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VIBRATION ANALYSIS OF A CANTILEVERED SQUARE PLATE BY THE STIFFNESS MATRIX METHOD

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The analytical development of the stiffness matrix of a triangular flat plate in bending is presented. The method used employs a stress assumption containing eleven constants as against six force/deflection coordinates in the supported element, the excess constants being adjusted so as to minimize a positive definite function involving edge stresses. The assumption is made so as to satisfy both the differential equations of equilibrium and compatibility. A square cantilevered plate of constant thickness is then simulated by a mesh of these triangles and vibration frequencies and locations of node lines computed. Results are compared with those of another author obtained by a classical solution employing Fourier's series.

INTRODUCTION

In recent years more and more use is being made of the stiffness matrix method for analyzing structures which is evidently due to an awakening realization that the method possesses an almost universal applicability, since with it one can analyze the most arbitrarily complicated structures and practically all types of analyses fall within the scope of its capability.

As is well known, the stiffness matrix method proceeds by simulating a given structure by a finite number of basic elements fastened together, the matrix of each element is computed and then added into an overall stiffness matrix from which the various desired results may be obtained. The key to a good representation of a structure therefore rests with having available the stiffness matrices of suitable basic elements. Elements of the bar and shear-panel type have been available for some time, (References 1 and 2) and more recently plates in bending for representing shell-type structures (Reference 3) and tetrahedrons to simulate solids (References 4 and 5) have made their appearance in the literature. This paper deals with the development and evaluation of a triangular plate in bending for thin shell-type structural analysis.

GENERAL BASIC ELEMENT ANALYSIS

A number of methods have been developed for obtaining the stiffness matrices of general basic elements (Reference 6). In most all cases an assumption is made either as regards the displacement distribution or the stress distribution within the element. Many of the methods require that certain of the matrices involved be square and nonsingular which amounts firstly, to imposing the restriction that the number of load/deflection coordinates in the developed stiffness matrix and the number of constants in the assumption be identical and secondly — the nonsingularity condition — to imposing a restriction on the assumption which can be very awkward to satisfy. More recently (References 7, 8 and 9), methods have been advanced which allow the number of constants in the assumption to exceed the number of coordinates in the desired stiffness matrix by an arbitrary amount thus giving added range to the applicability of

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the procedure. The element presented in this paper — a triangular plate in bending — is developed by use of such a method, a short outline of the method having been previously given in Reference 7.

STIFFNESS MATRIX OF TRIANGULAR PLATE IN BENDING

The load/deflection coordinate picture for this element is as shown in Figure 1. The P_i shown in this figure are transverse loads - the resultants of shearing forces in the z direction, while the double-headed arrows, — the M_i and T_i — are the resultants of clockwise bending and torsional moments along the edges of the plate, respectively.

A stress assumption - satisfying the differential equations of equilibrium (p. 229, Reference 10) and those of compatibility (pp. 7 and 230, Reference 10) — is made first. This assumption, involving 11 arbitrary constants, and which is intended to approximate the stress distribution in a small triangular portion of a thin plate in bending, is as follows:

$$
\begin{aligned}
\sigma_x &= (k_1 + k_2 x + k_3 y + k_4 x^2 + k_5 xy - k_4 y^2 + k_6 x^3/3 + k_7 x^2 y - k_6 xy^2 - k_7 y^3/3)\, z \\
\sigma_y &= (k_8 + k_9 x + k_{10} y - k_4 x^2 - k_5 xy + k_4 y^2 - k_6 x^3/3 - k_7 x^2 y + k_6 xy^2 + k_7 y^3/3)\, z \\
\sigma_z &= 0 \\
\tau_{xy} &= \Big(k_{11} + \frac{(k_3 - \nu k_{10})x}{1+\nu} + \frac{(k_9 - \nu k_2)y}{1+\nu} + k_5 x^2/2 - 2k_4 xy - k_5 y^2/2 \\
&\qquad + k_7 x^3/3 - k_6 x^2 y - k_7 xy^2 + k_6 y^3/3\Big)\, z \qquad (1) \\
\tau_{yz} &= (h^2 - z^2)\, \frac{k_3 + k_{10}}{2(1+\nu)} \\
\tau_{zx} &= (h^2 - z^2)\, \frac{k_2 + k_9}{2(1+\nu)}
\end{aligned}
$$

where ν = Poisson's ratio and the k_i are arbitrary constants.

