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CURVED TRIANGULAR ELEMENTS FOR THE ANALYSIS OF SHELLS

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This paper presents the development of a curved triangular element for the analysis of doubly-curved shells. This element satisfies the the compatibility requirements for the normal displacement w and the slopes $w_{,x}$ and $w_{,y}$ by assuming a complete (third-degree) polynomial for the displacement function. The membrane displacements u and v are also represented by complete cubic polynomials. A second, less refined, element is obtained by removing some of the degrees of freedom of the first element. Two examples demonstrate the convergence of the solution and show that, for a cylindrical shell, the accuracy is better than that obtained with flat elements, while, for a clamped hypar surface, the results are similar to those given by rectangular curved elements.

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SECTION I

INTRODUCTION

Although the finite element method has been well established for several years and has been used successfully to solve a large variety of problems, a satisfactory curved triangular element, highly desirable for the efficient analysis of shells of arbitrary shape, has not yet been introduced. Such an element is presented here and is shown to lead to excellent results.

Finite elements have been applied successfully to plane stress (References 1, 2, and 3) and plate bending problems (References 4, 5, and 6). The method has been extended to shells of revolution with the use of conical segments and curved meridional elements (References 7, and 8). Comparative studies have indicated that the curved elements lead to considerably better results.

Shells of arbitrary shape have been analyzed with flat triangular and quadrilateral elements (Reference 9). The rectangular elements are particularly convenient for translational shells with rectangular boundaries, but the flat triangular geometry is more powerful since it can be made to represent a general shell surface. A triangular element has been developed and used by Clough and Johnson (Reference 10), and by Carr (Reference 11).

The obvious geometrical errors introduced in representing a smooth curved surface as an angular assembly of plane elements, tend to decrease as the size of the elements decreases but, for highly and moderately curved surfaces, the number of operations required to arrive at a satisfactory solution may make the approach unrealistic and too time-consuming, even on the most modern computers. A curved element which would, for a given accuracy, permit a wider pattern and therefore require less unknowns appears highly desirable.

Connor and Brebbia (Reference 12) have introduced a curved rectangular element which, when applied to a clamped hypar shell, gives better accuracy than finite differences. Utku in Reference 13, proposed an element stiffness matrix for shallow curved triangular elements using linear functions to represent displacements. Expecting this last approach, no significant work on curved triangular elements has been reported so far.

The development of such a curved element is reported herein. The stiffness of the element is derived from the theory of shallow shells and the formulation is such that full compatibility is ensured for the bending state of stress while almost complete compatibility is obtained for the membrane state of stress.

The element is applied successfully to the analysis of shells of zero and negative Gaussian curvature and the results are shown to be slightly better than those obtained with flat elements.

SECTION II

STRAIN ENERGY FORMULATION

The stiffness matrix of the elements is derived from the principle of minimum potential energy, using the theory of shallow shells, which is more than accurate enough for the relatively small size of the elements considered.

DIFFERENTIAL GEOMETRY FOR A SHALLOW SURFACE

A set of orthogonal curvilinear coordinates ($\xi_1 \xi_2$) is chosen on the middle surface of the shell, the normal coordinate being represented by ξ_3. This is shown in Figure 1, along with a right-handed cartesian coordinate system (X, Y, Z).

According to Reissner (Reference 14), the differential surface element is expressed as

$$dA = \alpha_1 \alpha_2 d\xi_1 d\xi_2$$

where

$$\alpha_i^2 = \left(\frac{\delta \vec{r}}{\delta \xi_i} \quad \frac{\delta \vec{r}}{\delta \xi_i} \right) = (\vec{r}_{,i} \cdot \vec{r}_{,i})$$

and $\vec{r}(\xi_1, \xi_2)$ is the position vector of a point on the middle surface.

