ELEMENT STIFFNESS-MATRICES FOR BOUNDARY COMPATIBILITY AND FOR PRESCRIBED BOUNDARY STRESSES*

Theodore H.H. Pian**

Massachusetts Institute of Technology
Cambridge, Massachusetts

A method for the derivation of element stiffness matrices based on the principle of minimum complementary energy employing assumed stress distribution is described. Two different types of elements are considered: (1) one which is constrained at the entire boundary, and (2) one which has prescribed stresses along the boundary. The result shows that by using sufficiently large number of terms for the assumed stress distribution, a converging stiffness matrix can be obtained. Example calculations for rectangular panels in plane stress condition and for square plates in bending indicate an improvement in structural analysis when the element stiffness matrices satisfy complete boundary compatibility and boundary stress conditions.

INTRODUCTION

In the derivation of element stiffness matrices to be used in connection with the displacement method of matrix structural analysis, it is desirable that the following two conditions are satisfied: (1) the equilibrium of stress in the interior, and (2) displacement compatibility with the adjacent elements. However, for structure elements, in general, it is not possible to satisfy both conditions simultaneously. Thus, many of the simple methods suggested for the derivation of approximate stiffness matrix fall into two categories: (a) the use of a simple assumed stress distribution which satisfies condition (1) but the resulting displacements will violate condition (2), and (b) the use of a simple assumed displacement distribution which can be made to satisfy condition (2) but the corresponding stress distribution will, in general, violate condition (1). Comments on the various schemes and application to plane stress problems have been made by Gallagher (References 1 and 2). For problems concerning bending of plates, various authors (References 3, 4, and 5) have considered the choice of suitable displacement functions which satisfies the edge compatibility. For deriving the element stiffness matrices, the present author has considered (References 6 and 7) both minimum-potential-energy method and minimum-complementary-energy method which, in theory, can take into account unlimited number of terms in the displacement functions and stress functions, respectively and hence, progressively improved matrices can be obtained which will satisfy both conditions (1) and (2) above.

The present paper concerns mainly the use of assumed stress distributions and principle of minimum complementary energy for the derivations of stiffness matrices for rectangular elements in plane stress condition and in bending. The method can account for elements which require displacement compatibility at all four edges and for elements which are stress-free on some of their edges.

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**Associate Professor of Aeronautics and Astronautics.

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OUTLINE OF THE METHOD

The basic method which was described in Reference 7 is dealing with the following problem: For an element of the structure, a set of generalized displacements is chosen at the nodes and simple displacement patterns at the surface are assigned so that the compatibility condition with the neighboring elements can be satisfied. With the prescribed displacements at the boundary A_2, it is desirable to determine the stress distribution over the element. This problem can be solved by the principle of minimum complementary energy which may be stated as, (Reference 8)

$$\pi_C = U - \int_{A_2} u_i \, S_i \, dA = \text{minimum} \tag{1}$$

where U is the strain energy in terms of stress components σ_{ij}, u_i is the component of the prescribed displacement, and S_i is the component of surface force which is related to the stress components by

$$S_i = \sigma_{ij} \, n_j \tag{2}$$

where n_j is the direction cosine of the surface normal.

In applying this variational principle, one begins by expressing the stress distribution $\boldsymbol{\sigma}$ in terms of m undetermined stress coefficients $\boldsymbol{\beta}$ as follows

$$\boldsymbol{\sigma} = \mathbf{P} \, \boldsymbol{\beta} \tag{3}$$

where the terms of the matrix $\mathbf{P}$ are functions of the coordinates $x_i (i = 1,2,3)$. The number of elements in $\boldsymbol{\beta}$ is unlimited, but $\mathbf{P}$ and $\boldsymbol{\beta}$ must be so chosen that the stress distribution satisfies both the equations of equilibrium in the interior and any prescribed stress boundary (for example, stress-free conditions).

