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# THE EFFECTS OF ACTIVE COOLING ON THE AERODYNAMIC AND AEROTHERMODYNAMIC CHARACTERISTICS OF SLENDER BODIES OF REVOLUTION

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# FOREWORD

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The present effort was under the direction of Dr. B. M. Leadon, while the theoretical investigation was conducted by Dr. Jan Rosciszewski. Mr. F. Cochran was instrumental in the initial phases of the numerical computations and Mr. W. Holst was assigned this difficult task in its final stages of development. Mr. W. V. Carter was in charge of the design and fabrication of the models used in the experimental program and conducted the wind tunnel tests at AEDC. Also, the assistance of Mr. W. H. Gallaher in both the experimental and theoretical efforts is acknowledged.

## ABSTRACT

A new self consistent method has been developed for calculation of boundary layer with mass injection. The system of partial differential equations is reduced to a system of ordinary differential equations by integration over different strips. The velocity and enthalpy profiles are assumed to be expressed by series of error functions. A method of calculating the initial velocity and enthalpy profiles, based on an analysis of singularity, is presented herein. The calculation of the initial profiles is reduced to the solution of a system of transcendential equations. The initial derivatives are calculated from linearlized equations near the singular points and provide a smooth start of integration of the downstream equations. The present method was applied to the cases of a sharp edged body and a blunt body. In addition to the velocity and enthalpy profiles, the pressure distribution and shock layer thickness can be calculated from generalized Newtonian expressions developed as a part of this effort.

This technical report has been reviewed and is approved.

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# I. INTRODUCTION

The task of the present effort was to develop a general numerical solution for the boundary layer equations with mass injection which can be used for both sharp and blunt bodies. The basic difficulty, in solving the problem is associated with the singular nature of the equations at the initial section ($\xi=0$). Previous investigators have omitted the most difficult part of the problem by assuming similar solutions (infinite blowing velocity at $\xi=0$), and starting calculations downstream of the singular point. One therefore assumes rapid decay of the influence from initial profiles. This approach is normally applied with success to the case of a sharp body. However, such a procedure has not been successfully applied to the boundary layer on a blunt body.

In the present report, the multistrip method of integral relations was used to reduce the system of partial differential equations into a system of ordinary differential equations. Profiles of unknown parameters were assumed expressible as series of error functions with the coefficients depending in part upon longitudinal distance. Expressing the unknown parameters as series of error functions rather than polynomial provides a better fit and requires fewer terms in the series, thereby leading to a smaller number of differential equations. Provisions are made for different thicknesses of the unknown parameters: viscous, mass diffusion, and thermal boundary layers.

Despite the title the method is more general and not necessarily restricted to bodies of revolution. It can be used for an arbitrary body. The developed method anables to calculate the boundary layer with variable injection and variable surface temperature.

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## 2. THEORETICAL PROGRAM

### 2.1 General Equations for a Boundary Layer with Mass Injection

#### 2.1.1 Introduction

In solving the compressible boundary layer equations one often encounters serious computational difficulties. The most general finite difference schemes require enormous machine time and present some difficulty in getting started due to the singular point at the leading edge.

A great simplification is achieved by introducing the variable $\chi = \frac{1}{2\sqrt{x}} \int \frac{\rho}{\rho_0} dy$ which reduces the system of partial differential equations to ordinary differential equations. However, the resulting similarity of profiles holds only for certain distributions o: pressure and mass injection rate $\left( v_w \sim \frac{1}{\sqrt{x}} \right)$.

Non-similar solutions are obtained by the von Karman-Pohlhausen integral method in which the unknown velocity is expressed as a polynomial with unknown coefficients, which may depend upon the streamwise coordinate x, and the equations are integrated with respect to $y/\delta$ from zero to one, where y is the coordinate normal to the surface and $\delta$ is the boundary layer thickness (having, indeed, no rigorous meaning). The von Karman integral enables one to calculate $\delta$ as a function of x. To obtain additional integral conditions for use in determinining further terms in the assumed polynomials, one may multiply the differential equations by a certain function of the velocity before integration, thus finding higher moments of the von Karman relation. Another device for the same purpose used by Pallone (Ref. 1) and also by Rosciszewski (Ref. 2) in an unpublished paper, is the Dorodnitzin technique (Ref. 3) of integrating the differential equations in several strips from zero to various successive values of $y/\delta \leq 1$. Ultimately all of these integral methods reduce to the solution of a system of ordinary differential equations for the coefficients of the polynomials and $\delta$ which are functions of x. Pallone omitted the most difficult part of the problem taking the initial profile from the self-similar solution with infinite large blowing velocity at the leading edge to start the downstream integration. This procedure leads to the question regarding the influence of initial conditions and cannot be used in calculating such case as boundary layer at blunt body. A more consistent starting procedure was developed in Rosciszewski's paper based on analysis of singularity. In the present paper basic ideas of this analysis are repeated and developed further.

It is of interest to point out that Dorodnitzin (Ref. 4) and his school retreated from the strip method of the moment method in recent papers employing the integrals in terms of the Crocco variable from zero to $\frac{u}{U} = 1$. However this method requires that all flow parameters are single value functions of velocity. This leads to some difficulties when there is overshoot of some flow parameters (with blowing, this is possible), or when the thickness of viscous diffusion and thermal boundary layers are different. The present method does not lead to such difficulties.

In the present paper the velocity concentration and enthalpy distributions were assumed to be given by the functional series

$$\frac{u}{U} = \sum_{n=0}^{N} a_n(x)\, F_n\left[f(x)\int \frac{\rho}{\rho_0}\, dy\right]$$

$$\frac{h}{h_{01}} = \sum_{m=0}^{M} b_m(x)\, \Phi_m\left[\alpha \cdot f(x)\int \frac{\rho}{\rho_0}\, dy\right]$$

$$\frac{c}{c_1} = \sum_{r=0}^{R} g_r(x)\, \Psi_r\left[\beta\, f(x)\int \frac{\rho}{\rho_0}\, dy\right]$$

where $F_n$, $\Psi_m$ and $\psi_r$ are arbitrary functions, $a_n$, $g_r$, $b_m$ and f are the unknown functions of x, and $\alpha$ , $\beta$ are quantities which permit the viscous, mass diffusion and thermal layers to differ in thickness.

The choice of $F_n$, $\Psi_m$ and $\psi_r$ as error functions raised to the corresponding powers leads to very satisfactory results. The reason for this choice is that the error function is an elementary solution of the diffusion equation and as such is the correct first approximation to the velocity distribution. This may readily be seen from the integral form of the Blasius equation. The selection of higher powers of the error function for additional terms in the series, although suggested by the first therm, is less obvious.

On the one hand, it was found to be infeasible to deal with multiples of the argument instead of powers of the function, and, on the other, it is just as easy to deal with error functions as with exponentials, for example. No straightforward use can be made of orthogonality properties in the successive strip integral method in any case. Thus were we led to try this particular form of series.

Only three strip integrals were required to obtain initial profile in excellent agreement with exact similarity solution with $\hat{v}_\omega$ = 0 for the flat plate which is valid only for $\xi$ = 0. One obtains from these integrals a system of ordinary differential equations for $a_n$, $g_r$, $b_m$, $\alpha$, $\beta$ and f. By analysis of the singularity at x = 0 a system of transcendental equations is found for the initial values $a_{no}$, $g_{ro}$, $b_{mo}$, f, $\alpha_o$ and $\beta_o$.

The initial values of $a_n$ are obtained here once and for all for the case $\rho\mu$ = constant, and $V \neq o$ at x = o, which corresponds to the case of sharp-edged or pointed bodies with attached shock waves. The initial velocity profile is independent of concentration and total enthalpy profile, wall conductivity, and blowing distribution normal to the surface (unless the blowing rate is proportional to $x^{-1/2}$). The calculated initial velocity profiles using three strips and odd powers are compared with the Blasius solution and found to agree very well.

The calculation of the initial concentration and total enthalpy profile is more difficult because for Prandtl and Schmidt number different from unity the thicknesses of the viscous, concentration, and heat diffusion layers are unequal. The resulting initial enthalpy profiles depend upon the ratio of wall to free stream total enthalpy, the free stream Mach number and the Prandtl number, assuming Lewis number equal to unit. Therefore, for each new case the transcendental equations for $\alpha$ and $b_{mo}$ must be solved over again.

The solution of the system of transcendental equations is the most difficult part of the computational problem. This puts some limitation on the number of strips (unknowns $a_n$, $g_r$, $b_m$) which can be used. The assumption $\mu\rho$ = const. decouples the momentum from mass diffusion and energy equations. By this assumption mass diffusion equation is decoupled from the energy equation. There is a possible iteration procedure which would allow $\mu\rho$ to vary in this calculation. At any rate there is no difficulty in considering $\rho\mu$ variable down-stream.

### 2.1.2 Transformation of Laminar Boundary Layer Equations Into the Form of Incompressible Flow Equations

The Dorodnitzin-Stewartson transformation will reduce the equations of conservation of mass and momentum for the boundary layer to the form which they have for an incompressible flow.

The equations for the compressible boundary layer are

Conservation of mass

$$\frac{\partial(\rho u r^{\nu})}{\partial x} + \frac{\partial(\rho v r^{\nu})}{\partial y} = 0 \tag{1}$$

Conservation of momentum

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\mu(T)\frac{\partial u}{\partial y}\right] - \frac{\partial p}{\partial x} \tag{2}$$

Conservation of energy (Ref. 5)

$$\rho u \frac{\partial h_o}{\partial x} + \rho v \frac{\partial h_o}{\partial y} = \frac{\partial}{\partial y}\left[\frac{\mu}{P_r}\frac{\partial h_o}{\partial y} + \mu \frac{P_r - 1}{P_r} u \frac{\partial u}{\partial y}\right] + \tag{3}$$

$$\frac{\partial}{\partial y}\left[\sum_j \frac{\mu}{P_r}\left[L_{e_j} - 1\right](h_j - h_j^*)\frac{\partial c_j}{\partial y}\right]$$

Conservation of mass for a species j (mass diffusion equation)

$$\rho u\frac{\partial c_j}{\partial x}+\rho v\frac{\partial c_j}{\partial y}=\frac{\partial}{\partial y}\left[\left(\mu\frac{Le_j}{Pr}\frac{\partial c_j}{\partial y}\right]+m_j \qquad (4)$$

Diffusion-thermo and thermo-diffusion are neglected in these equations.

$r=r(x)$ - radius of the body (in the axisymmetric case)

$\nu \begin{cases} =1 & \text{- for axisymmetric case} \\ =0 & \text{- for plane flow} \end{cases}$

$\rho$ - denotes mean mass density of the mixture

$u, v$ - velocity components in x and y directions

$\mu$ - coefficient of viscosity

$Pr=\frac{c_p\mu}{\lambda}$ - Prandtl number ($\lambda$ coefficient of heat conduction)

$h_0=\sum_j h_j+\frac{u^2+v^2}{2}\cong\sum_j h_j+\frac{u^2}{2}$ - total enthalpy

$Le_j=\frac{\rho D_j c_{pj}}{\lambda}$ - Lewis Number ( $D_j$ - coefficient of mass diffusion of j-th component)

$h_j=\int c_{pj}dT$ enthalpy of j-th species not including enthalpy of formation

$h_j^*$ - heat of formation of j-th component

$c_j$ - mass fraction of j-th component

$m_j$ - rate of production of j-th component

We introduce the following transformation of coordinates

$$Y=\int_0^y\frac{\rho r^\nu}{\rho_\infty}dy, \quad X=\int_0^x r^\nu dx$$

where $\rho$ depends upon both x and y and $\rho_\infty$ denotes the reference density. Specifically, the following two operators apply:

$$\frac{\partial}{\partial x} = r^\nu \frac{\partial}{\partial X} + \frac{\partial}{\partial Y}\frac{\partial Y}{\partial x}$$

and

$$\frac{\partial}{\partial y} = r^\nu \frac{\rho}{\rho_\infty}\frac{\partial}{\partial Y} \tag{5}$$

Therefore, Eq. (1) becomes

$$r^\nu \frac{\partial \rho u r^\nu}{\partial X} + \frac{\partial \rho u r^\nu}{\partial Y}\frac{\partial Y}{\partial x} + \frac{\rho r^\nu}{\rho_\infty}\frac{\partial \rho v r^\nu}{\partial Y} = 0$$

or, upon introducing

$$\frac{u}{r^\nu}\frac{\partial Y}{\partial x} + \frac{\rho v}{\rho_\infty} = \hat{v} \tag{6}$$

and taking into account that

$$\frac{\partial}{\partial Y}\left(\frac{1}{r^\nu}\frac{\partial Y}{\partial x}\right) = \frac{\rho_\infty}{\rho r^{2\nu}}\frac{\partial}{\partial y}\left(\int_0^y \frac{1}{\rho_\infty}\frac{\partial \rho r^\nu}{\partial x}dy\right) =$$

$$\frac{1}{r^{2\nu}}\frac{1}{\rho}\frac{\partial \rho r^\nu}{\partial x} = \frac{1}{r^\nu \rho}\left(\frac{\partial \rho r^\nu}{\partial X} + \frac{1}{r}\frac{\partial \rho r^\nu}{\partial Y}\frac{\partial Y}{\partial x}\right)$$

the continuity of mass is expressed as

$$\frac{\partial u}{\partial X} + \frac{\partial \hat{v}}{\partial Y} = 0 \tag{7}$$

which is exactly the form for an incompressible flow.

In the same way the conservation of streamwise-momentum becomes

$$\rho u \frac{\partial u}{\partial X} + \rho u \frac{\partial u}{\partial Y} + \rho v \frac{\rho}{\rho_\infty}\frac{\partial u}{\partial Y} = \frac{\rho}{\rho_\infty}\frac{\partial}{\partial Y}\left(\frac{\rho}{\rho_\infty}\mu \frac{\partial u}{\partial Y}\right) - \frac{dp}{dX}$$

or, taking into account Eq. (6) and Eq. (7), we find

$$\frac{\partial u^2}{\partial X} + \frac{\partial u\hat{v}}{\partial Y} = \frac{1}{\rho_\infty}\frac{\partial}{\partial Y}\left(\frac{\rho\mu}{\rho_\infty}\frac{\partial u}{\partial Y}\right) - \frac{1}{\rho}\frac{dp}{dX} \tag{8}$$

This equation again has the form of the conservation of streamwise-momentum for an incompressible flow.

The usual estimates of the order of magnitude of terms in the equation of conservation of momentum normal to the surface again yield the information that $\partial P/\partial Y = 0$ for thin boundary layers.

Finally the energy equation is found to be

$$\frac{\partial u h_o}{\partial X} + \frac{\partial \hat{v} h_o}{\partial Y} = \frac{\partial}{\partial Y}\left[\frac{\rho\mu}{Pr}\frac{\partial h_o}{\partial Y} + \rho\mu\left(1-\frac{1}{Pr}\right)u\frac{\partial u}{\partial Y}\right]$$
$$+ \frac{\partial}{\partial Y}\left[\sum_j \frac{\rho\mu}{Pr}\left[Le_j - 1\right](h_j - h_j^*)\frac{\partial c_j}{\partial Y}\right] \tag{9}$$

For a flat plate and $\rho\mu = \text{const}$ ($\mu \sim T$) equations (7) and (8) are independent of equation (9).

Mass diffusion equation is

$$\frac{\partial(u c_j)}{\partial X} + \frac{\partial \hat{v} c_j}{\partial Y} = \frac{\partial}{\partial Y}\left[\rho\mu\frac{Le_j}{Pr}\frac{\partial c_j}{\partial Y}\right] + \frac{m_j}{\rho} \tag{10}$$

### 2.1.3. Integral Relations Method for a Compressible Boundary Layer with Mass Addition

We shall develop a non-similar method of calculation of the laminar compressible boundary layer with mass addition through a porous wall. This method is similar to the well-known Dorodnitzin method. The difference lies in the series representation of the velocity, enthalpy, and concentration profiles which are here expressed in powers of the error function rather than in powers of $\eta$. The initial value problem is also differently solved. The method will be useful for cases in which coolant injection is employed in an arbitrary distribution such as, for example, the case of injection at a finite rate in the nose region but in that region only.

We now introduce another change of variables

$$\begin{aligned} \eta &= f(\bar{X}) \cdot Y \\ \xi &= \bar{X} \end{aligned} \tag{11}$$

where

$$\bar{X} = \frac{X U_\infty}{\nu_\infty} \,, \quad \bar{Y} = \frac{Y U_\infty}{\nu_\infty}$$

and $\nu_\infty$ and $U_\infty$ denote reference kinematic viscosity and velocity, respectively.

$$\frac{\partial}{\partial \bar{X}} = \frac{\partial}{\partial \xi} + \frac{\eta f_\xi'(\xi)}{f(\xi)} \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial \bar{Y}} = f(\xi) \frac{\partial}{\partial \eta}$$

In Eq. (7) we have

$$\frac{\partial u}{\partial \xi} + f_\xi(\xi) \frac{\partial \hat{v}}{\partial \eta} + \frac{\eta f_\xi'(\xi)}{f(\xi)} \frac{\partial \bar{u}}{\partial \eta} = 0 \tag{12}$$

Equation (8) can be transformed into

$$\frac{\partial u^2}{\partial \xi} + \frac{\eta f_\xi'(\xi)}{f(\xi)} \frac{\partial u^2}{\partial \eta} + f(\xi) \frac{\partial u \hat{v}}{\partial \eta} = \frac{f(\xi)}{\rho_\infty} \frac{\partial}{\partial \eta} \left[ \frac{U_\infty \rho \mu}{\mu_\infty} f(\xi) \frac{\partial u}{\partial \eta} \right] - \frac{1}{\rho} \frac{dp}{d\xi} \tag{13}$$

Integrating equation (12) with respect to $\eta$ from zero to $\eta_i$ = const (Fig. 1) we have (integrating by parts and taking into account that at $\eta = 0, u = 0$);

$$\frac{d}{d\xi}\int_0^{\eta_i} u\,d\eta + \left[\frac{\eta u}{f} f'_{\xi}(\xi)\right]\Bigg|_{\eta=\eta_i} - \frac{f'(\xi)}{f(\xi)}\int_0^{\eta_i} u\,d\eta + f(\xi)\hat{v}\Big|_0^{\eta_i} = 0 \tag{14}$$

Similarly from Eq. (13) we obtain

$$\frac{d}{d\xi}\int_0^{\eta_i} u^2\,d\eta + \left[\frac{\eta u^2}{f} f'_{\xi}(\xi)\right]\Bigg|_{\eta=\eta_i} - \frac{f'(\xi)}{f(\xi)}\int_0^{\eta_i} u^2\,d\eta + \left(f(\xi)u\hat{v}\right)\Big|_0^{\eta_i} = \tag{15}$$

$$= \frac{f^2(\xi)U_\infty}{\rho_\infty \mu_\infty}\rho\mu\frac{\partial u}{\partial \eta}\Bigg|_0^{\eta_i} - \frac{dp}{d\xi}\int_0^{\eta_i}\frac{d\eta}{\rho}$$

Multiplying Eq. (14) through by $u_i = u|_{\eta=\eta_i}$ and subtracting from Eq. (15) we have

$$\frac{d}{d\xi}\int_0^{\eta_i} u^2\,d\eta - u_i\frac{d}{d\xi}\int_0^{\eta_i} u\,d\eta + \frac{f'}{f}\left[u_i\int_0^{\eta_i} u\,d\eta - \int_0^{\eta_i} u^2\,d\eta\right] +$$

$$+ f\hat{v}_w u_i = \frac{f^2 \mu\rho U_\infty}{\rho_\infty \mu_\infty}\frac{\partial u}{\partial \eta}\Bigg|_0^{\eta_i} - \frac{dp}{d\xi}\int_0^{\eta_i}\frac{d\eta}{\rho} \tag{16}$$

We shall express the velocity distribution in the form

$$\frac{u}{U} = a_1(\xi)\operatorname{erf}\eta + a_2(\xi)(\operatorname{erf}\eta)^2 + a_3(\xi)(\operatorname{erf}\eta)^3 + \ldots = \tag{17}$$

$$= \sum_{n=1}^{N} a_n(\xi)(\operatorname{erf}\eta)^n \,.$$

where $\operatorname{erf}\eta = \frac{2}{\sqrt{\pi}}\int_0^{\eta}\exp[-\eta^2]\,d\eta$ is known as the error function and is tabulated in many books. This representation of the velocity profile has an advantage over the Pohlhausen polynomial in that boundary conditions

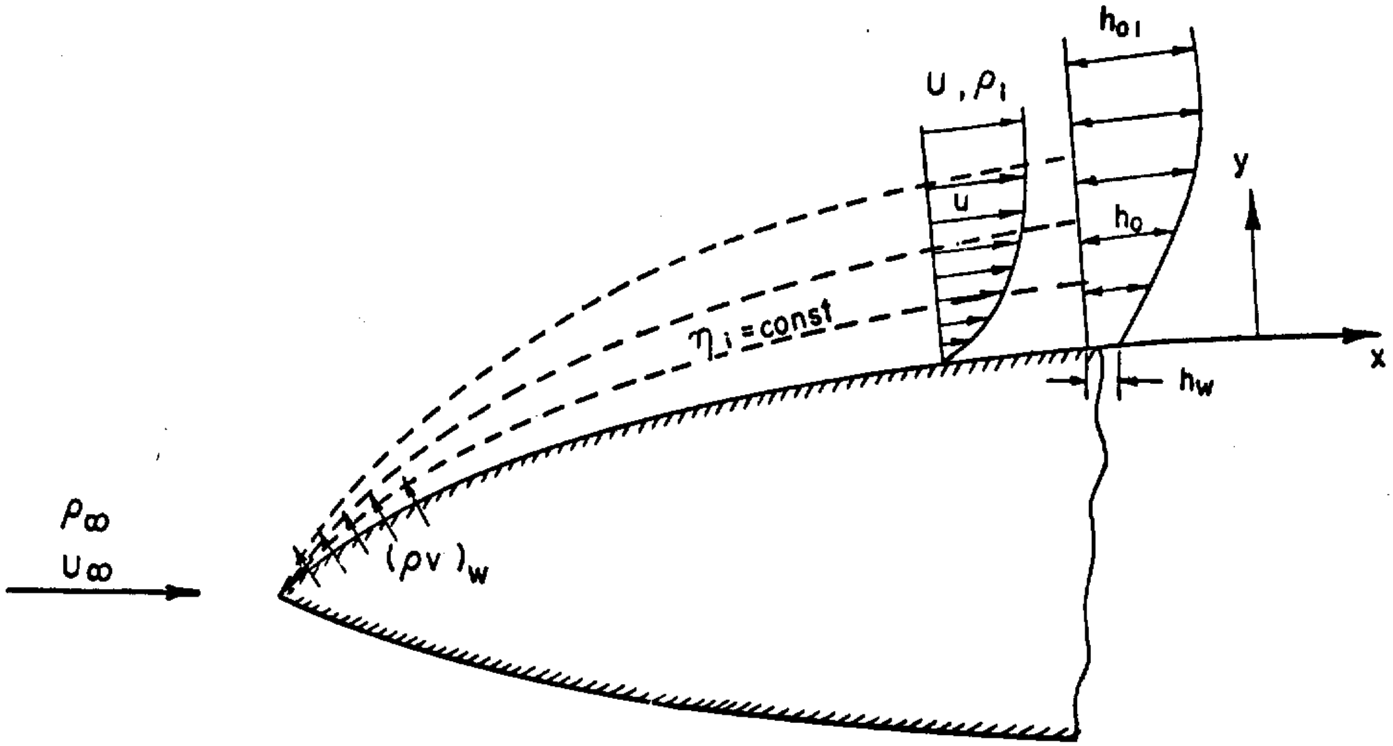


Figure 1. Schematic of a Slender Body Flow Field Showing the Coordinate System.

at $\eta=\infty \; \frac{\partial u}{\partial \eta}=0, \; \frac{\partial^2 u}{\partial \eta^2}=0$ are satisfied automatically. Moreover, $erf\,\eta$ is the elementary solution of the diffusion equation, and it is the first term in an expansion of the Blasius solution. Coefficients $a_1(\xi), a_2(\xi) \ldots .$ are unknown functions to be determined by Eqs. (16) with different $\eta_i$. In fact we have as many ordinary differential equations as necessary taking for example $\eta_i = 0.1, 0.2, \ldots .$

The outer boundary condition for the velocity distribution, $\left.\frac{u}{U}\right|_{\eta=\infty}=1$ is $(erf\,\infty=1)$

$$1=\sum_{n=1}^{N} a_n \tag{18}$$

Substituting expression (17) into the integral relation (16) we obtain

$$\begin{aligned}\sum_{n=1}^{N}\sum_{k=1}^{N}\left[2A_{1nk}(\eta_i)-A_{2nk}(\eta_i)\right]a_n\frac{da_k}{d\xi} &= -\left[\frac{1}{\bar U^2}\frac{d\bar U^2}{d\xi}-\frac{f'}{f}\right]\cdot \\ \sum_{n=1}^{N}\sum_{k=1}^{N} A_{1nk}(\eta_i)a_n a_k &+ \left[\frac{1}{\bar U}\frac{d\bar U}{d\xi}-\frac{f'}{f}\right]\sum_{n=1}^{N}\sum_{k=1}^{N} A_{2nk}(\eta_i)a_n a_k \\ -\frac{\hat v_\omega}{U} f\sum_{n=1}^{N} a_n(erf\,\eta_i)^n &+ \frac{\bar\rho_i f^2 \bar\mu_i \bar\rho_i}{U}\sum_{n=1}^{N} A_{3n}(\eta_i)a_n \\ -\frac{2}{\sqrt{\pi}}\frac{a_1\bar\rho_1\bar\mu_\omega}{U} f^2\bar\rho_\omega &- \frac{1}{\bar U^2}\frac{dp}{d\xi}\int_0^{\eta_i}\frac{d\eta}{\rho}\end{aligned} \tag{19}$$

where

$$A_{1nk}(\eta_i)=\int_0^{\eta_i}(erf\,\eta)^{n+k}d\eta, \quad A_{3n}(\eta_i)=\frac{2n}{\sqrt{\pi}}(erf\,\eta_i)^{n-1}\exp(-\eta_i^2)$$

$$A_{2nk}(\eta_i)=(erf\,\eta_i)^n\int_0^{\eta_i}(erf\,\eta)^k d\eta$$

$$\bar p=\frac{p}{\rho_\infty U_\infty^2}, \quad \bar\rho_i=\frac{\rho_i(\xi)}{\rho_\infty}, \quad \bar\mu_i=\frac{\mu_i}{\mu_\infty}, \quad \bar U=\frac{U}{U_\infty}$$

Expression (19) represents as many ordinary differential equations as necessary for unknowns $a_1, a_2, \ldots a_n$ and $f$.

By analogy with the expression (16) from (9) and (14) multiplied through by $h_{oi} = h_o|_{\eta=\eta_i}$ we obtain the following integral expression

$$\frac{d}{d\xi}\int_0^{\eta_i} u h_o d\eta - h_{oi}\frac{d}{d\xi}\int_0^{\eta_i} u d\eta + \frac{f'}{f}\left[h_{oi}\int_0^{\eta_i} u d\eta - \int_0^{\eta_i} u h_o d\eta\right] +$$

$$f(h_i - h_\omega)\hat{v}_\omega = f^2\left\{\mu_1\bar{\mu}\,\bar{\rho}\,\rho_1\frac{U_\infty}{P_r}\left[\frac{\partial h_o}{\partial \eta} + (P_r - 1)u\frac{\partial u}{\partial \eta}\right] + \right. \tag{20}$$

$$\left.\bar{\mu}_1\bar{\rho}_1\sum_j \frac{\bar{\rho}\bar{\mu}}{P_r}\left[Le_j - 1\right](h_j - h_j^*)\frac{\partial c_j}{\partial \eta}\right\}\Bigg|_0^{\eta_i}$$

The total enthalpy distribution is assumed in the form

$$\bar{h}_o \equiv \frac{h_o}{h_{o1}} = \sum_{m=1}^{M} b_m(\xi)(\mathrm{erf}\,\alpha\eta)^m \tag{21}$$

where $\alpha$ is a scale factor which accounts for different thicknesses of viscous and thermal layers.

For $P_r \ll 1$ viscous diffusion is very small as compared to thermal, and it is much better to consider $T$ or static enthalpy and $u$ explicitly rather than $h_o$ because the thickness of the thermal boundary layer is much greater than that of the viscous boundary layer. In that case the boundary layer can be split into two layers, one a thin viscous boundary layer where temperature is essentially constant $(h = h_\omega)$ and the other a thermal boundary layer in which viscous diffusion is not important $\mu\frac{\partial u}{\partial \eta} \cong 0$. However, at the present time we shall consider the case of Prandtl number close to one.

Substituting expression (21) into (20) we obtain

$$\sum_{n=1}^{N}\sum_{m=0}^{M} B_{1nm} a_n \frac{db_m}{d\xi} + \sum_{n=1}^{N}\sum_{m=0}^{M} B_{4nm} a_n b_m \frac{d\alpha}{d\xi} +$$

$$\sum_{n=1}^{N}\sum_{m=0}^{M} (B_{1nm} - B_{2nm}) b_m \frac{da_n}{d\xi} = -\left[\frac{1}{\bar{U}}\frac{d\bar{U}}{d\xi} - \frac{f'}{f}\right]\sum_{n=1}^{N}\sum_{m=0}^{M}\left[B_{1nm} - B_{2nm}\right] a_n b_m$$

$$-\frac{\hat{U}_\omega}{\bar{U}} f\left\{\sum_{m=0}^{M} b_m \operatorname{erf}(\alpha\eta_i)^m - b_0\right\} + f^2 \frac{\varrho_i}{\bar{U}}\left\{\sum_{m=0}^{M} B_{3m} b_m \cdot \frac{\mu_i \varrho_i}{Pr_i} -\right.$$

$$\frac{2\alpha}{\sqrt{\pi}}\left(\frac{\mu_\omega \varrho_\omega}{Pr_\omega}\right) b_1 + \frac{Pr-1}{h_{01}} \sum_{n=1}^{N}\sum_{k=1}^{N} A_{4nk} a_n a_k\Big\} + f^2\Big\{\sum_j (Le_j - 1)\Big(\frac{\mu_i \varrho_i}{Pr_i}\frac{c_{pj}}{c_p}\cdot$$

$$\left[\sum_{m=0}^{M} b_m \operatorname{erf}\alpha\eta_i - \frac{1}{2}\sum_{n=1}^{N}\sum_{k=1}^{N} a_n a_k (\operatorname{erf}\eta_i)^{n+k} - h_j^*\right]\sum_{r=0}^{R} G_{3r}\, g_{jr}$$

$$\left.\left. - \frac{2\beta}{\sqrt{\pi}} g_1 (b_0 - h_j^*)\left(\frac{\mu_\omega \varrho_\omega}{Pr_\omega}\frac{c_{pj}}{c_p}\right)\right)\right\} \qquad (22)$$

where

$$h_{01} = \frac{h_0}{\bar{U}^2}$$

$$B_{1nm}(\eta_i) = \int_0^{\eta_i} [\operatorname{erf}\eta]^n [\operatorname{erf}\alpha\eta]^m d\eta$$

$$B_{2nm}(\eta_i) = [\operatorname{erf}\alpha\eta_i]^m \int_0^{\eta_i} [\operatorname{erf}\eta]^n d\eta$$

$$B_{3m}(\eta_i) = m[\operatorname{erf}\alpha\eta_i]^{m-1} \frac{2\alpha}{\sqrt{\pi}} \exp[-\alpha^2\eta_i^2]$$

$$B_{4nm}(\eta_i) = \frac{2}{\sqrt{\pi}}\int_0^{\eta_i} (\operatorname{erf}\eta)^n m (\operatorname{erf}\alpha\eta)^{m-1} \exp[-\alpha^2\eta^2]\,\eta\, d\eta$$

$$A_{4nk}(\eta_i) = n[\operatorname{erf}\eta_i]^{n+k-1} \frac{2}{\sqrt{\pi}} \exp[-\eta_i^2]$$

$$G_{3r}(\eta_i) = r[\operatorname{erf}\beta\eta_i]^{r-1} \frac{2\beta}{\sqrt{\pi}} \exp[-\beta^2\eta_i^2]$$

These expressions have been tabulated (Table 1) for different numerical values of $\eta_i$.

