SESSION 5

DYNAMICS

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THE COMPUTATION OF DYNAMIC RESPONSE TIME HISTORIES OF A SYSTEM USING A MATRIX ITERATIVE TECHNIQUE

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Second order differential equations of motion in matrix form are written for solution of instantaneous accelerations for each degree of freedom in response to imposed conditions. These accelerations are used to establish velocity and displacement changes from initial conditions which are then summed to produce time histories. Two examples are given. The first traces out the motions of fluttering wing. The second simulates a closed, one dimensional acoustic chamber and demonstrates a resonant frequency response and speed of sound effects.

INTRODUCTION

This paper describes a computational procedure which has recently been programmed by Bell Aerosystems Company to utilize the stiffness matrix output from matrix structural analysis techniques in performing dynamic response analyses by digital methods. Since the program produces time histories utilizing a matrix iterative procedure it has been dubbed the MITH Program for Matrix Iterative Time History.

EQUATIONS OF MOTION

The equations of motion of a dynamic system are in essence a statement of the equilibrium of the internal and external forces on each element of a dynamic system. In matrix form they generally appear as shown in Equation 1.

$$\mathbf{M}\ddot{q} + \mathbf{D}\dot{q} + \mathbf{K}q = F(\ddot{q}, \dot{q}, q, t) \tag{1}$$

The left hand side of Equation 1 is frequently called the mathematical model of the system. Here the coefficient matrices **M**, **D** and **K** represent the mass, damping, and stiffness associated with each coordinate or degree of freedom of the system, and yield the internal forces of inertia damping and stiffness when multiplied by the accelerations $\ddot{q}$, velocities $\dot{q}$, and displacements q, respectively for the corresponding degrees of freedom.

The right hand side of Equation 1 describes the external force system acting on each degree of freedom. It may be any function of time depending upon the nature of the external force system to which the dynamic system is subjected.

Each line of the equation specifies the amplitude and relative phasing of the four forces (internal forces due to inertia, damping, and elastic restraint, and resultant of external

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forces) in equilibrium in the complex plane and acting on one element. Utilization of Equation 1 for the generation of time histories involves the real component of these vectors.

If the equations represent a linear system performing steady state sinusoidal oscillations at one frequency ω, then the vector force system rotates as a unit in the counter-clockwise direction with angular velocity ω, and the projections of each internal force vector on the real axis vs time provides a time history of acceleration, velocity and displacement of each degree of freedom of the system.

If the dynamic system is nonlinear or is acted upon by nonlinear forces, the complex vectors associated with time histories will not be of constant length or phasing relative to the other vectors. Since linear systems lend themselves readily to classical eigenfunction methods, we are concerned here rather with methods suitable for producing time histories of systems subjected to nonlinear forcing functions.

METHOD OF SOLUTION

The method of generation of time histories involves the determination of instantaneous acceleration over small intervals of time for each degree of freedom and then performing single and double integrations to obtain velocity and displacement changes over the same small interval. An iterative process is used to assure convergence to equilibrium conditions at each time increment. The velocity and displacement changes are then added to the initial starting values to obtain the desired time histories. The method in detail is as follows:

(a) Rewrite Equation 1 in the form of a solution of $\ddot{q}$. Thus,

$$\ddot{q} = M^{-1}(F(\ddot{q},\dot{q},q,t) - D\dot{q} - Kq) \qquad (2)$$

(b) Since we are dealing with a known physical system responding to a prescribed set of external forces, all coefficient matrices in Equation 2 are known and Equation 2 is used to compute accelerations at any starting time, t_0, for which we wish to prescribe starting values of velocity and displacement for each element. These may all be zero if the system is initially at rest or may be given any arbitrary combination of velocity and displacement we wish to impose on the system as starting conditions.

