_1(K_1^2+\alpha)\,\mathrm{sh}\,K_1\sin K_2]\sin K_2\zeta+[K_1(K_1^2+\alpha)\,\mathrm{sh}\,K_1\cos K_2-$$
$$-K_2(K_2^2-\alpha)\,\mathrm{ch}\,K_1\sin K_2]\cos K_2\zeta\}. \tag{27.18}$$

On the basis of the expansion (26.23) the formula for the determination of deflections of the rod in the zeroth approximation can be written in the abbreviated form;

$$u^*(\zeta,\tau)=a\varphi(\zeta)\cos\tau. \tag{27.19}$$

28. Determination of the frequency of vibrations in the first approximation

For the solution of the problem in the first approximation, we shall examine equation (26.31) in accordance with the expansions (26.23) — (26.25). We multiply the former once by $\varphi(\zeta)\sin\tau\,d\zeta\,d\tau$, and a second time by $\varphi(\zeta)\cos\tau\,d\zeta\,d\tau$ and integrate the two equations obtained along the whole length of the rod for one cycle of vibrations.

We obtain

$$\oint\int_0^1 \Big\{ \frac{\partial^4 u_1}{\partial\zeta^4} - \alpha_2 \frac{\partial^4 u_1}{\partial\zeta^2\,\partial\tau^2} + \alpha_1 \frac{\partial^2 u_1}{\partial\tau^2} - \alpha_1 \Delta_1 \varphi a \cos\tau +$$

$$+ \alpha_2 \Delta_1 \frac{d^2\varphi}{d\zeta^2} a \cos\tau + \alpha_3 \frac{\partial^4 u_1}{\partial\tau^4} + 2\alpha_3\omega_c^2 \Delta_1 \varphi a \cos\tau +$$

$$+ \frac{\alpha_4}{l^2} \frac{\partial^2}{\partial\zeta^2} \vec{\Phi}(\varphi a \cos\tau) + \frac{\alpha_4}{l} \frac{\partial}{\partial\zeta} \vec{\Psi}(\varphi a \cos\tau) -$$

$$- \alpha_4 q_1 \cos(\tau - \psi_0) \Big\} \varphi \sin\tau\, d\zeta\, d\tau = 0; \qquad (28.1)$$

$$\oint\int_0^1 \Big\{ \frac{\partial^4 u_1}{\partial\zeta^4} - \alpha_2 \frac{\partial^4 u_1}{\partial\zeta^2\,\partial\tau^2} + \alpha_1 \frac{\partial^2 u_1}{\partial\tau^2} - \alpha \Delta_1 \varphi a \cos\tau +$$

$$+ \alpha_2 \Delta_1 \frac{d^2\varphi}{d\zeta^2} a \cos\tau + \alpha_3 \frac{\partial^4 u_1}{\partial\tau^4} + 2\alpha_3\omega_c^2 \Delta_1 \varphi a \cos\tau +$$

$$+ \frac{\alpha_4}{l^2} \frac{\partial^2}{\partial\zeta^2} \vec{\Phi}\varphi(a \cos\tau) + \frac{\alpha_4}{l} \frac{\partial}{\partial\zeta} \vec{\Psi}(\varphi a \cos\tau) -$$

$$- \alpha_4 q_1 \cos(\tau - \psi_0) \Big\} \varphi \cos\tau\, d\zeta\, d\tau = 0. \qquad (23.2)$$

Integrating by parts with respect to ζ and τ, taking account of the boundary conditions of the bar and remembering also that function $u_1(\zeta,\tau)$ does not contain the fundamental harmonic, we can show that

$$\oint\int_0^1 \Big\{ \frac{\partial^4 u_1}{\partial\zeta^4} - \alpha_2 \frac{\partial^4 u_1}{\partial\zeta^2\,\partial\tau^2} + \alpha_1 \frac{\partial^2 u_1}{\partial\tau^2} + \alpha_3 \frac{\partial^4 u_1}{\partial\tau^4} \Big\} \varphi \sin\tau\, d\zeta\, d\tau = 0. \qquad (28.3)$$

$$\oint\int_0^1 \Big\{ \frac{\partial^4 u_1}{\partial\zeta^4} - \alpha_2 \frac{\partial^4 u_1}{\partial\zeta^2\,\partial\tau^2} + \alpha_1 \frac{\partial^2 u_1}{\partial\tau^2} + \alpha_3 \frac{\partial^4 u_1}{\partial\tau^4} \Big\} \varphi \cos\tau\, d\zeta\, d\tau = 0. \qquad (28.4)$$

In accordance with (28.1) and (28.2) we obtain

$$\oint\int_0^1 \Big\{ \Big[\alpha_2 \frac{d^2\varphi}{d\zeta^2} - (\alpha_1 - 2\alpha_3\omega_c^2)\varphi \Big] \Delta_1 a \cos\tau + \frac{\alpha_4}{l^2} \frac{\partial^2}{\partial\zeta^2} \vec{\Phi}(\varphi a \cos\tau) +$$

$$+ \frac{\alpha_4}{l} \frac{\partial}{\partial\zeta} \vec{\Psi}(\varphi_0 \cos\tau) - \alpha_4 q_1 \cos(\tau - \psi_0) \Big\} \varphi \sin\tau\, d\zeta\, d\tau = 0; \qquad (28.5)$$

$$\oint\int_0^1 \left\{\left[\alpha_2 \frac{d^2\varphi}{d\zeta^2} - (\alpha_1 - 2\alpha_3\omega_c^2)\varphi\right]\Delta_1 a \cos\tau + \frac{\alpha_4}{l^2}\frac{\partial}{\partial\zeta}\vec{\Phi}(\varphi a\cos\tau) + \right.$$

$$\left. + \frac{\alpha_4}{l}\frac{\partial}{\partial\zeta}\vec{\Psi}(\varphi a\cos\tau) - \alpha_4 q_1\cos(\tau-\psi_0)\right\}\varphi\cos\tau\, d\zeta\, d\tau = 0. \qquad (28.6)$$

From the last equations we find

$$\Delta_1 = \left\{\oint\int_0^1 \left[\alpha_2\frac{d^2\varphi}{d\zeta^2} - (\alpha_1 - 2\alpha_3\omega_c^2)\varphi\right]\varphi a\cos^2\tau\, d\zeta\, d\tau\right\}^{-1}\times$$

$$\times\oint\int_0^1\left[-\frac{\alpha_4}{l^2}\frac{\partial^2}{\partial\zeta^2}\vec{\Phi}(\varphi a\cos\tau) - \frac{\alpha_1}{l}\frac{\partial}{\partial\zeta}\vec{\Psi}(\varphi a\cos\tau + \right.$$

$$\left. + \alpha_4 q_1\cos(\tau-\psi_0)\right]\varphi\cos\tau\, d\zeta\, d\tau; \qquad (28.7)$$

$$\oint\int_0^1\left[\frac{\alpha_4}{l^2}\frac{\partial^2}{\partial\zeta}\vec{\Phi}(\varphi a\cos\tau) + \frac{\alpha_4}{l}\frac{\partial}{\partial\zeta}\vec{\Psi}(\varphi a\cos\tau)\right]\varphi\sin\tau\, d\zeta\, d\tau =$$

$$= \oint\int_0^1 \alpha_4 q_1\sin\psi_0\varphi\sin^2\tau\, d\zeta\, d\tau;$$

$$\sin\psi_0 = \left[q_1\pi\int_0^1\varphi\, d\zeta\right]^{-1}\left\{\oint\int_0^1\frac{1}{l^2}\left[\frac{\partial^2}{\partial\zeta^2}\vec{\Phi}(\varphi a\cos\tau) + \right.\right.$$

$$\left.\left. + \frac{1}{l}\frac{\partial}{\partial\zeta}\vec{\Psi}(\varphi a\cos\tau)\right]\varphi\sin\tau\, d\zeta\, d\tau. \qquad (28.8)$$

The square of frequencies in the first approximation can be found by the following formula using (26.24) and (28.7):

$$\omega_1^2 = \omega_c^2 + \varepsilon\Delta_1 = \omega_c^2 + \left\{\int_0^1\left[\alpha_2\frac{d^2\varphi}{d\zeta^2} - (\alpha_1 - 2\alpha_3\omega_c^2)\varphi\right]a\pi\varphi\, d\zeta\right\}^{-1}\times$$

$$\times\left[\alpha_4 q_1\pi\cos\psi_0\int_0^1\varphi d\zeta - \frac{\alpha_4}{l^2}\oint\int_0^1\frac{\partial^2}{\partial\zeta^2}\varepsilon\vec{\Phi}(\varphi a\cos\tau)\varphi\cos\tau\, d\zeta\, d\tau - \right.$$

$$\left. - \frac{\alpha_4}{l}\oint\int_0^1\frac{\partial}{\partial\zeta}\varepsilon\vec{\Psi}(\varphi a\cos\tau)\varphi\cos\tau\, d\zeta\, d\tau\right].$$

$$(28.9)$$

Multiplying the numerator and the denominator of the right side of the formula (28.8) by ε , we obtain

$$\sin\psi_0 = \left[q\pi\int_0^1 \varphi\, d\zeta\right]^{-1}\left[\frac{1}{l^2}\oint\int_0^1 \frac{\partial^2}{d\zeta^2}\varepsilon\vec{\vec{\Phi}}(\varphi a\cos\tau)\,\varphi\sin\tau\, d\zeta\, d\tau + \right.$$

$$\left. + \frac{1}{l}\oint\int_0^1 \frac{\partial}{\partial\zeta}\varepsilon\vec{\vec{\Psi}}(\varphi a\cos\tau)\,\varphi\sin\tau\, d\tau\, d\zeta\right]. \qquad (28.10)$$

From formulas (28.9) and (28.10) it is possible to construct a resonance curve for transverse vibrations of a short rod taking account of dissipation of energy in the material. Let us recall that the function $\varphi(\zeta)$ in the last formulas is determined by equation (27.1) and the expressions for the functionals $\vec{\vec{\Phi}}(\varphi_a\cos\tau)$ and $\vec{\vec{\Psi}}(\varphi_a\cos\tau)$ are written out according to formulas (26.14) and (26.19); in the latter the value of the deflection should be taken from the zeroth approximation, according to the formula (27.10).

Substituting in formula (26.14) the value of the deflection function in the zeroth approximation expressed in the dimensionless coordinates by formula (27.10), we obtain

$$\vec{\Phi}(\varphi a\cos\tau) = -\frac{E\eta}{\varkappa}\int_F\left\{\left[\left(\frac{a}{l}\,\frac{d^2\varphi}{d\zeta^2} + \frac{\varrho la}{kG}\,\omega_c^2\varphi\right) + \right.\right.$$

$$\left. + \left(\frac{a}{l}\,\frac{d^2\varphi}{d\zeta^2} + \omega_c^2\frac{al\varrho}{kG}\varphi\right)\cos\tau\right]^{\varkappa} z^{\varkappa} -$$

$$\left. - 2^{\varkappa-1}\left(\frac{a}{l}\,\frac{d^2\varphi}{d\zeta^2} + \frac{al\varrho}{kG}\,\omega_c^2\varphi\right)^{\varkappa} z^{\varkappa}\right\} z\, dF;$$

$$\overleftarrow{\Phi}(\varphi a\cos\tau) = \frac{E\eta}{\varkappa}\int_F\left\{\left[\left(\frac{a}{l}\,\frac{d^2\varphi}{d\zeta^2} + \frac{\varrho la}{kG}\,\omega_c^2\varphi\right) - \right.\right.$$

$$\left. - \left(\frac{a}{l}\,\frac{d^2\varphi}{d\zeta^2} + \frac{\varrho la}{kG}\,\omega_c^2\varphi\right)\cos\tau\right]^{\varkappa} z^{\varkappa} -$$

$$\left. - 2^{\varkappa-1}\left(\frac{a}{l}\,\frac{d^2\varphi}{d\zeta^2} + \frac{al\varrho}{kG}\,\omega_c^2\varphi\right)^{\varkappa} z^{\varkappa}\right\} z\, dF. \qquad (28.11)$$

Substituting the expressions (27.10) into formula (26.19) we find

$$\vec{\Psi}(\varphi a \cos\tau) = \frac{\nu a^n l^{2n} \varrho^n \omega_c^{2n} F^n}{n G^{n-1} I_y^n} \left(\int_{\zeta}^{1} \varphi \, d\zeta \right)^n \times$$

$$\times \left[(1+\cos\tau)^n - 2^{n-1} \right] \int_F \frac{S_{(z)}^n}{y^n} dF;$$

$$\overleftarrow{\Psi}(\varphi a \cos\tau) = \frac{-\nu a^n l^{2n} \varrho^n \omega_c^{2n} F^n}{n G^{n-1} I_y^n} \left(\int_{\zeta}^{1} \varphi \, d\zeta \right)^n \times$$

$$\times \left[(1-\cos\tau)^n - 2^{n-1} \right] \int_F \frac{S_{(z)}^n}{y^n} dF. \tag{28.12}$$

If we now introduce the expressions (28.11) and (28.12) into formulas (28.9) and (28.10) and also take account of the notations (26.21) and (26.23), the formulas for determining the square of frequency ω_I^2 in the first approximation and the sine of the angle of the phase shift $\sin\psi_0$ will take the form:

$$\omega_I^2 = \omega_c^2 + \left\{ a\pi \int_0^1 \left[\left(\frac{l^2 \varrho}{E} + \frac{l^2 \varrho}{kG} \right) \frac{d^2\varphi}{d\zeta^2} - \frac{l^2 \varrho}{E} \left(\frac{F l^2}{I_y} + \frac{2 \varrho l^2 \omega_c^2}{kG} \right) \varphi \right] \varphi \, d\zeta \right\}^{-1} \times$$

$$\times \left\{ \frac{q l^3 \pi \cos\psi_0}{EI} \int_0^1 \varphi \, d\zeta + \frac{l\eta}{I_y \varkappa} \int_0^1 \varphi \frac{\partial^2}{\partial \zeta^2} \left(\frac{a}{l} \frac{d^2\varphi}{d\zeta} + \frac{a l \varrho \omega_c^2}{kG} \varphi \right)^{\varkappa} d\zeta \times \right.$$

$$\times \left\{ \int_0^{\pi} \left[(1-\cos\tau)^{\varkappa} - 2^{\varkappa-1} \right] \cos\tau \, d\tau - \right.$$

$$\left. - \int_{\pi}^{2\pi} \left[(1+\cos\tau)^{\varkappa} - 2^{\varkappa-1} \right] \cos\tau \, d\tau \right\} \int_F z^{\varkappa+1} dF -$$

$$- \frac{\nu a^n l^{2n+3} \varrho^n \omega_c^{2n} F^n}{n G^{n-1} I_y^{n+1} E} \left\{ \int_{\pi}^{2\pi} \left[(1+\cos\tau)^n - 2^{n-1} \right] \cos\tau \, d\tau - \right.$$

$$\left. - \int_0^{\pi} \left[(1-\cos\tau)^n - 2^{n-1} \right] \cos\tau \, d\tau \right\} \times$$

$$\left. \times \int_0^1 \varphi \frac{\partial}{\partial \zeta} \left(\int_{\zeta}^{1} \varphi \, d\zeta \right) d\zeta \int_F \frac{S_{(z)}^n}{y^n} dF \right\}; \tag{28.13}$$

$$\sin\psi_0 = \left[q\pi \int_0^1 \varphi\, d\zeta \right]^{-1} \left\{ -\frac{E\eta}{l^2\varkappa} \int_0^1 \varphi \frac{\partial^2}{\partial\zeta^2} \left(\frac{a}{l} \frac{d^2\varphi}{d\zeta^2} + \right.\right.$$

$$\left. + \frac{al\varrho\omega_c^2}{kG}\varphi \right)^{\varkappa} d\zeta \left\{ \int_{\pi}^{2\pi} [(1+\cos\tau)^{\varkappa} - 2^{\varkappa-1}] \sin\tau\, d\tau + \right.$$

$$\left. + \int_0^{\pi} [(1-\cos\tau)^{\varkappa} - 2^{\varkappa-1}] \sin\tau\, d\tau \right\} \int_F z^{\varkappa+1}\, dF +$$

$$+ \frac{\nu a^n l^{2n-1} \varrho^n \omega_c^{2n} F^n}{nG^{n-1} I_y^n} \left\{ \int_{\pi}^{2\pi} [(1+\cos\tau)^n - 2^{n-1}] \sin\tau\, d\tau - \right.$$

$$\left. - \int_0^{\pi} [(1-\cos\tau)^n - 2^{n-1}] \sin\tau\, d\tau \right\} \times$$

$$\left. \times \int_0^1 \varphi \frac{\partial}{\partial\zeta} \left(\int_{\zeta}^1 \varphi\, d\zeta \right) d\zeta \int_F \frac{S_{(z)}^n}{y^n}\, dF \right\}. \tag{28.14}$$

Formulas (28.13) and (28.14) serve for the construction of the resonance curve for the vibrations of a short rod accounting for energy dissipation in the material.

29. Sample calculation

Let us construct resonance curves for a short cantilevered bar of rectangular cross-section, the dimensions of which are:

$$b = 1{,}2 \ \text{см},$$
$$h = 0{,}9 \ \text{см},$$
$$l = 2{,}7 \ \text{см},$$

the modulus elasticity in tension is $E = 2.1 \cdot 10^6$ kg/cm^2; the shear modulus $G = 8.07 \cdot 10^5$ kg/cm^2; the specific gravity $\gamma = 7.85 \cdot 10^{-3}$ kg/cm^3, the moment of inertia of the rod $I = 429 \cdot 10^{-4}$ cm^4, the cross-sectional area $F = 1.08$ cm^2, the section modulus in bending $W = 0.162$ cm^3.

The natural frequency of vibrations of the rod obtained from equation (27.14) equals

$$\omega_c = 7{,}105 \cdot 10^4 \ {}^1/\text{сек}.$$

According to (27.19) the deflection function is expressed numerically in the form

$$\varphi(\zeta)=0{,}558\ (-0{,}783\,\mathrm{sh}\,1{,}884\zeta+\mathrm{ch}\,1{,}884\zeta+ \\ +0{,}5371\sin 2{,}05843\zeta-\cos 2{,}0584\zeta).$$

On the basis of experimental data for Steel. 45, the parameters of the **hysteresis loop are**

in bending $\varkappa=1{,}4,\ \eta=1{,}64;$

in torsion $n=2{,}4,\ \nu=188.$

For the integrals in formulas (28.13) and (28.14) we find the following values:

$$\int_0^1\left[\left(\frac{\varrho l^2}{E}+\frac{\varrho l}{kG}\right)\frac{d^2\varphi}{d\zeta^2}-\frac{\varrho l}{E}\left(\frac{Fl^2}{I}+\frac{2\varrho l^3\omega_c^2}{kG}\right)\varphi\right]\varphi\,d\zeta=$$

$$=\int_0^1\left(1{,}36\cdot10^{-10}\frac{d^2\varphi}{d\zeta^2}-30{,}3\cdot10^{-10}\varphi\right)\varphi\,d\zeta=4{,}54\cdot10^{-10};$$

$$\int_0^1\varphi\,d\zeta=0{,}3775;$$

$$\int_0^1\varphi\frac{\partial^2}{\partial\zeta^2}\left(\frac{d^2\varphi}{d\zeta^2}+\frac{\varrho l^3\omega_c^2}{kG}\varphi\right)^{\varkappa}d\zeta=5{,}97;$$

$$\int_0^{\pi}[(1+\cos\tau)^{\varkappa}-2^{\varkappa-1}]\cos\tau\,d\tau=$$

$$=\int_0^{\pi}2^{\varkappa}\sin^{2\varkappa}\frac{\tau}{2}\left(2\cos^2\frac{\tau}{2}-1\right)d\tau=$$

$$=2^{\varkappa+2}\int_0^{\frac{\pi}{2}}\cos^{2(\varkappa+1)}z\,dz-2^{\varkappa+1}\int_0^{\frac{\pi}{2}}\cos^{2\varkappa}z\,dz=$$

$$=2^{\varkappa+2}\frac{\Gamma(0{,}5)\,\Gamma(\varkappa+1{,}5)}{2\Gamma(\varkappa+2)}-2^{\varkappa+1}\frac{\Gamma(0{,}5)\,\Gamma(\varkappa+0{,}5)}{2\Gamma(\varkappa+1)}.$$

Thus

$$\int_{\pi}^{2\pi}[(1-\cos\tau)^{\varkappa}-2^{\varkappa-1}]\cos\tau\,d\tau-\int_{0}^{\pi}[(1+\cos\tau)^{\varkappa}-2^{\varkappa-1}]\cos\tau\,d\tau=4{,}2252;$$

$$\int_{F} z^{\varkappa+1}\,dF=0{,}0467;$$

$$\int_{\pi}^{2\pi}[(1+\cos\tau)^{n}-2^{n-1}]\cos\tau\,d\tau-\int_{0}^{\pi}[(1-\cos\tau)^{n}-2^{n-1}]\cos\tau\,d\tau=8{,}0985;$$

$$\int_{0}^{1}\varphi\frac{\partial}{\partial\zeta}\left(\int_{\zeta}^{1}\varphi\,d\zeta\right)^{n}d\zeta=-0{,}0456;$$

$$\int_{(F)}\left(\frac{S_{(z)}}{y}\right)^{n}dF=2{,}2084\cdot10^{-3}.$$

The formulas for the construction of resonance curves, after substitution in them of all the known quantities, take the form

$$\left(\frac{\omega}{\omega_c}\right)^2=1-1{,}95\cdot10^{-5}\frac{q}{a}\cos\psi_0-1{,}63a^{0,4}-17{,}96a^{1,4}, \qquad (29.1)$$

$$\sin\psi_0=-\frac{10^6}{q}(0{,}1267a^{1,4}+1{,}337a^{2,4}). \qquad (29.2)$$

Table 12

$a\cdot10^3$	$\sin\psi_0$	$\cos\psi_0$	$\left(\frac{\omega}{\omega_c}\right)^2_л$	$\left(\frac{\omega}{\omega_c}\right)^2_п$	$\left(\frac{\omega}{\omega_c}\right)_л$	$\left(\frac{\omega}{\omega_c}\right)_п$
0,2	0,1051	0,994	0,170	1,720	0,410	1,316
0,3	0,1813	0,983	0,417	1,453	0,645	1,205
0,4	0,2767	0,961	0,551	1,301	0,742	1,145
0,5	0,3812	0,924	0,630	1,206	0,794	1,100
0,6	0.4920	0,871	0,684	1,136	0,827	1,065
0,7	0,5976	0,802	0,724	1,084	0,850	1,040
0,8	0,7276	0,686	0,764	1,032	0,874	1,015
0,9	0,8558	0,517	0,802	0,982	0,895	0,991
0,95	0,9280	0,371	0,828	0,950	0,909	0,975
0,98	0,9830	0,176	0,859	0,915	0,926	0,956
0,997	0,9999	0	0,886	0,886	0,941	0,941

The results of calculations from formulas (29.1) and (29.2) for various values of amplitude of the bar for the given exciting force q are shown on Table 12. Starting from a value of maximum stress at the place where the rod is clamped of $\sigma = 4000$ kg/cm^2, we obtain $a_{max} = 10^{-3}$, which corresponds to the value $q = 8.2$ kg/cm^2.

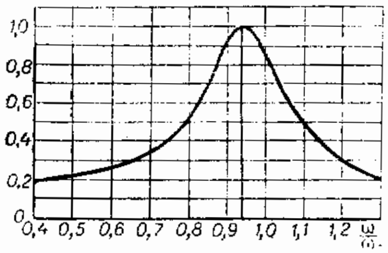

Fig. 18

For $q = 8$ kg/cm^2 equations (29.1) and (29.2) take the form

$$\left(\frac{\omega}{\omega_c}\right)^2 = 1 - 1{,}56\cdot 10^{-4}\frac{\cos\psi_0}{a} - 1{,}63\,a^{0,4} - 17{,}96\,a^{1,4};$$

$$\sin\psi_0 = 10^5\cdot 1{,}25\,(0{,}1267a^{1,4} + 1{,}337a^{2,4}).$$

The resonance curve of Fig. 18 is plotted in accordance with data in Table 12 where $a = 10^3\cdot a_{max}$.

Chapter VII

Transverse Vibrations of Turbine Blades of Variable Cross-Section in a Centrifugal Force Field

30. Derivation of the differential equation of vibrations

In the present chapter we examine transverse vibrations of turbine blades considering the tensile centrifugal forces and the dissipation of energy in the material.

In deriving the basic differential equation we shall examine a blade of arbitrary cross-section (Fig. 19), isolated from a group of vanes which are connected by a shroud.

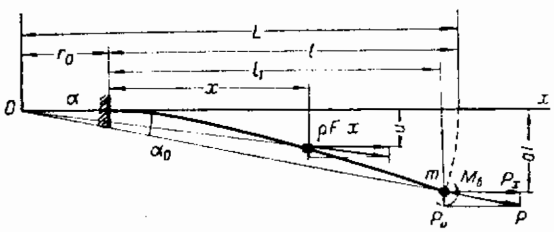

Fig. 19

We denote:

x — coordinate axis, coinciding with the blade axis;

l — length of the working section of the blade (effective length of the blade);

$\zeta = \frac{x}{l}$ — dimensionless coordinate;

$I(\zeta)$ — moment of inertia of area of a transverse cross-section of the blade;

$F(\zeta)$ — cross-sectional area of the blade;
λ — length of one segment of the shroud;
m — mass of one segment of the shroud;
E — modulus of elasticity of the material of the blade;
E_σ — modulus of elasticity of the material of the shroud;
I_σ — moment of inertia of area of a transverse cross-section of the shroud;
ρ — density of the material of the blade;
ω — constant angular velocity of the turbine rotor;
r_0 — outer radius of the disc;
u — dimensionless magnitude of the deflection of vibrating vane at a distance ζ from the origin of coordinates.