Again equations (22) should be solved* along with the boundary condition at $\eta=\infty$

$$\sum_{m=0}^{M} b_m = 1 \tag{23}$$

if $\rho\mu=$ const ($\mu\sim T$) and $\frac{dp}{d\xi}=0$ expression (19) is decoupled from Eq. (22). These expressions represent the system of ordinary differential equations for $a_{nj}, f, b_m, \alpha$ (with $n=1,\dots N; m=1,\dots M$).

From expressions (4) and (10) we obtain by analogy to Eq. (16) the following formula

$$\frac{d}{d\xi}\int_0^{\eta_i} u c_j d\eta - c_{ji}\frac{d}{d\xi}\int_0^{\eta_i} u_j d\eta + \frac{f'}{f}\left[c_{ji}\int_0^{\eta_i} u d\eta - \int_0^{\eta_i} u c_j d\eta\right]$$

$$+ f\hat{v}_w c_{ji} = f^2 \frac{\mu\rho U_\infty}{\rho_\infty \mu_\infty}\frac{Le_j}{Pr}\frac{\partial c_j}{\partial \eta}\Big|_0^{\eta_i} + \frac{\nu_\infty}{U_\infty^2}\int_0^{\eta_i}\frac{m_j}{\rho} d\eta$$

where $c_{ji} \equiv c_j\big|_{\eta=\eta_i}$ (24)

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*Knowing the approximate value of $\alpha$ ($\alpha_0=\sqrt{Pr}$) the Taylor expansion for $\alpha=\alpha_0+\Delta\alpha$ leads to considerable saving of the computing time.

$$B_{inm} = \int_0^{\eta_i}(\mathrm{erf}\,\eta)^n(\mathrm{erf}\,\alpha_0\eta)^m d\eta + \frac{2\Delta\alpha}{\sqrt{\pi}}\int_0^{\eta_i} m(\mathrm{erf}\,\eta)^n(\mathrm{erf}\,\alpha_0\eta)^{m-1}\exp[-\alpha_0^2\eta^2]\eta d\eta$$

$$+\frac{2m(m-1)}{\pi}\Delta\alpha^2\int_0^{\eta_i}(\mathrm{erf}\,\eta)^n(\mathrm{erf}\,\alpha_0\eta)^{m-2}\exp[-2\alpha^2\eta^2]\eta^2 d\eta$$

$$-\frac{2m\alpha_0}{\sqrt{\pi}}\Delta\alpha^2\int_0^{\eta_i}(\mathrm{erf}\,\eta)^n(\mathrm{erf}\,\alpha_0\eta)^{m-1}\exp[-\alpha^2\eta^2]\eta^3 d\eta + O(\Delta\alpha^3)$$

TABLE OF THE •A• COEFFICIENTS FOR N = 1, 3, 5

TABLE 1

| ETA | N | K | A1(N,K) | A2(N,K) | A4(N,K) | N | A3(N) |
|---|---|---|---|---|---|---|---|
| ETA = 0.30 | | | | | | | |
| | 1 | 1 | 0.11058938-01 | 0.16440849-01 | 0.33890019-00 | 1 | 0.10312609+01 |
| | 1 | 3 | 0.72401338-03 | 0.90112195-03 | 0.36599770-01 | | |
| | 1 | 5 | 0.56176397-04 | 0.65371512-04 | 0.39526185-02 | | |
| | 3 | 1 | 0.72401338-03 | 0.17755414-02 | 0.10979931-00 | 3 | 0.33411530-00 |
| | 3 | 3 | 0.56176397-04 | 0.97317314-04 | 0.11857855-01 | | |
| | 3 | 5 | 0.47365192-05 | 0.70598435-05 | 0.12805977-02 | | |
| | 5 | 1 | 0.56176397-04 | 0.19175087-03 | 0.19763093-01 | 5 | 0.60130371-01 |
| | 5 | 3 | 0.47365192-05 | 0.10509853-04 | 0.21343294-02 | | |
| | 5 | 5 | 0.41964709-06 | 0.76243289-06 | 0.23049844-03 | | |
| ETA = 0.40 | | | | | | | |
| | 1 | 1 | 0.25513018-01 | 0.37671999-01 | 0.41191703-00 | 1 | 0.96194132+00 |
| | 1 | 3 | 0.28609270-02 | 0.35499781-02 | 0.75595056-01 | | |
| | 1 | 5 | 0.37890967-03 | 0.44006434-03 | 0.13873212-01 | | |
| | 3 | 1 | 0.28609270-02 | 0.69135691-02 | 0.22610516-00 | 3 | 0.52938648+00 |
| | 3 | 3 | 0.37890967-03 | 0.65130883-03 | 0.41619636-01 | | |
| | 3 | 5 | 0.54446308-04 | 0.80760653-04 | 0.76380399-02 | | |
| | 5 | 1 | 0.37890967-03 | 0.12687789-02 | 0.69366060-01 | 5 | 0.16192176-00 |
| | 5 | 3 | 0.54446308-04 | 0.11952826-03 | 0.12730066-01 | | |
| | 5 | 5 | 0.82136698-05 | 0.14821203-04 | 0.23362231-02 | | |
| ETA = 0.50 | | | | | | | |
| | 1 | 1 | 0.48159266-01 | 0.70502580-01 | 0.45740609-00 | 1 | 0.87878260+00 |
| | 1 | 3 | 0.80533014-02 | 0.99476200-02 | 0.12392045-00 | | |
| | 1 | 5 | 0.15837658-02 | 0.18346553-02 | 0.33572526-01 | | |
| | 3 | 1 | 0.80533014-02 | 0.19100557-01 | 0.37176134-00 | 3 | 0.71423930+00 |
| | 3 | 3 | 0.15837658-02 | 0.26950090-02 | 0.10071757-00 | | |
| | 3 | 5 | 0.33723806-03 | 0.49704477-03 | 0.27286404-01 | | |
| | 5 | 1 | 0.15837658-02 | 0.51747227-02 | 0.16786262-00 | 5 | 0.32250282-00 |
| | 5 | 3 | 0.33723806-03 | 0.73013178-03 | 0.45477340-01 | | |
| | 5 | 5 | 0.75306367-04 | 0.13465936-03 | 0.12320720-01 | | |
| ETA = 0.60 | | | | | | | |
| | 1 | 1 | 0.79907575-01 | 0.11578684-00 | 0.47538156-00 | 1 | 0.78724347+00 |
| | 1 | 3 | 0.18206707-01 | 0.22372428-01 | 0.17334401-00 | | |
| | 1 | 5 | 0.48537641-02 | 0.56048014-02 | 0.63206498-01 | | |
| | 3 | 1 | 0.18206707-01 | 0.42720736-01 | 0.52003210+00 | 3 | 0.86118586+00 |
| | 3 | 3 | 0.48537641-02 | 0.81579247-02 | 0.18962549-00 | | |
| | 3 | 5 | 0.13977206-02 | 0.20437454-02 | 0.69145399-01 | | |
| | 5 | 1 | 0.48537641-02 | 0.15395449-01 | 0.31604748-00 | 5 | 0.52337408+00 |
| | 5 | 3 | 0.13977206-02 | 0.29747211-02 | 0.11524233-00 | | |
| | 5 | 5 | 0.42152493-03 | 0.74523521-03 | 0.42022184-01 | | |
| ETA = 0.80 | | | | | | | |
| | 1 | 1 | 0.17167350-00 | 0.24265525-00 | 0.44153955-00 | 1 | 0.59490581+00 |
| | 1 | 3 | 0.60891718-01 | 0.73861940-01 | 0.24316197-00 | | |
| | 1 | 5 | 0.24971084-01 | 0.28599753-01 | 0.13391268-00 | | |
| | 3 | 1 | 0.60891718-01 | 0.13363362-00 | 0.72948590+00 | 3 | 0.98300087+00 |
| | 3 | 3 | 0.24971084-01 | 0.40676798-01 | 0.40173803-00 | | |
| | 3 | 5 | 0.10996722-01 | 0.15750282-01 | 0.22124272-00 | | |
| | 5 | 1 | 0.24971084-01 | 0.73593894-01 | 0.66956339+00 | 5 | 0.90225375+00 |
| | 5 | 3 | 0.10996722-01 | 0.22401278-01 | 0.36873787-00 | | |
| | 5 | 5 | 0.50546696-02 | 0.86738993-02 | 0.20306908-00 | | |
| ETA = 1.00 | | | | | | | |
| | 1 | 1 | 0.29837598-00 | 0.40960755-00 | 0.34981152-00 | 1 | 0.41510750-00 |
| | 1 | 3 | 0.14158462-00 | 0.16902770-00 | 0.24841694-00 | | |
| | 1 | 5 | 0.76628359-01 | 0.86849333-01 | 0.17641207-00 | | |
| | 3 | 1 | 0.14158462-00 | 0.29088079-00 | 0.74525081+00 | 3 | 0.88435967+00 |
| | 3 | 3 | 0.76628359-01 | 0.12003420-00 | 0.52923621+00 | | |
| | 3 | 5 | 0.44232976-01 | 0.61675628-01 | 0.37583450-00 | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 5 | 1 | 0.76628359-01 | 0.20654757-00 | 0.88206035+00 | 5 | 0.10467061+01 |
| | 5 | 3 | 0.44232976-01 | 0.85241698-01 | 0.62639084+00 | | |
| | 5 | 5 | 0.26543132-01 | 0.43798645-01 | 0.44482038-00 | | |
| ETA = 1.20 | | | | | | | |
| | 1 | 1 | 0.45297115-00 | 0.60250042+00 | 0.24336728-00 | 1 | 0.26734436-00 |
| | 1 | 3 | 0.26131707-00 | 0.30640891-00 | 0.20167153-00 | | |
| | 1 | 5 | 0.16953996-00 | 0.18980788-00 | 0.16711944-00 | | |
| | 3 | 1 | 0.26131707-00 | 0.49927491-00 | 0.60501458+00 | 3 | 0.66462192+00 |
| | 3 | 3 | 0.16953996-00 | 0.25391232-00 | 0.50115832+00 | | |
| | 3 | 5 | 0.11646979-00 | 0.15728838-00 | 0.41546133-00 | | |
| | 5 | 1 | 0.16953996-00 | 0.41373488-00 | 0.83559719+00 | 5 | 0.91792204+00 |
| | 5 | 3 | 0.11646979-00 | 0.21040989-00 | 0.69243555+00 | | |
| | 5 | 5 | 0.82811558-01 | 0.13034039-00 | 0.57380157+00 | | |
| ETA = 1.30 | | | | | | | |
| | 1 | 1 | 0.53810510+00 | 0.70435927+00 | 0.19446788-00 | 1 | 0.20820799-00 |
| | 1 | 3 | 0.33381063-00 | 0.38775777-00 | 0.16964805-00 | | |
| | 1 | 5 | 0.23128349-00 | 0.25723445-00 | 0.14799596-00 | | |
| | 3 | 1 | 0.33381063-00 | 0.61446229+00 | 0.50894415+00 | 3 | 0.54490357+00 |
| | 3 | 3 | 0.23128349-00 | 0.33826847-00 | 0.44398789-00 | | |
| | 3 | 5 | 0.16906890 00 | 0.22440376-00 | 0.38732196-00 | | |
| | 5 | 1 | 0.23128349 00 | 0.53603881+00 | 0.73997983+00 | 5 | 0.79226305+00 |
| | 5 | 3 | 0.16906890-00 | 0.29509544-00 | 0.64553661+00 | | |
| | 5 | 5 | 0.12763034-00 | 0.19576323-00 | 0.56314713+00 | | |
| ETA = 1.40 | | | | | | | |
| | 1 | 1 | 0.62713809+00 | 0.80799632+00 | 0.15135779-00 | 1 | 0.15894172-00 |
| | 1 | 3 | 0.41308910-00 | 0.47534993-00 | 0.13725830-00 | | |
| | 1 | 5 | 0.30188487-00 | 0.33351149-00 | 0.12447222-00 | | |
| | 3 | 1 | 0.41308910-00 | 0.71277870+00 | 0.41177490-00 | 3 | 0.43240726-00 |
| | 3 | 3 | 0.30188487-00 | 0.43106946-00 | 0.37341665-00 | | |
| | 3 | 5 | 0.23195078-00 | 0.30244376-00 | 0.33863162-00 | | |
| | 5 | 1 | 0.30188487-00 | 0.66447252+00 | 0.62236109+00 | 5 | 0.65354507+00 |
| | 5 | 3 | 0.23195078-00 | 0.39091386-00 | 0.56438603+00 | | |
| | 5 | 5 | 0.18364373-00 | 0.27427009-00 | 0.51181154+00 | | |
| ETA = 4.00 | | | | | | | |
| | 1 | 1 | 0.32021297+01 | 0.34358222+01 | 0.12698234-06 | 1 | 0.12698235-06 |
| | 1 | 3 | 0.29642962+01 | 0.30621176+01 | 0.12698233-06 | | |
| | 1 | 5 | 0.28304639+01 | 0.28899756+01 | 0.12698232-06 | | |
| | 3 | 1 | 0.29642962+01 | 0.34358221+01 | 0.38094700-06 | 3 | 0.38094703-06 |
| | 3 | 3 | 0.28304639+01 | 0.30621175+01 | 0.38094699-06 | | |
| | 3 | 5 | 0.27389853+01 | 0.28899755+01 | 0.38094697-06 | | |
| | 5 | 1 | 0.28304639+01 | 0.34358219+01 | 0.63491164-06 | 5 | 0.63491169-06 |
| | 5 | 3 | 0.27389853+01 | 0.30621173+01 | 0.63491161-06 | | |
| | 5 | 5 | 0.26701494+01 | 0.28899754+01 | 0.63491158-06 | | |

TABLE OF THE "B" COEFFICIENTS FOR N = 1, 3, 5     M = 1, 2, 4     ALFA = 0.90000000+00

| ETA | N | M | B1(N,M) | B2(N,M) | B4(N,M) | B5(N,M) | M | B3(M) | B6(M) |
|---|---|---|---|---|---|---|---|---|---|
| ETA = 0.30 | 1 | 1 | 0.99864003-02 | 0.14879520-01 | 0.10778835-01 | 0.15744803-01 | 1 | 0.94414242+00 | 0.89609607+00 |
| | 3 | 1 | 0.65420674-03 | 0.81554562-03 | 0.70221724-03 | 0.86297169-03 | | | |
| | 5 | 1 | 0.50777884-04 | 0.59163413-04 | 0.54335256-04 | 0.62603917-04 | | | |
| | 1 | 2 | 0.22376402-02 | 0.44254445-02 | 0.48147695-02 | 0.93655910-02 | 2 | 0.56161083+00 | 0.11273009+01 |
| | 3 | 2 | 0.16247769-03 | 0.24255835-03 | 0.34819145-03 | 0.51332747-03 | | | |
| | 5 | 2 | 0.13228577-04 | 0.17596292-04 | 0.28279498-04 | 0.37239125-04 | | | |
| | 1 | 4 | 0.13271020-03 | 0.39146490-03 | 0.56878692-03 | 0.16569184-02 | 4 | 0.99357668-01 | 0.40970818-00 |
| | 3 | 4 | 0.10810939-04 | 0.21456167-04 | 0.46221659-04 | 0.90815594-04 | | | |
| | 5 | 4 | 0.93767983-06 | 0.15565286-05 | 0.40024195-05 | 0.65881789-05 | | | |
| ETA = 0.40 | 1 | 1 | 0.23096389-01 | 0.34236910-01 | 0.24378653-01 | 0.34866419-01 | 1 | 0.89209893+00 | 0.73429655+00 |
| | 3 | 1 | 0.25927490-02 | 0.32253676-02 | 0.27099364-02 | 0.32846720-02 | | | |
| | 5 | 1 | 0.34360014-03 | 0.39993745-03 | 0.35714383-03 | 0.40729104-03 | | | |
| | 1 | 2 | 0.67979450-02 | 0.13329452-01 | 0.14268287-01 | 0.27149076-01 | 2 | 0.69464151+00 | 0.12791809+01 |
| | 3 | 2 | 0.84426680-03 | 0.12557319-02 | 0.17593142-02 | 0.25576418-02 | | | |
| | 5 | 2 | 0.11727185-03 | 0.15570759-03 | 0.24330495-03 | 0.31714112-03 | | | |
| | 1 | 4 | 0.69389115-03 | 0.20204481-02 | 0.28917208-02 | 0.82303913-02 | 4 | 0.21058438-00 | 0.81670342+00 |
| | 3 | 4 | 0.96476321-04 | 0.19034100-03 | 0.40030146-03 | 0.77536312-03 | | | |
| | 5 | 4 | 0.14261334-04 | 0.23601805-04 | 0.58999020-04 | 0.96143071-04 | | | |
| ETA = 0.50 | 1 | 1 | 0.43732625-01 | 0.64404809-01 | 0.44867431-01 | 0.62411545-01 | 1 | 0.82937884+00 | 0.54831156+00 |
| | 3 | 1 | 0.73250353-02 | 0.90872499-02 | 0.74010292-02 | 0.88060087-02 | | | |
| | 5 | 1 | 0.14418686-02 | 0.16759758-02 | 0.14441973-02 | 0.16241061-02 | | | |
| | 1 | 2 | 0.15790008-01 | 0.30623308-01 | 0.32109414-01 | 0.59351095-01 | 2 | 0.78870891+00 | 0.12857234+01 |
| | 3 | 2 | 0.29191521-02 | 0.43208210-02 | 0.58697193-02 | 0.83741919-02 | | | |
| | 5 | 2 | 0.60167971-03 | 0.79689585-03 | 0.12015031-02 | 0.15444654-02 | | | |
| | 1 | 4 | 0.24176548-02 | 0.69234046-02 | 0.97210876-02 | 0.26836529-01 | 4 | 0.35662711-00 | 0.12725396+01 |
| | 3 | 4 | 0.49903417-03 | 0.97686350-03 | 0.19928230-02 | 0.37865222-02 | | | |
| | 5 | 4 | 0.10931955-03 | 0.18016448-03 | 0.43451981-03 | 0.69835430-03 | | | |
| ETA = 0.60 | 1 | 1 | 0.72824391-01 | 0.10640725-00 | 0.72196811-01 | 0.96982191-01 | 1 | 0.75867769+00 | 0.35135209-00 |
| | 3 | 1 | 0.16629767-01 | 0.20560094-01 | 0.16129025-01 | 0.18738977-01 | | | |
| | 5 | 1 | 0.44391187-02 | 0.51507706-02 | 0.42515519-02 | 0.46945394-02 | | | |
| | 1 | 2 | 0.30853648-01 | 0.59049535-01 | 0.60389502-01 | 0.10763840-00 | 2 | 0.84203972+00 | 0.11574137+01 |
| | 3 | 2 | 0.77550675-02 | 0.11409599-01 | 0.14935105-01 | 0.20797979-01 | | | |
| | 5 | 2 | 0.21650785-02 | 0.20581636-02 | 0.41280096-02 | 0.52103662-02 | | | |
| | 1 | 4 | 0.64799769-02 | 0.18104736-01 | 0.24950600-01 | 0.66296071-01 | 4 | 0.91862460+00 | 0.16582426+01 |
| | 3 | 4 | 0.18127079-02 | 0.35136695-02 | 0.69106346-02 | 0.12809780-01 | | | |
| | 5 | 4 | 0.53707012-03 | 0.88025402-03 | 0.20334028-02 | 0.32091409-02 | | | |
| ETA = 0.80 | 1 | 1 | 0.15776579-00 | 0.22608762-00 | 0.14366961-00 | 0.17576541-00 | 1 | 0.60472696+00 | -0.24726591-01 |
| | 3 | 1 | 0.56158731-01 | 0.68810914-01 | 0.49215344-01 | 0.53501320-01 | | | |
| | 5 | 1 | 0.23076890-01 | 0.26647066-01 | 0.19772725-01 | 0.20716008-01 | | | |
| | 1 | 2 | 0.84412721-01 | 0.15632445-00 | 0.15026751-00 | 0.24306005-00 | 2 | 0.83625639+00 | 0.61593033+00 |
| | 3 | 2 | 0.32856619-01 | 0.47583672-01 | 0.56840900-01 | 0.73985161-01 | | | |
| | 5 | 2 | 0.14079826-01 | 0.18424661-01 | 0.23921046-01 | 0.28647465-01 | | | |
| | 1 | 4 | 0.28017423-01 | 0.74735541-01 | 0.96036968-01 | 0.23240413-00 | 4 | 0.79959433+00 | 0.18321715+01 |
| | 3 | 4 | 0.12044332-01 | 0.22748785-01 | 0.40893315-01 | 0.70741599-01 | | | |
| | 5 | 4 | 0.54555354-02 | 0.88084558-02 | 0.18293852-01 | 0.27391540-01 | | | |
| ETA = 1.00 | 1 | 1 | 0.27673244-00 | 0.38734936-00 | 0.22705382-00 | 0.24398940-00 | 1 | 0.45177173-00 | -0.31122052-00 |
| | 3 | 1 | 0.13194825-00 | 0.15984269-00 | 0.10210481-00 | 0.10068410-00 | | | |
| | 5 | 1 | 0.71609782-01 | 0.82129915-01 | 0.53495886-01 | 0.51733217-01 | | | |
| | 1 | 2 | 0.17352723-00 | 0.30868198-00 | 0.27480309-00 | 0.38887443-00 | 2 | 0.72004142+00 | -0.42478088-01 |
| | 3 | 2 | 0.89793442-01 | 0.12737999-00 | 0.13611547-00 | 0.16047202-00 | | | |
| | 5 | 2 | 0.50643383-01 | 0.65450074-01 | 0.74613397-01 | 0.82453275-01 | | | |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | 4 | 0.78264554-01 | 0.19603252-00 | 0.23668629-00 | 0.49391958-00 | 4 | 0.91454344+00 | 0.10981800+01 |
| | 3 | 4 | 0.44331074-01 | 0.80894330-01 | 0.13037429-00 | 0.20381971-00 | | | |
| | 5 | 4 | 0.26303890-01 | 0.41564893-01 | 0.75835612-01 | 0.10472606-00 | | | |
| ETA = 1.20 | | | | | | | | | |
| | 1 | 1 | 0.42402538-00 | 0.57801984+00 | 0.30861789-00 | 0.27915243-00 | 1 | 0.31632718-00 | -0.46844538-00 |
| | 3 | 1 | 0.24604606-00 | 0.29395901-00 | 0.16507417-00 | 0.14196636-00 | | | |
| | 5 | 1 | 0.16016494-00 | 0.18209568-00 | 0.10220509-00 | 0.87942401-01 | | | |
| | 1 | 2 | 0.29702442-00 | 0.50479996+00 | 0.41138314-00 | 0.48758236-00 | 2 | 0.55251375+00 | -0.55137704+00 |
| | 3 | 2 | 0.18556868-00 | 0.25672215-00 | 0.24161957-00 | 0.24796594-00 | | | |
| | 5 | 2 | 0.12506245-00 | 0.15902895-00 | 0.15631916-00 | 0.15360484-00 | | | |
| | 1 | 4 | 0.16526067-00 | 0.38501040-00 | 0.42823172-00 | 0.74375711+00 | 4 | 0.84280331+00 | -0.27012213-01 |
| | 3 | 4 | 0.11195284-00 | 0.19580171-00 | 0.27876717-00 | 0.37824672-00 | | | |
| | 5 | 4 | 0.78965370-01 | 0.12129122-00 | 0.19101905-00 | 0.23430850-00 | | | |
| ETA = 1.30 | | | | | | | | | |
| | 1 | 1 | 0.50597654+00 | 0.68022171+00 | 0.34530382-00 | 0.28140802-00 | 1 | 0.25834014-00 | -0.49882608-00 |
| | 3 | 1 | 0.31583135-00 | 0.37446977-00 | 0.19629335-00 | 0.15491831-00 | | | |
| | 5 | 1 | 0.21960349-00 | 0.24841933-00 | 0.12877799-00 | 0.10277119-00 | | | |
| | 1 | 2 | 0.36982041-00 | 0.61356028+00 | 0.47653124-00 | 0.50766031+00 | 2 | 0.46604583-00 | -0.70707931+00 |
| | 3 | 2 | 0.24756651-00 | 0.33777189-00 | 0.29706722-00 | 0.27947276-00 | | | |
| | 5 | 2 | 0.17787538-00 | 0.22407434-00 | 0.20352117-00 | 0.18539931-00 | | | |
| | 1 | 4 | 0.22271534-00 | 0.49919558-00 | 0.53098378+00 | 0.82606972+00 | 4 | 0.75835425+00 | -0.52310327+00 |
| | 3 | 4 | 0.16089844-00 | 0.27481283-00 | 0.36624337-00 | 0.45476075-00 | | | |
| | 5 | 4 | 0.12067121-00 | 0.18230796-00 | 0.26550714-00 | 0.30168353-00 | | | |
| ETA = 1.40 | | | | | | | | | |
| | 1 | 1 | 0.59222529+00 | 0.78504567+00 | 0.37813798-00 | 0.27399314-00 | 1 | 0.20759251-00 | -0.50172801+00 |
| | 3 | 1 | 0.39263213-00 | 0.46184789-00 | 0.22551472-00 | 0.16119209-00 | | | |
| | 5 | 1 | 0.28799970-00 | 0.32403830-00 | 0.15478732-00 | 0.11309440-00 | | | |
| | 1 | 2 | 0.44866285-00 | 0.72635239+00 | 0.53653891+00 | 0.50701655+00 | 2 | 0.38414407-00 | -0.79436122+00 |
| | 3 | 2 | 0.31777795-00 | 0.42731822-00 | 0.35047652-00 | 0.29828140-00 | | | |
| | 5 | 2 | 0.24040851-00 | 0.29981184-00 | 0.25106360-00 | 0.20927799-00 | | | |
| | 1 | 4 | 0.28860904-00 | 0.62180220+00 | 0.63121569+00 | 0.86807454+00 | 4 | 0.65770178+00 | -0.90094760+00 |
| | 3 | 4 | 0.21958828-00 | 0.36581061-00 | 0.45546903-00 | 0.51069435+00 | | | |
| | 5 | 4 | 0.17295126-00 | 0.25665733-00 | 0.34494456-00 | 0.35830959-00 | | | |
| ETA = 4.00 | | | | | | | | | |
| | 1 | 1 | 0.31566386+01 | 0.34358210+01 | 0.51772273+00 | 0.36482896-04 | 1 | 0.23891374-05 | -0.66152558-04 |
| | 3 | 1 | 0.29336887+01 | 0.30621165+01 | 0.35976418-00 | 0.32514755-04 | | | |
| | 5 | 1 | 0.28068303+01 | 0.28899746+01 | 0.28401063-00 | 0.30686884-04 | | | |
| | 1 | 2 | 0.29910497+01 | 0.34358197+01 | 0.80626062+00 | 0.72965764-04 | 2 | 0.47782730-05 | -0.13230501-03 |
| | 3 | 2 | 0.28377885+01 | 0.30621154+01 | 0.61005385+00 | 0.65029486-04 | | | |
| | 5 | 2 | 0.27391145+01 | 0.28899734+01 | 0.50108384+00 | 0.61373746-04 | | | |
| | 1 | 4 | 0.27894699+01 | 0.34358171+01 | 0.11355228+01 | 0.14593142-03 | 4 | 0.95565385-05 | -0.26460962-03 |
| | 3 | 4 | 0.26998856+01 | 0.30621130+01 | 0.94143508+00 | 0.13005887-03 | | | |
| | 5 | 4 | 0.26336463+01 | 0.28899713+01 | 0.81361049+00 | 0.12274739-03 | | | |