(c) A first approximation to the velocity at t_1 is computed from

$$\dot{q}_{t_n} = \dot{q}_{t_{n-1}} + \frac{1}{2}(t_n - t_{n-1})(\ddot{q}_{t_{n-1}} + \ddot{q}_{t_n}) \qquad (3)$$

where for our initial computation

$$t_n = t_1 \quad \text{and} \quad t_{n-1} = t_0$$

and it is assumed that acceleration is constant over the time interval. This assumption is corrected by subsequent iterations as indicated in steps (e) through (h). The time period $t_n - t_{n-1}$ should be small (2-10 percent) compared to the period of the degree of freedom with highest natural frequency.

(d) A first approximation to the displacement at time t_1 is computed from

$$q_{t_n} = q_{t_{n-1}} + \frac{1}{2}(t_n - t_{n-1})\left\{\dot{q}_{t_{n-1}} + \dot{q}_{t_n}\right\} \qquad (4)$$

where again

$$t_n = t_1 \text{ and } t_{n-1} = t_0$$

(e) The error introduced by assuming that $\ddot{q}_{t_1} = \ddot{q}_{t_0}$ may now be reduced by recomputing the acceleration at time, t_1 by using Equation 2 with the values of velocity and displacement from steps (c) and (d) above.

(f) Recompute the velocities at time t_1 using Equation 3 with the values of acceleration from step (e) above.

(g) Recompute the displacements at time t_1 using Equation 4 with the values of velocity from step (f) above.

(h) Steps (e), (f) and (g) constitute a second iteration for the accelerations, velocities and displacements at time t_1 . Repeat these three steps as required until convergence has been obtained to the desired number of digits.

(i) Compute accelerations, velocities and displacements for time t_2 and subsequent time periods using steps (b) through (h) for each time interval to produce the required time histories.

ILLUSTRATIONS

The method is demonstrated on a fluttering wing configuration, and for a simulated one dimensional acoustic chamber.

Example A Fluttering Wing Configuration

The above formulations were applied to a cantilever wing model of 31-inch span and 18-inch chord having a center of gravity at 45.0 percent chord, elastic axis at 41.5 percent chord and a radius of gyration of 25 percent chord. By conventional flutter analysis methods this wing was found to have a flutter speed of 152 ft per second in its fundamental bending-torsion mode. The equations of motion for this system normally appear as

$$M\ddot{x} + D\dot{x} + Kx = A\ddot{x} + B\dot{x} + Cx \tag{5}$$

Rewritten in the form of Equation 2 we have

$$\ddot{x} = (M - A)^{-1}((B - D)\dot{x} + (C - K)x) \tag{6}$$

where for the subsonic bending-torsion case the following matrix definitions apply

$$x = \begin{bmatrix} h \\ \alpha \end{bmatrix}$$

where h = generalized coordinates for fundamental cantilever bending mode,

α = generalized coordinates for fundamental cantilever torsion mode.

$$\mathbf{M} = \begin{bmatrix} \int_0^{\ell} m f_h^2\,dx & \int_0^{\ell} S_\alpha f_h f_\alpha\,dx \\ \int_0^{\ell} S_\alpha f_h f_\alpha\,dx & \int_0^{\ell} I_\alpha f_\alpha^2\,dx \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} \int_0^{\ell} m f_h^2\,dx & 0 \\ 0 & \int_0^{\ell} I_\alpha f_\alpha^2\,dx \end{bmatrix} \begin{bmatrix} \omega_h^2 & 0 \\ 0 & \omega_\alpha^2 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} \int_0^{\ell} m f_h^2\,dx & 0 \\ 0 & \int_0^{\ell} I_\alpha f_\alpha^2\,dx \end{bmatrix} \begin{bmatrix} g_h \frac{\omega_h^2}{\omega} & 0 \\ 0 & g_\alpha \frac{\omega_\alpha^2}{\omega} \end{bmatrix}$$

$$\mathbf{A} = \pi\rho \begin{bmatrix} \int_0^{\ell} b^2 f_h^2\,dx & -\int_0^{\ell} b^3 a f_h f_\alpha\,dx \\ -\int_0^{\ell} b^3 a f_h f_\alpha\,dx & \int_0^{\ell} b^4 \left(a^2 + \frac{1}{8}\right) f_\alpha^2\,dx \end{bmatrix}$$