The point at which the vane is fixed to the periphery of the disc is taken as the origin of our coordinates.

The boundary conditions for the vane, considering the latter to be rigidly clamped at the rim of the disc, are the following:

for $\zeta = 0$

$$u(\zeta, t) = 0; \qquad u'(\zeta, t) = 0;$$

for $\zeta = 1$

$$u''(\zeta, t) = \frac{M_\sigma}{EI}; \quad u''(\zeta, t) = \frac{Q_\sigma}{EI}^{*}. \tag{30.1}$$

Here M_σ and Q_σ are the bending moment and shear force at the end of the blade, respectively; their values depend on the magnitude of deflection of the blade $u(\zeta, t)$ and on the stiffness of the shroud $E_\sigma I_\sigma$. In view of the complexity of the boundary conditions at the end of the vane, for $\zeta = 1$ we shall avail ourselves of the most general principle of

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*For conciseness, we shall use the notations:
$u'(\zeta,t) = \frac{\partial u(\zeta,t)}{\partial \zeta}$; $\dot{u}(\zeta,t) = \frac{\partial u(\zeta,t)}{\partial t}$

dynamics in setting up the fundamental equations—the variational principle of Ostrogradsky-Hamilton.

As is well known, for a conservative system the variation of the action integral of the conservative system upon going over from the actual trajectory to a neighboring one which has the same terminal points is zero. In other words, according to the principle of Ostrogradsky-Hamilton, the function $u(\zeta, t)$ which corresponds to the actual motion of the blade, must provide an extremum for the integral

$$H = \int_{t_1}^{t_2} (T - V)\, dt, \tag{30.2}$$

where T and V are the kinetic and potential energies of the system respectively.

The extremal value of the integral (30.2), which corresponds to the true motion may be obtained by equating the first variation of the integral (30.2) to zero; i.e.

$$\delta H = \delta \int_{t_1}^{t_2} (T - V)\, dt = 0. \tag{30.3}$$

We shall obtain the differential equation of vibration of the blade and the boundary conditions at its end from equation (30.3). Let us now determine the kinetic and potential energy of this elastic system.

1) The kinetic energy of the vane and shroud is:

$$T = \frac{1}{2} l^5 \rho \int_0^1 F(\zeta) \left[u(\zeta, t) \right]^2 d\zeta + \frac{1}{2} m l^2 \left[u(\zeta, t) \right]^2 . \tag{30.4}$$

2) The potential energy (strain energy) due to bending of the blade is equal to

$$V_p = \int_0^1 \frac{M_\zeta^2 \, d\zeta}{2 E l^3 I(\zeta)} = \frac{E l^3}{2} \int_0^1 I(\zeta) \left[u''(\zeta, t) \right]^2 d\zeta . \tag{30.5}$$

3) The strain energy due to bending of the shroud is

$$V_b = \frac{1}{2} M_b u'(1, t) = \frac{6 E_b I_b}{\lambda} \left[u'(1, t) \right]^2 , \tag{30.6}$$

where $M_б$ is the bending moment, exerted by the shroud on the blade, which may be calculated from the formula

$$M_б = \frac{1}{l} EI(1)\, u''(1, t) = \frac{12 E_б I_б}{\lambda} u'(1, t). \qquad (30.7)$$

4) The potential of the load resulting from the centrifugal force P on the shroud may be determined as the sum of the potential energies of its two components P_ζ and P_u, which act parallel to the coordinate axes. In accordance with Fig. 19, these components are equal to

$$P_\zeta = P \cos\alpha, \quad P_u = P \sin\alpha,$$

where

$$P = mL\omega^2.$$

Since the vibrations are quite small, we can assume

$$\sin\alpha = \frac{u(1, t)\, l}{L} \cong \alpha$$

$$P_\zeta = mL\omega^2, \qquad P_u = mu(1, t)\, l\, \omega^2.$$

The component P_ζ performs work through the displacement

$$l \int_0^1 \sqrt{1 + [u'(\zeta, t)]^2}\, d\zeta - l = \frac{1}{2}\, l \int_0^1 [u'(\zeta, t)]^2\, d\zeta.$$

The corresponding magnitude of potential energy is then equal to

$$V_{P_\zeta} = \frac{1}{2} mlL\omega^2 \int_0^1 [u'(\zeta, t)]^2\, d\zeta.$$

The component of the centrifugal force P_u, parallel to the axis of deflection u, decreases the potential energy of the system by an amount

$$V_{P_u} = \frac{1}{2} P_u u(1, t)\, l = \frac{1}{2} ml^2\omega^2 u^2(1, t).$$

Thus, the potential energy, caused by the centrifugal force on the shroud amounts to

$$V_P = V_{P_\zeta} - V_{P_u} = \frac{1}{2} mlL\omega^2 \int_0^1 [u'(1, t)]^2\, d\zeta - \frac{1}{2} ml^2\omega^2 u^2(1, t). \qquad (30.8)$$

5) The potential energy, brought about by the centrifugal forces on the blade itself is expressed as follows by analogy to the potential energy of the centrifugal force of the **shroud:**

$$V_{\text{пл}} = \frac{1}{2}\omega^2 l^4 \varrho \int_0^1 \left\{ F(\zeta)(r_0 + l\zeta) \int_0^\zeta [u'(\xi, t)]^2 \, d\xi \right\} d\zeta - - \frac{1}{2}\omega^2 l^5 \varrho \int_0^1 F(\zeta) u^2(\zeta \ t) \, d\zeta. \tag{30.9}$$

Substituting in equation (30.3) the expressions for the kinetic and potential energies of the elastic system and taking into account that

$$V = V_{\text{л}} + V_{\text{б}} + V_P + V_{\text{пл}},$$

we obtain

$$\delta \int_{t_1}^{t_2} \left\{ \frac{1}{2} l^5 \varrho \int_0^1 F(\zeta)[\dot{u}(\zeta, t)]^2 d\zeta + \frac{1}{2} l^2 m [\dot{u}(\zeta, t)]^2 - - \frac{1}{2} E l^3 \int_0^1 I(\zeta)[u''(\zeta, t)]^2 d\zeta - \frac{6 E_{\text{б}} I_{\text{б}}}{\lambda} [u'(1, t)]^2 - - \frac{1}{2} m l L \omega^2 \int_0^1 [u'(\zeta, t)]^2 d\zeta + \frac{1}{2} m l^2 \omega^2 u^2(1, t) - - \frac{1}{2} l^4 \varrho \omega^2 \int_0^1 \left[F(\zeta)(r_0 + l\zeta) \int_0^\zeta [u'(\xi, t)]^2 d\xi \right] d\zeta + + \frac{1}{2} l^5 \varrho \omega^2 \int_0^1 F(\zeta) u^2(\zeta, t) \, d\zeta \right\} dt = 0,$$

where ξ is a dummy variable.

Performing the variation of the expression we have obtained and removing derivatives from the expressions with the aid of integration by parts, remembering the conditions (30.1) at the fixed end of the blade, we find

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^1 \Big\{ & -l^5\varrho F(\zeta)\ddot u(\zeta,t) - El^3[I(\zeta)u''(\zeta,t)]'' + \\ & + mlL\omega^2 u''(\zeta,t) + l^4\varrho\omega^2\Big[u''(\zeta,t)\int_\zeta^1 F(\xi)(r_0+l\xi)\,d\xi + \\ & + F(\zeta)(r_0+l\zeta)u'(\zeta,t)\Big] + l^5\varrho\omega^2F(\zeta)u(\zeta,t)\Big\}\delta u(\zeta,t)\,d\zeta\,dt + \\ & + \int_{t_1}^{t_2}\Big\{-m_6 l\ddot u(1,t) - El^3[I(1)u''(1,t)-(I(1)u''(1,t)')] - \\ & - mlL\omega^2u'(1,t) + \frac{12E_6I_6}{\lambda}u'(1,t) + ml^2\omega^2u(1,t)\Big\}\delta u(1,t)\,dt = 0.\end{aligned} \tag{30.10}$$

Differentiating equation (30.10) with respect to ζ and t and dividing by $\delta u(\zeta,t)$, we obtain the differential equation for the transverse vibration of the blade

$$\begin{aligned}& El^3[I(\zeta)u''(\zeta,t)]'' - mlL\omega^2u''(\zeta,t) - l^4\varrho\omega^2\Big[u''(\zeta,t)\int_\zeta^1(r_0+l\xi)F(\xi)\,d\xi - \\ & - u'(\zeta,t)(r_0+l\zeta)F(\zeta)\Big] - l^5\varrho\omega^2F(\zeta)u(\zeta,t) + l^5\varrho F(\zeta)\ddot u(\zeta,t) = 0.\end{aligned} \tag{30.11}$$

Considering that the expression under the first integral sign in equation (30.10) is equal to zero, we obtain on the basis of (30.11), that the equality (30.10) is fulfilled if the function $u(\zeta,t)$ satisfies the condition

$$\begin{aligned}El^3[I(1)u''(1,t)]' = {} & El^3I(1)u''(1,t) + \\ & + mlL\omega^2u'(1,t) - \frac{12E_6I_6}{\lambda}u'(1,t) - \\ & - ml\omega^2u(1,t) + ml^2\ddot u(1,t).\end{aligned}$$

for $\zeta = 1$. Taking into account condition (30.7), we obtain

$$\begin{aligned}-El^3[I(1)u''(1,t)]' = {} & mlL\omega^2u'(1,t) - \\ & - ml^2\omega^2u(1,t) + ml^2\ddot u(1,t).\end{aligned} \tag{30.12}$$

Thus, the problem of free vibrations of a blade of variable cross-section without consideration of internal damping reduces to the integration of the differential equation (30.11) subject to the boundary conditions (30.1), (30.7), and (30.12).

In the case of forced vibrations with a disturbing force uniformly distributed along the blade, and varying harmonically

$$q = q_0 \cos \omega t,$$

and also taking into account the dissipation of energy in the material, the differential equation of vibration (30.11) must be supplemented by the appropriate terms. After this, we obtain the following equation of steady-state forced vibrations:

$$\begin{gathered} El^3 [I(\zeta) u''(\zeta, t)]'' - mlL\omega^2 u''(\zeta, t) - \\ - l^4 \varrho \omega^2 \left[u''(\zeta, t) \int_{\zeta}^{1} (r + l\xi) F(\xi) \, d\xi - \right. \\ \left. - u'(\zeta, t)(r_0 + l\zeta) F(\zeta) \right] - l^5 \varrho \omega^2 F(\zeta) u(\zeta, t) + l^5 \varrho F(\zeta) \ddot{u}(\zeta, t) + \\ + \varepsilon \Phi'' [u''(\zeta, t)] = \varepsilon q_0 l^2 \cos \omega t. \end{gathered} \tag{30.13}$$

Here $\varepsilon \Phi''[u''(\zeta, t)]$ is a functional, which takes account of energy dissipati n in the material on the basis of the hypothesis of dependence of energy dissipation on the stress amplitude. The magnitude of this term for steady-state vibrations of the system must be of the same order of smallness as the magnitude of the exciting force, which fact is indicated by the presence of the parameter ε as a factor in both cases.

Solving the obtained nonlinear differential equation (30.13), we can find the actual vibratory frequency of the elastic system in question and thereby construct a resonance curve.

In view of the impossibility of solving equation (30.13) exactly, we now use an approximate method of nonlinear mechanics for its solution; this method is bases on the asymptotic expansions in powers of the small parameter.

Utilizing this method, we can solve equation (30.13) to any degree of accuracy. However, in the "weakly nonlinear" problem in question; as shown in the first two chapters, the first approximation is sufficiently accurate for practical purposes.

31. Solution of the equation of vibrations in the zeroth and first approximation

We shall consider the solution of the following auxiliary equation as the solution in the zeroth approximation of equation (30.13):

$$\frac{\partial^2}{\partial\zeta^2}\left[I(\zeta)\frac{\partial^2 u(\zeta,t)}{\partial\zeta^2}\right]-\frac{mL\omega^2}{El^2}\frac{\partial^2 u(\zeta,t)}{\partial\zeta^2}-$$
$$-\frac{\varrho l\omega^2}{E}\left[\frac{\partial^2 u(\zeta,t)}{\partial\zeta^2}\int_\zeta^1(r_0+l\xi)F(\xi)\,d\xi-\frac{\partial u(\zeta,t)}{\partial\zeta}(r_0+l\zeta)F(\zeta)\right]-$$
$$-\frac{l^2\varrho\omega^2}{E}F(\zeta)u(\zeta,t)+\frac{l\varrho F(\zeta)}{E}\frac{\partial^2 u(\zeta,t)}{\partial t^2}=0, \qquad (31.1)$$

It is obtained from (30.13), by we setting $\epsilon = 0$. In the future, we shall call equation (31.1) the differential equation of undisturbed motion, or, more concisely, the "undisturbed" equation.

With the aid of the usual methods we can obtain the solution of equation (31.1), corresponding to normal vibrations,

$$u(\zeta,t)=\varphi_k a\cos(p_k t+\psi_k), \qquad (31.2)$$

with k = 1, 2, 3 . . . , where φ_k and p_k are the normal functions and natural frequencies respectively. In what follows we will find only the first frequency.

We begin construction of the approximate asymptotic formulas for the solution of the "disturbed" equation (30.13), by seeking the expression for the deflection function i.e. the solution of equation (31.13), in the form of the asymptotic series

$$u(\zeta,t)=\phi\, a\cos(pt+\psi)+\epsilon u_1(\zeta,t)+\epsilon^2 u_2(\zeta,t)+\cdots \tag{31.3}$$

It is assumed that $u_1(\zeta,t)$, $u_2(\zeta,t)$ are some periodic functions of t which have a period of 2π, and which do not contain the principle harmonics. The frequency of vibration p and the magnitude of the phase shift angle ψ are expressed in the following asymptotic series:

$$p^2 = p_c^2 + \epsilon\Delta_1 + \epsilon^2\Delta_2 + \cdots, \tag{31.4}$$

$$\psi = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \cdots, \tag{31.5}$$

where p_c is the natural frequency of vibrations determined from the undisturbed equation (31.1). Thus, the solution of equation (30.13) in the various approximations, reduces to the determination for the first approximation of $u_1(\zeta,t)$, Δ_1 and ψ_0, for the second approximation of $u_2(\zeta,t)$, Δ_2 and ψ_1 etc.

In order to determine the above quantities, let us substitute the series (31.3)—(31.5) into the perturbed differential equation (30.13) Equating coefficients of the same power to zero, in the equation obtained after substitution, we shall obtain the following system of differential equations:

$$\frac{d^2}{d\zeta^2}\left[I(\zeta)\frac{d^2\varphi(\zeta)}{d\zeta^2}\right]-\frac{mL\omega^2}{El^2}\frac{d^2\varphi(\zeta)}{d\zeta^2}-$$
$$-\frac{\varrho l\omega^2}{E}\left[\frac{d^2\varphi}{d\zeta^2}\int_\zeta^1(r_0+l\xi)F(\xi)\,d\xi-\frac{d\varphi}{d\zeta}(r_0+l\zeta)F(\zeta)\right]-$$
$$-\frac{\varrho l^2\omega^2}{E}F(\zeta)\varphi(\zeta)+\frac{l^4\varrho F(\zeta)}{E}p_c^2\varphi(\zeta)=0; \tag{31.6}$$

$$\frac{\partial^2}{\partial\zeta^2}\left[I(\zeta)\frac{\partial^2 u(\zeta,\tau)}{\partial\zeta^2}\right]-\frac{mL\omega^2}{El^3}\frac{\partial^2 u_1(\zeta,\tau)}{\partial\zeta^2}-$$
$$-\frac{\varrho l\omega^2}{E}\left[\frac{\partial^2 u_1(\zeta,\tau)}{\partial\zeta^2}\int\limits_{\zeta}^{1}(r_0+l\xi)F(\xi)\,d\xi-\frac{\partial u_1(\zeta,\tau)}{\partial\zeta}(r_0+l\zeta)F(\zeta)\right]$$
$$-\frac{l^2\varrho\omega^2}{E}F(\zeta)u_1(\zeta,\tau)+\frac{l^2\varrho F(\zeta)p_c^2}{E}\frac{\partial^2 u_1(\zeta,\tau)}{\partial\tau^2}+$$
$$+\frac{l^2\varrho F(\zeta)}{E}\Delta_1\varphi(\zeta)a\cos\tau+\frac{1}{El^3}\frac{\partial^2}{\partial\zeta^2}\vec{\Phi}\left[\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right]-$$
$$-\frac{q_0}{El}\cos(\tau-\psi_0)=0.$$

. . . .

. . . . (31.7)

Above we have introduced the new variable

$$\tau=pt+\psi. \tag{31.8}$$

In order to solve the problem in the **zeroth approximation**, it is necessary to examine equation (31.6) which represents the undisturbed equation of the form (31.1).

Considering the complexity of the differential equation (31.6), and the impossibility of finding its exact solution, it is expedient to apply an approximation which uses the series introduced in Chapter 4 in solving equation (21.1). We shall seek a solution of (31.6) in the form of the series

$$\varphi(\zeta)=A_0+A_1\zeta+A_2\zeta^2+A_3\zeta^3+\dots A_n\zeta^n, \tag{31.9}$$

where A_0, A_1, A_2; A_3 ...A_n are constants, determined from the following boundary conditions of equation (30.1):

$$[\varphi(\zeta)]_{\zeta=0}=0,\quad\left[\frac{d\varphi(\zeta)}{d\zeta}\right]_{\zeta=0}=0,$$

and also from (30.7) and (30.12)

$$\left[\frac{d^2\varphi(\zeta)}{d\zeta^2}\right]_{\zeta=1}=\frac{12E_6I_6}{EI(1)t_0^2}\frac{d\varphi(1)}{d\zeta};$$
$$\left[\frac{d^3\varphi(\zeta)}{d\zeta^3}\right]_{\zeta=1}=\frac{1}{El^3I(1)}\left[mlL\omega^2\frac{d\varphi(1)}{d\zeta}-ml^3\omega^2\varphi(1)-\right.$$
$$\left.-El^3\frac{d\varphi(1)}{d\zeta}\frac{d^2\varphi(1)}{d\zeta}-ml^3p_c^2\varphi(1)\right].$$

Without going into further details, in this solution in the zeroth approximation, which in principle does not differ from the solution of equation (21.1), let us examine the problem in the first approximation. With this in mind, and proceeding in a way analogous to that set forth in the previous chapters, we multiply equation (31.7) first by $\varphi \sin\tau \, d\tau \, d\zeta$ and a second time by $\varphi \cos\tau \, d\tau \, d\zeta$. The integrals along the whole length of the blade of both products we have obtained are equated to zero.

$$\int_0^{2\pi}\int_0^1 \Big\{ \frac{\partial}{d\zeta^2}\Big[I(\zeta)\frac{\partial^2 u_1(\zeta,\tau)}{\partial\zeta^2}\Big] - \frac{mL\omega^2}{El^2}\frac{\partial^2 u_1(\zeta,\tau)}{\partial\zeta^2} - \\ - \frac{\varrho l\omega^2}{E}\Big[\frac{\partial^2 u(\zeta,\tau)}{\partial\zeta^2}\int_\zeta^1 (r_0+l\xi)F(\xi)\,d\xi + \frac{\partial u_1(\zeta,\tau)}{d\zeta}(r_0+l\zeta)F(\zeta)\Big] - \\ - \frac{l^2\varrho\omega^2}{E}F(\zeta)\,u_1(\zeta,\tau) + \frac{1}{El^3}\frac{\partial^2}{\partial\zeta^2}\Phi\Big(\frac{d^2\varphi}{\partial\zeta^2}a\cos\tau\Big) - \\ - \frac{q_0}{El}\cos(\tau-\psi_0) + \frac{l\varrho F(\zeta)}{E}\Delta_1\varphi(\zeta)\,a\cos\tau + \\ + p_c^2\frac{l^2\varrho F(\zeta)}{E}\frac{\partial^2 u_1(\zeta,\tau)}{\partial\tau^2}\Big\}\varphi(\zeta)\sin\tau\,d\zeta\,d\tau = 0. \qquad (31.10)$$

$$\int_0^{2\pi}\int_0^1 \Big\{ \frac{\partial^2}{\partial\zeta^2}\Big[I(\zeta)\frac{\partial^2 u_1(\zeta,\tau)}{\partial\zeta^2} - \frac{mL\omega^2}{El^2}\frac{\partial^2 u_1(\zeta,\tau)}{\partial\zeta^2} - \\ - \frac{\varrho l\omega^2}{E}\Big[\frac{\partial^2 u(\zeta,\tau)}{\partial\zeta^2}\int_\zeta^1 (r_0+l\xi)F(\xi)\,d\xi + \frac{\partial u_1(\zeta,\tau)}{\partial\zeta}(r_0+l\zeta)F(\zeta)\Big] - \\ - \frac{l^2\varrho\omega^2}{E}F(\zeta)\,u_1(\zeta,\tau) + \frac{1}{El^3}\frac{\partial^2}{\partial\zeta^2}\Phi\Big(\frac{\partial^2\varphi(\zeta)}{\partial\zeta^2}a\cos\tau\Big) - \\ - \frac{q_0}{El}\cos(\tau-\psi_0) + \frac{l\varrho F(\zeta)}{E}\Delta_1\varphi(\zeta)\,a\cos\tau + \\ + p_c^2\frac{l^2\varrho F(\zeta)}{E}\frac{\partial^2 u_1(\zeta,\tau)}{\partial\tau^2}\Big\}\varphi(\zeta)\cos\tau\,d\zeta\,d\tau = 0. \qquad (31.11)$$

Since the function $u(\zeta,t)$ does not contain the principle harmonic, each of the last differential equations can be split into two equations of the following form:

$$\int_0^{2\pi}\int_0^1 \left\{ \frac{\partial^2}{\partial \zeta^2}\left[I(\zeta)\frac{\partial^2 u_1(\zeta,\tau)}{\partial \zeta^2}\right] - \frac{mL\omega^2}{El^2}\frac{\partial^2 u_1(\zeta,\tau)}{\partial \zeta^2} - \right.$$

$$- \frac{\varrho l\omega^2}{E}\left[\frac{\partial^2 u_1(\zeta,\tau)}{\partial \zeta^2}\int_\zeta^1 (r_0+l\xi)F(\xi)\,d\xi + \frac{\partial u_1(\zeta,t)}{\partial \zeta}(r_0+l\zeta)F(\zeta)\right] -$$

$$\left. - \frac{l^2\varrho\omega^2}{E}F(\zeta)u_1(\zeta,\tau) + p_c^2\frac{l^2\varrho F(\zeta)}{E}\frac{\partial^2 u_1(\zeta,\tau)}{\partial \tau^2}\right\}\varphi(\zeta)\sin\tau\, d\zeta\, d\tau = 0;$$

$$\int_0^{2\pi}\int_0^1 \left\{ \frac{1}{l^3}\frac{\partial^2}{\partial \zeta^2}\Phi\left(\frac{d^2\varphi}{d\zeta^2}a\cos\tau\right) - \frac{q_0}{l}\cos(\tau-\psi_0) + \right.$$

$$\left. + l^2\varrho F(\zeta)\Delta_1\varphi(\zeta)a\cos\tau\right\}\varphi(\zeta)\sin\tau\, d\zeta\, d\tau = 0; \qquad (31.12)$$

$$\int_0^{2\pi}\int_0^1 \left\{ \frac{\partial^2}{\partial \zeta^2}\left[I(\zeta)\frac{\partial^2 u_1(\zeta,\tau)}{\partial \zeta^2}\right] - \frac{mL\omega^2}{El^2}\frac{\partial^2 u_1(\zeta,\tau)}{\partial \zeta^2} - \right.$$

$$- \frac{\varrho l\omega^2}{E}\left[\frac{\partial^2 u_1(\zeta,\tau)}{d\zeta^2}\int_\zeta^1 (r_0+l\xi)F(\xi)\,d\xi + \frac{\partial u_1(\zeta,\tau)}{\partial \zeta}(r_0+l\zeta)F(\zeta)\right] -$$

$$\left. - \frac{l^2\varrho\omega^2}{E}F(\zeta)u_1(\zeta,\tau) + p_c^2\frac{l^2\varrho F(\zeta)}{E}\frac{\partial^2 u_1(\zeta,\tau)}{\partial \tau^2}\right\}\times$$

$$\times\varphi(\zeta)\cos\tau\, d\zeta\, d\tau = 0; \qquad (31.13)$$

$$\int_0^{2\pi}\int_0^1 \left\{ \frac{1}{l^3}\frac{\partial^2}{\partial \zeta^2}\Phi\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}a\cos\tau\right) - \frac{q_0}{l}\cos(\tau-\psi_0) + \right.$$

$$\left. + l\varrho F(\zeta)\Delta_1\varphi(\zeta)a\cos\tau\right\}\varphi(\zeta)\cos\tau\, d\zeta\, d\tau = 0. \qquad (31.14)$$

Solving the last equation for Δ_1, we find

$$\Delta_1 = \left[a\varrho l^3\int_0^1 F(\zeta)\varphi^2(\zeta)\,d\zeta\right]^{-1}\left\{\oint\int_0^1 \frac{1}{l^3}\frac{\partial^2}{\partial \zeta^2}\vec{\Phi}\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}a\cos\tau\right)\times\right.$$

$$\left.\times\varphi(\zeta)\cos\tau\, d\zeta\, d\tau - \frac{q_0}{l}\pi\cos\psi_0\int_0^1\varphi(\zeta)\,d\zeta\right\}.$$

The first integral in the curly brackets of the last equation may on the basis of (8.3), be expressed by the bending moment acting in the cross-section, i.e.