NORY = -1,
NT = 1,3,5,
●

TABLE OF THE *B* COEFFICIENTS FOR N = 1, 3, 5    M = 1, 3, 5    ALFA = 0.50000000+00

| | N | M | B1(N,M) | B2(N,M) | B4(N,M) | B5(N,M) | M | B3(M) | B6(M) |
|---|---|---|---|---|---|---|---|---|---|
| ETA = 0.30 | 1 | 1 | 0.99864003-02 | 0.14879520-01 | 0.10778835-01 | 0.15744803-01 | 1 | 0.94414242+00 | 0.89609607+00 |
| | 3 | 1 | 0.65420674-03 | 0.81554562-03 | 0.70721724-03 | 0.86297169-03 | | | |
| | 5 | 1 | 0.50777884-04 | 0.59163413-04 | 0.54335256-04 | 0.62603917-04 | | | |
| | 1 | 3 | 0.53413829-03 | 0.11162090-02 | 0.17199786-07 | 0.41782499-02 | 3 | 0.25055016-00 | 0.76804054+00 |
| | 3 | 3 | 0.41487509-04 | 0.72141337-04 | 0.13317971-03 | 0.22900961-03 | | | |
| | 5 | 3 | 0.35003760-05 | 0.52334628-05 | 0.11714294-04 | 0.16611406-04 | | | |
| | 1 | 5 | 0.33897043-04 | 0.11642890-03 | 0.10135255-03 | 0.61599768-03 | 5 | 0.36938509-01 | 0.19140506-00 |
| | 3 | 5 | 0.28612328-05 | 0.63814609-05 | 0.15277546-04 | 0.31762799-04 | | | |
| | 5 | 5 | 0.25369747-06 | 0.46294039-06 | 0.13527454-05 | 0.24493077-05 | | | |
| ETA = 0.40 | 1 | 1 | 0.23096389-01 | 0.34236910-01 | 0.24370653-01 | 0.34866419-01 | 1 | 0.89209893+00 | 0.73429655+00 |
| | 3 | 1 | 0.25927490-02 | 0.32253676-02 | 0.27099364-02 | 0.32846720-02 | | | |
| | 5 | 1 | 0.34360014-03 | 0.39993745-03 | 0.35714383-03 | 0.40729104-03 | | | |
| | 1 | 3 | 0.21294799-02 | 0.51895534-02 | 0.66766565-02 | 0.15854919-01 | 3 | 0.40566702-00 | 0.11601609+01 |
| | 3 | 3 | 0.28254824-03 | 0.48889393-03 | 0.88100715-03 | 0.14936495-02 | | | |
| | 5 | 3 | 0.40647376-04 | 0.60621615-04 | 0.12629082-03 | 0.18520877-03 | | | |
| | 1 | 5 | 0.23734705-03 | 0.78662080-03 | 0.12073927-02 | 0.40054213-02 | 5 | 0.10248348-00 | 0.50182662+00 |
| | 3 | 5 | 0.33451334-04 | 0.74105439-04 | 0.17321423-03 | 0.37734002-03 | | | |
| | 5 | 5 | 0.50532512-05 | 0.91888874-05 | 0.26101267-04 | 0.46789211-04 | | | |
| ETA = 0.50 | 1 | 1 | 0.43732625-01 | 0.64404809-01 | 0.44867431-01 | 0.62411545-01 | 1 | 0.82937884+00 | 0.54831156+00 |
| | 3 | 1 | 0.73250353-02 | 0.90872499-02 | 0.74010292-02 | 0.88060087-02 | | | |
| | 5 | 1 | 0.14418686-02 | 0.16759758-02 | 0.14441973-02 | 0.16241061-02 | | | |
| | 1 | 3 | 0.60603054-02 | 0.14560872-01 | 0.18365996-01 | 0.42330539-01 | 3 | 0.56252498+00 | 0.14621223+01 |
| | 3 | 3 | 0.11951008-02 | 0.20544713-02 | 0.35905999-02 | 0.59726624-02 | | | |
| | 5 | 3 | 0.25492898-03 | 0.37890939-03 | 0.76162279-03 | 0.11015476-02 | | | |
| | 1 | 5 | 0.99059439-03 | 0.32919521-02 | 0.49596125-02 | 0.15950347-01 | 5 | 0.21196207-00 | 0.96173877+00 |
| | 3 | 5 | 0.21155299-03 | 0.46448071-03 | 0.10537814-02 | 0.22505275-02 | | | |
| | 5 | 5 | 0.47337946-04 | 0.85664914-04 | 0.23476033-03 | 0.41506836-03 | | | |
| ETA = 0.60 | 1 | 1 | 0.72824391-01 | 0.10640725-00 | 0.72196811-01 | 0.96982191-01 | 1 | 0.75867769+00 | 0.35135209-00 |
| | 3 | 1 | 0.16629767-01 | 0.20560094-01 | 0.16129025-01 | 0.18738977-01 | | | |
| | 5 | 1 | 0.44391187-02 | 0.51507706-02 | 0.42515519-02 | 0.46945394-02 | | | |
| | 1 | 3 | 0.13875408-01 | 0.32768891-01 | 0.40356766-01 | 0.89599127-01 | 3 | 0.70092104+00 | 0.16022778+01 |
| | 3 | 3 | 0.37132278-02 | 0.63316317-02 | 0.10665943-01 | 0.17312416-01 | | | |
| | 5 | 3 | 0.10719139-02 | 0.15862175-02 | 0.30538921-02 | 0.43371533-02 | | | |
| | 1 | 5 | 0.31062124-02 | 0.10091420-01 | 0.14866341-01 | 0.45987845-01 | 5 | 0.35975628-00 | 0.14781702+01 |
| | 3 | 5 | 0.89813712-03 | 0.19498723-02 | 0.42637557-02 | 0.88858088-02 | | | |
| | 5 | 5 | 0.27163838-03 | 0.48848731-03 | 0.12821536-02 | 0.22260968-02 | | | |
| ETA = 0.80 | 1 | 1 | 0.15776579-00 | 0.22608762-00 | 0.14366961-00 | 0.17576543-00 | 1 | 0.60472696+00 | -0.24726591-01 |
| | 3 | 1 | 0.56158731-01 | 0.68818914-01 | 0.49215344-01 | 0.53501320-01 | | | |
| | 5 | 1 | 0.23076890-01 | 0.26647066-01 | 0.19772725-01 | 0.20716008-01 | | | |
| | 1 | 3 | 0.47775838-01 | 0.10808789-00 | 0.12545207-00 | 0.25208963-00 | 3 | 0.86732294+00 | 0.13130875+01 |
| | 3 | 3 | 0.19711044-01 | 0.32900923-01 | 0.50620310-01 | 0.76733676-01 | | | |
| | 5 | 3 | 0.87146828-02 | 0.12739419-01 | 0.22050295-01 | 0.29711705-01 | | | |
| | 1 | 5 | 0.16838811-01 | 0.51674622-01 | 0.72008567-01 | 0.70086487-00 | 5 | 0.69108242+00 | 0.21207923+01 |
| | 3 | 5 | 0.74640796-02 | 0.15729261-01 | 0.31454809-01 | 0.61141348-01 | | | |
| | 5 | 5 | 0.34468666-02 | 0.60904573-02 | 0.14372824-01 | 0.23674269-01 | | | |
| ETA = 1.00 | 1 | 1 | 0.27673244-00 | 0.38734936-00 | 0.22705382-00 | 0.24398940-00 | 1 | 0.45177173-00 | -0.31122052-00 |
| | 3 | 1 | 0.13194825-00 | 0.15984269-00 | 0.10210481-00 | 0.10068410-00 | | | |
| | 5 | 1 | 0.71609782-01 | 0.82129915-01 | 0.53495886-01 | 0.51733217-01 | | | |
| | 1 | 3 | 0.11463796-00 | 0.24599127-00 | 0.26535864-00 | 0.46484598-00 | 3 | 0.86071064+00 | 0.49138054-00 |
| | 3 | 3 | 0.62552934-01 | 0.10151019-00 | 0.13988137-00 | 0.19182226-00 | | | |
| | 5 | 3 | 0.36303383-01 | 0.92197676-01 | 0.79290657-01 | 0.98561568-01 | | | |

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| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 0.54659703-01 | 0.15621997-00 | 0.20327523-00 | 0.49201085-00 | 5 | 0.91100924+00 | 0.16677762+01 |
| 3 | 5 | 0.31833661-01 | 0.64465375-01 | 0.11568216-00 | 0.20303206-00 | | | |
| 5 | 5 | 0.19225036-01 | 0.33123415-01 | 0.68684893-01 | 0.10432135-00 | | | |
| ETA = 1.20 | | | | | | | | |
| 1 | 1 | 0.42402538-00 | 0.57801984+00 | 0.30861789-00 | 0.27915243-00 | 1 | 0.31632718-00 | -0.46844538-00 |
| 3 | 1 | 0.24604606-00 | 0.29395901-00 | 0.16507417-00 | 0.14196636-00 | | | |
| 5 | 1 | 0.16016494-00 | 0.18209568-00 | 0.10220509-00 | 0.87942401-01 | | | |
| 1 | 3 | 0.21825482-00 | 0.44085512-00 | 0.43680518-00 | 0.63872780+00 | 3 | 0.72378724+00 | -0.37274793-00 |
| 3 | 3 | 0.14300252-00 | 0.22420222-00 | 0.27255026-00 | 0.32483277-00 | | | |
| 5 | 3 | 0.98884726-01 | 0.13888418-00 | 0.18215173-00 | 0.20122073-00 | | | |
| 1 | 5 | 0.12775058-00 | 0.33623973-00 | 0.40403850-00 | 0.81192838+00 | 5 | 0.92005295+00 | 0.41484791-00 |
| 3 | 5 | 0.88710928-01 | 0.17099880-00 | 0.27139453-00 | 0.41291604-00 | | | |
| 5 | 5 | 0.63568116-01 | 0.10592682-00 | 0.18968652-00 | 0.25578474-00 | | | |
| ETA = 1.30 | | | | | | | | |
| 1 | 1 | 0.50597654+00 | 0.68022171+00 | 0.34530382-00 | 0.28140802-00 | 1 | 0.25834014-00 | -0.49882608-00 |
| 3 | 1 | 0.31583135-00 | 0.37446977-00 | 0.19629335-00 | 0.15491831-00 | | | |
| 5 | 1 | 0.21960349-00 | 0.24841933-00 | 0.12877799-00 | 0.10277119-00 | | | |
| 1 | 3 | 0.28292403-00 | 0.55343164+00 | 0.52358172+00 | 0.68686474+00 | 3 | 0.63056032+00 | -0.69581570+00 |
| 3 | 3 | 0.19808661-00 | 0.30467039-00 | 0.34641616-00 | 0.37812684-00 | | | |
| 5 | 3 | 0.14581464-00 | 0.20211516-00 | 0.24504167-00 | 0.25084539-00 | | | |
| 1 | 5 | 0.17879998-00 | 0.45027463-00 | 0.51811162+00 | 0.93139406+00 | 5 | 0.85504483+00 | -0.23606680-00 |
| 3 | 5 | 0.13220585-00 | 0.24788129-00 | 0.36852259-00 | 0.51274300+00 | | | |
| 5 | 5 | 0.10063459-00 | 0.16444186-00 | 0.27240479-00 | 0.34014834-00 | | | |
| ETA = 1.40 | | | | | | | | |
| 1 | 1 | 0.59222529+00 | 0.78504567+00 | 0.37813798-00 | 0.27349314-00 | 1 | 0.20759251-00 | -0.50172801+00 |
| 3 | 1 | 0.39263213-00 | 0.46184789-00 | 0.22551472-00 | 0.16119209-00 | | | |
| 5 | 1 | 0.28799970-00 | 0.32403830-00 | 0.15478732-00 | 0.11309440-00 | | | |
| 1 | 3 | 0.35499996-00 | 0.67204725+00 | 0.60583854+00 | 0.70366481+00 | 3 | 0.53313578+00 | -0.91638452+00 |
| 3 | 3 | 0.26227758-00 | 0.39537013-00 | 0.41963417-00 | 0.41397096-00 | | | |
| 5 | 3 | 0.20299033-00 | 0.27739666-00 | 0.31022716-00 | 0.79044724-00 | | | |
| 1 | 5 | 0.23904496-00 | 0.57531367+00 | 0.63262047+00 | 0.10039670+01 | 5 | 0.76066156+00 | -0.77650353+00 |
| 3 | 5 | 0.18586890-00 | 0.31846109-00 | 0.47046530-00 | 0.59064090+00 | | | |
| 5 | 5 | 0.14844076-00 | 0.23746856-00 | 0.36317162-00 | 0.41440109-00 | | | |
| ETA = 4.00 | | | | | | | | |
| 1 | 1 | 0.31566386+01 | 0.34358210+01 | 0.51772273+00 | 0.36487896-04 | 1 | 0.23891374-05 | -0.66152558-04 |
| 3 | 1 | 0.29336887+01 | 0.30621165+01 | 0.35976418-00 | 0.32514755-04 | | | |
| 5 | 1 | 0.28068303+01 | 0.28899746+01 | 0.28401063-00 | 0.30686884-04 | | | |
| 1 | 3 | 0.28762169+01 | 0.34358183+01 | 0.99692609+00 | 0.10944860-03 | 3 | 0.71674067-05 | -0.19845737-03 |
| 3 | 3 | 0.27620488+01 | 0.30621142+01 | 0.79675469+00 | 0.97544193-04 | | | |
| 5 | 3 | 0.26823301+01 | 0.28899724+01 | 0.67320602+00 | 0.92060584-04 | | | |
| 1 | 5 | 0.27203224+01 | 0.34358157+01 | 0.12425920+01 | 0.18241420-03 | 5 | 0.11945669-04 | -0.33076178-03 |
| 3 | 5 | 0.26474200+01 | 0.30621119+01 | 0.10586220+01 | 0.16257353-03 | | | |

The concentration profile will be assumed in the form of power series

$$\frac{C_j}{C_{oj}} = g_o(\xi) + g_{j1}(\xi)\operatorname{erf}\beta_j\eta + g_{j2}(\xi)(\operatorname{erf}\beta_j\eta)^2 + \ldots =$$

$$= \sum_{r=1}^{R} g_{jr}(\xi)(\operatorname{erf}\beta_j\eta)^r \tag{25}$$

where $\beta_j = \beta_j(\xi)$ coefficient which accounts for different thickness of viscous and mass diffusion layers.

Introducing expression (25) into Eq. (24) one obtains

$$\sum_{n=1}^{N}\sum_{r=0}^{R} G_{1nr} a_n \frac{dg_r}{d\xi} + \sum_{n=1}^{N}\sum_{r=0}^{R} G_{4nr} a_n g_r \frac{d\beta}{d\xi} +$$

$$+ \sum_{n=1}^{N}\sum_{r=0}^{R} (G_{1nr} - G_{2nr}) g_r \frac{dg_r}{d\xi} = -\left(\frac{1}{\bar{U}}\frac{d\bar{U}}{d\xi} - \frac{f'}{f}\right).$$

$$\sum_{n=1}^{N}\sum_{r=0}^{R} (G_{1nr} - G_{2nr}) g_{jr} a_n - \frac{1}{C_{oj}}\frac{dC_{oj}}{d\xi}\sum_{n=1}^{N}\sum_{r=0}^{R} G_{1nr} a_n g_{jr} +$$

$$f^2\left\{\frac{\bar{\rho}_1 \mu_i \rho_i}{\bar{U}\, S_{cj}} \sum_{r=0}^{R} G_{3r} g_r - \frac{2}{\sqrt{\pi}} \frac{g_{j1}\bar{\rho}_1}{S_{cj}} \frac{\bar{\mu}_\omega \bar{\rho}_\omega}{\bar{U}}\right\}$$

$$+ \frac{1}{\bar{U}^2}\int_0^{\eta_i} \frac{\bar{m}_j}{\bar{\rho}} d\eta \tag{26}$$

where

$$\bar{m}_j = \frac{m_j \nu_\infty}{U_\infty^2 \bar{\xi}_i C_{oj} \rho_\infty} \ , \quad S_{cj} = \frac{P_r}{L_{ej}} = \frac{\mu}{\rho D_j} \text{ — Schmidt Number}$$

$$G_{1nr}(\eta_i) = \int_0^{\eta_i} [\text{erf}\,\eta]^n [\text{erf}\,\beta_j \eta]^r d\eta$$

$$G_{2nr}(\eta_i) = [\text{erf}\,\beta_j \eta_i]^r \int_0^{\eta_i} [\text{erf}\,\eta]^n d\eta$$

$$G_{4nr}(\eta_i) = \frac{2}{\sqrt{\pi}} \int_0^{\eta_i} (\text{erf}\,\eta)^n r [\text{erf}\,\beta_j \eta]^{r-1} \exp[-\beta_j^2 \eta^2] \eta \, d\eta$$

It is interesting to point out that if the Schmidt number is equal to one quantity $\beta_j = 1$ and with mass source $m_j = 0$ (no dissociation) and $c_j = 0$ at $\eta = 0$ and $\bar{c}_j = 1$ for $\eta = 1$ $\left(\sum_{r=0}^{R} g_{jr} = 1\right)$ equations (26) then become identical with expression (19) for $\frac{dp}{d\xi} = 0$ (flat plate).

## 2.2 The Boundary Layer On a Sharp Nosed Body

### 2.2.1 Calculation of the Initial Profiles (at $\xi = 0$ )

To start calculations the initial values of $a_n = a_{n_0}$ and $b_m = b_{m_0}$, $c_j = c_{j_0}$ at $\xi = 0$ should be known. We shall present below the analysis for finding these values.

At $\xi = 0$ , the coefficients $a_{10}, \ldots a_{n_0}$ must be finite in order for the velocity to remain finite, but $\left.\frac{\partial u}{\partial \eta}\right|_{\eta=\xi=0} = \frac{2}{\sqrt{\pi}} a_1(\xi) f(\xi)$ must be infinitely large because at $\xi = 0$ the thickness of the boundary layer is zero and the velocity changes from zero at the wall to a finite value outside the boundary layer (velocity gradient $\sim \frac{\Delta u}{\delta} = \infty$ ). Therefore $f(\xi)$ must vary as follows

$$f(\xi) \sim \frac{1}{\xi^\beta}$$

where $\beta$ is an arbitrary power ( $\beta > 0$ ).

Dividing both sides of expression (19) through by $f^2$ and dropping all terms which fails to zero as $\xi$ approaches zero (assuming that $\hat{v}_\omega$ is finite at $\xi = 0$) we have

$$\overset{T}{\Phi} \equiv \frac{f'}{f^3}\sum_{n=1}^{N}\sum_{k=1}^{N}(A_{1nk}-A_{2nk})a_n a_k + \bar{\rho}_1 \frac{\bar{u}_i \bar{\rho}_i}{\bar{U}} \sum_{n=1}^{N} A_{3n} a_n - \frac{2}{\sqrt{\pi}} \frac{a_1 \bar{\rho}_1 \bar{\mu}_\omega \bar{\rho}_\omega}{\bar{U}} = 0 ,$$

or

$$\frac{1}{f^3}\frac{df}{d\xi} = \frac{\bar{\rho}_1 \mu_i \rho_i \sum_{n=1}^{N} A_{3n} a_n - \frac{2}{\sqrt{\pi}} a_1 \bar{\rho}_1 \bar{\mu}_\omega \rho_\omega}{\bar{U}\left\{\sum_{n=1}^{N}\sum_{k=1}^{N} A_{2nk} a_n a_k - \sum_{n=1}^{N}\sum_{k=1}^{N} A_{1nk} a_n a_k\right\}} \tag{27}$$

Taking into account that $a_n = a_{n_0} + \mathcal{E}a_n$ where $\mathcal{E}a_n \ll a_0$ for small $\xi$ we obtain upon separating variables and integrating with

$$f = \frac{K}{\sqrt{\xi}} \tag{28}$$

where

$$K = \sqrt{\frac{\bar{U}\left\{\sum_{n=1}^{N}\sum_{k=1}^{N}(A_{1nk}-A_{2nk})a_{n_0} a_{k_0}\right\}}{2\left(\bar{\rho}_1 \bar{\mu}_i \bar{\rho}_i \sum_{n=1}^{N} A_{3n} a_{n_0} - \frac{2}{\sqrt{\pi}} a_{10} \bar{\rho}_1 \bar{\mu}_\omega \bar{\rho}_\omega\right)}}$$

We see that necessarily $\beta = \frac{1}{2}$. This form of $f$ is the same as that used in the Blasius solution.

The quantity $f$ calculated from expression (27) should be the same for all $\eta_i$. This quantity represents in a certain sense the thickness of the boundary layer because $\operatorname{erf}\eta_i$ for $\eta_i = 2$ is close to one but $\eta = fY$. The greater the $f$ the smaller the $\bar{Y}_\delta$ (thickness of the boundary layer).

---

*and noting that $f^2\frac{d\bar{U}}{d\xi}$ and $f^2\frac{dp}{d\xi}$ vanish as $\xi$ approaches zero for pointed bodies as long as $\frac{dU}{d\xi}$ and $\frac{dp}{d\xi}$ are finite.

Therefore we obtain

$$\frac{\sum_{n=1}^{N}\sum_{k=1}^{N}\left[A_{1nk}(\eta_1)-A_{2nk}(\eta_1)\right]a_{no}a_{ko}}{\bar{\varsigma}_1\bar{\mu}\bar{\varsigma}\sum_{n=1}^{N}A_{3n}(\eta_1)a_{no}-\frac{2}{\sqrt{\pi}}a_{10}\bar{\varsigma}_1\bar{\mu}_\omega\bar{\varsigma}_\omega} =$$

$$\frac{\sum_{n=1}^{N}\sum_{k=1}^{N}\left[A_{1nk}(\eta_2)-A_{2nk}(\eta_2)\right]a_{no}a_{ko}}{\bar{\varsigma}_1\bar{\mu}\bar{\varsigma}\sum_{n=1}^{N}A_{3n}(\eta_2)a_{no}-\frac{2}{\sqrt{\pi}}a_{10}\bar{\varsigma}_1\bar{\mu}_\omega\bar{\varsigma}_\omega} = \ldots =$$

$$\frac{\sum_{n=1}^{N}\sum_{k=1}^{N}\left[A_{1nk}(\eta_q)-A_{2nk}(\eta_q)\right]a_{no}a_{ko}}{\bar{\varsigma}_1\bar{\mu}\bar{\varsigma}\sum_{n=1}^{N}A_{3n}(\eta_q)a_{no}-\frac{2}{\sqrt{\pi}}a_{10}\bar{\varsigma}_1\bar{\mu}_\omega\bar{\varsigma}_\omega} = \frac{2K_o^2}{\bar{U}_{o1}} \qquad (29)$$

where $q$ is the number of strips. Equations (29) form the system of transcendental equations for $a_{10}, a_{20}, \ldots a_{no}$. This system is supplemented by expression (18). We need as many strips as unknowns $a_{no}$. It is interesting to point out that $a_{10}, a_{20}, \ldots a_{no}, K$ do not depend on velocity gradient and injection given by $\hat{v}$ as long as $\bar{U} \neq X^n$ and $v \neq \infty$ at $\xi = 0$. The condition that $\bar{U} \sim X^n$ and $\hat{v} \sim \frac{1}{\sqrt{\xi}}$ are therefore the similarity conditions.

If $\varsigma\mu =$ const the system of equations (29) is decoupled from the similar system obtained from mass diffusion equation and energy equation.

From expression (22) the quantity $f$ could be found. Applying the similar limiting procedure as in the case of Eq. (19) it follows from equation (22) for $\xi \to 0$

$$f=\sqrt{\frac{\bar{U}_{10}\sum_{n=1}^{N}\sum_{m=0}^{M}(B_{1nm}-B_{2nm})a_{no}b_{mo}}{\xi 2\left\{\left(\bar{\mu}_i\bar{\rho}_i\bar{\rho}_{10}\frac{1}{P_r}\left[\sum_{m=0}^{M}b_{mo}B_{3m}-\frac{2}{\sqrt{\pi}}b_{10}+\frac{P_r-1}{h_{01}}\sum_{n=1}^{N}\sum_{k=1}^{N}A_{4nk}a_{no}a_{ko}\right]+\sum_j(Le_j-1)\left(\frac{\mu_i\rho_i}{P_{ri}}\frac{C_{pj}}{C_p}\left[\sum_{m=0}^{M}b_m\,\mathrm{erf}\,\alpha\eta_i-\frac{1}{2}\sum_{n=1}^{N}\sum_{k=1}^{N}a_n a_k(\mathrm{erf}\,\eta_i)^{n+k}-h_j^*\right]\sum_{r=0}^{R}G_{3r}g_{jr}-\frac{2B}{\pi}g_1(b_0-h_j^*)\left(\frac{\mu_w\rho_w}{P_{rw}}\frac{C_{pj}}{C_p}\right)\right)\right\}}} \tag{30}$$

Because again $f$ is a function of $\xi$ only, the above expression must be satisfied for all $\eta_i$. Substituting different $\eta_i$ one obtains the system of transcendental equations for coefficients $b_m$ and $\alpha$. This system has the form

$$\frac{2K^2}{U_o}=\frac{\sum_{n=1}^{N}\sum_{m=0}^{M}\left[B_{1nm}(\eta_1)-B_{2nm}(\eta_1)\right]a_{no}b_{mo}}{\left(\frac{\mu_1\bar{\rho}_1\bar{\rho}_{10}}{P_r}\left[\sum_{m=0}^{M}b_{mo}B_{3m}(\eta_1)-\frac{2}{\sqrt{\pi}}b_{10}+\frac{P_r-1}{h_{01}}\sum_{n=1}^{N}\sum_{k=1}^{N}A_{4nk}(\eta_1)a_{no}a_{ko}\right]+\sum_j(Le_j-1)\left(\frac{\mu_1\rho_1}{P_{r1}}\frac{C_{pj}}{C_p}\left[\sum_{m=0}^{M}b_m\,\mathrm{erf}\,\alpha\eta_1-\frac{1}{2}\sum_{n=1}^{N}\sum_{k=1}^{N}a_n a_k(\mathrm{erf}\,\eta_1)^{n+k}-h_j^*\right]\sum_{r=0}^{R}G_{3r}(\eta_1)g_{jr}-\frac{2B}{\pi}g_1(b_0-h_j^*)\frac{\mu_w\rho_w}{P_{rw}}\frac{C_{pj}}{C_p}\right)\right.}=$$

$$=\frac{\sum_{n=1}^{N}\sum_{m=0}^{M}\left[B_{1nm}(\eta_2)-B_{2nm}(\eta_2)\right]a_{no}b_{mo}}{\left(\frac{\mu_2\bar{\rho}_2\bar{\rho}_{10}}{P_r}\left[\sum_{m=0}^{M}b_{mo}B_{3m}(\eta_2)-\frac{2}{\sqrt{\pi}}b_{10}+\frac{P_r-1}{h_{01}}\sum_{n=1}^{N}\sum_{k=1}^{N}A_{4nk}(\eta_2)a_{no}a_{ko}\right]+\sum_j(Le_j-1)\left(\frac{\mu_2\rho_2}{P_{r2}}\frac{C_{pj}}{C_p}\left[\sum_{m=0}^{M}b_m\,\mathrm{erf}\,\alpha\eta_2-\frac{1}{2}\sum_{n=1}^{N}\sum_{k=1}^{N}a_n a_k(\mathrm{erf}\,\eta_2)^{n+k}-h_j^*\right]\sum_{r=0}^{R}G_{3r}(\eta_2)g_{jr}-\frac{2B}{\pi}g_1(b_0-h_j^*)\frac{\mu_w\rho_w}{P_{rw}}\frac{C_{pj}}{C_p}\right)\right.}= \tag{31}$$

$= \cdot \cdot \cdot \cdot \cdot \cdot$

The system of equations (31) requires the solution of the system of equations (29) and the analogous system for the initial mass concentration profile. However for $\rho\mu$ = const the last two systems are independent of enthalpy profile. If $Le_i = 1$ and Pr = const the concentration profile doesn't affect the initial enthalpy profile.

To find the initial concentration profile one must apply the same limiting procedure as before to eq. (26). One obtains

$$f = \sqrt{\frac{\bar{U}_{10} \sum_{n=1}^{N} \sum_{m=0}^{M} \left(G_{2nr}(\eta_i) - G_{1nr}(\eta_i)\right) g_{jr} a_n}{2\xi \left\{ \frac{\bar{\rho}_i \bar{\mu}_i \bar{\rho}_i}{P_{ri}} Le_i \sum_{r=1}^{R} G_{3r}(\eta_i) g_r - \frac{2 Le_\omega}{P_{r\omega} \sqrt{\pi}} g_1 \rho_1 \bar{\mu}_\omega \bar{\rho}_\omega \right\}}} \tag{32}$$

Substituting different numerical values for $\eta_i$ one obtains again the system of transcendental equations for coefficients $g_{jr_0}$ and $\beta_0$. This system has the form

$$\frac{2K^2}{\bar{U}_{10}} = \frac{\sum_{n=1}^{N} \sum_{m=0}^{M} \left[G_{2nr}(\eta_1) - G_{1nr}(\eta_1)\right] g_{jr_0} a_{n_0}}{\frac{\bar{\rho}_{10} \bar{\mu}_1 \bar{\rho}_1}{Sc_1} \sum_{r=1}^{R} G_{3r}(\eta_1) g_{1r} - \frac{2}{\sqrt{\pi}\, Sc_w} g_1 \bar{\rho}_{10} \bar{\mu}_\omega \bar{\rho}_\omega} =$$

$$= \frac{\sum_{n=1}^{N} \sum_{m=0}^{M} \left[G_{2nr}(\eta_2) - G_{1nr}(\eta_2)\right] g_{jr} a_n}{\frac{\bar{\rho}_{10} \bar{\mu}_2 \bar{\rho}_2}{Sc_2} \sum_{r=1}^{R} G_{3r}(\eta_2) g_{jr} - \frac{2}{\sqrt{\pi}\, Sc_\omega} g_1 \bar{\rho}_{10} \bar{\mu}_\omega \bar{\rho}_\omega} = \cdots \tag{33}$$

if $\rho_i \mu_i$ = const and $Sc_i$ = const system of transcendental equation requires only knowledge of the solution of eq. (29) but it is decoupled from the energy equations.

For $Sc = 1$ the initial concentration profile is the identical as the velocity distribution providing that $c_{j_w} = 0$ and $c_{j_{\eta=\infty}} = 1$

### 2.2.2. Example of a Flat Plate Boundary Layer ( $Le = 1$ )

We take $u = erf\,\eta$ and one strip $\eta_i = 4$ . From expression (24) evaluating $A_{1nk}, A_{2nk}, A_{3n}$ numerically, we obtain

$$f = 0.322 \sqrt{\frac{1}{\xi}} \tag{34}$$

or the coefficient of surface friction

$$C_f = 2 \frac{\partial \bar{u}}{\partial \eta}\Big|_{\eta=0} = 0.72 \sqrt{\frac{1}{\xi}}$$

The shear stress at the wall is

$$\tau_w = \mu \frac{\partial u}{\partial y}\Big|_{y=0} = 0.36 \mu U \sqrt{U/\nu x}$$

This is slightly smaller than the quantity given by the von Karman-Pohlhausen solution.

The velocity distribution obtained from the above formula is plotted in Figure 2 and compared with the exact solution (Low Ref 6) and with the three strip method.

Assuming the enthalpy distribution in the following form

$$\bar{h}_0 = \frac{h_0}{h_{01}} = b_0 + b_1 \operatorname{erf}(\alpha\eta) \qquad (35)$$

we obtain $\alpha = 0.829$ from formula (18) applied to one strip ( $\eta_i = 4$ ), taking $M_\infty = 5$ and $\bar{h}_w = b_0 = 0.05$ and $P_r = 0.72$ . Therefore

$$\frac{h_0}{h_{01}} = 0.05 + 0.95 \operatorname{erf}(0.829\eta)$$

The total enthalpy profile obtained using this formula is plotted in Figure 3.

The heat flux at the wall, $q_w = \left.\frac{\partial h_0}{\partial Y}\right|_{Y=0}$ agrees well with the exact solution, although there is deviation of total enthalpy at larger distances. The thickness of the thermal boundary layer is larger than that of the viscous layer by roughly $\frac{\delta_T}{\delta_v} = \frac{1}{\alpha} \cong 1.2$ .

Again for a flat plate ( $\rho\mu = 1$ ) the velocity profile may be taken in the form

$$\frac{u}{U} = a_1 \operatorname{erf}\eta + a_3 (\operatorname{erf}\eta)^3 + a_5 (\operatorname{erf}\eta)^5 \qquad (36)$$

Using three strips we obtain

$$K_0 = 0.25945 \quad a_1 = 1.1371 \quad a_3 = 0.2512, \; a_5 = -0.3883$$

which results in a velocity profile which is identical to that of Blasius' solution to about three significant figures. This latter result is also plotted in Figure 2.

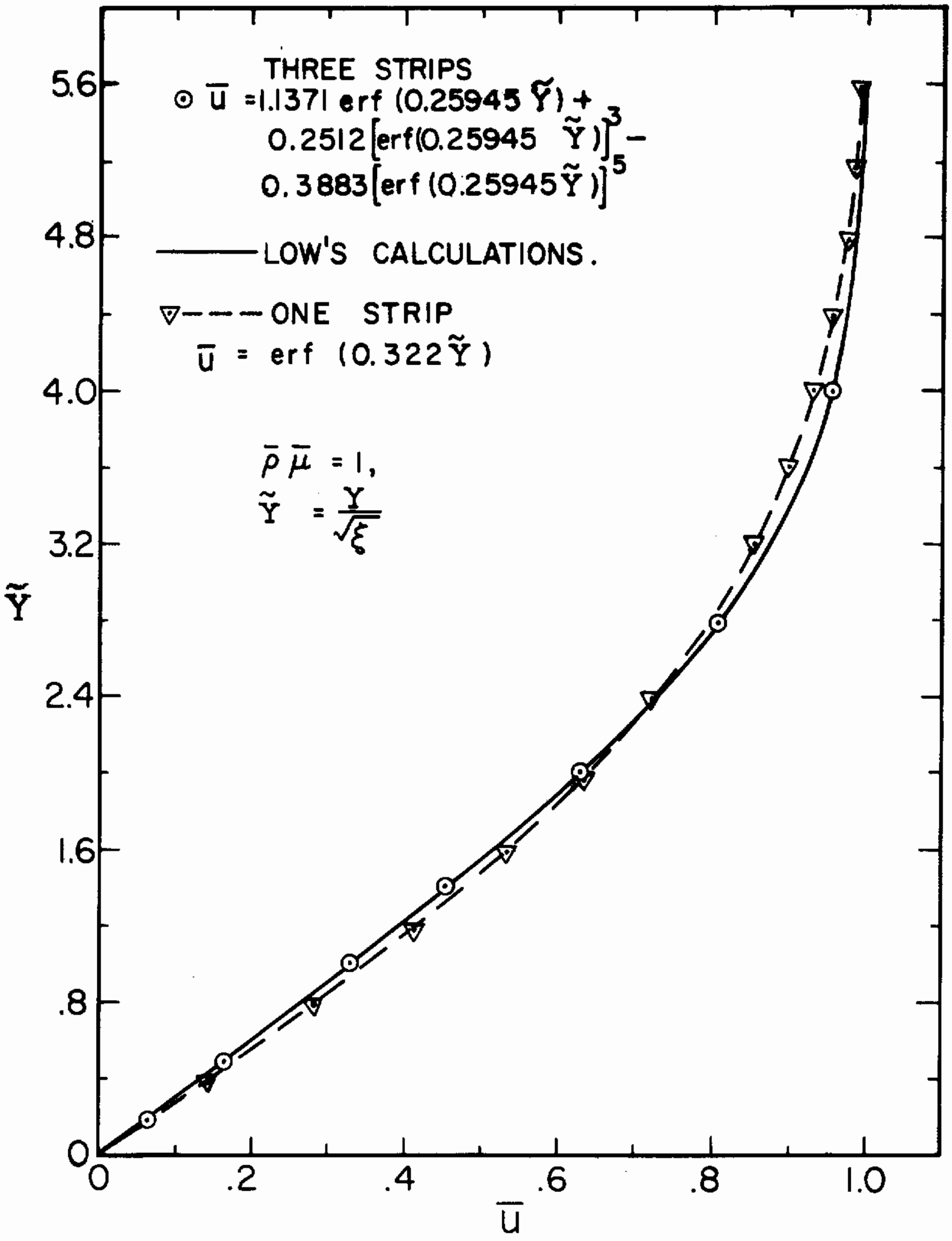



Figure 2 The initial velocity distribution using the present method compared to an exact solution (selfsimilar solution).

