FOREWORD

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ABSTRACT

The aim of the work reported has been to develop methods of solving random equations, that is, equations involving variates (random variables). The main difficulty of this task arises from the fact that no variate, if not degenerate, is invertible, or, algebraically expressed, even if the set V of variates is a commutative monoid under both addition and multiplication, it does not constitute a field.

For this purpose a set S of elements, called stochastic quantitites (for brevity, stochastics), of which V is a subset, has been constructed with the property that it constitutes a field. This implies that there exists for every element of it an inverse element relative to both the additive and multiplicative laws of composition, and thus it will be possible to compute with the stochastics just as easily as is done with the rational numbers with respect to the four fundamental operations +, -, •, :.

Considering a variate as a finite or infinite set of ordered pairs, denoted by $f(x)\ [x]$, where the first projection $f(x)$ is a real-valued, non-negative function, defined for a continuous set of values of x or for an at most denumerable set of points x_i and interpreted as a mass density or as discrete parts of a unit mass, respectively; and the second projection $[x]$ is anyone of the values that the variate can take, the notation of a stochastic is $f(x)\cdot j_z^n\ [x]$, where $f(x)$ is a real valued, positive or negative, function and the symbol $j_z^n\ [x]$ is defined, for n = 1, by $j_z\ [x] = (1/dz)\ [x] - (1/dz)\ [x + dz]$. Thus j_z may be interpreted as a duplex mass, composed of two infinitely large masses (1/dz) and - (1/dz) located at an infinitesimal distance dz from each other. For n = 2 we have $j_z^2\ [x] = (1/dz^2)\ [x] - (2/dz^2)\ [x + dz] + (1/dz^2)\ [x + 2\cdot dz]$ and j_z^2 may be interpreted as a triplex mass, composed of three infinitely large masses at an infinitesimal distance dz from each other, and so on for arbitrary values of n. Since, by definition, $j_z^o\ [x] = 1\ [x]$, the general expression includes the variates as a special case obtained by setting n = 0.

From the definition above it follows, if f(x) is a continuous function, that $f(x) \cdot j_z^n$ is equal to the ordinary derivative $d^n (fx))/dx^n$. Thus, j_z^n can be regarded not only as a multiplex mass but also as an operator. In the same way, j_x^{-n}, defined as the inverse of j_x^n, can be interpreted both as a mass distribution and as a repeated integration, further, the operator $d^n/dx^n = j^n$ may be defined, as is demonstrated, also for the general case that n^x is an arbitrary real number.

Since some problems leading to random equations have been presented, general properties of variates and multiplex stochastics are indicated. Based on the known laws of composition of variates, corresponding laws and some general theorems valid for multiplex stochastics have been deduced. Owing to the dual nature of the symbol j^n, simplified methods for composition and inversion of variates can be developed as is demonstrated. Finally, classification and some solutions of random equations and criteria for the existence of real roots are indicated.

This report has been reviewed and is approved.

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

SECTION I. INTRODUCTION

The ordinary algebra operates with letters, which are used as symbols for real or imaginary numbers and which are composed according to elementary laws of composition thus forming equations which involve known and unknown algebraic quantities. The fundamental theorem of algebra is the proposition that every algebraic equation has a root, and its main problem is to develop methods by which the roots can be determined. A decisive property of an algebraic quantity is that it can be specified by one single number.

There is, however, another class of quantities which can take a finite or an infinite set of values, to each of which is associated a real number. The specification of such a quantity, which will be called a stochastic quantity and which includes as a special case the random variables or the variates, requires more elaborate means than a single number. Equations involving known and unknown stochastic quantities composed by use of appropriate composition laws will be called stochastic equations, including the random equations which involve variates only. The importance of such equations and the need of methods for their solution is explained by the fact that many of the quantities involved in scientific and technical problems are random variables even if they for the sake of simplicity and due to the lack of suitable methods for their solution are - frequently quite improperly - considered to be algebraic. Some examples of problems leading to random equations will be presented.

The fundamental difference between the methods for solving these two types of equations, the algebraic and the stochastic, will be illustrated by a simple example.

Denoting algebraic quantities by small letters $a, b, c, \ldots,$ if known, and by $x, y, z, \ldots,$ if arbitrary or unknown, and stochastic quantities analogously by capital letters $A, B, C, \ldots,$ and $X, Y, Z, \ldots,$ the algebraic equation

$$a = b \cdot x + c \tag{1}$$

and the stochastic equation

$$A = B \cdot X + C \tag{2}$$

will be compared.

Since the solution consists in the separation of the unknown from the known quantities, the algebraic equation (1) is easily solved in two steps

$$a - c = b \cdot x + (c - c) = b \cdot x \qquad \text{and} \qquad (a - c)/b = b \cdot x/b = x \tag{3}$$

This procedure is based on the additive and multiplicative laws of composition of real numbers

$$x - x = 0 \qquad \text{and} \qquad x/x = 1 \tag{4}$$

These two simple laws of composition do not apply to variates or other stochastic quantities, since for any such quantity

$$X - X \neq 0 \qquad \text{and} \qquad X/X \neq 1 \tag{5}$$

The scope of the present investigation has been to develop methods for the solution of random equations. The leading idea was that it may be possible to find, for any given stochastic quantity X, four quantities denoted by X_+, X_-, X_x, $X_:$ which follow the laws of composition of variates and which satisfy the conditions

$$X + X_+ = 0 \qquad X - X_- = 0 \qquad X \cdot X_x = 1 \qquad X/X_: = 1 \tag{6}$$

Clearly, if such quantities, which will be called inverse components, have been found, the solution of equ.(2) is just as simple as that of equ.(1) as indicated by the formal solution of equ.(2)

$$X = (A + C_+) \cdot B_x \tag{7}$$

It is easy to prove that the inverse components can never be variates and that the introduction of a new class of stochastic quantities, which will be called multiplex stochastics, is required.

For the purpose of illustration, some examples of problems leading to random equations will be presented, after which the stochastic quantities will be defined and classified. Then their general properties and the laws of their composition will be demonstrated and finally methods of solving stochastic equations will be developed.

SECTION II. SOME PROBLEMS LEADING TO STOCHASTIC EQUATIONS

The most important application of the variates is related to the properties of elements of sets, viz., of individuals belonging to a population, say, their weights, lengths, colours, etc., taken as entities.

While some properties of a single individual, say its weight, can be specified by a single number, the same property of the individuals belonging to a population of, say, one thousand items will in the most complete form require a list giving all the individual weights. In most cases, it is, however, quite sufficient to know, for instance, that 175 items have a weight of 3 kg, 325 a weight of 2 kg, and 500 a weight of 1 kg. These figures are called the frequencies of the respective weights.

Dividing these figures by the total number 1,000 gives the relative frequencies 0.175, 0.325 and 0.500, which are also called the probabilities of the respective weights, for the reason that, if one single element is taken at random from the set, then there will be a probability of 17.5% of picking out an element having a weight of 3 kg. In cases where the set is infinitely large and the property in question may take any value within a finite or infinite interval, the probability has to be specified by a function $f(x)$, which defines the condition that the infinitesimal probability dP of drawing at random an element with a value x, belonging to the infinitesimal interval dx, is $dP = f(x)dx$. The function $f(x)$ is known as the density function (also frequency or probability function) of the variate X.

The density function provides all pertinent information on the variate. Consequently, the solution of a random equation consists in the determination of the unknown density functions from the known ones. Some examples will now be presented:

Example 1. Suppose that we have two sets, one consisting of a large number of items and the other of a large number of boxes. The weights of the boxes are represented by the variate B and those of the items by another variate C. If now the items are taken at random, one at the time, and enclosed one in each box, then the weights of the boxes with their contents is a new variate, denoted by A. We then have the random equation

$$A = B + C \tag{8}$$

This symbolic equation implies that, if we take at random one box from the set B, weigh it and find its weight to be b_i and one item from the set C, weigh it and find its weight to be c_i, then we can postulate that

$$a_i = b_i + c_i \tag{9}$$

is a random value from the set A, that is, the computed value a_i is a perfect substitute for the weight of a box with content actually taken at random from the set A.

This way of defining a random equation (which is not applicable to a stochastic equation in general) will be called the Monte-Carlo definition of the equation. Based on this definition, which provides also an experimental method of solving arbitrary random equations, mathematical laws of composition of variates will be deduced and extended to stochastic quantities in general.

The preceding procedure presumes that B and C are known and consequently that our equation may be written

$$X = B + C \tag{10}$$

This equation is easily solved by known methods, but let us examine another alternative

$$A = B + X \tag{11}$$

In this case, the variates A and B have been determined by weighing a sufficiently large number of boxes with and without contents and it is required to derive from this information the density function of the variate C.

If we now apply the Monte-Carlo method to this problem and take an element from the set A and find its weight to be a_i and a box from the set B and find its weight to be b_i, then it is evident that

$$x_i = a_i - b_i \tag{12}$$

does not provide a random value from the set C because, suppose that the least weight of the boxes is 2 kg and that of the items is 3 kg and that some of the boxes have a weight of 7 kg, then it may happen that we have an $a_i = 5$ kg and a $b_i = 7$ kg and thus $x_i = 5 - 7 = -2$ kg, which evidently is absurd.

The reason why the method fails in this particular case is that A and B are dependent variates, which will be denoted by

$$X = A(-)B \tag{13}$$

In an analogous way addition, multiplication, and division of dependent variates will be denoted by (+), (.), and (:), respectively.

Since equ.(12) cannot be applied without knowing the dependency between a_i and b_i, the solution of an equation even as simple as equ.(11) may be rather complicated, and still more if we take two items at random and put them into one and the same box, taken at random. The equation then takes the form

$$A = B + X + X \tag{14}$$

where the two letters X stand for two independent variates with identical density functions.

It should be noted that equ.(10), but not equ.(11), has always a real root; that is, a variate satisfying the equation. This difference between the two equations is an essential fact.

Example 2. This example is chosen to illustrate multiplication of two variates. Suppose that we have a large set of specimens. An individual specimen may have an ultimate strength s_i, cross-sectional area a_i, and specific strength b_i. Since all these values differ from item to item, we have three variates, related by

$$S = A \cdot B \tag{15}$$

Here A and B are, at least in most cases, independent variates. The variate S is easily determined from the known density functions of A and B, whereas it is much more difficult to determine B from S and A, since we have

$$X = B = S(:)A \tag{16}$$

and we do not know the dependence of S and A.

Example 3. This example is related to the important and frequently occurring problem of eliminating the influence of imperfect measuring devices used for experimental determination of variates. Let X denote the weight of a set of items which has to be determined by means of a spring balance. If the spring constant varies during the weighing procedure, it can be considered a variate denoted by B, and if the balance has a varying zero-error, this is another variate denoted by C. Then the actually observed weight A differs from the true weight X which has to be determined. Since B, C, and X are independent of each other, the correct random equation will be

$$A = B \cdot X + C \tag{17}$$

The variates B and C can be determined by a proper calibration. This equation is typical of a large number of measuring procedures.

Example 4. A random equation relating the fatigue life N of a specimen to the imposed pulsating load S may be put in the form

$$S = (S_u - S_e)(N/b + 1)^{-a} + S_e \tag{18}$$

where S_u = the ultimate tensile strength of the specimen, S_e = its fatigue limit, a and b constants (or more correctly expressed: degenerate variates). Since S, S_u, N, a, and b can be determined by independent experiments, the variate S_e is the unknown quantity.

The solution of this equation is a rather intricate problem, complicated by the circumstance that probably S_e is dependent of S_u and possibly a and b are non-degenerate variates. This problem gave, in fact, the initial incitement to the present investigations.

SECTION III. DEFINITIONS AND CLASSIFICATION OF STOCHASTIC QUANTITIES

Let X be a set, finite or infinite, of ordered pairs $f(x)[x]$ where x is a real number and $f(x)$ a real-valued function of x. It is convenient to visualize $f(x)$ as a mass density associated to the point x on the x-axis in an n-dimensional space R_n.

Let $P(x)$ be a function which is almost everywhere equal to zero, that is, it is equal to zero except in an at most denumerable set of points x_i where it takes finite values $P_i = P(x_i)$,

Let $p(x)$ be a function almost everywhere continuous, that is, except at the discontinuity points a_i where it has a finite jump (saltus) equal to $P_i' = p(a_i)$,

Let $p'(x) = d(p(x))/dx$ be a function continuous except in the discontinuity points b_i where it has finite jumps equal to $P_i'' = p'(b_i)$, etc.

Putting $f(x) = P(x_i)/dx + p(x)$, we specify the set X by

$$X = f(x)[x] = (P(x_i)/dx + p(x))[x] \qquad (19)$$

or, since $P(x)$ does not exist but for $x = x_i$,

$$X = (P_i/dx)[x_i] + p(x)[x] \qquad (20)$$

The function $f(x)$ will be called the density function of X and will be denoted

$$Df(X) = f(x) \qquad (21)$$

The notation P_i/dx implies that the density is infinitely large at points x_i but in such a way that there is a finite mass P_i within the infinitesimal interval dx. The notation $p(x)[x]$ implies that to each value x is associated a real number $p(x)$ (interpreted as a mass density). The function $f(x)$ will be regarded as a single object which can be moved in the space R_n. The notation $f(x)[x]$ denotes its initial location, while the notation $f(x)[x+a]$ implies that each value (real number) $f(x)$, initially associated to x, now is associated to the point $(x+a)$ on the x-axis, that is, the object $f(x)$ has been moved a distance a on the x-axis. A move in the arbitrary z-direction an infinitesimal distance dz will be denoted by $f(x)[x+dz]$.

On the particular conditions that

$$P_i \geq 0; \; p(x) \geq 0 \quad \text{and} \quad \Sigma P_i + \int_{-\infty}^{\infty} p(x)dx = 1 \qquad (22)$$

the set of ordered pairs of real numbers X will be called a variate (also random variable), because then the mass may be interpreted as a probability.

When there are no such restrictions imposed on the density function, the set X will be called a real stochastic quantity or, for brevity, a real stochastic.