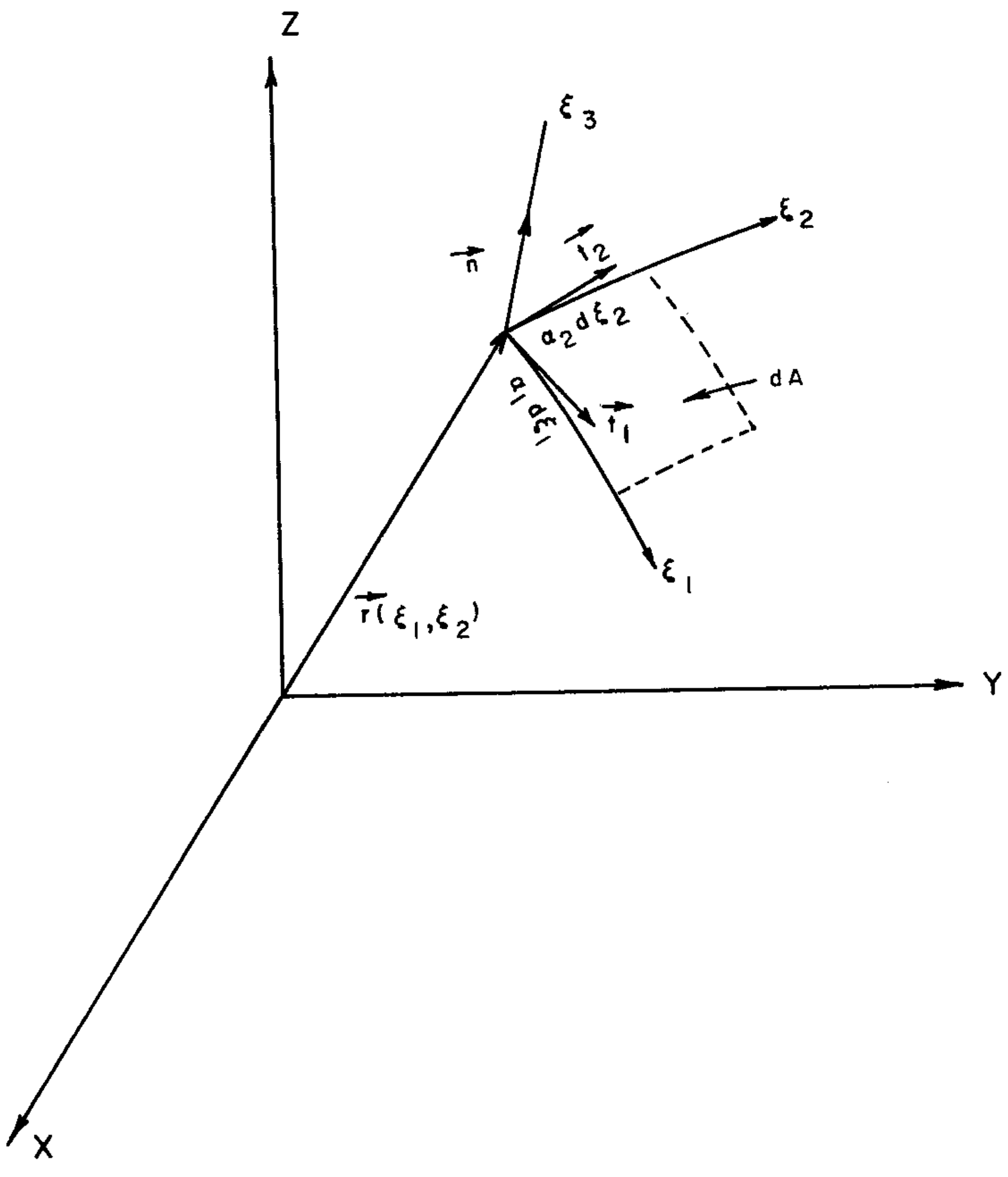


Figure 1. Differential Element

The tangent and normal vectors $\vec{t}_1, \vec{t}_2, \vec{n}$ are defined on the surface and the curvatures are obtained from:

$$\frac{1}{R_{ij}} = - \frac{1}{\alpha_i \alpha_j} \vec{n} \cdot \vec{r}_{ij}$$

$$\vec{n} = \left(\frac{1}{\alpha_1} \vec{r}_{,1} \right) \times \left(\frac{1}{\alpha_2} \vec{r}_{,2} \right)$$

The assumption of shallowness implies that:

$$z_{,x}^2 \ll 1, \quad z_{,y}^2 \ll 1$$

and leads to the following specialization of the above Equations (Reference 12)

$$\begin{aligned} \alpha_1 &\simeq \alpha_2 \simeq 1 \\ \frac{1}{R_{11}} &= - z_{,xx} \\ \frac{1}{R_{22}} &= - z_{,yy} \\ \frac{1}{R_{12}} &= - z_{,xy} \end{aligned} \tag{1}$$

STRAIN-DISPLACEMENT RELATIONSHIPS

The strain-displacement relationships for thin shells as given by Reissner are simplified for the shallow shell and expressed as follows in cartesian coordinates:

$$\begin{aligned} \epsilon_x &= e_x + \zeta K_x \\ \epsilon_y &= e_y + \zeta K_y \\ \gamma_{xy} &= e_{xy} + \zeta K_{xy} \end{aligned} \tag{2}$$

where

$$\begin{aligned} e_x &= u_{,x} - z_{,xx} w \\ e_y &= v_{,y} - z_{,yy} w \\ e_{xy} &= v_{,x} + u_{,y} - 2 z_{,xy} w \end{aligned} \tag{3}$$

$$\begin{aligned} K_x &\simeq - w_{,xx} \\ K_y &\simeq - w_{,yy} \\ K_{xy} &\simeq - 2 w_{,xy} \end{aligned} \tag{4}$$

and ζ is the thickness coordinate normal to the middle surface.

The strain energy of an isotropic linear elastic shell is given by Reference 15

$$U = \int_A \int \int_{-t/2}^{t/2} \frac{E}{2(1-\nu^2)} \left[\epsilon_x^2 + \epsilon_y^2 + 2\nu\epsilon_x \epsilon_y + \frac{1}{2}(1-\nu)\gamma_{xy}^2 \right] d\zeta \, dx \, dy \tag{5}$$

where

- t = thickness of the shell
- ν = Poisson's ratio
- E = modulus of elasticity

After substitution of Equations 2, 3, and 4 for ϵ_x, ϵ_y, γ_{xy} in the above expression and integration with respect to ζ, the strain energy can be separated into the membrane energy U_m and the bending energy U_b.

$$U = U_m + U_b$$

$$U_m = \frac{Et}{2(1-\nu^2)} \iint_A \left[e_x^2 + e_y^2 + 2\nu e_x e_y + \frac{1}{2}(1-\nu) e_{xy}^2 \right] dx \, dy \tag{6}$$

$$U_b = \frac{Et^3}{24(1-\nu^2)} \iint_A \left[K_x^2 + K_y^2 + 2\nu K_x K_y + \frac{1}{2}(1-\nu) K_{xy}^2 \right] dx \, dy$$

The potential energy is then written as:

$$\Phi = U - W$$

where W represents the work done by the external load system.

In the finite element method the total potential energy of a shell is expressed as:

$$\Phi = \sum_{k=1}^{n} \phi_k \tag{7}$$

where ϕ_k is the potential energy of the k^{th} element.

SECTION III

FLAT TRIANGULAR ELEMENTS

PLATE BENDING ELEMENT

A detailed bibliography of finite elements used to study the bending of plates is given by Clough and Tocher (Reference 6). A fully compatible element has been introduced by these authors, who satisfy the interelement compatibility requirements by imposing that, along an edge, there be a linear variation of the slope of the deflection surface normal to that edge.

In the development reported here, the normal deflection w is represented by a complete (ten-term) third-degree polynomial:

$$w = c_1 + c_2 x + c_3 y + c_4 xy + c_5 x^2 + c_6 y^2 + c_7 x^3 + c_8 x^2 y + c_9 xy^2 + c_{10} y^3 \tag{8}$$

In order to satisfy the compatibility requirements stated by Melosh (Reference 4) a given triangular element is first divided into three triangular subregions, as shown in Figure 2, which point "o" is assumed to be located at the center of gravity of the triangle. Three additional nodes are then located at the mid-points of the bounding sides of the triangle. A total of twelve degrees of freedom are then assigned to the triangle; three at each corner node and one (normal slope) at each mid-point node.

The normal deflection w is expressed in each subregion by an independent ten-parameter polynomial function. The displacement function for the complete triangle then involves 30 parameters, 18 of which are used up to satisfy the compatibility requirements between subregions, while the remaining 12 correspond to the 12 degrees of freedom of the complete triangle. The process of reduction from 30 parameters to 12 to obtain the 12 x 12 stiffness matrix is similar to that employed by Clough & Tocher (Reference 6).

It may be noted that there is no restriction here as to the choice of coordinates for each subregion. The actual orientation of the axes has been chosen such that analytical integration over the subregion is possible. The midpoint node may be eliminated, and the element consequently reduced to an HCT element, if the normal slope at a midpoint node is made equal to the average of the normal slopes at the two bounding nodes of that side.