$$\mathbf{B} = \pi\rho V \begin{bmatrix} 2\int_0^{\ell} b\, C(k) f_h^2\,dx & \int_0^{\ell} b^2\left[1 + 2C(k)\left(\frac{1}{2} - a\right) f_h f_\alpha\,dx\right. \\ -2\int_0^{\ell} b^2\left(a + \frac{1}{2}\right) C(k) f_h f_\alpha\,dx & -\int_0^{\ell} b^3\left(\frac{1}{2} - a\right)\left[2\left(\frac{1}{2} + a\right)C(k) - 1\right] f_\alpha^2\,dx \end{bmatrix}$$

$$\mathbf{C} = 2\pi\rho V^2 \begin{bmatrix} 0 & \int_0^{\ell} b\, C(k) f_h f_\alpha\,dx \\ 0 & -\int_0^{\ell} b^3 \left(a + \frac{1}{2}\right) C(k) f_\alpha^2\,dx \end{bmatrix}$$

*The aerodynamic matrices [A], [B] and [C] are for flight at subsonic speeds. It should be noted that the Theodorsen function C(k) makes the aerodynamic matrices, [B] and [C] complex. Only the real part need be retained after multiplying these coefficient matrices by the respective variables, $\dot{x}$ and x in complex form when computing acceleration using Equation 6. To obtain the complex form of the variables it is assumed that the system is performing essentially sinusoidal oscillations at some frequency ω. Then knowing the instantaneous real values of velocity and displacement $\dot{x}$ and x at a given instant of time the complex velocity and displacement will be given by

$$\dot{x}_{COMPLEX} = \dot{x} + i\omega x, \quad x_{COMPLEX} = x - \frac{i}{\omega}\dot{x}$$

Thus a value of ω must be used for aerodynamic force evaluation both to evaluate the Theodorsen function and to construct the complex form of the variables.

Since ω is actually an output to the analysis it will be desirable to use an estimated value for the time periods of the first half cycle of oscillation. The machine program may be written so that at the completion of one half-cycle of oscillation, a new frequency is computed as

$$\omega = \frac{\pi}{T/2} \tag{13}$$

where T/2 is the time required for one-half-cycle of oscillation as established from the time history. This new frequency is then used in the program for the time period of the next half-cycle of oscillation. For each subsequent half cycle of operation, a new frequency is computed and used. The effect of error in the original frequency estimate will have a minor effect on computed results, and convergence to the correct flutter frequency will be extremely rapid.

Two tests were made using the time history computational scheme on the model described. First the known flutter conditions of airspeed and bending torsion flutter amplitudes and phase angles were inserted as starting conditions to demonstrate that the program would continue to generate steady state oscillations with the proper frequency, amplitude ratio and phase angle. This was successfully demonstrated.

The second test repeated the first test except that an incorrect displacement amplitude ratio was inserted at the time t_0 starting condition to demonstrate the ability of the procedure to converge to the correct amplitude ratio. This was also successful. The results for this case have been summarized on Figure I.

This figure shows displacement time histories for the bending and twisting degrees of freedom over 5 cycles of oscillation in both graphical and numerical form. The numerical tabulation gives: (1), the absolute values of the input vectors and computed response vectors for each half-cycle peak; (2) the amplitude ratio of these vectors; and (3) the amplitude ratio from (2) as a fraction of the flutter analysis value.

Note that the imposed input amplitude ratio of 0.989 was 80 percent of the steady state flutter value obtained from flutter analysis. This is corrected to 96.9 percent of the steady state flutter value at the end of the first cycle, and is corrected to within 0.3 percent of the steady state flutter value at the end of three complete cycles of operation. The wing flutter frequency of 43.9 rad/sec which was established by conventional analysis procedures was generated accurately over each half-cycle in each degree of freedom.

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*This analysis is covered in Appendix 1.0 of AFOSR TN 60-1476 B. D'Ewart and R. Farrell (Air Force Office of Scientific Research Report).

