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QUADRATIC MATRIX EQUATIONS FOR DETERMINING VIBRATION MODES AND FREQUENCIES OF CONTINUOUS ELASTIC SYSTEMS

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The mathematical idealization of vibrating continuous elastic systems into discrete element systems, as used in matrix analysis, leads to the equation of motion in the form of an infinite matrix series whose coefficients involve ascending powers of the frequency. The equation of motion of the idealized discrete system is of the form $(\mathbf{A}-\omega^2\mathbf{B}-\omega^4\mathbf{C}-\ldots)\mathbf{q}=\mathbf{0}$ where the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}\ldots$ depend on the inertia and stiffness properties of the system, ω is the circular frequency, and $\mathbf{q}$ is the column matrix of discrete displacements. The conventional analysis uses only the first two terms of the matrix series and it leads to the characteristic equation of the form $|\mathbf{A}-\omega^2\mathbf{B}|=0$. If the next higher order term is retained, then the equation of motion is a quadratic matrix equation in ω^2 and the characteristic equation becomes $|\mathbf{A}-\omega^2\mathbf{B}-\omega^4\mathbf{C}|=0$. The formulation of the quadratic matrix equation of motion and its solution are discussed. Details of the method are presented for structures made up from bar and beam elements. Some typical numerical examples of the method are presented, including a comparison with the conventional eigenvalue solutions to demonstrate the considerable improvement in accuracy of the calculated vibration modes and frequencies when the term with frequency to the fourth power is retained in the equation of motion for the vibrating elastic system.

GENERAL THEORY

The essential feature of the matrix methods of structural analysis is that a continuous elastic system can be represented by an equivalent discrete element system having a finite number of degrees of freedom. In the discrete system the displacements are specified at points selected arbitrarily on the actual structure, and these displacements are then used to determine the equivalent elastic properties of the discrete element model representing the continuous system (see Figure 1) For static problems the determination of the equivalent elastic properties no special difficulty. The displacements $\mathbf{u}=\mathbf{u}(x,y,z)$ in the continuous system can be related to a finite number of displacements selected on the structure. This relationship may be expressed by the matrix equation

$$\mathbf{u} = \mathbf{a}\mathbf{U} \tag{1}$$

where

$$\mathbf{u} = \{u_x\ u_y\ u_z\} \tag{2}$$

represents displacements in the directions of x,y, and z axes,

$$\mathbf{U} = \{U_1\ U_2\ \cdots\ U_N\} \tag{3}$$

represents a column matrix of the N specified displacements and $\mathbf{a}$ is a rectangular matrix whose coefficients are functions of x,y,z. Naturally such a relationship is only applicable to linear systems. When the applied loads are time-dependent, no such simple relationship is

*Professor of Mechanics, Conference Co-chairman exactly possible since the displacements $\mathbf{u}$, being dependent on the previous history of the applied loading, can no longer be related to the instantaneous values of $\mathbf{U}$; however, if a large number of displacements $\mathbf{U}$ are considered then the static relationship $\mathbf{u} = \mathbf{a}\mathbf{U}$ is a good approximation provided $\mathbf{U}$ is determined from the equations of motion of the system. For harmonic vibrations the form of Equation 1 is preserved except that the coefficients in $\mathbf{a}$ depend also on the frequency of vibrations (Reference 1).

Equation 1 can be used to obtain the total strain-displacement relationship

$$\mathbf{e} = \mathbf{b}\mathbf{U} \tag{4}$$

where the coefficients in $\mathbf{b}$ are derived by differentiation of the matrix $\mathbf{a}$. If virtual displacements $\delta\mathbf{u}$ are imposed on the continuous system (see Figure 2), it follows from the Principle of Virtual Work and d'Alembert's principle that

$$\int_V \delta\boldsymbol{\epsilon}^T \boldsymbol{\sigma}\, dV = \int_S \delta\mathbf{u}^T \boldsymbol{\Phi}\, dS + \int_V \delta\mathbf{u}^T \mathbf{X}\, dV + \delta\mathbf{U}^T\mathbf{P} - \int_V \rho\,\delta\mathbf{u}^T \ddot{\mathbf{u}}\, dV \tag{5}$$

where $\delta\boldsymbol{\epsilon}$ are the virtual strains corresponding to $\delta\mathbf{u}$ and the remaining symbols are defined below:-

$$\boldsymbol{\sigma} = \left\{\sigma_{xx}\ \sigma_{xx} \cdots \sigma_{zx}\right\}, \text{ stresses} \tag{6}$$

$$\boldsymbol{\Phi} = \left\{\Phi_x\,\Phi_y\,\Phi_z\right\}, \text{ surface forces} \tag{7}$$

$$\mathbf{X} = \left\{X_x\ X_y\ X_z\right\}, \text{ body forces} \tag{8}$$

$$\mathbf{P} = \left\{P_1\ P_2 \cdots P_n\right\}, \text{ external forces (or moments) corresponding to the displacements } \mathbf{U} \tag{9}$$

$$\ddot{\mathbf{u}} = \left\{\ddot{u}_x\ \ddot{u}_y\ \ddot{u}_z\right\}, \text{ accelerations} \tag{10}$$

and

$$\rho = \rho(x,y,z), \text{ density} \tag{11}$$

Using Equation 1 it follows also that

$$\delta\mathbf{u} = \mathbf{a}\,\delta\mathbf{U} \tag{12}$$

and

$$\ddot{\mathbf{u}} = \mathbf{a}\ddot{\mathbf{U}} \tag{13}$$

Furthermore, since virtual displacements are taken at constant temperature, the virtual strains can be determined from Equation 4. Hence

$$\delta\boldsymbol{\epsilon} = \delta\mathbf{e} = \mathbf{b}\,\delta\mathbf{U} \tag{14}$$

The stresses $\boldsymbol{\sigma}$ are related to the total strains $\mathbf{e}$ through the generalized Hooke's law

$$\boldsymbol{\sigma} = \mathbf{C}\mathbf{e} + \mathbf{C}_T\,\alpha T \tag{15}$$

where the coefficients in $\mathbf{C}$ and $\mathbf{C}_T$ depend on elastic constants, α is the coefficient of thermal expansion and T is the temperature change.

Substitution of Equations 12 through 15 into Equation 5 leads to

$$\int_V \delta \mathbf{U}^T \mathbf{b}^T \mathbf{C}\,\mathbf{b}\, dV + \int_V \delta \mathbf{U}^T \mathbf{b}^T \mathbf{C}_T \alpha T\, dV = \int_S \delta \mathbf{U}^T \mathbf{a}^T \mathbf{\Phi}\, dS + \int_V \delta \mathbf{U}^T \mathbf{a}^T \mathbf{X}\, dV + \delta \mathbf{U}^T \mathbf{P} - \int \rho\, \delta \mathbf{U}^T \mathbf{a}^T \mathbf{a} \ddot{\mathbf{U}}\, dV \tag{16}$$

Since the virtual displacements δU are arbitrary Equation 16 will be satisfied provided

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{P} - \int_V \mathbf{b}^T \mathbf{C}_T \alpha T\, dV + \int_S \mathbf{a}^T \mathbf{\Phi}\, dS + \int_V \mathbf{a}^T \mathbf{X}\, dV \tag{17}$$

where

$$\mathbf{M} = \int_V \rho\, \mathbf{a}^T \mathbf{a}\, dV \tag{18}$$

represents the mass matrix of the equivalent discrete system and

$$\mathbf{K} = \int_V \mathbf{b}^T \mathbf{C}\,\mathbf{b}\, dV \tag{19}$$

is the stiffness matrix for the displacements $\mathbf{U}$. Equation 17 represents matrix equation of motion of the equivalent discrete system. The first term on the right hand side of Equation 17 is the column matrix of external forces in the directions of $\mathbf{U}$; the second term represents equivalent concentrated forces due to some specified temperature distribution; the third and fourth terms represent equivalent concentrated forces due to surface forces and body forces, respectively. Thus, Equation 17 serves not only to determine the discrete system inertia and stiffness properties but also to convert distributed loading into one consisting of discrete forces.

QUADRATIC EQUATIONS OF MOTION FOR A VIBRATING SYSTEM

For a harmonically vibrating system Equation 1 may be written as

$$\mathbf{u}(x,y,z,t) = \mathbf{a}(x,y,z;\omega)\,\mathbf{U}(t) \tag{20}$$

with

$$\mathbf{U}(t) = \mathbf{q}e^{i\omega t} \tag{21}$$

where ω is the circular frequency and $\mathbf{q}$ represents a column matrix of amplitudes of the displacements $\mathbf{U}$. The corresponding strains are then calculated from

$$\mathbf{e}(x,y,z,t) = \mathbf{b}(x,y,z;\omega)\,\mathbf{U}(t) \tag{22}$$

Thus both $\mathbf{a}$ and $\mathbf{b}$ are dependent here on the frequency ω.