We will now introduce a new concept called the derivative Y of a variate or of a stochastic, which will be denoted by $D(X)/dz$ and defined by

$$Y = D(X)/dz = \lim_{\Delta z \to 0}((f(x)[x] - f(x)[x+\Delta z])/\Delta z) \qquad (23)$$

If $dz = \Delta z \to 0$ is an interval which tends to zero (but never reaches it), equ.(23) may be written, for short,

$$Y = (f(x)[x] - f(x)[x+dz])/dz$$

Taking the two terms of $f(x)$ separately, we have, since P_i is a real number and $P_i[x_i]$ may be written $P_i \cdot 1[x_i]$,

$$D(P_i[x_i]dz = P_i(1[x_i] - 1[x_i + dz])/dz$$

With the notation

$$j_z[x_i] = (1[x_i] - 1[x_i + dz])/dz$$

we have

$$D(P_i[x_i])/dz = (P_i \cdot j_z)[x_i] \tag{24}$$

This result will, for brevity, be expressed by the statement that the derivative of a real number P_i is

$$D(P_i)/dz = P_i \cdot j_z \tag{25}$$

keeping always in mind that the real number has to be located anywhere in the space R_n.

Applying the limiting process to $P_i \cdot j_z$ results in a second derivative

$$D(P_i \cdot j_z)/dz = D^2(P_i)/dz^2 = P_i \cdot j_z^2 \tag{26}$$

and by further repetitions in general

$$D^n(P_i)/dz^n = P_i \cdot j_z^n \tag{27}$$

In the particular case $n = 0$ we have

$$j_z^o[x_i] = 1[x_i] \tag{28}$$

or, for short,

$$j_z^o = 1$$

The symbol j_z^n will be defined for n equal to any real number, positive of negative, as will be demonstrated in Section 5.

The second term $p(x)[x]$ derivated becomes

$$D(p(x)[x])/dz = (p(x)[x] - p(x)[x + dz])dz$$

and, since $p(x)$ is a real number,

$$D(p(x)[x])/dz = p(x) \cdot j_z[x] \qquad (29)$$

or, for brevity,

$$D(p(x))/dz = p(x) \cdot j_z \qquad (30)$$

keeping in mind that the object $p(x)$ has to be located anywhere in the space R_n and has, in the limiting process, been moved the infinitesimal distance dz in the z-direction.

The important, particular case that $z = x$ in equ.(30) and the effect of discontinuities in $p(x)$ will be discussed in Section 5.

Applying the derivation procedure to $p(x) \cdot j_z$ results in a second derivative

$$D^2(p(x))/dz^2 = p(x) \cdot j_z^2 \qquad (31)$$

and by further repetitions in general

$$D^n(p(x))/dz^n = p(x) \cdot j_z^n \qquad (32)$$

Combining equs.(27) and (32) we have

$$D^n(f(x))/dz^n = (P_i/dx + p(x)) \cdot j_z^n \qquad (33)$$

where $f(x)$ stands for $f(x)[x]$ and j_z^n for $j_z^n[x_i]$ and $j_z^n[x]$, respectively.

Just as P_i is a real number which may be interpreted as a concentrated mass and $p(x)$ is another real number which may be interpreted as a mass density, $P_i \cdot j_z^n$ and $p(x) \cdot j_z^n$ may be called multiplex numbers and interpreted as a <u>multiplex mass</u> and a <u>multiplex mass density</u> associated to the points x on the straight x-line.

The set of ordered pairs $X \cdot j_z^n = f(x) \cdot j_z^n[x]$, where $f(x)$ is a real-valued function will be called a <u>multiplex stochastic</u>.

The stochastic quantities may thus be classified into: Real stochastics, including as a particular case the variates, and the multiplex stochastics. Each of these classes may be subdivided into discrete, continuous, and mixed stochastics.

In the particular case that the stochastic quantity consists of one single ordered pair, it will be called degenerate (degenerate variate, degenerate real stochastic, degenerate multiplex stochastic).

According to the conditions (22), a degenerate variate must be denoted by $1[x_i]$, and a degenerate stochastic by $k \cdot j_z^n[x_i]$ where k is a real number, positive or negative. The case $n = 0$ corresponds to the degenerate real stochastic $k[x_i]$, where $k \neq 1$.

It is evident that the derivative of a variate can never be a variate but sometimes a real stochastic.

SECTION IV. GENERAL PROPERTIES OF VARIATES

4.1 Degenerate and Discrete Variates

A degenerate variate X is defined by the condition that it takes only one single value x_i, which implies that there is 100% probability that it takes this value, or that the mass distribution consists of a unit mass located at the point x_i. It will be denoted

$$X = (1[x_i]) \tag{34}$$

The particular case that $x_i = 0$ will sometimes be denoted by $X = 0$. This notation does not imply that X disappears, and should correctly be denoted by $X = (1[0])$.

Comparing the two degenerate stochastics $(1[0])$ and $(0[1])$, the first notation implies that the probability of the value 0 is 100%, while the second one implies that there is no probability at all, that the real stochastic takes the value 1 and sofar no other value, so this notation actually implies that the quantity disappears.

In the same way, $X = 1$ should correctly be written $X = 1[1]$, but the notations $X = 0$ and $X = 1$ will be accepted, when no confusion can arise.

The most simple non-degenerate variate is that one which takes two values, x_1 and x_2 with the probabilities P_1 and $P_2 = 1 - P_1$ respectively. It will be denoted by

$$X = (P_1[x_1] + P_2[x_2]) \tag{35}$$

Care must be taken to distinguish this notation from

$$X = (P_1[x_1]) + (P_2[x_2]) \tag{36}$$

which implies the sum of two degenerate stochastics.

If $x_2 \rightarrow x_1$, then $X \rightarrow (1[x_1])$, that is, to a degenerate variate, which may, for short, be written $X = x_1$.

4.2 Continuous Variates

If the density function $f(x)$ does not include any discrete, infinitely large values, it will be denoted by $p(x)$. The mass within an infinitesimal interval dx then represents the probability dP of obtaining, at random, a value belonging to this interval.

Thus

$$dP = p(x)\,dx \tag{37}$$

In the same way, the mass content of the interval (a,b), corresponding to the probability P_{ab} of finding, at random, a value belonging to this interval, is

$$P_{ab} = \int_a^b p(x)\,dx \tag{38}$$

The function $F(x)$ defined by

$$F(x) = \int_{-\infty}^{x} p(x)\,dx \tag{39}$$

is equal to the probability that X takes a value equal to or less than x, denoted $\mathrm{Prob}(X \leq x)$. This function is known as the cumulative distribution function and will be denoted by

$$\mathrm{Cdf}(X) = F(x) \tag{40}$$

4.3 Bounded Variates

The variates may, from another aspect, be classified into bounded and unbounded variates. The importance of the bounded variates is due to the fact that all properties of concrete objects, represented by a variate, for instance, length, weight, material strength, fatigue life, times, etc., are bounded (as never taking negative values), even if they are frequently assumed to be normal variates, which have no finite bounds.

In analogy with the notation x^+, known as the positive part of x and defined by

$$x^+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{41}$$

we will use the notation

$$(x - b)^{+a} = \begin{cases} (x - b)^a & \text{if } x - b \geq 0 \\ 0 & \text{if } x - b \leq 0 \end{cases} \tag{42}$$

where $a \geq 0$.

A variate bounded from below has a density function which is equal to zero for $x - b \leq 0$. The value b is called the lower bound. A variate bounded from above has a density function which is zero for $b - x \leq 0$. The value b is called the upper bound of the variate.

The essential difference between bounded and unbounded continuous variates consists in the property that the density function of a bounded variate has at least one discontinuity and consequently higher multiplex derivatives, while that of an unbounded variate has no discontinuities at all.

Two classes of bounded variates of practical importance will now be examined: The first class includes all variates having a density function which can be developed into a series of the form

$$f(x) = \Sigma k_a (x - b)^{+a} \tag{43}$$

where the real number $a \geq 0$. It should be noted that this function and its derivatives have no discontinuities except possibly at the lower bound b.

As an example of such a density function we will take

$$f(x) = m \, . \, x^{+(m-1)} \, . \, e^{-x^{+m}} \qquad (m \geq 1) \tag{44}$$

which may be developed into the series

$$f(x) = m \, . \, \Sigma(-1)^n \, . \, x^{+(n \, . \, m + m - 1)} / n! \tag{45}$$

where the integer $n \geq 0$.

In particular we have for

$$\underline{m=1} \quad f(x) = x^{+0}/0! - x^{+1}/1! + x^{+2}/2! - \quad = \Sigma(-1)^n . x^{+n}/n! = e^{-x^+} \qquad (46)$$

$$\underline{m=2} \quad f(x) = 2(x^{+1}/0! - x^{+3}/1! + x^{+5}/2! - \quad = 2\Sigma(-1)^n . x^{+(2n+1)}/n! \qquad (47)$$

and, for illustration purpose,

$$\underline{m=1.5} \quad f(x) = 1.5(x^{+0.5}/0! - x^{+2.0}/1! + x^{+3.5}/2! - \quad =$$
$$= 1.5\Sigma(-1)^n..x^{+(1.5n+0.5)}/n! \qquad (48)$$

The <u>second class</u> includes variates which have a density function of the form

$$f(x) = \Sigma k_i (x - b_i)^{+n} \quad (n \geq 0) \qquad (49)$$

This function has the property, useful for composition and solution of equations, that itself and all its derivatives up to the $(n-1)$th are continuous. The discontinuities are located at the points b_i. The lowest value of b_i is the lower bound of the variate.

<u>The rectangular variate</u> V_o (also denoted $R(a,b)$)

$$V_o = (k_1(x-a)^{+0} + k_2(x-b)^{+0})[x], \quad \text{where} \quad k_1 = -k_2 = 1/(b-a) \qquad (50)$$

The density function has discontinuities at a and b equal to $1/(b-a)$ and $-1/(b-a)$, respectively.

<u>The triangular variate</u> V_1

$$V_1 = (k_1(x-a)^{+1} + k_2(x-b)^{+1} + k_3(x-c)^{+1})[x], \quad \text{where} \qquad (51)$$

$k_1 = 2/(b-a)(c-a)$; $k_2 = 2/(c-b)(a-b)$; $k_3 = 2/(a-c)(b-c)$

For the symmetrical case $a = -c$; $b = 0$, we have

$$V_1 = (((x+c)^{+1} - 2(x)^{+1} + (x-c)^{+1})/c^2)\ [x] \qquad (52)$$

The density function is continuous but its first derivative has discontinuities at $-c$ and c equal to $1/c^2$ and at 0 equal to $2/c^2$.

<u>The parabolic variate</u> V_2

$$V_2 = (k_1(x+a)^{+2} + k_2(x+b)^{+2} + k_3(x-c)^{+2} + k_4(x-d)^{+2})[x] \qquad (53)$$

For the symmetrical case $d = a$; $c = b$, we have

$$k_1 = -k_4 = 3/2a(a^2 - b^2) \text{ and } -k_2 = k_3 = 3/2b(a^2 - b^2) \quad (54)$$

The density function and its first derivative are continuous but the second derivative has discontinuities at $-a$, $-b$, b, a equal to $3/a(a^2 - b^2)$; $-3/b(a^2 - b^2)$; $3/b(a^2 - b^2)$, and $-3/a(a^2 - b^2)$. The density functions of V_0, V_1, and V_2 are presented in Fig.1.

4.4 Approximation of Continuous Variates

Any continuous variate can be approximated by a discrete variate composed of the masses of each of a sufficiently large set of intervals of equal length located at the mid-point of the interval.

Another method, which in some cases is preferable, consists in taking the intervals, not of equal length, but of equal mass content and to locate these concentrated masses in the center of gravity of each interval. This procedure is easy to perform, by use of the distribution function $F(x)$ defined by equ.(39).

Denoting the inverse function of F by F^{-1} we have

$$F(F^{-1}(x)) = x \quad (55)$$

If m is the total number of intervals, then the mass $1/m$ has to be located at the m points

$$x_i = F^{-1}(i/2m) \quad (i = 1, \ 3, \ \ldots, \ (2m-1)) \quad (56)$$

As an example take the distribution function

$$P = F(x) = 1 - e^{-x^{1/\alpha}} \quad (57)$$

which will be approximated by the mass $1/m$ located at the m points

$$x_i = F^{-1}(P) = (-\log(1-P))^\alpha = (-\log((2m - i)/2m))^\alpha \quad (58)$$

It is worth mentioning that every variate, experimentally determined, is a discrete variate according to the first alternative. Suppose, for instance, that X represents the weights of the elements of a large set and that these weights have been determined with an accuracy of $\pm$ 0.5 kg. The only actual information obtained are the observed numbers of elements belonging to each interval, that is, the experimentally determined density function is a discrete one. Now it may be assumed that the density be constant within each interval, or linear, or parabolic. These assumptions lead to the density function (49), taking n equal to 0, 1, and 2, respectively. In any case, every function which is continuous can be represented to any desired degree of accuracy by means of a sufficient number of intervals but this number decreases considerably with n for a preassigned degree of accuracy.

The second alternative of approximating the density function is easily obtained from the experimental data. Taking, for instance, the number of intervals equal to 10, then, after a proper smoothing procedure, the relative number of observations equal to and less than 5%, 15%, 25%, etc., provides the corresponding values x_i of equ.(56).

Another way of approximating continuous variates consists in the use of a power series defined in X^+ as indicated by equ. (43). This method is of particular interest when the variate represents some material property, for example, the strength or the fatigue life of a specimen, where only the lower part of the distribution function (39) is of practical interest, that is, for low probabilities of failure. From purely physical reasons, such variates must be bounded from below.

We then assume the distribution to be

$$F(x) = \Sigma\, k_n (x - x_u)^{+n} \tag{59}$$

where x_u is the lower bound of the variate.

As an example, let us take the Rayleigh variate with $x_u = 0$, defined by equ.(47). Thus

$$P = F(x^+) = 1 - e^{-x^{+2}} \tag{60}$$

The corresponding power series is

$$F(x) = x^{+2} - x^{+4}/2 + x^{+6}/6 - x^{+8}/24 \tag{61}$$

Since $F(x)$ is the probability that $X \leq x$, any approximation by use of the first term x^{+2} only can be used with an error of less than 0.5% up to a probability of 10%, which for a designer may be an excessively high percent of failure. By use of the two first terms the function $F(x)$ can be approximated up to 30% with an error of less than 0.04%.