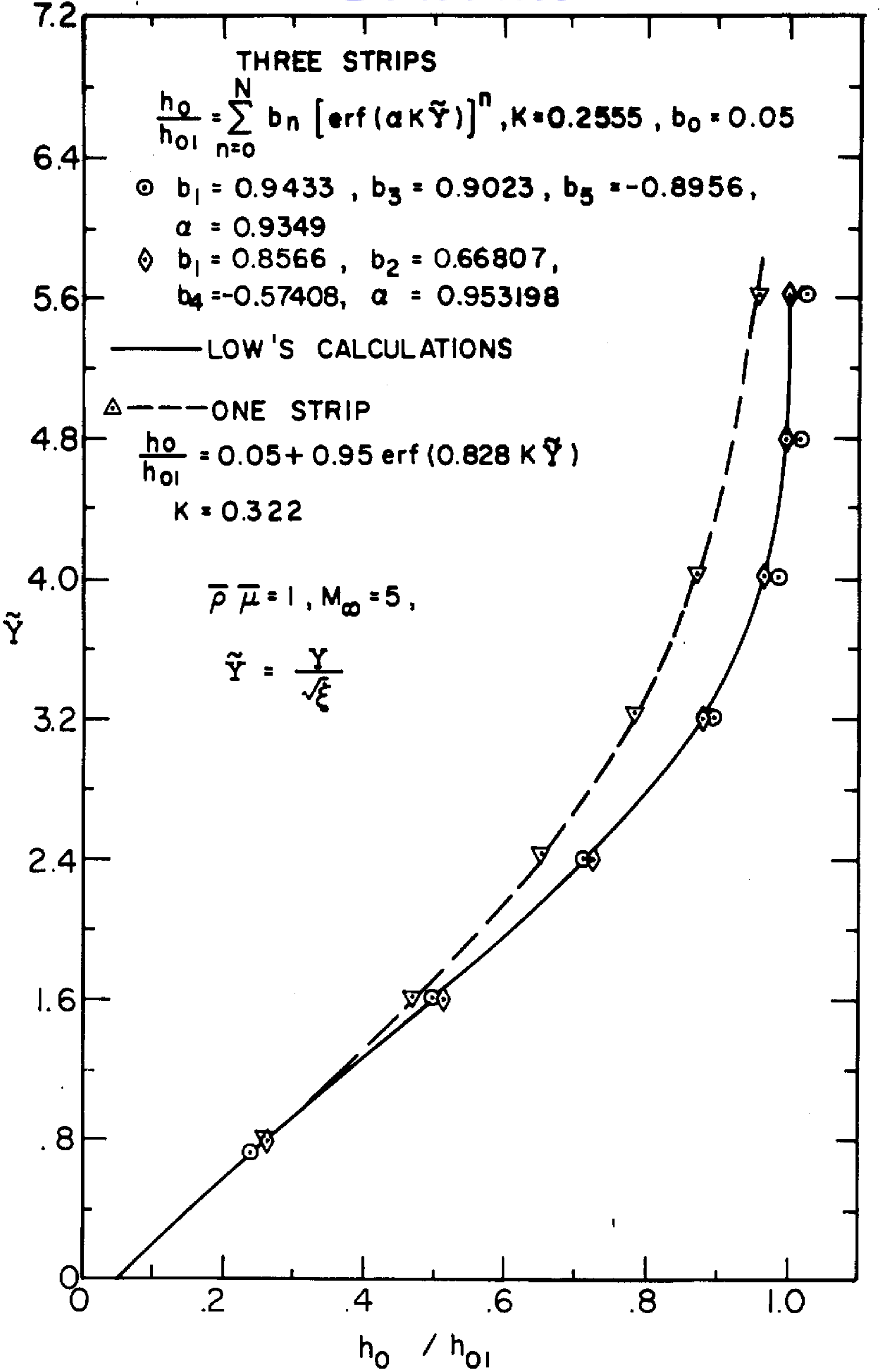



Figure 3. The initial total enthalpy profile obtained by the present method and a self similar solution.

It is interesting to note that as long as $\rho\mu$ = const., the initial velocity profile is independent of temperature or total enthalpy profile and wall conductivity. Neither does it depend upon blowing rate as long as $\hat{v}_\omega$ is not proportional to $\frac{1}{\sqrt{\xi}}$.

The calculation of the initial total enthalpy profile is more difficult because for $P_r \neq 1$ the thicknesses of the viscous and thermal diffusion layers are not equal. The coefficient $\alpha$ in (21) which accounts for that fact enters under the integral sign in the expression for $B_{lnm}$. The result is that the solution of the system of transcendental equations (27) requires a prohibitive amount of computer time. The substitution $\alpha = \alpha_0 + \Delta\alpha$ with $\alpha_0 = \sqrt{P_r}$ removes $\Delta\alpha$ from under the integral sign and results in a substantial saving of computer time. The total enthalpy curves are, however, not as general the velocity profile, since they depend upon the ratio of the wall enthalpy to the total enthalpy of the free stream and also upon the free stream Mach number. The assumed series may be taken, for example, with either odd or even powers of the error function. Both give good agreement with Blasius' solution. This indicates that the result is only slightly dependent of the form of the series. It is not the case for an assumed polynomial profile.

Using three strips one obtains for the total enthalpy profile (assuming $\bar{h}_\omega = 0.05,\ P_r = 0.72,\ M_\infty = 5,\ \bar{\rho}\bar{\mu} = 1$ )

$$\frac{h}{h_{01}} = 0.05 + 0.8566\,\mathrm{erfc}\,\eta + 0.66807[\mathrm{erfc}\,\eta]^2 - 0.57468[\mathrm{erfc}\,\eta]^4 \tag{37}$$

$\alpha_0 = 0.953198$

or
$$\frac{h}{h_{01}} = 0.05 + 0.9433\,\mathrm{erf}(\alpha\eta) + 0.9023[\mathrm{erf}(\alpha\eta)]^3 - 0.8956[\mathrm{erf}\,\alpha\eta]^5 \quad \text{with } \alpha = 0.9349 \tag{38}$$

For $\rho\mu = C$ these calculations are still valid but $K_1 = \frac{K}{C}$. The results of these calculations are shown in **Figure 2.**

2.2.3 <u>First step solution (Calculations of derivaties at singular point)</u>

In the way indicated above one can obtain the initial (starting profile) for the velocity and enthalpy distribution. However, the next step cannot be calculated from equation (19) because of the singularity $(\frac{0}{0})$ at $\xi = 0$. Therefore the first step must be obtained separately by analysis of the singularity. As we show in Appendix A, equations (19), and (22) for small $\xi$ represent the linear system of equations for $\frac{da_n}{d\xi}$, $\frac{dK}{d\xi}$ . The left hand side of these equations represent a matrix proportional to $f_0 = \frac{K_0}{\sqrt{\xi}}$. Near the singular point all derivatives are inversely proportional to $\sqrt{\xi}$ .

$$\frac{da_n}{d\xi} = \frac{C_a}{\sqrt{\xi}}$$

where Ca coefficient of proportionality. Therefore, integrating, one obtains

$$a_n = a_{n_0} + 2C_a\sqrt{\xi} = a_{n_0} + 2\left.\frac{da_n}{d\xi}\right|_{\xi} \xi$$

or, differentiating with respect to , one obtains

$$\frac{da_n}{d\xi} = 2\frac{da_n}{d\xi} + 2\xi\frac{d^2a_n}{d\xi^2}$$

or

$$\frac{d^2a_n}{d\xi^2} = -\frac{1}{2\xi}\frac{dK}{d\xi}$$

The factor 2 in expressions (39) agrees with a general property of parabolas.

If $\hat{v}_\omega = 0$ but $U \neq$ const the solution near the singular point has a form

$$\frac{da_n}{d\xi} \sim \xi$$

$$a_n = a_{n_0} + \left.\frac{da_n}{d\xi}\right|_{\xi} \cdot \frac{\xi}{2}$$

The result of substituting $a_n = a_{n_0} + \Delta a_n \ldots etc$ into expression (19) is as follows (see Appendix A)

$$\begin{aligned}
&2\sum_{n=1}^{N}\sum_{k=1}^{N}\left[2A_{1nk}(\eta_i) - A_{2nk}(\eta_i)\right]a_{n_0}\frac{da_k}{d\xi} - \frac{2K_0^2\bar{\rho}_1\bar{\mu}_i\bar{\rho}_i}{U}\sum_{n=1}^{N}A_{3n}(\eta_i)\frac{da_n}{d\xi} \\
&-\frac{4K_0^2\bar{\rho}_1\bar{\mu}_\omega\bar{\rho}_\omega}{\sqrt{\pi}\,U}\frac{da_1}{d\xi} - \sum_{n=1}^{N}\sum_{k=1}^{N}A_{2nk}a_{k_0}\frac{da_n}{d\xi} - \frac{3}{K_0}\left[\sum_{n=1}^{N}\sum_{k=1}^{N}\left[A_{1nk} - A_{2nk}\right]a_{n_0}a_{k_0}\right]\cdot \\
&\frac{dK}{d\xi} = -\frac{1}{U^2}\frac{dU^2}{d\xi}\sum_{n=1}^{N}\sum_{k=1}^{N}A_{1nk}(\eta_i)a_{n_0}a_{k_0} + \frac{1}{U}\frac{dU}{d\xi}\sum_{n=1}^{N}\sum_{k=1}^{N}A_{2nk}(\eta_i)a_{n_0}a_{k_0} \\
&-\frac{\hat{v}_\omega}{U}\frac{K_0}{\sqrt{\xi}}\sum_{n=1}^{N}a_n\left[\operatorname{erf}\eta_i\right]^n - \frac{1}{\bar{\rho}_1U^2}\frac{dp}{d\xi}\int_0^{\eta_i}\frac{d\eta}{\rho} + \frac{d}{d\xi}\left(\frac{\bar{\rho}_1\bar{\mu}_i\bar{\rho}_i}{U}\right)\sum_{1}^{N}A_{3n}(\eta_i)a_{n_0}K_0^2 \\
&-\frac{d}{d\xi}\left(\frac{\bar{\rho}_1\bar{\mu}_\omega\bar{\rho}_\omega}{U}\right)\frac{2a_1}{\sqrt{\pi}}K_0^2 + \Delta
\end{aligned} \tag{39}$$

where $\Delta$ contains $\xi$ in powers 1/2, 1, 1 1/2, 2 etc.

A similar procedure (Appendix B) leads to the system of algebraic equations for initial deviatives in enthalpy expression.

$$\sum_{n=1}^{N}\sum_{m=0}^{M} a_{n_0}(2B_{1nm}-B_{2nm})\frac{db_m}{d\xi}+\sum_{n=1}^{N}\sum_{m=0}^{M} a_n b_m(2B_{4nm}-B_{5nm})\frac{d\alpha}{d\xi}$$

$$-2K_0^2\frac{\bar{\mu}\bar{\rho}\bar{\rho}_1}{UP_r}\left\{\sum_{m=0}^{M} b_m B_{6m}\frac{d\alpha}{d\xi}+\sum_{m=0}^{M} B_{3m}\frac{db_m}{d\xi}-\frac{2\alpha}{\sqrt{\pi}}\frac{db_1}{d\xi}-\frac{2b_1}{\sqrt{\pi}}\frac{d\alpha}{d\xi}\right\}=$$

$$=-2\left[\sum_{n=1}^{N}\sum_{m=0}^{M} b_{m_0}(B_{1nm}-B_{2nm})\frac{da_n}{d\xi}\right]+\frac{2K_0^2}{UP_r}\bar{\rho}\bar{\mu}\bar{\rho}_1\frac{P_r-1}{h_{01}}\sum_{n=1}^{N}\sum_{k=1}^{N} A_{4nk}\left(a_k\frac{da_n}{d\xi}+a_{n_0}\frac{da_k}{d\xi}\right)$$

$$-\frac{1}{U}\frac{dU}{d\xi}\sum_{n=1}^{N}\sum_{m=0}^{M} a_{n_0}b_{m_0}(B_{1nm}-B_{2nm})-\frac{\hat{v}_w}{U}\frac{K_0}{\sqrt{\xi}}\left\{\sum_{m=0}^{M} b_m(\mathrm{erf}\,\alpha\eta_i)^m-b_0\right\}$$

$$+\frac{3}{K_0}\sum_{n=1}^{N}\sum_{m=0}^{M} a_{n_0}b_{m_0}(B_{1nm}-B_{2nm})\frac{dK}{d\xi}+$$

$$+\frac{d}{d\xi}\left(\frac{K^2\bar{\mu}\bar{\rho}\bar{\rho}_1}{U^2P_r}\right)\left\{\sum_{m=0}^{M} b_{m_0}B_{3m}-\frac{2\alpha b_1}{\sqrt{\pi}}+\frac{P_r-1}{h_{01}}\sum_{n=1}^{N}\sum_{k=1}^{N} A_{4nk}a_n a_k\right\} \qquad (40)$$

where

$$B_{5nm}=\frac{2}{\sqrt{\pi}}m\left[\mathrm{erf}(\alpha_0\eta_i)\right]^{m-1}\eta\, e^{-\alpha^2\eta_i^2}\int_0^{\eta_i}\left[\mathrm{erf}(\eta)\right]^m d\eta$$

$$B_{6m}=\frac{2}{\sqrt{\pi}}m\left[\mathrm{erf}\,\alpha_0\eta_i\right]^{m-1}e^{-\alpha_0^2\eta_i^2}\left[1-2\alpha_0^2\eta_i^2+\frac{2\alpha}{\sqrt{\pi}}\eta_i(m-1)\,\mathrm{erf}(\alpha\eta_i)^{-1}\right]$$

If $\hat{v}_w=0$ (no mass injection) and $U=$ const (flat plate), and $\rho\mu=$ const., the solution must be

$$\frac{da_k}{d\xi}=\frac{db_m}{d\xi}=\frac{d\alpha}{d\xi}=0$$

This is, therefore a self-similar solution (not depending on $\xi$ but on $\eta$ only).

As indicated by equations (39) and (40) the deviation from the self-similar solution due to blowing ( $\hat{v}_\omega = 0$ ) is greater than that due to pressure gradient ( $U =$ const. ).

For $\hat{v}_\omega$ = const. the transformation $\bar{\xi} = \hat{v}_\omega^2 \xi$ makes expressions (36), (37) independent of the actual blowing rate (the coordinate $\xi$ being stretched).

After calculating the first step from equations (42) and (43), the expressions (19) and (22) should be used. The linear expression with $\Delta = 0$ gives $\frac{d a_n}{d \xi}$ etc. and then $\Delta$ can be calculated in next approximation. The solution of the first step and of further steps for the downstream characteristics of the boundary layer is accomplished by the solution of systems of ordinary differential which do not require inordinate amounts of computer time. Standard matrix inversion methods exist enabling the solution of such a problem to be obtained quite readily.

The result of calculation of flow over a flat plate with constant blowing and constant temperature at $P_r = 0.72$ are presented in Figures 4a and 4b.

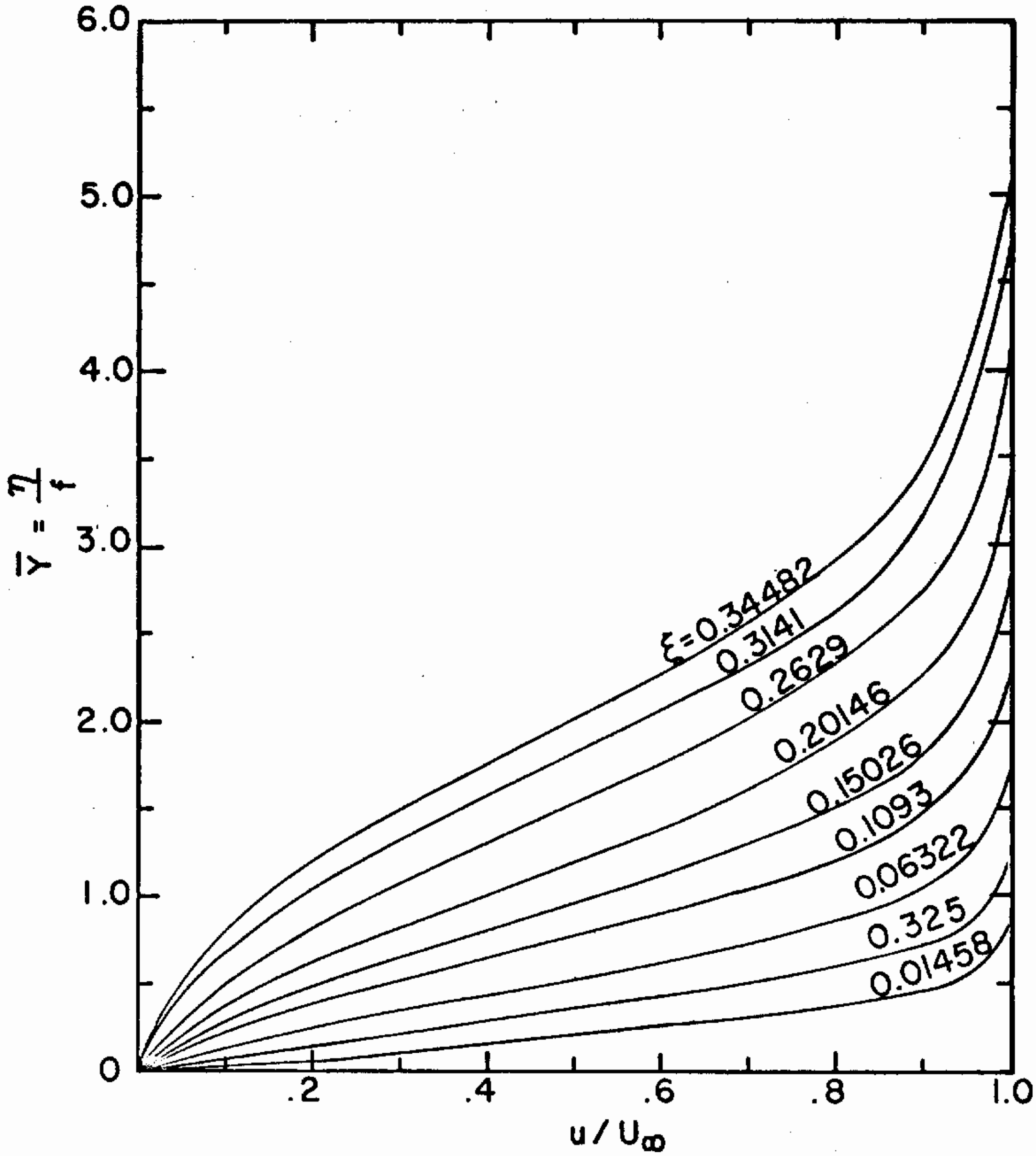



Figure 4a. Velocity Profiles on a Flat Plate for the Present Method.

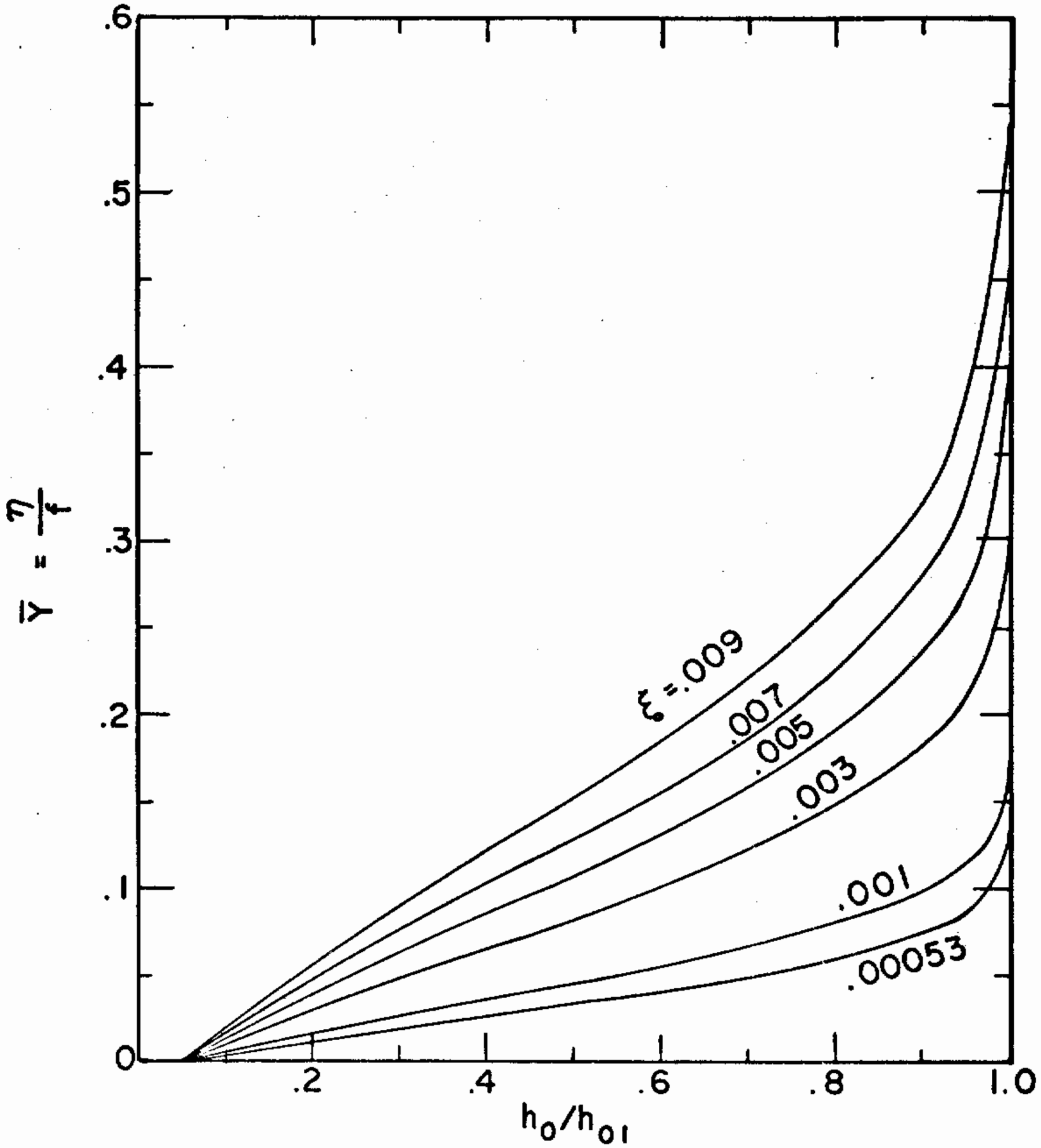



Figure 4b. Enthalpy Profiles on a Flat Plate for the Present Method.

## 2.3 The Boundary Layer on a Blunt Body

A solution of the blunt body boundary layer [was given in ref. (7)] assuming $\frac{dU}{d\xi}$ = const. and constant blowing velocity. The similar solution obtained in this way is applicable near the stagnation point.

In the present paper the differential equations are integrated using a multistrip method and assuming the distributions of the unknown flow parameters to be expanded in series. This results in a statement of the problem which is non-similar and allows the variation of $\frac{dU}{d\xi}$ and the blowing velocity $\hat{v}_w$ . Therefore the solution is not restricted to the vicinity of stagnation point.

The main difficulty in solving the problem is associated with the singularity at the stagnation point ( $U=0$ ). The analysis of this singularity leads to a system of transcendental equations for the initial velocity and enthalpy distributions whose solutions agree well with the similarity solutions. But in addition linearation of these equations gives the initial streamwise derivatives which allows the smooth extension of the integral curves downstream of the singular point. Thus a difficulty of the similar solution method is overcome without an arbitrary matching procedure to change to the necessarily non-similar characteristics of the downstream solutions.

The system of equations was developed in the Part 2.1 assuming velocity and total enthalpy distributions in the form

$$\frac{u}{U} = \sum_{n=1}^{N} a_n (\operatorname{erf} \eta)^n$$

and

$$\frac{h}{h_{01}} = \sum_{m=0}^{M} b_m (\operatorname{erf} \alpha \eta)^m$$

This particular form of finite series was chosen for several reasons. First, it was desired to limit the required number of integral strips and hence the number of terms in the series. To do so with polynomials would have limited the smoothness of the curves, because polynomials always introduce inflection points and thus generally require more terms for adequate representations. Secondly, the error function, is the first order solution of the Blasius equation and therefore correctly represents conditions near the outer edge of the boundary layer. Thirdly, high speed computers can handle powers of the error function as readily as trigonometric functions, for example, so to preserve smoothness and edge features higher powers of these functions were chosen. Orthogonality considerations were irrelevant in the multiple strip technique.

In any case it must be emphasized that this particular choice of expansion is incidental to the principal purpose of this paper which is to show how the singular point problem may be analyzed in a self-consistent fashion and extended smoothly downstream.

2.3.1 Basic System of Equations

By neglecting mass diffusion[x] the resulting equations are (eg. 19 and 22 of Part 2.1)

$$\sum_{n=1}^{N}\sum_{k=1}^{N}\left[2A_{1nk}-A_{2nk}\right]a_n\frac{da_k}{d\xi}=-\frac{1}{U}\frac{dU}{d\xi}\sum_{n=1}^{N}\sum_{k=1}^{N}\left[2A_{1nk}-A_{2nk}\right]a_n a_k$$

$$+\frac{f'}{f}\sum_{n=1}^{N}\sum_{k=1}^{N}\left(A_{1nk}-A_{2nk}\right)a_n a_k-\frac{1}{U}\Big[\hat{v}_w f\sum_{n=1}^{N}a_n(\mathrm{erf}\,\eta_i)^n- \qquad (41)$$

$$\bar{\rho}_1\bar{\mu}_1\bar{\mu}_i\bar{\rho}_i f^2\sum_{n=1}^{N}A_{3n}a_n+\frac{2}{\sqrt{\pi}}\bar{\rho}_1\bar{\mu}_1\bar{\rho}_w\bar{\mu}_w f^2-\frac{dU}{d\xi}\int_0^{\eta_i}\frac{d\eta}{\rho}$$

$$\sum_{n=1}^{N}a_n=1 \qquad (42)$$

and

$$\sum_{n=1}^{N}\sum_{m=0}^{M}B_{1nm}a_n\frac{db_m}{d\xi}+\sum_{n=1}^{N}\sum_{m=0}^{M}B_{4nm}a_n b_m\frac{d\alpha}{d\xi}+\sum_{n=1}^{N}\sum_{m=0}^{M}\left(B_{1nm}-B_{2nm}\right)b_m\frac{da_n}{d\xi}$$

$$=\left[\frac{1}{U}\frac{dU}{d\xi}-\frac{f'}{f}\right]\sum_{n=1}^{N}\sum_{m=0}^{M}\left[B_{2nm}-B_{1nm}\right]a_n b_m-\frac{1}{U}\Big\{v_w\hat{f}\Big[\sum_{m=0}^{M}b_m(\mathrm{erf}\,\alpha\eta)^m-b_0\Big]$$

$$-\frac{f^2\bar{\mu}\bar{\rho}\bar{\mu}_1\bar{\rho}_1}{Pr}\Big[\sum_{m=0}^{M}B_{3m}b_m-\frac{2\alpha}{\sqrt{\pi}}b_1+\frac{Pr-1}{h_0}U^2\sum_{n=0}^{N}\sum_{k=1}^{N}A_{4nk}a_n a_k\Big]\Big\} \qquad (43)$$

$$\sum_{m=0}^{M}b_m=1 \qquad (44)$$

---

[x] The biggest complication associated with including mass diffusion is associated with the increase of the number of equations in the system of transcendental equations for the initial profile.

where

$$A_{1nk} = \int_0^{\eta_i} (\operatorname{erf}\eta)^{n+k} d\eta \ , \quad A_{2nk} = (\operatorname{erf}\eta)^n \int_0^{\eta_i} (\operatorname{erf}\eta)^k d\eta$$

$$A_{3n}(\eta_i) = \frac{2n}{\sqrt{\pi}} (\operatorname{erf}\eta_i)^{n-1} \exp(-\eta_i^2)$$

$$\bar{p} = \frac{p}{\rho_r U_r^2} \ , \ \bar{\rho}_1 = \frac{\rho_1}{\rho_r} \ , \ \bar{\mu}_i = \frac{\mu_i}{\mu_r} \ , \ \bar{U} = \frac{U}{U_r} \ , \ \xi = \frac{U_r X}{\nu_r}$$

$U_r$, $\nu_r$ etc. are reference velocity, kinematic viscosity, etc., respectively.

$$\bar{h}_o = \frac{h_{o1}}{U_r^2}$$

$$B_{1nm}(\eta_i) = \int_0^{\eta_i} [\operatorname{erf}\eta]^n [\operatorname{erf}\alpha\eta]^m d\eta$$

$$B_{2nm}(\eta_i) = [\operatorname{erf}\alpha\eta_i]^m \int_0^{\eta_i} [\operatorname{erf}\eta]^n d\eta$$

$$B_{3m}(\eta_i) = m[\operatorname{erf}\alpha\eta_i]^{m-1} \frac{2\alpha}{\sqrt{\pi}} \exp[-\alpha^2\eta_i^2]$$

$$B_{4mm}(\eta_i) = \frac{2m}{\sqrt{\pi}} \int_0^{\eta_i} (\operatorname{erf}\eta)^n (\operatorname{erf}\alpha\eta)^{m-1} \exp[-\alpha^2\eta^2]\eta\, d\eta$$

$$\frac{1}{\rho} = \frac{\rho_1}{\rho} = \frac{\rho_1 RT}{p_1} = \rho_1 \frac{\gamma-1}{\gamma} \frac{h}{p_1} = \rho_1 \frac{\gamma-1}{\gamma} \frac{h_o - \frac{u^2}{2}}{p_1} =$$

$$= \bar{\rho}_1 \frac{\gamma-1}{\gamma} \frac{\bar{h}_{o1}\bar{h}_o - \bar{U}^2 \frac{\bar{u}^2}{2}}{\bar{p}_1} \qquad (45)$$

$$\bar{\rho}_1 = \frac{\rho_1}{\rho_r} \, , \quad \bar{h}_{01} = \frac{h_0}{U_r^2} \quad , \quad \bar{U} = \frac{U}{U_r}$$

$$\bar{P}_1 = \frac{P_1}{\rho \; U_r^2}$$

therefore

$$\int \frac{d\eta}{\bar{\rho}} = \frac{\bar{\rho}_1}{\bar{P}_1} \frac{\gamma - 1}{\gamma} \left[ \bar{h}_{01} \sum_{m=0}^{M} B_{8m} b_m - \frac{\bar{U}^2}{2} \sum_{n=1}^{N} \sum_{k=1}^{N} A_{1nk} \, a_n a_k \right] \tag{46}$$

where

$$B_{8m} = \int_0^{\eta_i} (\operatorname{erf} \alpha \eta)^m d\eta$$

$$B_{9m} = \int_0^{\eta_i} (\operatorname{erf} \alpha \eta)^{m-1} \eta \; \exp[-\alpha^2 \eta^2] d\eta$$