When the stress-strain relations

$$\boldsymbol{\epsilon} = \mathbf{N} \, \boldsymbol{\sigma} \tag{4}$$

are introduced, one can express the internal strain energy as

$$U = \frac{1}{2} \int_V \boldsymbol{\sigma} \, \mathbf{N} \, \boldsymbol{\sigma} \, dV \tag{5}$$

or

$$U = \frac{1}{2} \boldsymbol{\beta}^T \mathbf{H} \, \boldsymbol{\beta} \tag{6}$$

where $\mathbf{H}$ is a symmetric matrix determined by

$$\mathbf{H} = \int_V \mathbf{P}^T \, \mathbf{N} \, \mathbf{P} \, dV \tag{7}$$

The prescribed displacements at the boundary A_2 are given in terms of the n generalized displacement q at the nodes in the form of

$$\mathbf{u} = \mathbf{L} \, \mathbf{q} \tag{8}$$

where the terms in the matrix **L** contain coordinates of the surface. The surface force **S** can be expressed in terms of the stresses $\boldsymbol{\sigma}$ by means of Equation 2 and hence, can be related to the undetermined stress coefficients $\boldsymbol{\beta}$, i.e.,

$$\mathbf{S} = \mathbf{R}\boldsymbol{\beta} \tag{9}$$

where the terms in **R** also contain the coordinates on the surface.

The total complementary energy is then

$$\pi_C = \frac{1}{2}\,\boldsymbol{\beta}^T \mathbf{H}\,\boldsymbol{\beta} - \boldsymbol{\beta}^T\,\mathbf{T}\,\mathbf{q} \tag{10}$$

where

$$\mathbf{T} = \int_{A_2} \mathbf{R}^T\,\mathbf{L}\,dA \tag{11}$$

The condition of minimum complementary energy, i.e., $\partial\pi_C/\partial\beta_i = 0$ ($i = 1, \ldots, m$), yields

$$\mathbf{H}\,\boldsymbol{\beta} = \mathbf{T}\mathbf{q} \tag{12}$$

Equation 12 can be solved for $\boldsymbol{\beta}$, that is,

$$\boldsymbol{\beta} = \mathbf{H}^{-1}\,\mathbf{T}\,\mathbf{q} \tag{13}$$

Substituting this expression into Equation 6, one obtains

$$U = \frac{1}{2}\,\mathbf{q}^T\,\mathbf{T}^T\,\mathbf{H}^{-1}\,\mathbf{T}\,\mathbf{q} \tag{14}$$

Since the strain energy U can also be expressed in terms of the element stiffness matrix [k] as

$$U = \frac{1}{2}\,\mathbf{q}^T\,\mathbf{k}\,\mathbf{q} \tag{15}$$

one can conclude

$$\mathbf{k} = \mathbf{T}^T\,\mathbf{H}^{-1}\,\mathbf{T} \tag{16}$$

It is seen that the corresponding column of generalized forces **Q** is given by

$$\mathbf{Q} = \mathbf{k}\,\mathbf{q} \tag{17}$$

Substituting **k** from Equation 16 and then using Equation 13, one finds

$$\mathbf{Q} = \mathbf{T}^T\boldsymbol{\beta} \tag{18}$$

The matrix $\mathbf{T}^T$ thus relates the equivalent generalized forces **Q** and the assumed stress coefficients $\boldsymbol{\beta}$.

RECTANGULAR ELEMENT UNDER PLANE-STRESS CONDITION

To illustrate the method outlined above, a rectangular element of a dimension a x b, under plane-stress condition is considered. Figure 1 shows the eight generalized forces and the corresponding displacements. The stiffness matrices for such an element using various numbers of β (up to β_{10}) have been given in Reference 7. Here in this investigation, results are