$$\varepsilon \oint \int_0^1 \frac{\partial^2}{\partial \zeta^2} \vec{\Phi} \left(\frac{d^2 \varphi(\zeta)}{d\zeta^2} a \cos \tau \right) \varphi(\zeta) \cos \tau \, d\zeta \, d\tau =$$

$$= \oint \int_0^1 \frac{\partial^2}{\partial \zeta^2} M \left(\frac{d^2 \varphi(\zeta)}{\partial \zeta^2} a \cos \tau \right) \varphi(\zeta) \cos \tau \, d\zeta \, d\tau =$$

$$= l^3 \oint \int_0^1 \frac{\partial^2}{\partial \zeta^2} \left[EI(\zeta) \frac{d^2 \varphi(\zeta)}{\partial \zeta^2} a \cos \tau \right] \varphi(\zeta) \cos \tau \, d\zeta \, d\tau .$$

Then the square of the frequency in the first approximation will be

$$p^2 = p_c^2 + \varepsilon \Delta_1 = p_c^2 + \left(a \varrho l^4 \int_0^1 F(\zeta) \varphi^2(\zeta) \, d\zeta \right)^{-1} \times$$

$$\times \oint \int_0^1 \frac{1}{l^3} \frac{\partial^2}{\partial \zeta^2} M \left(\frac{d^2 \varphi(\zeta)}{d\zeta^2} a \cos \tau \right) \varphi(\zeta) \cos \tau \, d\zeta \, d\tau -$$

$$- a \pi E \int_0^1 \frac{\partial^2}{\partial \zeta^2} \left[I(\zeta) \frac{d^2 \varphi(\zeta)}{d\zeta^2} \right] \varphi(\zeta) \, d\zeta -$$

$$- \frac{q_0 \pi}{l} \cos \psi_0 \int_0^1 \varphi(\zeta) \, d\zeta \Big\} . \qquad (31.15)$$

The sine of the phase shift angle in the first approximation, as in the preceding case, is found from equation (31.11), i.e.

$$\sin \psi_0 = \left[\frac{q_0 \pi}{l} \int_0^1 \varphi(\zeta) \, d\zeta \right]^{-1} \left[\oint \int_0^1 \frac{\partial^2}{\partial \zeta^2} M \left(\frac{d^2 \varphi(\zeta)}{\partial \zeta^2} a \cos \tau \right) \times \right.$$

$$\left. \times \varphi(\zeta) \sin \tau \, d\zeta \, d\tau \right] \frac{1}{l^3} . \qquad (31.16)$$

Employing formulas (31.15) and (31.16), we can construct a resonance curve for vibrations of the bar (blade) in question taking into account the dissipation of energy in the material. For this purpose it is necessary to calculate the double integral of expressions containing the bending moments. Proceeding analogously to our previous computations, i.e. as with equations (16.11)—(16.18), we can write

$$\oint\int_0^1 \frac{\partial^2}{\partial\zeta^2} M\left(\frac{d^2\varphi(\zeta)}{\partial\zeta^2} a\cos\tau\right)\varphi(\zeta)\cos\tau\, d\zeta\, d\tau =$$

$$= Ea\pi\int_0^1 l^3\frac{\partial^2}{\partial\zeta^2}\left(I(\zeta)\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)\varphi(\zeta)\,d\tau + 2E\frac{\nu}{l^n n}a^n\int_0^\pi(1-\cos\tau)^n\cos\tau\,d\tau\times$$

$$\times\int_0^1\left[\frac{\partial^2}{\partial\zeta^2}\left(\frac{d^2\varphi(\zeta)}{\partial\zeta^2}\right)^n\varphi(\zeta)\iint_F z^{n+1}\,dz\,dy\right]d\zeta; \tag{31.17}$$

$$\oint\int_0^1 \frac{\partial^2}{\partial\zeta^2} M\left(\frac{d^2\varphi(\zeta)}{\partial\zeta^2} a\cos\tau\right)\varphi(\zeta)\sin\tau\, d\zeta\, d\tau =$$

$$= \frac{\nu}{n}Ea^n\frac{2^{n+1}(n-1)}{l^n(n+1)}\int_0^1\left\{\frac{\partial^2}{\partial\zeta^2}\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\varphi(\zeta)\iint_F z^{n+1}\,dz\,dy\right\}d\zeta. \tag{31.18}$$

Substituting expressions (31.17) and (31.18) in the formula (31.15) and (31.16), we obtain

$$p^2 = p_c^2 + \left[a\varrho l^4\int_0^1 F(\zeta)\varphi^2(\zeta)\,d\zeta\right]^{-1}\left\{\frac{2E\nu a^n}{nl^{n+3}}\int_0^\pi(1-\cos\tau)^n\cos\tau\,d\tau\times\right.$$

$$\times\int_0^1\left[\frac{\partial^2}{\partial\zeta^2}\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\varphi(\zeta)\iint_F z^{n+1}\,dz\,dy\right]d\zeta -$$

$$\left. - q_0\pi l\cos\psi_0\int_0^1\varphi(\zeta)\,d\zeta\right\}; \tag{31.19}$$

$$\sin\psi_0 = \left[\frac{q_0\pi}{l}\int_0^1\varphi(\zeta)\,d\zeta\right]^{-1}\frac{2^{n+1}\nu Ea^n(n-1)}{n(n+1)l^{n+3}}\times$$

$$\times\int_0^1\left\{\frac{\partial^2}{\partial\zeta^2}\left(\frac{d^2\varphi(\zeta)}{d\zeta^2}\right)^n\varphi(\zeta)\iint_F z^{n+1}\,dz\,dy\right\}d\zeta. \tag{31.20}$$

The magnitude of the deflection of the bar may be determined by the following formula, on the basis of equation (31.3)

$$u(\zeta, t)=\varphi(\zeta)\, a \cos(pt+\psi_0). \qquad (31.21)$$

As before, we limit our consideration to the first approximation, which for technical purposes gives sufficient accuracy.

Chapter VIII

Torsional Vibrations of Bars

32. Derivation of basic equations

In the present chapter, formulas for the construction of a resonance curve of the torsional vibrations for circular shafts, with an allowance for the dissipation of energy in the material, are derived. As above, we base our approach on an experimentally established dependence of the energy dissipation on the magnitude of the stress in the material, and on the utilization of the method of the small parameter.

Let us examine steady-state vibrations, which are maintained by a periodic external exciting force.

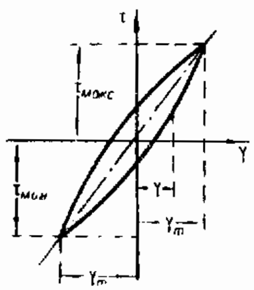

Fig. 20

Let us assume, following N. N. Davidenkov, that the true modulus of elasticity in shear of the material for the rising and falling branches of the hysteresis loop, (Fig. 20) may be expressed by the following formulas:

$$\frac{\overrightarrow{d\tau}}{d\gamma} = G\left[1 - \nu(\gamma_m + \gamma)^k\right],$$

$$\frac{\overleftarrow{d\tau}}{d\gamma} = G\left[1 + \nu(\gamma_m - \gamma)^k\right], \qquad (32.1)$$

where $\vec{\tau}$ and $\overleftarrow{\tau}$ are shearing stresses determined for the rising and falling branches of the hysteresis loop respectively;

G — modulus of elasticity in shear;

γ_m — amplitude of shear at a given point;

γ — magnitude of shear at any aribtrary time;

ν and k — constants of the material, experimentally determined.

The expressions (32.1) satisfy the following conditions arising from the symmetry of the hysteresis loop:

$$\left[\frac{d\vec{\tau}}{d\gamma}\right]_{\gamma=\gamma_m}=\left[\frac{d\overleftarrow{\tau}}{d\gamma}\right]_{\gamma=\gamma_m};\quad \left[\frac{d\vec{\tau}}{d\gamma}\right]_{\gamma=-\gamma_m}=\left[\frac{d\overleftarrow{\tau}}{d\gamma}\right]_{\gamma=-\gamma_m}$$

Integrating expressions (32.1) and determining the constants of integration from the condition that $\vec{\tau}=\overleftarrow{\tau}$ for $\gamma=\gamma_m$, we obtain the dependence between the shear stresses and the shear strains for both branches of the hysteresis loop

$$\vec{\tau}=G\left\{\gamma-\frac{\nu}{n}[(\gamma_m+\gamma)^n-2^{n-1}\gamma_m^n]\right\},$$

$$\overleftarrow{\tau}=G\left\{\gamma+\frac{\nu}{n}[(\gamma_m-\gamma)^n-2^{n-1}\gamma_m^n]\right\}, \qquad (32.2)$$

where $n=k+1$.

On the basis of the above statements, to determine the torsional moment we should proceed not from Hooke's law, but rather from the relation (32.2).

The shear strain of an element of the bar, located at a distance r from the axis, is determined by the formulas

$$\gamma=r\frac{d\phi}{dx}=r\phi'$$

and

$$\gamma_m=r\left(\frac{d\phi}{dx}\right)_{max}=r\phi'_m$$

Substituting the expression for γ and γ_m in the relation (32.2), we obtain

$$\vec{\tau} = rG\left\{\varphi' - \frac{\nu r^{n-1}}{n}\left[(\varphi'_m + \varphi')^n - 2^{n-1}\varphi_m'^n\right]\right\},$$
$$\overleftarrow{\tau} = rG\left\{\varphi' + \frac{\nu r^{n-1}}{n}\left[(\varphi'_m - \varphi')^n - 2^{n-1}\varphi_m'^n\right]\right\}. \qquad (32.3)$$

The value of the moment of the shear stresses acting in the cross-section of the bar, about its axis, is determined in the following manner:

$$\vec{M} = \int_F \tau r\, dF = \int_0^{r_m} \tau r 2\pi r\, dr =$$
$$= \frac{2\pi r_m^4}{4} G\left\{\varphi' \mp \frac{4\nu r^{n-1}}{n(n+3)}\left[(\varphi'_m \pm \varphi')^n - 2^{n-1}\varphi_m'^n\right]\right\}$$

or

$$\vec{M} = GI_p\left\{\varphi' - \frac{4\nu r^{n-1}}{n(n+3)}\left[(\varphi_m + \varphi')^n - 2^{n-1}\varphi_m'^n\right]\right\},$$
$$\overleftarrow{M} = GI_p\left\{\varphi' + \frac{4\nu r^{n-1}}{n(n+3)}\left[(\varphi'_m - \varphi')^n - 2^{n-1}\varphi_m'^n\right]\right\}, \qquad (32.4)$$

where

$$I_p = \frac{\pi r_m^4}{2}.$$

Setting $\nu = 0$, we obtain the usual relationship.

$$\vec{M} = \overleftarrow{M} = M = GI_p\varphi'.$$

On the basis of relation (32.4), we can form the differential equation of the torsional vibrations of the bar (Fig. 21). Applying d'Alembert's principle and examining the conditions of equilibrium of an element of the rod of length dx, we find

$$-M + M + \frac{\partial M}{\partial x}dx - \int_F \varrho r^2\, dF \frac{\partial^2 \varphi}{\partial t^2}dx = 0,$$

whence

$$\frac{\partial M}{\partial x} = \varrho I_p \frac{\partial^2 \varphi(x,t)}{\partial t^2}. \qquad (32.5)$$

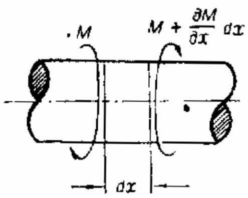

Fig. 21

Substituting in equation (32.5) the expression (32.4) for the torsional moment, we find

$$\frac{\partial}{\partial x} GI_p \left\{ \frac{\partial \varphi(x,t)}{\partial x} - \frac{\nu}{n}\left[\left(\frac{d\varphi_m(x)}{dx} + \frac{\partial \varphi(x,t)}{\partial x}\right)^n - 2^{n-1}\left(\frac{d\varphi_m(x)}{dx}\right)^n\right]\frac{4r^{n-1}}{n+3}\right\} = \varrho I_p \frac{\partial^2 \varphi(x,t)}{\partial t^2},$$

$$\frac{\partial}{\partial x} GI_p \left\{ \frac{\partial \varphi(x,t)}{\partial x} + \frac{\nu}{n}\left[\left(\frac{d\varphi_m(x)}{dx} - \frac{\partial \varphi(x,t)}{\partial x}\right)^n - 2^{n-1}\left(\frac{d\varphi_m(x)}{dx}\right)^n\right]\frac{4r^{n-1}}{n+3}\right\} = \varrho I_p \frac{\partial^2 \varphi(x,t)}{\partial t^2},$$

(32.6)

where $\frac{d\varphi_m(x)}{dx}$ is the maximum value (amplitude) of the relative angle of twist;

$\frac{\partial \varphi(x,t)}{\partial x}$ is the value of the relative angle of twist at time t .

For a bar of uniform cross-section, equations (32.6), may be transformed in the following manner:

$$\frac{\partial^2 \varphi(x,t)}{\partial x^2} - \frac{4r^{n-1}\nu}{n(n+3)}\frac{\partial}{\partial x}\left[\left(\frac{d\varphi_m(x)}{dx} + \frac{\partial \varphi(x,t)}{\partial x}\right)^n - 2^{n-1}\left(\frac{d\varphi_m(x)}{dx}\right)^n\right] - k^2 \frac{\partial^2 \varphi(x,t)}{\partial t^2} = 0,$$

$$\frac{\partial^2 \varphi(x,t)}{\partial x^2} + \frac{4r^{n-1}\nu}{n(n+3)}\frac{\partial}{\partial x}\left[\left(\frac{d\varphi_m(x)}{dx} - \frac{\partial \varphi(x,t)}{dx}\right)^n - 2^{n-1}\left(\frac{d\varphi_m(x)}{dx}\right)^n\right] - k^2 \frac{\partial^2 \varphi(x,t)}{dt^2} = 0,$$

(32.7)

where $k^2 = \frac{\varrho}{G}\ \sec^2/cm^2$.

If we denote the second terms of the two equations by

$$\varepsilon \frac{\partial}{\partial x}\vec{\Phi}(\varphi', t) = -\frac{\partial}{\partial x}\left[\left(\frac{d\varphi_m(x)}{dx} + \frac{\partial \varphi(x,t)}{\partial x}\right)^n - 2^{n-1}\left(\frac{d\varphi_m(x)}{dx}\right)^n\right]\frac{4\nu r^{n-1}}{n(n+3)},$$

(32.8) cont.

$$\varepsilon \frac{\partial}{\partial x} \overleftarrow{\Phi}(\varphi', t) = + \frac{\partial}{\partial x}\left[\left(\frac{d\varphi_m(x)}{dx} - \frac{\partial \varphi(x, t)}{\partial x}\right)^n - \right.$$
$$\left. - 2^{n-1}\left(\frac{d\varphi_m(x)}{dx}\right)^n\right]\frac{4\nu r^{n-1}}{n(n+3)}, \tag{32.8}$$

then equation (32.7) may be rewritten in the following form:

$$\frac{\partial^2 \varphi(x, t)}{\partial x^2} + \varepsilon \frac{\partial}{\partial x} \vec{\Phi}(\varphi', t) - k^2 \frac{\partial^2 \varphi(x, t)}{\partial t^2} = 0,$$
$$\frac{\partial^2 \varphi(x, t)}{\partial x^2} + \varepsilon \frac{\partial}{\partial x} \overleftarrow{\Phi}(\varphi', t) - k^2 \frac{\partial^2 \varphi(x, t)}{\partial t^2} = 0. \tag{32.9}$$

In order to maintain the steady-state vibrations of a non-conservative system, we require the action of an external disturbing force. Since the dissipation of energy in the material is small, the disturbing force must also be small. This "smallness" can be characterized by the introduction of a small parameter as a factor of the term expressing the exciting force, i.e.

$$\varepsilon q = q_0 \frac{1}{c\text{м}^2}.$$

Then the differential equations of forced vibrations may be written in the form

$$\frac{\partial^2 \varphi(x, t)}{\partial x^2} + \varepsilon \frac{\partial}{\partial x} \vec{\Phi}(\varphi', t) - k^2 \frac{\partial^2 \varphi(x, t)}{\partial t^2} = \varepsilon q \cos \omega t;$$
$$\frac{\partial^2 \varphi(x, t)}{\partial x^2} + \varepsilon \frac{\partial}{\partial x} \overleftarrow{\Phi}(\varphi', t) - k^2 \frac{\partial^2 \varphi(x, t)}{\partial t^2} = \varepsilon q \cos \omega t. \tag{32.10}$$

We look for the solution of equation (32.10) in the form of series expansion in powers of the small parameter. We shall represent in series form the function of the angle of rotation $\varphi(x,t)$, frequency of the vibrations of the bar and the magnitude of the phase shift

$$\varphi(x, t) = \psi(x)\varphi_0 \cos(\omega t + \alpha) + \varepsilon\varphi_1(x, t) + \varepsilon^2\varphi_2(x, t) + \ldots, \quad (32.11)$$

$$\omega = \omega_c + \varepsilon\Delta_1 + \varepsilon^2\Delta_2 + \ldots, \quad (32.12)$$

$$\alpha = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \ldots, \quad (32.13)$$

where φ_0 is the amplitude of the angle of twist.

ω_c is the natural frequency of vibrations of the bar.

Let us now introduce a new variable

$$\theta = \omega t + \alpha \quad (32.14)$$

and transform $\cos \omega t = \cos(\theta - \alpha)$, utilizing the expression (32.13) for

$$\cos(\theta - \alpha) = \cos(\theta - \alpha_0)\cos\varepsilon(\alpha_1 + \varepsilon\alpha_2 + \ldots) + \\ + \sin(\theta - \alpha_0)\sin\varepsilon(\alpha_1 + \varepsilon\alpha_2 + \ldots). \quad (32.15)$$

Then, let us also express

$\cos\varepsilon(\alpha_1 + \varepsilon\alpha_2 + \ldots)$ and $\sin\varepsilon(\alpha_1 + \varepsilon\alpha_2 + \ldots)$

by the series

$$\cos\varepsilon(\alpha_1 + \varepsilon\alpha_2 + \ldots) = 1 - \frac{\varepsilon^2(\alpha_1 + \varepsilon\alpha_2 + \ldots)}{2!},$$

$$\sin\varepsilon(\alpha_1 + \varepsilon\alpha_2 + \ldots) = \varepsilon(\alpha_1 + \varepsilon\alpha_2 + \ldots) - \frac{\varepsilon^3(\alpha_1 + \varepsilon\alpha_2 + \ldots)}{3!}$$

and substitute their values in formula (32.15). Neglecting the terms containing the small parameter ε, to a power higher than the second we obtain approximately

$$\cos(\theta - \alpha) = \cos(\theta - \alpha_0) + \varepsilon\alpha_1 \sin(\theta - \alpha_0) + \\ + \varepsilon^2\left[\alpha_2 \sin(\theta - \alpha_0) - \frac{\alpha_1^2}{2}\cos(\theta - \alpha_0)\right]. \quad (32.16)$$

Substituting the expansions (32.11)—(32.13) in equation (32.10), and also taking into account the change of variable (32.14) and the expression (32.15), we obtain

$$\frac{d^2\psi(x)}{dx^2}\varphi_0\cos\theta+\varepsilon\frac{\partial^2\varphi_1(x,\theta)}{\partial x^2}+\varepsilon^2\frac{\partial^2\varphi_2(x,\theta)}{\partial x^2}+$$
$$+\varepsilon\frac{\partial}{\partial x}\vec{\Phi}[\psi'(x)\varphi_0\cos\theta+\varepsilon\varphi_1'(x,t)+\varepsilon^2\varphi_2'(x,\theta)+\ldots]-$$
$$-k^2\left[(\omega_c^2+\varepsilon\Delta_1+\varepsilon^2\Delta_2+\ldots)\left(-\varphi_0\psi(x)\cos\theta+\varepsilon\frac{\partial^2\varphi_1(x,\theta)}{\partial\theta^2}+\right.\right.$$
$$\left.\left.+\varepsilon^2\frac{\partial^2\varphi_2(x,\theta)}{\partial\theta^2}+\ldots\right)\right]-\varepsilon q[\cos(\theta-\alpha_0)+\varepsilon\alpha_1\sin(\theta-\alpha_0)]+$$
$$+\varepsilon^2\left[\alpha_2\sin(\theta-\alpha_0)-\frac{\alpha_1^2}{2}\cos(\theta-\alpha_0)\right]=0. \tag{32.17}$$

Grouping the terms of the last equation, containing the small parameter terms ϵ of the same power, and equating to zero the expressions which multiply the various powers of ϵ, we obtain the following system of differential equations:

$$\frac{d^2\psi(x)}{dx}\varphi_0\cos\theta+k^2\omega_c^2\varphi_0\psi(x)\cos\theta=0, \tag{32.18}$$

$$\frac{\partial^2\varphi_1(x,\theta)}{\partial x^2}+\frac{\partial^2}{\partial x^2}\vec{\Phi}[\psi(x)\varphi_0\cos\theta]+k^2\Delta_1\psi(x)\varphi_0\cos\theta-$$
$$-k^2\omega_c^2\frac{\partial^2\varphi_1(x,\theta)}{\partial\theta^2}-q\cos(\theta-\alpha_0)=0, \tag{32.19}$$

$$\frac{\partial^2\varphi_2(x,\theta)}{\partial x^2}+\frac{\partial^2}{\partial x^2}\vec{\Psi}(x,\theta)+k^2\psi(x)\Delta_2\varphi_0\cos\theta-$$
$$-k^2\Delta_1\frac{\partial^2\varphi_1(x,\theta)}{\partial\theta^2}-k^2\omega_c^2\frac{\partial^2\varphi_2(x,\theta)}{\partial\theta^2}=0. \tag{32.20}$$

After cancelling $\varphi_0\cos\theta$ in equation (32.18) and introducing the new notation $p^2=k^2\omega_c^2$ equation (32.18) takes on the following form:

$$\frac{d^2\psi(x)}{dx^2}+p^2\psi(x)=0. \tag{32.21}$$

Integrating the system of equations (32.18)—(32.20), we shall obtain the solution of the problem of the torsional vibrations of the bar taking account of the dissipation of energy in the material, in the zeroth, first and second approximations.

33. The solution of the problem in the zeroth approximation

To solve the problem in the zeroth approximation (with an accuracy to the zeroth power of the small parameter), it is necessary to solve the differential equation (32.21) which is a homogeneous, ordinary, linear equation of the second order. We write the general solution of this equation as

$$\psi(x) = A\cos px + B\sin px, \tag{33.1}$$

where A and B are constants, which must be determined from the boundary conditions. For a bar having both its ends free, and in the absence of concentrated masses, the boundary conditions are

$$\psi(0) = 0, \quad \psi(l) = 0. \qquad (33.2)^*$$

Then from (33.1) we will obtain the frequency equation

$$\sin pl = 0, \tag{33.3}$$

whence

$$pl = i\pi.$$

Considering that

$$p^2 = k^2\omega_c^2, \tag{33.4}$$

we obtain

$$\sin k\omega_c l = 0,$$

---

*These are the conditions for fixed ends, not free ends. (Trans.)

hence

$$k\omega_c l = i\pi, \tag{33.5}$$

where i is an integer. Assuming $i = 1, 2, 3 \ldots,$ we obtain the frequencies of the different modes of vibrations. The frequency of the gravest mode of vibrations is equal to

$$\omega_c = \frac{\pi}{kl}$$

or

$$\omega_c = \frac{\pi}{l}\sqrt{\frac{G}{\varrho}}. \tag{33.6}$$

The formula for this mode of vibrations is of the form

$$\psi(x) = B \sin px = \sin \frac{\pi}{l} x. \tag{33.7}$$

The function of the angle of twist in the zeroth approximation is

$$\varphi(x, \theta) = \varphi_0 \psi(x) \cos \theta = B\varphi_0 \sin \frac{\pi}{l} x \cos \theta. \tag{33.8}$$

Denoting

$$B\varphi_0 = C,$$

we rewrite the last equation

$$\varphi(x, \theta) = C \sin \frac{\pi}{l} x \cos \theta. \tag{33.9}$$

The constant C is determined from the conditions that the center of the rod $\left(x=\frac{l}{2}\right)$ at the initial instant, with $\theta=0$, the angle of twist will have the maximum value φ_0, i.e.

$$\varphi\left(\frac{l}{2},\theta\right)=C\sin\frac{\pi l}{l\cdot 2}\cos 0=\varphi_0. \tag{33.10}$$

Hence

$$C=B\varphi_0=\varphi_0;\quad B=1.$$

The angle of twist at any cross-section of the rod is determined from the expression

$$\varphi(x,\theta)=\varphi_0\sin\frac{\pi}{l}x\cos\theta. \tag{33.11}$$

34. Determination of frequencies of vibration and of the phase shift in the first approximation

To solve the problem in the first approximation in accordance with expansions (32.11) and (32.12) we turn to equation (32.19), from which we shall find Δ_1 and $\sin\alpha_0$. To examine the balance of vibrational energy of the rod let us multiply equation (32.19) first by $\psi(x)\sin\theta\,dx\,d\theta$, and again by $\psi(x)\cos\theta\,dx\,d\theta$ and equate to zero the integrals of both products along the whole length for one cycle. We obtain

$$1)\quad \oint\int_0^l\left\{\frac{\partial^2\varphi_1(x,\theta)}{\partial x^2}+\frac{\partial}{\partial x}\vec{\vec{\Phi}}[\psi'(x)\,\varphi_0\cos\theta)]+k^2\Delta_1\psi(x)\,\varphi_0\cos\theta-\right.$$

$$\left.-k^2\omega_c^2\frac{\partial^2\varphi_1(x,\theta)}{\partial\theta^2}-q\cos(\theta-\alpha_0)\right\}\psi(x)\sin\theta\,dx\,d\theta=0; \tag{34.1}$$

$$2)\quad \oint\int_0^l \left\{ \frac{\partial^2\varphi_1(x,\theta)}{\partial x^2} + \frac{\partial}{\partial x}\overset{\rightleftarrows}{\Phi}[\psi'(x)\varphi_0\cos\theta] + k^2\Delta_1\psi(x)\varphi_0\cos\theta - \right.$$

$$\left. - k^2\omega_c^2\frac{\partial^2\varphi_1(x,\theta)}{\partial\theta^2} - q\cos(\theta-\alpha_0)\right\}\psi(x)\cos\theta\,dx\,d\theta = 0. \qquad (34.2)$$

Here it turns out that

$$\int_0^{2\pi}\int_0^l \left\{ \frac{\partial^2\varphi_1(x,\theta)}{\partial x^2} - k^2\omega_c^2\frac{\partial^2\varphi_1(x,\theta)}{\partial\theta^2}\right\}\psi(x)\sin\theta\,dx\,d\theta = 0. \qquad (34.3)$$

We can justify this equality by integrating by parts, the first term with respect to x and the second with respect to θ, taking into account here the conditions at the ends of the bar (33.2).

$$\int_0^l \frac{\partial^2\varphi_1(x,\theta)}{\partial x^2}\psi(x)\,dx = \psi(x)\frac{\partial\varphi_1(x,\theta)}{\partial x}\Big|_0^l - \int_0^l \frac{\partial\varphi_1(x,\theta)}{\partial x}\frac{\partial\psi(x)}{\partial x}\,dx =$$

$$= 0 - \left[\varphi_1(x,\theta)\frac{\partial\psi(x)}{\partial x}\Big|_0^l - \int_0^l \varphi_1(x,\theta)\frac{\partial^2\psi(x)}{\partial x^2}\,\partial x = \right.$$

$$= \int_0^l \frac{\partial^2\psi(x)}{\partial x^2}\varphi_1(x,\theta)\,dx; \qquad (34.4)$$

$$\int_0^{2\pi}\frac{\partial^2\phi_1(x,\theta)}{\partial\theta^2}\sin\theta\,d\theta = \frac{\partial\phi_1(x,\theta)}{\partial\theta}\sin\theta\Big|_0^{2\pi} - \int_0^{2\pi}\frac{\partial\phi_1(x,\theta)}{\partial\theta}\cos\theta\,d\theta =$$

$$= 0 - \left[\phi_1(x,\theta)\cos\theta\Big|_0^{2\pi} - \int_0^{2\pi}\phi_1(x,\theta)(-\sin\theta)d\theta\right] =$$

$$= -\int_0^{2\pi}\phi_1(x,\theta)\sin\theta\,d\theta. \qquad (34.5)$$

Substituting (34.3) and (34.4) in (34.2), we obtain

$$\int_0^{2\pi}\int_0^{l}\left\{\frac{\partial^2\psi(x)}{\partial x^2} + k^2\omega_c^2\psi(x)\right\}\phi_1(x,\theta)\sin\theta\,dx\,d\theta = 0.$$

For, according to (32.21), the expression in the curly brackets is equal to zero, which proves the correctness of the equation (34.3).