Some of the pertinent functions are given below

$$F(x) = x^{+2} \qquad - x^{+4}/2$$
$$f(x) = 2x^{+} \qquad - 2x^{+3}$$
$$f'(x) = 2x^{+0} \qquad - 6x^{+2}$$
$$f''(x) = 2/dx \qquad - 12x^{+} \qquad (62)$$
$$f'''(x) = 2j/dx \qquad - 12x^{+0}$$
$$f''''(x) = 2j^{2}/dx \qquad - 12/dx \qquad (63)$$

From this table it may be found that, taking the first term only, the second derivative $f''(x)$ is represented by a mass 2 concentrated in the point $x = 0$, and taking the two first terms, the fourth derivative $f''''(x)$ is represented by two masses concentrated in $x = 0$, one multiplex $= 2j_x^2$ and the other a real negative $= -12$. These results makes a considerable simplification of compositions and determination of inverse components, that is, of the solution of equations involving the Rayleigh variate, as will be demonstrated in the following.

SECTION V. GENERAL PROPERTIES OF MULTIPLEX STOCHASTICS

The multiplex number j_z was in Section 3 defined by a limiting process resulting in the derivative of the ordered pair $k[x_i]$

$$D(k[x_i])/dz = kj_z[x_i] = (k[x_i] - k[x_i + dz])/dz \qquad (64)$$

This derivative may be geometrically interpreted as a transformation of the real number (mass) k into a duplex number (mass) kj_z, which is composed of two infinitely large numbers (masses) k/dz and $-k/dz$ located at an infinitesimal distance dz from each other.

Applying the limiting process to the ordered pair $kj_z[x_i]$, the second derivative becomes

$$D^2(k[x_i])/dz^2 = D(kj_z)/dz = kj_z(1[x_i] - 1[x_i + dz])/dz$$

or

$$D^2(k[x_i])/dz^2 = kj_z \cdot j_z[x_i] = kj_z^2[x_i] \qquad (65)$$

This derivative may be geometrically interpreted as three infinitely large masses $1/dz^2$, $-2/dz^2$, $1/dz^2$, located at the infinitesimal distance dz from each other. Strictly, we should have four masses $1/dz^2$, $-1/dz^2$, $-1/dz^2$, $1/dz^2$ at the distance dz from each other. It can, however, be proved that, from the view-point of composition, these two mass distributions are equivalent and thus that the initial distance Δz which tends to zero and reaches the infinitesimal distance dz may not only tend to but also reach the value $\Delta z = 0$ for the two mid-masses. This rule applies to all masses of equal sign.

The mass distributions for the different derivative $j_x^n[0]$ are presented in the following table

Multiplex masses	Location of mass-elements 0	dx	2dx	3dx	4dx
j^0	1	-	-	-	-
$j \cdot dx$	1	-1	-	-	-
$j^2 \cdot dx^2$	1	-2	1	-	-
$j^3 \cdot dx^3$	1	-3	3	-1	-
$j^4 \cdot dx^4$	1	-4	6	-4	1

All derivatives, except j^0, have a direction, in the preceding denoted by z, which for higher derivatives may change in the successive steps.

As an example we take the second derivative

$$D^2(k[x_i])/dx \cdot dy = kj_{xy}^2[x_i] \qquad (66)$$

which may be geometrically interpreted as a mass distribution composed of two masses $k/dx \cdot dy$ and two masses $-k/dx \cdot dy$ located at $(x_i, 0)$; $(x_i + dx, dy)$ and $(x_i + dx, 0)$; (x_i, dy), respectively.

In the preceding, we have discussed the <u>derivatives of degenerate stochastics</u> (including variates).

The extension of this concept to discrete stochastics is immediate.

Regarding a discrete stochastic as a sum of degenerate stochastics, located at a set of points x_i, that is,

$$X = \Sigma(k_i \cdot j_z^n[x_i]) \tag{67}$$

it follows that

$$D^m(X)/dz^m = \Sigma(k_i \cdot j_z^{m+n}[x_i]) \tag{68}$$

We will now examine the concept of <u>derivatives of a continuous stochastic</u>

$$X = f(x)[x] \tag{69}$$

including the variates, if $f(x)$ satisfies the conditions of equ.(22).

The symbol $f(x)[x]$ implies an infinite set of ordered pairs. To each real number x is associated another real number $f(x)$, interpreted as a mass density. The function $f(x)$ is regarded as a single object having an initial location from which it can be moved without changing its shape.

Thus $f(x)[x+b]$ implies that the real number $f(x)$, initially associated to x, now is associated to $(x+b)$, that is, the object $f(x)$ has been moved a distance b on the x-axis. A move of $f(x)$ a distance dz in the arbitrary direction z will thus be denoted by $f(x)[x+dz]$.

Considering that the symbol $f(x)$ denotes a real number, obtained by applying a set of composition laws to x, it follows that

$$f(x)[x+b] = f(x-b)[x] \tag{70}$$

and also that

$$f(x)[x/b] = f(bx)[x] \tag{71}$$

In general, for any transformation

$$z = g(x)\ ;\ \ x = g^{-1}(z) \tag{72}$$

where $g(x)$ is a real-valued function, which is finite and uniquely defined for all real x, we have

$$f(x)[g(x)] = f(g^{-1}(z)[z] \tag{73}$$

Mathematically, $f(x)[z]$ and $f(z)[z]$ are identical objects, since both x and z denotes any arbitrarily chosen real numbers.

By specification of different x-axis and z-axis in the space R_n, a distinction may be imposed on them.

If we now denote by Y the derivative of the continuous stochastic X of equ.(69)

$$Y = D(X)/dz \tag{74}$$

then it follows from the preceding that

$$Y = D(f(x)[x])/dz = f(x)\ .\ j_z[x] \tag{75}$$

and, in general

$$Y = D^n(X)/dx^n = f(x)\ .\ j_z^n[x] \tag{76}$$

In the important, particular case that $z = x$, it follows, by definition, that

$$D(f(x)[x])/dx = f(x)\ .\ j_x = (f(x)[x] - f(x)[x + dx])/dx$$

and by equ.(70) that

$$f(x)\ .\ j_x[x] = (f(x)[x] - f(x - dx)[x])/dx = f'(x)[x]$$

Hence, for any continuous function $f(x)$ we have

$$f(x)\ .\ j_x = df(x)/dx = f'(x) \tag{77}$$

and in general

$$f(x)\ .\ j_x^n = d^n f(x)/dx^n \tag{78}$$

where the integer $n \geq 0$, that is, equal to the ordinary derivatives.

In particular

$$f(x) \cdot j_x^o = d^o f(x)/dx^o = f(x) \tag{79}$$

Equ.(78) which is valid for positive integers only, will now be generalized by putting

$$f(x) \cdot j_x^a = d^a(f(x))/dx^a \tag{80}$$

where a is an arbitrary real number.

For the particular case $f(x) = k \cdot x^n$ we will derive a formula for $d^a(x^n)/dx^a$ which satisfies the following conditions:

(1) $$d^a(x^n)/dx^a = k(n,a) \cdot x^{m(n,a)}$$

that is, k and m are uniquely determined by n and a.

(2) $$d^b(d^a(x^n)/dx^a)/dx^b = k(n,\ a+b) \cdot x^{m(n,\ a+b)}$$

that is, k and m are uniquely determined by n and $(a+b)$.

(3) If n, a, and b are integers, the formulas must be identical with the ordinary formulas of derivatives.

The first condition is satisfied by $m = n - a$, and so is the second condition, since $(n-a) - b = n - (a-b)$, and also the third condition. Thus, we may put

$$d^a(x^n)/dx^a = k \cdot x^{n-a}$$

The expression $k = f(n)/f(n-a)$ obviously satisfies the first condition, and also the second one, since

$$k = (f(n)/f(n-a)) \cdot (f(n-a)/f(n-a+b) = f(n)/f(n-a+b)$$

The third condition is satisfied by $f(x) = x!$, since

$$\begin{aligned} k = n!/(n-a)! &= 1 \quad \text{if } a = 0 \\ &= n \quad \text{if } a = 1 \\ &= n(n-1) \text{ if } a = 2 \text{ etc.} \end{aligned}$$

Accordingly, the derivative will be defined by

$$d^a(x^n)/dx^a = x^n \cdot j_x^a = (n!/(n-a)!) \cdot x^{n-a} \qquad (n-a \geq 0) \qquad (81)$$

For any continuous function $f(x)$ which can be expanded into a series

$$f(x) = \Sigma k_n \cdot x^n \qquad (82)$$

we have

$$d^a f(x)/dx^a = f(x) \cdot j_x^a = \Sigma k_n (n!/(n-a)!) x^{n-a} \qquad (83)$$

where n and a are arbitrary real numbers and $(n-a) \geq 0$.

Equ.(83) will now be applied to density functions which are bounded from below.

As the first example we take

$$f(x) = k(x-b)^{+(n+a)} \qquad (84)$$

where n is a positive integer. We then have

$$\begin{aligned} f(x) \cdot j^a &= k((n+a)!/n!)(x-b)^{+n} \\ f(x) \cdot j^{n+a} &= k((n+a)!/0!)(x-b)^{+0} \\ f(x) \cdot j^{n+1+a} &= k((n+a)!/dx)[b] \\ f(x) \cdot j^{n+2+a} &= k((n+a)!j/dx)[b] \end{aligned} \qquad (85)$$

The two last expressions interpreted as density functions take infinite values at $x = b$, but in such a way that there is a finite mass $k(n+a)!$ and $k(n+a)!\,j$, respectively, located in this point. When no confusion can arise, the term dx will be omitted.

Applying equs.(85) to the density function (46)

$$f(x) = e^{-x^+} = x^{+0}/0! - x^{+1}/1! + x^{+2}/2! - .. \qquad (86)$$

we have

$$\begin{aligned} e^{-x^+} \cdot j &= 1[0] - x^{+0}/0! + x^{+1}/1! - \\ e^{-x^+} \cdot j^n &= (j^{n-1} - j^{n-2} + j^{n-3} - .. \quad)[0] \end{aligned} \qquad (87)$$

and multiplying both members by j^{-n}

$$e^{-x^+} = (j^{-1} - j^{-2} + j^{-3} - .. \quad)[0] \qquad (88)$$

Applying equs.(85) to the density function (48)

$$f(x) = 1.5(x^{+0.5}/0! - x^{+2.0}/1! + x^{+3.5}/2! - .. \tag{89}$$

we have in the same way

$$f(x) = 1.5(j^{-1.5} . 0.5!/0! - j^{-3.0} . 2.0!/1! + j^{-4.5} . 3.5!/2! - ..) \tag{90}$$

We will now examine the concept of an integral of a stochastic X, which will be denoted by

$$I^n(X) . dx^n$$

and defined, as being the inverse of $D^n(X)/dx^n$, by

$$I^n(D^n(X)/dx^n)dx^n = X \tag{91}$$

Since $D^n(X)/dx^n = X . j_x^n$ we have

$$I^n(X . j_x^n) . dx^n = X$$

which motivates the notation

$$I^n(X)dx^n = X . j_x^{-n}$$

By use of the preceding formulas, some useful relations will be derived. From equ.(81) it follows that

$$(x-b)^{+a} . j_x^a = a!(x-b)^{+0}$$

and

$$(x-b)^{+a} . j^{a+1} = a![b]$$

(omitting the term dx)

Multiplying both members by $j^{-(a+1)}$ we have

$$(x-b)^{+a} = a!j^{-(a+1)}[b] \quad \text{and} \quad j^{-a}[b] = (x-b)^{+(a-1)}/(a-1)! \tag{92}$$

and in particular for $b = 0$

$$x^{+a} = a!j^{-(a+1)}[0] \qquad \text{and} \quad j^{-a}[0] = x^{+(a-1)}/(a-1)! \tag{93}$$

By use of these formulas the transformation of the series (86) into (88) and (89) into (90) is easily performed.

The integration can be extended to the n-dimensional space. Taking, for instance,

$$k \cdot j_x^{-1}[a] \cdot j_y^{-1}[b] = k(x-a)^{+0} \cdot (y-b)^{+0} \tag{94}$$

which represents a mass distributed over the surface $(x > a,\ \ y > b)$ with the constant density k.

SECTION VI. COMPOSITION OF VARIATES

6.1 General Laws of Composition

Let the symbol $\ast$ in the composite

$$X = A \ast B \tag{95}$$

denote an arbitrary law of composition.

Then,

$$x_i = a_i \ast b_i \tag{96}$$

is a random value from the set X.

By repetition of this procedure, a random sample of any size can be produced and an approximate value of the integrated density function of X, denoted by $C\,df(X) = F(x)$ is obtained by counting the number (n) of values, which are equal to or less than x and divide this number by the total number (N) of computed values.

Thus

$$P = F(x) = n/N \tag{97}$$

The deviation of the observed value n/N from the true value $F(x)$ can be as small as desired by a sufficient increase of N.

The practical application of this method, which is called the Monte-Carlo method, can be carried out in the following way: Let F_a and F_b denote the distribution functions of A and B and their inverse functions be denoted by F_a^{-1} and F_b^{-1}. Since $P_1 = F_a(a)$ and $P_2 = F_b(b)$ we have

$$a = F_a^{-1}(P_1) \quad \text{and} \quad b = F_b^{-1}(P_2) \tag{98}$$

The probabilities P_1 and P_2 are uniformly distributed over the interval $(0,1)$, that is, they are rectangular variates defined by equ.(50). Thus, by taking independently of each other two random numbers r_1 and r_2 from $R(0,1)$ two independent values

$$a_i = F_a^{-1}(r_1) \quad \text{and} \quad b_i = F_b^{-1}(r_2) \tag{99}$$

are obtained. The values r_1, r_2 can be taken from a Random Number Table or they can be produced by a digital computer.