This can be expressed in matrix form as

$$\boldsymbol{\sigma} = \mathbf{U}\mathbf{k} \qquad (2)$$

where

$$
\begin{aligned}
\boldsymbol{\sigma} &= \mathrm{col}(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}) \\
\mathbf{k} &= \mathrm{col}(k_1, k_2, \cdots k_{11})
\end{aligned}
$$

The deflections corresponding to loads T_3, M_3, P_3 are now fixed at zero so as to support the element. The remaining loads are then obtained by integrating the appropriate stresses over the edges of the plate. The formulas are as follows:

$$
\begin{aligned}
P_1 &= 2\int_0^{x_1}\int_0^{h} (-\tau_{yz})\, dx\, dz \\
M_1 &= 2\int_0^{x_1}\int_0^{h} (-\sigma_y)\, z\, dz\, dx
\end{aligned} \qquad (3)
$$

$$
\begin{aligned}
T_1 &= 2\int_0^{x_1}\int_0^{h}(\ \tau_{xy}\ )\, z\,dz\,dx \\
P_2 &= 2\int_0^{y_3}\int_0^{h}(-\tau_{zx})\,dz\,dy + 2\int_0^{x_3}\int_0^{h}(\tau_{yz})\,dz\,dx \\
M_2 &= 2\int_0^{h}\int_0^{\ell_{23}}(c_1^2\sigma_x + 2s_1c_1\tau_{xy} + s_1^2\sigma_y)\,z\,dz\,ds \\
T_2 &= 2\int_0^{h}\int_0^{\ell_{23}}(s_1c_1\sigma_x + (s_1^2 - c_1^2)\tau_{xy} - s_1c_1\sigma_y)\,z\,dz\,ds
\end{aligned}
\tag{3}
$$

where h = 1/2 thickness of the plate (assumed constant) and $c_i = \cos\alpha_i$ and $s_i = \sin\alpha$ with the α_i as shown in Figure 2. Integration along the side 2, 3, of length ℓ_{23}, is indicated by ds.

Inserting the assumed expressions for the stresses (Equations 1), and carrying out the integrations, gives in matrix form:

$$\bar{p} = V k \tag{4}$$

where

$$\bar{p} = \mathrm{col}\,(P_1, M_1, T_1, P_2, M_2, T_2)$$

the matrix **V**, defined by the above, is too large and complicated to be shown in this paper.

Let m be the order of the stiffness matrix of the supported element, (in this case m = 6) and let r be the number (=11) of constants in the stress assumption, there are therefore 5 excess constants. For any given loading these excess constants are adjusted so as to minimize the function

$$\overline{W} = \frac{1}{2}\int_S \bar{\sigma}^T\bar{\sigma}\, dS \tag{5}$$

where $\boldsymbol{\sigma} = \mathrm{col}\{\sigma_n, \tau_n\}$ and σ_n, τ_n are the normal and tangential shearing stresses on the edges of the plate, $\oint_S$ indicating integration over the plate edges. Let the matrix T transform from x, y, z coordinates to surface coordinates. Then

$$\bar{\sigma} = \mathbf{T}\boldsymbol{\sigma} \tag{6}$$

where, for this case - for side i -

$$\mathbf{T}_i = \begin{bmatrix} c_i^2 & s_i^2 & 0 & 2c_is_i & 0 & 0 \\ s_ic_i & -s_ic_i & 0 & (s_i^2 - c_i^2) & 0 & 0 \end{bmatrix} \tag{7}$$

Inserting Equations 2 and 6 into 5 gives

$$\overline{W} = \frac{1}{2}k^T B k \tag{8}$$

where

$$\mathbf{B} = \oint_S U^T T^T T U \, dS = \left[b_{\alpha,\beta} \right] \tag{9}$$

is an (r x r) matrix.