From Equation 21 it follows immediately that

$$\ddot{\mathbf{U}} = -\omega^2 \mathbf{q} e^{i\omega t} = -\omega^2 \mathbf{U} \tag{23}$$

Substituting therefore Equations 21 and 23 into Equation 17 with its right-hand side equal to zero (no external loading present) and then cancelling the exponential factor $e^{i\omega t}$, one obtains

$$(-\omega^2 \mathbf{M} + \mathbf{K})\mathbf{q} = 0 \tag{24}$$

Equation 24 has nonzero solution for $\mathbf{q}$ provided that the determinant formed by the coefficients of $(-\omega^2 \mathbf{M} + \mathbf{K})$ is equal to zero, i.e.

$$|-\omega^2 \mathbf{M} + \mathbf{K}| = 0 \tag{25}$$

It should, however, be noted that both $\mathbf{M}$ and $\mathbf{K}$ are now dependent on the frequency ω. Thus no direct eigenvalue solution is possible for Equation 24, but this situation can be remedied if the matrix $\mathbf{a}$ is expanded into an infinite series in ascending powers of ω. It will be assumed that

$$\mathbf{a} = \sum_{r=0}^{\infty} \omega^r \mathbf{a}_r \tag{26}$$

such that $\mathbf{a}_r = 0$ for $r \geq 1$ at the points where $\mathbf{U}$ displacements are specified; thus $\mathbf{a}_0$ will represent static displacements due to unit values of $\mathbf{U}$. It follows therefore that the matrix $\mathbf{b}$ can also be expanded into a similar series, i.e.

$$\mathbf{b} = \sum_{r=0}^{\infty} \omega^r \mathbf{b}_r \tag{27}$$

It will be demonstrated later that when the series 26 and 27 are derived and substituted into Equations 18 and 19, then the mass and stiffness matrices are of the form

$$\mathbf{M} = \mathbf{M}_0 + \omega^2 \mathbf{M}_2 + \cdots \tag{28}$$

and

$$\mathbf{K} = \mathbf{K}_0 + \omega^4 \mathbf{K}_4 + \cdots \tag{29}$$

$\mathbf{M}_0$ is the mass matrix based on static displacements while $\mathbf{M}_2$ is the first term in the expansion for $\mathbf{M}$ depending on ω. Similarly, $\mathbf{K}_0$ is the stiffness matrix based on static displacements and $\mathbf{K}_4$ is the first frequency dependent term in $\mathbf{K}$. Substitution of Equations 28 and 29 into Equations 24 and 25 leads to the equation of motion

$$(\mathbf{K}_0 - \omega^2 \mathbf{M}_0 - \omega^4(\mathbf{M}_2 - \mathbf{K}_4) - \cdots)\mathbf{q} = 0 \tag{30}$$

and the characteristic equation

$$|\mathbf{K}_0 - \omega^2 \mathbf{M}_0 - \omega^4(\mathbf{M}_2 - \mathbf{K}_4) - \cdots| = 0 \tag{31}$$

For practical calculations, it appears that there is no need to go beyond the term with ω^4 and consequently only quadratic equations of the form

$$(\mathbf{A} - \omega^2 \mathbf{B} - \omega^4 \mathbf{C})\mathbf{q} = 0 \tag{30a}$$

and the characteristic determinants

$$| \mathbf{A} - \omega^2 \mathbf{B} - \omega^4 \mathbf{C} | = 0 \tag{31a}$$

will be considered in subsequent analysis.

The expansions of **M** and **K** for a single structural element lend themselves to the standard congruent transformation procedures for assembling the total mass and stiffness matrices. If there are any actual concentrated masses present, these are simply added to the corresponding diagonal terms in $\mathbf{M}_O$.

EQUIVALENT MASS AND STIFFNESS MATRICES FOR BAR ELEMENTS

The equation of motion for a bar element (see Figure 3) is given by

$$c^2 \frac{\partial^2 u_x}{\partial x^2} - \ddot{u}_x = 0 \tag{32}$$

where

$$c^2 = E/\rho \tag{33}$$

and E is the Young's modulus. Assuming the solution of Equation 32 to be of the form

$$u_x = \mathbf{a}\mathbf{U} = \mathbf{a}\mathbf{q}\, e^{i\omega t} \tag{34}$$

with

$$\mathbf{U} = \left\{ U_1 \ U_2 \right\} = \left\{ q_1 \ q_2 \right\} e^{i\omega t} \tag{35}$$

it can be demonstrated that

$$\mathbf{a} = \left[(\cos \omega x/c - \cot \omega \ell/c \ \sin \omega x/c)(\operatorname{cosec} \omega \ell/c \ \sin \omega x/c) \right] \tag{36}$$

where ℓ is the length of the bar element. The strain e for this case is determined from

$$e = \frac{du_x}{dx} = \frac{d\mathbf{a}}{dx} \mathbf{U} \tag{37}$$

Hence

$$\begin{aligned} \mathbf{b} &= d\mathbf{a}/dx \\ &= \left[-\frac{\omega}{c}(\sin \omega x/c + \cot \omega \ell/c \ \cos \omega x/c) \frac{\omega}{c}(\operatorname{cosec} \omega \ell/c \ \cos \omega x/c) \right] \end{aligned} \tag{38}$$

Substituting Equations 36 and 38 into Equations 18 and 19 and then integrating over the whole volume of the element one obtains

$$\mathbf{M} = \rho \frac{A\ell}{2} \frac{c}{\omega\ell} \operatorname{cosec} \frac{\omega\ell}{c} \begin{bmatrix} (\frac{\omega\ell}{c} \operatorname{cosec} \frac{\omega\ell}{c} - \cos \omega \frac{\ell}{c})(1 - \frac{\omega\ell}{c} \cot \omega \frac{\ell}{c}) \\ (1 - \frac{\omega\ell}{c} \cot \frac{\omega\ell}{c})(\frac{\omega\ell}{c} \operatorname{cosec} \frac{\omega\ell}{c} - \cos \omega \frac{\ell}{c}) \end{bmatrix} \quad (39)$$

$$\mathbf{K} = \frac{AE}{2\ell} \frac{\omega\ell}{c} \operatorname{cosec} \frac{\omega\ell}{c} \begin{bmatrix} (\frac{\omega\ell}{c} \operatorname{cosec} \frac{\omega\ell}{c} + \cos \frac{\omega\ell}{c}) - (1 + \frac{\omega\ell}{c} \cot \omega \frac{\ell}{c}) \\ -(1 + \frac{\omega\ell}{c} \cot \frac{\omega\ell}{c})(\frac{\omega\ell}{c} \operatorname{cosec} \frac{\omega\ell}{c} + \cos \omega \frac{\ell}{c}) \end{bmatrix} \quad (40)$$

where the symbol A has been introduced to represent the cross-sectional area of the bar element. The coefficients in matrices **M** and **K** are here functions of the circular frequency ω. Since the frequency is unknown initially these matrices cannot be evaluated numerically. Each coefficient in Equations 39 and 40 could be expanded into a series in ascending powers of ω in order to formulate matrix expressions of the form given by Equations 28 and 29. Although this method has been tried out, it is not recommended since the formal way described below is more expedient.

Using Equation 26 the displacements u_x in a vibrating bar element are given by

$$u_x = \left(\sum_{r=0}^{\infty} \omega^r \mathbf{a}_r\right) \mathbf{q}\, e^{i\omega t} \quad (41)$$

Substituting Equation 41 into equation of motion 32

$$c^2\left(\sum_{r=0}^{\infty} \omega^r \mathbf{a}_r''\right) \mathbf{q} e^{i\omega t} + \omega^2\left(\sum_{r=0}^{\infty} \omega^r \mathbf{a}_r\right) \mathbf{q} e^{i\omega t} = 0 \quad (42)$$

where primes denote differentiation with respect to x. Equating to zero coefficients of the same powers of ω in Equation 42, the following equations are obtained:

$$\mathbf{a}_0'' = 0 \quad (43)$$

$$\mathbf{a}_1'' = 0 \quad (44)$$

$$c^2 \mathbf{a}_2'' = -\mathbf{a}_0 \quad (45)$$

$$c^2 \mathbf{a}_3'' = -\mathbf{a}_1 \quad \text{etc} \quad (46)$$

Equations 43 through 45 can be integrated directly. Only the first matrix term $\mathbf{a}_0$ is used to satisfy the boundary conditions that $u_x = U_1$ at $x = 0$ and $u_x = U_2$ at $x = \ell$, while the remaining terms $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ . . . must all vanish at $x = 0$ and ℓ. Hence

$$a_0 = \left[(1-\xi)\ \xi\right];\ \xi = x/\ell \tag{47}$$

$$a_1 = 0 \tag{48}$$

$$a_2 = \rho\frac{\ell^2}{6E}\left[(2\xi - 3\xi^2 + \xi^3)(\xi - \xi^3)\right] \tag{49}$$

$$a_3 = 0 \text{ etc} \tag{50}$$

The matrix a_0 represents the static displacement distribution due to unit values of the bar and displacements U_1 and U_2.

