PERTURBATIONS ON NATURAL MODES DUE TO NONPROPORTIONALITY OF VISCOUS DAMPING

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The complex eigenvectors and eigenvalues of multi-degree-of-freedom system with moderately nonproportional viscous damping are approximated by a second-order perturbation method, in terms of the natural frequencies and mode shapes of the counterpart undamped system and the actual nonproportional damping matrix. Only the nonproportionality, not the overall level of damping itself, is assumed to be either moderate or weak. This new method can be particularly advantageous when designing, or when identifying, the system damping. Either task requires reanalysis of an eigenproblem of nonproportionally damped system each time that a different damping matrix is considered. The proposed technique requires only the smaller eigenproblem of counterpart undamped system to be analyzed directly, and only once. All the necessary explicit formulas are listed.

1. PROPORTIONAL VS. NONPROPORTIONAL DAMPING

Representing the mass of the discretized system by matrix **M**, the stiffness by **K**, and the damping by **C**, all of size nxn when there are n degrees of freedom, the equation of non-gyroscopic motion subject to external forces represented by vector **f**, may be set up as in Eq. 1 below.

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \tag{1}$$

It is assumed that the coordinates **x** have been so selected that the matrices **M** and **K** are positive definite. Being considered are cases where **C** is positive definite and the overall damping level may be high but still subcritical. The latter condition may be checked a priori, for instance, by the criterion of Inman and Andry [1].

Counterpart Undamped System Were the system undamped and freely vibrating (Eq. 2), the natural frequencies ω_{0j} and mode shapes y_{0j} (j=1,2,...,n) could be identified as in Eq. 3. Note that $i = \sqrt{-1}$.

$$M \ddot{x} + K x = 0 \tag{2}$$

$$x_j = y_{0j} \exp(i\, \omega_{0j} t) \tag{3}$$

Consideration of Eq. 3 in Eq. 2 leads to the eigenvalue problem, or eigenproblem, described by Eq. 4. In the latter context, ω_{0j} are eigenvalues and y_{0j} are eigenvectors.

$$(-\omega_{0j}^2 M + K)\, y_{0j} = 0\,, \quad j = 1, 2, \ldots, r, \ldots n \tag{4}$$

$$y_{0k}^T M y_{0j} = \delta_{jk} \tag{5}$$

$$y_{0k}^T K y_{0j} = \omega_{0j}^2\, \delta_{jk} \tag{6}$$

It is assumed herein that the eigenvectors y_{0j} are normalized such that the orthogonality properties are expressible as Eqs. 5-6. δ_{jk} is Kronecker delta. Eq. 5 is a very common and convenient choice of normalization in computer implementation of classical modal analysis.

Counterpart Proportionally Damped System Were the system damped such that C is of a form C_p that satisfies Eq. 7, which is Caughey and O'Kelly's proportionality criterion [2], the free damped vibration and associated eigenproblem would be described by Eqs. 8-12:

$$C_p M^{-1} K = K M^{-1} C_p \tag{7}$$

$$M \ddot{x} + C_p \dot{x} + K x = 0 \tag{8}$$

$$x_j = y_{0j} \exp(\lambda_{0j} t) \tag{9}$$

$$(\lambda_{0j}^2 M + \lambda_{0j} C_p + K)\, y_{0j} = 0\,, \quad j = 1, 2, \ldots, r, \ldots n, \ldots, 2n \tag{10}$$

$$\lambda_{0j} = -\,\omega_{0j}\, \xi_{0j} + i\, \omega_{0j} \sqrt{1 - \xi_{0j}^2}\,, \quad j = 1, 2, \ldots, r, \ldots n \tag{11}$$

$$\xi_{0j} = y_{0j}^T C_p y_{0j} / 2\, \omega_{0j} \tag{12}$$

ω_{0j} in Eqs. 11-12 are the natural frequencies of the counterpart undamped system (Eq. 4). With M, K and C_p as specified after Eq. 1, each eigenvalue λ_0 is complex with negative real part; i.e., both natural frequency ω_0 and damping ratio ξ_0 are positive. An ordering is assumed here such that the (n+r)-th eigenvalue is conjugate of the r-th.