The minimization of Equation 8 holding the loads constant is an isoperimetric problem in the calculus of variations. (Reference 11). Hence one writes as the function to be minimized

$$\phi = \frac{1}{2} \sum_{\alpha,\beta} b_{\alpha,\beta} k_\alpha k_\beta + \sum_{\alpha,\beta} \lambda_\alpha (V_{\alpha,\beta} k_\beta - \bar{p}_\alpha) \tag{10}$$

where the λ_α are Lagrange multipliers, ($\alpha = 1, 2, \ldots, m$). Setting the partials with respect to k_γ and λ_γ to zero gives - on noting that $\mathbf{B}$ is symmetrical -

$$\frac{\partial \phi}{\partial k_\gamma} = \sum_{\alpha=1}^{r} b_{\alpha,\gamma} k_\alpha + \sum_{\alpha=1}^{m} \lambda_\alpha V_{\alpha,\gamma} = 0 \qquad \gamma = 1, 2, \cdots, r \tag{11}$$

$$\frac{\partial \phi}{\partial \lambda_\gamma} = \sum_{\beta=1}^{r} V_{\gamma,\beta} k_\beta - \bar{p}_\gamma = 0 \qquad \gamma = 1, 2, \cdots, m \tag{12}$$

Returning to matrix notation and letting $\Gamma = \mathrm{col}\ (\lambda_1, \lambda_2, \ldots \lambda_m)$ transforms Equation 11 into

$$\mathbf{B}k + V^T \Gamma = 0 \tag{13}$$

Equation 12 becoming Equation 4 again.

The matrix $\mathbf{B}$ is shown to be nonsingular in Reference 7 hence may be inverted. Transposing Equation 13 and solving for k gives

$$k = -\mathbf{B}^{-1} V^T \Gamma \tag{14}$$

Substituting this into Equation 4 and premultiplying the result by $(V\mathbf{B}^{-1} V^T)^{-1}$ and substituting into Equation 14 gives

$$k = M \bar{p} \tag{15}$$

where

$$M = \mathbf{B}^{-1} V^T (V \mathbf{B}^{-1} V^T)^{-1} \tag{16}$$

the nonsingularity of $(V\mathbf{B}^{-1} V^T)$ being also proved in Reference 7.

By Reference 12 the flexibility matrix of the supported element is

$$F_{ij} = \int_{V_0} \sigma_{(i)}^T \, \epsilon_{(j)} \, dV_0 \tag{17}$$

where $\sigma_{(i)}$ is the stress vector at a point in the element caused by a unit load at coordinate i, and $\epsilon_{(j)} = \mathrm{col}\ (\epsilon_x^j, \epsilon_y^j, \epsilon_z^j, \gamma_{xy}^j, \gamma_{yz}^j, \gamma_{zx}^j)$ is the strain vector caused by unit load at j, the integration being taken over the volume V_0 of the element, in this case 2h times the area of the triangle.

Expressing the customary relation between stress and strain in matrix notation as

$$\epsilon = N\sigma \tag{18}$$

and using Equations 2 and 15 converts Equation 17 into

$$F_{ij} = e_i^T M^T G M e_j \tag{19}$$

where

$$G = \int_{V_0} U^T N U dV_0 \quad \text{and} \quad e_i$$

is a column vector with a one in the ith place and zero elsewhere. This amounts to

$$F = M^T G M \tag{20}$$

By inversion of Equation 20 — the nonsingularity of G and hence F being also proved in Reference 7 — one obtains the 6 x 6 stiffness matrix of the supported element:

$$\bar{S} = F^{-1} \tag{21}$$

In order to obtain the 9 x 9 stiffness matrix S of the unsupported element, the transformation matrix H is defined satisfying

$$\begin{bmatrix} \bar{p} \\ \hdashline p_f \end{bmatrix} = H \bar{p} \tag{22}$$

where p_f = support loads determined from equilibrium equations as linear combinations of the applied loads p so that the stiffness matrix of the element sought is

$$S = H \bar{S} H^T \tag{23}$$

The integrations in the above are accomplished by use of numerical formulas which are exact in each case up to the degree of function involved, all calculations being performed upon a high speed digital computer.

EVALUATION STUDIES

For purposes of evaluation the vibration frequencies and mode shapes of a square cantilevered plate of constant thickness were calculated by the stiffness matrix method using the above element. The triangles were arranged in groups of four, each group forming a square as shown in Figure 3, the masses being concentrated at the mid-points of the contained sides. The node-line results for a 144 triangle representation are shown in Figure 4 together with the node-line locations calculated in Reference 13 which employs a Fourier series solution and is considered to be exact. Also in Table I the corresponding frequencies obtained in Reference 13 are shown together with those obtained by the stiffness matrix method for a variety of simulations, employing 16, 64, and 144 triangles successively.