Thus, the differential equation (34.1) splits into two equations, equation (34.3) and the following:

$$\oint\int_0^l \left\{ k^2\Delta_1\psi(x)\varphi_0\cos\theta - q\cos(\theta-\alpha_0) + + \frac{\partial}{\partial x}\vec{\vec{\Phi}}(\psi'(x)\varphi_0\cos\theta)\right\}\psi(x)\sin\theta\,dx\,d\theta = 0. \tag{34.6}$$

Analogously, equation (34.2) splits into the two following equations:

$$\int_0^{2\pi}\int_0^l \left\{ \frac{\partial^2\phi_1(x,\theta)}{\partial x^2} - k^2\omega^2\frac{\partial^2\phi(x,\theta)}{\partial\theta^2}\right\}\psi(x)\cos\theta\,dx\,d\theta = 0, \tag{34.7}$$

$$\oint\int_0^l \left\{ k^2\Delta_1\psi(x)\phi_0\cos\theta - q\cos(\theta-\alpha) + + \frac{\partial}{\partial x}\vec{\vec{\Phi}}\left[\psi'(x)\phi\cos\theta\right]\right\}\psi(x)\cos\theta\,dx\,d\theta = 0. \tag{34.8}$$

Solving equation (34.8) for Δ_1 , we obtain

$$\Delta_1 = \left[\int_0^l \pi k^2\phi_0\psi^2(x)\,dx\right]^{-1}\left\{q\pi\cos\alpha_0\int_0^l\psi(x)\,dx - -\oint\int_0^l\frac{\partial}{\partial x}\vec{\vec{\Phi}}\left[\psi'(x)\phi_0\cos\theta\right]\psi(x)\cos\theta\,dx\,d\theta\right\}$$

Keeping in mind, that

$$\psi(x) = \sin\frac{\pi}{l}x,$$

$$\int_0^l\psi^2(x)\,dx = \int_0^l\sin^2\frac{\pi x}{l}\,dx = \frac{l}{2},$$

$$\int_0^l\psi(x)\,dx = \int_0^l\sin\frac{\pi x}{l}\,dx = \frac{2l}{\pi},$$

we obtain the following expression for Δ_1

$$\Delta_1=\frac{2}{\pi k^2 l\varphi_0}\left\{-\oint\int_0^l \frac{\partial}{\partial x}\vec{\Phi}[\psi'(x)\varphi_0\cos\theta]\,\psi(x)\cos\theta\,dx\,d\theta+2ql\cos\alpha_0\right\}. \tag{34.9}$$

From equation (34.6) we determine the sine of the phase shift angle

$$-2lq\sin\alpha_0+\oint\int_0^l \frac{\partial}{\partial x}\vec{\Phi}(\psi'(x)\varphi_0\cos\theta)\,\psi(x)\sin\theta\,dx\,d\theta=0,$$

$$\sin\alpha_0=\frac{1}{2lq}\oint\int_0^l \frac{\partial}{\partial x}\vec{\Phi}(\psi'(x)\varphi_0\cos\theta)\,\psi(x)\sin\theta\,dx\,d\theta. \tag{34.10}$$

To calculate Δ_1 and $\sin\alpha_0$ it is necessary to calculate the double integrals

$$1)\quad \oint\int_0^l \frac{\partial}{\partial x}\vec{\Phi}(\psi'(x)\varphi_0\cos\theta)\,\psi(x)\cos\theta\,dx\,d\theta;$$

$$2)\quad \oint\int_0^l \frac{\partial}{\partial x}\vec{\Phi}(\psi'(x)\varphi_0\cos\theta)\,\psi(x)\sin\theta\,dx\,d\theta. \tag{34.11}$$

The expression for the functional (32.7) has the following form in the first approximation:

$$\varepsilon\frac{\partial}{\partial x}\vec{\Phi}(\psi',t)=-\frac{\nu}{n}\frac{\partial}{\partial x}\left[(1+\cos\theta)^n\varphi_0^n\left(\frac{d\psi(x)}{dx}\right)^n-2^{n-1}\varphi_0^n\left(\frac{d\psi(x)}{dx}\right)^n\right]\times$$
$$\times\frac{4r^{n-1}}{n+3}=\nu\varphi_0^n\left(\frac{d\psi(x)}{dx}\right)^{n-1}\left(\frac{d^2\psi(x)}{dx^2}\right)[(1+\cos\theta)^n-2^{n-1}]\frac{4r^{n-1}}{n+3},$$

$$\varepsilon\frac{\partial}{\partial x}\overleftarrow{\Phi}(\psi',t)=\frac{\nu}{n}\frac{\partial}{\partial x}\left[(1-\cos\theta)^n\varphi_0^n\left(\frac{d^2\psi(x)}{dx^2}\right)^n-2^{n-1}\varphi_0^n\left(\frac{d\psi(x)}{dx}\right)^n\right]\times$$
$$\times\frac{4r^{n-1}}{n+3}=\nu\varphi_0^n\left(\frac{d\psi(x)}{dx}\right)^{n-1}\left(\frac{d^2\psi(x)}{dx^2}\right)[(1-\cos\theta)^n-2^{n-1}]\frac{4r^{n-1}}{n+3}.$$

Substituting in the last equation the value $\psi(x)=\sin\frac{\pi}{l}x$, we find

$$\varepsilon\frac{\partial}{\partial x}\vec{\Phi}(\psi'(x)\varphi_0\cos\theta)=$$

$$=\frac{-4\nu r^{n-1}\pi^{n+1}\varphi_0^n}{(n+3)l^{n+1}}[(1+\cos\theta)^n-2^{n-1}]\cos^{n-1}\frac{\pi x}{l}\left(-\sin\frac{\pi x}{l}\right);$$

$$\varepsilon\frac{\partial}{\partial x}\overleftarrow{\Phi}(\psi'(x)\varphi_0\cos\theta)=$$

$$=\frac{4\nu r^{n-1}\pi^{n+1}\varphi_0^n}{(n+3)l^{n+1}}[(1-\cos\theta)^n-2^{n-1}]\cos^{n-1}\frac{\pi x}{l}\left(-\sin\frac{\pi x}{l}\right).$$

Bearing in mind the last expression, we transform the integrals (34.11) into the form

$$1)\quad\oint\int_0^l\varepsilon\frac{\partial}{\partial x}\overleftrightarrow{\Phi}[\psi'(x)\varphi_0\cos\theta]\psi(x)\cos\theta\,dx\,d\theta=$$

$$=\int_\pi^{2\pi}\int_0^l\frac{4\nu\pi^{n+1}r^{n-1}\varphi_0^n}{(n+3)l^{n+1}}[(1-\cos\theta)^n-2^{n-1}]\cos^{n-1}\frac{\pi x}{l}\sin^2\frac{\pi x}{l}\cos\theta\,dx\,d\theta+$$

$$+\int_0^\pi\int_0^l-\frac{4\pi^{n+1}r^{n-1}\varphi_0^n}{(n+3)l^{n+1}}[(1+\cos\theta)^n-2^{n-1}]\cos^{n-1}\frac{\pi x}{l}\sin^2\frac{\pi x}{l}\cos\theta\,dx\,d\theta=$$

$$=\frac{8\nu r^{n-1}\pi^{n+1}\varphi_0^n}{(n+3)l^{n+1}}\int_0^l\cos^{n-1}\frac{\pi x}{l}\sin^2\frac{\pi x}{l}dx\int_0^\pi[(1-\cos\theta)^n-2^{n-1}]\cos\theta\,d\theta;$$

$$2)\quad\oint\int_0^l\varepsilon\frac{\partial}{\partial x}\overleftrightarrow{\Phi}(\psi'(x)\varphi_0\cos\theta)\psi(x)\sin\theta\,dx\,d\theta=$$

$$=\int_\pi^{2\pi}\int_0^l\frac{4\nu r^{n-1}\pi^{n+1}\varphi_0^n}{(n+3)l^{n+1}}[(1-\cos\theta)^n-2^{n-1}]\cos^{n-1}\frac{\pi x}{l}\sin\frac{\pi x}{l}\sin\theta\,dx\,d\theta-$$

$$-\int_0^\pi\int_0^l\frac{4\nu r^{n-1}\pi^{n+1}\varphi_0^n}{(n+3)l^{n+1}}[(1+\cos\theta)^n-2^{n-1}]\cos^{n-1}\frac{\pi x}{l}\sin^2\frac{\pi x}{l}\sin\theta\,dx\,d\theta=$$

$$=-\frac{8\nu r^{n-1}\pi^{n+1}\varphi_0^n}{(n+3)l^{n+1}}\int_0^l\cos^{n-1}\frac{\pi x}{l}\sin^2\frac{\pi x}{l}dx\int_0^\pi[(1-\cos\theta)^n-2^{n-1}]\sin\theta\,d\theta. \qquad (34.12)$$

The increment to the square of the frequency, on the basis of (34.9) and (34.12), is determined by the formula

$$\varepsilon\Delta_1=\frac{2}{\pi k^2 l\varphi_0}\left\{\frac{8\nu r^{n-1}\pi^{n+1}\varphi_0^n}{(n+3)l^{n+1}}\int_0^l\cos^{n-1}\frac{\pi x}{l}\sin^2\frac{\pi x}{l}\,dx\times\right.$$
$$\left.\times\int_0^\pi[(1-\cos\theta)^n-2^{n-1}]\cos\theta\,d\theta+2q_0 l\cos\alpha_0\right\}=\frac{16\nu\pi^n r^{n-1}\varphi_0^{n-1}}{k^2(n+3)l^{n+2}}\times$$
$$\times\int_0^l\cos^{n-1}\frac{\pi x}{l}\sin^2\frac{\pi x}{l}\,dx\int_0^\pi[(1-\cos\theta)^n-2^{n-1}]\cos\theta\,d\theta,$$

and the square of the frequency in the first approximation is, according to (32.11),

$$\omega^2=\omega_c^2+\varepsilon\Delta_1=\frac{\pi^2}{l^2k^2}+\frac{16\nu\pi^n r^{n-1}\varphi_0^{n-1}}{k^2(n+3)l^{n+2}}\times$$
$$\times\int_0^l\cos^{n-1}\frac{\pi x}{l}\sin^2\frac{\pi x}{l}\,dx\int_0^\pi[(1-\cos\theta)^n-2^{n-1}]\cos\theta\,d\theta+2q_0 l\cos\alpha_0, \tag{34.13}$$

whence we obtain a more convenient formula for the purposes of calculation

$$\frac{\omega^2}{\omega_c^2}=1+\frac{16\nu\pi^{n-2}r^{n-1}\varphi_0^{n-1}}{(n+3)l^n}\int_0^l\cos^{n-1}\frac{\pi x}{l}\sin^2\frac{\pi x}{l}\times$$
$$\times\int_0^\pi[(1-\cos\theta)^n-2^{n-1}]\cos\theta\,d\theta+\frac{4q_0l^2}{\pi^3\varphi_0}\cos\alpha_0. \tag{34.14}$$

The phase shift angle in the first approximation, on the basis of (34.10) and (34.12), is determined by the formula

$$\sin\alpha_0=\frac{1}{2lq_0}\,\frac{-8r^{n-1}\nu\pi^{n+1}\varphi_0^n}{(n+3)l^{n+1}}\times$$
$$\times\int_0^l\cos^{n-1}\frac{\pi x}{l}\sin^2\frac{\pi x}{l}\int_0^\pi[(1-\cos\theta)^n-2^{n-1}]\sin\theta\,d\theta.$$

After several transformations we obtain

$$\sin\alpha_0 = -\frac{4r^{n-1}\pi^{n+1}\nu\varphi_0^n}{g_0(n+3)l^{n+2}}\int_0^l \cos^{n-1}\frac{\pi x}{l}\sin^2\frac{\pi x}{l}\times$$
$$\times\int_0^\pi[(1-\cos\theta)^n-2^{n-1}]\sin\theta\,d\vartheta. \tag{34.15}$$

We can now construct a resonance curve of forced torsional vibrations of an elastic bar, considering the dissipation of energy in the material from formulas (34.14) and (34.15).

35. Sample calculation

Let us apply the formulas (34.14) and (34.15) which have been derived to construct resonance curves.

Let a bar with diameter d = 20 mm and length l = 200 mm execute forced torsional vibrations. The bar is made of brass with parameters n = 3 and ν = 143 found experimentally.

We shall take an amplitude of the disturbing force which ensures a maximum shearing stress in the bar during torsional vibrations of the order of 1000 kg/cm^2 g_0 = 0.12458 $\cdot$ $10^6\pi^4$cm^{-2}. The maximum angle of twist of the bar, assuming that the disturbing force acts at the central transverse cross-section, will be $(\varphi_0)_{max}$ = 0.8 $\cdot$ 10^{-2}.

Introducing the notation $\varphi_0 = a \cdot 10^{-2} l$ in what follows for an arbitrary amplitude of the maximum angle of twist and taking values of a = 0.2; 0.4; 0.5; 0.6; 0.7; and 0.8 we shall calculate by means of formulas (34.14) and (34.15) the values of frequency for the case in question. The results are recorded in tables 13 and 14.

Table 13

$\frac{\varphi_0}{l}$	$\sin\alpha_0$	$\sin^2\alpha_0$	$\cos^2\alpha_0$	$\cos\alpha_0$	$0{,}09006\frac{1}{a^2}$	$0{,}031312\frac{1}{a}$
0	0	0	1	±1,000	0	∞
$0{,}2\cdot10^{-2}$	0,25	0,063	0,9375	±0,968	0,00360	0,15656
$0{,}4\cdot10^{-2}$	0,50	0,250	0,75	±0,866	0,01441	0,07828
$0{,}5\cdot10^{-2}$	0,63	0,391	0,609375	±0,781	0,02252	0,06262
$0{,}6\cdot10^{-2}$	0,75	0,563	0,4375	±0,661	0,03242	0,05219
$0{,}7\cdot10^{-2}$	0,88	0,766	0,234375	±0,484	0,04413	0,04473
$0{,}8\cdot10^{-2}$	1,00	1,000	0	±0,000	0,05764	0,03914

Table 14

$\frac{\varphi_0}{l}$	$0{,}031312\,\frac{l}{a}\cos\alpha_0$	$\left(\frac{\omega}{\omega_c}\right)^2_{пр}$	$\left(\frac{\omega}{\omega_c}\right)^2_{л}$	$\left(\frac{\omega}{\omega_c}\right)_{пр}$	$\left(\frac{\omega}{\omega_c}\right)_{л}$
0	±	+	−	+	−
$0{,}2\cdot10^{-2}$	±0,15159	1,14799	0,84481	1,07144	0,91913
$0{,}4\cdot10^{-2}$	±0,05779	1,05338	0,91780	1,02634	0,95802
$0{,}5\cdot10^{-2}$	±0,04889	1,02637	0,92360	1,01310	0,96364
$0{,}6\cdot10^{-2}$	±0,03452	1,00210	0,93306	1,00105	0,96595
$0{,}7\cdot10^{-2}$	±0,02166	0,97753	0,93422	0,98870	0,96655
$0{,}8\cdot10^{-2}$	±0,00000	0,94236	0,94236	0,97075	0,97075

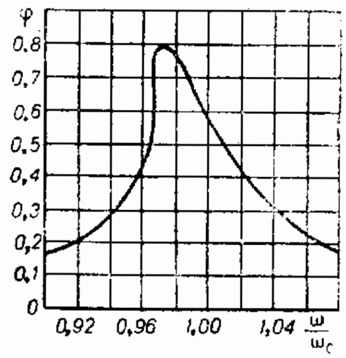

Fig. 22

Fig. 22 shows a resonance curve constructed from the data in Table 14 (here we take $\varphi = \frac{10^2}{l}\varphi_0$). In examining the curve it is seen that the allowance made for dissipation of energy in the material in the investigated case of torsional vibrations, involves a considerable nonlinearity.

36. Torsional vibrations of bars with concentrated masses

The method presented above may be applied also to the investigation of vibrations of an elastic cylindrical bar with concentrated masses. Let us examine the torsional vibrations of a shaft with discs on its ends (Fig. 23). On the basis of formulas (32.2), (32.10) and (32.13), we start from the expression

$$\varphi(x, \theta) = \varphi_0 \psi(x) \cos\theta = (A\cos px + B\sin px)\cos\theta. \tag{36.1}$$

We determine integration constants from the conditions at the ends. Taking into account the presence of discs at the ends of the shaft which possess rotational inertia, we obtain

$$I_1\left[\frac{\partial^2\varphi(x,\theta)}{\partial\theta^2}\right]_{x=0}=GI_p\left[\frac{\partial\varphi(x,\theta)}{\partial x}\right]_{x=0},$$
$$I_2\left[\frac{\partial^2\varphi(x,\theta)}{\partial\theta^2}\right]_{x=l}=-GI_p\left[\frac{\partial\varphi(x,\theta)}{\partial x}\right]_{x=l},$$

(36.2)

where I_1 and I_2 are the moments of inertia of the discs, l is the length of the shaft. Substituting (36.1) in equation (36.2) we obtain

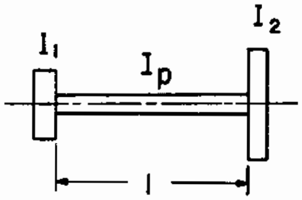

Fig. 23

$$I_1A\cos\theta=GI_pBp\cos\theta,$$
$$-I_2(A\cos pl+B\sin pl)\cos\theta=$$
$$=-GI_p(-Ap\sin pl+Bp\cos pl)\cos\theta,$$

After division by $\cos\theta$ and some transformations, the last equation may be rewritten in the following form:

$$AI_1+BGI_pp=0,$$
$$A(GI_pp\sin pl-I_2\cos pl)+B(-GI_pp\cos pl+I_2\sin pl). \quad (36.3)$$

This system has a solution for A and B , different from zero, if the determinant of this system reduces to zero, i.e.

$$\begin{vmatrix} I_1; & GI_pp \\ GI_pp\sin pl-I_2\cos pl; & (-GI_pp\cos pl+I_2\sin pl)\end{vmatrix}=0,$$

whence, writing the obtained determinant in expanded form, we obtain the following equation for the frequency:

$$-I_1GI_pp\cos pl+I_1I_2\sin pl-G^2I_p^2p^2\sin pl+$$
$$+GI_pI_2p\cos pl=0$$

or

$$G^2I_pp^2\sin pl+(I_1-I_2)GI_pp\cos pl-I_1I_2\sin pl=0.$$

(36.4)

Introducing the additional notations

$$\alpha = \frac{I_1 - I_2}{GI_p},$$

$$\beta = \frac{I_1 I_2}{G^2 I_p^2}$$

and dividing equation (36.4) by $\sin pl$, we obtain the final frequency equation in the form

$$p^2 + \alpha p \operatorname{ctg} pl - \beta = 0. \qquad (36.5)$$

The solution of this transcendental equation, for the natural frequency p may be obtained by a graphical method. After the determination of the natural frequency of vibration from equation (36.5), and after the solution of the system (36.3), it is possible to find the values of the constants of integration A and B, and hence the function of the angle of twist in the zeroth approximation. From the determined function of the angle of twist and the natural frequency of vibration, we can calculate the frequency of vibration of the system and the magnitude of the phase shift in the first and the successive approximations by adhering to the accepted scheme of calculations.

In conclusion we should point out that in the present investigation only the frequencies of the first order were examined.

We should also note that the derived formulas in the first approximation of the investigated class of nonlinear vibrations, where nonlinearity is determined by the hysteretic losses in the material, allow us to obtain the solution with high degree of accuracy, quite acceptable for practical purposes. Therefore, in the present investigation, we have restricted ourselves to the examination of the first approximation only, although to obtain the formulas of the second approximation does not pose any difficulties in principle; it is only necessary to further develop these investigations based on equation (8.18).

The results cited in the present work refer only to the influence of the hystersis type of dissipation on the vibrations, without consideration of external losses. However, as shown in the investigations of Guye, Rowett, and Geiger*, the magnitude of energy dissipated in the material during torsional vibration, for instance of crankshafts of internal combustion engines, very often comprised not less than 2/3 of the total loss of energy. In the light of this fact the proposed method of analyzing torsional vibrations, with an allowance for the dissipation of energy in the material, may be of some theoretical interest and has practical significance.

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*For references, see Bibliography of the Material Damping Field, WADC 56-180 (Trans.)

Chapter IX

Experimental Methods of Determining the Dissipation of Energy in the Material During Forced Vibrations

37. The object of the experimental investigations

The theoretical investigations set forth above were based on hypotheses according to which the dissipation of energy in the material during vibrations depends on the stress amplitude. The amount of energy dissipated per unit volume per cycle of vibration is given by the area of the hysteresis loop.

Taking expressions for the true modulus of elasticity at any instant for the ascending and descending motions according to formulas proposed by N. N. Davidenkov and then integrating these expressions, we have obtained formulas relating the stress σ and strain ξ . These relations could have been replaced by others; the particular form is of no real significance, as has been repeatedly pointed out by us and by other authors too. It is well known that for the same stress amplitudes hysteresis loops of different materials will differ from each other both in area and in shape. Hence, if we try to express analytically the equations of curves which form the hysteresis loop, it is necessary to introduce several constants which we have to determine experimentally into the functional relation $\sigma = f(\xi)$ In the relation we have cited (6.2), these constants are ν and n , which subequently entered all the formulas. The above-mentioned parameters for different materials may be obtained directly from the hysteresis loop, determined on the basis of appropriate experiments. However, the suggested method entails considerable difficulties connected with the high precision of measurements. Besides, the investigations of hysteresis loops, which have been performed up to the present time, dealt only with static tests of the materials.* It is doubtful whether parameters found on the basis of static experiments can be utilized in the dynamic calculations of vibrations of elastic systems. It is apparent that it is more correct to determine these parameters from experiments conducted under dynamic conditions, corresponding to the working conditions of the elastic vibrational systems in question.

Since we do not have any reliable experimental data at our disposal on the investigation of the hysteresis loop even for static experiments — to say nothing of data in dynamic experiments — we consider it very important to give our attention to experimental determination of the parameters of the hysteresis loop and to the study of other factors which characterize energy dissipation in material in vibrations of rods.

A description of experimental arrangements and apparatuses developed by the author is presented in this chapter, together with an account of the method of experimental investigations of energy dissipation in the material, and several results that were obtained from these investigations.

We shall examine several methods of determining the logarithmic decrement of damping and of the hysteresis loop parameters in steady-state as well as free vibrations. In the last case, we will examine transverse, torsional and longitudinal vibrations.

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*One case is known to us, when, with the help of a special optical device on the Schenk machine used for a fatigue test in twisting, it was possible to obtain an image of the hysteresis loop on a ground-glass screen during the dynamic operation. However, the results of the tests were not published.