The strains in the bar element can be calculated from

$$e = \frac{\partial u_x}{\partial x} = \left(\sum_{r=0}^{\infty} \omega^r a_r'\right) q e^{i\omega t} = \left(\sum_{r=0}^{\infty} \omega^r b_r\right) q e^{i\omega t} \tag{51}$$

Hence

$$b_0 = \frac{da_0}{dx} = \frac{1}{\ell}\left[-1\ \ 1\right] \tag{52}$$

$$b_1 = \frac{da_1}{dx} = 0 \tag{53}$$

$$b_2 = \frac{da_2}{dx} = \rho\frac{\ell}{6E}\left[(2-6\xi+3\xi^2)(1 - 3\xi^2)\right] \tag{54}$$

$$b_3 = \frac{da_3}{dx} = 0 \text{ etc} \tag{55}$$

Using Equations 18, 26 and Equations 47 through 50 it can be shown that

$$M_0 = \rho\frac{A\ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{56}$$

and

$$M_2 = 2\rho^2\frac{A\ell^3}{45E}\begin{bmatrix} 1 & 7/8 \\ 7/8 & 1 \end{bmatrix} \tag{57}$$

Similarly, Equations 19, 27 and Equations 52 through 55 lead to

$$K_0 = \frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{58}$$

$$K_4 = \rho^2\frac{A\ell^3}{45E}\begin{bmatrix} 1 & 7/8 \\ 7/8 & 1 \end{bmatrix} \tag{59}$$

The matrices $\mathbf{M}_0$ and $\mathbf{K}_0$ can be recognized as the equivalent mass and stiffness derived from the static displacement distribution.

EQUIVALENT MASS AND STIFFNESS MATRICES FOR BEAM ELEMENTS

The equation of motion of a beam element (see Figure 4) in the transverse direction is given by

$$c^4 \frac{\partial^4 u_y}{\partial x^4} + \ddot{u}_y = 0 \tag{60}$$

where

$$c^4 = EI/\rho A \tag{61}$$

and I is the moment of inertia of the beam cross-section. For simplicity of presentation shear deformations will be neglected, but if required these can be accounted for without any special difficulties. In addition to the transverse displacements u_y the beam element undergoing transverse vibrations will have displacements u_x which in accordance with Engineering Bending theory can be calculated from

$$u_x = -\frac{\partial u_y}{\partial x} y = -\ell \frac{\partial u_y}{\partial x} \eta; \quad \eta = y/\ell \tag{62}$$

As before the displacements $\mathbf{u}$ are expanded in ascending powers of ω so that

$$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \left(\begin{bmatrix} a_{0x} \\ a_{0y} \end{bmatrix} + \omega \begin{bmatrix} a_{1x} \\ a_{1y} \end{bmatrix} + \omega^2 \begin{bmatrix} a_{2x} \\ a_{2y} \end{bmatrix} + \cdots \right) \mathbf{U}$$
$$= (\mathbf{a}_0 + \omega \mathbf{a}_1 + \omega^2 \mathbf{a}_2 + \cdots)\, \mathbf{U} = \mathbf{a} \mathbf{q} e^{i\omega t} \tag{63}$$

where

$$\mathbf{U} = \{U_1 \; U_2 \; U_3 \; U_4\} = \{q_1 \; q_2 \; q_3 \; q_4\}\, e^{i\omega t} \tag{64}$$

Hence

$$u_x = \left(\sum_{r=0}^{\infty} \omega^r a_{rx}\right) q e^{i\omega t} = a_x\, q\, e^{i\omega t} \tag{65}$$

$$u_y = \left(\sum_{r=0}^{\infty} \omega^r a_{ry}\right) q e^{i\omega t} = a_y\, q\, e^{i\omega t} \tag{66}$$

Substituting Equation 66 into Equation of motion 60

$$c^4 \left(\sum_{r=0}^{\infty} \omega^r a_{ry}^{iv}\right) q e^{i\omega t} - \omega^2 \left(\sum_{r=0}^{\infty} \omega^r a_{ry}\right) q e^{i\omega t} = 0 \tag{67}$$

Equating to zero coefficients of the same powers of ω in Equation 67 the following equations are obtained:

$$a_{oy}^{IV} = 0 \tag{68}$$

$$a_{1y}^{IV} = 0 \tag{69}$$

$$c^4 a_{2y}^{IV} = a_{oy} \tag{70}$$

$$c^4 a_{3y}^{IV} = a_{1y} \cdots \text{etc} \tag{71}$$

Solving Equations 68 through 71 it can be demonstrated that

$$a_{oy} = \left[(1-3\xi^2 + 2\xi^3)(\xi - 2\xi^2 + \xi^3)\ell(3\xi^2 - 2\xi^3)(-\xi^2 + \xi^3)\ell \right] \tag{72}$$

$$a_{1y} = 0 \tag{73}$$

$$a_{2y} = \frac{\rho A \ell^4}{2520 EI}\left[(66\xi^2 - 156\xi^3 + 105\xi^4 - 21\xi^6 + 6\xi^7)(12\xi^2 - 22\xi^3 + 21\xi^5 - 14\xi^6 + 3\xi^7)\ell \right.$$
$$\left. (39\xi^2 - 54\xi^3 + 21\xi^6 - 6\xi^7)(-9\xi^2 + 13\xi^3 - 7\xi^6 + 3\xi^7)\ell \right] \tag{74}$$

$$a_{3y} = 0 \cdots \text{etc} \tag{75}$$

The matrix a_{oy} represents static transverse deflection distribution due to unit values of $U_1 \ldots . U_4$. The remaining matrices in a are determined from Equation 62. This gives

$$a_{ox} = \left[6(\xi - \xi^2)\eta \; (-1 + 4\xi - 3\xi^2)\ell\eta \; 6(-\xi + \xi^2)\eta \; (2\xi - 3\xi^2)\ell\eta \right] \tag{76}$$

$$a_{1x} = 0 \tag{77}$$

$$a_{2x} = \frac{\rho A \ell^4}{2520 EI}\left[(-132\xi + 468\xi^2 - 420\xi^3 + 126\xi^5 - 42\xi^6)\eta \; (-24\xi + 66\xi^2 - 105\xi^4 + 84\xi^5 \right.$$
$$\left. -21\xi^6)\ell\eta \; (-78\xi + 162\xi^2 - 126\xi^5 + 42\xi^6)\eta \; (18\xi - 39\xi^2 + 42\xi^5 - 21\xi^6)\ell\eta \right] \tag{78}$$

$$a_{3x} = 0 \cdots \text{etc} \tag{79}$$

The strains in the beam element are derived from

$$e = \frac{\partial u_x}{\partial x} = -\frac{\partial^2 u_y}{\partial x^2} y = -\ell \frac{\partial^2 u_y}{\partial x^2} \eta$$
$$= (b_0 + \omega b_1 + \omega^2 b_2 + \cdots) U = bU \tag{80}$$

Hence using Equations 72 through 75 and Equation 80 it follows that

$$b_0 = -\ell\eta a''_{0y}$$

$$= -\frac{\eta}{\ell}\left[(-6+12\xi)(-4+6\xi)\,\ell(6-12\xi)(-2+6\xi)\,\ell\right] \tag{81}$$

$$b_1 = -\ell\eta a''_{1y} = 0 \tag{82}$$

$$b_2 = -\ell\eta a''_{2y} \tag{83}$$

$$= \frac{-\rho A\eta\ell^3}{420EI}\Big[(22-156\xi+210\xi^2-105\xi^4+42\xi^5)(4-22\xi+70\xi^3$$

$$-70\xi^4+21\xi^5)\,\ell(13-54\xi+105\xi^4-42\xi^5)(-3+13\xi-35\xi^4+21\xi^5)\,\ell\Big] \tag{84}$$

$$b_3 = -\ell\eta a''_{3y} = 0 \cdots \text{etc} \tag{85}$$

The next step is to substitute the calculated series for a into Equation 18 to determine the equivalent mass matrix **M**. The first two component matrices appearing in the expansion of Equation 28 have been calculated and are given by Equations 86 and 87.