This method is applicable also to composition of any number of independent variates and to any type of composites, for example,

$$X = A^B \tag{100}$$

which defines a random value

$$x_i = F_a^{-1}(r_1)^{F_b^{-1}(r_2)} \tag{101}$$

The disadvantage of this method depends on the large number of values x_i required even for a moderate accuracy of the estimated function $F(x)$. Furthermore, this method cannot be applied to stochastics in general, because the values of a stochastic other than a variate are not associated to a probability. Another method will therefore be demonstrated in the sequel, but before then the concept functions of variates will be examined.

6.2 Functions of Variates

The variate X is said to be a function of another variate A

$$X = f(A) \tag{102}$$

if it is a composite involving one single variate and constants composed according to some preassigned law. It can be defined by the Monte-Carlo method.

However, care must be taken to distinguish the constants from real numbers as will be demonstrated on the linear function

$$X = b \cdot A + c \tag{103}$$

Postulating that the sum or the product of a real number and a variate can never be a variate, except in the particular case that the real number is a positive integer n and in the two expressions, where n signifies the number of elements of the sum and the product,

$$n \cdot A = A + A + \quad \text{and} \quad A^n = A \cdot A \ldots \tag{104}$$

the constants in equ.(103) must be degenerate variates and the correct notation of equ.(103) is

$$X = (1[b]) \cdot A + (1[c]) \tag{105}$$

In most cases no confusion can arise through the notation used in equ.(103), except when b is a positive integer. The difference between $X_1 = 2A$ and $X_2 = (1/2/) \cdot A$ will be demonstrated by the Monte-Carlo method. The first symbol implies

$$x_i = F_a^{-1}(r_1) + F_a^{-1}(r_2) \tag{106}$$

where r_1 and r_2 must be independent random numbers, while the second symbol implies

$$x_i = 2 \cdot F_a^{-1}(r_1) \tag{107}$$

The second interpretation is that one of equ.(103).

Considering that

$$\mathrm{Prob}(A \leq a_i) = P_i = f(a_i) \tag{108}$$

it follows that, on the condition that $b > 0$, there is, in any random sample from X of equ.(103) exactly the same number n of a-values equal to or less than a_i as there is x-values equal to or less than $x_i = b \cdot a_i + c$, or

$$\mathrm{Prob}(X \leq x_i) = \mathrm{Prob}(A \leq a_i) = P_i \tag{109}$$

Hence,

$$\begin{aligned} \mathrm{Cdf}(X) &= F((x - c)/b) \\ \mathrm{Df}(X) &= f((x - c)/b)b \end{aligned} \tag{110}$$

On the condition that $b < 0$, it follows in the same way that

$$\mathrm{Prob}(X \leq x_i) = \mathrm{Prob}(A > a_i) = 1 - P_i \tag{111}$$

and consequently

$$\begin{aligned} \mathrm{Cdf}(X) &= 1 - F((x - c)/b) \\ \mathrm{Df}(X) &= -f((x - c)/b)/b \end{aligned} \tag{112}$$

In the particular case that $b = -1;\ c = 0$, we have

$$X = -A \qquad Df(X) = f(-a) \tag{113}$$

which implies that, if the density function of A is symmetric with respect to $x = 0$ as, for instance, the normal distribution, then

$$A = -A \tag{114}$$

which does not imply that $A = 0$.

6.3 Composition of Variates

The usual method of composing variates, which in many cases results in closed expressions for the density functions and which will be adapted to stochastics in general, will now be demonstrated.

Let a unit mass, representing a probability, be distributed over the (x,y)-plane in such a way that the mass $f(x) \cdot f(y)dx \cdot dy$ is allotted to the two-dimensional interval $dx \cdot dy$. If now a curve $z_i = x * y =$ constant is drawn as demonstrated in Fig.2 and all values on one side of this curve are equal to or less than z_i, then $F(z_i)$ is equal to the total mass allotted to this side of the curve.

The mass distribution over the (x,y)-plane is uniquely determined by the two density functions $f(x)$ and $f(y)$ and is independent of the law of composition $*$, while the curves for constant values z_i are uniquely determined by the law in question but independent of the variates. Curves corresponding to the particular cases: $z = x + y = 3$, $z = x - y = 2$, $z = x \cdot y = 4$, and $z = x/y = 1/2$ are drawn in Fig.2.

This method will now be applied first to degenerate and discrete variates and subsequently to continuous variates.

6.3.1 Degenerate and Discrete Variates

Let $A = (1[a])$ and $B = (1[b])$ be two degenerate variates. As demonstrated in Fig.3 we have one unit mass located at the point $\underline{a}$ on the a-axis and one unit mass at the point $\underline{b}$ on the b-axis. The joint density field consists of one unit mass at the point (a,b) in the (a,b)-plane. The z-axis may serve as axis both for addition and multiplication but with different scales. We have the mass distributions for the sum and the product

$A + B = (1[a]) + (1[b]) = (1[a+b])$ and $A \cdot B = (1[a]) \cdot (1[b]) = (1[a.b])$

It is readily seen that the resulting ordered pairs are obtained in both cases by multiplying the first projections of 1[a] and 1[b] and by adding the second projections in the sum and by multiplying them in the product.

As the second example we take the two discrete variates

$X = ((3/4)[1] + (1/4)[2])$ and $Y = ((1/4)[1] + (2/4)[2] + (1/4)[3])$

The joint density field is composed of six concentrated masses as demonstrated in Fig.4.

By geometrical construction we have

$$X + Y = (3[2] + 7[3] + 5[4] + 1[5])/16 \qquad (115)$$

and

$$X \cdot Y = (3[1] + 7[2] + 3[3] + 2[4] + 1[6])/16$$

The sum of the first projections must for all variates be equal to unity. Without graphical constructions these result can be obtained by the rule of multiplying the first projections and to sum the second projections for addition and to multiply them for multiplication according to the following schemes:

$$\begin{aligned} (\Sigma a_i[x_i]) + (\Sigma b_j[x_j]) &= \Sigma a_i \cdot b_j[x_i + x_j] \\ (\Sigma a_i[x_i]) \cdot (\Sigma b_j[x_j]) &= \Sigma a_i \cdot b_j[x_i \cdot x_j] \end{aligned} \qquad (116)$$

6.3.2 Continuous Variates

Random Addition and Subtraction

Putting $X = A + B$, the relation between a and b for a constant value of x is $b = x - a$ and the x-curves are straight lines with a 45° slope, as indicated in Fig.5. The bisector to the a- and b-axes may serve as the $(a+b)$-axis, because just as $F(a_i)$ is equal to the mass allotted to the left-hand region of a perpendicular to the a-axis and going through the point a_i and likewise for $F(b_i)$, thus $F(x_i)$ is equal to the mass on the left-hand side of a perpendicular to the bisector and going to the point x_i. It is to be noted that the scale of the (a + b)-axis differs from those of the a- and b-axis.

In a similar way, the (a-b)-axis is indicated in Fig.5. It should be pointed out that the difference $(A - B)$ can be considered a sum of A and $-B$, the density of which is $f(-b)$, as indicated in equ.(113).

The additive law of composition will be demonstrated on a simple example: Let $A = R(4,6)$ and $B = R(1,4)$ be two rectangular variates. From equ.(50) it follows that $f(a) = 1/2$ for $4 \leq a \leq 6$ and $f(b) = 1/3$ for $1 \leq b \leq 4$. The joint density field is equal to the shaded rectangle in Fig.6 and the surface density is constant $p = 1/6$. The density functions $Df(A + B)$ and $Df(A - B)$ are obtained from the condition that $f(x) \,.\, dx$ is equal to the mass located within a small strip dx perpendicular to the axis.

Random Multiplication

Putting $X = A \,.\, B$, the relation between a and b for a constant value x is $b = x/a$, and the x-curves are hyperbolas, as indicated in Fig.7. Also in this case, the bisector can be used as the x-axis but with a non-uniform scale. All positive products $(a \,.\, b)$ are located within the quadrants I and III.

The distribution function $F(x)$ is equal to the mass located between the two hyperbolas going through the points x for positive products and outside the two hyperbolas going through the points $-x$ for negative products.

If this method is applied to the random square $X = A^2 = A \,.\, A$, where $A = R(1,2)$, then the joint density field is equal to the shaded square in Fig.7 with a density $p = 1$. By integration, it is found that

$$F(x) = x \,.\, \log x - x + 1 \;;\quad f(x) = \log x \quad \text{for} \quad 1 \leq x \leq 2$$
$$F(x) = x \,.\, \log 4/x + x - 3 \;;\quad f(x) = \log 4/x \quad \text{for} \quad 2 \leq x \leq 4 \qquad (117)$$

Random Division

Putting $X = B : A$, the relation between a and b for a constant value of x is $b = x \cdot a$ and the x-curves are straight lines going through the origin. In Fig.10 the curves for $x = 0.5$ and $x = 2$ are indicated. A vartical line through the point $a = 1$ is an appropriate x-axis with the same scale as those of the a- and b-axes. The distribution function $F(x)$ is equal to the mass within the quadrants I and IV which is located below the x-line together with the mass within the quadrants II and III which is located above this line, as indicated by two archs in the figure for $b:a = 0.5$.

Applied to the ratio $X = A : A$, where $A = R(1,2)$, the distribution and density functions are:

$$F(x) = 2x - 2 + 1/2x\,; \quad f(x) = 2 - 1/2x^2 \qquad \text{for } 0.5 \leq x \leq 1.0$$
$$F(x) = -x/2 + 3 - 2/x\,; \quad f(x) = -1/2 + 2/x^2 \qquad \text{for } 1.0 \leq x \leq 2.0 \tag{118}$$

SECTION VII COMPOSITION OF MULTIPLEX STOCHASTICS

7.1 General Laws of Composition

The composition of two variates X and Y consists, as demonstrated in the preceding paragraph, of two procedures: the formation of a two-dimensional density field, which is uniquely defined by the density functions of X and Y, and the determination of the mass distribution within this field in relation to a set of curves, which are uniquely defined by the law of composition. This mass distribution provides the density function (or the distribution function) of the composite $Z = X * Y$.

These two procedures will be maintained for the composition of multiplex stochastics.

7.1.1 Formation of the density field

In the case of two discrete variates, denoted by $X = (P_i[x_i])$ and $Y = (P_j[y_j])$, where x_i and y_j are two at most denumerable set of points, the density field is composed of infinitely large densities $P_i \cdot P_j / dx \cdot dy$ associated to the points (x_i, x_j), or, there is a finite mass $P_i \cdot P_j$ located at each of these points.

This law will be applied also to the multiplex stochastics $(P_i \cdot j_u^n)[x_i]$ and $(P_j \cdot j_v^m)[y_j]$. Thus, to each point (x_i, x_j) will be associated a finite multiplex mass $P_i \cdot P_j \cdot j_u^n \cdot j_v^m$. The subscripts u and v indicate the directions of the multiplex units and they may or may not be equal to the x- or the y-directions.

In the case of continuous variates, denoted by $X = f_1(x)[x]$ and $Y = f_2(y)[y]$, the density field is formed in such a way that to each two-dimensional interval $dx \cdot dy$ is associated a mass $f_1(x) \cdot f_2(y) \cdot dx \cdot dy$, that is, to each point (x,y) is associated a finite mass density $f_1(x) \cdot f_2(y)$.

This law will be applied also to the continuous multiplex stochastics $f_1(x) \cdot j_u^n[x]$ and $f_2(y) \cdot j_v^m[y]$. Thus, to each point (x,y) is associated a finite mass density $f_1(x) \cdot f_2(y) \cdot j_u^n \cdot j_v^m$.

7.1.2 The mass content of an infinitesimal strip

From the field, formed according to the preceding procedure, the density function $f(z)$ of $Z = X * Y$ will be determined by the condition that $f(z) \cdot dz$ is equal to the mass located between the two curves $x * y = z$ and $x * y = z + dz$, that is, the mass content of the infinitesimal strip dz.

This law, which has already been demonstrated by means of various compositions of variates will be maintained for multiplex density fields. It is of importance for the practical application that in the case of continuous multiplex fields, the masses enclosed between the two curves z and $z + dz$ may be real masses. This statement will be proved for a density field consisting of a single line over which is distributed a multiplex mass with a density $f(x) \cdot j_z$.

As indicated in Fig.9, the symbol $f(x) \cdot j_z$ signifies two real mass distributions with the densities $f(x)/\Delta z$ and $-f(x)/\Delta z$, the latter moved in the z-direction a distance Δz. The angle between the z-and the y-axes is α. An infinitesimal strip dx, intersecting the x-axis at an angle β, as indicated in the figure, contains a pair of real mass elements, the sum of which is called the tangential mass dP_β and is equal to

$$dP_\beta = ((f(x + \Delta x)/\Delta z - f(x)/\Delta z)dx$$

Since

$$\Delta x = (tg\alpha + tg\beta)\Delta y \quad \text{and} \quad \Delta y = \cos\alpha . \Delta z$$

we have

$$\Delta x = (\sin\alpha + \cos\alpha \, . \, tg\beta\,)\Delta z \tag{119}$$

and

$$P_\beta = dP_\beta/dx = ((f(x+\Delta x) - f(x))/\Delta x \, . \, (\sin\alpha + \cos\alpha . \, tg\beta)$$

or, if $\Delta x \to 0$

$$P_\beta = (\sin\beta + \cos\alpha \, . \, tg\beta) \, . \, f'(x) \tag{120}$$

The tangential mass associated to the interval (a,b) is

$$P_{ab} = \int_a^b P_\beta \, dx = (\sin\alpha + \cos\alpha \, . \, tg\beta)(f(b) - f(a)) \tag{121}$$

If the density function $f(x)$ has a discontinuity with a jump equal to P_i' at a point x_i belonging to the interval (a,b) and we let this interval tend to zero, then there will be a finite mass

$$P_\beta(x_i) = P_i'(\sin\alpha + \cos\alpha \, . \, tg\beta) \tag{122}$$

located at the point x_i.