It should be emphasized that while the convergence of the time histories to the known exact solution is considered an excellent check on the practicality of the method, the rate of convergence is not such a check. This is because the computational process merely attempts to compute what is happening to the physical system as it responds at each instant of time to the conditions imposed.

Example B One Dimensional Acoustic Chamber

Chamber Simulation

This example involves a 40-foot tube closed at both ends as shown in Figure 2. The tube has been divided into 40 stations with simulation of the air as a concentrated mass at the center of each station connected in series with massless springs. The effective spring rate of air is determined as shown in Figure 3 for isentropic conditions. Since a one-foot square cross section has been used, the mass at each station is 0.002378 slugs/ft^3 assuming standard atmospheric conditions.

This model provides a convenient check case since the stiffness influence coefficients for the idealization described can be written by inspection as shown in Figure 3, and exact values for the speed of sound and natural frequencies are readily obtained as shown in Figure 4 for the acoustic chamber.

1st Test Case — Natural Frequency Check

In the two check cases presented here we have not used external forces on the system but have instead generated the time histories of the system responding to prescribed input conditions at time t = 0.

In the first case we have checked the ability of the program to trace out the time history of the model in its fundamental mode. As indicated in Figure 5 this was done by letting starting accelerations and displacements by zero, with a half-sine wave velocity distribution over the tube length with center stations at peak values of 100 ft/second.

Figure 6 shows time history results after 1/4 cycle of oscillation. It was found that 1/4 cycle of oscillations was executed in 401.70 time periods giving a frequency of 13.964 cps as compared to the exact value of 13.946 cps. It is interesting to note further that the displacement shape for the forty stations was a half-sine wave accurate to seven digits although only 5 digit accuracy was used in the input velocities.

2nd Test Case — Speed of Sound Computation

In this case a disturbance was introduced at the left end of the tube by giving station No. 1 an initial velocity of 100 ft/second at zero displacement. All other stations were given zero velocities and displacements at time zero. Starting conditions are summarized in Figure 7. The progress of the wave-front down the tube is indicated in the velocity distribution plots of Figure 8 for a number of time points.

In each of the curves shown the wave-front is at the station with zero velocity (maximum amplitude) marked X. Having identified the wave-front location as a function of time, the speed of the wave-front was computed between a number of points down the tube as shown and plotted in Figure 9. The computed values are low but appear to be approaching the correct value asymptotically. It seems likely that a different choice of input disturbances might be found that would more rapidly generate a wave-front with a more correct traveling speed.