The essence of damping proportionality is that the mode shapes $\mathbf{y}_{0j}$ of the counterpart undamped system (Eq. 4) are preserved as eigenvectors even of the damped system (Eq. 10). $\mathbf{y}_{0j+n}$ and $\mathbf{y}_{0j}$ are identical, as a consequence of the ordering of their respective λ_{0j}.

Complications due to Damping Nonproportionality From the above introduction, it is apparent why the hypothesis of proportional damping is convenient. Conceptually, it has the advantage that the real eigenvectors have the familiar interpretation as mode shapes. Computationally, iterative numerical algorithm to solve the quadratic eigenproblem of Eq. 10 is unnecessary; the eigenvalues and eigenvectors are directly expressible in terms of ω_{0j}, $\mathbf{y}_{0j}$ and $\mathbf{C}_p$, as pointed out in the preceding two paragraphs.

Much as the proportionality hypothesis is convenient, however, it has to be abandoned in certain cases. For example, confidence in both modelling and testing of structural elements or substructures in some applications has grown to a level where the assembled or complete structure, materially nonhomogenous as it is, cannot but be modelled with nonproportional damping, unless **C** turns out to be actually proportional. Also, when experimentally identifying the damping of existing structure, it is more general and hence arguably better to hypothesize that **C** may be nonproportional. Thirdly, when designing damping into the structure, the optimally efficient distribution may correspond to a nonproportional **C**.

Foss [3] more than 30 years ago pointed out that a generalized modal analysis can be applied to nonproportionally damped systems. The idea is summarized below.

Were the nonproportionally damped system freely vibrating, the free damped vibration and associated quadratic eigenproblem would be described by Eqs. 13-15 below. Note the formal analogies between (9) and (14), and between (10) and (15).

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \tag{13}$$

$$\mathbf{x}_j = \mathbf{y}_j \exp(\lambda_j t) \tag{14}$$

$$(\lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K})\, \mathbf{y}_j = \mathbf{0}, \quad j = 1, 2, \ldots, r, \ldots, n, \ldots, 2n \tag{15}$$

Like Eq. 10, Eq. 15 has 2n pairs of complex eigenvalue λ and eigenvector $\mathbf{y}$. The same ordering is assumed here for both eigenproblems. λ_j may also be expressed in form analogous to Eq. 11:

$$\lambda_j = -\omega_j \xi_j + i\,\omega_j \sqrt{1-\xi_j^2} \tag{16}$$

By this analogy, ω_j may be called pseudo natural frequency, and ξ_j, pseudo damping ratio of mode j. Unlike Eq. 10, however, the eigenvectors of Eq. 15 are complex and cannot be as readily interpreted as physical shapes.

Computationally, the eigenproblem of Eq. 15 is much more demanding than Eq. 10 [3]. Nevertheless, the complex eigenvectors provide a set of base vectors through which a coordinate transformation enables the uncoupling of the second-order differential equations implied in Eq. 1, into first-order differential equations. As pointed out by Foss, the dynamic response $\mathbf{x}(t)$ may be obtained by a generalized modal superposition:

$$\mathbf{x}(t) = 2\,\mathrm{Re}\sum_{j=1}^{n} P_j(t)\;\mathbf{y}_j$$
$$= \sum_{j=1}^{n} (\,2\,\mathrm{Re}\,P_j)(\,\mathrm{Re}\,\mathbf{y}_j) - (\,2\,\mathrm{Im}\,P_j)(\,\mathrm{Im}\,\mathbf{y}_j) \tag{17}$$

where Re and Im stand for "real part of" and "imaginary part of", respectively. The scalar function P_j, which might be called modal complex coordinate or modal participation function, is:

$$P_j = \mathbf{y}_j^T \exp(\lambda_j t)\,[\,\textstyle\int_0^t \mathbf{f}(\tau)\exp(-\lambda_j\tau)\,d\tau\; +$$
$$\lambda_j\mathbf{M}\mathbf{x}_0 + \mathbf{C}\mathbf{x}_0 + \mathbf{M}\dot{\mathbf{x}}_0\,]\;/\;[\,\mathbf{y}_j^T\,(\,2\lambda_j\mathbf{M} + \mathbf{C}\,)\,\mathbf{y}_j\,] \tag{18}$$

where effects of initial displacement $\mathbf{x}_0$ and velocity $\dot{\mathbf{x}}_0$ have been included. Modal uncoupling is demonstrated by Eq. 18, whereby the complex participation functions are obtained independently for each mode.