2.3.2 Calculation of the Velocities and Enthalpy Distribution in the Stagnation Point

Multiplying equations (41) through by $U$ for $U \to 0$ (stagnation point) we get ( $\frac{\partial u}{\partial \eta} \neq \infty$ at $\xi = 0, \eta = 0$; f and f' are not infinite as opposed to the case of a sharp body for which $U \neq 0$ at $\xi = 0$ , but they are finite in this case).

$$\Phi_I \equiv D_1 f^2 + D_2 f + D_3 = 0 \tag{47}$$

where

$$D_1(\eta_i) = \bar{\rho}_1 \bar{\mu}_1 \bar{\rho}_i \bar{\mu}_i \sum_{n=1}^{N} A_{3n}(\eta_i) a_{n_0} - \frac{2}{\sqrt{\pi}} a_0 \bar{\rho}_1 \bar{\mu}_1 \bar{\rho}_\omega \bar{\mu}_\omega$$

$$D_2(\eta_i) = -\hat{v}_\omega \sum_{n=1}^{N} a_{n_0} (\operatorname{erf} \eta)^n$$

$$D_3(\eta_i) = \left.\frac{dU}{d\xi}\right|_0 \left\{ \sum_{n=1}^{N} \sum_{k=1}^{N} \left[ A_{2nk}(\eta_i) - 2A_{1nk}(\eta_i) \right] a_{n_0} a_{k_0} + \int_0^{\eta_i} \frac{d\eta}{\rho} \right\}$$

$D_1$ can be positive or negative (velocity profile with inflection)

$D_2$ must be negative for

$D_3$ must be positive

because $f$ must be positive

$$f = \frac{-D_2 \mp \sqrt{D_2^2 - 4D_1 D_3}}{2D_1} \tag{48}$$

Because $f$ is a function of $\xi$ only $f \equiv f(\xi)$ we have the system of equations for the initial profile

$$\frac{D_2(\eta_i) \pm \sqrt{D_2^2 - 4D_1 D_3}}{2D_1(\eta_i)} = \frac{D_2(\eta_{i+1}) \pm \sqrt{D_2^2(\eta_{i+1}) - 4D_1(\eta_{i+1}) D_3(\eta_{i+1})}}{2D_1(\eta_{i+1})} =$$

$$= \cdots \tag{49}$$

One obtains as many equations as the number of strips used. An additional condition is given as equation (42)

In a similar way one obtains from equation (43)

$$\Phi_{II} = E_1 f^2 + E_2 f + E_3 = 0 \tag{50}$$

with

$$E_1 = \frac{\overline{\mu}\,\overline{\varrho}}{P_r}\overline{\mu}_1\overline{\varrho}_1 \left\{ \sum_{m=0}^{M} B_{3m}(\eta_i) b_{mo} - \frac{2\alpha}{\sqrt{\pi}} b_{10} + \frac{P_r - 1}{\overline{h}_{o1}} \sum_{n=1}^{N}\sum_{k=1}^{N} A_{4nk} a_{no} a_{ko} \right\}$$

$$E_2 = -\hat{U}_\omega \left\{ \sum_{m=0}^{M} b_{mo} (\mathrm{erf}\,\alpha\eta_i)^m - b_o \right\}$$

$$E_3 = -\left.\frac{dU}{d\xi}\right|_o \sum_{n=1}^{N}\sum_{m=0}^{M} (B_{1nm} - B_{2nm}) a_{no} b_{mo}$$

Therefore,

$$f = \frac{-E_2 \mp \sqrt{E_2^2 - 4E_1E_3}}{2E_1} \tag{51}$$

Because $f$ must be the same as that given by equation (48), the system of eqs (49) will be supplemented by

$$f = \frac{-E_2(\eta_i) \mp \sqrt{E_2^2(\eta_i) - 4E_1(\eta_i)E_2(\eta_i)}}{2E_1(\eta_i)} = \frac{-E_2(\eta_{i+1}) \mp \sqrt{E_2^2(\eta_{i+1}) - 4E_1(\eta_i)E_2(\eta_i)}}{2E_1(\eta_{i+1})} \tag{52}$$

and, as before, an additional relation is

$$\sum_{m=0}^{M} b_{m_0} = 1$$

Expressions (49), (52) are coupled because the integral in $D_3$ depends on $b_{m_0}$ according to expression (46). Therefore these expressions must be solved simultaneously. This leads to complications because of limitations of the numerical technique of solving the system of transcendental equations.

However knowing an initial guess from the solution of the related problem when similarity obtains $\frac{dU}{d\xi} = const$ one can solve such a system. In fact knowledge of the exact solution is not necessary because of the possibility of introducing a compensating term $\Phi_0(\eta_i)$ resulting from the fact that $\Phi_{I,II}$ is not exactly zero.

It follows from expressions (47) and (50) that the solution depends on the ratio of $\frac{v_w}{\sqrt{\frac{dU}{d\xi}\big|_0}}$. The transformation $\hat{f} = \frac{f}{\sqrt{\frac{dU}{d\xi}\big|_0}}$ eliminates $\frac{dU}{d\xi}\Big|_0$ and therefore the direct influence of Reynolds number.

Initial profiles calculated from the systems (49) and (52) and compared with result of reference 8.

2.3.3 Calculation of the Initial Derivaties

At the singular point with $U = 0$ and $\Phi_I = 0$ according to equation (41) equation (47) includes the term $\frac{\Phi_I}{U}$.

There appears therefore a singularity $\left(\frac{0}{0}\right)$ which will be analyzed by taking small increments in all parameters depending on $\xi$.

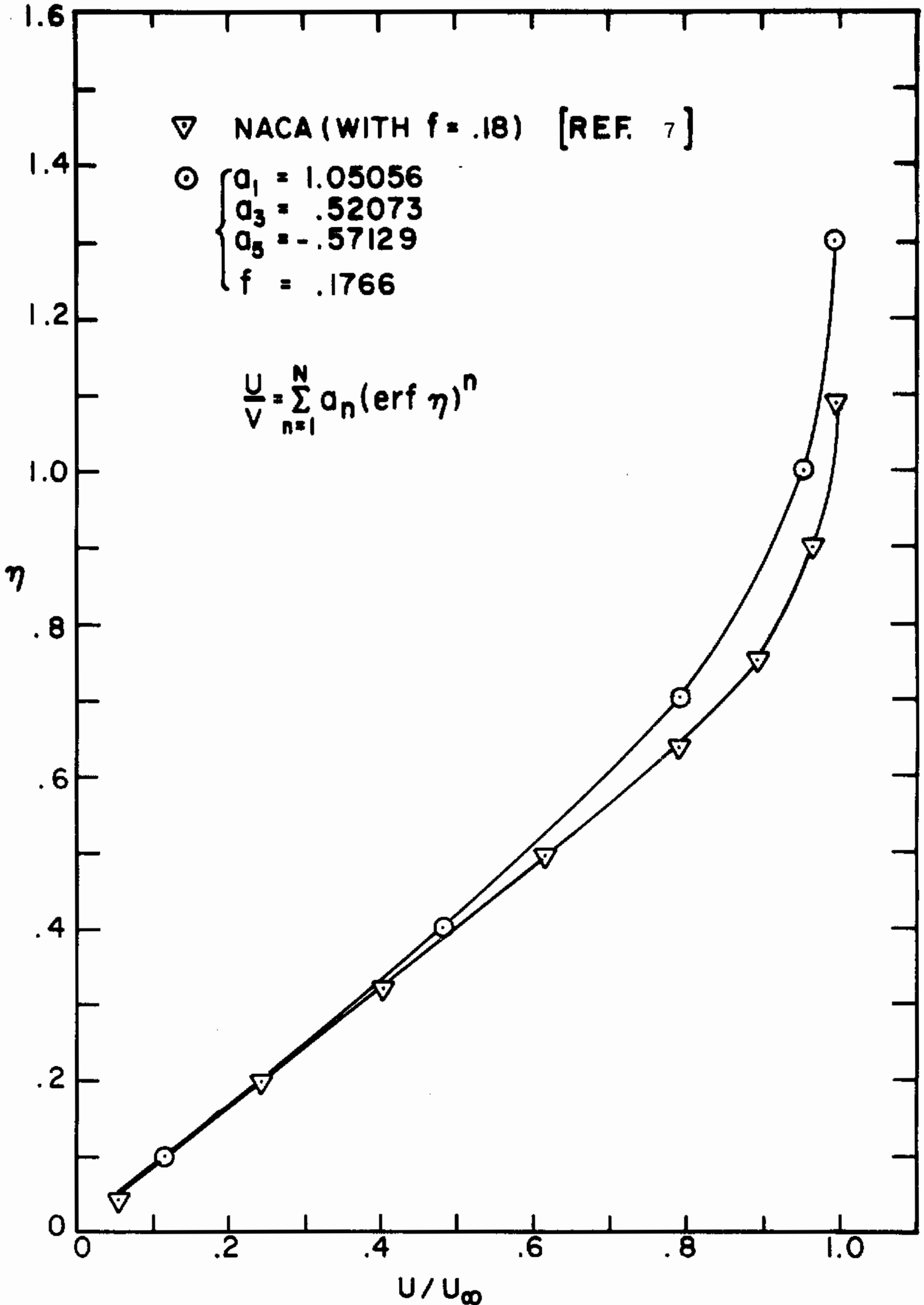



Figure 5a.
Velocity Distribution at Blunt Body Stagnation Point.

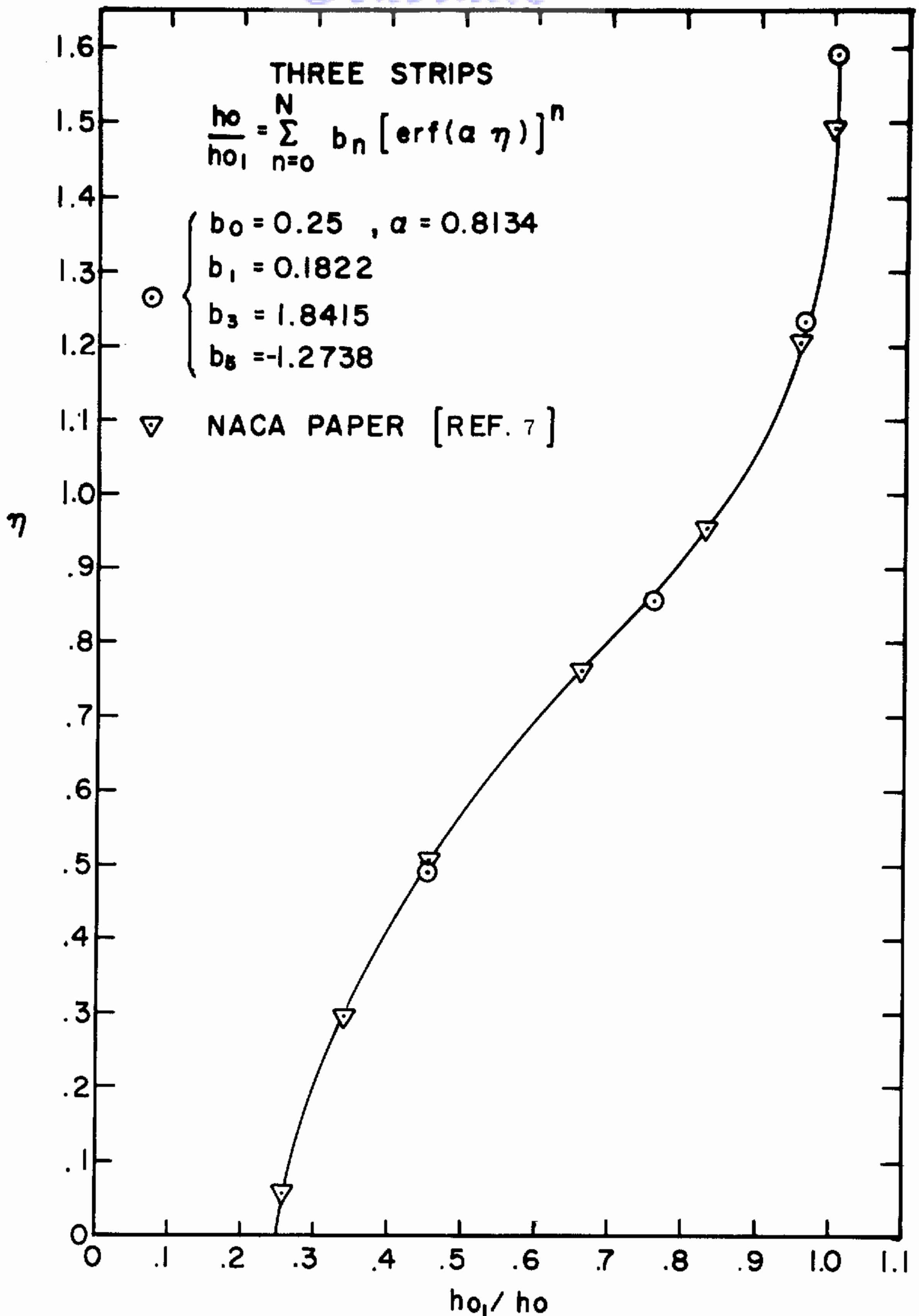



Figure 5b

Total Enthalpy Distribution at Blunt Body Stagnation Point.

$$
\begin{aligned}
a &= a_{n_0} + \Delta a_n \quad \text{with } \Delta a_n \ll a_{n_0} \\
\frac{dU}{d\xi} &= \left.\frac{dU}{d\xi}\right|_0 + \Delta\frac{dU}{d\xi} \\
U = U_0 &= \left.\frac{dU}{d\xi}\right|_0 \xi + O(\xi)
\end{aligned}
\tag{53}
$$

Neglecting second order terms, taking into account equation (46) with $h_{01}$ = const (Bernoulli integral), and assuming $\bar{\rho}_1\mu_1 = K_1 p$ and $\rho_i\mu_i$ = const, one obtains from equation (41)

$$
\begin{aligned}
&\sum_{n=1}^{N}\sum_{k=1}^{N}\left[2A_{1nk}-A_{2nk}\right]a_{n_0}\frac{da_k}{d\xi} + \frac{1}{\xi}\sum_{n=1}^{N}\sum_{k=1}^{N}\left[2A_{1nk}-A_{2nk}\right]\left[\Delta a_n a_{k_0} + \Delta a_k a_{n_0}\right] \\
&-\frac{f'}{f_0}\sum_{n=1}^{N}\sum_{k=1}^{N}\left(A_{1nk}-A_{2nk}\right)a_{n_0}a_{k_0} + \frac{1}{\xi\left.\frac{dU}{d\xi}\right|_0}\Bigg\{\Delta f\,\hat{v}_{w_0}\sum_{n=1}^{N}a_{n_0}(\operatorname{erf}\eta_i)^n + \\
&+f_0\,\hat{v}_{w_0}\sum_{n=1}^{N}\Delta a_n(\operatorname{erf}\eta_i)^n - \bar{\rho}_{10}\bar{\mu}_{10}\,\bar{\rho}_i\bar{\mu}_i\,2f_0\Delta f\sum_{n=1}^{N}A_{3n}a_{n_0} \\
&-\bar{\rho}_{10}\bar{\mu}_{10}\bar{\mu}_i\bar{\rho}_i f_0^2\sum_{n=1}^{N}A_{3n}\Delta a_n + \frac{2\Delta a_1}{\sqrt{\pi}}\bar{\rho}_1\bar{\mu}_1\,\bar{\rho}_w\bar{\mu}_w f_0^2 + \frac{4a_{10}}{\sqrt{\pi}}\bar{\rho}_1\bar{\mu}_1\bar{\rho}_w\bar{\mu}_w f_0\Delta f\Bigg\} \\
&-\frac{1}{\xi}\frac{\rho_{10}}{p_{10}}\frac{\gamma-1}{\gamma}\bar{h}_{01}\sum_{m=0}^{M}\left[B_{8m_0}\Delta b_m + \Delta B_{8m}b_{m_0}\right] = \\
&-\frac{1}{\xi\left.\frac{dU}{d\xi}\right|_0}\Delta\frac{dU}{d\xi}\sum_{n=1}^{N}\sum_{k=1}^{N}\left(2A_{1nk}-A_{2nk}\right)a_{n_0}a_{k_0} - \frac{1}{\left.\frac{dU}{d\xi}\right|_0\xi}\Delta v_w f_0\sum_{n=1}^{N}a_{n_0}(\operatorname{erf}\eta_i)^n \\
&-\frac{1}{\xi}\frac{\rho_{10}}{p_{10}}\frac{\gamma-1}{\gamma}\frac{U^2}{2}\sum_{n=1}^{N}\sum_{k=1}^{N}A_{1nk}a_{n_0}a_{k_0} + \frac{1}{\xi}\frac{\gamma-1}{\gamma}\frac{\rho_{10}}{p_{10}}\left(\frac{\Delta\rho_1}{\rho_{10}} - \frac{\Delta p_1}{p_{10}} + \frac{1}{\left.\frac{dU}{d\xi}\right|_0}\Delta\frac{dU}{d\xi}\right) \\
&\left[h_{01}\sum_{m=0}^{M}B_{8m_0}b_{m_0}\right] = \frac{K_1\Delta p\,\bar{\mu}_1}{\xi\left.\frac{dU}{d\xi}\right|_0}f_0\left[\sum_{n=1}^{N}A_{3n}a_{n_0} - \frac{2}{\sqrt{\pi}}a_{10}\right]
\end{aligned}
\tag{54}
$$

A similar way the system of equations (43) gives $\left(\frac{1}{\bar{h}_0} = O(U^2)\right)$

$$\sum_{n=1}^{N}\sum_{m=0}^{M} a_{n_0} B_{1nm_0}\frac{db_m}{d\xi} + \sum_{n=1}^{N}\sum_{m=0}^{M} B_{4nm}\, a_{n_0}\, b_{m_0}\frac{d\alpha}{d\xi} + \sum_{n=1}^{N}\sum_{m=0}^{M}(B_{1nm_0}-B_{2nm})\, b_{m_0}\frac{da_n}{d\xi}$$

$$+\frac{f'}{f_0}\sum_{n=1}^{N}\sum_{m=0}^{M}[B_{2nm}-B_{1nm}]\,a_{n_0} b_{m_0} - \frac{1}{\xi}\sum_{n=1}^{N}\sum_{m=0}^{M}\{[B_{2nm_0}-B_{1nm_0}](a_{n_0}\Delta b_m + b_{m_0}\Delta a_n)$$

$$+(\Delta B_{2nm}-\Delta B_{1nm})\,a_{n_0} b_{m_0}]\} + \frac{1}{\frac{dU}{d\xi}\big|_0 \xi}\{\hat{v}_{w_0}\Delta f[\sum_{m=0}^{M} b_{m_0}(\mathrm{erf}\,\alpha_0\eta)^m - b_0]$$

$$+\hat{v}_{w_0} f_0[\sum_{m=0}^{M}\Delta b_m(\mathrm{erf}\,\alpha_0\eta_i)^m + \sum_{m=0}^{M} b_{m_0}\frac{2m}{\sqrt{\pi}}\eta_i \exp[-\alpha_0\eta_i](\mathrm{erf}\,\alpha_0\eta_i)^{m-1}\Delta\alpha]$$

$$-\frac{2f_0\Delta f(\bar{\mu}\bar{\varrho}\,\bar{\varrho}_1(\bar{\mu}_1}{P_r}[\sum_{m=0}^{M} B_{3m} b_{m_0} - \frac{2\alpha_0}{\sqrt{\pi}} b_{10}] - \frac{f_0^2(\bar{\mu}\bar{\varrho}(\bar{\mu}_1\bar{\varrho}_1}{P_r}[\sum_{m=0}^{M}\Delta B_{3m} b_{m_0} +$$

$$+\sum_{m=0}^{M} B_{3m_0}\Delta b_m - \frac{2\alpha_0}{\sqrt{\pi}}\Delta b_1 - \frac{2\Delta\alpha}{\sqrt{\pi}} b_{10}]\} = \frac{1}{\frac{dU}{d\xi}\big|_0\cdot\xi}\Delta\frac{dU}{d\xi}\sum_{n=1}^{N}\sum_{m=0}^{M}[B_{2nm_0}-B_{1nm_0}]\,a_{n_0} b_{m_0}$$

$$-\frac{1}{\frac{dU}{d\xi}\big|_0 \xi}\{\Delta v_w f_0[\sum_{m=0}^{M} b_{m_0}(\mathrm{erf}\,\alpha_0\eta)^m - b_0] - \frac{P_r-1}{\bar{h}_0}U^2\sum_{n=0}^{N}\sum_{k=1}^{N} A_{4nk}\, a_{n_0} a_{k_0}$$

$$\frac{f_0^2(\bar{\mu}\bar{\varrho}(\bar{\mu}_1\bar{\varrho}_1}{P_r} - K_1\Delta P f_0^2(\frac{\bar{\mu}_1\bar{\varrho}_1}{P_r}[\sum_{m=0}^{M} B_{3m} b_{m_0} - \frac{2\alpha}{\sqrt{\pi}} b_{10}]\}$$

Assuming a certain pressure distribution $U=U(\xi)$ and blowing distribution $\hat{v}_w = \hat{v}_w(\xi)$ , one can express $\frac{\Delta a_n}{\xi}$ , $\frac{\Delta f}{\xi}$ etc., by derivatives proper multiplying factors. We shall do this for a Newtonian pressure distribution.

### 2.3.4 Newtonian Pressure Distribution

According to Newtonian Theory the pressure distribution is given by

$$p_1 = \rho_\infty U_\infty^2 K^2 \cos^2\beta \tag{56}$$

$\rho_\infty$, $U_\infty$ are velocity and density of the undisturbed flow, $\beta$ the angle of the element of surface with the free stream direction, $K^2$, a constant coefficient which can be calculated for a blunt body using normal shock relations and assuming isentropic compression between the shock wave and the stagnation point (for $\gamma = 1.4$, $K^2 = .92$ ).

Taking the Bernoulli integral

$$\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \frac{U^2}{2} = \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0} \cong \frac{U_\infty^2}{2} \tag{57}$$

and assuming the isentropic relationship between $p_1$ and $\rho_1$

$$\frac{p_1}{p_0} = \left(\frac{\rho_1}{\rho_0}\right)^\gamma \tag{58}$$

one obtains

$$U = \sqrt{U_\infty^2 - \frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}\left(\frac{p_1}{p_0}\right)^{\frac{\gamma-1}{\gamma}}} = U_\infty\sqrt{1-\left(\frac{p_1}{p_0}\right)^{\frac{\gamma-1}{\gamma}}}$$

where $U_\infty$ is a reference velocity $(U_r = U_\infty)$

According to expression (56)

$$\bar{U} = \sqrt{1 - (\cos^2\beta)^{\frac{\gamma-1}{\gamma}}} \tag{59a}$$

or for small $\beta$ (59b)

$$\bar{U} = \sqrt{\frac{\gamma-1}{\gamma}} \sin\beta \left(1 + \frac{\sin^2\beta}{4\gamma}\right)$$

and from expression (58)

$$\frac{\rho_1}{\rho_0} = (\cos\beta)^{2/\gamma} \tag{60}$$

where, according to expression (56) and (57),

$$\rho_0 = \frac{2\gamma}{\gamma-1} K^2 \rho_\infty \tag{61}$$

If one takes stagnation density as a reference density the dimensionless density is:

$$\bar{\rho}_1 = \frac{\rho_1}{\rho_0}$$

For small $\beta$

$$\bar{\rho}_1 = \left[1 - \frac{1}{\gamma}\sin^2\beta + \frac{1-\gamma}{2\gamma^2}\sin^4\beta\right] \tag{62}$$

and the density derivative is

$$\frac{d\bar{\rho}_1}{d\xi} = -\frac{2}{\gamma}(\cos\beta)^{\frac{2-\gamma}{\gamma}} \sin\beta \frac{d\beta}{d\xi} \tag{63a}$$

or for small

$$\frac{d\varrho_1}{d\xi} = -\frac{2}{\gamma}\sin\beta\left(1-\frac{2-\gamma}{2\gamma}\sin^2\beta\right)\frac{d\beta}{d\xi} \tag{63b}$$

The derivative of velocity is

$$\frac{dU}{d\xi} = \frac{\gamma-1}{\gamma\sqrt{1-(\cos^2\beta)^{\frac{\gamma-1}{\gamma}}}}(\cos\beta)^{\frac{\gamma-2}{\gamma}}\sin\beta\,\frac{d\beta}{d\xi} \tag{64}$$

or for small $\beta$

$$\frac{dU}{d\xi} = \sqrt{\frac{\gamma-1}{\gamma}}\cos\beta\left(1+\frac{3\sin^2\beta}{4\gamma}\right)\frac{d\beta}{d\xi} \tag{65}$$

and

$$\frac{d^2U}{d\xi^2} = \sqrt{\frac{\gamma-1}{\gamma}}\left\{\sin\beta\left[\frac{3\cos^2\beta}{2\gamma}-\frac{3\sin^2\beta}{4\gamma}-1\right]\left(\frac{d\beta}{d\xi}\right)^2 + \cos\beta\left(1+\frac{3\sin^2\beta}{4\gamma}\right)\frac{d^2\beta}{d\xi^2}\right\} \tag{66}$$

Also

$$\frac{d\bar{\varrho}_1\bar{\mu}_1}{d\xi} = \frac{d\,p_1/p_{10}}{d\xi} = 2\cos\beta\frac{d\beta}{d\xi}\sin\beta \tag{67}$$

For a spherical nose ( $r_0$ sphere radius)

$$\frac{d\beta}{d\xi} = \frac{d\,s/r_0}{d(s/\nu_0)U_\infty} = \frac{\nu_0}{r_0 U_\infty} = \frac{1}{Re}. \tag{68}$$

where $\nu_0$ is the kinematic viscosity at the stagnation point.

$Re$ - Reynolds number

The dimensionless pressure is $\left(\frac{\rho_0}{\rho_\infty} = \frac{2K^2\gamma}{\gamma-1}\right)$

$$\bar{p}_1 = \frac{p_1}{\rho_0 U_\infty^2} = \frac{\rho_\infty}{\rho_0} K^2 \cos^2\beta = \frac{\gamma-1}{2\gamma}\cos^2\beta \tag{69}$$

### 2.3.5 Calculation of Initial Derivatives with Newtonian Pressure Distribution

If in equations (54) one substitutes expressions (61), (65), (66), etc., one obtains $(\beta \ll 1)$

$$\Delta \frac{dU}{d\xi} = \sqrt{\frac{\gamma-1}{\gamma}}\left(\frac{3}{4\gamma}-\frac{1}{2}\right)\beta^2\frac{d\beta}{d\xi} = \left(\frac{3}{2\gamma}-1\right)\frac{\beta^2}{2}\sqrt{\frac{\gamma-1}{\gamma}}\frac{d\beta}{d\xi}$$

and

$$\frac{\Delta \frac{dU}{d\xi}}{\xi} = \frac{1}{\xi}\left(\frac{3}{2\gamma}-1\right)\frac{\beta^2}{2}\frac{d\beta}{d\xi}\sqrt{\frac{\gamma-1}{\gamma}} \tag{71}$$

For a spherical nose $\quad \beta = \varkappa\xi \quad$ ($\varkappa$ = constant)

and

$$\frac{\Delta \frac{dU}{d\xi}}{\xi} = \frac{1}{2}\frac{d^2U}{d\xi^2} \sim \xi \tag{71}$$

Taking into account that

$$\frac{\Delta a_n}{\xi} \sim \frac{da_n}{d\xi}$$

$$\frac{\Delta f}{\xi} \sim \frac{df}{d\xi} \tag{72}$$

From equation (54) and (55), assuming that $\Delta v_\omega \sim \xi^2 + O(\xi^3)$ (i.e. $\Delta v$ is not linearly proportional to $\xi$ ) it follows that the right hand side is proportional to $\xi$ .

Therefore

$$\frac{da_n}{d\xi} \sim \xi$$

or $\frac{da_n}{d\xi} = K_2 \xi$ , where $K_2$ = const, and upon intergrating

$$a_n = a_{n0} + K_2 \frac{\xi^2}{2} = a_{n0} + \frac{da_n}{d\xi} \frac{\xi}{2} \qquad (73)$$

Similarily

$$f = f_0 + \frac{df}{d\xi} \frac{\xi}{2}$$

$$b_m = b_{m0} + \frac{db_m}{d\xi} \frac{\xi}{2} \qquad (74)$$

where the derivatives $\frac{df}{d\xi}$ etc. are calculated at point $\xi > 0$.