Let us now apply this formula to the particular case that $f(x)$ is the density function of a rectangular stochastic with the mass equal to P (=1 for a variate) uniformly distributed over the interval $(a_i, a_i + \Delta x)$. Then

$$f(x) = \begin{cases} P/\Delta x & \text{in the interval} \\ 0 & \text{outside} \end{cases}$$

and the tangential mass distribution is composed of one concentrated mass $(P/\Delta x)(\sin\alpha + \cos\alpha \, . \, tg\beta)$ at $x = a_i$ and another $-(P/\Delta x)(\sin\alpha + \cos\alpha \, . \, tg\beta)$ at $(a_i + x)$.

If now $\Delta x \to 0$, keeping P constant, then, by definition, the tangential mass distribution tends to a concentrated mass $P \, . \, j_x(\sin\alpha + \cos\alpha \, . \, tg\beta)$. It can thus be concluded that the tangential mass P_β corresponding to a concentrated mass $P \, . \, j_z$ is a mass

$$P_\beta = P \cdot j_x(\sin\alpha + \cos\alpha \cdot tg\beta) \tag{123}$$

From equs.(25) and (33) it follows that the three alternatives corresponding to equs. (120), (122), and (123) are all included in the expression

$$P_\beta = (\sin\alpha + \cos\alpha \cdot tg\beta) \cdot f(x) \cdot j_x \tag{124}$$

From this expression it follows that the tangential mass is depending on the angle β, except in the case that $\cos\alpha = 0$, which corresponds to the linear distribution $f(x) \cdot j_x$. This result is obvious, since $f(x) \cdot j_x$ signifies a real mass distribution and a real mass has - contrary to a multiplex mass - no direction.

For the other extreme case, $\cos\alpha = 1$, we have for the mass distribution with the density $f(x) \cdot j_y$ a tangential mass density

$$p_\beta = f(x) \cdot j_x \cdot tg\beta \tag{125}$$

Let us now examine a density field consisting of a single line over which is distributed a mass with the density $f(x) \cdot j_z^2$

From Fig.10 it may be found that

$$p_\beta = (f(x+2\Delta x) - 2f(x+\Delta x) + f(x))/\Delta z^2$$

and by equ.(119)

$$p_\beta = (\sin\alpha + \cos\alpha \cdot tg\beta)^2 \cdot f''(x) \tag{126}$$

Repeating this procedure, we have the tangential mass density corresponding to a linear density $f(x) \cdot j_z^n$ to be

$$p_\beta = (\sin\alpha + \cos\alpha \cdot tg\beta)^n \cdot f(x) \cdot j_x^n \tag{127}$$

This formula includes also the cases that $f(x)$ has discontinuity and infinity points.

As a particular case we have for $n = 0$

$$p_\beta = f(x) \tag{128}$$

that is, p_β is independent of β, since $j_z^o = 1$ has no direction.

Equ.(127) provides the fundamental tool for composition of a degenerate or discrete multiplex stochastic and a variate or other real stochastic.

As an example, the addition of the variate $f(x)[x]$ and the stochastic $(k \cdot j^n_z[y_i])$ will be demonstrated.

Since the additive law of composition requires that $tg\beta = 1$ and $(\sin\alpha + \cos\alpha \cdot tg\beta) = 1$ both for $\alpha = 0$ and $\alpha = 90^\circ$, it follows from equ.(125) that

$$Z = (f(x)[x]) + (k . j^n_x[y_i]) = (f(x)[x]) + (k . j^n_y[y_i]) = ((k . f(x) . j^n_x)[x + y_i]) \quad (129)$$

or by equ.(70)

$$Z = (k \cdot f(x - y_i) \cdot j^n_x[x]) \quad (130)$$

7.2 General Theorems of Multiplex Composition

Simplified methods for composition of variates and other stochastics can be developed by use of some theorems which will be presented.

Let X and Y be two stochastics and their arbitrary composite $Z = X * Y$. This equation will be put in the form

$$f(z)[z] = f(x)[x] + f(y)[y] \quad (131)$$

The density functions $f(z)$, $f(x)$ and $f(y)$ which are arbitrary functions differing from each other, will now be subjected to various changes and the effect on the density function of the composite will be stated.

Theorem 1. If one of the elements of an arbitrary composite is multiplied by a real number, then the composite shall be multiplied by the same number, that is,

$$k_1 \cdot f(x)[x] * k_2 \cdot f(y)[y] = k_1 \cdot k_2 \cdot f(z)[z] \quad (132)$$

In particular, if $k_1 = k_2^{-1} = k$,

$$k \cdot f(x)[x] * k^{-1} \cdot f(y)[y] = f(z)[z] \quad (133)$$

Theorem 2. a) If one of the elements of a sum is moved a certain distance, then the sum moves the same distance, that is,

$$f(x)[x+b] + f(y)[y+c] = f(z)[z+b+c] \tag{134}$$

In particular, if $c = -b$,

$$f(x)[x+b] + f(y)[y-b] = f(z)[z] \tag{135}$$

Equ.(135) can be geometrically interpreted as a move of the density field in a direction perpendicular to the z-axis.

b) If one of the elements of a product is expanded (contracted), then the product will be correspondingly expanded (contracted).

$$(f(x)[b \cdot x] \cdot (f(y)[c \cdot y]) = f(z)[z \cdot b \cdot c] \tag{136}$$

In particular, if $c = b^{-1}$,

$$(f(x)[b \cdot x]) \cdot (f(y)[y \cdot b^{-1}]) = f(z)[z] \tag{137}$$

Equ.(137) can be geometrically interpreted as a move of the density field along the curves $z = x \cdot y =$ constant.

As a general law, it can be stated that moving of masses along the z-curves does not affect the composite.

Theorem 3. If the density function of one of the elements of a sum is composed of two functions, as indicated by

$$(f_1(x) + f_2(x))[x] \ast f(y)[y] = f(z)[z] \tag{138}$$

then

$$f(z) = Df(X_1 \ast Y) + Df(X_2 \ast Y) \tag{139}$$

where

$$X_1 = f_1(x)[x] \quad ; \quad X_2 = f_2(x)[x] \tag{140}$$

Care must be taken to distinguish $(f_1(x) + f_2(x))[x]$ from $(f_1(x)[x]) + (f_2(x)[x])$.

Theorem 4. If one of the elements of a sum is multiplied by the multiplex number $k \cdot j_x^n$ or $k \cdot j_y^n$, then the sum shall be multiplied by the multiplex number $k \cdot j_z^n$, that is,

$$k_1 \cdot f(x) \cdot j_x^m[x] + k_2 \cdot f(y)\, j_y^n[y] = k_1 \cdot k_2 \cdot f(z)\, j_z^{m+n}\, [2] \tag{141}$$

This theorem can be proved by use of the three preceding theorems in the following way:

By definition

$$f(x) \cdot j_x = (f(x)[x] - f(x)[x + dx])/dx = ((f(x) - f(x - dx))/dx)[x]$$

Hence,

$$(f(x)/dx - f(x - dx)/dx)[x] + f(y)[y] = (f(z)/dz - f(z - dz)/dz)[z] =$$

$$= f'(z)[z] = f(z) \cdot j_z[z]$$

Thus

$$f(x) \cdot j_x[x] + f(y)[y] = f(z) \cdot j_z[z] \qquad (142)$$

and for higher derivatives, positive and negative, by repetition of this procedure.

7.3 Simplified Methods for Composition of Continuous Variates

The preceding theorems can be used to transform by derivation the elements of the composite into discrete stochastics, which are much easier to compose than the originally. After this procedure has been performed, the final result is obtained by inverse transformations.

This method will be demonstrated by means of three examples.

As the first example is taken the addition of the rectangular variates $X = R(4,6)$ and $Y = R(1,4)$. This addition has already been performed geometrically, as demonstrated in Fig.6.

Since $f(x)$ has two discontinuities: $1/2$ and $-1/2$ at $x = 4$ and $x = 6$, respectively, and $f(y)$ has two discontinuities: $1/3$ and $-1/3$ at $y = 1$ and $y = 4$, respectively, and $f'(x) = f'(y) = 0$ elsewhere, it follows that

$$f(x) \cdot j_x = (1/2)[4] - (1/2)[6] = (1[4] - 1[6])/2$$

$$f(y) \cdot j_y = (1/3[1] - (1/3)[4] = (1[1] - 1[4])/3$$

Applying Theorem 4 and considering equ.(116), the following scheme is convenient to use

$2 \cdot f(x) \cdot j_x \cdot dx$	$3 \cdot f(y) \cdot j_y \cdot dy$	$6 \cdot f(z) \cdot j_z^2 \cdot dz^2$
$1[4] - 1[6]$	$1[1] - 1[4]$	$1[5] - 1[7] - 1[8] + 1[10]$

Thus,

$$6 \cdot f(z) \cdot j_z^2 \cdot dz^2 = 1[5] - 1[7] - 1[8] + 1[10]$$

and

$$f(z) \cdot dz^2 = (1[5] - 1[7] - 1[8] + 1[10]) \cdot j_z^{-2}/6$$

and by equ.(92)

$$f(z) = ((x-5)^{+1} - (x-7)^{+1} - (x-8)^{+1} + (x-10)^{+1})/6 \qquad (143)$$

This equation provides the mathematical representation of the graph $f(a+b)$, presented in Fig.7.

As the second example the multiplication of the rectangular variate $X = R(1,2)$ by itself has been chosen, that is,

$$Z = X \cdot X = X^2$$

The easiest way to proceed is to transform the multiplication into an addition by the new variables: $x' = \log x$; $y' = \log y$; $z' = \log z$. Then $f(x')$ and $f(y')$ have discontinuities 1 and -1 at $x' = y' = \log 1 = 0$ and $= \log 2$, respectively, and the scheme becomes

$f(x') \cdot j_x \cdot dx$	$f(y') \cdot j_y \cdot dy$	$f(z') \cdot j_z^2 \cdot dz^2$
$1[0] - 1[\log 2]$	$1[0] - 1[\log 2]$	$1[0] - 2[\log 2] + 1[2 \log 2]$

Hence,

$$f(z') \cdot j_z^2 \cdot dz^2 = 1[0] - 2[\log 2] + 1[2 \log 2]$$

and, as in the preceding example,

$$f(z) = (z')^{+1} - 2(z' - \log 2)^{+1} + (z' - 2\log 2)^{+1}$$
$$= (\log z)^{+1} - 2(\log z - \log 2)^{+1} + (\log z - 2\log 2)^{+1}$$

Thus,

$$\begin{aligned} f(z) &= \log z && \text{if} \quad 1 \leq z \leq 2 \\ &= \log z \; - 2\log z + 2\log 2 = \log 4/z && \text{if} \quad 2 \leq z \leq 4 \qquad (144) \\ &= \log 4/z + \log z/4 = 0 && \text{if} \quad 4 \leq z \end{aligned}$$

This result is identical with that given in equ.(117), which was obtained by geometrical construction.

As the third example the addition of two Rayleigh variates, approximated for low probabilities, will be presented.

Using the first term of equ.(61) only, and assuming the lower limits to be b and c, we have

$$f(x) = (x - b)^{+2} \quad \text{and} \quad f(y) = (x - c)^{+2}$$

The scheme then becomes

$f(x) \cdot j_x^3 \cdot dx$	$f(y) \cdot j_y^3 \cdot dy$	$f(z) \cdot j_z^6 \cdot dz^2$
2[b]	2[c]	4[b + c]

Thus

$$f(z) \cdot j_z^6 \cdot dz^2 = 4[b + c] \quad ; \quad f(z) = 4[b + c] \cdot j_z^{-6}/dz^2$$

and by equ.(92)

$$f(z) = (z - b - c)^{+5} \qquad (145)$$

7.4 Inverse Components

The usefulness of inverse components has been indicated in the Introduction and their definitions are given in equ.(6). Some examples of such stochastics, corresponding to a few simple variates and relating to addition, multiplication, and division, will be presented.

It will be found that the determination of inverse components can be performed either geometrically or analytically. Both methods will be demonstrated for comparison.

7.4.1 Addition

The additive inverse component, called the inverse addendum and denoted by X_+ is defined by

$$X + X_+ = 0 \tag{146}$$

where the right-hand member stands for 1[0].

(a) Let X be the discrete variate

$$X = ((3/4)[1] + (1/4)[2]) \tag{147}$$

that is, the mass distribution consists of a mass 3/4 concentrated in $x = 1$ and another mass 1/4 concentrated in $x = 2$.

As demonstrated in Fig.11, a mass $P = 4/3$ at the point $x = -1$ will, together with X, establish a density field composed of two masses 1 and 1/3 at the points $x = 1$ and $z = 2$ on the line $x_+ = -1$. The second mass must be compensated by a mass $-4/9$ at the point $x_+ = 0$, and so on.

Thus,

$$X_+ = (4/3)[-1] - (4/9)[0] + (4/27)[1] - (4/81)[2] + \dots \tag{148}$$

This result may be analytically derived according to the following scheme:

(4) X	(1/4) . X_+	$X + X_+$
3[1] + 1[2]	(1/3)[-1] -(1/9)[0] (1/27)]1] -(1/81)[2]	1[0] + (1/3)[1] - (1/3)[1] - (1/9)[2] + (1/9)[2] + (1/27)[3] -(1/27)[3]-

It is of interest to note that the term 1[0] can be obtained also in the following way

(4) X	(1/4) X_+	$X + X_+$
3[1] + 1[2]	1[-2] -3[-3]	1[0] + 3[- 1] - 3[-1] - 9[- 2]

Thus

$$X_+ = 4[-2] - 12[-3] + 36[-4] - 108[-5] + \dots \tag{149}$$

is another inverse addendum of the variate (147). Even if the masses increase infinitely, it can be used just as equ.(148) in connection with bounded variates, as will be proved by the following example.

Taking, for instance, the variate, defined by equ.(115), which certainly contains X, and adding to i X_+, defined by equ.(148), we have the scheme

X_+	(16) . (Y + Y)	(16) .(X_+ + X + Y)
(4/3)[-1] - (4/9)[0] + (4/27)[1] - (4/81)[2]	3[2] + 7[3] + 5[4] + 1[5]	4[1] - (4/3)[2] + (4/9)[3] - +(28/3)[2]-(28/3)[3] + +(20/3)[3] -

Thus,

$$X_+ + (X + Y) = (1/4)[1] + (2/4)[2] + (1/4)[3] = Y \tag{150}$$

In the same way,

X_+	(16) . (X + Y)	(16) . (X_+ + X + Y)
4[-2] - 12[-3] + 36[-4] - 108[-5] +	3[2] + 7[3] + 5[4] + 1[5]	4[3] + 20[2] + 28[1] + 12[0] + ... -12[2] - 60[1] - 84[0] + ... + 36[1] + 180[0] - ... - 108[0] + ...