The generalized modal analysis method of Eq. 17, although mathematically well established, did not find early extensive application in structural engineering practice. Both computationally and conceptually, it is more complicated than the classical modal analysis of proportionally damped systems.

Many studies have since been published that assume the complex eigenvectors and eigenvalues to be known and concentrate the efforts on efficiently and accurately calculating the equivalent of P_j of Eq. 18. That is not to forget, however, that the computational effort required in solving Eq. 15 itself, can be much more than the requirement of the eigenproblem of the counterpart undamped system (Eq. 4), and certainly more than that of the counterpart proportionally damped system (Eq. 10). While each of **M**, **C** and **K** is of size nxn, numerical algorithms to solve

Eq. 15 actually solve the eigenproblem of a 2nx2n matrix. Techniques of reducing both storage and computing time should be much welcome, particularly when designing or when identifying the system damping. Either task requires reanalysis of a quadratic eigenproblem each time that a different damping matrix is considered.

Some perturbation techniques have been proposed for lightly damped systems [4-6] that may avoid increasing the eigenproblem size from nxn to 2nx2n. Chung and Lee [7], applying the technique of Meirovitch and Ryland [6], proposed to use a counterpart proportionally damped system as the unperturbed system in obtaining the eigenproperties. The present authors recently proposed [8-10] a general second-order perturbation technique assuming that the nonproportionality is moderate, and derived explicit approximate formulas for the perturbations on frequencies, modal damping ratios, and nonproportionally damped "modes". The approach is equivalent in order, but different in formulation from Chung and Lee's.

Details of the method are presented below and in the cited references. Computational and conceptual advantages over "exact" solution of Eq. 15 are pointed out where most relevant.

2. MODERATE NONPROPORTIONALITY AS PERTURBATION

The eigenvalues ω_{0j} and mode shapes $\mathbf{y}_{0j}$ of the counterpart undamped system (Eqs. 4-6) are assumed to be known. Modal matrix $\mathbf{Y}_0$ is defined such that its j-th column is $\mathbf{y}_{0j}$. Transforming the damping matrix $\mathbf{C}$ using the modal matrix $\mathbf{Y}_0$ as in Eq. 19 below, and separating the diagonal and off-diagonal elements, it is possible to uniquely identify the counterpart proportional damping matrix $\mathbf{C}_p$ (Eq. 20) and damping nonproportionality matrix $\mathbf{C}_n$ (Eq. 21):

$$\mathbf{Y}_0^T \mathbf{C} \mathbf{Y}_0 = \text{diag}\,[\,2\,\omega_{0j}\,\xi_{0j}\,] + \text{offdiag}\,\tilde{\mathbf{C}} \tag{19}$$

$$\mathbf{C}_p = \mathbf{Y}_0\,\mathbf{M}\,\text{diag}\,[\,2\,\omega_{0j}\,\xi_{0j}\,]\,\mathbf{M}\,\mathbf{Y}_0^T \tag{20}$$

$$\mathbf{C}_n = \mathbf{Y}_0\,\mathbf{M}\,\text{offdiag}\,\tilde{\mathbf{C}}\,\mathbf{M}\,\mathbf{Y}_0^T \tag{21}$$

When the nonproportionality is moderate, as being considered here, the norm of $\mathbf{C}_n$ is one order smaller than the corresponding norm of $\mathbf{C}_p$. The quadratic eigenproblem of Eq. 15 may now be rewritten as:

$$(\lambda_j^2\,\mathbf{M} + \lambda_j(\mathbf{C}_p + \mathbf{C}_n) + \mathbf{K})\,\mathbf{y}_j = \mathbf{0} \tag{22}$$

where $\mathbf{C}_n$ is a perturbation due to damping nonproportionality.

The Unperturbed System From Eq. 22, neglecting C_n, the unperturbed (or zero-order perturbed) eigenproblem of Eq. 23 below is obtained, which is identical to Eq. 10:

$$(\lambda_j^2 M + \lambda_j C_p + K) y_j = 0 \qquad (23)$$

with solutions known from Eqs. 4, 5, 11, 12, 24 and 25.

$$\lambda_j = \lambda_{0j} \qquad (24)$$

$$y_j = y_{0j} \qquad (25)$$

C_p in Eq. 12 need not be set up explicitly; it is replaced by C in actual calculation of (unperturbed) ξ_{0j}. Note that the eigenvectors (Eq. 25) are real, while the eigenvalues (Eq. 24) are complex. λ_{j+n} and λ_j are conjugates; y_{0j+n} and y_{0j} are identical.