Calculating from equations (73) and (74) $\Delta a_n = a_n - a_{n0} = \frac{1}{2} \frac{da_n}{d\xi} \xi$ etc., and substituting into expression (54), and (55) one obtains ( $A_{1nk} = A_{1kn}$, $\left.\frac{dv}{d\xi}\right|_0 \equiv c_1$ )

$$
\begin{aligned}
&\sum_{n=1}^{N}\sum_{k=1}^{N}\left[4A_{1nk}-\frac{3}{2}A_{2nk}\right]a_{no}\frac{da_k}{d\xi}-\frac{1}{2}\sum_{n=1}^{N}A_{2nk}\,a_{ko}\frac{da_n}{d\xi}+\\
&\frac{1}{C_1}\left\{\frac{\hat{v}_\omega}{2}f_o\sum_{n=1}^{N}[\mathrm{erf}\,\eta_i]^n\frac{da_n}{d\xi}-\frac{\bar{\varrho}_{10}\bar{\mu}_{10}\bar{\mu}_i\bar{\varrho}_i}{2}f_o^2\sum_{n=1}^{N}A_{3n}\frac{da_n}{d\xi}+\right.\\
&\left.+\frac{\bar{\varrho}_{10}\bar{\mu}_{10}\bar{\varrho}_\omega\bar{\mu}_\omega f_o^2}{\sqrt{\pi}}\frac{da_1}{d\xi}\right\}+\frac{df}{d\xi}\left\{\frac{1}{C_1}\left[\frac{\hat{v}_\omega}{2}\sum_{n=1}^{N}a_{no}[\mathrm{erf}\,\eta_i]^n-\right.\right.\\
&\left.\left.-\bar{\varrho}_1\bar{\mu}_1\bar{\varrho}_i\bar{\mu}_i f_o\sum_{n=1}^{N}A_{3n}a_{no}+\frac{2}{\sqrt{\pi}}a_{10}\varrho_{10}\mu_{10}f_o\bar{\mu}_\omega\bar{\varrho}_\omega\right]-\frac{1}{f_o}\sum_{n=1}^{N}\sum_{k=1}^{N}(A_{1nk}-A_{2nk})a_{no}a_{ko}\right\}\\
&-\frac{\varrho_{10}}{p_{10}}\frac{\gamma-1}{2\gamma}\bar{h}_o\left[\sum_{m=0}^{M}B_{8mo}\frac{db_m}{d\xi}+\frac{2}{\sqrt{\pi}}\sum_{m=0}^{M}m\,B_{9m}b_{mo}\frac{d\alpha}{d\xi}\right]=\\
&\frac{d^2U}{d\xi^2}\frac{1}{2C_1}\left\{\sum_{n=1}^{N}\sum_{k=1}^{N}(A_{2nk}-2A_{1nk})a_{no}a_{ko}+\sum_{m=0}^{M}B_{8mo}b_{mo}\right\}\\
&-\frac{1}{C}\frac{\Delta v_\omega}{\xi}f_o\sum_{n=1}^{N}a_{no}(\mathrm{erf}\,\eta_i)^n-\frac{\gamma-1}{2\gamma}\frac{\bar{\varrho}_{10}}{\bar{p}_{10}^2}\bar{h}_o\frac{dp}{d\xi}\sum_{m=0}^{M}B_{8m}b_m\\
&-\frac{K_1\bar{\mu}_i\bar{\varrho}_i}{C_1}\frac{f_o^2}{2}\left[\sum_{n=1}^{N}A_{3n}a_{no}-\frac{2}{\sqrt{\pi}}a_{10}\right]\frac{dp}{d\xi}+\frac{\gamma-1}{2\gamma}\frac{1}{\bar{p}_{10}}\bar{h}_o\sum_{m=0}^{M}B_{8m}b_{mo}\frac{d\bar{\varrho}_1}{d\xi}\\
&-\frac{\gamma-1}{\gamma}\frac{\varrho_{10}}{p_{10}}\frac{\bar{U}}{2}\sum_{n=1}^{N}\sum_{k=1}^{N}A_{1nk}a_{no}a_{ko}\frac{dU}{d\xi}
\end{aligned}
\tag{75}
$$

where

$$B_{8m}=\int_0^{\eta_i}(\mathrm{erf}\,\alpha\eta)^m\,d\eta$$

$$B_{9m}=\int_0^{\eta_i}(\mathrm{erf}\,\alpha\eta)^{m-1}\exp[-\alpha^2\eta^2]\,\eta\,d\eta$$

and

$$\sum_{n=1}^{N}\sum_{m=0}^{M} B_{1nm} a_{n0} \frac{db_m}{d\xi} + \frac{1}{2}\sum_{n=1}^{N}\sum_{m=0}^{M}\left[B_{1nm0} - B_{2nm0}\right]\frac{db_m}{d\xi} +$$
$$+ \frac{\hat{v}_{\omega 0} f_0}{2C_1}\sum_{m=0}^{M}\left[\operatorname{erf}(\alpha \eta_i)\right]^m \frac{db_m}{d\xi} - \frac{f_0^2 \left(\bar{\mu}_{10}\bar{\varrho}_{10}\left(\bar{\mu}_i\bar{\varrho}_i\right.\right.}{2C_1 P_r}\left\{\sum_{m=0}^{M} B_{3m0}\frac{db_m}{d\xi}\right.$$
$$\left. -\frac{2\alpha_0}{\sqrt{\pi}}\frac{db_1}{d\xi}\right\} - \frac{\hat{v}_{\omega 0} f_0}{2C_1}\frac{db_0}{d\xi} + \left\{\sum_{n=1}^{N}\sum_{m=0}^{M} a_{n0} b_{m0} B_{4nm} + \frac{v_\omega f_0}{2C_1}\cdot\right.$$
$$\sum_{m=0}^{M} b_{m0}\, m\left[\operatorname{erf}(\alpha_0 \eta)\right]^{m-1}\frac{2\eta}{\sqrt{\pi}}\exp\left[-\alpha^2\eta_i^2\right] - \frac{f_0^2\left(\bar{\mu}_{10}\bar{\varrho}_{10}\left(\bar{\mu}_i\bar{\varrho}_i\right.\right.}{2C_1 P_r}\Big[$$
$$\left.\sum_{m=0}^{M} B_{7m} b_{m0} - \frac{2b_{10}}{\sqrt{\pi}}\right] + \frac{1}{2}\sum_{n=1}^{N}\sum_{m=0}^{M}\left(B_{4nm} - B_{6nm}\right)a_{n0} b_m\Big\}\frac{d\alpha}{d\xi} +$$
$$+ \frac{3}{2}\sum_{n=1}^{N}\sum_{m=0}^{M}\left(B_{1nm0} - B_{2nm}\right) b_{m0}\frac{da_n}{d\xi} - \left\{\frac{f_0\left(\bar{\mu}_{10}\bar{\varrho}_{10}\left(\bar{\mu}_i\bar{\varrho}_i\right.\right.}{P_r C_1}\left[\sum_{m=1}^{M} B_{3m0} b_{m0}\right.\right.$$
$$\left. -\frac{2\alpha b_{10}}{\sqrt{\pi}}\right] - \frac{\hat{v}_{\omega 0}}{2C_1}\left[\sum_{m=0}^{M} b_{m0}\left[\operatorname{erf}(\alpha\eta)\right]^m - b_0\right] + \frac{1}{f_0}\sum_{m=0}^{M}\sum_{n=1}^{N}\left(B_{1nm0} - \right.$$
$$\left.\left. - B_{2nm0}\right) a_{n0} b_{m0}\right\}\frac{df}{d\xi} = \frac{1}{2C_1}\frac{d^2U}{d\xi^2}\sum_{n=1}^{N}\sum_{m=0}^{M}\left(B_{2nm0} - B_{1nm0}\right) a_{n0} b_{m0}$$
$$-\frac{1}{C_1}\frac{\Delta v_\omega}{\xi} f_0\left[\sum_{m=0}^{M} b_{m0}(\operatorname{erf}\alpha_0\eta)^n - b_0\right] + \frac{K_1\left(\bar{\mu}_i\bar{\varrho}_i\right.}{C_1 P_r}\frac{f_0^2}{2}\left[\sum_{n=1}^{N} B_{3m} b_{m0} - \right.$$
$$\left.\frac{2\alpha}{\sqrt{\pi}} b_{10}\right]\frac{dp}{d\xi} - \frac{P_r - 1}{h_0} U \sum_{n=1}^{N}\sum_{k=1}^{N} A_{4nk} a_{n0} a_{k0} \cdot \frac{f_0^2 \bar{\mu}_i\bar{\varrho}_i\left(\mu_1\varrho_1\right.}{P_r} \tag{76}$$

Equations (75) and (76) are a linear system of equations for the initial derivatives of theunknown quantities at point $\xi$.

## 2.4 Inviscid Hypersonic Flow About an Arbitrary Body

### 2.4.1 Introduction

Hypersonic flow about yawed bodies presents a difficult analytical problem, and exact numerical solutions, even neglecting real gas effects, require a great expenditure of computer time. Laborious methods of calculation do not necessarily give better agreement with experimental pressure data than does simple Newtonian theory, nor do they encourage one to attack more complex problems. Newtonian theory does not provide characteristics of the flow other than pressure distribution.

The purpose of the present work is to develop a theory based upon the fundamental conservation laws in the integral form and upon certain simplifying assumptions which will give the Newtonian result plus centrifugal correction for pressure and at the same time will give the local average velocity and the shock layer thickness. To accomplish this an average velocity parallel to the body with a zero azimuthal component and a constant average density is assumed. These assumptions may be released and certain variations of these quantities in a direction normal to the body may be taken into account. The possibility of this refinement is, of course, an advantage of an integral method.

The results are applied to a number of simple flows for comparison with the theoretical and experimental results of others.

### 2.4.2 Development of The General Equations

The fundamental conservation laws are applied to the flow through the control segment shown in Figure 6 while assuming zero transverse velocity ($w = 0$).

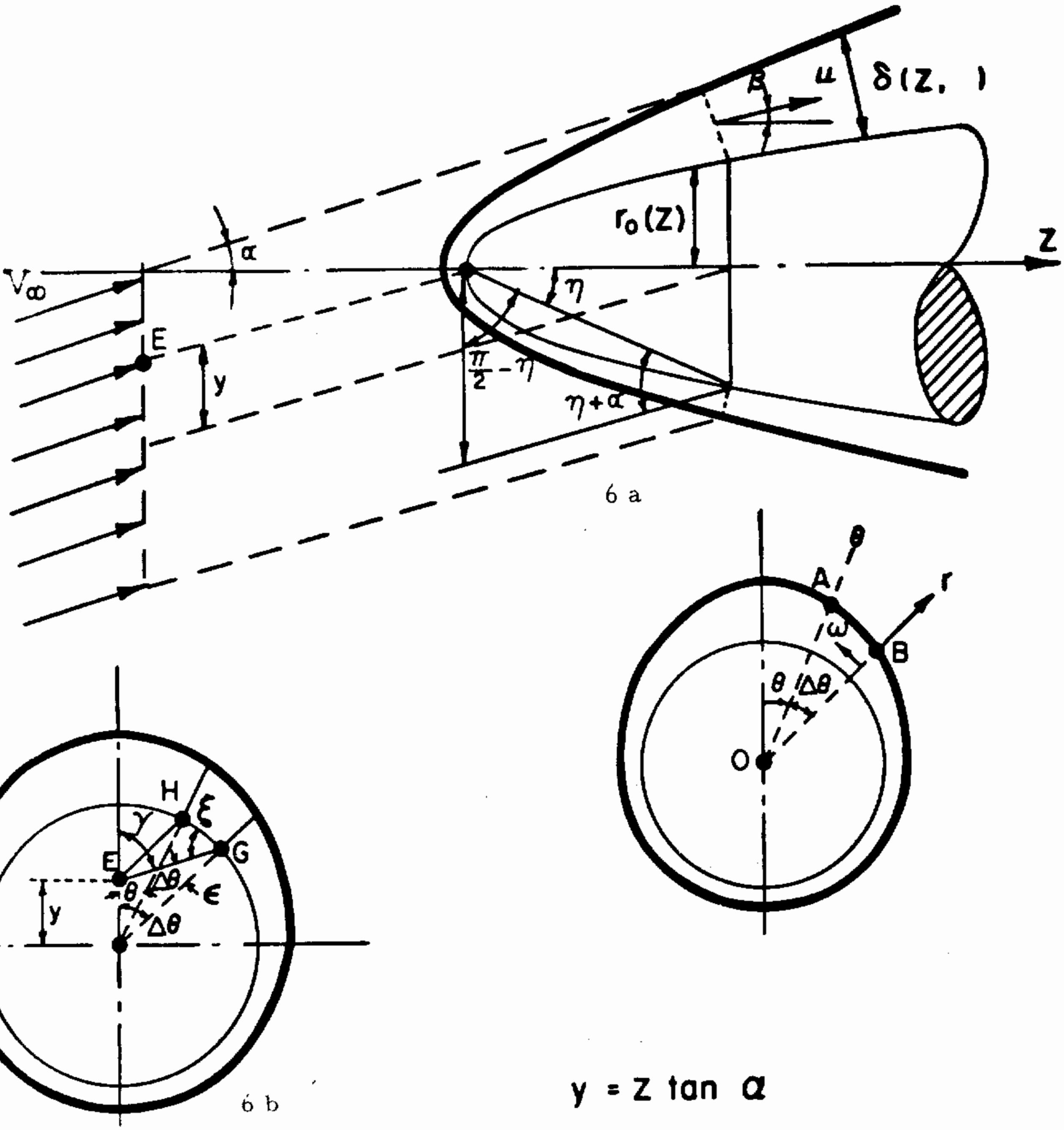



Figure 6. The Control Segments Used for the Solution of Hypersonic Inviscid Flow About an Arbitrary Body.

We write the conservation of mass as

$$\rho_\infty V_\infty \Delta S \cos\alpha = \int_0^{\theta+\Delta\theta}\int_\theta^{\delta} \rho u (r_0 + y\cos\beta)\, dy\, d\theta \tag{77}$$

where

$$\Delta S = \frac{r_0^2}{2\sin\gamma}\sin\theta\cos(\gamma-\theta)\Delta\theta + r_0\delta\cos\beta\,\Delta\theta \equiv S_\theta\,\Delta\theta + r_0\delta\cos\beta\,\Delta\theta$$

which defines $S_\theta$.

For vanishing $\Delta\theta$ and setting $\rho/\rho_\infty = \bar{\rho}$ and $u/v_\infty = \bar{u}$, this may be written

$$\cos\alpha\left[\frac{r_0^2}{2\sin\gamma}\sin\theta\cos(\gamma-\theta) + r_0\delta\cos\beta\right] = \int_0^{\delta} \bar{\rho}\bar{u}(r_0 + y\cos\beta)\, dy \tag{78}$$

If we now assume $\bar{\rho}\bar{u} = f(r_0)$ and $S/R \ll 1$, where R is the local normal intercept length measured from the surface to the reference axis ($R = r_0/\cos\beta$), this may be further simplied to:

$$\cos\alpha\left(\frac{r_0^2}{2\sin\gamma}\right)\sin\theta\cos(\gamma-\theta) = \bar{\rho}\bar{u}\, r_0\,\delta \tag{79}$$

The corresponding integral form of the law of conservation of axial momentum is

$$\rho_\infty V_\infty^2 \cos^2\alpha\,\Delta S - \int_\theta^{\theta+\Delta\theta}\int_0^{\delta} \rho u^2 \cos\beta(r_0 + y\cos\beta)\, dy\, d\theta =$$

$$= \int_\theta^{\theta+\Delta\theta}\int_0^{r_0} p_w\, r_0\, dr_0\, d\theta$$

Assuming in addition here that the normal velocity component is zero ($v = 0$), differentiating with respect to $r_o$ and allowing $\Delta\theta$ to tend to zero, we obtain

$$\cos^2\alpha\,\frac{r_0}{\sin\gamma}\sin\theta\cos(\gamma-\theta)\left(1 - \frac{r_0\cos\theta}{2\cos(\gamma-\theta)\sin\gamma}\frac{d\gamma}{dr_0}\right) =$$

$$= \frac{\partial}{\partial r_0}\int_0^{\delta} \rho u^2 \cos\beta\,(r_0 + y\cos\beta)\, dy + p_w\, r_0 \tag{80}$$

Now we introduce the notation $\bar{p}_w = p_w/\rho_\infty V_\infty^2$ and again neglect $\delta/R$ compared to unity to obtain with the help of Equation (79):

$$\cos^2\alpha \, S_{r\theta} = \frac{d}{dr_0}\left(u \cos\beta \cos\alpha \, S_\theta\right) + \bar{p}_w r_0 \tag{81}$$

where

$$S_{r\theta} = \frac{dS_\theta}{dr_0} \tag{82}$$

$S_\theta$ is derived in Appendix C and given in Equation (78) for bodies of "nearly circular" cross-section (bodies whose transverse radius of curvature is everywhere of the same order of magnitude as the mean radius of the section). The corresponding relationships for arbitrary shapes are derived in Appendix D. It is desirable to carry Equation (81) a step further as

$$\cos^2\alpha = \bar{u} \cos\beta \cos\alpha + \frac{\bar{p}_w r_0}{S_{r\theta}} + \frac{S_\theta}{S_{r\theta}} \cos\alpha \left(\cos\beta \frac{du}{dr_0} - \bar{u} \sin\beta \frac{d\beta}{dr_0}\right) \tag{83}$$

In the same manner the law of conservation of radial momentum leads to

$$\cos\theta \cos\alpha \sin\alpha = -\frac{r_0 \bar{p}_w \cot\beta}{S_{r\theta}} + \bar{u} \cos\alpha \sin\beta + \frac{S_\theta}{S_{r\theta}} \cos\alpha \left(\sin\beta \frac{d\bar{u}}{dr_0} + \bar{u} \cos\beta \frac{d\beta}{dr_0}\right) \tag{84}$$

Eliminating $\bar{p}_w$ from the Equations (83) and (84), we obtain

$$\frac{d\bar{u}}{dr_0} = \frac{S_{r\theta}}{S_\theta}\left(-\bar{u} + \cos\alpha \cos\beta + \cos\theta \sin\alpha \sin\beta\right) \tag{85}$$

Now pressure can be calculated from Equation (83):

$$\bar{p}_w = \frac{S_{r\theta}}{r_0}\left(\cos^2\alpha \sin^2\beta - \cos\theta \sin\alpha \cos\alpha \sin\beta \cos\beta + \frac{S_\theta}{S_{r\theta}} \bar{u} \cos\alpha \sin\beta \frac{d\beta}{dr_0}\right) \tag{86}$$

From Equation(79) the thickness of the shock layer is found to be given by

$$\frac{\delta}{R} = \frac{\cos\alpha \cos\beta}{2 \sin\gamma} \sin\theta \, \frac{\cos(\gamma-\theta)}{\bar{\rho}\,\bar{u}} \tag{87}$$

where $R = r_o/\cos\beta$ as mentioned before. A suitable average density might be the arithmetic mean

$$\bar{\rho} = \frac{1}{2}(\bar{\rho}_w + \bar{\rho}_s) \tag{88}$$

where $\bar{\rho}_s = (\gamma+1)/(\gamma-1)$ is the dimensionless density at the shock in the limit as $M_\infty \to \infty$. The density at the wall, $\rho_w$, can be found from the Bernoulli equation

$$\frac{p_w}{\bar{\rho}_w} = \frac{\gamma-1}{2\gamma}(1-\bar{u}^2) \tag{89}$$

Equations (85), (86), (87), (88), and (89) enable one to calculate approximate values for the pressure, density and velocity distributions over arbitrary bodies at angles of attack. These equations were derived with the aid of simplifying assumptions which we summarize here:

a) Zero normal and azimuthal velocity components,

b) Small shock layer thickness relative to the length $R = r_o/\cos\beta$,

c) $\bar{\rho}$ and $\bar{u}$ independent of $y$.

For less restrictive assumptions, Equations (79), (83), (84), and (89)

**(with $\bar{u}$ interpreted as $\bar{u}_w$) must be used for the same (presumably refined) information.**

**Now we shall apply the simpler set of equations to several typical flows.**

### 2.4.3 Bodies at Zero Angle of Attack

In this case $\alpha$, $\gamma$-$\theta$ and $\frac{d\gamma}{dr_o}$ all vanish. Moreover, if the body has geometrically similar cross-sections, $\frac{d\chi}{dr_o}$ also vanishes. Therefore, for a large class of bodies one can integrate Equation 85 to obtain

$$\bar{u} = \frac{2}{r_o^2}\int_0^{r_o} r\cos\beta\, dr \tag{90}$$

In a similar way for plane flow one obtains

$$\bar{u} = \frac{1}{r_o}\int_0^{r_o} \cos\beta\, dr \tag{91}$$

or in general

$$\bar{u} = \frac{j}{r_o^j}\int_0^{r_o} r^{j-1}\cos\beta\, dr \tag{92}$$

where j = 1 for plane cases and j = 2 for bodies of geometrically similar cross-section.

The temperature, found from the energy equation, is

$$\bar{T} = \frac{1-\bar{u}^2}{2} \tag{93}$$

where

$$\bar{T} = c_p T / V_\infty^2$$

The density, found from the equation of state, is

$$\bar{\rho}_w = \frac{\gamma \bar{p}_w}{(\gamma-1)\bar{T}} \tag{94}$$

The shock layer thickness, found from the continuity equation, is

$$\delta = \frac{r_0}{j\,\bar{\rho}\,\bar{u}} \tag{95}$$

In general the integration in Equation (92) will have to be performed numerically. The special class of bodies for which $\beta$ = constant, i.e., wedges and cones, allows a completely analytical solution:

$$\bar{u} = \cos\beta \tag{96}$$

$$\bar{p}_w = \sin^2\beta \tag{97}$$

$$T = \frac{1}{2}\sin^2\beta \tag{98}$$

$$\rho_w = 2\gamma/(\gamma-1)$$

$$\bar{\rho} = (\bar{\rho}_w + \bar{\rho}_s)/2 = \frac{3\gamma+1}{2(\gamma-1)} \tag{99}$$

and

$$\delta = \frac{2}{j}\,\frac{(\gamma-1)\,r_0}{(3\gamma+1)\cos\beta} \tag{100}$$

We note that only $\delta$ depends upon j.

### 2.4.4 Wedges and Arbitrary Cones at Angles of Attack

Clearly the above results for a wedge at zero angle of attack are unchanged by choosing instead a positive attitude. The value of $\beta$ must accommodate the combined inclination of the normal and the angle of attack. Similarly, as we have seen, several features of the flow about cones of arbitrary cross-section at zero angle of attack may be calculated in a simple way. Since those results depend only upon $\beta$ and $r_0$ and $\beta$ may always be redefined with respect to the free stream direction, we may replace $\beta$

by $\beta'$ and use Equations (96) through (100) for flow over wedges and arbitrary cones at angles of attack. Derivations are given in Appendix E.

The value of $\beta'$ is related to $\varkappa$ and $\alpha$ by the equation

$$\sin\beta' = \frac{-\sin\alpha\cos\varkappa + (r_o/z)\cos(\theta-\varkappa)}{\sqrt{1+(r_o/z)^2\cos^2(\theta-\varkappa)}}$$

where $\varkappa$ is the angle of inclination of the surface normal with respect to the vertical when projected upon the plane of cross-section. This angle is given by

$$\varkappa = \theta - \tan^{-1}\left(\frac{1}{r_o}\frac{dr_o}{d\theta}\right)$$

The value of $\beta$ is obtained from the expression above for $\beta'$ by letting $\alpha = 0$.

### 2.4.5 Applications to Common Body Shapes

We now apply the above solutions to several simple body shapes. The following sections will consider a wedge, a cylinder, a hemisphere, a yawed circular cone and an elliptical cone at zero and at a positive angle of attack.

Wedge

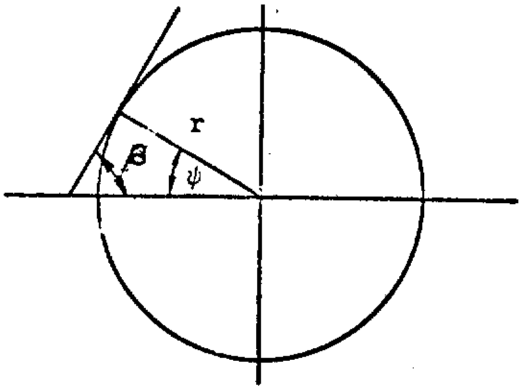
The expressions found already for $\alpha = 0$ (i.e., Equations (96) through (100) with $j = 1$) constitute the solution for a wedge and they may be interpreted directly for $\alpha \neq 0$. The expression for the velocity, Equation (9) is exact. The exact expression for the dimensionless pressure behind an oblique shock at angle $\phi$ with respect to the free stream is

$$\bar{p} = \frac{2\sin\Phi}{\gamma-1} - \frac{\gamma-1}{\gamma+1}\bar{p}_\infty$$

For hypersonic conditions, the second term may be dropped compared to the first term, and the shock angle approaches the body angle, $\phi \to \beta$. This then shows that Expression (97) gives the pressure distribution which is exact for a gas having $\gamma = 1$, and identifies the theory with other Newtonian perfect gas models (see, for example, Reference 8).

Cylinder

For a cylinder of radius r, $\beta$ is related to $\psi$ as shown in the sketch below.




The following results are obtained:

$$\beta = \frac{\pi}{2} - \psi, \quad L = r, \quad Y = r \sin\psi, \quad j = 1$$
$$u = \frac{1}{2} \sin\psi \tag{101}$$

$$p = 1 - \frac{3}{2} \sin^2\psi = p_{NEWTONIAN} - \frac{1}{2} \sin^2\psi = p_{NEWT.-BUSSEMANN} \tag{102}$$

$$T = \frac{1}{2}\left(1 - \frac{1}{4} \sin^2\psi\right) \tag{103}$$

$$\rho_w = \frac{2\gamma}{\gamma - 1} \, \frac{1 - \frac{3}{2} \sin^2\psi}{1 - \frac{1}{4} \sin^2\psi} \tag{104}$$

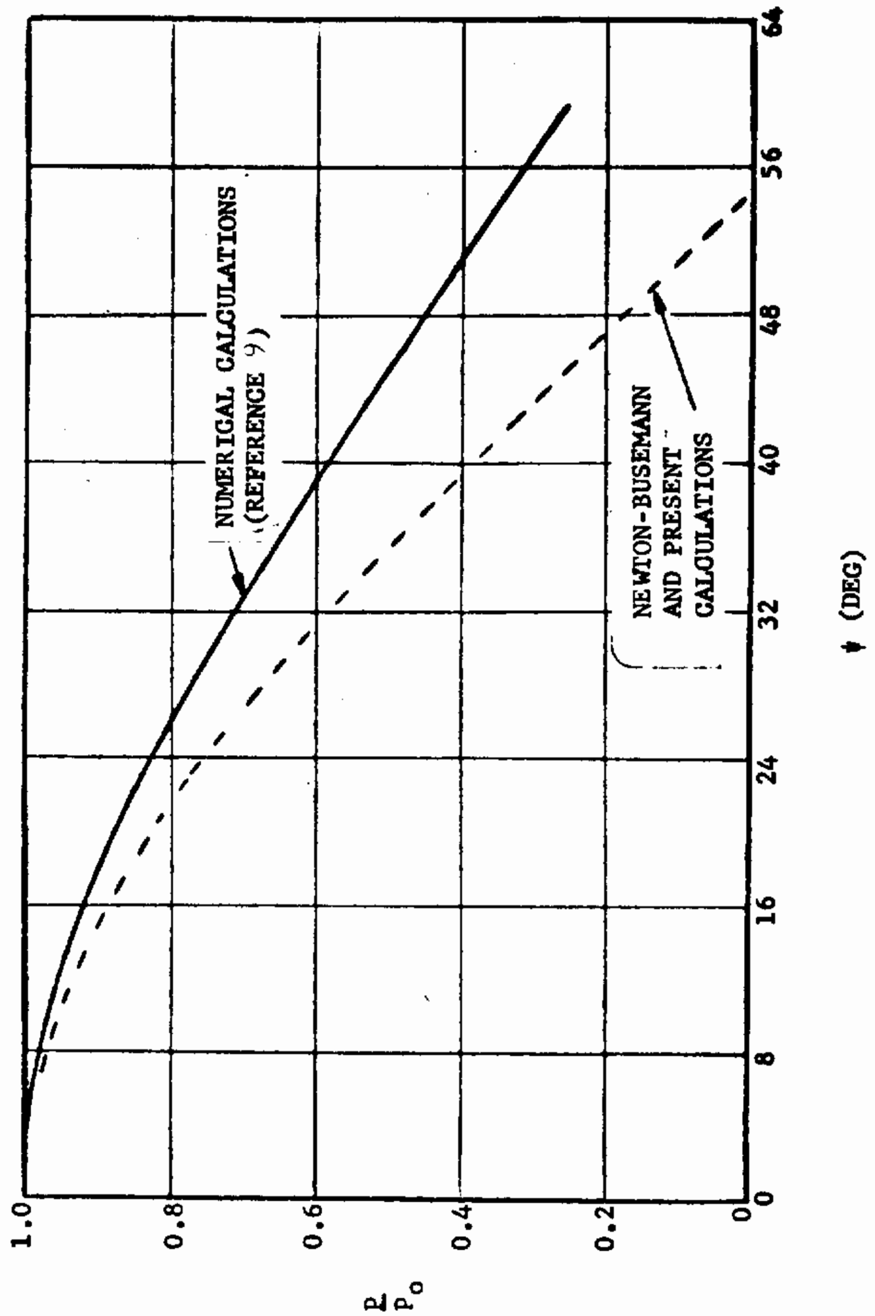



Figure 7. Pressure Distribution on Cylinder

$$\delta = \frac{4R}{\bar{\rho}_\omega + \frac{\gamma+1}{\gamma-1}} \tag{105}$$

The pressure distribution given by Eq.(112) is plotted in Fig. 7 along with the numerical results given in Ref. 9. It may be seen that the present calculations give exactly the Newton-Busemann results (where the centrifugal correction has been made).

Hemisphere

For a hemisphere of radius r, the following results are obtained (see the above sketch for cylinder);

$$u = \frac{2}{3}\sin\psi \tag{106}$$

$$p_\omega = 1 - \frac{4}{3}\sin^2\psi \tag{107}$$

$$T_\omega = \frac{1}{2}\left(1 - \frac{4}{9}\sin^2\psi\right) \tag{108}$$

$$\rho_\omega = \frac{2\gamma}{\gamma-1}\,\frac{1-\frac{4}{3}\sin^2\psi}{1-\frac{4}{9}\sin^2\psi} \tag{109}$$

$$\delta = \frac{3}{2}\,\frac{1}{\rho_\omega + \frac{\gamma+1}{\gamma-1}} \tag{110}$$

The pressure distribution given by Eq.(107) and the shock layer thickness given by Eq.(110) are plotted in Figs. 8 and 9, respectively, along with the numerical results given in Ref. 10.

Circular cone at an angle of attack

In this case $\beta'$ is given by

$$\sin\beta' = \frac{-\sin\alpha\cos\theta + (r_o/z)\cos\alpha}{\sqrt{1+(r_o/z)^2}} \tag{111}$$

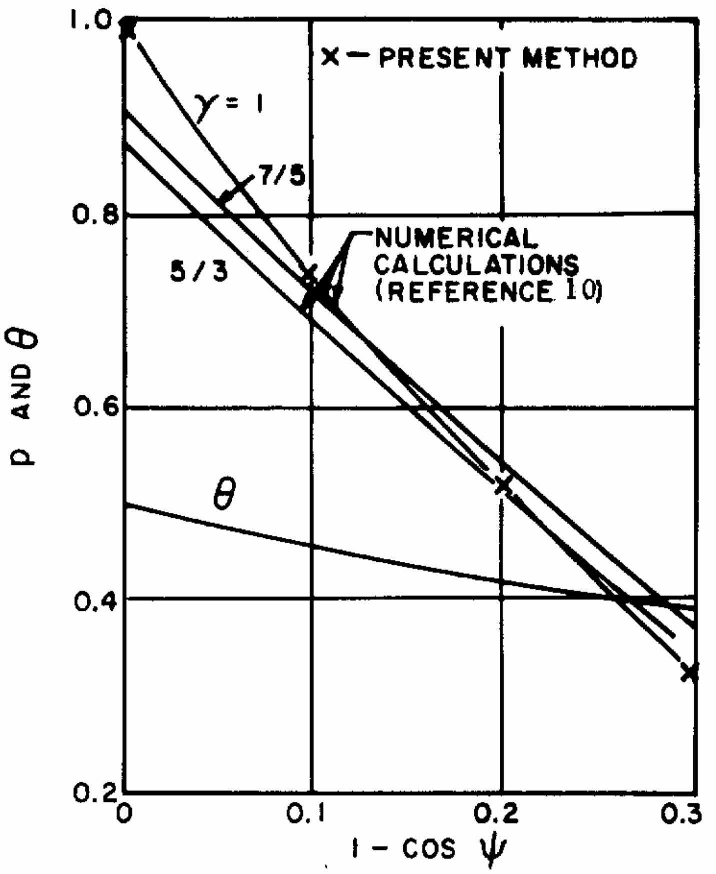



FIGURE 8 : PRESSURE DISTRIBUTION ON HEMISPHERE.