38. Set-up for the experimental investigation of energy dissipation in the material in steady-state transverse vibrations of bars

In the experimental investigation of energy dissipation in the material, we have employed a specially designed assembly, for the purpose of investigating energy dissipation in the material during steady-state transverse vibrations of cantilevered samples, excited by forced periodic rotation of the fixed end. The general view of the vibrational assembly[1] is shown in Fig. 24[2], and the schematic drawing in Fig. 25; the apparatus is composed of the following basic parts: 1—sample being tested, 2—case, 3—mirror, 4—vibrator, 5—flexible shaft, 6—packing, 7—small gear, 8—electric motor, 9—large gear, 10—flywheel, 11—main shaft, 12—strain gauge.

The shaft of the vibrational assembly, which is the most important part rests on two bearings placed in the frame, in such a manner that one of its ends with special thickening, in which the sample is clamped, overhangs like a cantilever. In the central part between the bearings, the shaft has two symmetrically placed extensions on each side. Mechanical vibrators are placed in ball bearings at the ends of each pair of these extensions. These vibrators are eccentrically placed masses, fastened to the axes which are rotated by means of a gear by a d.c. electric motor. The unbalanced vibrator masses are placed at an angle of 180°, relative to each other, and, due to this, the inertia forces of these masses which arise during synchronous action of the vibrators, provide an alternating couple, which generates the periodic turning of the shaft of the vibrational assembly, first in one direction, then in the other; i.e. they transmit torsional vibrations to the shaft. The reactions of the unbalanced

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[1]Engineer F. S. Semko took an active part in the design of the apparatus.

[2]Refer to rear of book for Fig. 24.

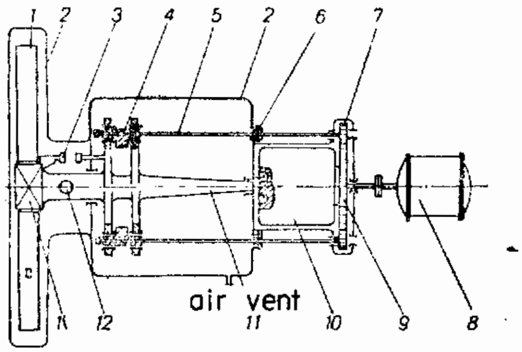

Fig. 25

masses of both vibrators balance each other, and are not transmitted to the bearings of the main shaft of the apparatus. During periodic forced turning of the shaft through a small angle (1 — 2°) back and forth, each of the two cantilevered samples, clamped at the grip which is at the end of the cantilevered part of the main shaft, will execute forced transverse vibrations. Placing the grip for the sample beyond the bearing frame, on the end of the cantilevered portion of the shaft allows the cantilevered part of the shaft between the bearing and the clamp to be utilized as a dynomometer. In this way, we also eliminate the effect on the twisting action of the latter by the friction forces of the bearings and of the other connections between the bearings. If we neglect the resistance of the air, then during the steady-state motion of the vibrational assembly, the angle of twist of the dynamometer is determined by the work of the inertia forces of the sample and clamp, which is associated with reversible processes, and also by the work required for the dissipation of energy in the material, which goes toward the irreversible processes and which must be continually supplied from outside.

The bars used as samples for the investigation of damping in the material with the help of the above described vibrational assembly were chosen of prismatic form with transverse cross-section 15 x 30 mm and 400 mm length.

The vibrational assembly provides for simultaneous investigation of two samples. In order to assure rigid clamping, both samples are prepared from the same piece of material (Fig. 26)*. Such a pair of samples connected by thickened ends, or one sample with two working sections, is attached with the help of special conical wedges and bolts to the clamping head of the shaft of the vibrational assembly. Special measures were taken to guarantee a sufficiently rigid clamping of the samples, so that during vibrations, additional losses would not take place on account of friction in the connection of the sample to the clamping arrangement of the shaft. These measures consisted of making the flexural stiffness of the section of the specimen at the clamp 13 times as large as that of the sample itself.

The motion of the vibrators is accomplished by a d.c. electric motor, which, with the help of a rheostat, permits variation of the number of revolutions per minute within wide limits, and hence also of the frequency of forced vibrations of the shaft. The rotating speed of the vibrator, because of the gear with a gear ratio of 50:14 can attain 10,000 revolutions/min. (166 cycles/sec).

The exciting alternating moment which generates the periodic turning of the vibro-assembly shaft may be regulated both by the variation of number of revolutions and by the variation of the amount of unbalance of the masses of the vibrator. The latter is accomplished by turning the eccentric masses through some angle.

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*Refer to rear of book for Fig. 26.

The free rotation of the shaft of the assembly either to one side or the other through an angle up to 3° is achieved, because the small vibrator shafts running from the gear, and having lengths of 400 mm and made of flexible steel wire 2.5 mm in diameter, do not disturb the above mentioned vibrations of the shaft of the assembly. The stability of rotational vibrations of the vibro-assembly shaft is ensured by supporting the shaft at the points where the bearings of the vibrator are placed by two spiral springs which are set against the upper lid of the welded body of the vibrational assembly. A uniform and steady running of the assembly is achieved because the electric motor is equipped with a massive flywheel.

39. Method of experimental determination of energy dissipation in the material, based on direct measurement of power of the electric motor of the vibrational assembly

The energy dissipation in the material during vibrations is characterized by the loop of elastic hysteresis formed as a result of change in direction of loading of the material. The equations of the loop contain the parameters v and n.

The magnitude of the power input, which goes into energy dissipation in the material, allows us to determine the geometric parameters v and n, of the hysteresis loop, and also the logarithmic decrement of damping or the relative energy dissipation in the material during vibrations.

The power that is consumed by the internal losses in the material comprises only a part of the easily measured total power. The rest of the power is consumed by the external losses, which accompany the working of the vibrational assembly. In order to separate from the total power of the motor that part which is consumed by energy dissipation in the material, we proceed as follows. We measure the power of the electric motor, which sets the vibrators in motion, during the testing of specimens made of "ideally" elastic material, i.e. of material, that practically does not dissipate any energy. In this way, we determine the power that is consumed by the external losses and also that part of it which goes into overcoming the inertia of the whole vibrating system. As such an "ideal" material we have chosen a ball-bearing steel ShKh 15, in which, according to data from literature, and in particular according to the experiments of Föppl-Buseman* (curve 7 on Fig. 27) and Föppl -Pertz* (curve 6 on Fig. 28), which were conducted during the investigation of material damping as a function of shear stresses, the energy dissipation is so insignificant that we can neglect it for practical purposes. The curves $\delta = f_1(\tau_0)$ of Fig. 27 are given for the following materials:

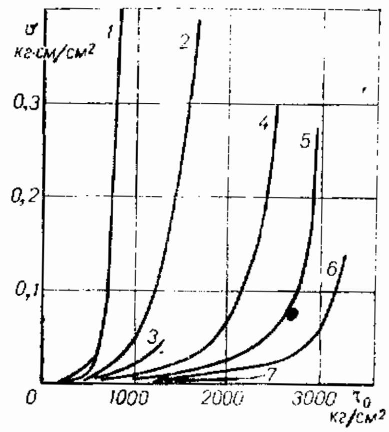

Fig. 27

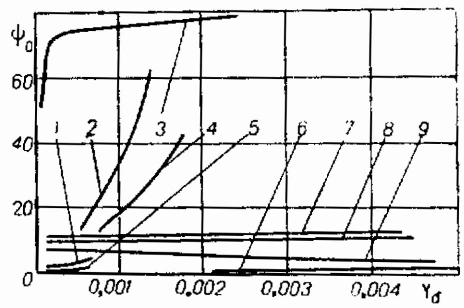

Fig. 28

1 — copper, 2 — soft steel, 3 — aluminum bronze, 4 — tempered steel, 5 — high strength structural steel, 6 — steel for crankshafts, 7 — ball-bearing steel. The curves in Fig. 28 $\psi_0 = f_2(\gamma_0)$ are given for the following materials: 1 — glass, 2 — copper, 3 — trolit, 4 — steel, 5 — porcelain, 6 — ball-bearing steel, 7 — pine wood, 8 — beech wood, 9 — trolon. Then we choose a steel specimen whose damping we wish to investigate. The steel taken for

*See WADC Technical Report 56-180 (Trans.)

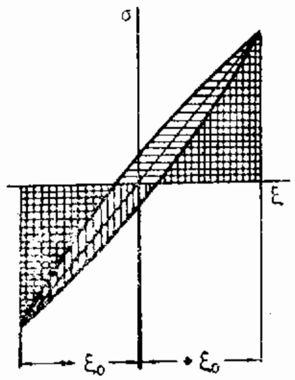

Fig. 29

this purpose is St 20, i.e. soft steel, in which, according to the same data, the energy dissipation is immeasurably greater than in the ball-bearing steel.

If we let W_i stand for the magnitude of power of the electric motor at the motor terminals during the investigation of the samples made from ideally elastic material, and if W the corresponding power supplied to the motor during the testing of the samples made of steel which possess damping, then the energy consumed by the dissipation in the material, is equal to

$$W_s = W - W_i. \tag{39.1}$$

On the other hand, we measure the elongation of the outer fibers of the specimen. The magnitude of the energy dissipated per unit volume, as was pointed out, is determined by the area of the hysteresis loop, and on the basis of (6.2), the energy dissipation can be determined as a function of the maximum amplitude of strain in the following manner (Fig. 29).

$$A = \int_{-\xi_0}^{+\xi_0} \vec{\sigma}\, d\xi - \int_{-\xi}^{+\xi_0} \overleftarrow{\sigma}\, d\xi = \frac{2^{n+1}(n-1)\nu E \xi_0^{n+1}}{n(n+1)}. \tag{39.2}$$

The quantity ξ represents the amplitude of strain of any fiber in the cross-section of the bar. Now by measuring the strain along the length of the bar, we obtain the maximum value of these amplitudes for a given cross-section, $\xi_{o\,max}$. Hence if we take a linear distribution of strain with height through the cross-section of a rectangular bar, then the expression for the strain of any fiber with height in this cross-section, in terms of the measured deformation on the surface (Fig. 5) will be

$$\xi_0 = \frac{2\,\xi_{o\,max}\, z}{h}. \tag{39.3}$$

By representing, on the basis of experiments, the curve of the distribution of strain of the outer fibers along the length of the specimen (Fig. 30) as a function of the coordinate

$$\xi_{o_{max}} = \bar{\xi}_o(x) \;, \tag{39.4}$$

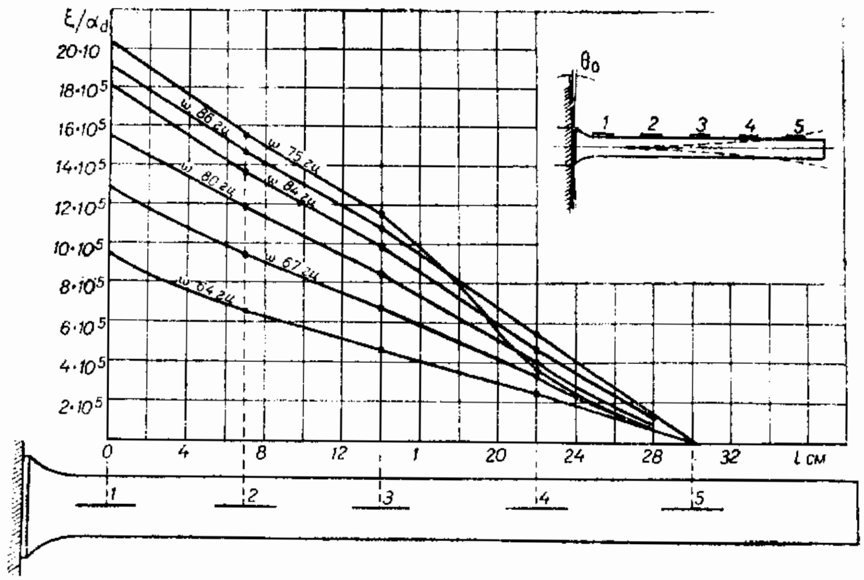

Fig. 30

we can obtain an expression for the amount of energy for a prismatic bar of width b and height h by using (39.1) — (39.4), and taking into account here the two specimens which are investigated at the same time:

$$W_s = 4\bar{\omega}\int_0^l \int_0^{\frac{h}{2}} \frac{2^{n+1}(n-1)\nu E}{n(n+1)} \left[\frac{2\bar{\xi}_0(x)}{h} z\right]^{n+1} b\, dz\, dx, \tag{39.5}$$

where $\bar{\omega} = \frac{\pi \bar{n}}{30}$ is the excitation frequency.

Equation (39.5) may be transformed into the following form:

$$W_s = \bar{\omega}\int_0^l \frac{2^{n+2}(n-1)\nu Ebh}{n(n+1)(n+2)} [\bar{\xi}(x)]^{n+1} dx. \tag{39.6}$$

Equation (39.6) contains the unknown quantities v and n, which are sought.

We obtain the second equation which is needed, if we measure the motor power and plot the curve of strain of the sample $\xi_0(x)$ for a different angular speed of the electric motor.

40. Experimental method of determining energy dissipation in the material, based on the measurement of the angle of twist of the shaft of the apparatus

In order to check the method of experimental investigations set forth in the previous paragraph, and also to verify the results obtained, we propose another method, based on the measurement of the torsional deformation of the shaft of the apparatus. In this instance also we perform comparative tests of samples of the steel under investigation, and also of a steel in which the damping can be neglected.

For the case of a uniformly rotating shaft, the power transmitted by the shaft is equal to

$$W = M_{\kappa p}\overline{\omega}, \tag{40.1}$$

where $M_{\kappa p}$ is the twisting moment,

ω is the angular speed of the shaft.

In the present case of alternating rotation, the twisting moment transmitted by the shaft at any instant is determined by the formula

$$M_{\kappa p} = M^0_{\kappa p} \cos \omega t, \tag{40.2}$$

and the corresponding angle of twist of the shaft is equal to

$$\varphi = \varphi_0 \cos \omega t. \tag{40.3}$$

where $M^0_{\kappa p}$ and φ_0 are the amplitudes of the torque and of the angle of twist of the shaft.

The increment of potential energy, stored in the shaft between the cross-section adjacent to the specimen clamp and the cross-section at the point of action of the mechanical vibrators (i.e. in the whole length of the dynamometer), may be determined by analogy to equation (40.1)

$$dW = M_{\kappa p}\frac{d\varphi}{dt}dt.$$

The potential energy stored by the shaft during a quarter period is equal to

$$W_{\frac{T}{4}} = \int_0^{\frac{\pi}{2\omega}} M_{\kappa p}\frac{d\varphi}{dt}dt = -M_{\kappa p}\varphi_0\int_0^{\frac{\pi}{2\omega}}\sin\omega t\cos\omega t\,dt = -\frac{M^0_{\kappa p}\varphi_0}{2}$$

and for an entire period

$$W_T = 2M^0_{\kappa p}\varphi_0. \tag{40.4}$$

However, it is necessary to keep in mind that the energy stored by the shaft during the first quarter of the cycle is released during the subsequent quarter of the cycle.

Due to this reversibility, the work done for one cycle of vibrations reduces to zero. But, as is well known, the energy consumed in one cycle by the electric motor, which generates these vibrations of the shaft is determined by formula (40.4).

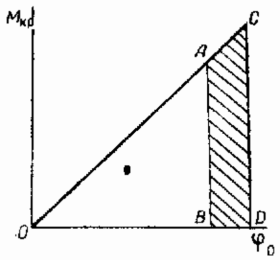

Fig. 31

Let us suppose that tests are performed on an ideally elastic sample which does not dissipate energy in the material. Let the twisting of the shaft (amplitude of the angle) measured in the dynamic operation, be equal to φ_0, and let the torque determined by means of static calibrations be equal to $M^0_{\kappa p}$. We plot in the coordinates $M^{\circ}_{\kappa p} - \varphi_0$ the diagram shown in Fig. 31. On this diagram the area of the triangle OAB expresses the magnitude of the potential energy stored by the shaft during a quarter period of the vibration of the specimen. It is evident then, that the resistance brought about by the inertia of the specimen, the clamping device, and also other losses, will necessitate the application of this torsional moment by the vibrators, which will twist the shaft through an angle φ_0 .

Now let us suppose that we are testing a sample possessing internal damping. We measure the magnitude of the amplitude of the angle of twist $\overline{\varphi}_0$ and corresponding to this angle we measure the torsional moment $\overline{M}_{\kappa p}$. Then plotting the obtained results on the same diagram (Fig. 31) we obtain the triangle OCD, whose area differs from the area of the first triangle by the size of the loop ACDB (shaded part of the figure).

Since the dimensions and shapes of the samples in both instances are exactly the same, and since the elastic properties of steel ShKh 15 and St 20 are also the same, the source of the additional resistance of the vibrating specimens (resulting in an increased twisting moment for the tests of St 20 specimens) is the energy dissipation in the material.

The magnitude of the dissipation of energy of the two samples for one cycle, is determined by four times the area of the trapezoid ACDB in Fig. 31, i.e.

$$W_p = 4\left(\frac{M^{\circ}_{\kappa p}\varphi_0}{2} - \frac{M_{\kappa p}\varphi_0}{2}\right) = 2\,(M^{\circ}_{\kappa p}\varphi_0 - M_{\kappa p}\varphi_0). \qquad (40.5)$$

Thus the method of determining the magnitude of energy dissipation in the material of a sample, made of a certain steel, is the following. The angle of twist of the shaft $\overline{\varphi}$ is measured in a test of specimens of the steel under study. Next, for the same vibration frequency, the angle of twist of identical samples made from an "ideal" steel free from internal friction is measured. Then, on the basis of data for static calibration data of the shaft, from the measured angles of twist, we can determine the corresponding magnitudes of the torsion moments $\overline{M}^{0}_{\kappa p}$ and $M^{0}_{\kappa p}$. Finally, by formula (40.5) we calculate the magnitude of the dissipation of energy in the material of the two samples of the steel under investigation for one cycle of vibrations. Multiplying W_p by the frequency of vibrations, we can obtain the magnitude of energy dissipation in the material of the two samples in one second, i.e.

$$W_s = \omega W_p = 2\omega(\overline{M}^{0}_{\kappa p}\overline{\varphi}_0 - M^{0}_{\kappa p}\varphi_0). \tag{40.6}$$

Having obtained the magnitude of the power W_s, we perform further operations to determine the coefficients ν and n, according to a scheme analogous to the one described in the previous paragraph. For this purpose, it is necessary to use equation (39.6), since the single equation (40.6) is not sufficient to determine ν and n; it is necessary, as it was pointed out above, to have two different values of W_s and correspondingly — two equations for the strain curves $\xi(\tau)$, obtained from the experiments under different loading, and in our case — for different forced vibration frequencies of the bar under study. To obtain reliable results it is convenient to have several such systems of equations, whose solutions, with respect to ν and n, would enable us to take the averaged values of the parameter.

41. Experimental method of determining the angle of twist of the shaft of the vibro-assembly

In determining experimentally the amplitude of the angle of twist, we apply the following two methods. The first, optical, method is the following: the twisting of the shaft is measured using mirrors (Fig. 32), mounted at the end cross-sections of the part of the shaft which is the dynamometer (one mirror at the vibrators, the other at the specimen clamp), and special light sources. The light sources used for this purpose are "Filmosto" projection apparatuses. The ray of light from the source is directed at the mirror which is fixed in such a way, that after reflection from the mirror, the ray falls on a screen, giving a bright "point", or more correctly a bright spot. During the experiments, these bright spots from the two mirrors mounted on the dynamometer part of the shaft at a certain distance from each other, due to the turning of the shaft (dynamometer), and hence, also of the mirrors attached on it, will give a bright strip on the screen representing the double amplitude of the vibrations of the reflected ray. If the distance from the mirrors to the screen is considerable (in our experiments it was about 5.2 meters), and the angle of rotation of the section of the shaft together with the mirrors that are connected with it, is small, then we can determine the angle from the formula

$$\varphi \approx \mathrm{tg}\,\varphi \approx \frac{a}{L}, \qquad (41.1)$$

where a is the half-amplitude of the vibration measured on the screen, L is the base.

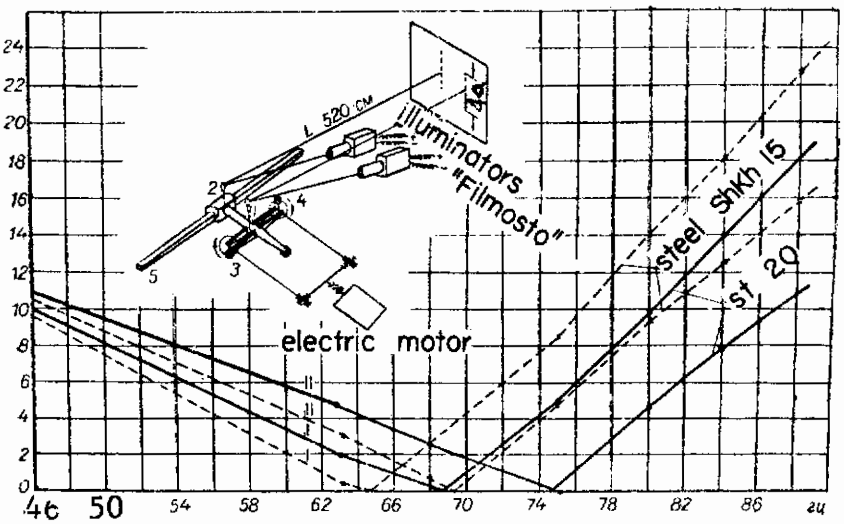

Fig. 32

The relative angle of turning of sections 1 and 2 is equal to the angle of twist of the part of the shaft 1—2.

$$\varphi_0 = \varphi_1 - \varphi_2. \tag{41.2}$$

For a large base this method is the simplest and the most reliable. Therefore, we apply it not only to dynamic measurements, but also to the measurements of the angle of twist in static calibration.

The second method of determining the angle of twist of the shaft consists of measuring the strains of the outer fibers of the shaft at an angle of 45° to the shaft using resistance strain gauges. The readings of the strain gauges are recorded by a bifilar oscillograph in the form of oscillograms which characterize the variation of strain of the shaft with time. These measurements are performed in tests of samples made of different steels (St 20 and ShKh 15) at different frequencies of vibration. Once we have the strain oscillograms of the shaft in twisting recorded during the dynamic operation, and also the data of a calibration of the strain gauges from the measured results of the angle of twist of the shaft by the optical method it is possible to determine the magnitude of this angle for different vibration frequencies. The application of this method, in conjunction with the optical method, is convenient inasmuch as it permits us to follow the process of the variation of deformation of the shaft with time.

42. The determination of the parameters ν and n of the hysteresis loop based on the phase shift between the load and the strains of the bar

In spite of the accuracy and the simplicity of the method of determining the parameters ν and n based on the measurement of energy dissipation, this method cannot be applied in the investigation of energy dissipation in all materials. For, in order to measure directly the vibration energy dissipated in the material of the system, in addition to the prepared sample of the investigated material, it is also necessary to test an exactly similar sample prepared from analogous material (similar in elastic properties and specific gravity), but having a negligible amount of damping.

Therefore, the above method may be applied to the investigation of energy dissipation in those materials for which, by suitable choice of chemical composition and heat treatment, we can obtain specimens possessing practically the same specific gravity and elastic properties, but which have great difference in damping capacity in vibration. We can obtain such "ideal" steels, for instance: ball-bearing steel, spring steel, etc., which dissipate in the material so little energy, that for practical purposes we can neglect it.

We now describe another method. Since the magnitude of the phase shift between the stresses and the strains is a rather sensitive measure of energy dissipation in the material, it is natural then to utilize the measurement of this phase shift for the experimental determination of the damping characteristics of the material.

Our vibro-assembly allows us to obtain the experimental data necessary for the measurement of the magnitude of the phase shift without difficulty. This is most conveniently accomplished by means of a direct phase shift on the oscillogram. With the help of the oscillograph, we can obtain such oscillograms by simultaneously recording on sensitized film the torsional strain of the shaft which corresponds to the external loading, and the flexural strain of the sample under investigation. In this case it is expedient to measure the strain in the specimen at the places where it attains its maximum value, i.e. at those points, where according to our assumptions, we will have the maximum energy dissipation in the material.

Thus, we measure on the oscillogram the magnitude of the phase shift between the normal strain of the shaft during twist, measured at an angle of 45° to its axis, and the normal strain of the bar due to bending at frequencies close to resonance, Under such conditions there is a noticeable displacement of phases between the above strains. The required parameters ν and n which characterize the energy dissipation in the material, are then determined by the formulas

$$\sin\psi = \frac{-12(n-1)a^n h^{n-1}\nu}{n(n+1)(n+2)\theta_0 k^2 \pi}\int_0^l \left(\frac{d^2\varphi}{dx^2}\right)^{n+1} dx; \tag{42.1}$$

$$\left(\frac{\omega}{\omega_c}\right)^2 = 1 - \frac{4\theta_0}{alk^2}\left[\frac{(n+1)\sin\psi}{2^n(n-1)}\int_0^{\pi}(1-\cos\tau)^n\cos\tau\, d\tau + \cos\psi\right] \tag{42.2}$$

In order to determine ν and n by formulas (42.1) and (42.2) it is necessary to have the measured amplitude of the angle of rotation of the fixed cross-section of the specimen θ_0 . This can be done by the optical method described above. Moreover, it is also necessary to measure the amplitude of vibration at the end of the sample. It is most convenient to measure this amplitude on an oscillogram, recorded on the film which is moved by the clock mechanism of the Geiger oscillograph by a special writing device fixed to the vibrating sample.

43. Measurement of the strains of the vibrating specimen

The strain of outer fibers along the bar were measured with the aid of wire resistance gauges. The principle of operation of these strain gauges, as is well known, is based on the variation of the ohmic resistance of a metallic wire or metallic tape due to strain.