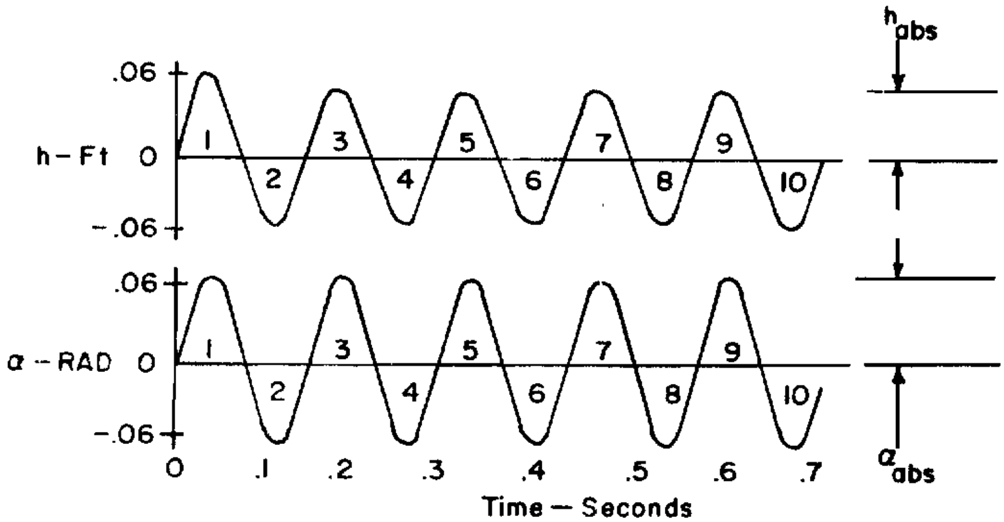


Figure 1. Time History Analysis Output Displacement Summary

TABLE I

TABULATION OF ABSOLUTE VALUES OF h, α AND $\frac{h}{\alpha}$

Cycle Designation	$h_{absolute}$ Ft	$\alpha_{absolute}$ Rad	$\frac{\alpha}{h}\frac{abs}{abs}$ Rad/Ft	Amplitude Ratio as a Fraction of Flutter Analysis Value
Input	.0625	.0618	0.989	0.80
1	.06338781	.06695492	1.056	0.86
2	-.05714757	-.06767146	1.184	0.969
3	.05407092	.06691427	1.238	1.005
4	-.05321080	-.06627186	1.245	1.012
5	.05322346	.06598823	1.240	1.007
6	-.05338031	-.06591754	1.235	1.003
7	.05346337	.06591588	1.233	1.002
8	-.05350172	-.06591591	1.232	1.001
9	.05350932	-.06590546	1.232	1.001
10	-.05350683	-.06588378	1.231	1.000
ω = 43.9 Rad/Sec Observed at Each Cycle				

Chamber Description

Chamber Length: 40.00 ft.

Chamber Cross Sectional Area: $= 1.0\ ft^2$

Idealization: Divided into 40 stations of one ft. length with point mass at center of each station

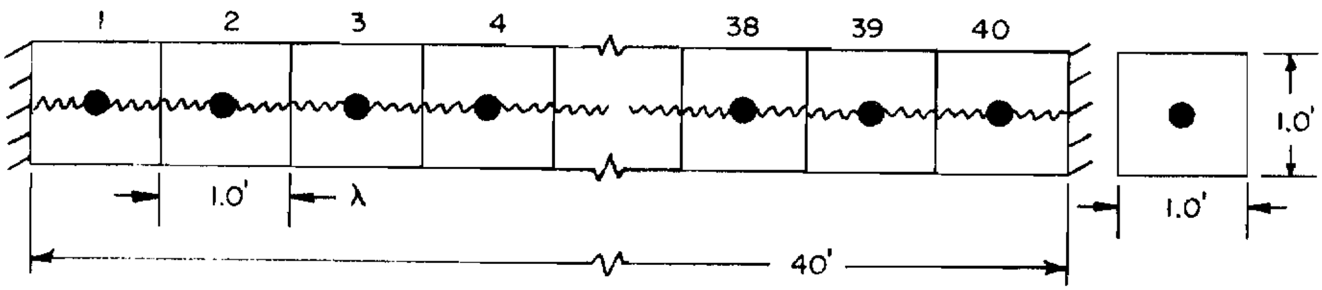


Figure 2. Simulation of One-Dimensional Acoustic Chamber

Physical Properties

$K = \frac{PAk}{\lambda}$, Isentropic Spring Rate

$= 2960$ Lbs/Ft.

$M =$ Mass of 1 Ft^3 of Air at Std. atmosphere

$= .002378$ Slugs/Ft^3

where

$P = 2116$ Lbs/Ft^2

$A = 1.0\ Ft^2$

$k = \frac{C_p}{C_v} = 1.4$

= specific heat ratio

$\lambda = 1.0'$

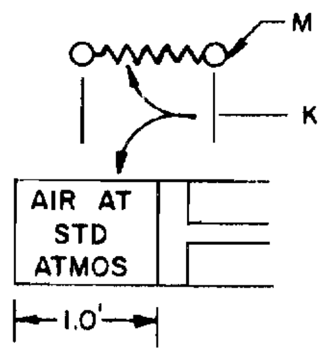


Figure 3. Simulation of One-Dimensional Acoustic Chamber

Equations of Motion

$$.002378 \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \ddot{q} + 2960 \begin{bmatrix} 3 & -1 & 0 & 0 & & \\ -1 & 2 & -1 & 0 & & \\ 0 & -1 & 2 & -1 & & \\ & & & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & 0-1 & 3 \end{bmatrix} q = 0$$