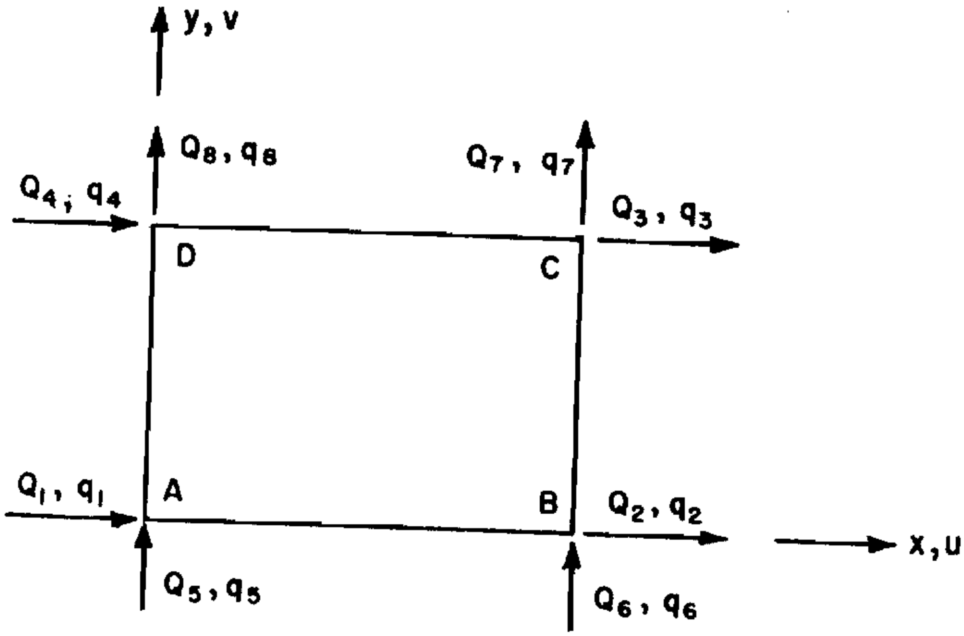

Figure 1. Generalized Forces and Displacements of a Rectangular Element in Plane Stress Condition

obtained using more terms in the assumed stress functions. For the present problem in the stress matrix $\{\sigma\}$ contains only three elements, that is, $\boldsymbol{\sigma} = \{\sigma_x, \sigma_y, \tau\}$, and by expressing these in terms of polynomials of x- and y-coordinates, the corresponding **P** matrix (see Equation 3) has been determined as shown in Table 1. Here, terms including all cubic functions are used and a total of 18 independent β's are required. It can be easily verified that the equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0 \; ; \; \frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{20}$$

are satisfied.

The boundary force matrix **S** consists of eight elements representing the x and y components of the boundary forces along the four edges. They are:

$$\begin{aligned} \mathbf{S} = & -\tau_{AB}(x), \; -\sigma_{y_{AB}}(x), \; \sigma_{x_{BC}}(y), \; \tau_{BC}(y), \\ & \tau_{DC}(x), \; \sigma_{y_{DC}}(x), \; -\sigma_{x_{AD}}(y), \; -\tau_{AD}(y) \end{aligned} \tag{21}$$

The corresponding boundary displacement matrix is

$$\begin{aligned} \mathbf{u} = & u_{AB}(x), \; V_{AB}(x), \; u_{BC}(y), \; V_{BC}(y), \\ & u_{DC}(x), \; V_{DC}(x), \; u_{AD}(y), \; V_{AD}(y) \end{aligned} \tag{22}$$

The matrix **R** which relates **S** and $\boldsymbol{\beta}$, and **L** which relates the displacements **u** and the eight generalized coordinates **q** are also listed in Table 1. It is seen that the displacement along any edge is linearly related only to the nodal displacements at the ends of that edge. Thus, the boundary compatibility between neighboring elements is insured. The matrix **N** for the stress-strain relation is

$$\mathbf{N} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \tag{23}$$

With **P**, **N**, **R** and **L**, the matrices **H** and **T** can be obtained by integration and the element stiffness-matrix can be determined by means of Equation 16.

Stiffness matrices of a square element with thickness equal to t and Poisson's ratio equal to 1/3 have been evaluated using different number of terms in the stress-function series. Using terms up to β_5 (i. e., three constant terms and two linear terms) the resulting matrix is

$$Et \begin{bmatrix} .45833 & -.27083 & -.29167 & .10417 & .18750 & 0 & -.18750 & 0 \\ & .45833 & .10417 & -.29167 & 0 & -.18750 & & .18750 \\ & & .45833 & .27083 & -.18750 & 0 & .18750 & 0 \\ & & & .45833 & 0 & .18750 & 0 & -.18750 \\ & & & & .45833 & .10417 & -.29167 & -.27083 \\ & \text{Symmetric} & & & & .45833 & -.27083 & -.29167 \\ & & & & & & .45833 & .10417 \\ & & & & & & & .45833 \end{bmatrix}$$