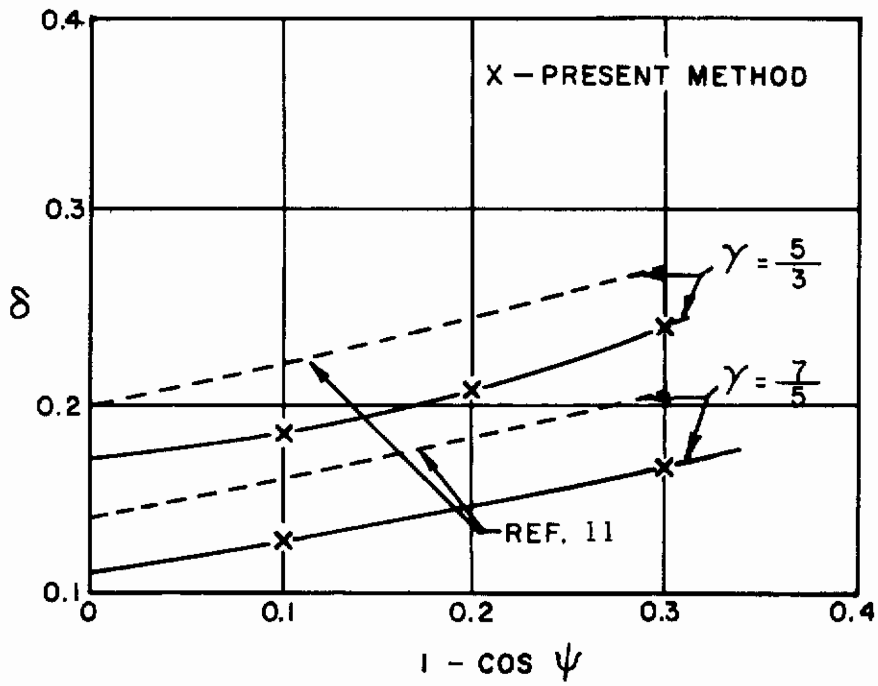



FIGURE 9 : SHOCK LAYER THICKNESS FOR HEMISPHERE.

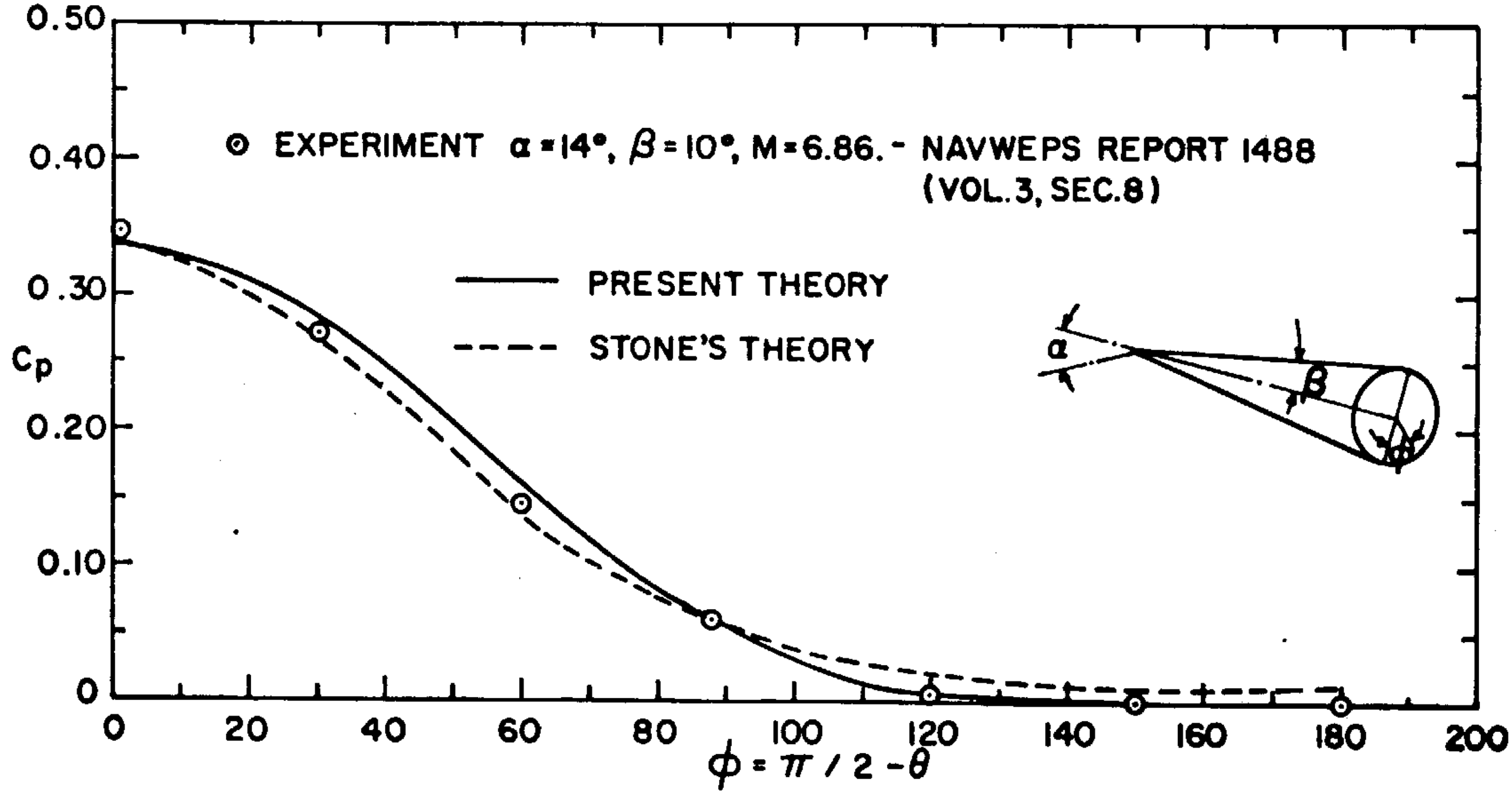



Figure 10a. Pressure Distributions Calculated from the Present Theory and Compared with Experimental and Theoretical Results

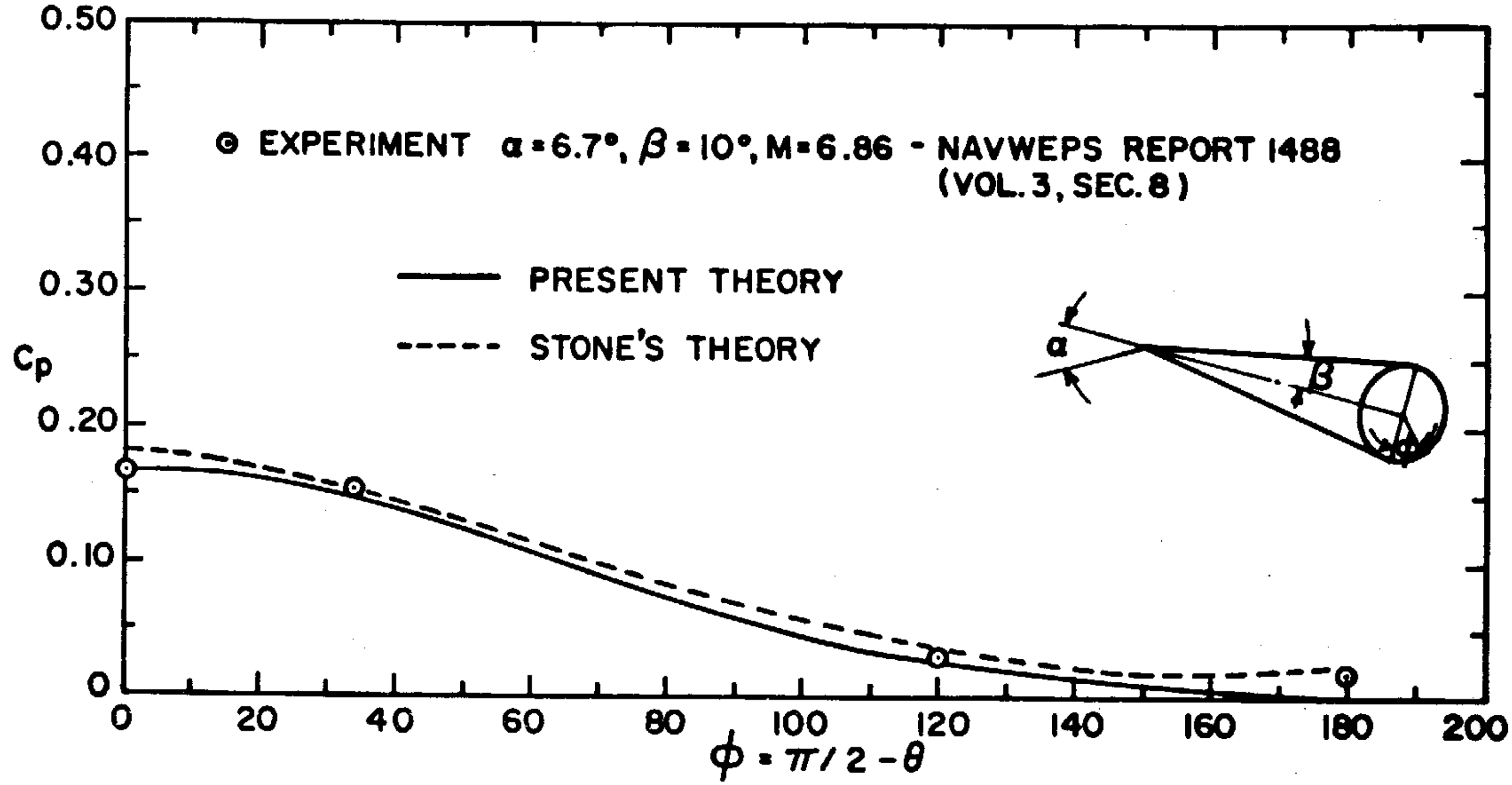



**Figure** 10b. **Pressure Distribution Calculated from the Present Theory and Compared with Experimental and Other Theoretical Results.**

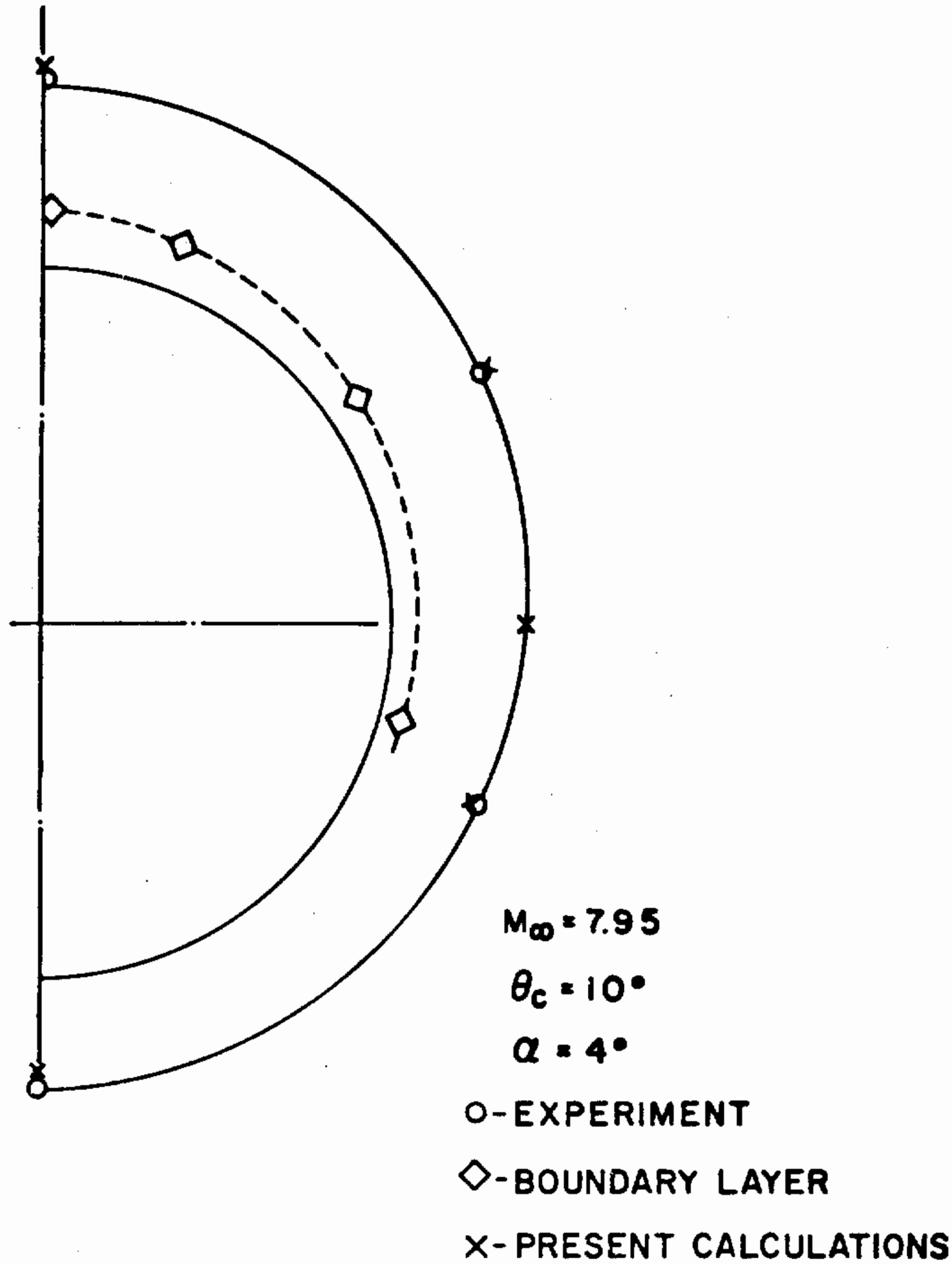



Figure 11a. Shock Layer Thickness Data for a Conical Body (Reference 11) Compared to The Present Theory

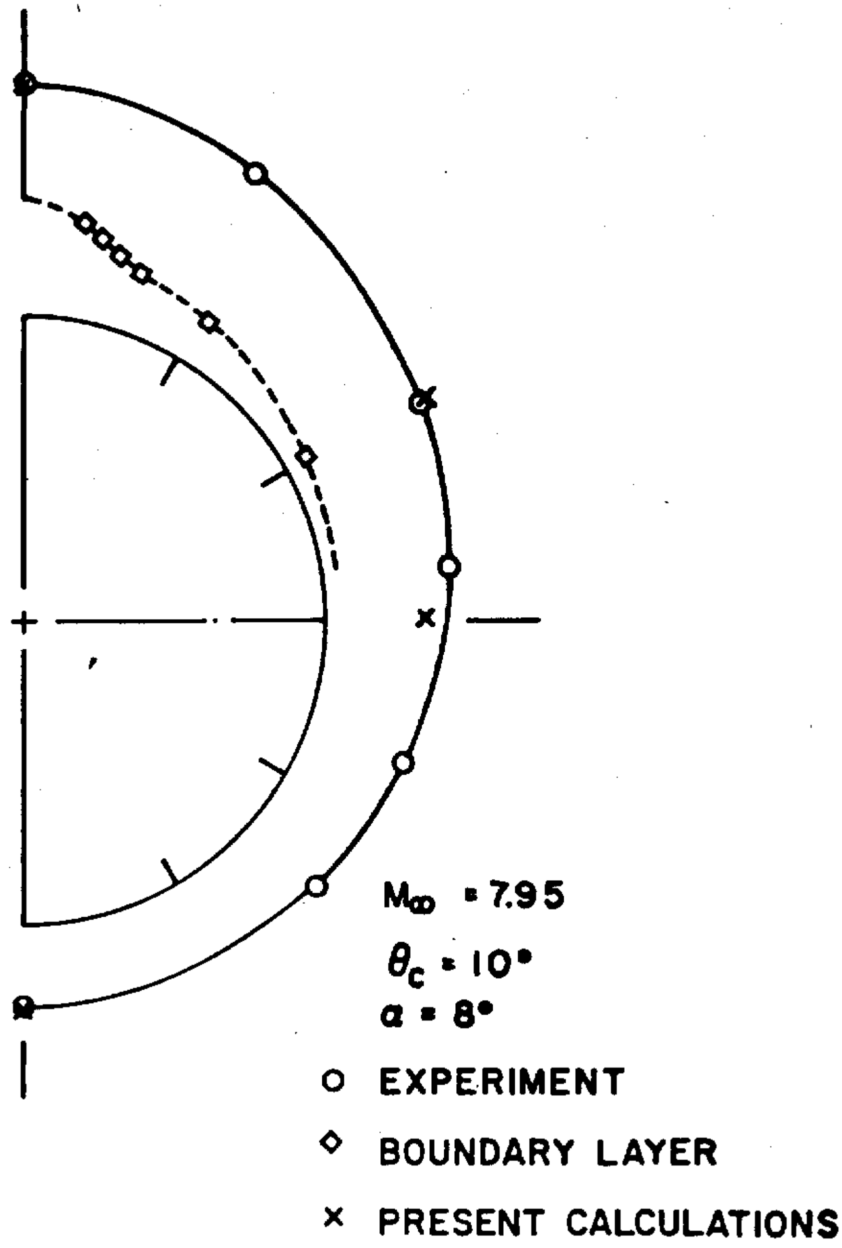



Figure 11b Shock Layer Thickness Data for a Conical Body (Reference 11) Compared to the Present Theory.

## 2.5 Calculation of Skin Friction and Heat Transfer at the Wall

The calculation of skin friction and heat transfer is easily obtained using the method of integral relations and the series representation of velocity and total enthalpy. The skin friction and heat flux will be obtained as functions of the longitudinal distance $\xi$. The skin friction (tangential stress) is given by the expression

$$\tau = \mu \frac{\partial u}{\partial y}\Big|_{y=0} = \left(\frac{r}{R_o}\right)^{\nu} \frac{\mu_\omega \rho_\omega}{\rho_\infty} \frac{U U_\infty}{\nu_\infty} f(\xi) \frac{\partial \bar{u}}{\partial \eta}\Big|_{\eta=0} \tag{112}$$

where $U$ is the potential flow velocity given by Newtonian theory or numerical results, $f(\xi)$, and $a_1$ are given as a result of solving the systems of differential equations (printed computer output), and

is given and a function of wall temperature.

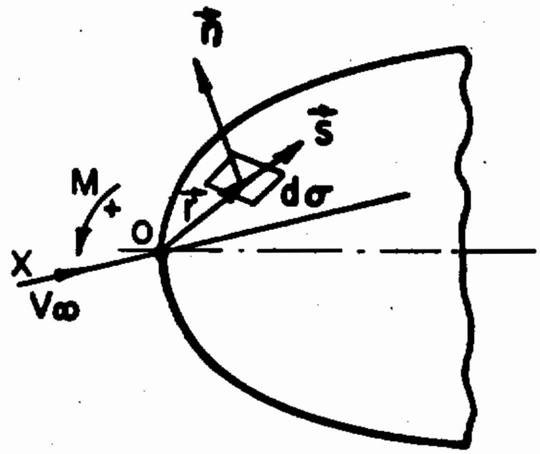
The heat flux is obtained similarly

$$q = \left(\frac{r}{R_o}\right)^{\nu} \frac{1}{Pr} \frac{\rho_\omega \mu_\omega}{\rho_\infty \mu_\infty} \rho_\infty U^2 U_\infty \bar{h}_{o}, b_1 f(\xi) \tag{113}$$

where $b_1$ is obtained from the solution of the system of differential equations (printed computer output), and $Pr = \frac{C_{p_\omega} \mu_\omega}{\lambda_\omega}$ is the Prandtl number.

## 2.6 Calculation of Forces Acting on the Body

**Force acting on the frontal area of the body results from the pressure distribution and the skin friction.**




**The force is therefore equal to**

$$\vec{P} = -\iint_{\sigma} p\vec{n}\,d\sigma + \iint_{\sigma} \tau\vec{s}\,d\sigma$$

**where** $\sigma$ **is the surface of the body, p is the pressure given by potential flow theory.**

$\vec{n}$ **is the unit normal vector** directed **outside to the surface** $\sigma$**.**

**is the unit vector streamwise tangential to the surface and**

$\tau$ **denotes tangential stress calculated from eq. (112).**

Component of force (drag) in the direction x (undisturbed flow direction).

$$P_x = -\iint_{\sigma} p\cos(n,x)\,d\sigma + \iint_{\sigma} \tau\vec{s}\,d\sigma \qquad \textbf{(114)}$$

where cos (n, x) and cos (s, x) are given by the body geometry and the angle of **attack** .

**Lift is equal to**

$$P_L = -\iint_{\sigma} p\sin(n,x)\,d\sigma + \iint_{\sigma} \tau\sin(s,x)\,d\sigma$$

**Pitching moment**

$$M = -\iint_{\sigma}\left[p(\vec{r}\times\vec{n})_z - \tau(\vec{r}\times\vec{s})_z\right]d\sigma$$

where the subscript z denotes the component along the axis perpendicular to the plane passing through the axis of the body and velocity vector $V_\infty$ .

## 2.7 Real Gas Effects

If one assumes thermodynamic equilibrium, the effect of exitation of vibrational degrees of freedom, electronic exitation and dissociation results in a variable $C_p$ and gas constant R . ($R = R_o (1+x)$ where x(T) denotes degree of dissociation).

Since the properties of a given gas are known functions of temperature, it is no problem to input these real gas properties in places of the ideal gas properties. Naturally, one must do this for each gas separately.

## 3. DISCUSSION

A method of solution for the case of mass injection into a boundary layer has been outlined in the present report. The system of partial differential equations were reduced to a system of ordinary differential equations by integration over different strips. The velocity and enthalpy profiles were assumed to be expressed by a series of error functions. The system of equations were solved starting from the section $\xi = 0$ without additional assumptions as to initial profiles. The initial profiles were obtained by an analysis of singularity at $\xi = 0$ and were reduced to the solution of a simultaneous system of transcendental equations. The present solution of the transcendental equations uses a GD/Convair Subroutine called NON II, which results in satisfactory convergence. In the blunt body case, velocity and enthalpy profiles are coupled and, therefore simultaneous solution of eight equations was required. For given input parameters (ratio of enthalpy at the body surface to free stream enthalpy and the injection parameter) one can use existing solutions as the initial guess and attempt to converge the system of equations with perturbations of input parameters. In this way, tables of initial profiles can be calculated.

In the present report a lineralized system of equations was derived to calculate the initial derivatives of the coefficients used in the power series expressions for each unknown profile. The linear system of equations provides smooth transition to the downstream equations. It is also possible to obtain initial derivaties in tabular form, leaving only the solution of the downstream flow.

Example calculations have been presented for cases of sharp and blunt bodies. The results indicates no trouble starting the computation near the singular point and that smooth behavior of the downsteam boundary layer parameters was obtained. Therefore, it is possible to state that the present method of calculation is accurate (multistrip), self-consistent, and adequate for sharp and blunt nosed bodies. However, to reduce the present method to a fairly simple calculation, one should calculate tables of initial profiles and initial derivatives for a wide variety of input parameters. The initial profiles and derivatives could be calculated by a small perturbation of the input parameters in the transcential equations using existing solutions as initial guesses.

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## APPENDIX A

## CALCULATIONS OF INITIAL DERIVATIVES OF COEFFICIENTS IN VELOCITY PROFILE

$$\lim_{\xi \to \Delta\xi} f^2 \Phi \qquad \Delta\xi \ll 1 \qquad f = \frac{K}{\sqrt{\xi}}$$

From equation (19), it follows that $\frac{dK}{d\xi} \sim \frac{1}{\sqrt{\xi}}$ , say $\frac{dK}{d\xi} = \frac{C}{\sqrt{\xi}}$

and, integrating, we get

$$K = K_o + 2C\sqrt{\xi} = K_o + 2\left.\frac{dK}{d\xi}\right|_{\xi=\Delta\xi} \cdot \Delta\xi$$

Now

$$f = \frac{K}{\sqrt{\xi}} = \frac{K_o}{\sqrt{\xi}} + 2\left.\frac{dK}{d\xi}\right|_{\xi=\Delta\xi}$$

$$f' = -\frac{K_o}{2\xi^{3/2}} + \frac{1}{\sqrt{\xi}}\frac{dK}{d\xi} + 2\sqrt{\xi}\frac{d^2K}{d\xi^2}$$

and

$$\frac{d^2K}{d\xi^2} = \frac{d}{d\xi}\left(\frac{C}{\sqrt{\xi}}\right) = -\frac{C}{2\xi^{3/2}} = -\frac{1}{2\xi}\frac{dK}{d\xi}$$

Therefore,

$$f' = -\frac{K_o}{2\xi^{3/2}}$$

$$f^3 = \frac{K_o^3}{\xi^{3/2}}\left(1 + \frac{6}{K_o}\xi\frac{dK}{d\xi} + \frac{12}{K_o^2}\left(\frac{dK}{d\xi}\right)^2\xi^2 + \frac{8}{K_o^3}\left(\frac{dK}{d\xi}\right)^3\xi^3\right)$$

and consequently,

$$\frac{f'}{f^3} = -\frac{1}{2K_o^2}\,\frac{1}{1 + \frac{6}{K_o}\frac{dK}{d\xi}\xi + \frac{12}{K_o^2}\left(\frac{dK}{d\xi}\right)^2\xi^2 + \frac{8}{K_o^3}\left(\frac{dK}{d\xi}\right)^3\xi^3}$$

or using the expansion

$$\frac{1}{1+\epsilon} = 1 - \epsilon + \epsilon^2 + \cdots \qquad \text{where } \epsilon \ll 1$$

$$\frac{f'}{f^3} = -\frac{1}{2K_o^2}\left(1 - \frac{6}{K_o}\xi\frac{dK}{d\xi} + \frac{24}{K_o^2}\left(\frac{dK}{d\xi}\right)^2\xi^2 - \frac{80}{K_o^3}\left(\frac{dK}{d\xi}\right)^3\xi^3 + \cdots\right.$$

Now we may write

$$\lim_{\xi \to \Delta\xi} f^2 \Phi = \lim \left[ \frac{K_o^2}{\xi} + 4\frac{dK}{d\xi}K_o + 4\left(\frac{dK}{d\xi}\right)^2 \xi \right] (\Delta\Phi + \Phi_o)$$

where

$$\Phi_o + \Delta\Phi = \frac{1}{2K_o^2}\left(1 - \frac{6}{K_o}\xi\frac{dK}{d\xi} + \frac{24}{K_o^2}\left(\frac{dK}{d\xi}\right)^2\xi^2 - \frac{80}{K_o^3}\left(\frac{dK}{d\xi}\right)^3\xi^3\right) \cdot$$

$$\sum_{n=1}^{N}\sum_{k=1}^{N}(A_{2nk} - A_{1nk})\left[a_{no}a_{ko} + \Delta a_n a_{ko} + \Delta a_k a_{no} + \Delta a_n \Delta a_k\right]$$

$$+ \frac{\bar{\rho}_1\sqrt{\bar{\mu}_i\bar{\rho}_i}}{U}\sum_{n=1}^{N}A_{3n}(a_{no} + \Delta a_n) - \frac{2}{\sqrt{\pi}}\frac{a_{10} + \Delta a_1}{U}\bar{\rho}_1\sqrt{\bar{\mu}_\omega\bar{\rho}_\omega}$$

We could take into account that

$$\Phi_o = \frac{1}{2K_o^2}\sum_{n=1}^{N}\sum_{k=1}^{N}(A_{2nk} - A_{1nk})a_{no}a_{ko} + \frac{\bar{\rho}_1\sqrt{\bar{\mu}_i\bar{\rho}_i}}{U}\sum_{n=1}^{N}A_{3n}a_{no}$$

$$-\frac{2}{\sqrt{\pi}}\frac{a_{10}}{U}\bar{\rho}_1\sqrt{\bar{\mu}_\omega\bar{\rho}_\omega} = 0$$

(from this condition the system of transcendental equations for $a_{no}$, $a_{ko}$ and $K_o$ was obtained). In fact, $\Phi_o$ is not equal exactly zero because of an inevitable error in numerical calculation. This puts a lower limit on $\Delta\xi$ because must be $\frac{\Phi_o}{\Delta\xi} \ll 1$ and $\Delta\xi$ cannot be taken arbitrarily small to satisfy this condition. However, we shall include the term $\frac{\Delta\Phi_o}{\xi}$ as a small error.