Speed of Wave Propagation

$$C = \sqrt{\frac{K'}{M'}}$$

where

C = speed of wave propagation through material of bulk modulus K' and mass density M'

for the above case

$$K \equiv K' = 2960 \text{ lbs/ft}^2$$

$$M \equiv M' = .002378 \frac{\text{slugs}}{\text{ft}^3}$$

Thus

$$C = \sqrt{\frac{2960.00}{.002378}} = 1115.68 \text{ ft/sec}$$

Natural Frequency of Closed Tube

$$f = \frac{cn}{2l},$$

For the first mode

$$f = \frac{1115.68\ (1)}{2\ (40.00)}$$

$$f = 13.9460 \text{ cps}$$

where

f = natural freq. -cps.

c = speed of sound-ft/sec

n = mode number

l = tube length-ft

Figure 4. Exact Dynamic Characteristics of Acoustic Chamber Representation of Figure 2

ACCELERATIONS: a = 0.0 ft/sec^2 at all stations

VELOCITIES: Sine wave velocity distribution as shown below in Figure 5 with 5 digit accuracy

DISPLACEMENTS: Δ = 0.0 at all stations

TIME INCREMENT: .00004475 sec.

PERIOD OF TIME COVERED: 1/2-cycle of 1st mode or 800 time increments

CONVERGENCE CRITERIA: Agreement to 0.10 percent between iterations

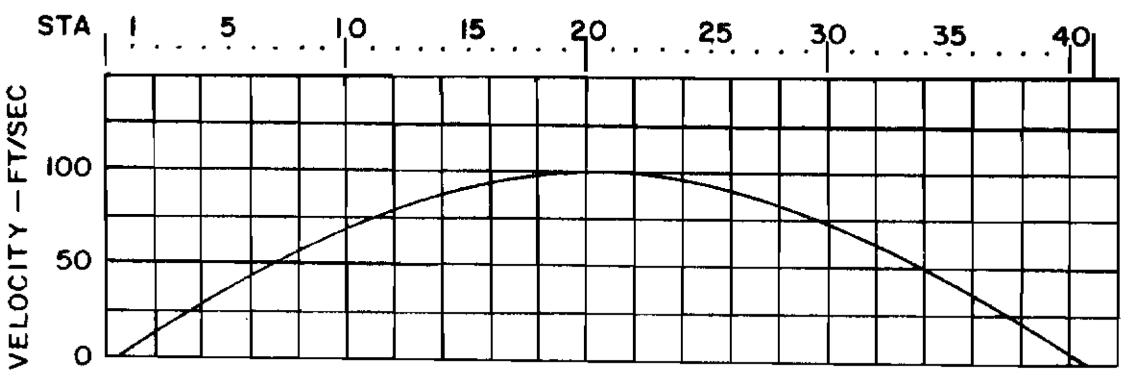


Figure 5. Acoustic Chamber - 1st Test Case Fundamental Mode Computation

INPUT

Initial Conditions at t = 0

A. Frequency Check

Time increments required for 1/4 cycle = 401.7

Length of one time period = 0.00004475 sec.

Time period for 1/4 cycle = $401.7 \times .00004475 = \frac{T}{4}$

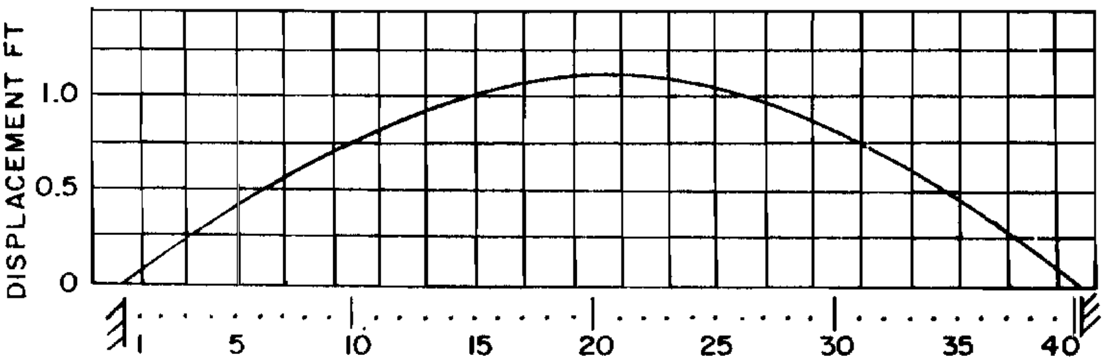
= .0179031 sec.