$$M_0 = \frac{\rho A\ell}{420}\begin{bmatrix} 156 & & \text{Symmetric} & \\ 22\ell & 4\ell^2 & & \\ 54 & 13\ell & 156 & \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix}$$

$$+\frac{\rho A\ell}{30}\left(\frac{r}{\ell}\right)^2\begin{bmatrix} 36 & & \text{Symmetric} & \\ 3\ell & 4\ell^2 & & \\ -36 & -3\ell & 36 & \\ 3\ell & -\ell^2 & -3\ell & 4\ell^2 \end{bmatrix} \tag{86}$$

$$M_2 = \frac{(\rho A\ell)^2\ell^3}{EI}\begin{bmatrix} 0.729746 & & \text{Symmetric} & \\ 0.153233\ell & 0.0325248\ell^2 & & \\ 0.659142 & 0.144386\ell & 0.729746 & \\ -0.144386\ell & -0.0314082\ell^2 & -0.153233\ell & 0.0325248\ell^2 \end{bmatrix}\times 10^{-3}$$

$$+\frac{(\rho A\ell)^2\ell^3}{EI}\left(\frac{r}{\ell}\right)^2\begin{bmatrix} 0.317460 & & \text{Symmetric} & \\ 0.793651\ell & 0.317460\ell^2 & & \\ -0.317460 & 0.595238\ell & 0.317460 & \\ -0.595238\ell & -0.277778\ell^2 & -0.793651\ell & 0.317460\ell^2 \end{bmatrix}\times 10^{-3} \tag{87}$$

where r is the radius of gyration of the beam cross-section. The first matrix term in Equations 86 and 87 represents the translational inertia of the beam element while the second term represents the rotatory inertia effects. The stiffness matrix is determined from the calculated series for b and Equation 19. The first two component matrices appearing in the expansion of Equation 29 are given by Equations 88 and 89.

$$K_0 = \frac{EI}{\ell^3}\begin{bmatrix} 12 & & \text{Symmetric} & \\ 6\ell & 4\ell & & \\ -12 & -6\ell & 12 & \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix} \qquad (88)$$

$$K_4 = (\rho A \ell)^2 \frac{\ell^3}{EI}\begin{bmatrix} 0.364872 & & \text{Symmetric} & \\ 0.0766162\ell & 0.0162624\ell^2 & & \\ 0.329571 & 0.0721933\ell & 0.364872 & \\ -0.0721933\ell & -0.0157041\ell^2 & -0.0766162\ell & 0.0162624\ell^2 \end{bmatrix} \times 10^{-3} \qquad (89)$$

The matrices $\mathbf{M}_0$ and $\mathbf{K}_0$ represent the equivalent mass and stiffness based on the static displacement distribution in a beam element (References 2, 3, 4, and 5).

EIGENVALUES AND EIGENVECTORS OF THE QUADRATIC MATRIX EQUATION

Iterative Solution (Method i)

The quadratic matrix equation

$$(\mathbf{A} - \omega^2 \mathbf{B} - \omega^4 \mathbf{C})\, \mathbf{q} = 0 \qquad (30a)$$

will be assumed to have eigenvectors $\mathbf{p}_1, \mathbf{p}_2 \ldots \mathbf{p}_n$ corresponding to positive (including zero) eigenvalues $\omega_1, \omega_2 \ldots \omega_n$. Introducing

$$\Omega^2 = \lfloor \omega_1^2 \; \omega_2^2 \cdots \omega_n^2 \rfloor \qquad (90)$$

$$\mathbf{p} = [\mathbf{p}_1 \; \mathbf{p}_2 \cdots \mathbf{p}_n] \qquad (91)$$

Equation 30a can be written as

$$\mathbf{A}\mathbf{p} - \mathbf{B}\mathbf{p}\Omega^2 - \mathbf{C}\mathbf{p}\Omega^4 = 0 \qquad (92)$$

Postmultiplying Equation 92 by $\mathbf{p}^{-1}$ one obtains

$$\mathbf{A} - \mathbf{B}\mathbf{p}\Omega^2\mathbf{p}^{-1} - \mathbf{C}\mathbf{p}\Omega^4\mathbf{p}^{-1} = 0 \qquad (93)$$

Introducing new matrix

$$\mathbf{E} = p\,\Omega^2 p^{-1} \tag{94}$$

Equation 93 can be transformed into

$$\mathbf{E}^2 = C^{-1}A - C^{-1}B\mathbf{E}$$

This latter equation is then used for an iterative solution. By letting

$$\mathbf{E} = \mathbf{E}_0 + \Delta\mathbf{E} \tag{96}$$

where

$$\mathbf{E}_0 = p_0\,\Omega_0^2\,p_0^{-1} \tag{97}$$

is obtained from the conventional solution eigenvectors p_0 and eigenvalues Ω_0^2 satisfying

$$A p_0 - B p_0 \Omega_0^2 = 0 \tag{98}$$

Equation 95 becomes

$$(\mathbf{E}_0 + \Delta\mathbf{E})\mathbf{E} = C^{-1}A - C^{-1}B\mathbf{E} \tag{99}$$

Hence

$$\mathbf{E} \cong (\mathbf{E}_0 + C^{-1}B)^{-1} C^{-1}A \tag{100}$$

The iterative loop is then established from

$$\mathbf{E}_{r+1} \cong (\mathbf{E}_r + C^{-1}B)^{-1} C^{-1}A$$

and once satisfactory convergence has been achieved for E the quadratic equation eigenvalues and eigenvectors are determined from the conventional eigenvalue equation

$$\mathbf{E}p - p\,\Omega^2 = 0 \tag{101}$$

or

$$(\mathbf{E} - \omega^2 I)q = 0 \tag{102}$$

The above method has been programmed for the IBM 7094 computer. The computer program was prepared by Mr. F.O. Young of the Digital Computation Division, Wright-Patterson AFB, Ohio, and was successfully used for the illustrative problems discussed in the subsequent section. One drawback of this method is that occasionally, since the quadratic equation has both positive and negative eigenvalues, the iteration procedure will produce some negative eigenvalues in place of a few highest eigenvalues (frequencies). If these frequencies are required alternative methods must be used for these cases. This usually occurs when E_0 differs greatly from the final value of $\mathbf{E}$.

Direct Eigenvalue Solution (Reference 6) (Method ii)

Introducing

$$\dot{q} = \omega^2 q \tag{103}$$

Equation 30a can be manipulated into

$$C^{-1} A q - C^{-1} B \dot{q} - \omega^2 \dot{q} = 0 . \tag{104}$$

Equations 103 and 104 can now be combined into one matrix equation

$$\begin{bmatrix} -\omega^2 I & I \\ C^{-1} A & -(C^{-1} B + \omega^2 I) \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = 0 \tag{105}$$

or

$$\left(\begin{bmatrix} 0 & I_n \\ C^{-1} A & -C^{-1} B \end{bmatrix} - \omega^2 I_{2n} \right) \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = 0 \tag{105a}$$

Since Equation 105a is of a standard form, available eigenvalue and eigenvector programs can be used. The only disadvantage of this method is that it involves 2n unknown components in each eigenvector as compared with n components in the original idealized system.

Direct Evaluation of the Determinant (Method iii)

If necessary the quadratic equation eigenvalues can also be found numerically from the solution of the determinant

$$| A - \omega^2 B - \omega^4 C | = 0 \tag{106}$$

NUMERICAL RESULTS

In order to demonstrate the considerable improvement in accuracy of the vibration modes and frequencies obtained from the quadratic equations, three examples have been considered for which exact solutions are readily available. The three examples are: longitudinal vibrations of free-free bar, and fixed-free bar, and transverse vibrations of cantilever beam.

Ratios of frequencies of vibration of a free-free bar determined from the quadratic equation over the exact frequencies are shown in Table 1, for number of elements varying from 1 to 10. For comparison, the corresponding ratios obtained from the conventional analysis are also presented. This table indicates clearly that considerable improvement in accuracy is obtained when the quadratic equations are used. In Figure 5 the percentage errors are plotted against the number of elements for the 1st and 5th modes in order to show more clearly the general trends. A perusal of Table 1 and Figure 5 reveals that the percentage error in frequencies is reduced by almost an order of magnitude when quadratic equations are used instead of the conventional eigenvalue equations. Furthermore, the rate of decrease in percentage error when the number of elements is increased is considerably greater for quadratic equation solutions; thus, the convergence to the true frequency values is much faster with quadratic equations.

Tables 2 and 3 and also Figures 6 and 7 present the results of computations for a fixed-free bar and a cantilever beam. The general conclusions are also valid, although the improvement in accuracy in the case of a cantilever beam is not as dramatic as for the free-free and fixed-free bars. As example of the improvement in the calculated mode shapes 1st and 2nd modes for a fixed-free bar are plotted in Figure 8.