DISCUSSION OF RESULTS

From Table I it is apparent that, as the mesh of the representation grows finer the frequencies calculated tend to converge toward the exact values. Also, from Figure 4, it is noted that the approximation of the first four symmetric modes is good but that for the 4th anti-symmetric mode is relatively inaccurate. A possible explanation lies in the fact that — as result of the stress assumption, Equation 1, which makes τ_{yz} and τ_{zx} independent of x and y — the torsion on the edge of the element, i. e. the T coordinate in Figure 1, can only be carried by shearing stresses τ_{xy} in the plane of the plate and no transverse shear contributes to this moment, its integral being zero. Hence, as result, this element is better equipped to carry bending moments than twisting torques. Another feature of the simulation which may contribute toward the above discrepancy is that concentrated masses are used, thus neglecting moments of inertia of those portions of the plate adjacent to mass concentrations. Doubtless the use of mass matrices (Reference 14) would give superior results in this respect for any given representation.

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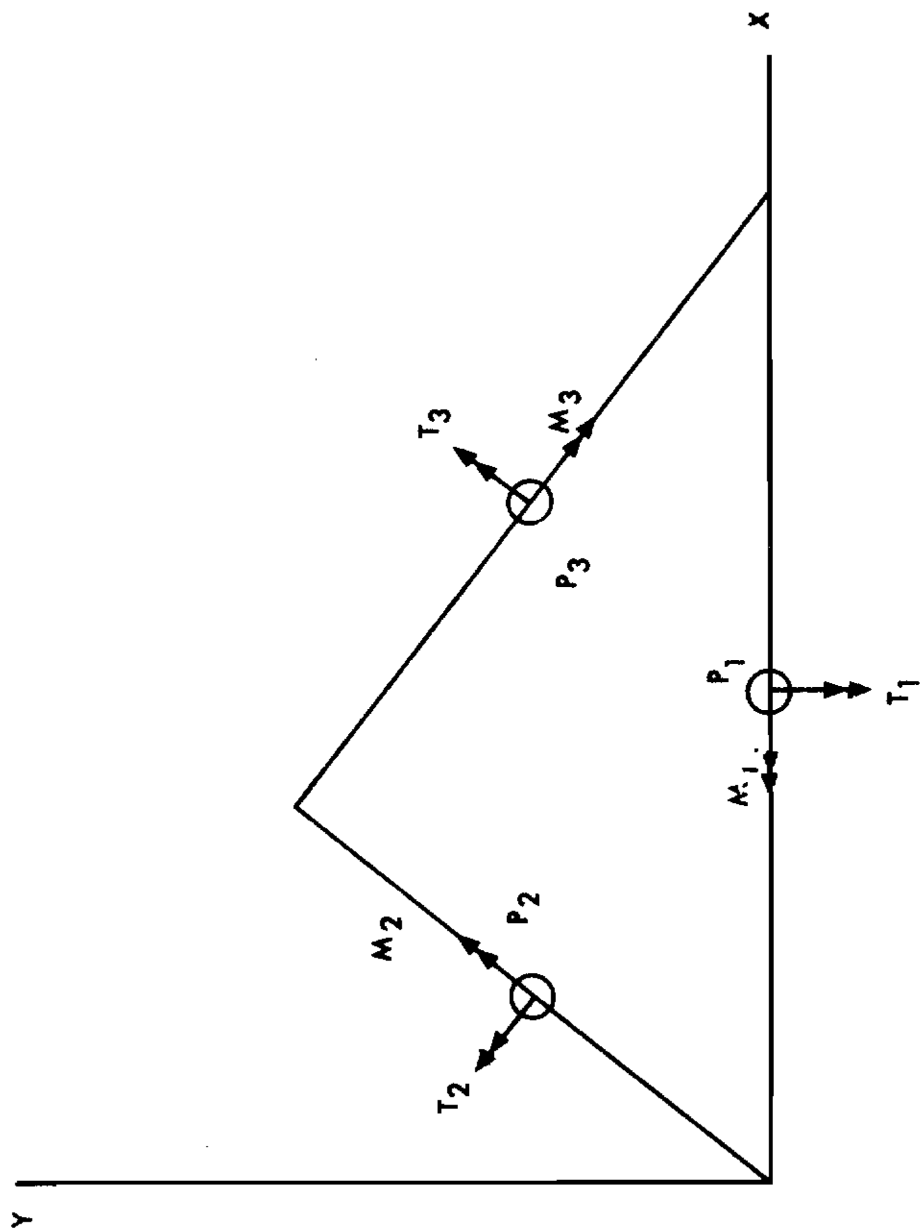