Thus,

$$X_+ + (X + Y) = (1/4)[3] + (2/4)[2] + (1/4)[1] + 0[0] + 0[-1] + \dots \qquad (151)$$

which is identical with equ.(150).

The first addendum is bounded from below and the second one is bounded from above. The former should be used in connection with variates bounded from below and the latter in connection with variates bounded from above.

The sum of the masses P_i of X_+ in equ.(148) is

$$\Sigma P_i = a_o/(1 - q) = (4/3)(1 + 1/3) = 1 \qquad (152)$$

Consequently, if the mass (1/4) of equ.(147) is moved until it coincides with the mass (3/4), then $X \rightarrow 1[1]$ while $X_+ \rightarrow 1[-1]$, that is, degenerate variates follow the same additive law of composition as real numbers.

The preceding method of computing inverse addenda can be extended to any variate of the discrete type.

(b) Let X be the rectangular variate

$$R(a,b) = (k(x-a)^{+0} - k(x-b)^{+0})[x] \qquad (153)$$

The inverse addendum can be geometrically determined, as demonstrated in Fig.12. If the density $p = 1/(b-a) = 1/h$, then a duplex mass $(1/p) \cdot j_x$ at the point $x_+ = -a$ will, together with X, establish a density field composed of two masses 1 and -1 at the points $x = a$ and $x = b$ on the line $x_+ = -a$, respectively. The second mass is compensated by another duplex mass $(1/p) \cdot j_x$ at the point $x = h - a$, and so on. The inverse addendum thus becomes

$$R_+(a,b) = (1/h)(1[-a] + 1[h-a] + 1[2h - a] + \dots) \cdot j_x \qquad (154)$$

It follows from equ.(125) that j_y can be substituted for j_x in equ.(154). This addendum is bounded from below. Another addendum, which is bounded from above, can be determined, viz.,

$$R_+(a,b) = (-1/h)(1[-b] + 1[-b-h] + 1[-b - 2h] + \dots) \cdot j_x \qquad (155)$$

Equs.(154) and (155) can also be deduced analytically, as demonstrated in the following example.

(c) Let X be the triangular variate (Cf.equ.(52))

$$T = (x-1)^{+1} - 2(x-2)^{+1} + (x-3)^{+1} \tag{156}$$

It follows by equ.(92) that, (omitting, for brevity, the factor 1/dx),

$$T \cdot j_x^2 = 1[1] - 2[2] + 1[3] \tag{157}$$

The computing scheme becomes

$T \cdot j_x^2$	$T_+ \cdot j_x^{-2}$	$T + T_+$
1[1] - 2[2] + 1[3]	1[-1] 2[0]	1[0] - 2[1] + 1[2] +2[1] - 4[2] + 2[3] + 3[2] - 6[3] + + 4[3] -

Thus,

$$T_+ = (1[-1] + 2[0] + 3[1] + \ldots) \cdot j_x^2 \tag{158}$$

The other addendum that is bounded from above is

$$T_+ = (1[-3] + 2[-4] + 3[-5] + . \; .) \cdot j_x^2 \tag{159}$$

Also a third addendum, which is unbounded, can be derived.

From the preceding examples it may be found that it is easy to determine the inverse addendum to any polynomial variate of the type defined by equ.(49). It will be of the discrete type and be composed of multiplex masses $k \cdot j_x^{n+1}$. Evidently, any variate of the continuous type can, with any desired degree of accuracy, be approximated by equ.(49).

(d) Let X be the exponential variate

$$X = e^{-x^+}[x] \tag{160}$$

Considering equ.(88) the computing scheme becomes

X	X_+	$X + X_+$
$j^{-1} - j^{-2} + j^{-3} - \dots$	j 1	$1[0] - j^{-1} + j^{-2} - j^{-3} + \dots$ $j^{-1} - j^{-2} + j^{-3} - \dots$

Thus

$$X_+ = (j+1)[0] \tag{161}$$

In an analogous way the inverse addendum of any variate having a density function which can be developed into the series indicated in equ.(43) can easily be determined. The procedure consists in a transformation by means of equ.(92) of the density function into a series

$$f(x) = \Sigma k_n \cdot j^{-n} \tag{162}$$

which is treated according to the computing scheme above. In general, an infinite series will result.

7.4.2 Multiplication

The multiplicative inverse component of an arbitrary variate X, called the inverse multiplier and denoted by X_x is defined by

$$X \cdot X_x = 1 \tag{163}$$

where the right-hand member stands for $1[1]$.

The inverse multiplier of a discrete variate can be determined by means of a computing scheme in analogy with that used for addition with the modification that the second projections of the ordered pairs must be multiplied, as indicated in equ.(116).

(a) Let X be the discrete variate, defined by equ.(147). The computing scheme becomes

$(4)\cdot X$	$(1/4)\cdot X_x$	$X\cdot X_x$
3[1] + 1[2]	(1/3)[1]	1[1] + (1/3)[2]
	-(1/9)[2]	- (1/3)[2] - (1/9)[4]
	(1/27)[4]	+ (1/9)[4]+(1/27)[8]
		-(1/27)[8]

Thus,

$$X_x = (4/3)[1] - (4/9[2] + (4/27)[4] - (4/81)[8] + \dots \qquad (164)$$

Comparing this expression with equ.(148), it is found that the masses are the same but with different locations.

(b) Let X be the rectangular variate, defined by equ.(150), putting $a = 1$ and $b = 2$.

Then

$$R(1,2)\cdot j_x = 1[1] - 1[2] \qquad (165)$$

and the computing scheme becomes

$R\cdot j_x$	$R_x\cdot j_x^{-1}$	$R\cdot R_x$
1[1] - 1[2]	1[1]	1[1] - 1[2]
	1[2]	1[2] - 1[4]
	1[4]	1[4] - 1[8]

Thus,

$$R_x(1,2) = (1[1] + 1[2] + 1[4] + 1[8] + \dots)\cdot j_x \qquad (166)$$

This inverse multiplier is bounded from below. Another inverse multiplier, bounded from above, can be determined in a similar way, being

$$R_x(1,2)\,(-(1[0.5]+1[0.25]+1[0.125]+\ldots)\cdot j_x \qquad (167)$$

It should be noted that it is possible to deduce an inverse multiplier which is composed of j_y - masses, but then $tg\beta \neq 1$ which makes the formulas somewhat more complicated.

7.4.3 Division

The inverse component of an arbitrary variate X, called the inverse divisor and denoted $X_:$, is defined by

$$X : X_: = 1 \qquad (168)$$

where the right-hand member stands for 1[1].

The inverse divisor of discrete and continuous variates can be determined by means of computing schemes demonstrated below.

(a) Let X be the discrete variate, defined by equ.(147). The corresponding scheme for division then becomes:

(4) . X	(1/4) . $X_:$	X : $X_:$
3[1] + 1[2]	(1/3)[1] -(1/9)[0.5] (1/27[0.25]	1[1] + (1/3)[2] - (1/3)[2] - (1/9)[4] (1/9)[4]

Thus,

$$X_: = (4/3)[1] - (4/9)[0.5] + (4/27)[0.25] - \ldots \qquad (169)$$

(b) Let X be the rectangular variate R(1,2) defined by equ. (165). The corresponding scheme becomes:

R . j_x	$R_: \cdot j_x^{-1}$	R : $R_:$
1[1] - 1[2]	1[1] 1[0 . 5] 1[0 . 25]	1[1] - 1[2] 1[2] - 1[4] 1[4] - 1[8]

Thus,

$$R_s(1,2) = (1[1] + 1[0.5] + 1[0.25] + \ldots) \cdot j_x \qquad (170)$$

This inverse divisor is bounded from above. In a similar way, another inverse divisor, bounded from below, is obtained, being

$$R_s(1,2) = -(1[2] + 1[4] + 1[8] + \ldots) \cdot j_x \qquad (171)$$

SECTION VIII. SOLUTION OF STOCHASTIC EQUATIONS

8.1 General Considerations

Any equation involving one or more stochastic quantities A, B, C with known density functions and one or more stochastic quantities X, Y, Z with unknown density functions will be called a stochastic equation. The solution of such an equation consists in the determination of the unknown density functions from the known ones.

If all the known quantities are variates, degenerate or non-degenerate, the equation will be called a random equation. Such an equation will be said to have a real root if the equation can be satisfied by substituting variates for the unknown stochastics. Criteria for the existence of real roots will be discussed below.

Considering that any composite of variates is another variate and can never be a real number, and a degenerate variate only on the condition that all elements are degenerate variates, it follows that none of the members of a random equation can be a real number or a degenerate variate. Each member must involve at least one non-degenerate variate.

If a random equation involves only one unknown variate, it will be said to be of the first degree. If two of the elements are unknown, of the second degree, and so on. It should, however, be noted that an equation including the terms $X^2 = X \cdot X$ and $2X = X + X$ is of the second degree, in spite of the fact that there is only one unknown density function.

Various types of equations will now be discussed.

8.2 Random Equations of the First Degree

Considering that any composite of variates obtained by a combination of elementary laws of composition is itself a variate, any equation of the first degree can be put in the form

$$A \cdot X + B = C \cdot X + D \tag{172}$$

This equation defines the stochastic X, in some cases being a variate. It can be given two somewhat different significations. Applying the Monte-Carlo definition it may be required that, if we take at random four values a_i, b_i, c_i, and d_i,

$$x_i = (b_i - d_i)/(c_i - a_i) \tag{173}$$

should be a random value from the set X. Another interpretation of equ.(172) would be that, if X has been determined and we take at random a large number of values x_i, a_i, and b_i and independently another set of values x_j, c_j, and d_j, then it is required that these two random samples should correspond to the same variate.

A formal solution of equ.(172) can be deduced according to the following scheme.

Adding D_+ to each member, we have

$$A \cdot X + B + D_+ = C \cdot X$$

and, since $X \cdot X_x = 1[1]$,

$$A \cdot X + (B + D_+) \cdot (X \cdot X_x) = C \cdot X = X(A + (B + D_+) \cdot X_x)$$

Thus, on the condition that $C \neq 1[0]$,

$$A + (B + D_+) \cdot X_x = C$$

Hence,

$$(B + D_+) \cdot X_x = C + A_+$$

and finally

$$X = (B + D_+) \cdot (C + A_+)_x \tag{174}$$

Some particular cases of this solution will be examined:

(a) Supposing A, B, C, D to be degenerate variates $1[a]$, $1[b]$, $1[c]$, $1[d]$, and considering that then

$$D_+ = 1[-d] \; ; \; A_+ = 1[-a] \; ; \; (B + D_+) = 1[b - d] \; ; \; (C + A_+) = 1[c - a]$$

and $(C + A_+)_x = 1[(c - a)^{-1}]$, we have

$$X = 1[x] = 1[(b - d) : (c - a)] \tag{175}$$

that is, there is 100% probability that X takes the value

$$X = (b - d)/(c - a) \tag{176}$$

This result corresponds to the first interpretation of equ.(172).

(b) Let $A = 1[0]$; $D = 1[0]$, $C = 1[1]$, then $A_+ = 1[-1]$ and $(A_+)_x = 1[-1]$ and

$$X = B \,.\, (1[1]) = B \tag{177}$$

(c) If $B = D$ and $C \neq A$, then $(B + D_+) = 1[0]$ and $X = 1[0]$

(d) If $A = 1[0]$, then $X = (B + D_+) \,.\, C_x$

(e) If $A = C$; $B = D$, then $X = (1[0]) \,.\, (1[0])_x = (1[0]/ \,.\, (1[\infty])$.

This result corresponds to the case $A \,.\, X + B = A \,.\, X + B$ which is satisfied by any X.

8.3 Random Equations of the Second and Higher Degrees

General solutions of this type of equations have not been found.

In the particular cases that

$$A \,.\, (2 \,.\, X) + B = C(2 \,.\, X) + D \tag{178}$$

and

$$A \,.\, X^2 + B = C \,.\, X^2 + D \tag{179}$$

where

$$2X = X + X \quad \text{and} \quad X^2 = X \cdot X \tag{180}$$

the equations can, however, be solved by substituting $2X = Y$ and $X^2 = Z$. After Y and Z have been determined according to the method presented in the preceding paragraph, the variate X can be determined by successive computations, if discrete, and by approximations, if continuous.

8.4 Random Equations Involving Dependent Variates

By means of the preceding methods some problems relating to the composition of dependent variates can be analysed. Let us take the following practical problem: If a specimen is subjected to pulsating load, there will, after a certain number of cycles, appear a visible fatigue crack. This number, denoted by I, varies from specimen to specimen and is a variate which can be determined by means of a sufficiently large number of tests. A continuation of the fatigue test will, after another number of cycles, result in final failure. This number, denoted by N, is called the fatigue life of the specimen. It is another variate which also can be experimentally determined.

There is a reason to believe that a specimen which has a long initiation period I will frequently, if not ever, have a long fatigue life N. This plausible statement can be mathematically expressed by the approach

$$\begin{aligned} N &= X + Z \\ I &= Y + Z \end{aligned} \tag{181}$$

where X, Y, and Z are independent variates. For the i^{th} specimen they take the values x_i, y_i, z_i. Clearly, the value z_i of the i^{th} specimen and the value z_j of the j^{th} specimen are quite independent of each other, so N and I, experimentally determined by a set of independent fatigue tests, are independent variates.