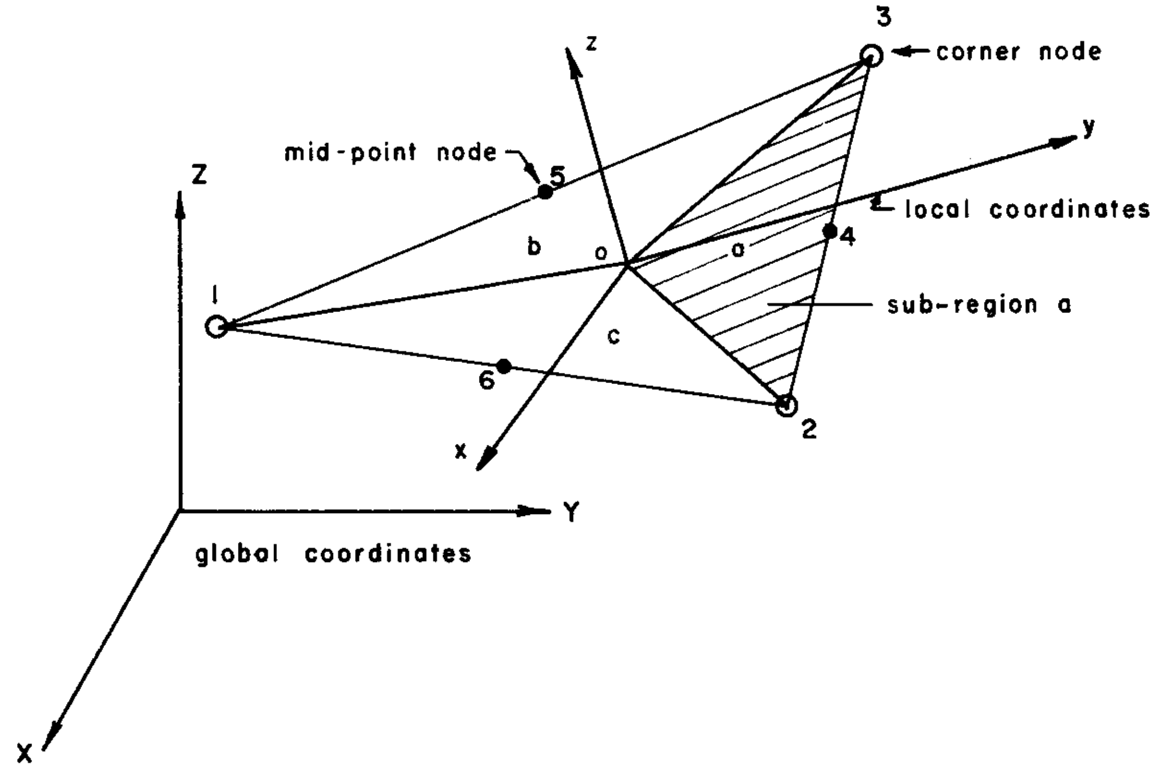


Figure 2. Projection of Curved Element on Local Reference Plane

MEMBRANE ELEMENT

The plane stress problem may be considered to be a special case of the membrane problem, and the corresponding element will be referred to as the membrane element for convenience.

Turner, Wilson, Clough and Johnson, and Felippa, (References 1, 2, 10, and 3) have used the displacement components u and v for plane stress problems. A more refined element is studied by Carr (Reference 11) who introduces higher-order displacement components ($u, u,_x$ $u,_y$ $v, v,_x, v,_y$). Carr has employed this better representation in analyzing curved shells through flat elements and has obtained results better than those of Johnson et al.

In this paper, the tangential displacement components are defined by $u, u,_x, u,_y, v, v,_x$ and $v,_y$ and are represented by independent complete third-degree polynomials for u and v :

$$u = a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2 + a_6 y^2 + a_7 x^3 + a_8 x^2 y + a_9 xy^2 + a_{10} y^3$$
$$v = b_1 + b_2 x + b_3 y + b_4 xy + b_5 x^2 + b_6 y^2 + b_7 x^3 + b_8 x^2 y + b_9 xy^2 + b_{10} y^3 \quad (9)$$

The procedure by which the 24 x 24 stiffness matrix is obtained for this membrane element is identical to that used previously for the bending element. Six degrees of freedom are assigned to each corner node, and two ($u,_n$, $v,_n$) to each midpoint node at the complete triangle (Figure 3).

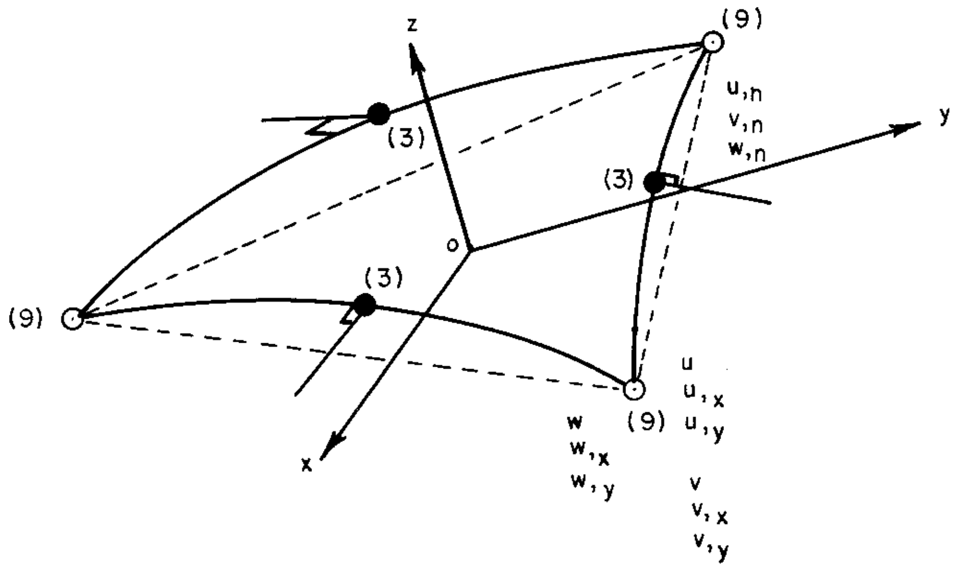


Figure 3. The 36 Degrees of Freedom Element

SECTION IV

CURVED TRIANGULAR ELEMENTS

In the analysis of shells by flat elements, a significant shortcoming lies in the impossibility of coupling the in-plane strains with the normal deflection within an element; the expressions used for strains are

$$\begin{aligned} e_x &= u_{,x} \\ e_y &= v_{,y} \\ e_{xy} &= v_{,x} + u_{,y} \end{aligned} \tag{10}$$

The membrane and the bending stiffness matrices are determined separately and are then superimposed to give the total stiffness matrix of the element, as illustrated in Figure 4a.