Denoting the strain by $\xi = \frac{\Delta l}{l}$ and the corresponding change of ohmic resistance $r = \frac{\Delta R}{R}$, we obtain the coefficient of proportionality $\eta = \frac{r}{\xi}$, also called the sensitivity of the strain gauge, which is determined by the conductivity of the given material. In our experiments we have utilized gauges made of constantan wires, the working length of which were 20-30 mm. According to the experimental data of the Structural Mechanics Institute of the Academy of Science of the Ukr. SSR, for constantan the coefficient $\eta = 1.8 \div 2.2$.

The wire resistance gauge, which is a constantan wire 0.02-0.05 mm in diameter, is pasted on paper in the form of loops. Two leads made of copper wire 0.1-0.2 mm in diameter, which connect the resistance strain gauge to the measuring circuit, are soldered to the ends of the constantan wire. The protective paper is glued on top. Carbinol, bakelite or some other glue is used to glue the strain gauge to the surface of the specimen, the strain of which is to be measured. During the loading of the sample, a strain gauge attached in this manner will experience strains (tensile or compressive), which are identical to the strains of the outer fibers of the deformable surface of the specimen. Using the fact that the gauge responds to strain by changing its ohmic resistance, we can record these changes electrically in the form of an oscillogram. The scale of these oscillograms may be established by suitable calibration. Owing to their negligible weight, these wire strain gauge resistors have no inertia; therefore, their utilization in the measurement of the deformation of samples in the dynamic regime has been found to be convenient.

Certain difficulties are connected with the calibration of resistance strain gauges. Usually when we employ wire strain gauges for dynamic measurements we use the data of a static calibration. However, as our investigations have shown, the characteristics of resistance strain gauges obtained on the basis of static calibration do not coincide with the data obtained in dynamic operation. Therefore, in our measurements we did not restrict ourselves only to static calibration of the strain gauges but we introduced corrections, characterized by a certain dynamic coefficient, which was determined on the basis of the measurement of amplitude of vibration of the bar by the method described above.

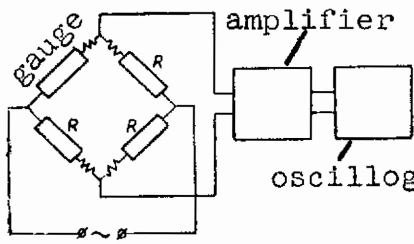

Fig. 33

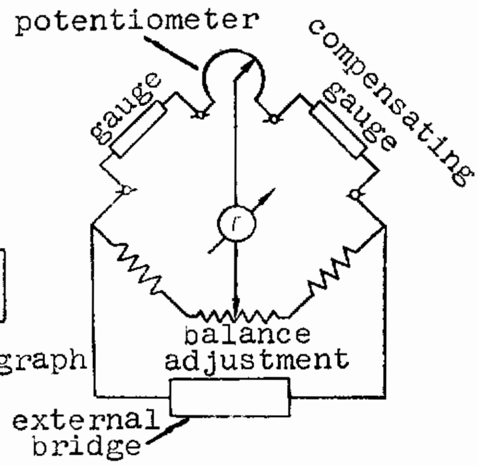

Fig. 34

During static calibration of the resistance strain gauges, the latter were connected to an a.c. electric circuit, (Fig. 33).

In order to measure the variable component of stress for the frequencies of 50-100 cycles per sec. the resistance strain gauges were connected in the electric network shown in Fig. 34. In this case a d.c. current source was used and the gauges were connected to the amplifier in parallel with a condenser. The oscillograms of strain at the outer fibers at different points along the length of the sample were recorded with the help of a bifilar oscillograph.

We can determine the magnification factor of the resistance strain gauge on the basis of the relation:

$$\sigma = \frac{E\Delta}{l\varrho}, \tag{43.1}$$

where σ is the stress
E is the modulus of elasticity of the material
l is the length of gauge
ϱ is the magnification factor of the gauge
Δ is the number of subdivisions on the dial.

Let M denote the bending moment acting at some particular cross-section, and let W denote the section modulus of the bar. Then the maximum stress in bending at a given cross-section is

$$\sigma = \frac{M}{W}. \tag{43.2}$$

From (43.1) and (43.2) the magnification factor of the gauge is

$$\varrho = \frac{EW\Delta_{cp}}{Ml}. \tag{43.3}$$

Utilizing formula (43.3) and average values of the readings on the oscillograph scale (Δ_{cp}) obtained on the basis of data

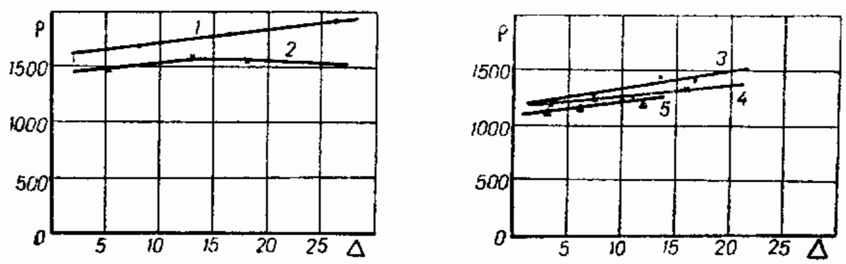

Fig. 35

of static experiments, we determined the magnification factor ϱ of the strain gauges in terms of Δ_{cp}. The graph of magnification factors as a function of deformation is shown in Fig. 35.

44. The characteristics of the materials and samples utilized in the experiments

Before giving an account of the results of experimental data we obtained in the investigation of energy dissipation in the material of bars with constant cross-section for transverse vibrations, we shall first describe the characteristics of the material and the shapes of the specimens. The samples which were tested were made of St 20, whose chemical composition is given in Table 15 and whose basic mechanical characteristics are given in Table 16.

Table 15

C	Mn	Si	Ni	Cr	P	S	S+P
0,15–0,25	0,35–0,65	0,17–0,37	0,30	0,30	<0,045	<0,045	<0,08

Table 16

Unit elongation %	Coefficient of Plasticity %	Nominal Strength kg/mm^2	Yield Limit kg/mm^2	Actual failure stress kg/mm^2	Fictitious failure strength kg/mm^2
29.0	64.0	43.5	25.3	70.3	25.3

The Brinell hardness number of St 20 under a 3000 kgs load, and employing a hardened steel ball bearing 10 mm in diameter is H_B = 121. The fatigue limit is σ_{-1} = 19.7 kg/mm^2.

The sample with 1.5 x 3 cm cross-section and length 40 cm is shown in Fig. 26.

45. Results of strain measurement for the sample in dynamic tests

By the method described in #43 we obtained oscillograms of the strain of outer fibers of the sample. The oscillograms of each strain gauge resistor were recorded on separate strips of film. Sections of separate oscillograms are shown in the photograph in Fig. 36.* The oscillograms show the variation of the strains of the outer fibers of the sample as functions of the loading, which in our case was determined by the frequency of the vibrations. At resonance of the elastic system, which occurred at a frequency of 75 cycles/sec, the strains of the bar sharply increased, which can be clearly seen on the oscillograms.

Utilizing the oscillograms obtained, with the help of a tool-making microscope (magnification = 30) we obtained the values of double amplitudes of the strain of external fibers of the sample on the oscillograms. From the measured double amplitudes on the oscillogram which we obtained for different vibration frequencies of the sample and the gauge factors obtained on the basis of static calibration and represented graphically in Fig. 35, we determined the magnitudes of the average strains of the outer fibers of the specimen at the gauge locations. The results of the experiments are shown in Table 17.

Table 17

No. of Gauge	Base length of gauge mm	Frequency of Vibrations cycles/sec	64	67	75	80	84	88
1	31	ϱ	1640	1650	1700	1660	1670	1690
		$2\varrho l\xi$	9,6	13,2	21,4	15,5	18,8	19,9
		$\xi \cdot 10^5$	9,44	12,95	20,30	15,05	18,15	19,00
2	30	ϱ	1420	1420	1420	1420	1420	1420
		$2\varrho l\xi$	5,6	8,0	13,2	10,1	11,2	12,85
		$\xi \cdot 10^5$	6,56	9,40	14,95	11,85	12,7	14,55
3	32	ϱ	950	960	980	970	980	980
		$2\varrho l\xi$	2,8	4,2	7,3	5,3	6,2	6,8
		$\xi \cdot 10^5$	4,60	6,85	11,60	8,52	9,90	10,82
4	31	ϱ	1200	1200	1200	1210	1220	1220
		$2\varrho l\xi$	1,84	2,5	2,6	2,9	3,6	4,0
		$\xi \cdot 10^5$	2,48	3,35	3,50	3,85	4,75	5,65
5	31	ϱ	110	1100	1100	1100	1100	1100
		$2\varrho l\xi$	0	0	0	0	0	0
		$\xi \cdot 10^5$	0	0	0	0	0	0

*Refer to rear of book for Fig. 36.

According to the data in Table 17, graphs characterizing the change of the strains along the length of the bar, for different frequencies of forced vibrations of the bar were constructed, Fig. 30. Due to the fact that the strains were obtained on the basis of static calibration, they will not correspond to the actual values of strain occurring in the vibrating bar. In order to obtain the true strain values it is necessary to introduce a correcting dynamic coefficient, which can be obtained on the basis of the results of amplitude of the vibrations at the end of the bar, which was also measured.

46. The results of measurement of energy dissipation in the material

The magnitude of energy dissipation in the material may be obtained by separating it from the total measured energy consumed by the vibrations of the sample during the steady-state. For this purpose it is necessary to subtract the mechanical losses in the bearings of the gears and vibrators, the losses in the connections of the mechanical system, aerodynamic losses, etc. from the electric motor power, which is consumed in the vibrations of the specimens prepared from the steel under study. Besides, it is also necessary to subtract from the total power, the power required for the vibrations of masses of the specimens themselves, and also of the masses of the vibro-assembly parts which take part in the vibrations.

Instead of using a specimen made of steel, St 20, under study we might place in the vibro-assembly a sample which possesses all the properties of St 20, with the exception of the property of dissipating energy in the material. Then, measuring the power of the motor during the testing of such an "ideal" sample at the same frequency we can obtain that power which must be subtracted from the total power measured during the testing of the sample of St 20 in order to obtain the magnitude of the dissipation of energy in the latter.

As has been already pointed out, ball-bearing steel of the type ShKh 15, from which we prepared an exactly similar sample as the sample of the investigated St 20 may be considered to be such an "ideal" steel.

The chemical composition of steel ShKh 15 is given in Table 18.

Table 18

C	Mn	Si	Cr	Ni	S	P
0,95–1,10	0,20–0,40	0,15–0,35	1,30–1,65	0,20	0,020	0,027

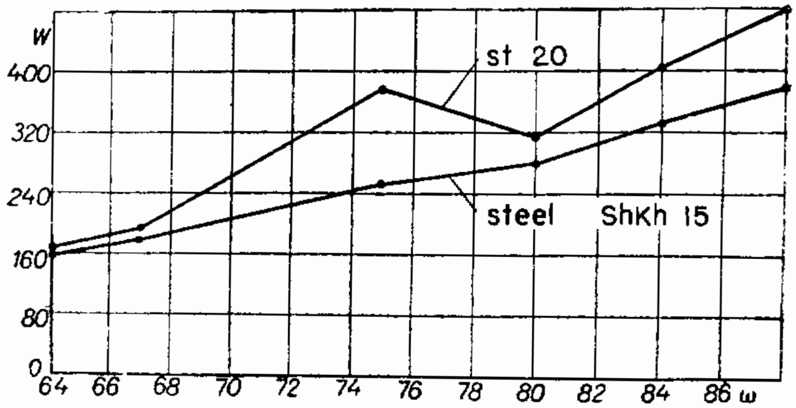

Fig. 37

The specimen prepared from steel ShKh 15, was heat treated (quenched from a temperature of 800°C and then tempered up to 200°C), after which its Rockwell hardness turned out to be 60 units. The amperage and the voltage were measured during the experiment at the terminals of the d.c. motor, which brings about the forced vibrations of the shaft by driving the mechanical vibrators. From the results of these measurements we determined the power consumed by the electric motor.

The magnitude of energy dissipation in the material of the specimens made of St 20, obtained as the power difference during the testing of the sample made of St 20 and of steel ShKh 15, is given in Table 19.

The graphs of $W = f(\omega)$ are shown in Fig. 37.

Table 19

ω	64	67	75	80	84	88
$W_{St\ 20}$	167	193	378	310	406	474
$W_{ShKh\ 15}$	160	181	259	277	338	376
W_s, $(W_{St\ 20} - W_{ShKh\ 15})$	7	12	119	33	68	98
W_s, kgcm/sec	71,4	122	1215	336	694	100
$\frac{W_s}{\omega}$, kgcm/cycle	1,12	1,82	16,20	4,20	8,26	11,35

47. The determination of energy dissipation in the specimens made of St 20 on the basis of the angles of twist of the shaft

On the basis of the method described in #40, we obtained the angles of twist of the vibro-assembly shaft. The scheme for measuring the angles of twist of the shaft with the help of mirrors is shown in Fig. 32. From the results of measurements, we constructed graphs in Fig. 32 of the dependence of the double amplitudes of rotation of the cross-section of the shaft (at the points of mounting of the mirrors) as a function of frequency of vibrations. Knowing the ordinate values of the obtained curves and knowing the base, we were able to determine the relative angles of rotation of the end sections of the shaft.

For the base we chose (distance from the mirror to the screen) L = 520 cm, the angles of twist of the shaft may be determined by the formula

$$\varphi \approx \operatorname{tg} \varphi = \frac{a}{L}, \tag{47.1}$$

where a is the half-amplitude of the motion of the ray on the screen (Fig. 32).

Accoring to the graphs shown in Fig. 32, the appropriate angles of twist of the shaft during the testing of the samples, made of St 20, are given in Table 20, and of steel ShKh 15 — in Table 21.

Table 20

ω	46	64	68	75	80	84	88
a	0,50	1,35	1,10	2,40	2,60	3,05	3,64
$\varphi \cdot 10^3$	0,962	2,69	2,12	4,61	5,00	5,86	7,00

Table 21

ω	46	64	68	75	80	84	88
a	0,045	1,30	1,00	1,85	1,45	2.85	3,40
$\varphi \cdot 10^3$	0,865	2,50	1,92	3,56	4,71	5,48	6,54

Static calibration was performed in order to determine the magnitudes of twisting moments corresponding to the angles of twist. In this connection, the measurement of the twist of the shaft was performed by the same optical method and with the same base as in the case of the dynamic investigations.

On the basis of experimental data, the difference between the maximum half-amplitude of deviation of the rays on the screen, reflected from the mirrors 1 and 2 (Fig. 32), during maximum loading $M_{\kappa p}^{\alpha \tau}$ = 2955 kg cm amounted to Δ_{max} = 4.3 — 2.7 = 1.6 cm. Then, the relative angle of twist of the part of shaft between the cross-section at the points of installation of the mirrors is

$$\varphi_{\text{ст}} \cong \operatorname{tg} \varphi_{\text{ст}} = \frac{1,6}{520} = 0,00308 \text{ rad.}$$

The magnitude of the coefficient of proportionality between the twisting moment and the corresponding angle of twist is equal to

$$k = \frac{M_{\text{кр}}^{\text{ст}}}{\varphi_{\text{ст}}} = \frac{2955}{0,00308} = 9,6 \cdot 10^5 \text{ kg cm/rad}$$

The magnitudes of the twisting moments corresponding to the different angles of twist are determined by the formula

$$M_{\text{кр}} = k\varphi. \tag{47.2}$$

On the basis of (47.2) and the data in Table 20 and 21, the magnitudes of the twisting moments for different vibration frequencies of the samples (Table 22) are calculated for different experiments on the samples.

Table 22

Frequency	ω	46	64	68	78	80	84	88
St 20	$\varphi_0 \cdot 10^3$	0,962	2,60	2,12	4,61	5,00	0,00586	0,00700
	M_0	923	2495	2035	4425	4800	5520	6710
ShKh 15	$\varphi_0 \cdot 10^3$	0 865	2,50	1,92	3,27	4,71	0,00548	0,00654
	M_0	830	2400	1845	3140	4520	5260	6280

On the basis of the reasons set forth in #40, the magnitude of the potential energy, stored by the shaft during the complete vibrational period of the investigated shafts can be determined by the formula

$$W_{\text{T}} = 2M_0\varphi_0, \tag{47.3}$$

where M_0 and φ_0 are corresponding amplitudes of vibration of the twisting moment and the angle of twist of the shaft.

Utilizing formula (47.3) and the magnitudes of the measured angles of twist and their corresponding twisting moments during the testing of the samples made of St 20 and S_hKh 15 (Table 22), we determine the amount of energy which is consumed in the testing of the above mentioned steels during different vibration frequencies of the samples. Then, the difference in energy consumed in the vibrations of the specimens made of St 20 and of ShKh 15 at the same frequencies will give us the magnitude of energy dissipation in the material of the investigated samples made of St 20 at the given frequency. We recall that two samples are tested simultaneously.

The values of power calculated by the above method are given in Table 23; the last line of this table gives the magnitude of the energy dissipation in the material of the two samples made of St 20, found for different frequencies.

Table 23

ω	46	63	68	75	80	84	88
$W_{St\ 20}$, kg cm/sec	1,72	12,90	8,54	40,80	48,00	65,90	94,00
$W_{ShKh\ 15}$, kg cm/sec	1,38	12,00	7,09	24,3	42,60	57,60	82,20
$W = W_{St\ 20} - W_{ShKh\ 15}$,	0,34	0,90	1,55	16,5	5,4	8,3	11,8

To conclude this paragraph we give some data on the measurement of strains with the help of resistance strain gauges. Utilizing the method, described in #39 and #43, we photographed oscillograms characterizing the strains in torsion of the shaft during the testing of the samples made of St 20 and ShKh 15 at different vibration frequencies.

Comparing the data of static calibration of the strain gauge from the oscillogram with the static calibration results of the optical method of measuring the twist of the shaft, it is established that an angle of twist of the shaft

$$\varphi = 0{,}001015$$

corresponds to Δ = 1 mm on the oscillogram, recorded according to the readings of the resistance strain gauge. **In this way, from the measurements of the oscillograms we** found the value of the power transmitted by the shaft at different frequencies of vibrations of the samples made of St 20 and ShKh 15. The data relating to St 20 is given in Table 24, and to steel ShKh 15 in Table 25.

Neglecting the energy dissipation of steel ShKh 15 we found the magnitude of energy dissipation in St 20 by subtracting the corresponding values shown in the last columns of Tables 24 and 25. The results are shown in Table 26.

Table 24

ω, cycles/sec.	64	68	75	80	84
Δ, m.m.	2,11	2,67	4,35	4,15	4,55
ϕ, rad.	0,00212	0,00271	0,00442	0,00421	0,00462
M, kg. cm.	2040	2600	4240	4040	4440
$W=2\phi M$, kg.cm/cycle	8,7	14,1	37,5	34,1	41,1

Table 25

ω, cycles/sec.	64	68	75	80	84
Δ, mm	2,05	2,50	3,27	3,84	4,07
ϕ, rad	0,00208	0,00254	0,00332	0,00390	0,00413
M, kg cm	1995	2440	3160	3740	3960
$W=2\phi M$, kg.cm/cycle	8,30	12,4	21,5	29,1	32,7

Table 26

ω, cycles/sec	64	68	75	80	84
W, kg cm/cycle	0,4	1,7	16,0	5,0	8,4
$N = \omega W$ kg cm/sec	26	115	1200	400	706

48. Determination of the parameters ν and n, characterizing the dissipation of energy in the material

Before we turn to the determination of the hysteresis loop constants ν and n in terms of the magnitude of energy dissipation during transverse vibrations of samples made of St 20, we present a tabular summary of values of energy dissipation found experimentally by the different methods of investigation. The data shown in Table 27 corresponds to a pair of samples made of St 20 , of rectangular cross-sections 1.5 x 3 cm and 40 cm length.

Comparing the magnitudes of energy dissipation during vibrations for one cycle, obtained experimentally for the different methods of measurement, we come to the conclusion that all three methods give quite close values. These values coincide especially well at higher frequencies of vibration, including the critical frequency (75 cycles/sec) i.e. in the instances, which are of greatest practical interest. **The basic characteristics of the vibrational** process should be determined from precisely these data which correspond to such frequencies.

Table 27

Frequency of vibrations cycles/sec	63	64	67	68	75	80	84	88
From direct measurement of the power to the motor	-	1.12	1.82	-	16.20	4.20	8.26	11.35
From the angle of twist of the shaft measured optically	0.90	-	-	1.55	16.5	5.4	8.3	11.8
From the angle of twist of the shaft measured by resistance strain gauges	-	0.4	-	1.70	16.0	8.0	8.4	-

In order to determine the quantities ν and n we shall utilize formula (39.6). Examining the magnitude of the energy dissipated by the two samples during one cycle of vibrations, we rewrite (39.6) in the following form:

$$W_s' = \int_0^l \frac{2^{n+2} E\nu bh(n-1)}{n(n+1)(n+2)} [\bar{\xi}(x)]^{n+1} dx. \tag{48.1}$$

As has been already pointed out above, it is necessary to set up an equation of the type (48.1) for the two values of W_s and the corresponding strains. In this way we obtain a system of two equations.

$$W_1 = \frac{E\nu bh(n-1)2^{n+2}}{n(n+1)(n+2)} \int_0^l [\bar{\xi}_1(x)]^{n+1} dx,$$

$$W_2 = \frac{E\nu bh(n-1)2^{n+2}}{n(n+1)(n+2)} \int_0^l [\bar{\xi}_2(x)]^{n+1} dx. \tag{48.2}$$

Eliminating v , we find

$$\frac{\int_0^l [\bar{\xi}_1(x)]^{n+1}\,dx}{\int_0^l [\bar{\xi}_2(x)]^{n+1}\,dx} = \frac{W_1}{W_2}. \qquad (48.3)$$

Let us determine the coefficients $\varkappa$ and v for the case of forced vibrations at frequencies ω_{I} = 80 cycles/sec and ω_{II} = 88 cycles/sec. The magnitudes of energy dissipation at these frequencies, found on the basis of measurement of the power input to the electric motor are shown in Table 27 (first line) and are equal to

$$W_1 = 4{,}2 \qquad\qquad W_2 = 11{,}35$$

Substituting the values of W_1 and W_2 into formula (48.3), we obtain

$$\frac{\int_0^l [\bar{\xi}_1(x)]^{n+1}\,dx}{\int_0^l [\bar{\xi}_2(x)]^{n+1}\,dx} = \frac{11{,}35}{4{,}2} = 2{,}7. \qquad (48.4)$$

The functions $\xi_1(x)$ and $\xi_2(x)$ characterize the variation of the strain along the specimen during the vibration. The equations of these experimental curves (Fig. 30) may be expressed approximately in the following form:

$$\bar{\xi}_1(x) = 10^{-5}(20-0{,}613\,x)\,\alpha_{II},$$
$$\bar{\xi}_2(x) = 10^{-5}(16{,}5-0{,}523\,x)\,\alpha\ . \qquad (48.5)$$

where α_{II} is a certain coefficient.

Substituting the last expression in formula (48.4) and taking the limits of integration in accordance with the actual length of the stressed portion of the sample l = 30 cm, we obtain

$$\frac{\int_0^{30} (20-0{,}613x)^{n+1}\,dx}{\int_0^{30} (16{,}5-0{,}523x)^{n+1}\,dx} = 2{,}7. \qquad (48.6)$$

Solving equation (48.6), we find the value of the constant

$$n \approx 3.$$

If we introduce into the calculation two experimental curves from the family, shown in Fig. 30, then the value of the parameter n turns out to be close to 3. Taking n = 3 and utilizing equation (48.1) we find

$$\nu = \frac{n(n+1)(n+2)W_s}{2^{n+2}(n-1)bhE\int_0^l [\bar{\xi}(x)]^{n+1}dx} \tag{48.7}$$

In determining the value of ν by formula (48.7) the value of the energy W_s and the equations of the curve of strain are taken for a frequency of vibration of the specimen of ω = 88 cycles/sec.

In selecting the equation for the curve of strain we shall proceed from the experimental curves of Fig. 30, introducing here a dynamic coefficient. In determining the latter we shall utilize formula (9.9) which contains the vibration amplitude a, determined in the experiment.

On the basis of the equation (9.9) obtained in Chapter 2 for the deflection curve of the axis of the vibrating bar and the relationships (10.15), the equation of the curve of strain of the outer fibers $\xi_0(x)$ may be written in the following form:

$$\xi_0(x) = \frac{ahk^2}{4\sin kl\,\mathrm{sh}\,kl}[(\cos kl + \mathrm{ch}\,kl)(\mathrm{ch}\,kx + \cos kx) + (\sin kl - \mathrm{sh}\,kl)(\mathrm{sh}\,kx + \sin kx)]. \tag{48.8}$$

With the length of the bar* l = 40.5 cm according to the data in #9.