$$\text{Fundamental frequency} = \frac{1}{T} = \frac{1}{(4)\ (.0179031)}$$

= 13.964 cps

Exact frequency = 13.946 cps (From Figure 3)

B. Displacement Shape Check



Displacement Shape Observations

(1) Displacement shape is sine wave accurate to 7 digits

(2) Displacements of corresponding stations on left- and right-sides check to 7 digits.

Figure 6. Acoustic Chamber - 1st Test Case-Fundamental Mode Computation OUTPUT Computed Results After 1/4 Cycle

VELOCITY

Station 1 + 100 ft/sec (toward sta. 40)

All other stations 0.0 ft/sec.

DISPLACEMENT

At all stations 0.0 ft.

DAMPING AND EXTERNAL FORCES

At all stations 0.0

TIME INCREMENT Δt = 0.00004475 sec.

CONVERGENCE CRITERIA 0.1 percent between iterations

Figure 7. Acoustic Chamber - 2nd Test Case Speed of Sound Determination Initial Conditions at Time t = 0

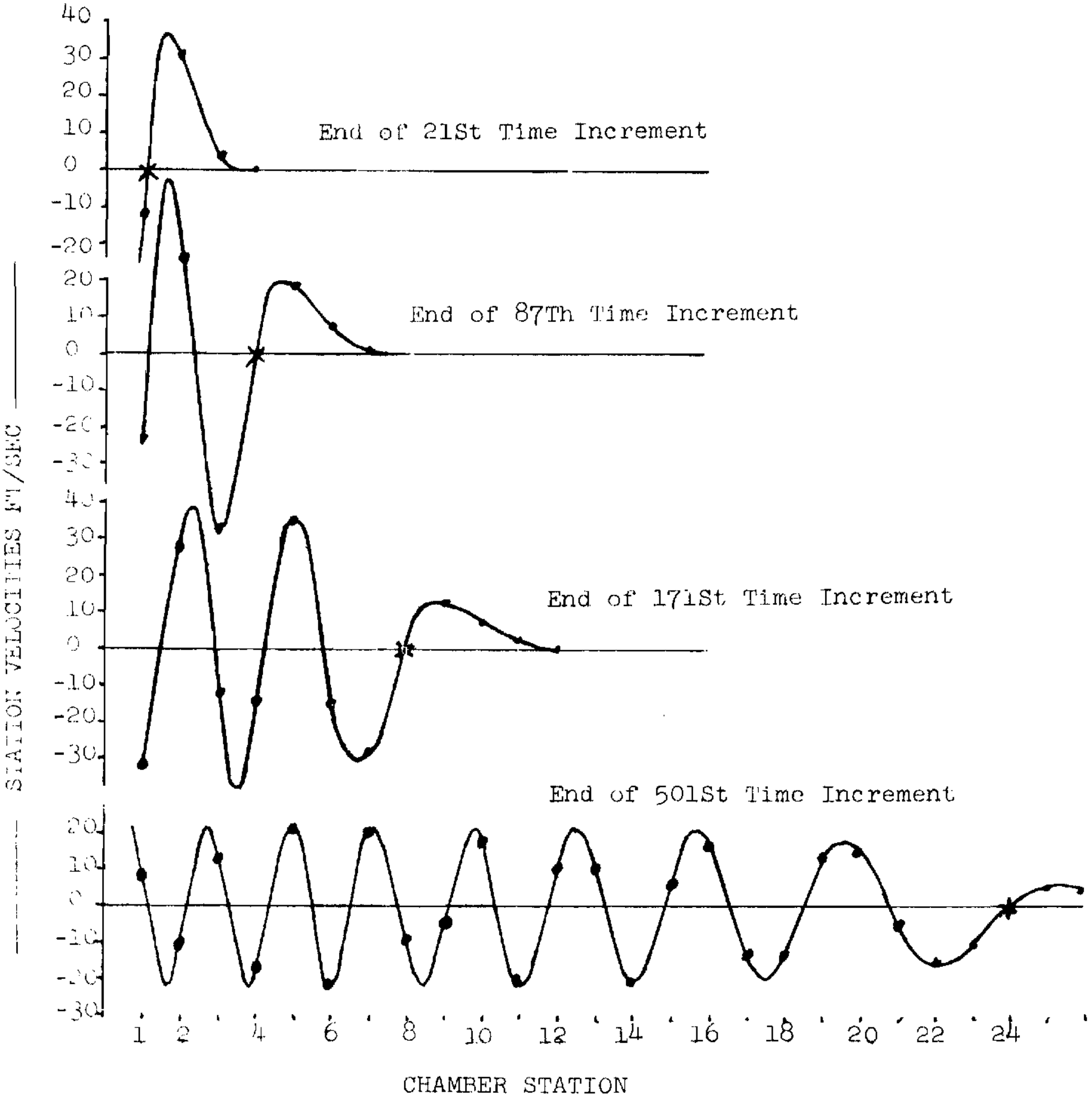


Figure 8. Acoustic Chamber - 2nd Test Case Speed of Sound Determination
Station Velocities at Times Indicated

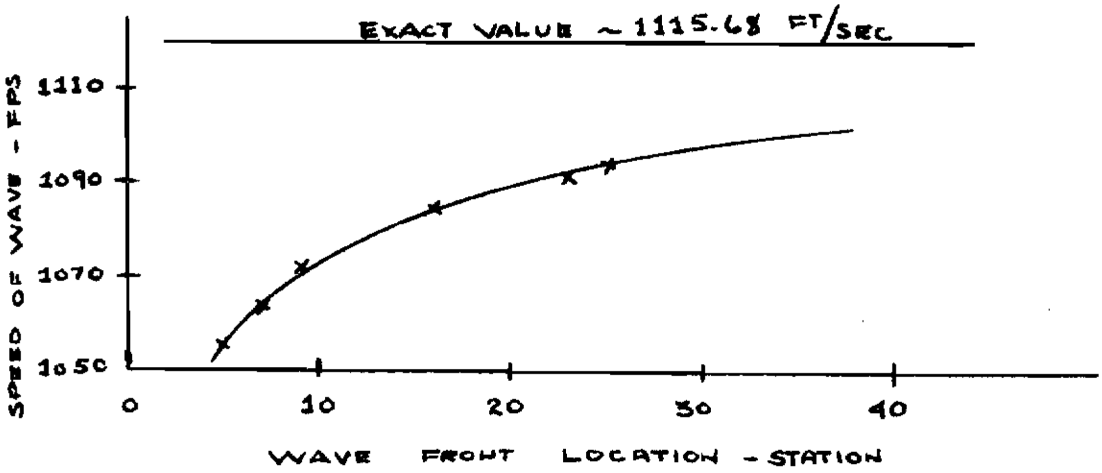


Figure 9. Acoustic Chamber 2nd Test Case Speed of Sound Determination

Time Increment	Location of Wave Front Peak Amp Sta.	Elapsed Time Sec.	Distance Traveled Ft.	Speed of Wave Front
87	4			
		.00190187	2.000	1056.2
129.5	6			
		.0018795	2.000	1064.1
171.5	8			
		.0018652	2.000	1072.3
213.18	10			
		.0110564	12.000	1085.3
460.25	22			
		.0018307	2.000	1092.5
501.16	24			
		.0018276	2.000	1094.3
542.00	26			