Second-order Perturbations When the eigenproblem is perturbed by C_n, the eigenvalues and eigenvectors are assumed to be perturbed in the following forms:

$$\lambda_j = \lambda_{0j} + \lambda_{1j} + \lambda_{2j} \qquad (26)$$

$$y_j = y_{0j} + y_{1j} + y_{2j} \qquad (27)$$

$$y_{1j} = \sum_{k=1}^{n} a_{jk}(1 - \delta_{jk})\, y_{0k} \qquad (28)$$

$$y_{2j} = \sum_{k=1}^{n} b_{jk}(1 - \delta_{jk})\, y_{0k} \qquad (29)$$

where the first of two subscripts in Eqs. 26-27 indicates the order of perturbation. In Eqs. 28-29 for y_{1j} and y_{2j}, it is not necessary to include k=j, i.e. y_{0j}. The vector set $y_{01}, y_{02}, \ldots, y_{0n}$ constitutes a complete vector space, in terms of which the expansion of y_j can be written; however y_{0j} is already included in the expansion (Eq. 23) as the first term.

The perturbations λ_{1j} and λ_{2j}, and perturbation coefficients a_{jk} and b_{jk} are obtainable by: substitution of Eqs. 26-27 into Eq. 22; grouping of terms of the same order of magnitude to yield three separate matrix equations, namely zero-order (Eq. 30), first order (Eq. 31) and second-order (Eq. 32); and application of ortho-normalization properties of Eqs. 5-6 and expansions Eq. 28-29.

$$(\lambda_{0j}^2 M + \lambda_{0j} C_p + K)\, y_{0j} = 0 \qquad (30)$$

$$(\lambda_{0j}^2 M + \lambda_{0j} C_p + K)\, y_{1j} = -(2\lambda_{0j}\lambda_{1j} M + \lambda_{1j} C_p + \lambda_{0j} C_n)\, y_{0j} \qquad (31)$$

$$(\lambda_{0j}^2 M + \lambda_{0j} C_p + K)\, y_{2j} = -(2\lambda_{0j}\lambda_{1j} M + \lambda_{1j} C_p + \lambda_{0j} C_n)\, y_{1j} - ((2\lambda_{0j}\lambda_{2j} + \lambda_{1j}^2) M + \lambda_{2j} C_p + \lambda_{1j} C_n)\, y_{0j} \tag{32}$$

Eq. 30 is identical to Eq. 10. As for Eq. 31, after some tedious but straightforward matrix algebra, it can be reduced to formulas for λ_{1j} and a_{jk}; likewise Eq. 32 yields formulas for λ_{2j} and b_{jk}. Denoting the elements of $\tilde{C}$ as $\tilde{c}_{jk}$, the complex perturbations may be expressed as:

$$\tilde{c}_{jk} = y_{0k}^T\, C\, y_{0j} \tag{33}$$

$$\lambda_{1j} = 0 \tag{34}$$

$$\lambda_{2j} = -\lambda_{0j} \sum_{k=1}^{n} a_{jk}(1 - \delta_{jk})\, \tilde{c}_{jk} \,/\, 2\,(\lambda_{0j} + \omega_{0j}\,\xi_{0j}) \tag{35}$$

$$a_{jk} = \lambda_{0j}\, \tilde{c}_{jk} \,/\, (\lambda_{0k} - \lambda_{0j})\,(\lambda_{0k} + 2\,\omega_{0k}\,\xi_{0k} + \lambda_{0j}) \tag{36}$$

$$b_{jk} = \lambda_{0j} \sum_{l=1}^{n} a_{jl}(1 - \delta_{jl})\, \tilde{c}_{kl} \,/\, (\lambda_{0k} - \lambda_{0j})\,(\lambda_{0k} + 2\,\omega_{0k}\xi_{0k} + \lambda_{0j}) \tag{37}$$

The denominators of Eqs. 36 and 37 indicate that eigenvector perturbations are particularly large when both $\omega_{0j} \approx \omega_{0k}$ and $\xi_{0j} \approx \xi_{0k}$.