In the curved element, the presence of curvature introduces the desired coupling of the in-plane strains with the normal displacement; this can be seen in Equation 3. The total element stiffness matrix is then obtained as shown in Figure 4b. We note that, for a shallow shell, the influence of membrane displacements on the bending strains is negligible.

ELEMENT GEOMETRY

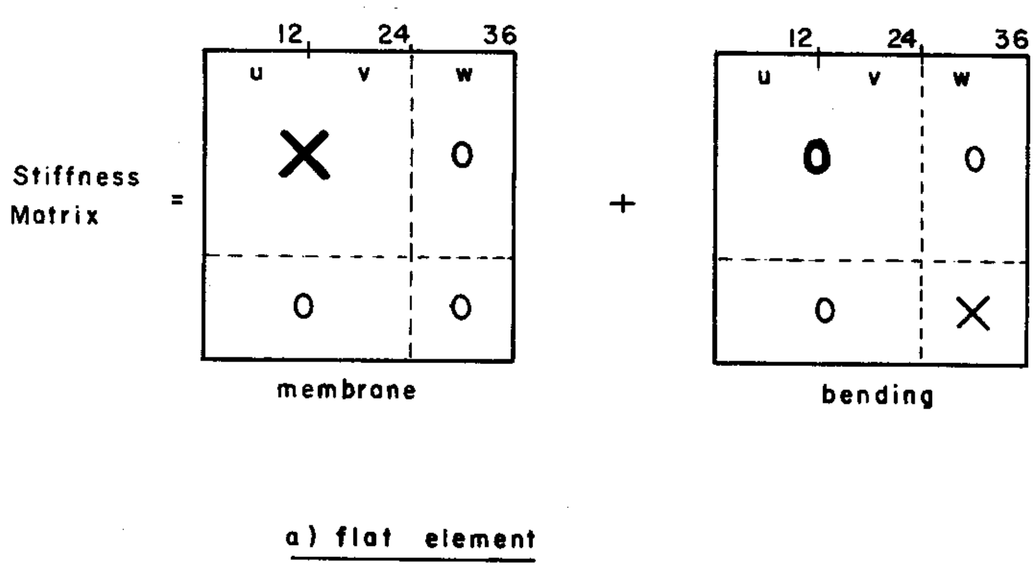
The corner nodes of the curved triangular elements are located on the middle surface of the shell according to the desired pattern and the planar triangular projection of an element is defined by the straight lines joining these nodes.

The actual geometry of the elements is then specified by assuming constant curvatures within any element, that is the actual middle surface of the shell is replaced, within the element, by a quadratic surface

$$z(x,y) = h_1 x^2 + h_2 y^2 + h_3 xy + h_4 x + h_5 y + h_6 \tag{11}$$

The six parameters h_i are determined by making the assumed surface coincide with the true surface at six points determined as follows: three of the points are the corner nodes themselves, while the other three are points on the true surface whose projections on the global X-Y plane are the midpoints of the projection, on the same plane, of the triangular base of the element (Figure 5). Although this representation does not ensure continuity of slopes and curvatures between adjacent elements, it is accurate enough in this first stage of the development and simple to use. The stiffness matrix of an element is referenced to its local coordinate axes o-x and o-y, located in the plane of the base triangle "1-2-3" with point o at the center of gravity of the triangle; the z-axis is taken as pointing outwards from the surface.

It is noted that both the local and the global coordinates are right-handed cartesian systems.



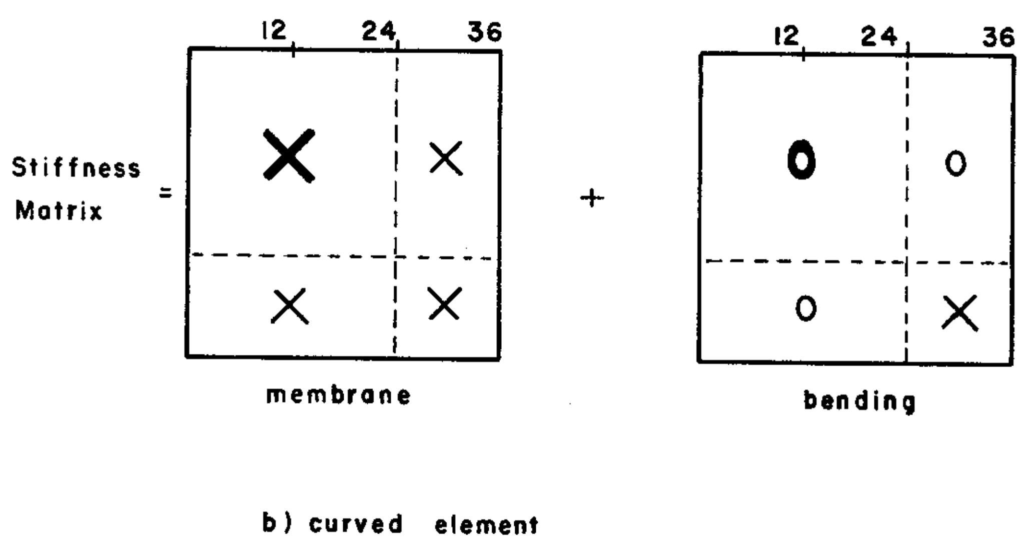


Figure 4. Configuration of the Element Stiffness Matrix

STIFFNESS MATRIX

The displacement function used here for u, v and w fully satisfies the compatibility requirements for flat elements. For the curved element, however, the inclusion of all rigid body displacements, a necessary conditions for full compatibility, could be obtained only if the displacement functions were expressed in terms of surface curvilinear coordinates.

For the shallow element, expressed in terms of cartesian coordinates, the displacement functions for u and v do not include all rigid body displacements and complete compatibility is not obtained. But, as pointed out by Connor and Brebbia (Reference 12), Stricklin et al (Reference 16) have shown that this simplification is permissible, and the solution will still converge to the true solution.