Using terms up to β_{12} (i.e., all constant, linear and quadratic terms) the result is:

$$Et \begin{bmatrix} .46875 & -.28125 & -.28125 & .09375 & .18750 & 0 & -.18750 & 0 \\ & .46875 & .09375 & -.28125 & 0 & -.18750 & & .18750 \\ & & .46875 & -.28125 & -.18750 & 0 & .18750 & 0 \\ & & & 46875 & 0 & .18750 & 0 & -.18750 \\ & & & & .46875 & .09375 & -.28125 & -.28125 \\ & \text{Symmetric} & & & & .46875 & -.28125 & -.28125 \\ & & & & & & .46875 & .09375 \\ & & & & & & & .46875 \end{bmatrix}$$

Using terms up to β_{18} (i.e., with the addition of all cubic terms) the result is:

$$
Et \begin{bmatrix}
.47229 & -.28479 & -.27771 & .09021 & .18750 & 0 & -.18750 & 0 \\
 & .47229 & .09021 & -.27771 & 0 & -.18750 & 0 & .19750 \\
 & & .47229 & -.28479 & -.18750 & 0 & .18750 & 0 \\
 & & & .47229 & 0 & .18750 & 0 & -.18750 \\
 & & & & .47229 & .09021 & -.27771 & -.28479 \\
 & \text{Symmetric} & & & & .47229 & -.28479 & -.27771 \\
 & & & & & & .47229 & .09021 \\
 & & & & & & & .47229
\end{bmatrix}
$$

It is seen that when the number of β terms is increased, the values of the diagonal elements are also increased. This may be explained by the fact that additional constraint at the boundary yields a more rigid stiffness matrix. It should be remarked that the matrix obtained by using terms up to β_{12} is identical to those obtained by using terms up to β_7, β_9 and β_{10}. In Reference 7, when the present author had calculated the stiffness matrices up to only β_{10} terms, a statement was made that a converging result was reached using only seven terms of the series. Apparently, when the complete cubic terms of the series are included, the resulting matrix becomes still more rigid. It is of interest to note that this new result is indeed very close to that derived by the potential energy approach using ten terms in the assumed displacement function (Reference 7). The effect of additional terms in the assumed displacement function, however, is to yield a more flexible stiffness matrix.

For square elements with partially free boundary conditions, the stiffness matrices can be evaluated by the same procedure except that the **P** and **R** matrices should have fewer columns. For example, for an element with the left-hand edge under stress-free conditions, β_1, β_2, β_5, β_6, β_8, β_{11}, β_{13} and β_{14} should be eliminated. For an element with both the left-hand and lower edges under stress-free conditions, only β_{10}, β_{16}, and β_{17} need be taken into account.

EXAMPLE ANALYSIS

To see if the introduction of the special element stiffness matrices for partial stress-free boundaries can improve the deflection and stress predictions in structural analyses, an example analysis was carried out. The problem is the same one used in References 1 and 9. It consists of a long rectangular strip loaded at the midpoint of each end by a 2-lb. concentrated force (Figure 2). Reference 9 has provided analytical solutions for the deflection distribution along the centerline and the normal and shear stress distributions at various sections.

The stress distribution at the section five inches from the loaded end is used in the present comparison. In the finite element analysis only a quarter of the panel is considered. This quarter panel is reduced to 48 elements which are either square panels or rectangular panels of one by two aspect ratio (Figure 2). For the convenience of stress evaluation, the elements are so arranged that the reference section is located along the center of four square elements.

The results of two different calculations are presented here. For the first one, all the stiffness matrices used correspond to elements with all four edges constrained while for the second case, stiffness matrices of the corresponding stress-free boundary conditions are used for the elements along the two free edges.