On the other hand, for a given specimen a large value of z_i implies a large value both of N and of I, so there is a correlation between N and I, if they belong to the same specimen. The propagation time T_1 measured for each individual specimen will be

$$T_1 = N(-) I = X - Y \tag{182}$$

while the propagation time T_2 measured as the difference between the independent variates N and I, taken at random from different specimens, will be

$$T_2 = N - I = X - Y + Z - Z \tag{183}$$

From equ.(182) we have the inverse addendum

$$(N(-)I)_+ = (X - Y)_+ \tag{184}$$

and adding equs.(183) and (184)

$$Z - Z = (N-I) + (N(-)I)_+ \tag{185}$$

Since N, I, and $(N(-)I)$ are independent variates, which have been experimentally determined, the difference $(Z - Z)$ can be computed and from this difference the variate Z can be determined at least by approximation methods. It is readily seen that the right-hand member of equ.(185) is equal to $1[0]$, if there is no correlation between N and I within the same specimen. The easiest criterion of this correlation consists, however, in a comparison between the variances of T_1 and T_2, because

$$2 \operatorname{Var} Z = \operatorname{Var} T_2 - \operatorname{Var} T_1 \tag{186}$$

Another type of dependency will be illustrated by the following example: If H is a property of a specimen, say, its hardness, and T another property, say, its ultimate strength, then there may be a correlation between these properties of each individual specimen. We may thus try the approach

$$\begin{aligned} T &= X \cdot Z \\ H &= Y \cdot Z \end{aligned} \tag{187}$$

where X, Y, Z are independent variates.

Suppose now that T and H have been experimentally determined by separate tests. Then T and H are two independent variates, and the ratio

$$T : H = (X \cdot Z) : (Y \cdot Z) = (X : Y) \cdot (Z : Z) \tag{188}$$

If, however, this ratio is computed from values belonging to the same specimen, then we have

$$T(:)H = (X \,.\, Z)(:)(Y \,.\, Z) = X : Y \tag{189}$$

From equs.(188) and (189) we have

$$Z : Z = (T : H) \,.\, (T(:)H)_x \tag{190}$$

where the right-hand member is composed of the experimentally determined variates T, H, and $(T(:)H)$.

8.5 Criteria for the Existence of Real Roots

General criteria are difficult to find, but some particular cases will be discussed.

Let us compare the two equations

$$X + B = C \tag{191}$$

and

$$Y + C = B \tag{192}$$

Adding X_+ to both members of equ.(191)

$$Y + C = B = C + X_+$$

Since the sum of two stochastics is uniquely determined by its elements, it follows that

$$Y = X_+ \tag{193}$$

From equ.(193) it can be concluded that if X is a variate, then Y is certainly not a variate. Thus, if equ.(191) has a real root, then equ.(192) has no real root.

This criterion can be extended to the more general case

$$A \,.\, X + B = C \,.\, X + D \tag{194}$$

and

$$A \,.\, Y + D = C \,.\, Y + B \tag{195}$$

By addition of equs.(194) and (195), we have

$$A(X+Y)+(B+D)=C(X+Y)+(B+D)$$

Adding $(B+D)_+$ to each member we have

$$A(X+Y)=C(X+Y)$$

and, if $A \neq C$

$$X+Y=1[0] \qquad (196)$$

Consequently, X and Y cannot be variates both of them, and only one of the equations (194) and (195) can have a real root.

Further, comparing

$$A \cdot X+B=C \cdot X+D \qquad (197)$$

and

$$C \cdot Y+B=A \cdot Y+D \qquad (198)$$

we have, after some easy calculations,

$$B+D_+=C \cdot X+(A \cdot X)_+=A \cdot Y+(C \cdot Y)_+$$

and

$$C \cdot X+C \cdot Y=A \cdot X+A \cdot Y$$

which may be written

$$C(X+Y)=A(X+Y) \qquad (199)$$

Thus, if $A \neq C$,

$$X+Y=1[0] \qquad (200)$$

and X and Y cannot be variates both of them, that is, at most one of the equs.(197) and (198) can have a real root.

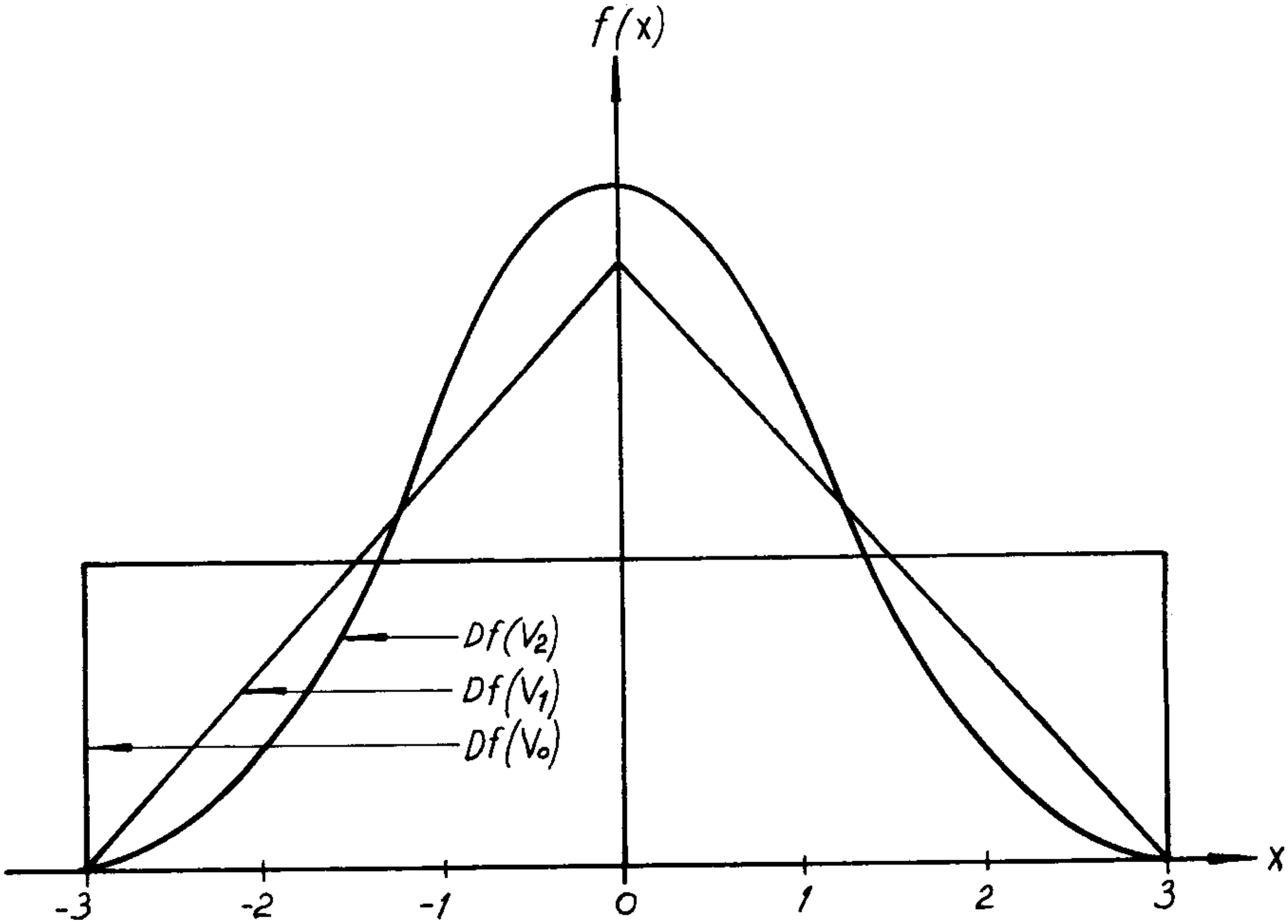

Fig.1. Density functions of the rectangular (V_0), the triangular (V_1), and the parabolic variate (V_2).

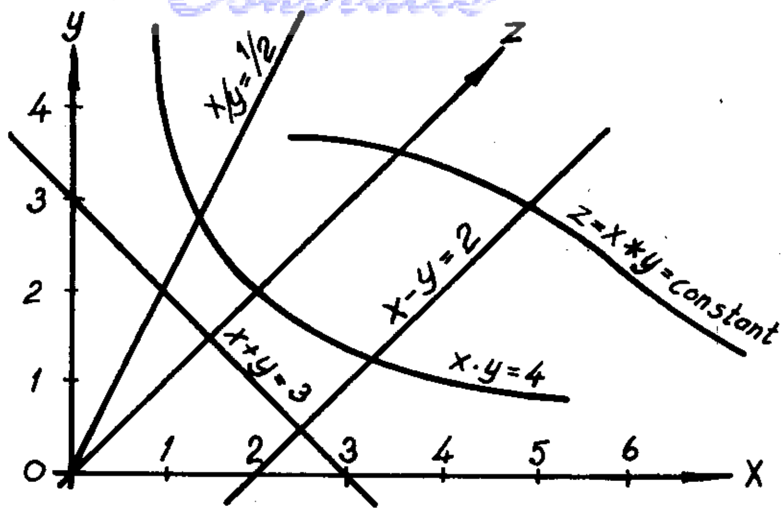

Fig.2. Composition of variates.

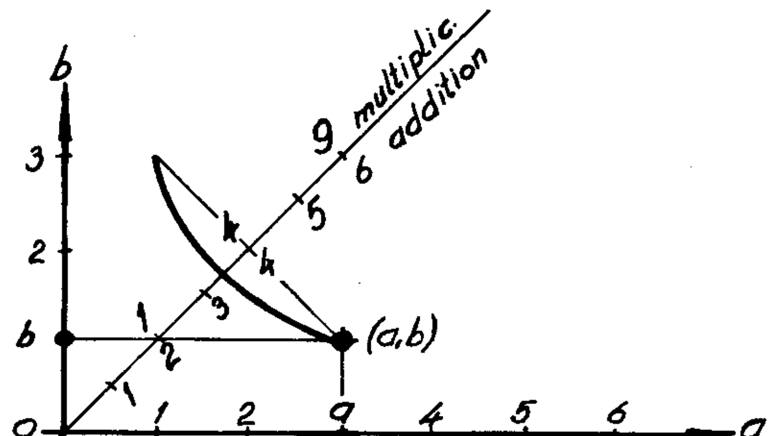

Fig.3. Addition and multiplication of degenerate variates.

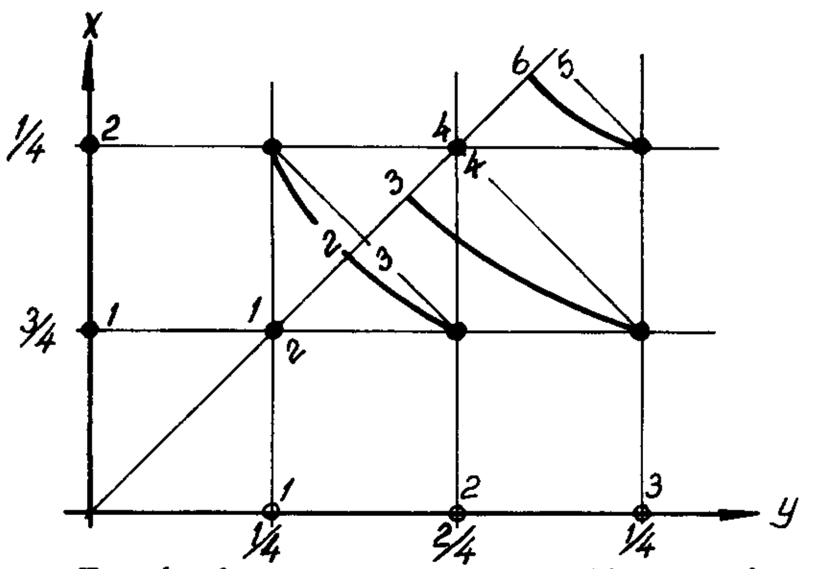

Fig.4. Addition and multiplication of discrete variates.

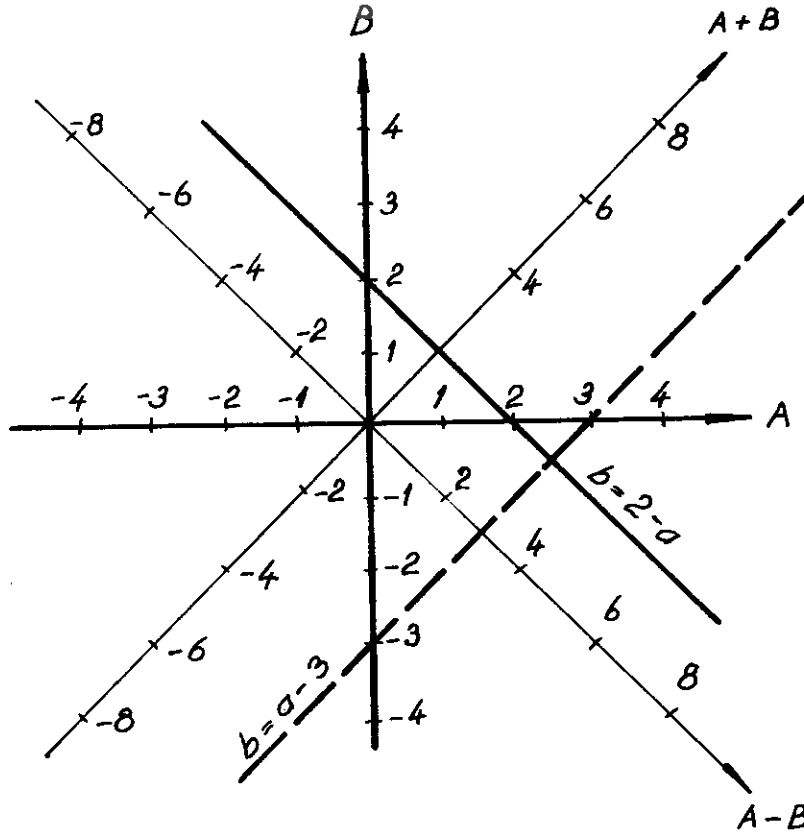

Fig. 5. Addition and subtraction of variates.