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*The actual length of the samples was equal to 40.5 cm. The part of the sample which experienced a change in longitudinal strain of the fibers had a length of l = 30 cm.

$$k = \frac{1{,}8751}{40{,}5} = 0{,}0462988.$$

We rewrite equation (48.8) in the form

$$\xi_0(x) = 0{,}0003528\,[\,2{,}23(\operatorname{sh} kx - \sin kx) + + 3{,}0375(\operatorname{ch} kx + \cos kx)\,\frac{ah}{2}. \qquad (48.9)$$

Comparing the value of the maximum strain of the fibers at the root of the sample according to formula (48.8) for the vibration amplitude at the end of the bar a = 0.8 cm, corresponding to the vibration frequency of 88 cycles, with the value of strain, found experimentally (second curve from the top in Fig. 30), we find

$$\frac{\xi_0}{a} = 1{,}62 \cdot 10^{-3}, \quad \frac{\bar{\xi}}{a_{\text{Д}}} = 2 \cdot 10^{-4}, \quad \xi_0 = 0{,}8 \cdot 1{,}62 \cdot 10^{-3}, \quad \bar{\xi} = a_{\text{Д}} \cdot 2 \cdot 10^{-4},$$

but since $\xi = \bar{\xi}$

$$a_{\text{Д}} = \frac{8 \cdot 1{,}62}{2} = 6{,}48.$$

Then according to (48.5) the equation of the curve $\bar{\xi}(x)$ which must be substituted in the formula (48.7) in order to determine the parameter v, will take on the final form

$$\bar{\xi}(x) = 10^{-3}(1{,}3 - 0{,}0397\,x). \qquad (48.10)$$

We first calculate the value of the integral

$$\int_0^l \left[\xi(x)\right]^{n+1} dx = 10^{-12}\int_0^{30}(1{,}3 - 0{,}0397\,x)^4 dx = 0{,}188 \cdot 10^{-10}.$$

Substituting all the known quantities into the formula (48.7) for a magnitude of energy dissipation in the material, corresponding to the values ω = 88 cycles/sec, W_s = 11.35 kg cm we obtain the value of the parameter of the hysteresis loop

$$\nu = \frac{n(n+1)(n+2)W_s}{2^{n+2}(n-1)bhE\int\limits_0^l[\bar{\xi}(x)]^{n+1}\,dx} =$$

$$= \frac{3\cdot 4\cdot 5\cdot 11{,}35}{32\cdot 2\cdot 3\cdot 1{,}5\cdot 2{,}08\cdot 10^6\cdot 0{,}188\cdot 10^{-10}} \approx 6\cdot 10^4.$$

Thus the values of the geometric parameters of the hysteresis loop for St 20 were found to be equal to

$$n = 3;\quad \nu = 6\cdot 10^4.$$

Chapter X

Experimental Methods of Investigating the Dissipation of Energy in the Material During Free Vibrations

49. Logarithmic decrement of damping

The logarithmic decrement of damping is the value of the natural logarithm of the ratio of two consecutive (adjacent) amplitudes of vibration

$$\delta = \ln \frac{a_k}{a_{k+1}}. \tag{49.1}$$

If we assume that the logarithmic decrement of damping does not depend on the absolute magnitude of amplitudes, then formula (49.1) may be expressed in the form

$$\delta = \frac{1}{z} \ln \frac{a_k}{a_{k+z}}, \tag{49.2}$$

where a and a_{k+z} are the amplitudes of vibration at the beginning and at the end of the interval, z is the number of vibrations in the interval.

It is well-known that the ratio of the energy dissipated in one period of vibration to the stored energy at the beginning of the given period, is equal to twice the logarithmic decrement

$$\frac{\Delta W}{W} = 1 - e^{-2\delta} \approx 2\delta. \tag{49.3}$$

Therefore, the magnitude of the logarithmic decrement of damping of vibrations may be calculated from the magnitude of energy dissipation in the material. For, from the known magnitude of the area of the hysteresis loop A, which gives the dissipation of energy in the material per unit volume, and also knowing the magnitude of the potential energy W, stored in the same unit volume of the material for the same amplitude of strain, we determine on the basis of (49.3) the magnitude of the logarithmic decrement by the formula

$$\delta = \frac{A}{2W}. \tag{49.4}$$

After substituting the values of A and W in formula (49.4) we have

$$A = \frac{2^{n+1}(n-1)E\nu\xi_0^{n+1}}{n(n+1)}, \tag{49.5}$$

$$W = \frac{E\xi_0^2}{2}.$$

Hence,

$$\delta = \frac{2^{n+1}\nu(n-1)\xi_0^{n-1}}{n(n+1)}.$$

The final form of formula (49.6) for the determination of the true logarithmic decrement of St 20, with the parameters $n = 3.0$ and $\nu = 6 \cdot 10^4$ which have been found, is

$$\delta = 16 \cdot 10^4 \xi_0^2. \tag{49.7}$$

Thus if we know the magnitude of the strain ξ_0 at a given point of the vibrational system, then by formula (49.7) we can calculate the magnitude of the logarithmic decrement determined by the dissipation of energy in the material.

In practice we usually work with the average magnitude of the logarithmic decrement, which depends on the dissipation of energy in the material and on the other losses which accompany the vibrations of the elastic system. This magnitude is determined from experiments on cantilevered specimens by the measurement of consecutive free vibrations.

Sometimes by the elimination of the external losses we determine the magnitude of a logarithmic decrement which depends only on the dissipation of energy in the material. Then we reduce the average magnitude of the logarithmic decrement to its true magnitude by expressing it as a function of the stresses which arise at the extreme fibers. We shall compare the decrement of damping obtained on the basis of the hysteresis loop parameters ν and n, which we have determined from the experiments of steady-state forced vibrations of samples with the decrement of damping for the same material (St 20), found from the experiments on free vibrations generally accepted in engineering practice.

An oscillogram of strain during free vibrations was recorded with the help of an oscillograph and a wire resistance strain gauge placed at the critical cross-section of the sample. This allowed us to obtain the damping curve of the amplitude of oscillation of the strain at the outer fibers of the specimen. Moreover, we recorded oscillograms of amplitude of deflection of the sample during damped vibrations. To accomplish this we used a Geiger oscillograph. The initial impulse of loading of the cantilever specimen which was gripped at one end was accomplished by the sudden removal of a load suspended at the free end of the specimen. Here the magnitude of the load was quite well-defined; therefore, the initial amplitude on the oscillogram gave the scale of stress deflection of the specimen.

In order to eliminate the influence of external losses (air friction, losses due to operating the recording mechanism of the Geiger oscillograph, etc.) we proceeded as follows. Along with the recording of the oscillogram of the amplitude of the free vibrations of the investigated sample made of St 20 we also recorded the oscillogram of amplitude of vibrations of a sample made of ShKh 15. In both instances the experimental conditions were identical. It turned out that the initial amplitudes of the oscillogram obtained for the sample made of steel ShKh 15, for an initial stress of 1100 kg/cm^2 at the critical cross-section were greater than the corresponding amplitudes for the sample made of St 20 by about 30%.

Photographs of oscillographs corresponding to the two steels are shown in Fig. 38.* In order to determine the magnitude of the logarithmic decrement of the sample made of St 20 we measured the amplitudes on the part of the oscillograms having the maximum values, starting with the first amplitude. The results of these measurements for the first 16 double amplitudes are given in Table 28.

Table 28

Amplitude	1	2	3	4	5	6	7	8
$2a$	19.3	18.4	17.3	16.3	15.7	15	14.4	13.8
$\frac{a_z}{a_{z+1}}$	1.050	1.063	1.060	1.040	1.047	1.041	1.042	1.038
Amplitude	9	10	11	12	13	14	15	16
$2a$	13.3	12.8	12.4	12.0	11.6	11.1	10.7	10.3
$\frac{a_z}{a_{z+1}}$	1.040	1.032	1.033	1.034	1.045	1.038	1.039	-

On the basis of the data in Table 28 the average value of the ratios of the preceding amplitude to the following one for the first seven amplitudes at the beginning of the oscillogram, where considerable stresses occur, is equal to

$$\left(\frac{a_z}{a_{z+1}}\right)_{cp} = 1{,}050.$$

*At the end of the book.

The average logarithmic decrement of damping is, according to (49.1):

$$\delta'_{cp} = \ln\left(\frac{a_z}{a_{z+1}}\right)_{cp} = 0{,}0484. \qquad (49.8)$$

We now determine the average logarithmic decrement of damping of the vibrations of a specimen made of steel ShKh 15. On the oscillogram which refers to this steel we measure the amplitudes on the parts corresponding to the maximum values of the amplitudes on the oscillogram for the sample made of St 20. The results of these measurements are given in Table 29. The magnitude of the average logarithmic decrement of damping of free vibrations of the sample made of steel ShKh 15 on the basis of the date given in Table 29 is equal to

$$(\delta_{cp})_{\text{ШХ } 15} = \ln\left(\frac{a_z}{a_{z+1}}\right)_{cp} = \ln 1{,}035 = 0{,}0345. \qquad (49.9)$$

Table 29

Amplitude	5	6	7	8	9	10	11	12
$2a$	19.8	19.2	18.5	17.9	17.3	16.7	16.1	15.6
$\frac{a_z}{a_{z+1}}$	1.031	1.038	1.034	1.034	1.036	1.037	1.031	1.040
Amplitude	13	14	15	16	17	18	19	20
$2a$	15.0	14.5	14.0	13.5	13.1	12.7	18.3	11.9
$\frac{a_z}{a_{z+1}}$	1.034	1.035	1.036	1.030	1.031	1.032	1.033	-

If we neglect the dissipation of energy in the steel ShKh 15, then the magnitude of the logarithmic decrement will be determined only by the external losses. However, the magnitude of the logarithmic decrement δ_{cp}, obtained in the experiments on the sample made of steel St 20, is determined both by the internal dissipation of energy in this steel and by all the external losses of energy. The problem now consists in determining the magnitude of the dissipation of energy in St 20 by utilizing the values of the decrements δ'_{cp} and $(\delta_{cp})_{ShKh\,15}$.

Let us, first of all, determine the amount of energy in the sample at the initial moment of free vibrations. For this purpose we find the total magnitude of the potential energy in the sample at its maximum deviation from the average position. This magnitude, expressed per unit volume, is equal to

$$W_1 = \frac{E\xi_0^2}{2}.$$

The magnitude of the potential energy in the total volume of the sample when the law of the variation of deformation, both with respect to the height of the cross-section and along the length of the sample is known, will be

$$W = \frac{E}{2}\int_0^l 2\int_0^{\frac{h}{2}} \left(\frac{2\xi_0(x)\,z}{h}\right)^2 b\,dx\,dz = \frac{Ehb}{6}\int_0^l [\xi_0(x)]^2\,dx, \qquad (49.10)$$

where

$$\xi_0(x) = \frac{h}{2}\,\frac{d^2u(x,0)}{dx^2} = \frac{ah}{2}\,\frac{d^2\varphi}{dx^2}. \qquad (49.11)$$

Earlier we had

$$\frac{d^2\varphi}{dx^2} = 3{,}529\cdot 10^{-4}\,[\,2{,}23\,(\mathrm{sh}\,kx - \sin kx) + 3{,}038\,(\mathrm{ch}\,kx + \cos kx)\,],$$

from which, with

$$k = \frac{kl}{l} = \frac{1{,}8751}{40{,}5} = 0{,}0463$$

we have

$$\int_0^l \left(\frac{d^2\varphi}{dx^2}\right)^2 dx = 3{,}824\cdot 10^{-4}\ \text{см}^{-3}.$$

Hence

$$W = 3{,}824 \cdot 10^{-4} \frac{Eh^3ba^2}{24}$$

Substituting the values of E and b and carrying out the calculations, we obtain

$$W = 336 \cdot a^2 \text{ kg cm}, \tag{49.12}$$

where a is the amplitude of the vibration of the end of the bar, cm.

According to the oscillogram recorded with a magnification of three times at the initial instant, the amplitude of vibrations of the section of the specimen at a distance of 8 cm from its end amounted to 0.32 cm. The amplitude of the end of the specimen was equal to 0.378 cm. Substituting this amplitude in formula (49.12) we obtain

$$W = 336 \cdot 0{,}378^2 = 48 \text{ kg cm}.$$

The total amount of energy dissipation resulting from the external and internal resistances during the vibrations of the sample made of St 20, on the basis of formulas (49.3) and (49.8), is equal to

$$\Delta W_{\Sigma} = 2\delta'_{cp} W = 2 \cdot 0{,}0484 \cdot 48 = 4{,}62 \text{ kg cm}.$$

In an analogous way we determine the energy required for the external resistance by using the logarithmic decrement obtained for the damping of the vibrations of the specimen made of ShKh 15.

According to (49.3) and (49.9) we obtain

$$\Delta W_{B} = 2(\delta_{cp})_{ШХ15} W = 2 \cdot 0{,}0345 \cdot 48 = 3{,}31 \text{ kg cm}.$$

The energy dissipation in the material of one specimen for one cycle of vibrations is equal to

$$\Delta W_t = \Delta W_{\Sigma} - \Delta W_{B} = 4{,}62 - 3{,}31 = 1{,}31 \text{ kg cm}. \quad (49.13)$$

Then the average logarithmic decrement of damping of the sample made of St 20 caused by the internal losses in the material is

$$\delta_{\text{Ст.}20} = \frac{\Delta W_t}{2W} = \frac{1{,}31}{2 \cdot 48} \approx 0{,}014^{*}$$

or

$$\delta_{\text{Ст.}20} = 1{,}4\%.$$

50. Energy dissipation for the vibrations of groups of bars

In the present paragraph we give experimental data on a study of energy dissipation in a group of prismatic bars, the conditions of support and interconnection of which is analogous to that which actually occurs in real groups of the turbine blades.

It is very important to know energy dissipation in the vibrations of such machines as gas and steam turbines, where the strength of the blades, for instance, are mostly determined by their ability to absorb energy during the vibrations which inevitably arise in the operation of the turbines.

The dissipation of energy in the vibrations of assemblies consists of the energy dissipation in the material of the system resulting from the imperfect elasticity of the material,

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*In the present case we obtained the value of the average logarithmic decrement of the bar in question made of St 20; on the other hand, formula (49.7) allows us to determine the value of the true decrement, corresponding to the dissipation of energy of the stressed material at a given point.

and the dissipation of energy resulting from the external resistance. Included in the latter are the energy dissipation due to friction with the external medium and to friction in the connections of the elements of the vibrating system (points of attachment to a support, points of connection of the parts of the system, etc.).

Taking into account our meager knowledge of the capacity to dissipate energy in individual bars, to say nothing of systems of bars connected as a group, we made an attempt to investigate the damping of vibrations in groups of prismatic bars. The problem which is posed consists of the following:

1) to determine the magnitude of the decrements of damping as a function of the magnitude of the stress for a single bar rigidly fixed at one end, and also to determine the decrement of damping of vibrations of a group, composed of similar bars with the same fixity at one end and connected on the opposite end by a band (shroud, lashing) with a rivet;

2) to investigate the damping of vibrations of the same groups of bars for the case, when, besides being riveted the band is also welded on;

3) to compare the damping characteristics obtained for bars under the different conditions of support indicated above, and to analyze their influence in the magnitude of decrement;

4) to establish the relationship of the magnitudes of energy losses resulting from energy dissipation in the material and the dissipation in the connection of the group bars.

To solve these problems we designed and constructed a special experimental set-up, which provides for the possibility of rigid clamping of the separate bars and of a group composed of six bars. The apparatus also provided for excitation of both forced and free vibrations of the bars. A photograph of the over-all view of the set-up is shown in Fig. 39.*

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*At end of book.

The base of the vibration assembly is a massive body, made up of a block with a lengthwise groove milled out and with two heavy bars connected to the sides of the block. The opposite ends of the heavy bars are connected by a stiff cross piece which serves as a support for the tightening jack.

The base of the apparatus is attached to the foundation with the help of welded pedestals with openings for bolts. The pedestals also brace the stand on which we mount electromagnets for exciting the vibrations of the bars.

The samples used for testing were prismatic bars made of St 5 with a rectangular cross-section 30 x 10 mm for a designed length of 300 mm.

The cross-section of the steel band (shroud) was 30 x 4 mm.

The ends of the bars under investigation were fitted into the groove of the block (in a holder) up to a depth of 30 mm and were tightened by the jack in the self-strained scheme, as shown in Fig. 39. The distance between adjacent bars was taken to be 30 mm.

The excitation of vibrations of the separate bars was accomplished with the help of electromagnets (Fig. 39), as well as by stretching and the subsequent sudden breaking of a wire which is connected to the upper part of the group of bars and which carries a weight on its other end and passes over a pulley (Fig. 40).*

The measurement of the amplitudes of vibration was carried out with the help of the oscillograms which were recorded by a trifilar oscillograph, with the utilization of wire resistance strain gauges connected by the special bridge scheme shown in Fig. 33. The strain gauges 30 mm long were pasted at the most highly stressed parts of the bar, i.e. near the holder.

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*At the end of the book.

A sample oscillogram of free vibrations of the individually fixed cantilevered bars, for the excitation of vibrations with the help of the electromagnets with an initial amplitude corresponding to the resonant state is shown in Fig. 41.*

The maximum amplitude of the deflection at the end of the bar was equal to a_{max} = 6 mm, and the stress σ_{max} = 3000 $kgcm^2$.

The magnitude of stresses was determined with the help of an extensometer of the Hugenberger type mounted at the base of the bar (Fig. 39).*

A sample of the recorded oscillograms of free vibrations of groups of bars is shown in Fig. 42.*

From the data of the oscillogram for damped vibrations and initial amplitudes of maximum deflection of the bars, it was possible to determine the magnitude of the average logarithmic decrement of damped vibrations for different amplitudes, both for separate bars, and also for a system of bars connected as a group by a shroud.

The average logarithmic decrement of vibrations dampings, for the period of time during which the sample executed vibrations, while the amplitude of the vibrations changed from the value a_k to the value a_{k+z}, is determined by the well-known formula

$$\delta_{cp} = \frac{1}{z} \ln \frac{a_k}{a_{k+z}}.$$

The amplitudes a_k and a_{k+z} were determined directly from the recorded oscillograms. The results of the reduction of data of the oscillograms of the separate bars and groups of them for various end conditions are graphically illustrated in Figs. 43, 44, and 45.

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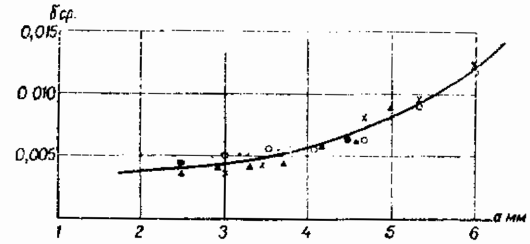

Fig. 43

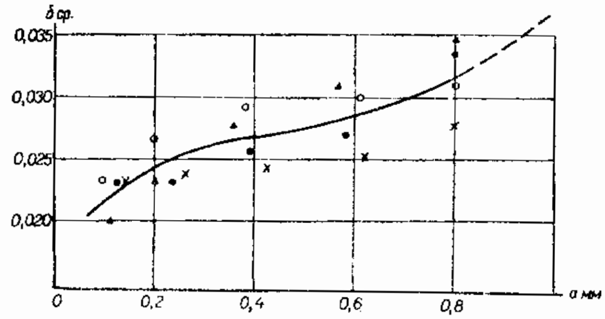

Fig. 44

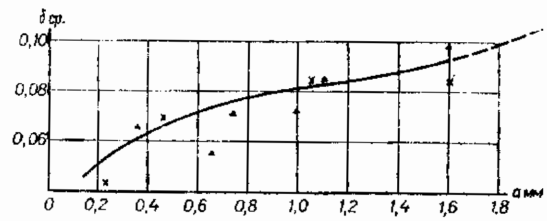

Fig. 45

Fig. 43 shows the curve of the variation of the magnitude of logarithmic decrement of damping with the amplitude of a single bar. Fig. 44 shows the variation of the logarithmic decrement of damping of groups of bars for the case of rigidly fixed ends of the shafts and with a riveted shroud for the electromagnetic excitation of vibrations from an amplitude of 0.8 mm.

Fig. 45 shows the graph of the variation of logarithmic decrement for the same group composed of six bars, with insufficiently rigid clamping in the rigid holder, for the case of the excitation of vibration of the group by means of bending it with an initial amplitude of 1.6 mm at the end.

Fig. 46 shows an analogous graph of $\delta_{cp} = f(a)$, plotted on the basis of reduction of oscillograms of the free vibrations of a group of bars rigidly fixed in the holder and having the belt (shroud) rigidly attached by electric welding.

The bars and the groups of bars had the following natural frequencies:

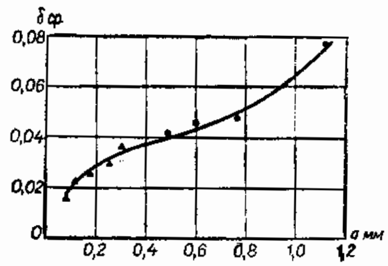

Fig. 46

a) rigidly fixed bar, 81 cycle/sec

b) group made up of six specimens for the case of a riveted belt. 107 cycles/sec

c) the same block not quite rigidly fixed in the holder, 105 cycles/sec

d) the same block, rigidly fixed in the holder, but with a welded belt, 130 cycles/sec.

For comparison of the values of the average logarithmic decrements of damping obtained for individual bars and for groups of the same bars for the given three cases of interconnection for the same initial displacements at the end of the bars a tabular summary is given in Table 30.

Table 30

Deflection at the end of the bar a_{max}, mm.	Boundary Conditions			
	Single bar, fixed at one end in an arbor, free at the other end	Group of Six Bars		
		Rigidly fixed in an arbor with a riveted shroud	Non-rigidly fixed in an arbor, with riveted shroud	Rigidly fixed in an arbor, with a welded shroud
0.2	0.0030	0.0244	0.050	0.0280
0.4	0.0031	0.0270	0.064	-
0.6	0.0032	0.0286	0.072	-
0.8	0.0033	0.0316	0.077	-
1.0	0.0034	0.0362	0.107	-
6.0	0.0100	-	-	-

Comparing the magnitude of energy dissipation in the vibration of a group of bars with the magnitude of energy dissipation of vibration of a single bar for the same deflection at the end of the shaft equal to 1 mm, which corresponds to stresses of about 500 kg/cm^2, at the base, we find that energy dissipation in the single shaft, in particular

$$\frac{(\delta_{cp})_{bar}}{(\delta_{cp})_{single\ shaft}} = \frac{0{,}0362}{0{,}0034} = 10{,}6$$

Thus, we confirm the results of investigations of damping in vibrations of groups of blades of steam turbines, obtained at the Institute of Structural Mechanics Academy of Science Ukr. SSR by A. D. Kovalenko.* Regarding the dissipation of energy in the groups of bars with insufficiently rigid clamping of the ends of the bars in the holder, we found that this depends on the degree of the rigidity of gripping. For our assumed case of clamping, for which the frequency of vibration was lowered by 2 cycles/sec (from 107 to 105), we found that the logarithmic decrement of damping more than doubled in comparison with the decrement of damping of the same group, but with the rigid clamping of the group of bars in the holder.

We should call attention to the paradoxical phenomenon of the increase of logarithmic decrement of damping of the group of bars with a welded band in comparison with the logarithmic decrement of damping of the group with a riveted band. This phenomenon may be explained in the following manner. In our experiments the ratio of the moments of inertia of the bars and the band for the case of rigid connection of the belt with the bars by electric welding makes the group of bars and the band a monolithic structure, a

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*A. D. Kovalenko, Investigation of damping in the vibration of groups of blades of steam turbines, published by Academy of Science Ukr. SSR, Collection of reports on the dynamic strength of machine parts, M.—L., 1946.

rigid frame with rigid joints at the band. In the vibrations of the group, even for the case of small initial amplitudes (of the order of 0.2 mm) high stresses arise in the band which lead to a great amount of energy dissipation in the material of the band.

51. Apparatus for the investigation of energy dissipation in the material during torsional vibrations of bars

Lately there has been noticed an increasing interest of investigators in the questions of energy dissipation for various types of vibrations of elastic systems. As a result, there have appeared a series of interesting theoretical and experimental works on the problems of damping. However, we do not know of any works which deals with the question of the dependence of energy dissipation in the material on the character of the stress distribution in sections of different structural form.

The investigation of this question appears to us quite important both from the point of view of establishing optimum structural forms of those vibrating parts of machines in which the damping due to energy dissipation in the material is of decisive significance, as well as from the point of view of establishing a method of theoretical consideration of energy dissipation in the material of the bars of different cross-sectional forms.

The problem of studying the energy dissipation in the material in torsional vibrations is particularly urgent.

Members of the department of strength of materials at the Kiev Polytechnic Institute designed and constructed a special apparatus intended for studying energy dissipation in the material during torsional vibrations.* The base of

*N. E. Solomentsev, lecturer in the department of strength of materials, K.P.I., was engaged in the construction of the apparatus under our supervision.

the apparatus, a sketch of which is shown in Fig. 47, is the rigid frame 1 with dimensions 580 x 530 mm, suspended from the ceiling by a relatively thin steel wire of 2.5 mm diameter. In order to increase the mass of the frame, two weights of 30 kg each are symmetrically and rigidly connected to the lower part of the frame. A special clamp was constructed in the upper part of the frame and rigidly connected with the frame by bolts. This clamp serves to tighten the tested vertical sample 2 by its thickened, cylindrically shaped upper end.

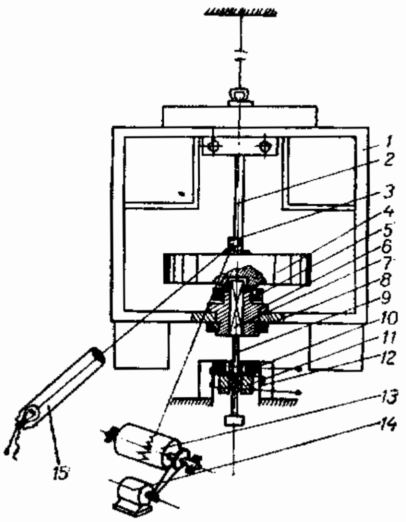

Fig. 47

A massive steel disc 4, 270 mm in diameter and 75 mm in thickness, is rigidly connected to the lower part of the sample which is in the form of a conical head 50 mm in length. If necessary, the size of the disc may be varied.

The working length of the sample is 300 mm. The cross-section may be of different dimensions and shapes. The maximum diameter of the tested circular sample is equal to 20 mm.

The initial twisting of the tested elastic sample up to the required amplitude, is accomplished by turning the disc 4, with the help of the rod 9, whose rectangular end 7 enters simultaneously into the sleeve 6 which is rigidly connected with the lower cross frame by the nut 8. The disc is secured in this position.