All calculated values in Tables 1, 2, and 3 have been obtained from the iterative solution (method i) of the quadratic equations with the exception of the results marked by an asterisk. For these cases the solution converged to negative values instead of high positive eigenvalues (frequencies), and a direct solution (method ii) had to be used.

It has been demonstrated that the frequencies and mode shapes calculated from the quadratic equation $(\mathbf{K}_0 - \omega^2 \mathbf{M}_0 - \omega^4(\mathbf{M}_2 - \mathbf{K}_4))\,\mathbf{q} = \mathbf{O}$ are considerably more accurate than the conventional eigenvalue solutions obtained from $(\mathbf{K}_0 - \omega^2 \mathbf{M}_0)\,\mathbf{q}_0 = \mathbf{O}$. This new method appears to be particularly advantageous for longitudinal vibrations of bars when approximately the same accuracy is achieved with n degrees of freedom in the quadratic equation as with 2n degrees in the conventional equation. This improvement in accuracy is achieved, of course, at the expense of more complicated matrix operations necessary to compute the solution of the quadratic equation; however, this disadvantage is largely offset by the reduced number of specified displacements and discrete elements in the idealized system.

REFERENCES

1. Przemieniecki, J.S., Generalization of the Unit Displacement Theorem with Applications to Dynamics, Air Force Conference on Matrix Methods of Structural Analysis, Wright-Patterson AFB, Ohio, 27-28 October 1964.

2. Archer, J.S., Consistent Mass Matrix for Distributed Mass Systems, Space Technology Laboratories, Inc., Redondo Beach, Calif., Report EM 13-4, February 1963.

3. Archer, J.S., Consistent Matrix Formulations for Structural Analysis Using Influence Coefficient Technique, Space Technology Laboratories, Inc., Redondo Beach, Calif., Report EM 13-24, November 1963; and AIAA Paper No. 64-488 presented at the First AIAA Annual Meeting, 29 June to 2 July 1964.

4. McCalley, R.B., Mass Lumping for Beams, Report DIG/SA 63-68, General Electric Co., Knolls Atomic Power Laboratory, Schenectady, N.Y., 1 July 1963.

5. McCalley, R.B., Rotary Inertia Correction for Mass Matrices, Report DIG/SA 63-73, General Electric Company, Knolls Atomic Power Laboratory, Schenectady, N.Y., 9 July 1963.

6. Buckingham, R.A., Numerical Methods, Sir Isaac Pitman and Sons, Ltd., London, pp. 387-389, 1957.

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NUMERICAL RESULTS

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REFERENCES

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2. Archer, J.S., Consistent Mass Matrix for Distributed Mass Systems, Space Technology Laboratories, Inc., Redondo Beach, Calif., Report EM 13-4, February 1963.

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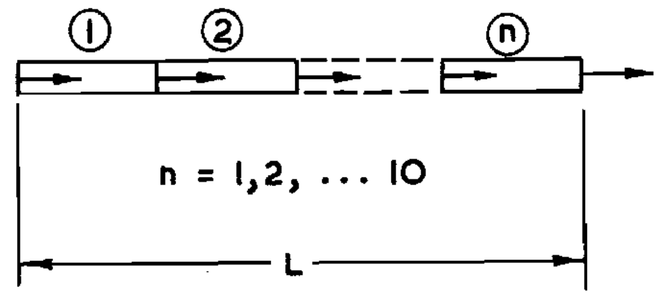
6. Buckingham, R.A., Numerical Methods, Sir Isaac Pitman and Sons, Ltd., London, pp. 387-389, 1957.

TABLE 1

Ratios of ω/ω_{exact} for Longitudinal Vibrations of a Free-Free Bar
(Numbers in parenthesis represent values obtained from conventional analysis)

n	FREQUENCY NUMBER *									
	1	2	3	4	5	6	7	8	9	10
1	1.019 (1.103)									
2	1.019 (1.103)	1.019 (1.103)								
3	1.004 (1.046)	1.050 (1.170)	1.019 (1.103)							
4	1.002 (1.026)	1.019 (1.103)	1.069 (1.195)	1.019 (1.103)						
5	1.001 (1.017)	1.009 (1.066)	1.036 (1.143)	1.079 (1.201)	1.019 (1.103)					
6	1.000 (1.012)	1.004 (1.046)	1.019 (1.103)	1.050 (1.170)	1.083 (1.200)	1.019 (1.103)				
7	1.000 (1.008)	1.002 (1.034)	1.011 (1.076)	1.030 (1.132)	1.061 (1.186)	1.084 (1.196)	1.019 (1.103)			
8	1.000 (1.006)	1.002 (1.026)	1.007 (1.058)	1.019 (1.103)	1.041 (1.154)	1.069 (1.195)	1.083 (1.192)	1.019 (1.103)		
9	1.000 (1.005)	1.001 (1.020)	1.004 (1.046)	1.013 (1.082)	1.027 (1.125)	1.050 (1.170)	1.075 (1.199)	1.081 (1.186)	1.019 (1.103)	
10	1.000 (1.004)	1.001 (1.017)	1.003 (1.037)	1.009 (1.066)	1.019 (1.103)	1.036 (1.143)	1.057 (1.181)	1.079 (1.201)	1.079 (1.182)	1.019 (1.103)



*Rigid body (zero) frequency is not counted in this sequence

TABLE 2

Ratios of ω/ω exact for Longitudinal Vibrations of a Fixed-Free Bar
(Numbers in parenthesis represent values obtained from conventional analysis)

n	FREQUENCY NUMBER 1	2	3	4	5	6	7	8	9	10
1	1.019 (1.103)									
2	1.002 (1.026)	1.069) (1.195)								
3	1.000 (1.012)	1.019 (1.103)	1.083 (1.200)							
4	1.000 (1.006)	1.007 (1.058)	1.041 (1.154)	1.083 (1.191)						
5	1.000 (1.004)	1.003 (1.037)	1.019 (1.103)	1.057 (1.181)	1.079 (1.182)					
6	1.000 (1.003)	1.002 (1.026)	1.010 (1.072)	1.032 (1.137)	1.069 (1.195)	1.075 (1.173)				
7	1.000 (1.002)	1.001 (1.019)	1.006 (1.053)	1.019 (1.103)	1.044 (1.161)	1.076 (1.200)	1.070 (1.166)			
8	1.000 (1.002)	1.001 (1.015)	1.003 (1.041)	1.012 (1.079)	1.029 (1.128)	1.054 (1.177)	1.080 (1.201)	1.066 (1.159)		
9	1.000 (1.001)	1.000 (1.012)	1.002 (1.032)	1.008 (1.063)	1.019 (1.103)	1.038 (1.148)	1.062 (1.188)	1.083 (1.200)	1.062 (1.154)	
10	1.000 (1.001)	1.000 (1.009)	1.002 (1.026)	1.005 (1.051)	1.013 (1.084)	1.027 (1.123)	1.046 (1.163)	1.069 (1.195)	1.084 (1.198)	1.059 (1.150)

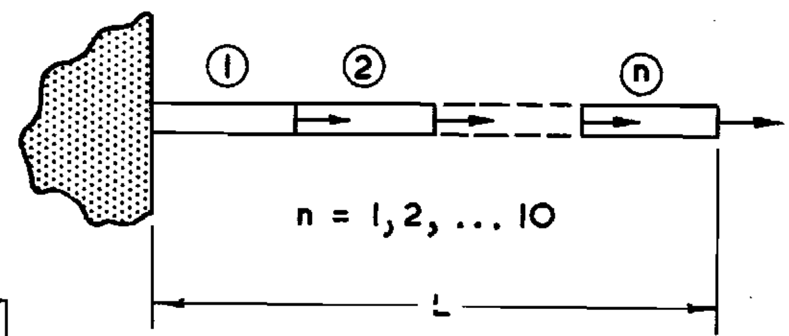
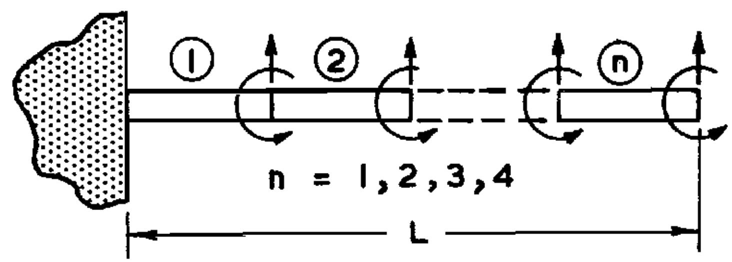