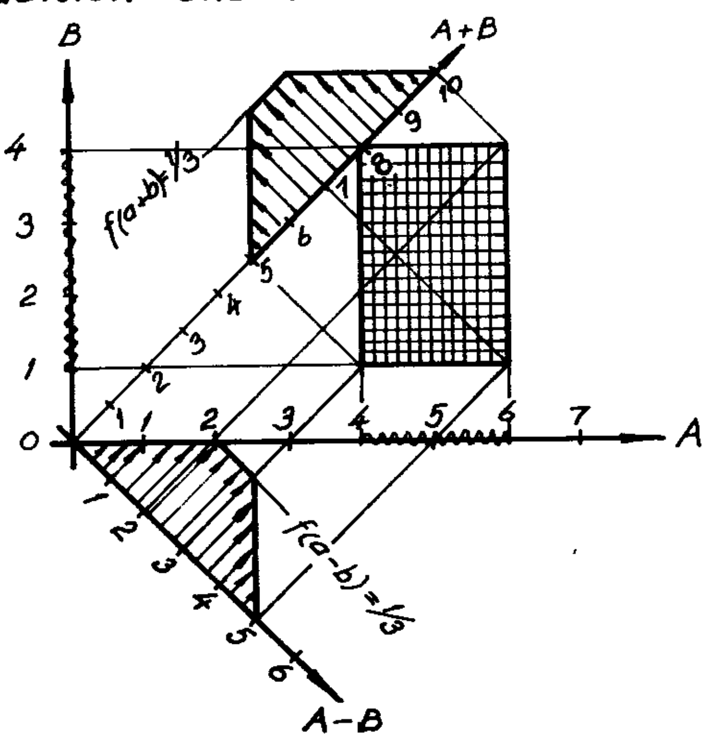

Fig. 6. Addition and subtraction of the rectangular variates R(4,6) and R(1,4).

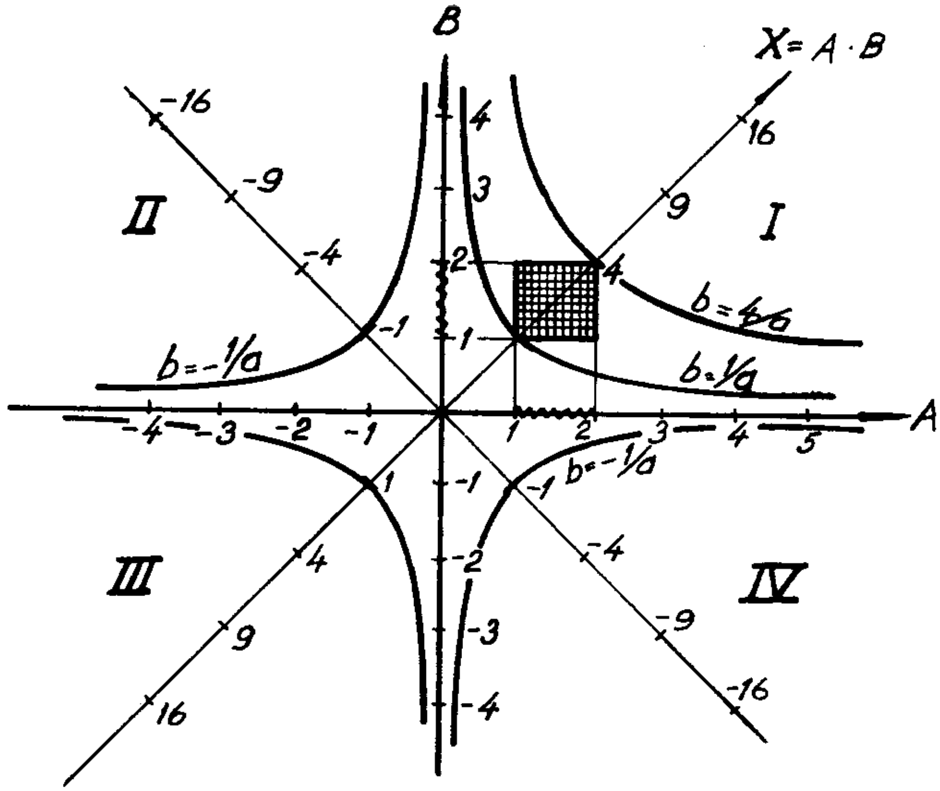

Fig.7. Multiplication of the rectangular variate R(1,2) by itself.

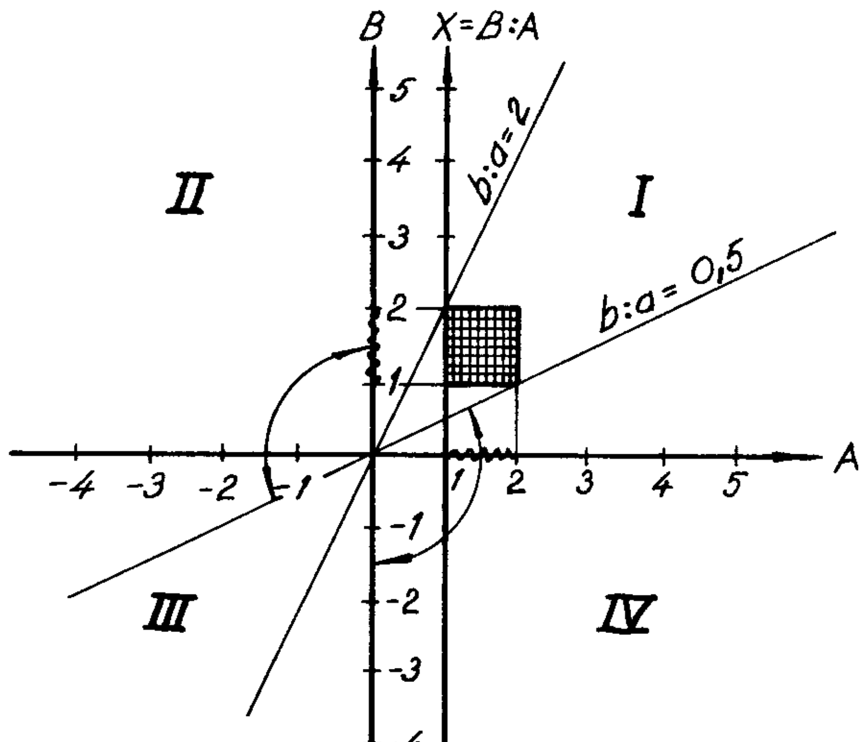

Fig.8. Division of the rectangular variate R(1, 2) by itself.

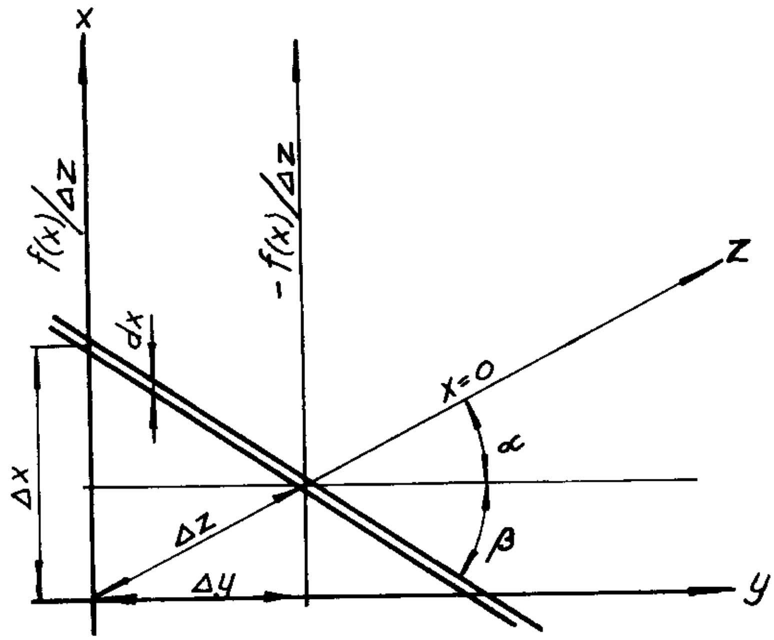

Fig. 9. Mass content of a strip intersecting a duplex mass distribution.

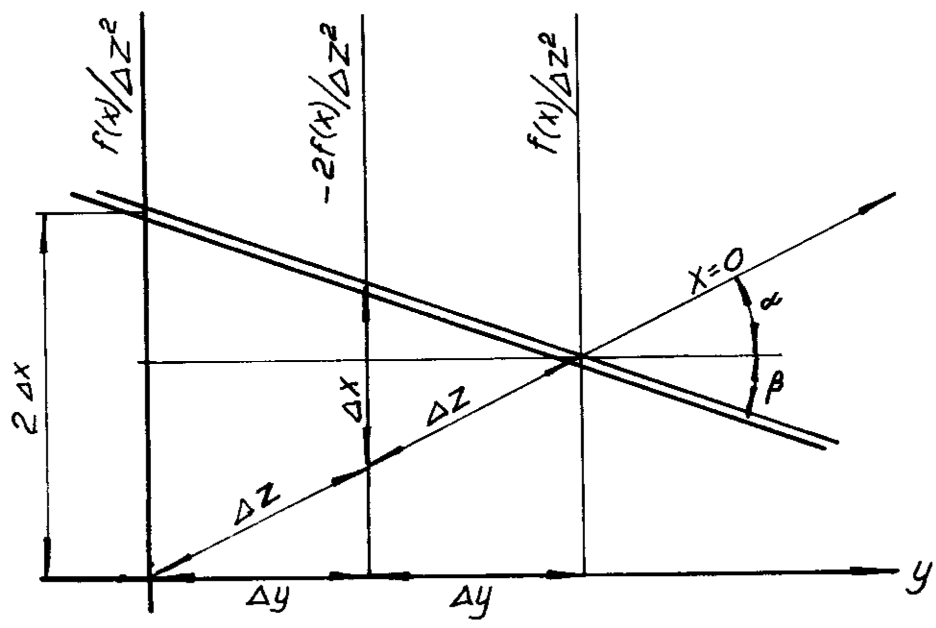

Fig. 10. Mass content of a strip intersecting a triplex mass distribution.

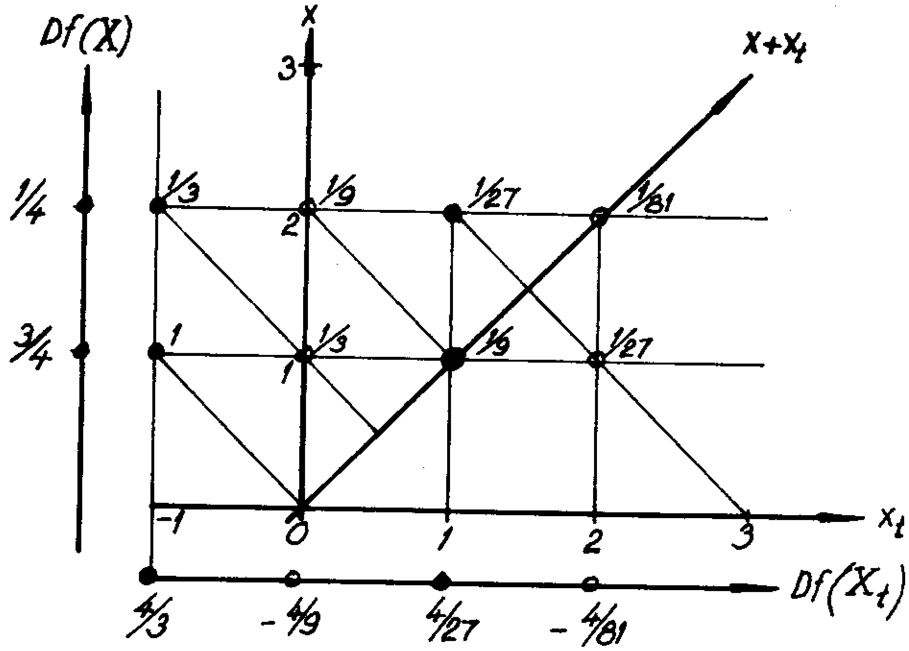

Fig. 11. Inverse addendum of a discrete variate.

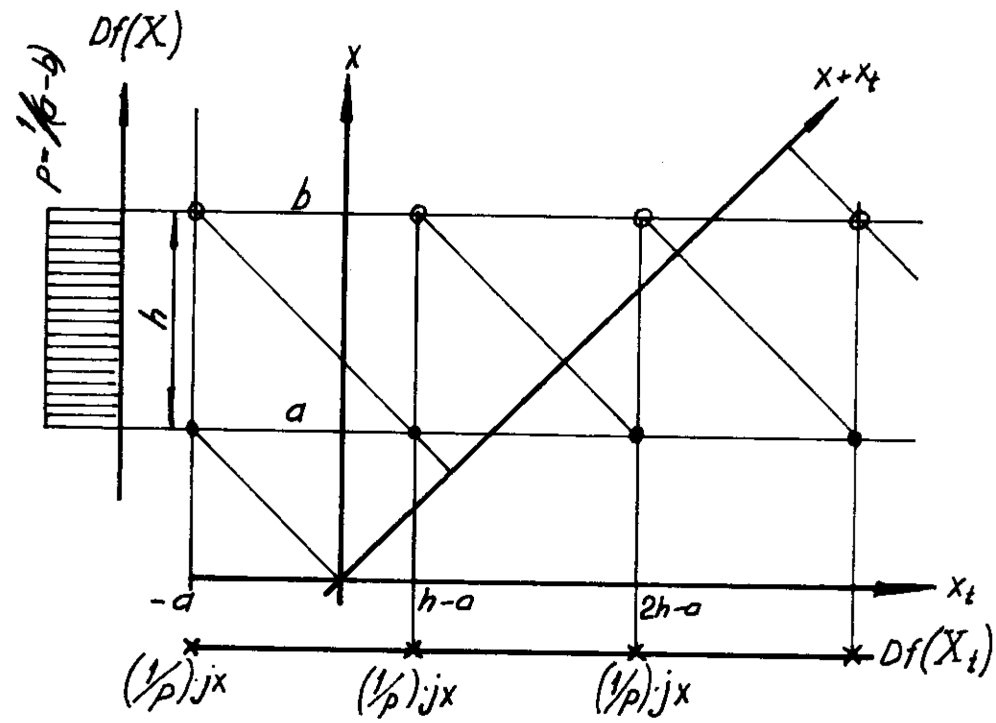

Fig. 12. Inverse addendum of a rectangular variate.