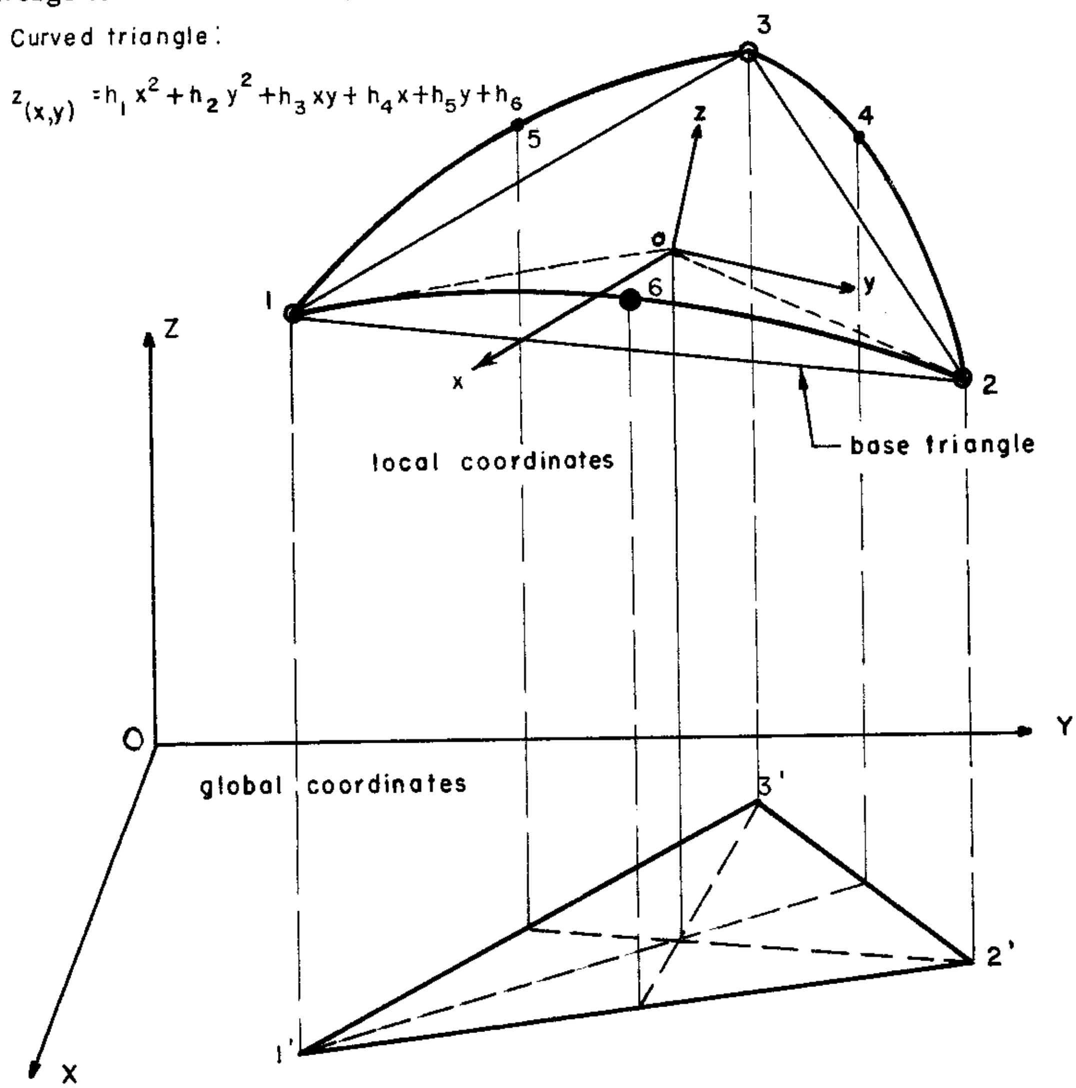


Figure 5. Element Geometry

SECTION V

ASSEMBLY OF ELEMENTS AND SOLUTION

COORDINATE TRANSFORMATIONS

The total stiffness matrix of the shell is built from the stiffness matrices of the finite elements but, before assembly, the matrices of the individual elements must be transformed and expressed in terms of a common coordinate system. In general, this common system can be:

(a) global coordinates (X.Y.Z)

(b) tangent plane coordinates

It is emphasized that for the curved element the stiffness matrix, which relates quantities defined on the surface, is inherently expressed in terms of a common coordinate system, the surface coordinates. It is then possible, with an appropriate choice of local coordinates, to avoid the coordinate transformation completely.

For flat elements, the tangent plane transformation is used. The components of displacement are rearranged as follows:

$$u, v, w, \theta_x, \theta_y, \theta_z, u_{,x}, v_{,y}, \gamma_{xy}$$

$$\text{where } \theta_x = w_{,y} \qquad \theta_y = -w_{,x} \qquad \theta_z = (v_{,x} - u_{,y})/2$$

$$\gamma_{xy} = v_{,x} + u_{,y}$$

For convenience, the in-plane strains $u_{,x}$, $v_{,y}$, γ_{xy} are kept in the local coordinate system.

SOLUTION

The flow graph methods (Reference 17) are applied to obtain the required solution. The complete stiffness matrix is automatically assembled from the element stiffness matrices. It is then modified to include the prescribed boundary conditions; this is done by striking out appropriate rows and columns. It must be noted that the elements of the complete matrix are, in general, submatrices which may correspond to a node or a set of nodes.

The flow graph methods* make it possible to take full advantage of the topological properties of the matrix and so reduce the computation time storage requirements. A given matrix may be composed of banded matrices and sparse matrices and the computer program used here makes use of these properties to minimize the solution time.

SECTION VI

NUMERICAL RESULTS

The accuracy and adequacy of the curved elements is established by applying them to the analysis of two shells previously studied by other researchers by both the finite element and the finite difference methods and comparing results.

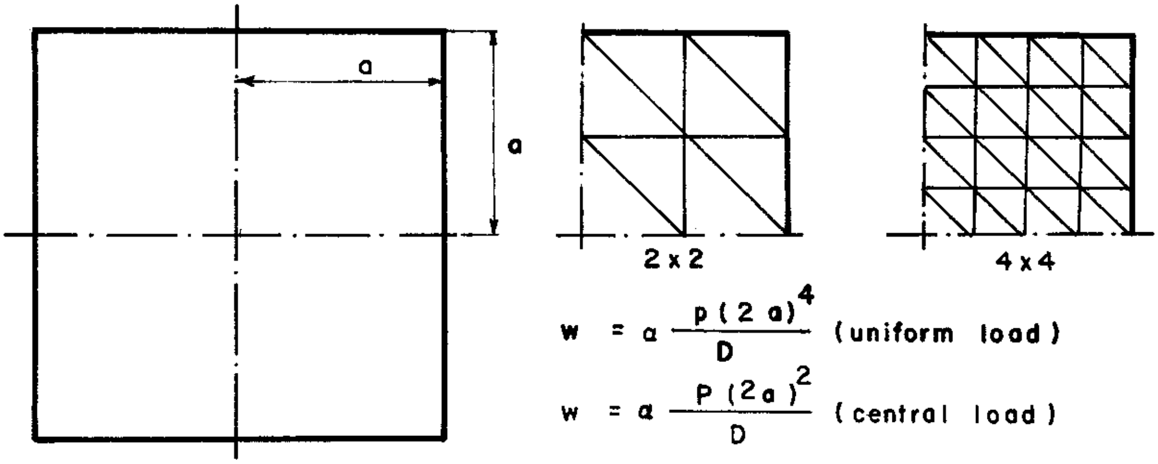
As a corollary, results obtained for clamped and simply-supported plates using the flat triangular element with midpoint nodes are compared with corresponding results given by the HCT element (Reference 6) (Figure 6).