TABLE 3

Ratios of ω/ω_{exact} for Transverse Vibrations of a Straight Cantilever Beam
(Numbers in parenthesis represent values obtained from conventional analysis)

n	FREQUENCY NUMBER 1	2	3	4	5	6	7	8
1	1.000 (1.005)	1.259 (1.580)						
2	1.000 (1.001)	1.000 (1.009)	1.063 (1.218)	1.503* (1.804)				
3	1.000 (1.000)	1.000 (1.003)	1.001 (1.013)	1.039 (1.164)	1.139 (1.325)	1.486* (1.768)		
4	1.000 (1.000)	1.000 (1.001)	1.000 (1.008)	1.001 (1.015)	1.030 (1.142)	1.080 (1.227)	1.190* (1.393)	1.443* (1.717)



* frequencies obtained using direct solution (method ii)

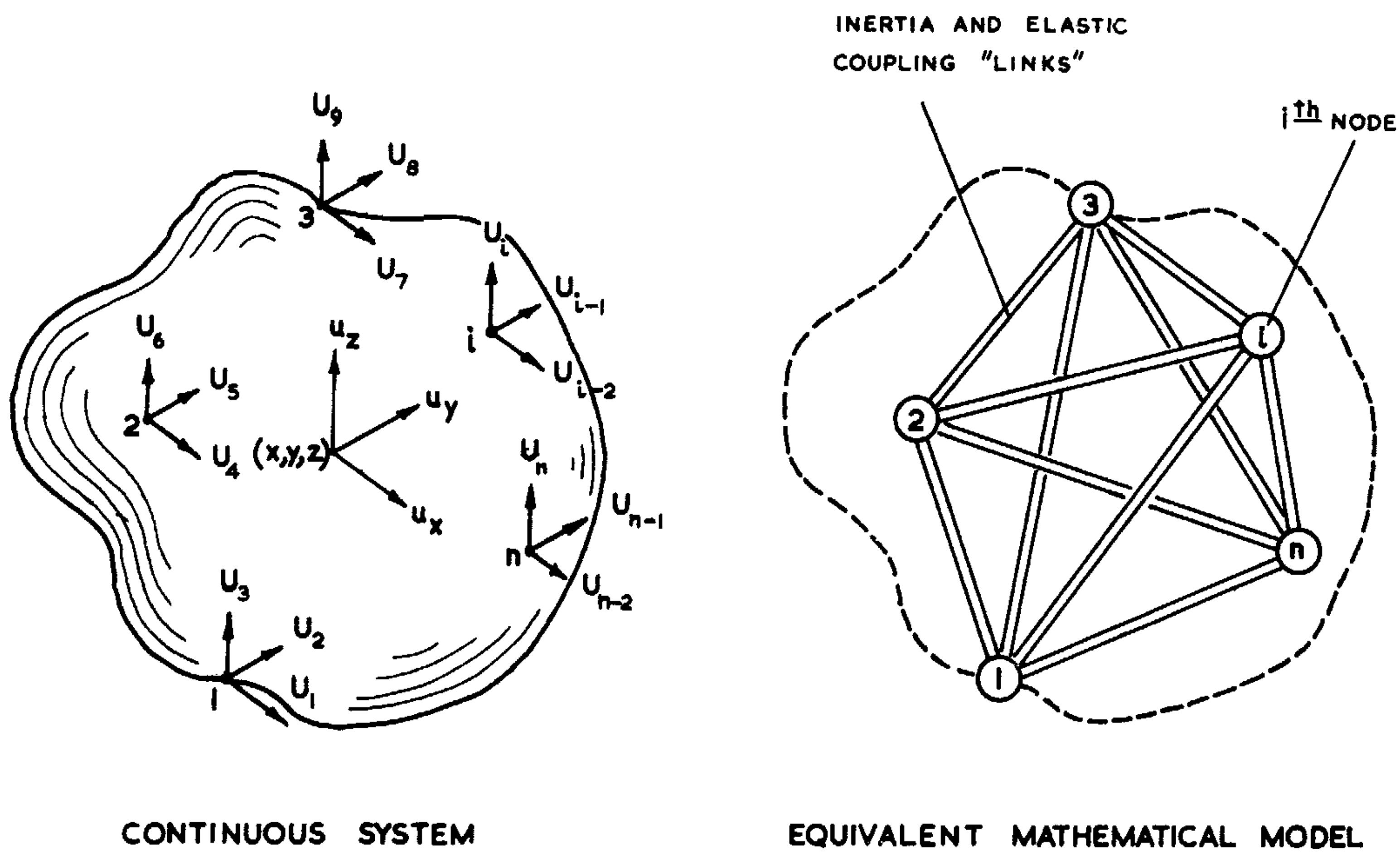