The approximated (perturbed) complex eigenvalues and eigenvectors may be rewritten explicitly in terms of their respective real and imaginary parts. The forms in Eqs. 38-40 below are so chosen that the real-valued perturbations may take on some physical interpretation. For example, α may be identified as nonproportionality-induced perturbation of natural frequency.

$$\omega_j = \omega_{0j}\sqrt{1 + \alpha_j} \tag{38}$$

$$\xi_j = \xi_{0j}\sqrt{1 + \beta_j} \tag{39}$$

$$y_j = y_{0j} + \sum_{k=1}^{n} \zeta_{jk}\, y_{0k} + i \sum_{k=1}^{n} \eta_{jk}\, y_{0k} \tag{40}$$

$$\mathrm{Re}\; y_j = y_{0j} + \sum_{k=1}^{n} \zeta_{jk}\, y_{0k} \tag{40a}$$

$$\mathrm{Im}\; y_j = \sum_{k=1}^{n} \eta_{jk}\, y_{0k} \tag{40b}$$

α_j and β_j are nonproportionality-induced perturbations of natural frequency and modal damping, respectively. As for the eigenvector, Eq. 40 states that an eigenvector being complex is equivalent to damping-induced "coupling" of natural modes. As the perturbations ζ_{jk} and η_{jk} are generally not the same for all pairs of j and k, the relative values of these perturbations indicate which natural modes of the counterpart undamped system are significantly coupled due to damping nonproportionality. This can be a useful new way of understanding the complex eigenvectors.

The formulas for α_j, β_j, ζ_{jk} and η_{jk} are summarized below. For compactness of expressions, Eqs. 41-45 are introduced as definitions.

$$\sigma_{oj} = \omega_{oj}\,\xi_{oj} \tag{41}$$

$$\phi_{oj} = \omega_{oj}\sqrt{1-\xi_{oj}^2} \tag{42}$$

$$R_{jk} = [\{(\sigma_{ok}-\sigma_{oj})^2 + (\phi_{ok}^2-\phi_{oj}^2)\}\sigma_{oj} - \{2(\sigma_{ok}-\sigma_{oj})\phi_{oj}\}\phi_{oj}]/D_{jk} \tag{43}$$

$$I_{jk} = -[\{(\sigma_{ok}-\sigma_{oj})^2 + (\phi_{ok}^2-\phi_{oj}^2)\}\phi_{oj} + \{2(\sigma_{ok}-\sigma_{oj})\phi_{oj}\}\sigma_{oj}]/D_{jk} \tag{44}$$

$$D_{jk} = \{(\sigma_{ok}-\sigma_{oj})^2 + (\phi_{ok}^2-\phi_{oj}^2)\}^2 + \{2(\sigma_{ok}-\sigma_{oj})\phi_{oj}\}^2 \tag{45}$$

$$\gamma_j = \sum_{k=1}^{n}(R_{jk} - I_{jk}\,\sigma_{oj}/\phi_{oj})\,\tilde{c}_{jk}^2 / 2\sigma_{oj} \tag{46}$$

$$\kappa_j = -\sum_{k=1}^{n}(R_{jk}\,\sigma_{oj}/\phi_{oj} + I_{jk})\,\tilde{c}_{jk}^2 / 2\phi_{oj} \tag{47}$$

$$\alpha_j = \xi_{oj}^2\,[(1+\gamma_j)^2 - (1+\kappa_j)^2] + (2\kappa_j + \kappa_j^2) \tag{48}$$

$$\beta_j = (\gamma_j^2 + 2\gamma_j - \alpha_j)/(1+\alpha_j) \tag{49}$$

$$\zeta_{jk} = R_{jk}\tilde{c}_{jk} + \sum_{l=1}^{n}(R_{jk}R_{jl} - I_{jk}I_{jl})\,\tilde{c}_{jl}\,\tilde{c}_{kl} \tag{50}$$

$$\eta_{jk} = I_{jk}\tilde{c}_{jk} + \sum_{l=1}^{n}(R_{jk}I_{jl} + I_{jk}R_{jl})\,\tilde{c}_{jl}\,\tilde{c}_{kl} \tag{51}$$

3. FURTHER DISCUSSIONS

With Eqs. 4-5, 11-12 (using C in place of C_p), 26-29, and 33-37, the complex eigenvectors and eigenvalues of Eq. 15 have been expressed in terms of the real eigenvectors, or mode shapes, and real eigenvalues, or natural frequencies, of Eq. 4. Eqs. 38-51 give the explicit approximate formulas for the perturbations on natural frequencies, damping ratios, and mode shapes.