CIRCULAR CYLINDER

The shell considered is an open circular cylinder (zero Gaussian curvature), loaded by its own weight, simply supported by diaphragms at the ends and entirely free along the sides (Figure 7). Double symmetry permits taking into consideration only one quarter of the shell.

Three different meshes (2 x 3, 4 x 5, 8 x 12) are studied for each element to demonstrate the monotonic convergence to the true values. Results for these cases are presented in Figure 8 along with data previously published by others.

* These methods have been used recently to solve a system of 10,000 equations on an IBM 360/40 with two work disks.



$$w = \alpha \frac{p(2a)^4}{D} \quad \text{(uniform load)}$$

$$w = \alpha \frac{P(2a)^2}{D} \quad \text{(central load)}$$

Values of α for deflection at center

Mesh size	Clamped		Simply supported	
	Central load	Uniform	Central load	Uniform
2 x 2	.004897	.001160	.011280	.003695
4 x 4	.005490	.001256	.011515	.003968
6 x 6	.005580	.001260	.011604	.004060

Triangular element with mid-point nodes.

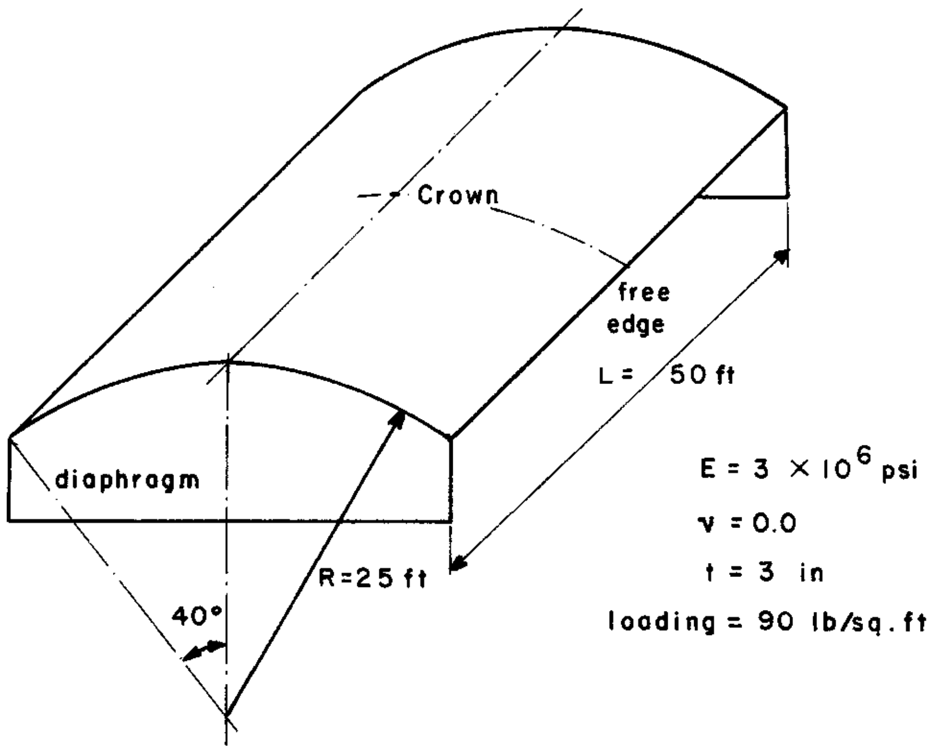
Exact solution Timoshenko	.00560	.00126	.01160	.00406

Comparaison H.C.T.–Triangular element with mid-point nodes

2 x 2	.004610	.001104	.010388	.003490
4 x 4	.005312	.001222	.011266	.003889
6 x 6	.005472	.001247	.011443	.003994

HCT reproduced.

Figure 6. Results for Plate Bending



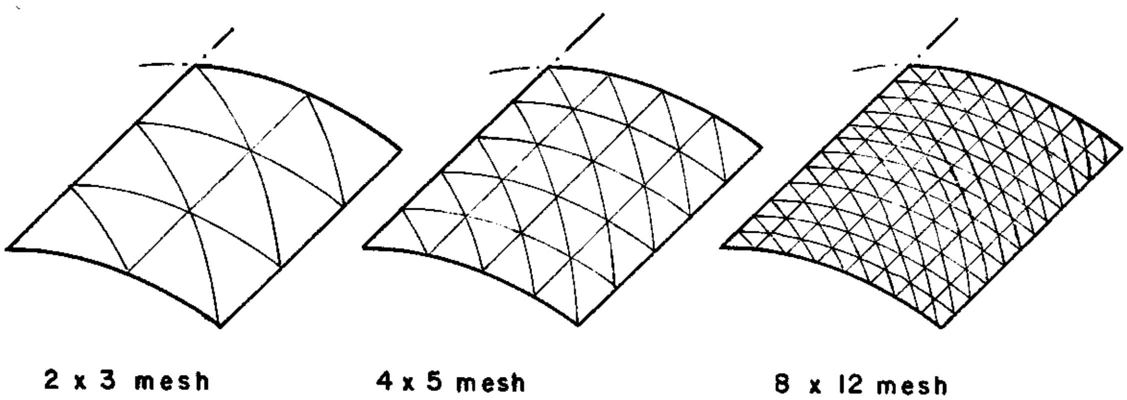


Figure 7. Circular Cylinder

Element type	Vertical deflection w	Mesh size 2 x 3	4 x 5	8 x 12	Ref. 18
Curved A *	Free edge at midspan	2.65	3.56	3.71	
	Crown	-	−0.48	−0.524	
Curved B *	Free edge at midspan	3.88	3.78	—	
	Crown	−0.529	−0.537	—	
Ref. 11	Free edge at midspan	—	3.61	3.663	3.696
	Crown	—	−0.517	−0.547	−0.552

* A. Curved element with 27 degrees of freedom. (midpoint nodes eliminated)