Figure 1. Mathematical Model Used in Matrix Methods of Structural Analysis

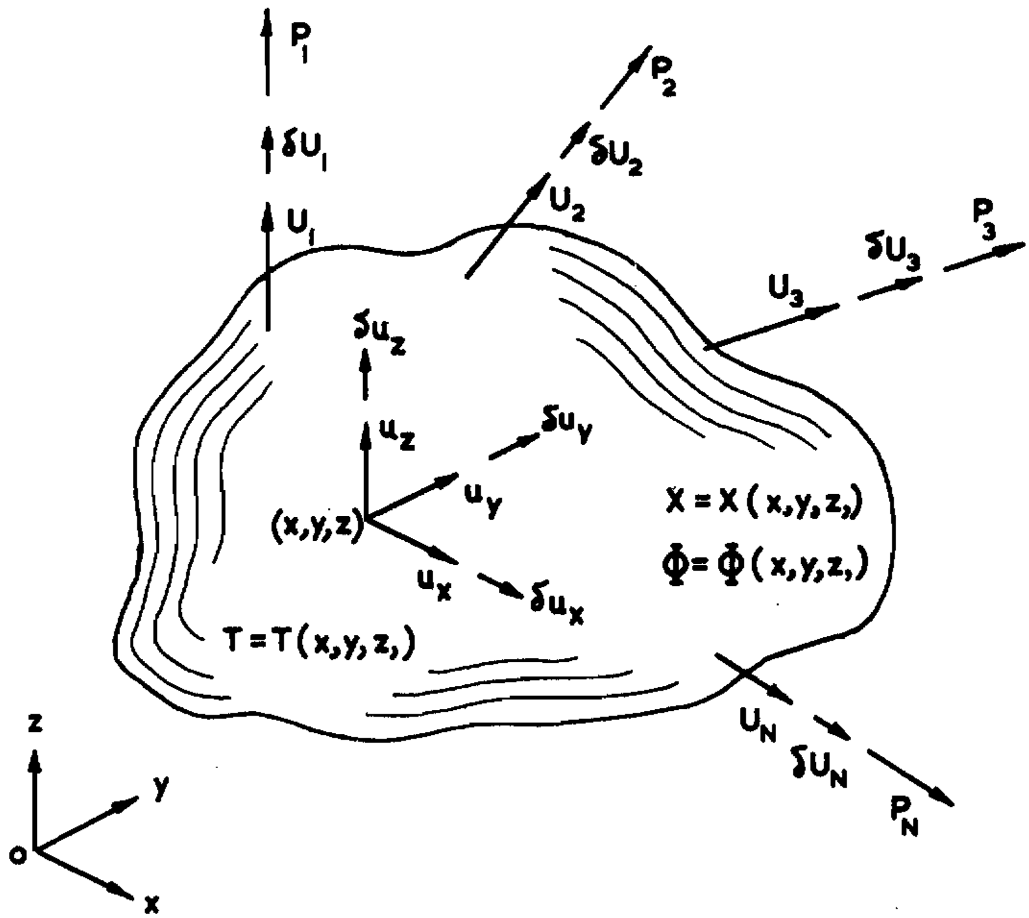


Figure 2. Virtual Displacements

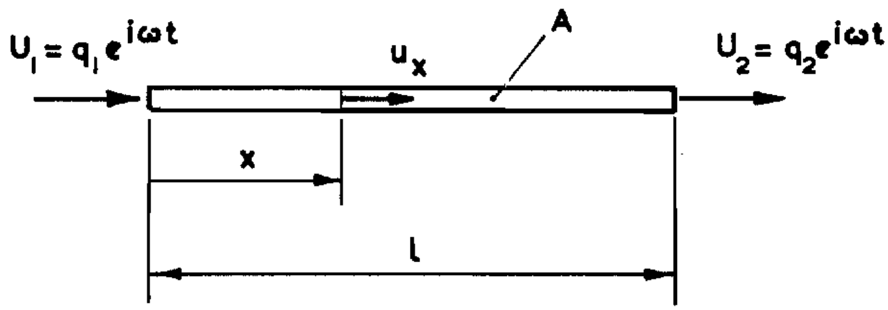


Fig. 3. Bar Element

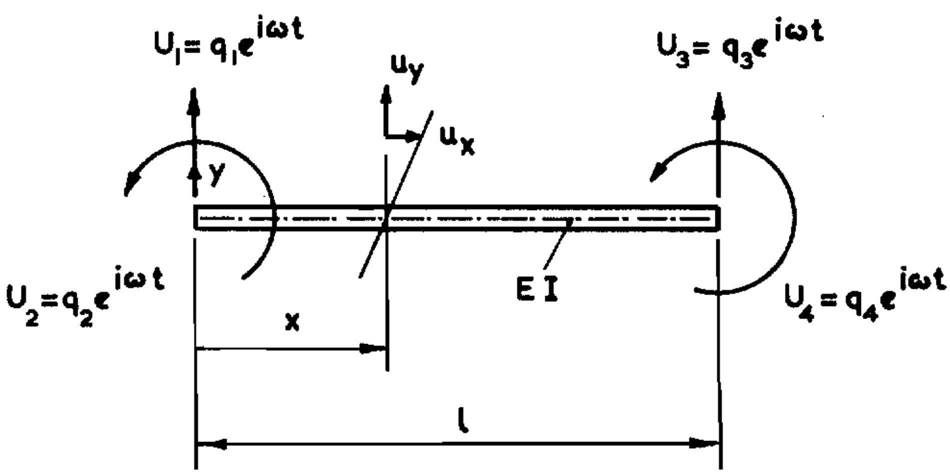


Figure 4. Beam Element

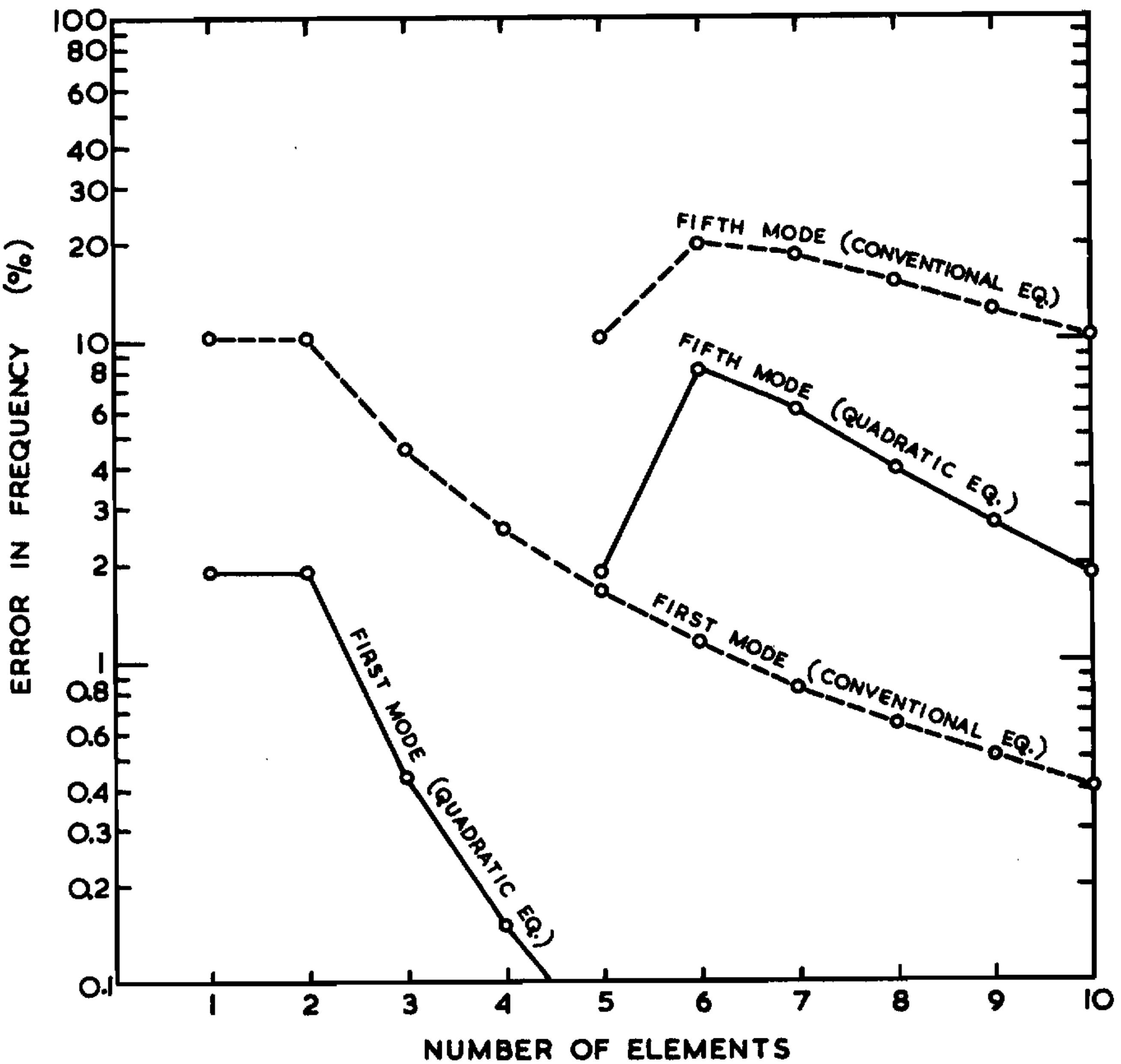


Figure 5. Variations of Percentage Error in Frequencies of a Free - Free Bar Calculated from Quadratic and Conventional Equations

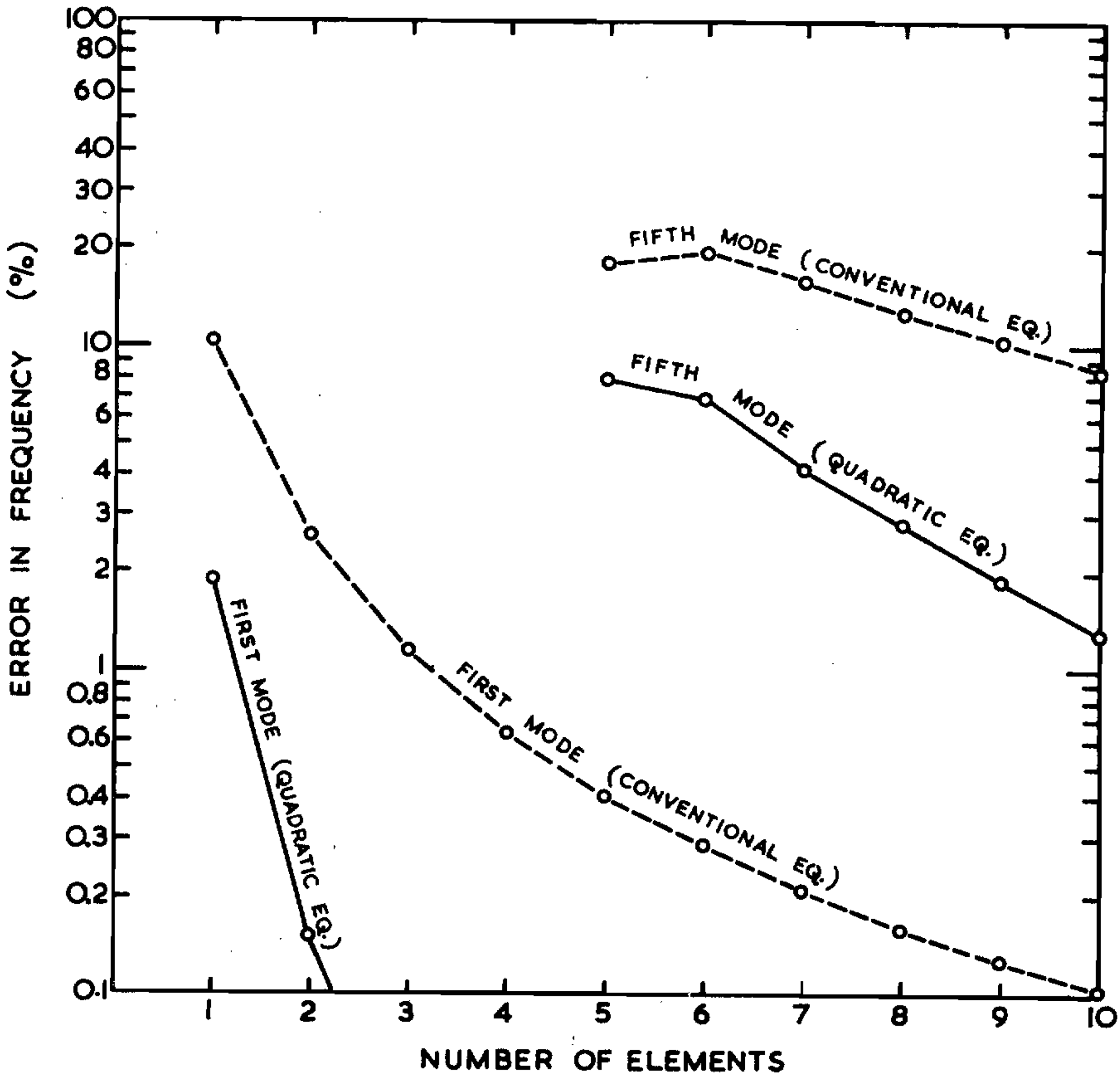


Figure 6. Variation of Percentage Error in Frequencies of a Fixed - Free Bar Calculated from Quadratic and Conventional Equations