The latter equations may appear cumbersome; but they are in fact explicit formulas ready for computer coding. These may be added easily to standard subroutines that are originally intended for the eigenvalue problem of Eq. 4 subject to eigenvector normalization of Eq. 5. Unlike in numerical algorithms to solve Eq. 15, no iterations are required except in the solution of Eq. 4 itself. This can mean a big reduction in the required numerical calculations, especially when several eigenproblems have to be analyzed with the same **M** and **K**, but different **C**'s.

For two-degree-of-freedom (2DOF) and three-degree-of-freedom (3DOF) systems, even the solution of the counterpart undamped eigenproblem (Eq. 4) can be obtained in closed form, allowing completely explicit approximate formulas for the pseudo natural frequencies, pseudo modal damping ratios, and complex modes. Such explicit approximate formulas for close-coupled 2DOF system have been reported by the authors [10].

Numerical examples and parametric studies are found in References [8], [9], and [10], with discussions of the accuracy of the present method. It has been shown through examples that the absolute values of the perturbations α_j, β_j, ζ_{jk} and η_{jk} indirectly serve as indicator of potential error due to the approximation inherent in the method.

It has also been shown through simple examples [8] that while the nonproportionality being considered by the method is moderate at most, the response error due to disregard of such moderate nonproportionality can be very significant.

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LIST OF SYMBOLS

Matrices

$\underline{\mathbf{C}}$ = damping matrix
$\mathbf{C}$ = offdiagonal matrix from transformation of $\mathbf{C}$ by $\mathbf{Y}_0$
$\mathbf{C}_n$ = nonproportional part of $\mathbf{C}$
$\mathbf{C}_p$ = proportional part of $\mathbf{C}$
$\mathbf{K}$ = stiffness matrix
$\mathbf{M}$ = mass or inertia matrix
$\mathbf{Y}_0$ = modal matrix where column j is mode shape $\mathbf{y}_{0j}$

Vectors

$\mathbf{f}(t)$ = external force
$\mathbf{x}_0$ = initial displacement
$\dot{\mathbf{x}}_0$ = initial velocity
$\mathbf{x}(t)$ = displacement
$\dot{\mathbf{x}}(t)$ = velocity
$\ddot{\mathbf{x}}(t)$ = acceleration
$\mathbf{y}_j$ = complex j-th eigenvector
$\mathbf{y}_{0j}$ = j-th mode, mode shape, or real eigenvector
$\mathbf{y}_{1j}$ = complex first-order perturbation on j-th mode
$\mathbf{y}_{2j}$ = complex second-order perturbation on j-th mode

Common scalars

i = unit imaginary number
t = time

Scalars pertaining to mode j

- $P_j(t)$ = complex coordinate or participation function
- α_j = perturbation on natural frequency
- β_j = perturbation on damping ratio
- γ_j = perturbation paired with κ_j (Eq. 46)
- κ_j = perturbation paired with γ_j (Eq. 47)
- λ_j = complex perturbed eigenvalue
- λ_{0j} = complex unperturbed eigenvalue
- λ_{1j} = complex first-order perturbation on eigenvalue
- λ_{2j} = complex second-order perturbation on eigenvalue
- ξ_{0j} = damping ratio when proportionally damped
- ξ_j = pseudo damping ratio
- σ_{0j} = absolute value of real part of λ_{0j} (Eq. 41)
- ϕ_{0j} = imaginary part of λ_{0j} (Eq. 42)
- ω_{0j} = natural frequency
- ω_j = pseudo natural frequency

Scalars relating modes j and k

- D_{jk} = (Eq. 45)
- I_{jk} = (Eq. 44)
- R_{jk} = (Eq. 43)
- a_{jk} = complex coefficient of first-order perturbation on j-th mode
- b_{jk} = complex coefficient of second-order perturbation on j-th mode
- $\tilde{c}_{jk}$ = element of $\tilde{C}$ (Eq. 33)
- δ_{jk} = Kronecker delta
- ζ_{jk} = perturbation coefficient on real part of j-th mode
- η_{jk} = perturbation coefficient on imaginary part of j-th mode