B. Curved element with 36 degrees of freedom.

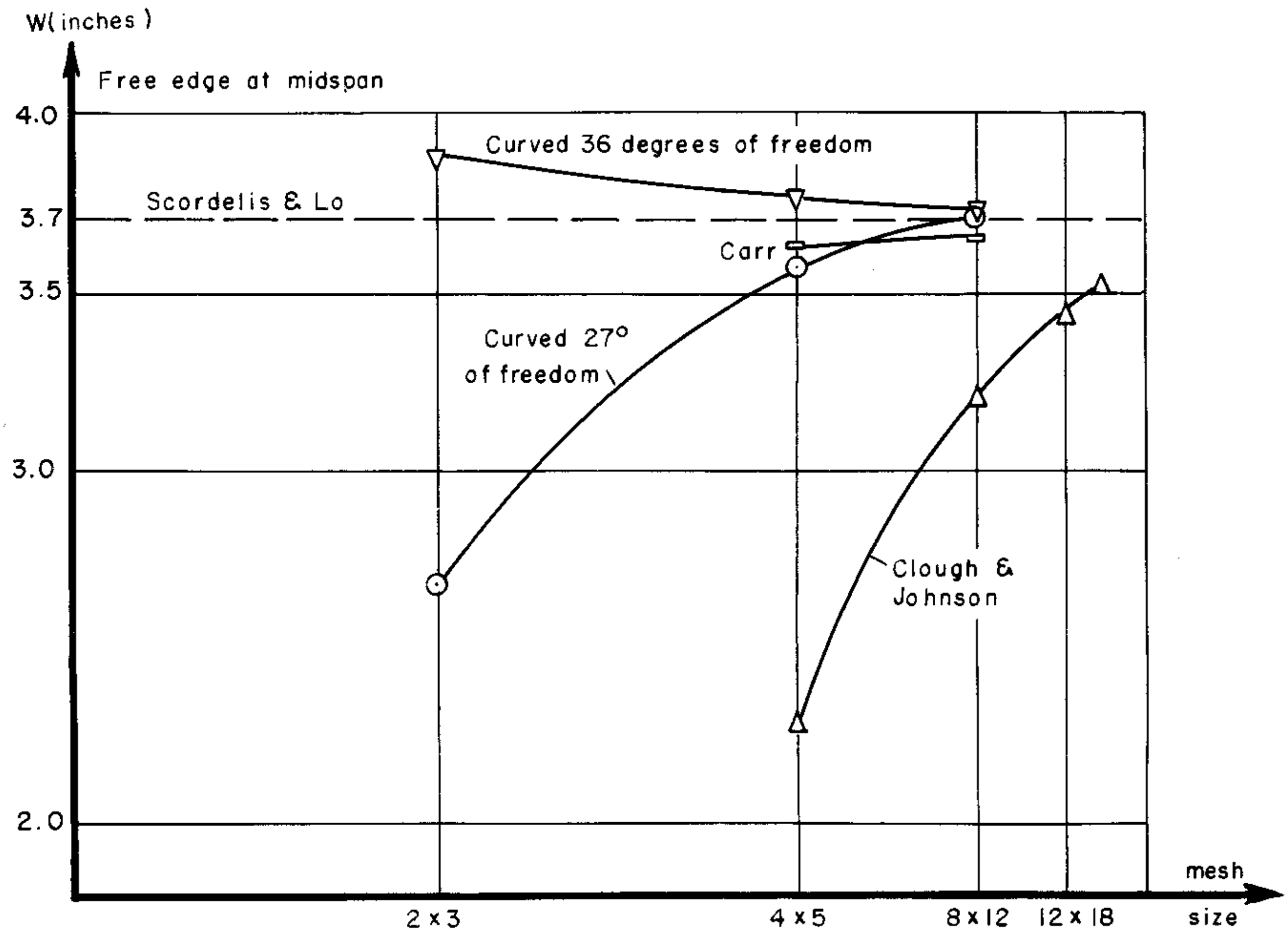


Figure 8. Comparison of Results for Circular Cylinder

CLAMPED HYPERBOLIC PARABOLOID

A hypar surface clamped on all edges and loaded by a uniform normal pressure "p" (Figure 9) is analyzed with three meshes (2 x 2, 4 x 4, 8 x 8) with both the element with midpoint nodes and the element without midpoint nodes. Results for both cases are given in Figures 10 and 11, together with those obtained by Connor and Brebbia (Reference 12), and compared to other solutions reported by Brebbia (Reference 19) and Chetty and Tottenham (Reference 20).

For the two shells studied, it is seen that the convergence is excellent and that the results come very close to the best solution known to date.