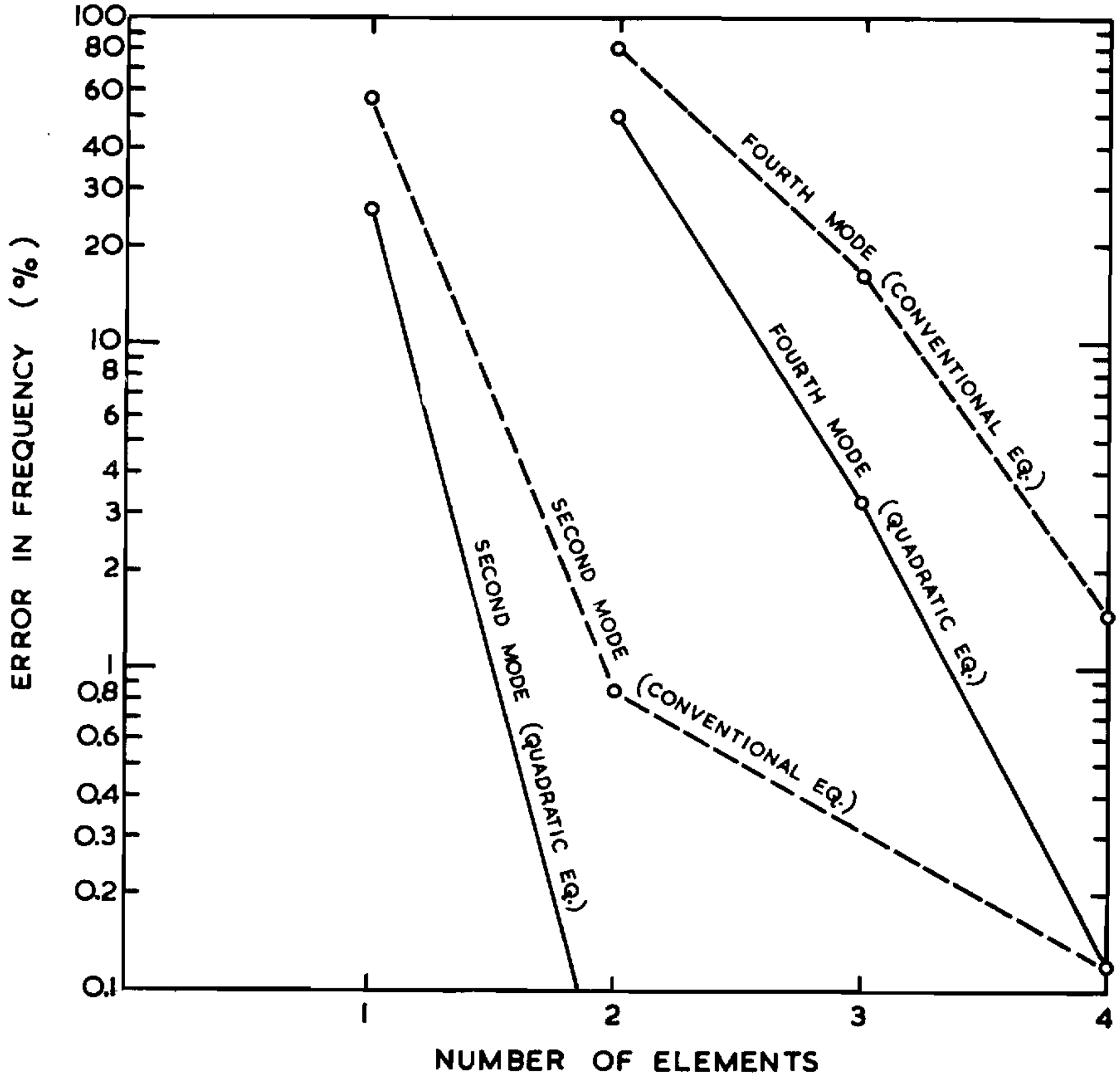


Figure 7. Variation of Percentage Error in Frequencies of a Uniform Cantilever Beam Calculated from Quadratic and Conventional Equations

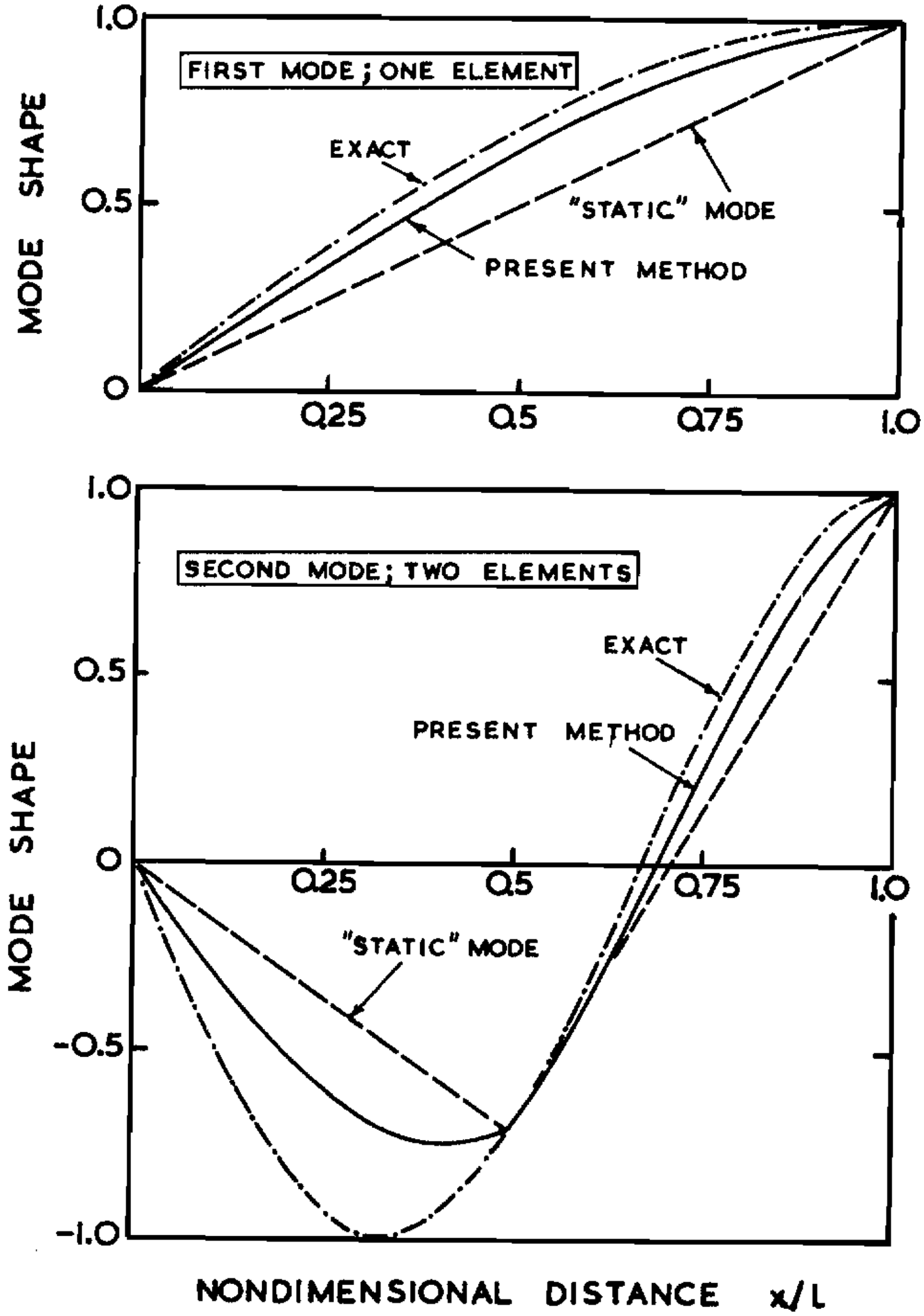


Figure 8. Typical Mode Shapes for a Fixed - Free Bar