Figure 1. Loads on Triangular Plate

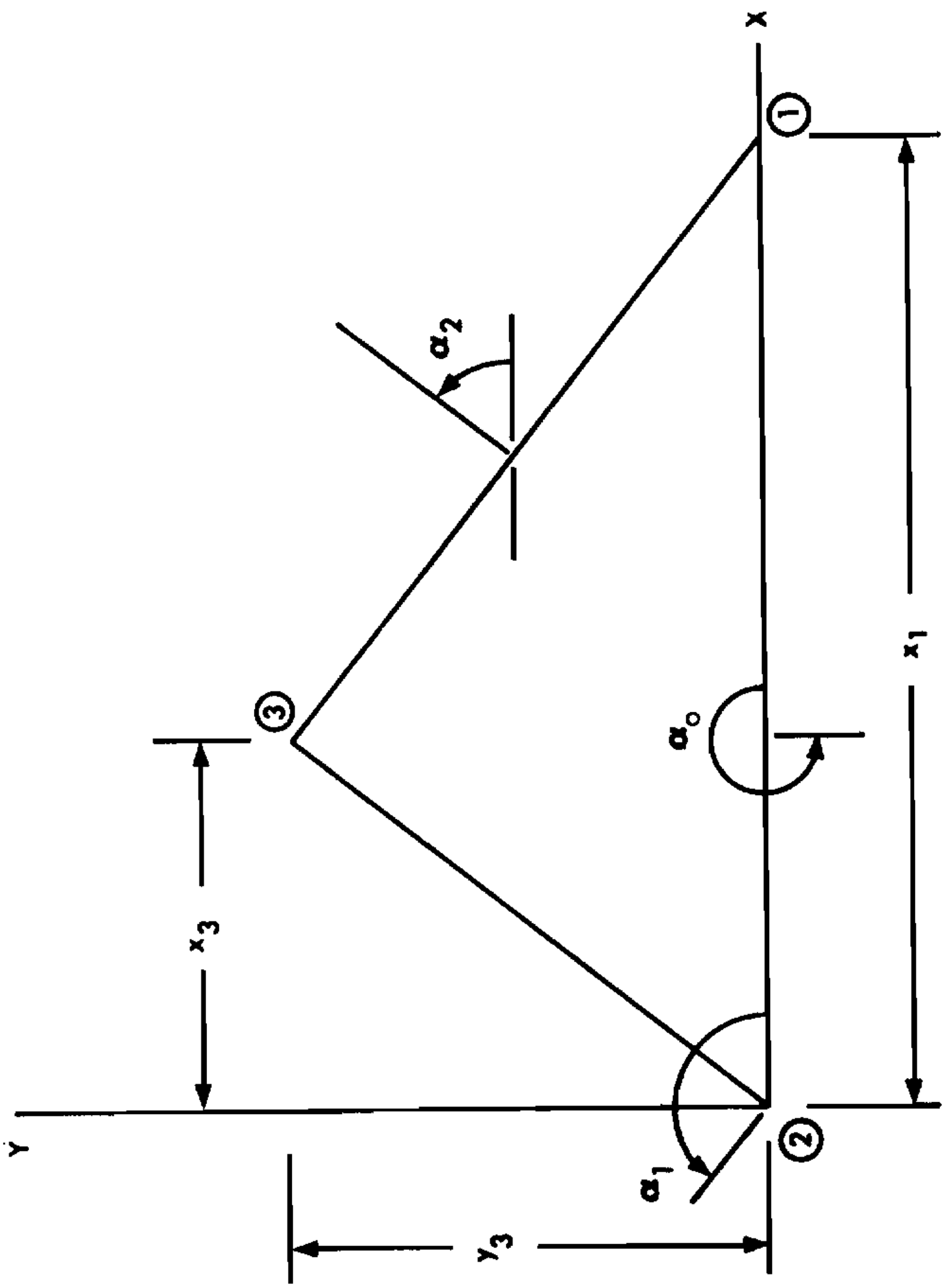

Figure 2. Geometry of Triangular Plate

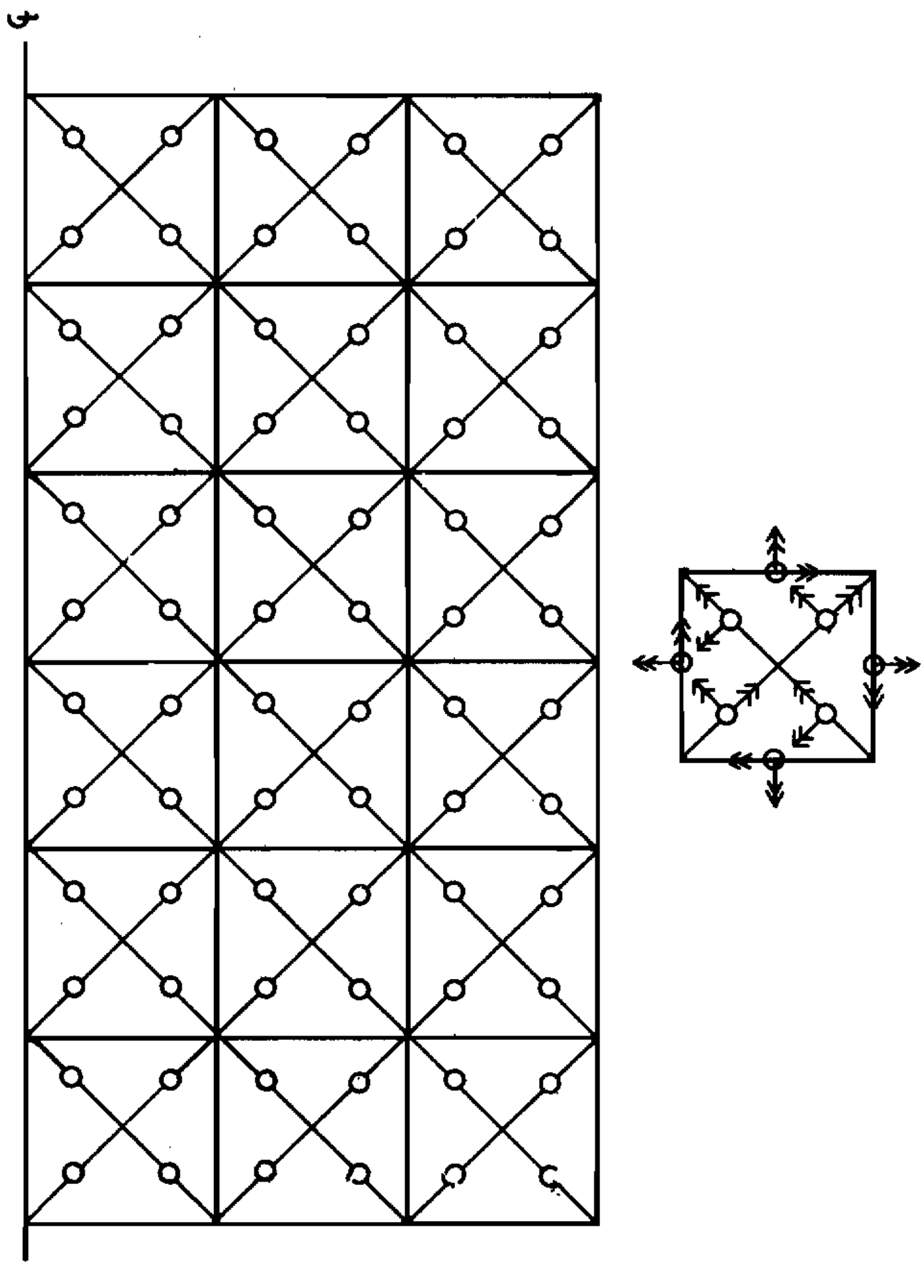

Figure 3. Typical Element

Table I

SQUARE PLATE FREQUENCIES (RAD/SEC)

SYMMETRIC

Mode	Claassen & Thorne	144 ∇'s	% Error	64 ∇'s	% Error	16 ∇'s	% Error
1	36.152	36.802	+1.80	36.002	- .415	35.692	- 1.27
2	221.660	219.75	- .862	218.07	-1.62	210.00	- 5.26
3	283.626	278.56	-1.79	276.39	-2.55	264.91	- 6.60
4	565.196	532.25	-5.83	510.20	-9.73	365.32	-35.4
5	638.158	625.71	-1.95	608.25	-4.69		