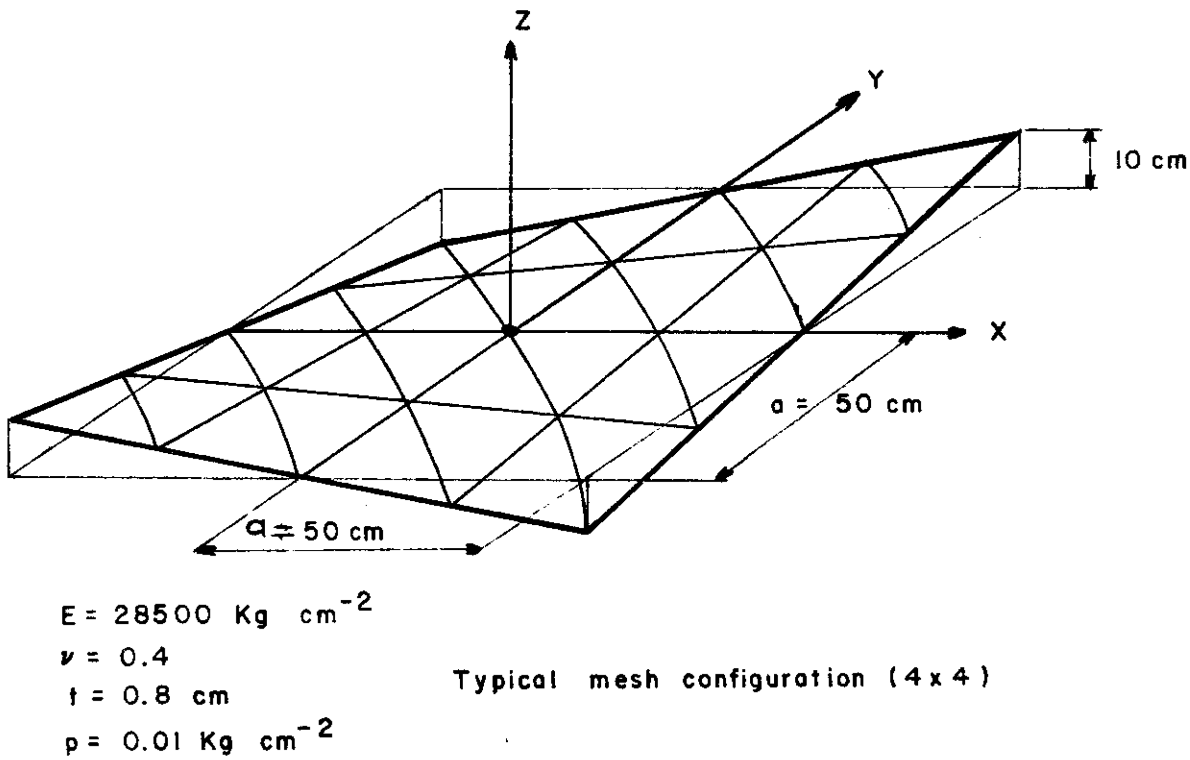


Figure 9. Clamped Hypar

Element type \ Mesh size	2 x 2	4 x 4	8 x 8
A *	—	.0345	.0283
B	.044	.0275	.0265

* A. Curved element with 27 degrees of freedom
  B. Curved element with 36 degrees of freedon

Figure 10. Vertical Deflection at Center

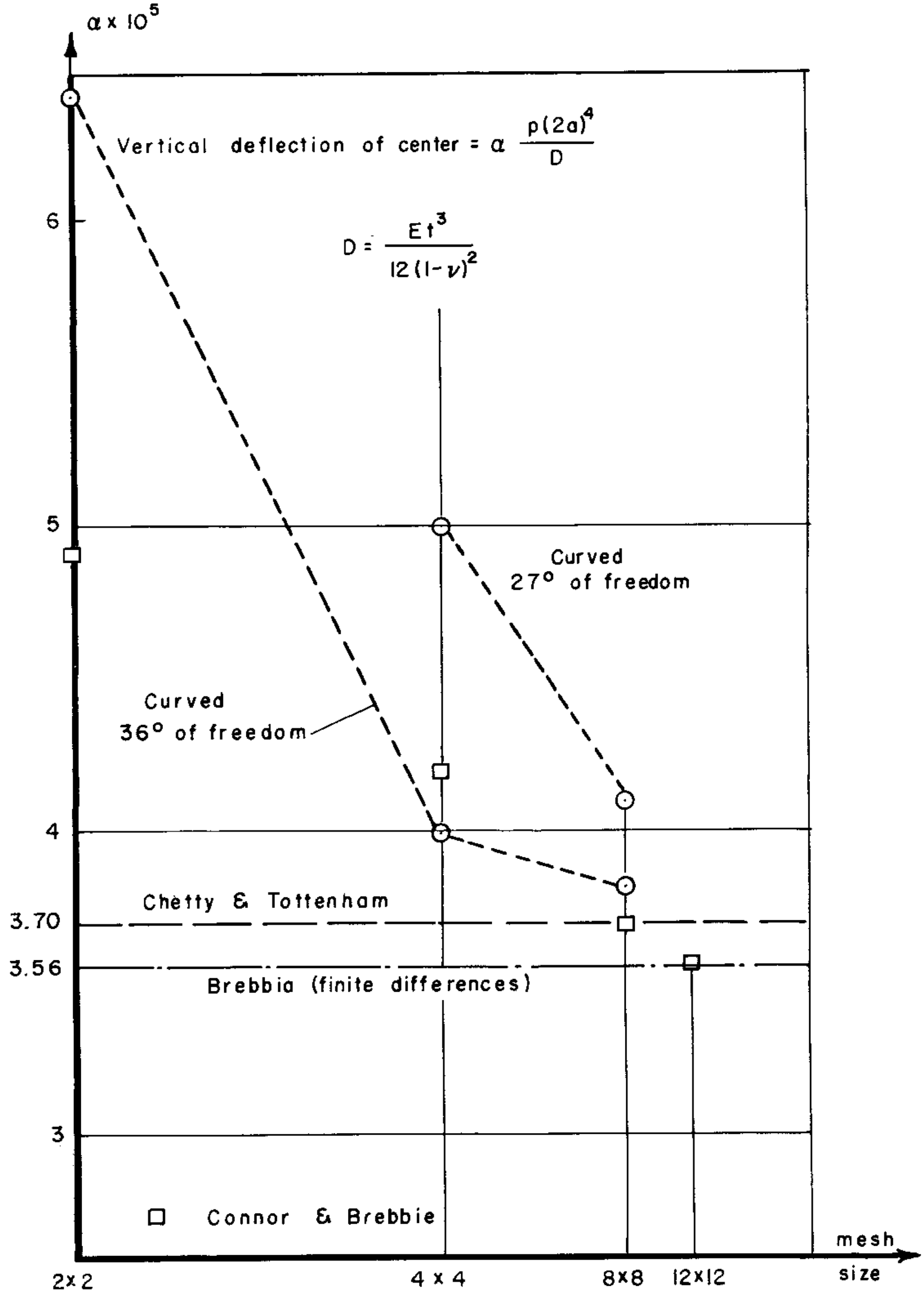


Figure 11. Comparison of Vertical Deflection at Center

SECTION VII

CONCLUSION

The results presented in the preceding article, though too few to permit general conclusions, clearly indicate that the curved element leads to results which converge monotonically to the exact solution and is a promising improvement over the flat element for the analysis of shells.

One slight shortcoming of the solution presented lies in the exclusive use of concentrated loads instead of the consistent loading systems that could be obtained by energy principles; the authors feel that correcting this omission, task, would permit attaining even better results.

Another interesting consideration will be to examine the performance of the element in terms of the prediction of stresses, which are of foremost importance in the majority of engineering problems. In this regard the high number of degrees of freedom used for the element will be of definite advantage since membrane stresses will be obtained directly in terms of the unknowns $u,_x$, $u,_y$, $v,_x$ and $v,_y$, thereby reducing significantly the computing effort required to obtain stresses.

More important however, it is thought that the curved element presented here is only a first step in the development of a more refined curved element that would permit cutting the shell on a very wide pattern, with as few unknowns as possible. Such a large finite element would be very useful for the efficient analysis of the usual single or continuous shells, but is essential for the use of incremental methods in the study of shells made of nonlinear materials as well as the dynamic and stability analysis of various shells.

There will always be a balance to be maintained between the time spend in determining the properties of the elements and that spent in the actual solution, but it seems fairly obvious that, with incremental methods, the fewer the elements, the faster the solution will be.

ACKNOWLEDGEMENT

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SECTION VIII

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