For such a "loading", the lower part of the frame of the apparatus is set on stationary supports with the help of a hoist so that the supporting wire is completely unloaded. In order to prevent bending of the specimen at the time of initial twisting, the sample is positioned in the vertical direction with the help of a special movable sleeve 5, held by the threaded projection of the upper part of the lower cross frame and by its upper part which grips cylindrical projection of the disc. Then the supports are removed, and the frame with the twisted sample is suspended by the long thin (2.5 mm diameter) wire. To excite the disc's vibrations, the piston rod holding the sample in the twisted state, is instantaneously released and thrown back by the electromagnetic device 10-12 which is connected to the frame.

The oscillograph recording of the torsional vibrations of the specimen is accomplished by an optical method with the help of a light ray 15, reflected from the mirror 3, which is mounted on the lower end of the sample (next to the disc), and rotating drum 13 with the light sensitive paper, which is rotated by the motor 14. The scheme of the above optical assembly is analogous to the one described in #52. A photograph of the general view of the apparatus is shown in Fig. 48.*

A number of investigations were conducted on the above mentioned assembly. We will give only the results of the experiments in the investigation of torsional vibrations of solid and hollow bars. Experiments were performed on two bars made of brass, one solid bar with circular cross-section 7 mm in diameter and the other with a hollow circular cross-section of outside diameter 8 mm and 6 mm internal diameter. The moments of resistance** the torsion of the two bars are almost identical, for the solid bar $W_{pc} = 0.074$ cm^3 and for the hollow bar $W_{pT} = 0.0685$ cm^3.

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**By "moment of resistance" is meant the polar moment of inertia divided by the radius. This quantity is exactly analogous to the section modulus of a beam, in that stress, here maximum shear stress = torque/moment of resistance. (Trans.)

Attached to the bar, is the massive disc, guaranteeing a natural frequency of torsional vibrations of 4 cycles/sec for the case of a solid bar and 5 cycles/sec for the case of a hollow bar.

The magnitude of the logarithmic decrement of damping in both instances was determined by the measurement of amplitudes directly on the oscillograms of free torsional vibrations of the system.

On the basis of reduced data of the oscillograms, graphs are constructed of the dependence of the average logarithmic decrement of damping on the magnitude of the maximum shear stresses which arise in the bar during torsional vibrations. These graphs are shown in Fig. 49.

From the comparison of the curves obtained for the above two cases, it follows that for the same maximum stress on the periphery, the energy dissipation in the material of the sample with the hollow cross-section, in which the distribution of tangential stresses is

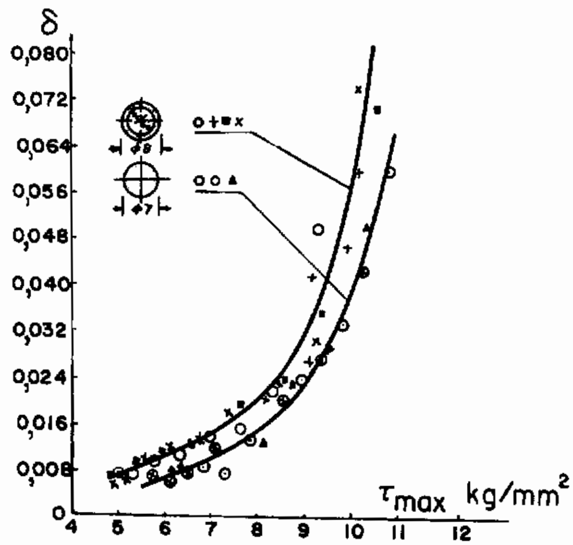

Fig. 49

almost uniform and close to the maximum exceeds, on the average, 1.5 times the magnitude of energy dissipation in the material of the sample with solid cross-section in which the distribution of stresses varies linearly from a maximum on the periphery of the cross-section to zero at the center. These results also point out that the hollow cross-sections are more efficient from the point of view of total utilization of damping properties of the material.

The results obtained also justify the hypothesis that energy dissipation in the material depends on the magnitude of the stresses.

52. Vacuum apparatus for studying energy dissipation in the material

In the present paragraph we present a description of the construction and the principle of operation of a new type of vibration apparatus devised for the investigation of energy dissipation in the material in transverse vibrations of two-dimensional specimens under the conditions of pure bending, and which permits conducting investigations in a vacuum at both normal and high temperatures.

In the light of the necessity of obtaining experimental data characterizing the damping properties of the material, particularly of those utilized in the turbine blades, it is very important to obtain these data under conditions which are close to the actual working conditions of the parts in question.

For this purpose, we developed and constructed a special apparatus

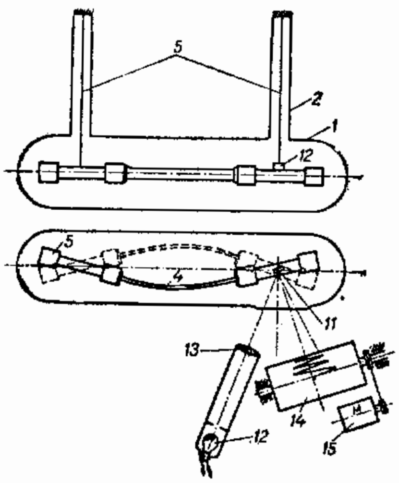

Fig. 50

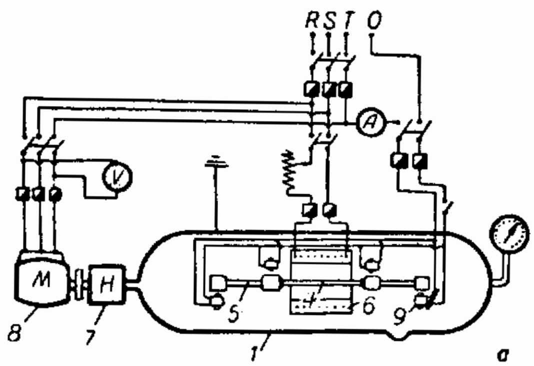

Fig. 51

which has a number of advantages and additional possibilities in comparison with the existing apparatuses.

In striving to obtain the necessary data which would be of practical usefulness in the analysis of the blade vibrations of turbines, the operation of our experimental assembly was based on the method of the determination of energy dissipation in the material of free transverse vibrations of the investigated sample.

The second requirement demanded of the apparatus is the guarantee of the possibility of conducting experiments under high stresses, i.e. under stresses which we encounter in the actual turbine blades.

The next important requirement of the assembly is to reduce to a minimum the external losses during the vibrations of the tested sample. With this goal in mind, in order to eliminate air resistance during the vibrations, the tested specimen was placed in a special chamber from which the air was removed by a displacement vacuum pump. In order to minimize the losses in the connections of the specimen (at the points where it is clamped) with the immovable elements of the assembly (the base), relative to which vibrational motions take place, the sample is suspended by thin wires.

The last requirement which had to be fulfilled in the construction of the assembly was to enable us to test the specimens at high temperatures. For this purpose, the sample was inserted in a special heating oven, placed in the interior of the evacuated chamber of the apparatus.

The designed and constructed vacuum apparatus, satisfying all the above mentioned requirements (schematically shown in Figs. 50 and 51, and by photographs in Figs. 52* and 53*) is composed of the following basic parts:

*At the end of the book.

1—steel case (chamber) in the form of a two-piece cylindrical vessel designed to hold the specimen with the test oven;

2—pipes soldered to the frame and intended to surround the supporting wires;

3—base, made of soldered pipes, on which the case is placed (Fig. 52);

4—plane sample to be tested, with two weights connected to it by special grips;

5—thin steel wires, on which the sample with the weights is suspended;

6—tubular heating oven placed in the interior of the evacuated chamber for the purpose of heating the tested sample;

7—displacement vacuum pump;

8—electric motor connected to the pump;

9—electromagnets placed in the interior of the frame and used to excite the vibrations of the specimen by the formation of an initial pure bending by a pair of magnetic forces, which act on the weights, placed at the ends of the sample;

10—electrical control board (Fig. 52), with electrical connections to the electric motor of the vacuum pump, to the electromagnets, to the heating oven, and to the wire leads of the thermostat placed in the oven. Several measuring devices are also installed on the electrical board; vacuum-meter, ammeter, voltmeter, galvanometer for controlling the temperature of the test oven with the help of a thermocouple, and also a special electrical contact device which are elastic contacts placed in the interior of a pipe made of insulating material and a solid metallic sphere falling in the interior of the pipe which switches on momentarily the electromagnets which excite vibrations of the specimen.

In order to record the vibrations of the tested samples, the apparatus is provided with an optical device, a sketch of which is also shown in Fig. 50. This device consists of the following parts:

11—mirror placed at one of the points of connection of the sample to a wire;

12—light source;

13—a system of converging lenses by which the ray of light is made to fall on the mirror;

14—drum with attached photo sensitive paper or photo film, on which the oscillogram is recorded by the ray of light reflected from the mirror and projected on the drum as a point.

15—electric motor rotating the drum during the recording of the oscillograms.

Since the mirror is positioned inside the frame, while the source of light is outside the frame, a small glass window is installed in the lid of the frame through which the ray of light passes which falls on the mirror and is then reflected back from it.

From the description of the construction of the vacuum vibro-assembly it follows that the suspension of the sample with weights in the vacuum frame by thin steel wires guarantees practically the total elimination of loss of energy in the support of the sample, as well as losses due to the resistance of the medium. For, we can neglect the energy dissipation in the material which is twisted through an angle of $5-10^{\circ}$, for wires 0.2—0.4 mm in diameter and of 450 mm length, in comparison with the losses in the material of the investigated sample of 300 x 30 x 3 mm dimensions, with an initial maximum amplitude of normal stresses of 2000—3000 kg/cm^2. We can also neglect energy dissipation resulting from losses connected with the resistance of the medium, since the experiment with the sample is carried out at vacuum conditions at a pressure of the order of 10^{-1} mm of mercury.

It should be noted that in order to eliminate the absorption of energy at the connection points of the weights and the samples, the ends of the latter are considerably thickened (Fig. 54). Therefore, the ends of the samples entering into the tightening grips of the weights (Fig. 55), are made quite rigid, and have very little stress in comparison with the highly stressed section of the sample in which energy dissipation is investigated.

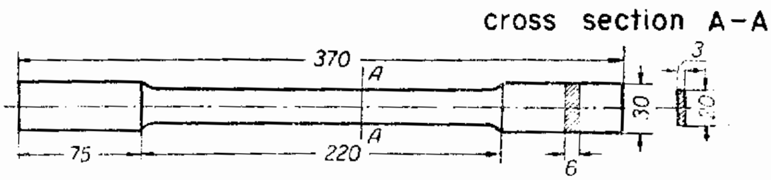

Fig. 54

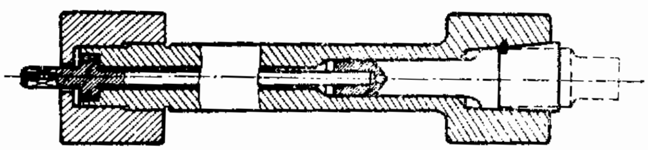

Fig. 55

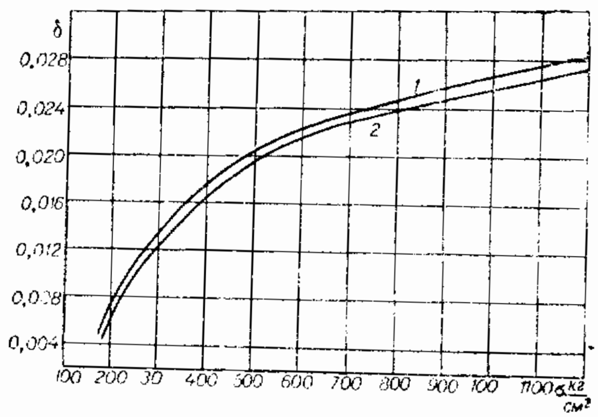

Fig. 57

One of the oscillograms which was obtained from this vacuum vibration apparatus in an investigation of energy dissipation in a material used in manufacture of turbine blades is shown in Fig. 56.* The initial stress amplitude was 25 kg/mm^2; the temperature was $20^{o}C$. In Fig. 57 curves are shown of the logarithmic decrement of samples of one of the types of steel used for turbine blades as a function of stress. These were obtained on the apparatus described.**

Curve 1 was obtained for experiments in vacua and curve 2 for experiments in the atmosphere. The experiments were carried out at room temperature. It follows from examination of the curves given that at the frequencies at which the experiments were carried out (7 cycles/sec), the effect of energy loss due to air is not significant and amounts to about 5% of the loss due to energy dissipation in the steel.

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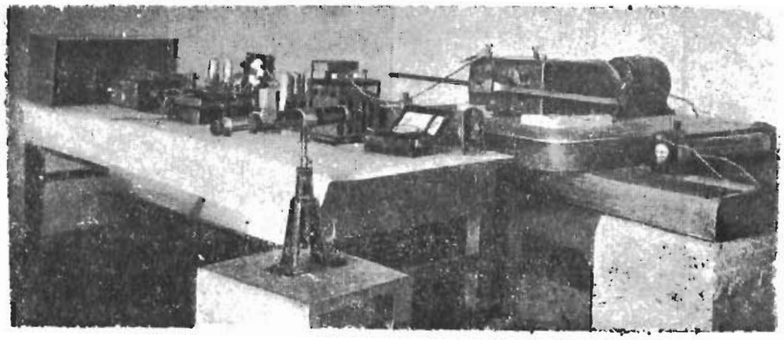
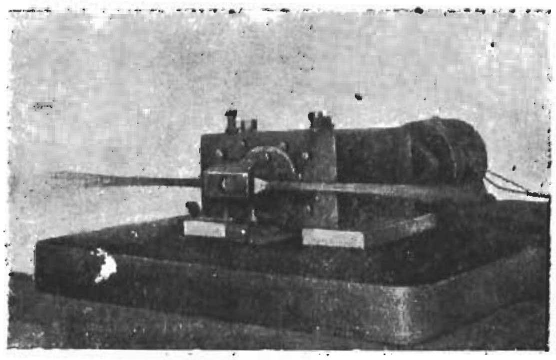
**The experimental work was done by V. V. Khilchevsky, graduate student, under our supervision.



(a)



(b)

Fig. 24

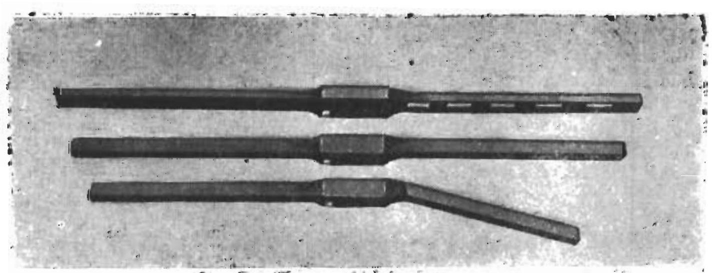


Fig. 26

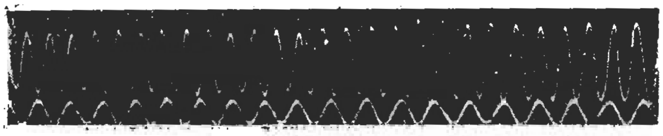


Fig. 36

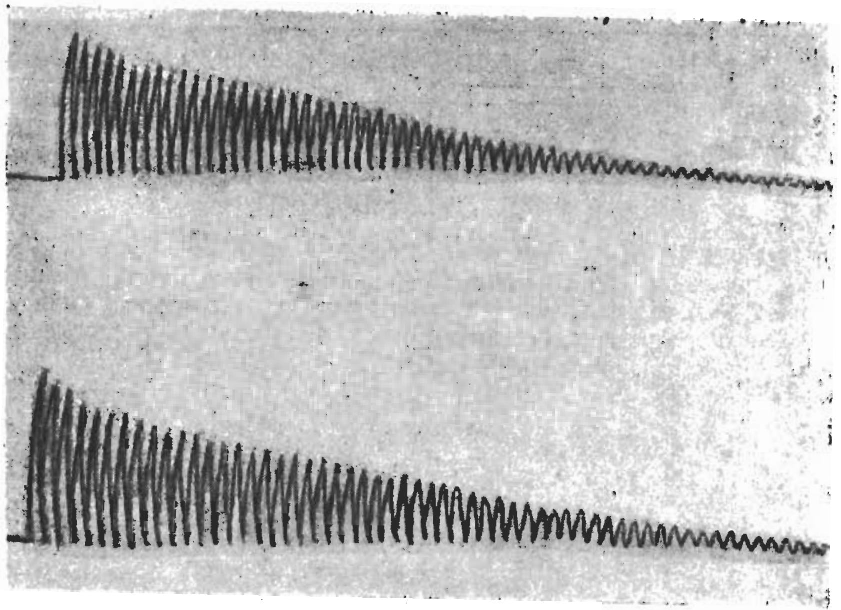
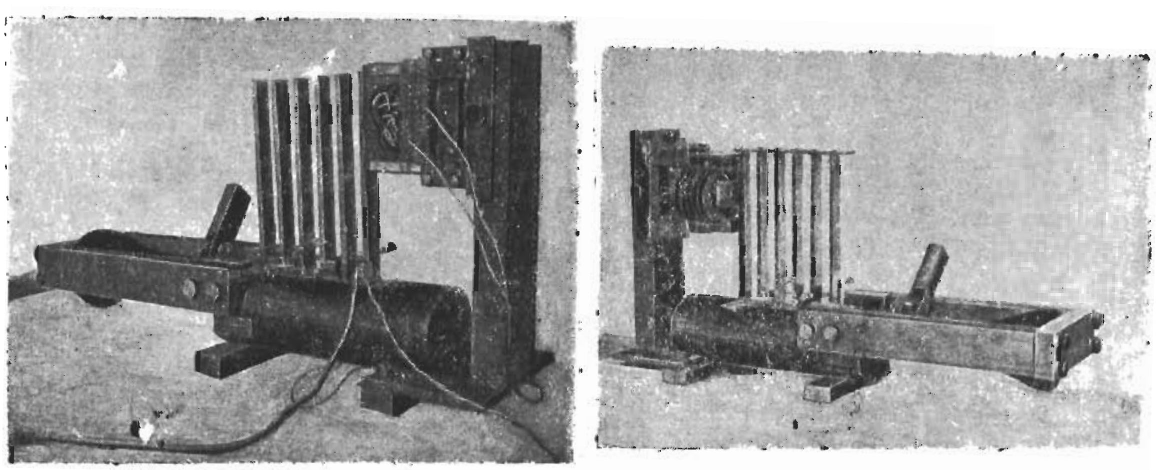


Fig. 38



(a) (b)

Fig. 39

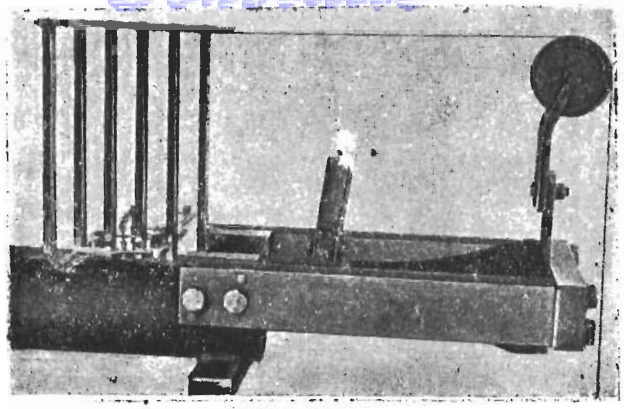


Fig. 40

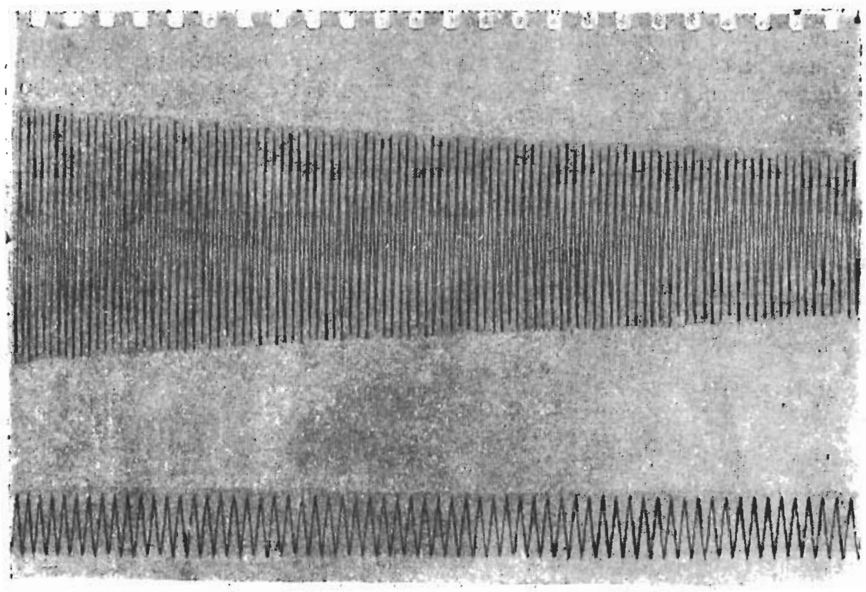


Fig. 41

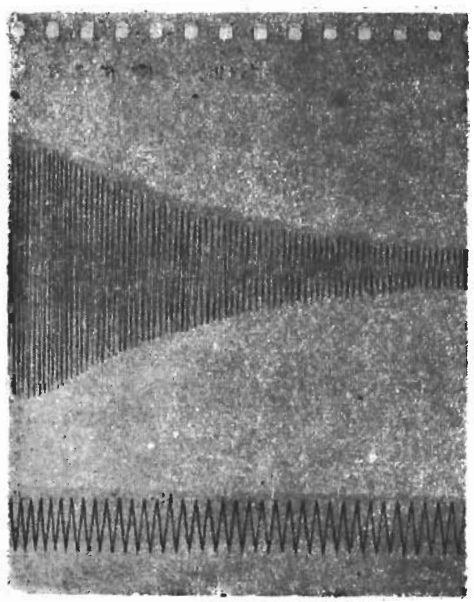


Fig. 42

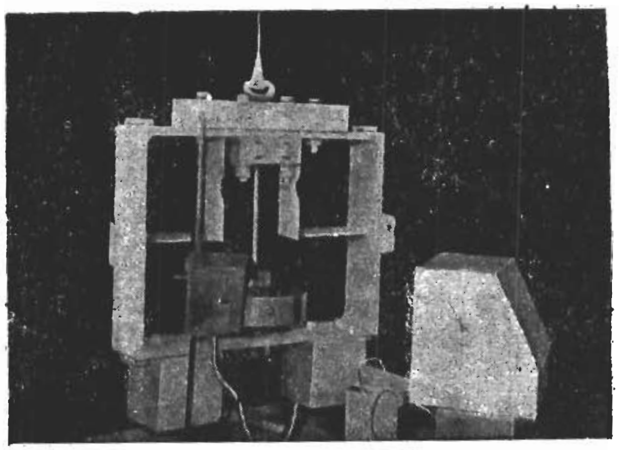


Fig. 48

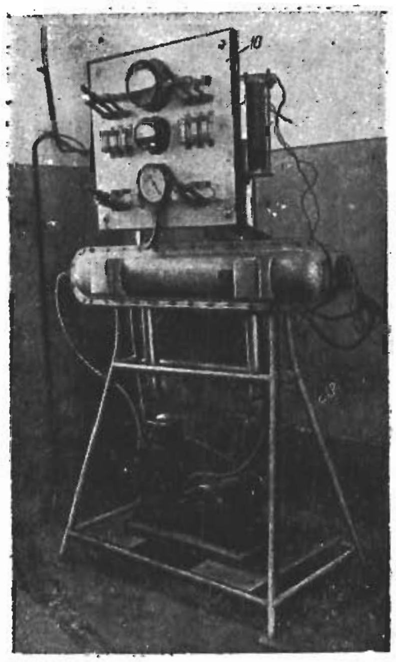


Fig. 52

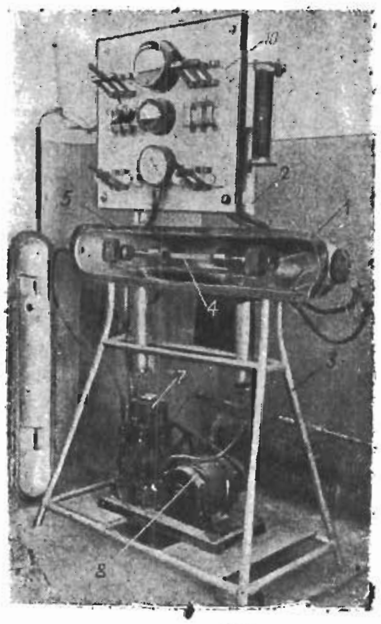


Fig. 53

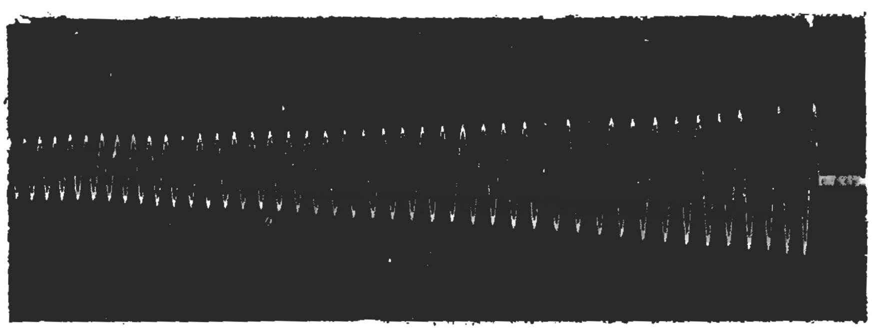


Fig. 56

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