ANTI-SYMMETRIC

Mode	Claassen & Thorne	144 ∇'s	% Error	64 ∇'s	% Error	16 ∇'s	% Error
1	88.993	85.822	-3.56	84.278	-5.30	78.596	-11.7
2	323.703	310.01	-4.23	301.59	-6.83	263.51	-18.6
3	667.959	656.41	-1.73	638.85	-4.36		
4	739.894	710.71	-3.94	685.59	-7.34		

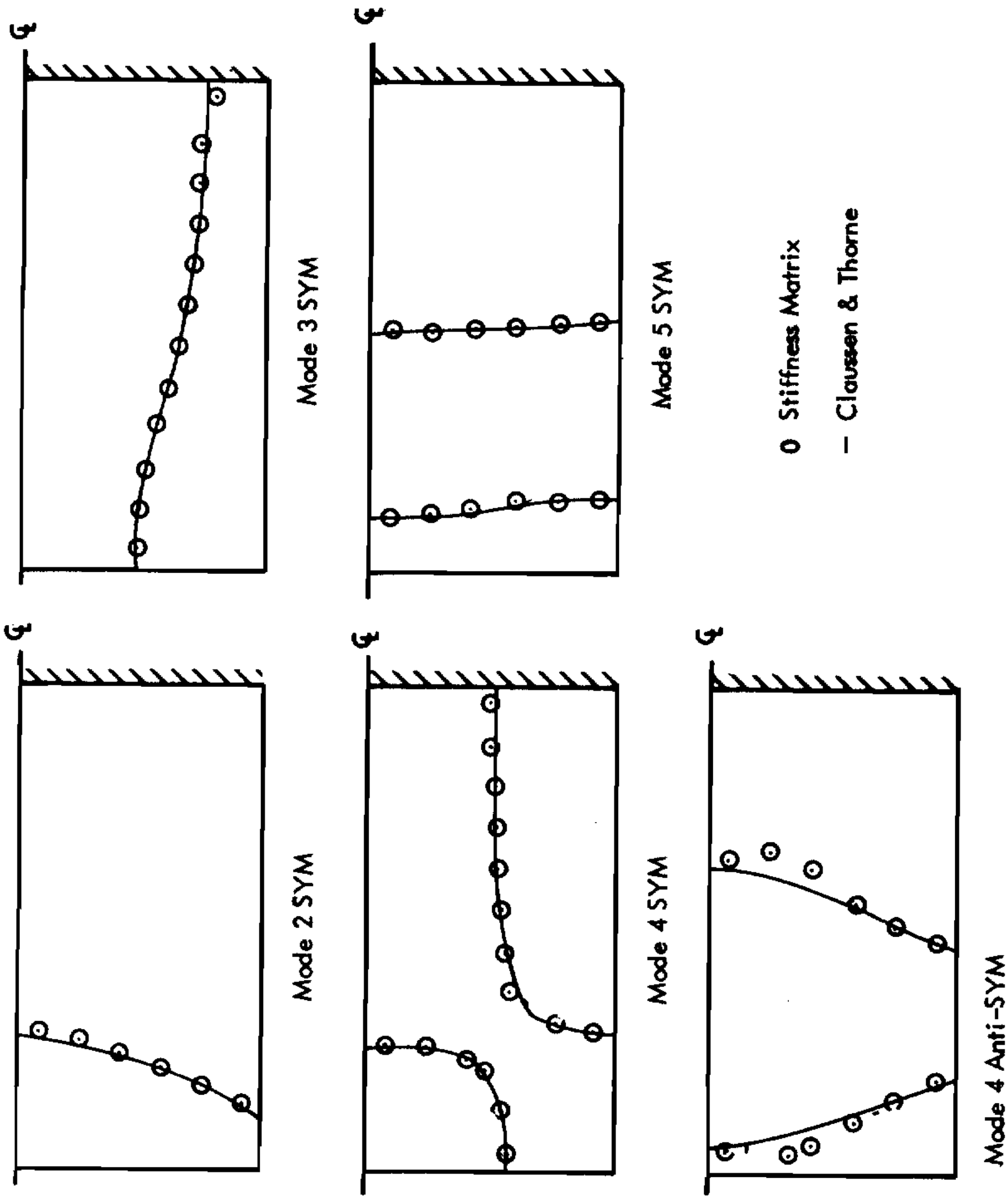

Figure 4. Comparison of Node Lines