From equation (A8) and expression (A7) one gets

$$\lim_{\xi \to \Delta\xi} f^2 \Phi = \lim \Big[ \frac{K_o^2}{\xi} + 4\frac{dK}{d\xi}K_o + 4\Big(\frac{dK}{d\xi}\Big)^2 \xi \Big] \Big\{ \frac{1}{2K_o^2}\Big[ -\frac{6\xi}{K_o}\frac{dK}{d\xi} + \frac{24}{K_o^2}\Big(\frac{dK}{d\xi}\Big)^2 \xi^2$$

$$- \frac{80}{K_o^3}\Big(\frac{dK}{d\xi}\Big)^3 \xi^3 \Big] \sum_{n=1}^{N}\sum_{k=1}^{N} (A_{2nk} - A_{1nk}) \Big[ a_{no}a_{ko} + \Delta a_n a_{ko} + \Delta a_k a_{no} + \Delta a_n \Delta a_k \Big]$$

$$+ \frac{1}{2K_o} \sum_{n=1}^{N}\sum_{k=1}^{N} (A_{2nk} - A_{1nk}) \Big[ \Delta a_n a_{ko} + \Delta a_k a_{no} + \Delta a_k \Delta a_n \Big] +$$

$$+ \bar{\varrho}_1 \sqrt{\mu_i \varrho_i} \sum_{n=1}^{N} A_{3n} \Delta a_n - \frac{2}{\sqrt{\pi}} \frac{\Delta a_1 \varrho_1 \sqrt{\mu_\omega \varrho_\omega}}{U} + \Phi_o \Big\}$$

Introducing

$$\Delta a_n = 2\frac{da_n}{d\xi}\xi \quad , \quad \Delta a_k = 2\frac{da_k}{d\xi}\xi$$

one obtains

$$\lim_{\xi \to \Delta\xi} f^2 \Phi = \sum_{n=1}^{N}\sum_{k=1}^{N} (A_{2nk} - A_{1nk}) \Big( a_{ko}\frac{da_n}{d\xi} + a_{no}\frac{da_k}{d\xi} \Big) +$$

$$- \frac{3}{K_o}\frac{dK}{d\xi} \sum_{n=1}^{N}\sum_{k=1}^{N} (A_{2nk} - A_{1nk}) a_{no} a_{ko}$$

$$+ 2K_o^2 \frac{\varrho_1 \sqrt{\mu_i \varrho_i}}{U} \sum_{n=1}^{N} A_{3n} \frac{da_n}{d\xi} - \frac{4K_o^2}{\sqrt{\pi}} \frac{\varrho_1 \sqrt{\mu_\omega \varrho_\omega}}{U} \frac{da_1}{d\xi} + \Delta_1$$

where

$$\Delta_1 = \frac{K_o^2 \Phi_o}{\xi} + \left[4K_o\frac{dK}{d\xi} + 4\left(\frac{dK}{d\xi}\right)^2 \xi\right]\Big\{\Phi_o + 2\xi\Big(\frac{\rho_1\mu_i\rho_i}{U}\sum_{n=1}^{N} A_{3n}\frac{da_n}{d\xi}$$

$$-\frac{2}{\sqrt{\pi}}\,\frac{\rho_1\mu_\omega\rho_\omega}{U}\frac{da_1}{d\xi}\Big) + \frac{\xi}{K_o^2}\sum_{n=1}^{N}\sum_{k=1}^{N}(A_{2nk}-A_{1nk})\Big(\frac{da_n}{d\xi}a_{ko} + \frac{da_k}{d\xi}a_{no}$$

$$+2\frac{da_n}{d\xi}\frac{da_k}{d\xi}\xi\Big)\Big\} + 2\sum_{n=1}^{N}\sum_{k=1}^{N}(A_{2nk}-A_{1nk})\xi\frac{da_n}{d\xi}\frac{da_k}{d\xi}$$

$$-\xi^2\left(\frac{dK}{d\xi}\right)^3\frac{8}{K_o^3}\left[1-\frac{2\xi}{K_o}\frac{dK}{d\xi}\right]\sum_{n=1}^{N}\sum_{k=1}^{N}(A_{2nk}-A_{1nk})\Big[a_{no}a_{ko} +$$

$$+2\xi\frac{da_n}{d\xi}a_{ko} + 2\xi\frac{da_k}{d\xi}a_{no} + 4\xi^2\frac{da_k}{d\xi}\frac{da_n}{d\xi}\Big]$$

The other small order terms as follows from expression (19) are

$$\Delta_2 = -2\sum_{n=1}^{N}\sum_{k=1}^{N}\left[2A_{1nk}-A_{2nk}\right]\xi\frac{da_n}{d\xi}\frac{da_k}{d\xi} + \frac{2}{U^2}\frac{dU^2}{d\xi}\xi\sum_{n=1}^{N}\sum_{k=1}^{N}A_{1nk}\Big($$

$$a_{no}\frac{da_k}{d\xi} + a_{ko}\frac{da_n}{d\xi} + 2\xi\frac{da_n}{d\xi}\frac{da_k}{d\xi}\Big) + \frac{2\xi}{U}\frac{dU}{d\xi}\sum_{n=1}^{N}\sum_{k=1}^{N}A_{2nk}\Big(a_{no}\frac{da_k}{d\xi} +$$

$$+a_{ko}\frac{da_n}{d\xi} + 2\xi\frac{da_n}{d\xi}\frac{da_k}{d\xi}\Big) - \frac{\hat{v}_\omega}{U}2\frac{dK}{d\xi}\sqrt{\xi}\sum_{n=1}^{N}\Big(a_{no} + 2\xi\frac{da_n}{d\xi}\Big)\left[\operatorname{erf}\eta_i\right]^n$$

$$-\frac{2\hat{v}_\omega}{U}K_o\sqrt{\xi}\sum_{n=1}^{N}\frac{da_n}{d\xi}\left[\operatorname{erf}\eta_i\right]^n$$

$$\Delta = \Delta_1 + \Delta_2$$

## APPENDIX B

## CALCULATIONS OF INITIAL DERIVATIVES OF COEFFICIENTS IN THE ENTHALPY EXPRESSION

It is convenient to separate into a group those terms of equation (19) which do not vanish individually when each term of that equation is divided by $f^2$ and $\xi$ is allowed to approach zero. Accordingly, we have

$$\Phi_h = -\frac{f'}{f^3}\sum_{n=1}^{N}\sum_{k=1}^{N}(B_{2nm}-B_{1nm})a_n b_m + \frac{1}{P_r}(\bar{\mu}_i \bar{\varsigma}_i \bar{\varsigma}_1\Big[\sum_{m=1}^{M} B_{3m}b_1 - \frac{2\alpha}{\sqrt{\pi}}b_1 + \frac{P_r-1}{h_{01}}\sum_{n=1}^{N}\sum_{k=1}^{N} A_{4nk}a_n a_k\Big], \tag{B1}$$

or

$$\Phi_h = -\frac{f'}{f^3}\sum_{n=1}^{N}\sum_{k=1}^{N}(B_{2nm_0}-B_{1nm_0}+\Delta B_{2nm}-\Delta B_{1nm})(a_{n0}+\Delta a_n)(b_{m0}+\Delta b_m)$$
$$+\frac{1}{P_r}(\bar{\mu}_i \bar{\varsigma}_i \bar{\varsigma}_1\Big[\sum_{n=1}^{N}(b_m+\Delta b_m)(B_{3m}+\Delta B_{3m}) - \frac{2(\alpha+\Delta\alpha)}{\sqrt{\pi}}(b_1+\Delta b_1)$$
$$+\frac{P_r-1}{h_{01}}\sum_{n=1}^{N}\sum_{k=1}^{N} A_{4nk}(a_{n0}+\Delta a_n)(a_{k0}+\Delta a_k) \tag{B2}$$

Now in studying the limiting form of equation (19) for small positive $\xi$ the group $\Phi$ is multiplied by $f^2$ and the first order expansions of all functions of $\xi$ are taken. Thus for $0<\xi\ll 1$

$$f^2\Phi_h = -\frac{f'}{f}\sum_{n=1}^{N}\sum_{k=1}^{N}(B_{2nm_0}-B_{1nm_0}-\Delta B_{2nm}-\Delta B_{1nm})(a_{n0}+\Delta a_n)\cdot$$
$$(b_{m0}+\Delta b)+\frac{f^2}{P_r}(\mu_i \varsigma_i \varsigma_1\Big[\sum_{n=1}^{N}(b_{m0}+\Delta b_m)(B_{3m}+\Delta B_{3m}) - \frac{2(\alpha+\Delta\alpha)}{\sqrt{\pi}}(b_1+\Delta b_1)$$
$$+\frac{P_r-1}{h_{01}}\sum_{n=1}^{N}\sum_{k=1}^{N} A_{4nk}(a_{n0}+\Delta a_n)(a_{k0}+\Delta a_k) \tag{B3}$$

Where from the discussion in the text we have demonstrated that for very small positive $\xi$, K is a parabolic function of $\xi$ and the following relationships are valid:

$$f' = -\frac{K_0}{2\xi^{3/2}} \qquad f = \frac{K_0}{\sqrt{\xi}} + 2\frac{dK}{d\xi}\sqrt{\xi}$$

$$f^2 = \frac{K_0^2}{\xi} + 4K_0\frac{dK}{d\xi} + 4\xi\left(\frac{dK}{d\xi}\right)^2$$

$$\frac{1}{f} = \frac{\sqrt{\xi}}{K_0}\,\frac{1}{1+\frac{2}{K_0}\frac{dK}{d\xi}\xi} = \frac{\sqrt{\xi}}{K_0}\left(1 - \frac{2}{K_0}\frac{dK}{d\xi}\xi + \frac{4}{K_0^2}\left(\frac{dK}{d\xi}\right)^2\xi^2 - \frac{8}{K_0^3}\left(\frac{dK}{d\xi}\right)^3\xi^3 + \dots\right)$$

$$\frac{f'}{f} = -\frac{1}{2\xi}\left(1 - \frac{2}{K_0}\frac{dK}{d\xi}\xi + \frac{4}{K_0^2}\left(\frac{dK}{d\xi}\right)^2\xi^2 - \frac{8}{K_0^3}\left(\frac{dK}{d\xi}\right)^3\xi^3 + \dots\right) \tag{B4}$$

With these expressions the group $f^2\Phi$ becomes, for $0<\xi\ll 1$

$$f^2\Phi_h = \frac{1}{2\xi}\left(1 - \frac{2}{K_0}\frac{dK}{d\xi}\xi + \frac{4}{K_0^2}\left(\frac{dK}{d\xi}\right)^2\xi^2 - \frac{8}{K_0^3}\left(\frac{dK}{d\xi}\right)^3\xi^3 + \dots\right)\cdot$$

$$\sum_{n=1}^{N}\sum_{k=1}^{N}\Big[\big[(B_{2nm_0} - B_{1nm_0})a_{n0}b_{m0} + (\Delta B_{2nm} - \Delta B_{1nm})a_{n0}b_{m0} + (B_{2nm_0} - B_{1nm_0})$$

$$(a_{n0}\Delta b_m + b_{m0}\Delta a_n)\big] + \sum_{n=1}^{N}\sum_{k=1}^{N}(B_{2nm_0} - B_{1nm_0})\Delta a_n\Delta b_m + \sum_{n=1}^{N}\sum_{m=0}^{M}(\Delta B_{2nm} - \Delta B_{1nm})\cdot$$

$$(a_{n0}\Delta b_m + b_{m0}\Delta a_n + \Delta a_n\Delta b_m)\Big\} + \frac{K_0^2}{\xi} + 4K_0\left(\frac{dK}{d\xi}\right)^2\frac{1}{P_r}(\mu_i\rho_i\rho_1\Big\{$$

$$\sum_{m=0}^{M} b_m B_{3m_0} - \frac{2\alpha_0}{\sqrt{\pi}}b_{10} + \frac{P_r - 1}{h_{01}}\sum_{n=1}^{N}\sum_{k=1}^{N} A_{4nk}\,a_{n0}a_{k0} +$$

$$+\sum_{m=0}^{M}\left(b_{m0}\Delta B_{3m_0} + B_{3m_0}\Delta b_m + \Delta b_m\Delta B_{3m}\right) - \frac{2\alpha_0\Delta b_1}{\sqrt{\pi}}$$

$$-\frac{2b_1\Delta\alpha}{\sqrt{\pi}} - \frac{2\Delta b_1\Delta\alpha}{\sqrt{\pi}} + \frac{P_r - 1}{h_{01}}\sum_{n=1}^{N}\sum_{k=1}^{N} A_{4nk}(a_{n0}\Delta a_k +$$

$$+ a_{k0}\Delta a_n + \Delta a_k\Delta a_n)\Big\} \tag{B5}$$

Making specific use of the parabolic form of K, $a_n$, $b_m$ and $\alpha$ we may express their values at the point $\xi$ in terms of their derivatives there and the value of the function at $\xi = 0$ thereby avoiding the singular derivative which would occur in a Maclaurin expansion (typically $\alpha = \alpha_0 + 2\left.\frac{d\alpha}{d\xi}\right|_\xi \cdot \xi$. Again for $0 < \xi \ll 1$

$$
\begin{aligned}
f^2\Phi_h = {} & \sum_{n=1}^{N}\sum_{m=1}^{M}\left[B_{2nm_0} - B_{1nm_0}\right]\left(a_{n0}\frac{db_m}{d\xi} + b_{m0}\frac{da_n}{d\xi}\right) + \\
& + \sum_{n=1}^{N}\sum_{k=1}^{N}\left(B_{4nm} - B_{6nm}\right)a_0 b_m \frac{d\alpha}{d\xi} + 2K_0^2\left(\bar{\mu}\bar{\varrho}\frac{\varrho_1}{UP_r}\left\{\sum_{m=0}^{M} B_{3m}\frac{db_m}{d\xi} + \right.\right. \\
& + \sum_{m=0}^{M} b_{m0} B_{5m}\frac{d\alpha}{d\xi} - \frac{2\alpha_0}{\sqrt{\pi}}\frac{db_1}{d\xi} + \frac{P_r - 1}{h_{01}}\left[\sum_{n=1}^{N}\sum_{k=1}^{N} A_{4nk}\left(a_{k0}\frac{da_n}{d\xi} + a_{n0}\frac{da_k}{d\xi}\right)\right]\Bigg\} \\
& - \frac{3}{K_0}\sum_{n=1}^{N}\sum_{m=0}^{M}\left[B_{2nm_0} - B_{1nm_0}\right]a_{n0}\, b_{m0}\frac{d\alpha}{d\xi} + \Delta_1
\end{aligned}
\tag{B6}
$$

where

$$
\begin{aligned}
\Delta_1 = {} & \frac{K_0^2\Phi_{h0}}{\xi} + 4K_0\frac{dK}{d\xi}\Phi_{h0} - \frac{1}{K_0}\frac{dK}{d\xi}\left\{\sum_{n=1}^{N}\sum_{m=0}^{M}\left(\Delta B_{2nm} - \Delta B_{1nm}\right)a_{n0} b_{m0}\right. \\
& + \left(B_{2nm_0} - B_{1nm_0}\right)\left(a_{n0}\Delta b_m + b_{m0}\Delta a_n + \Delta a_n \Delta b_m\right] + \sum_{n=1}^{N}\sum_{m=0}^{M}\left(\Delta B_{2nm} - \Delta B_{1nm}\right)\Big( \\
& \left. a_{n0}\Delta b_m + b_{m0}\Delta a_n + \Delta a_n \Delta b_m\Big)\right\} + \frac{2}{K_0^2}\left(\frac{dK}{d\xi}\right)^2\xi\left[1 - \frac{2}{K_0}\frac{dK}{d\xi}\xi\right]\sum_{n=1}^{N}\sum_{m=0}^{M}\Big( \\
& B_{2nm_0} - B_{1nm_0} + \Delta B_{2nm} - \Delta B_{1nm}\Big)\left(a_{n0} + \Delta a_n\right)\left(b_{m0} + \Delta b_m\right) + \\
& \frac{1}{2\xi}\left\{\sum_{n=1}^{N}\sum_{m=0}^{M}\left(B_{2nm_0} - B_{1nm_0}\right)\Delta a_n \Delta b_m + \sum_{n=1}^{N}\sum_{m=0}^{M}\left(\Delta B_{2nm} - \Delta B_{1nm}\right)\Big(a_{n0}\Delta b_m + \right. \\
& \left. + b_{m0}\Delta a_n + \Delta a_n \Delta b_m\Big)\right\} + \frac{K_0^2}{\xi}\frac{1}{P_r}\left(\bar{\mu}_i\bar{\varrho}_i\bar{\varrho}_1\left\{\sum_{n=1}^{N}\sum_{k=1}^{N} A_{4nk}\Delta a_k \Delta a_n\frac{P_r - 1}{h_{01}} - \frac{2\Delta b_1 \Delta\alpha}{\sqrt{\pi}} + \right.\right. \\
& \left. + \sum_{m=1}^{M}\Delta B_{3m_0}\Delta b_m\right\} + 4K_0\frac{dK}{d\xi}\left(1 + \frac{\xi}{K_0}\frac{dK}{d\xi}\right)\frac{1}{P_r}\left(\bar{\mu}_i\bar{\varrho}_i\varrho_1\left\{\sum_{m=0}^{M}\Big(b_{m0}\Delta B_{3m_0} + B_{3m_0}\Delta b_m + \right.\right. \\
& + \Delta B_{3m}\Delta b_m - \frac{2\alpha_0\Delta b_1}{\sqrt{\pi}} - \frac{2b_{10}\Delta\alpha}{\sqrt{\pi}} - \frac{2\Delta b_1\Delta\alpha}{\sqrt{\pi}} + \frac{P_r - 1}{h_{01}}\sum_{n=1}^{N}\sum_{k=1}^{N} A_{4nk}\Big(a_{n0}\Delta a_k + a_{k0}\Delta a_n + \\
& \Delta a_k \Delta a_n + 4\xi\left(\frac{dK}{d\xi}\right)^2\frac{1}{P_r}\left(\bar{\mu}_i\bar{\varrho}_i\bar{\varrho}_1\left\{\sum_{m=1}^{M} B_{3m_0} b_{m0} - \frac{2\alpha}{\sqrt{\pi}}b_{10} + \frac{P_r - 1}{h_{01}}\sum_{n=1}^{N}\sum_{k=1}^{N} A_{4nk}a_{n0}a_{k0}\right\}\right.
\end{aligned}
\tag{B7}
$$

Now, if the remaining terms of equation (19) are also expressed in these terms valid for small positive $\xi$, we obtain additional terms which may be included for the sake of completeness:

$$
\begin{aligned}
\Delta_2 = &-\sum_{n=1}^{N}\sum_{m=0}^{M}\left[(B_{1nm}+\Delta B_{1nm})\Delta a_n + \Delta B_{1nm}\, a_0\right]\frac{db_m}{d\xi} \\
&-\sum_{n=1}^{N}\sum_{m=0}^{M} B_{4nm}\left[a_{n_0}\Delta b_m + \Delta a_n\, b_{m_0} + \Delta a_n \Delta b_m\right]\frac{d\alpha}{d\xi} - \sum_{n=1}^{N}\sum_{m=0}^{M}\Delta B_4(a_{n_0}+\Delta a_n)\cdot \\
&(b_{m_0}+\Delta b_m)\frac{d\alpha}{d\xi} - \sum_{n=1}^{N}\sum_{m=0}^{M}\left[(B_{1nm_0}-B_{2nm_0})\Delta b_m + (\Delta B_{1nm}-\Delta B_{2nm})(b_{m_0}+\Delta b_m)\right]\frac{da_n}{d\xi} \\
&-2\frac{\hat{v}_\omega}{U}\frac{dK}{d\xi}\sqrt{\xi}\left\{\sum_{m=1}^{M}(b_{m_0}+\Delta b_m)\left(\operatorname{erf}\left((\alpha_0+\Delta\alpha)\eta_i\right)\right)^m\right\} - \frac{\hat{v}_\omega}{U}\frac{K_0}{\sqrt{\xi}}\Bigg[\sum_{m=1}^{M} \\
&\Delta b_m\left(\operatorname{erf}\left((\alpha_0+\Delta\alpha)\eta_i\right)\right)^m - \frac{2\hat{v}_\omega}{\sqrt{\pi}\,U}\frac{K_0}{\sqrt{\xi}}\sum_{m=1}^{M} b_{m_0}\, m\left(\operatorname{erf}\alpha_0\eta_i\right)^{m-1}\left[\exp(-\alpha_0^2\eta_i^2)\right]\Delta\alpha\,\eta_i
\end{aligned}
\tag{B8}
$$

where

$$\Delta a_k = 2\frac{da_k}{d\xi}\xi$$

$$\Delta b_m = 2\frac{db_m}{d\xi}\xi$$

$$\Delta\alpha = 2\frac{d\alpha}{d\xi}\xi$$

# APPENDIX C

## GEOMETRIC RELATIONSHIPS FOR A BLUNT BODY

Geometric relationships for body shapes of Figs. 6a and 6b used in the derivation of Eq. 82 are given here. Relationships for general body shapes are given in Appendix D.

$$S = S_{EGH} + r_0(\theta_0 - \theta)\delta \qquad \text{(C-1)}$$

$$S_{EGH} = \frac{EG \cdot HG}{2} \sin\xi = \frac{EG \cdot HG \cos\epsilon}{2}$$

$$HG = r_0(\theta_0 - \theta)$$

$$\epsilon = \gamma - \theta_0$$

$$EG = \frac{r_0}{\sin\gamma} \sin\theta_0$$

$$S_{EGH} = \frac{r_0^2}{2\sin\gamma} \sin\theta_0 \cos(\gamma - \theta) \cdot (\theta - \theta_0) \qquad \text{(C-2)}$$

$$\left.\frac{\partial S_{EGH}}{\partial \theta}\right|_{\theta = \theta_0} = \lim_{\Delta\theta \to 0} \frac{S_{EGH}}{\Delta\theta} = \frac{r_0^2}{2\sin\gamma} \sin\theta_0 \cos(\gamma - \theta_0) \qquad \text{(C-3)}$$

$$\frac{\partial^2 S_{EGH}}{\partial\theta\,\partial r_0} = \frac{r_0}{\sin\gamma} \sin\theta_0 \cos(\gamma - \theta_0)\left[1 - \frac{r_0}{2}\left(\tan(\gamma - \theta_0) + \cot\gamma\right)\frac{d\gamma}{dr_0}\right] =$$

$$= \frac{r_0}{\sin\gamma} \sin\theta_0 \cos(\gamma - \theta_0)\left[1 - \frac{r_0}{2}\,\frac{\cos\theta_0}{\cos(\gamma - \theta_0)\sin\gamma}\,\frac{d\gamma}{dr_0}\right] \qquad \text{(C-4)}$$

$$\frac{1}{\cos\theta_0 - \tan\gamma \sin\theta_0} = \frac{1}{\dfrac{\sin(\eta + \alpha)}{\sin\eta \cos\alpha} - 1} = \frac{1}{\tan\alpha \cot\eta}$$

$$\cot\gamma = \cot\theta_0 - \tan\alpha \cot\eta \frac{1}{\sin\theta_0} \equiv X$$

$$\gamma = \operatorname{arc\,cot} X \qquad \text{(C-5)}$$

$$\frac{d\gamma}{dr_0} = \frac{1}{1 + X^2}\left[-\frac{1}{\sin^2\eta}\frac{d\eta}{dr_0} \tan\alpha \frac{1}{\sin\theta_0}\right] \qquad \text{(C-6)}$$

$$\tan\eta = \frac{r_0}{x} \qquad \text{(C-7)}$$

# APPENDIX D

## GEOMETRIC RELATIONSHIP FOR A GENERAL BODY SHAPE

Geometric relationships for a general body shape as shown in schematic of Figure 32.

$$S_{EGH} = \frac{EG \cdot GH}{2} \sin\xi$$

$$EG = \frac{r_0}{\sin\gamma} \sin\theta_0$$

$$HG = r_0(\theta - \theta_0) = \frac{r_0 \, \Delta\theta}{\cos(\theta - \varkappa)}$$

$$\xi = \frac{\pi}{2} - \theta_0 + \varkappa - \epsilon$$

$$\epsilon = \gamma - \theta_0$$

$$\xi = \frac{\pi}{2} + \varkappa - \gamma$$

$$S_{EGH} = \frac{r_0^2}{2\sin\gamma} \sin\theta_0 \frac{\cos(\gamma - \varkappa)}{\cos(\theta - \varkappa)} (\theta - \theta_0)$$

$$\frac{\partial S_{EGH}}{\partial \theta} = \frac{r_0^2}{2\sin\gamma} \sin\theta_0 \frac{\cos(\gamma - \varkappa)}{\cos(\theta - \varkappa)}$$

$$\frac{\partial^2 S'_{EGH}}{\partial\theta \, \partial r_0} = \frac{r_0(\theta_0)}{\sin\gamma} \sin\theta_0 \frac{\cos(\gamma - \varkappa)}{\cos(\theta - \varkappa)} \left[1 - \frac{r_0}{2}\left(\tan(\gamma - \varkappa) + \cot\gamma\right)\frac{\partial\gamma}{\partial r_0} + \right.$$

$$\left. + \frac{r_0}{2} \frac{\tan\gamma - \varkappa}{\tan\varkappa - \theta} \frac{\partial\varkappa}{\partial r_0}\right.$$

$$\tan\varkappa = -\frac{dy}{dx}, \qquad y = r\cos\theta, \; x = r\sin\theta$$

$$\frac{dy}{dx} \equiv Y = \frac{\frac{1}{r_0}\frac{\partial r_0}{\partial\theta} - \tan\theta}{\frac{1}{r_0}\frac{\partial r_0}{\partial\theta}\tan\theta - 1}$$

$$\frac{\partial\varkappa}{\partial r_0} = \frac{1}{1 + Y^2} \frac{\partial Y}{\partial z} \frac{dz}{dr_0} = \frac{\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial r}{\partial\theta}\right)}{\left(\frac{1}{r}\frac{\partial r}{\partial\theta} - \tan\theta\right)^2 \cos^2\theta} = \frac{\sec^2\theta\,(2r_{0\theta}^2 + r_0^2)}{r_{0\theta}\,(r_{0\theta}\tan\theta + r_0)^2}$$

$= 0$ for bodies having similar cross-sections

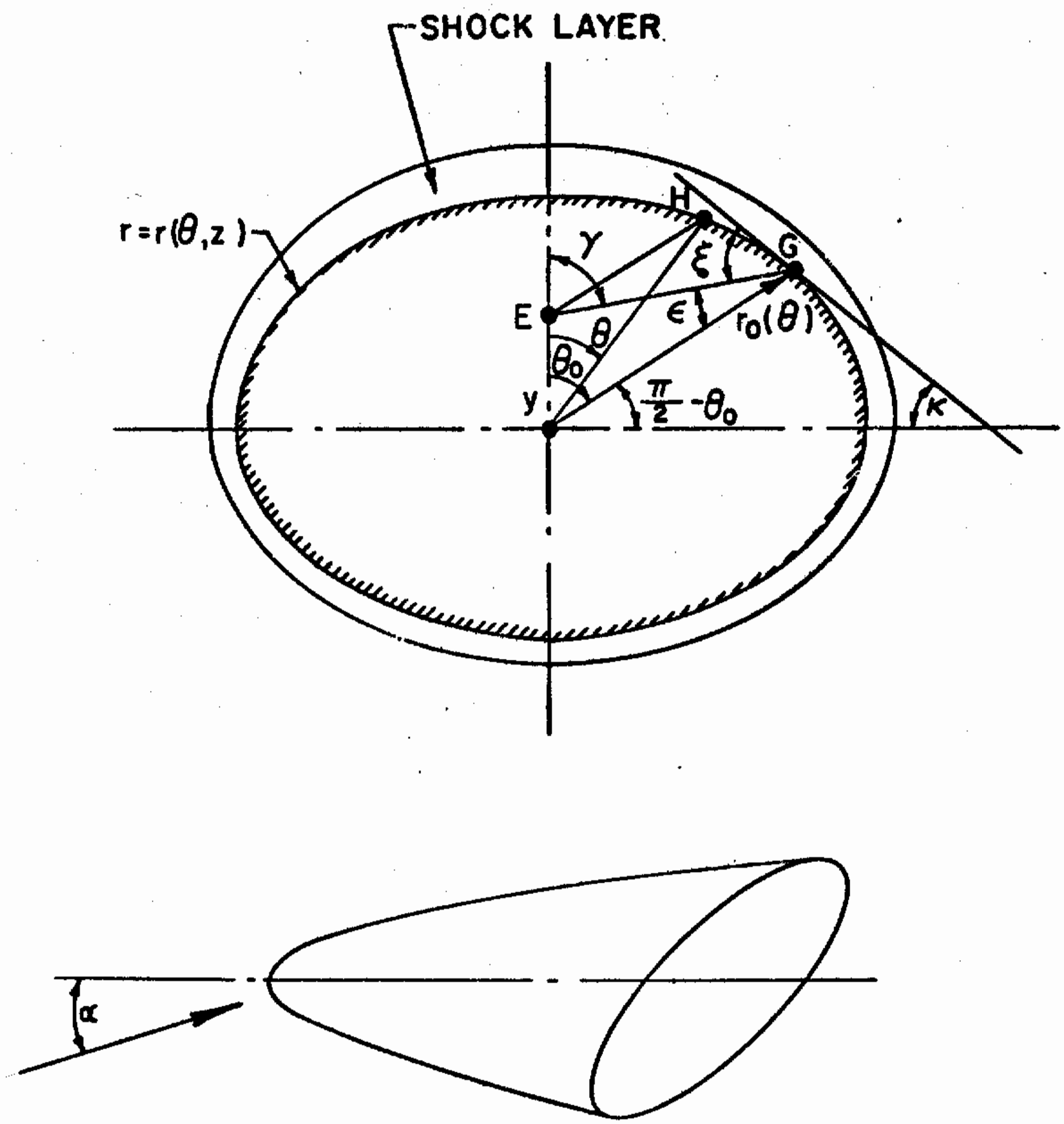
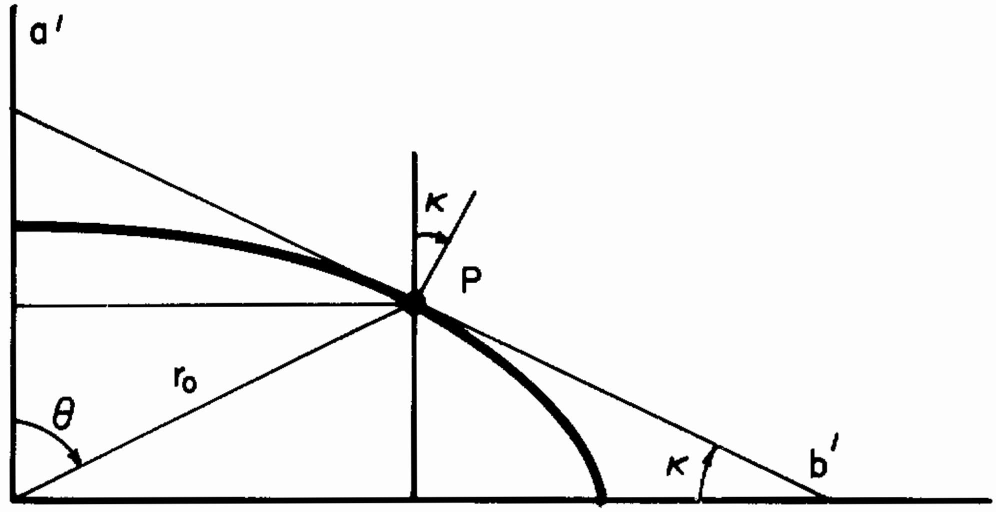



Figure 12. Geometric Relationships for a General Body Shape

# APPENDIX E

## LOCAL SURFACE INCLINATION FOR A CONE OF ARBITRARY SHAPE




A cone of arbitrary shape extends generally in the negative z-direction and has its apex at the origin of coordinates.

We may write several geometrical relationships for the purpose of expressing the inclination of the local surface normal to a plane set normal to the free stream direction.

In the plane z = const

$$\frac{1}{r_o}\frac{dr_o}{d\theta} = \tan(\theta - \kappa) \tag{E-1}$$

Drawing vectors from the apex of the cone to the point P and from b' to a' we have

$$\bar{r} = \bar{i}\, r_0 \sin\theta + \bar{j}\, r_0 \cos\theta - \bar{k}\, z \tag{E-2}$$

$$\bar{t} = \bar{i}\left(-r_0 \cos(\theta - \varkappa)/\sin\varkappa\right) - \bar{j}\,\frac{r_0 \cos(\theta - \varkappa)}{\cos\varkappa} \tag{E-3}$$

The local unit outward normal vector to the cone at P is

$$\bar{n} = \frac{\bar{r} \times \bar{t}}{|\bar{r} \times \bar{t}|} = \frac{\bar{i}\sin\varkappa + \bar{j}\cos\varkappa + \bar{k}\,\frac{r_0}{z}\cos(\theta - \varkappa)}{\sqrt{1 + \left(\frac{r_0}{z}\right)^2 \cos^2(\theta - \varkappa)}} \tag{E-4}$$

The unit forward normal to the plane through P and set normal to the z-axis is simply $\bar{k}$. But the unit normal to a plane through P and set normal to the remote stream direction is

$$\bar{n}_\omega = -\bar{j}\sin\alpha + \bar{k}\cos\alpha \tag{E-5}$$

The scalar product of $\bar{n}$ and $\bar{n}_w$, since they are unit vectors, gives directly the cosine of the included angle, but $\beta'$ is the complement of the included angle so we have simply

$$\sin\beta' = (\bar{n} \cdot \bar{n}_\omega) = \frac{-\sin\alpha\cos\varkappa + \frac{r_0}{z}\cos\alpha\cos(\theta - \varkappa)}{\sqrt{1 + \left(\frac{r_0}{z}\right)^2 \cos^2(\theta - \varkappa)}} \tag{E-6}$$

At zero angle of attack this reduces to

$$\sin\beta = \frac{\frac{r_o}{z}\cos(\theta - \varkappa)}{\sqrt{1+\left(\frac{r_o}{z}\right)^2\cos^2(\theta-\varkappa)}} \tag{E-7}$$

since $\varkappa = 0$. Now Equations (85), (86), (87), (88), and (89) may be used to compute the flow characteristics. Figures (10) and (11) present calculations of pressure distributions and shock layer shape for a yawed circular cone. Because the velocity is close to $V_{max}$ in Equation (11) $\bar{u}$ has been set equal to unity, and $\bar{\rho}$ to $\bar{\rho}_s = (\gamma+1)/(\gamma-1)$.

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A new self consistent method has been developed for calculation of boundary layer with mass injection. The system of partial differential equations is reduced to a system of ordinary differential equations by integration over different strips. The velocity and enthalpy profiles are assumed to be expressed by series of error functions. A method of calculating the initial velocity and enthalpy profiles, based on an analysis of singularity, is presented herein. The calculation of the initial profiles is reduced to the solution of a system of transcendential equations. The initial derivatives are calculated from linearlized equations near the singular points and provide a smooth start of integration of the downstream equations. The present method was applied to the cases of a sharp edged body and a blunt body. In addition to the velocity and enthalpy profiles, the pressure distribution and shock layer thickness can be calculated from generalized Newtonian expressions developed as a part of this effort.

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