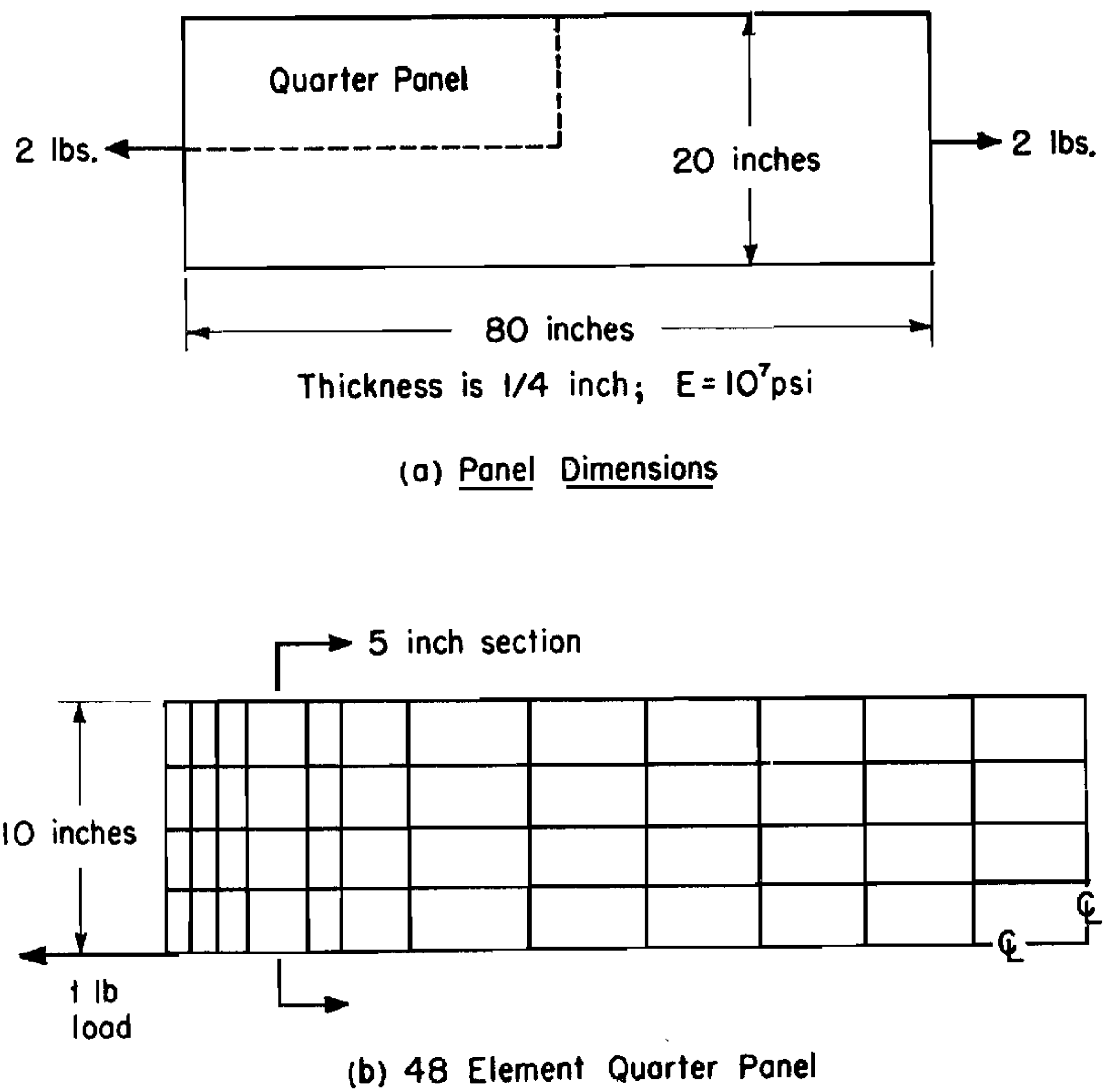

Figure 2. Rectangular Strip Under Concentrated Loads at the Two Ends

Figure 3 shows two finite-element solutions for the axial displacement along the horizontal centerline of the panel when the vertical centerline is held fixed. The analytical solution is available except for the region adjacent to the loaded end. It is seen that the introduction of the stress-free stiffness matrices has improved the overall deflection calculation.

The finite-element matrix analysis consists of mainly the calculation of the generalized displacements $\mathbf{q}$. Knowing $\mathbf{q}$ the stress distribution can be determined by using Equations 3 and 13. Figures 4 and 5 present comparisons of normal and shear stresses along the reference section. Again, the introduction of stress-free stiffness matrices appears to yield a more accurate stress prediction.

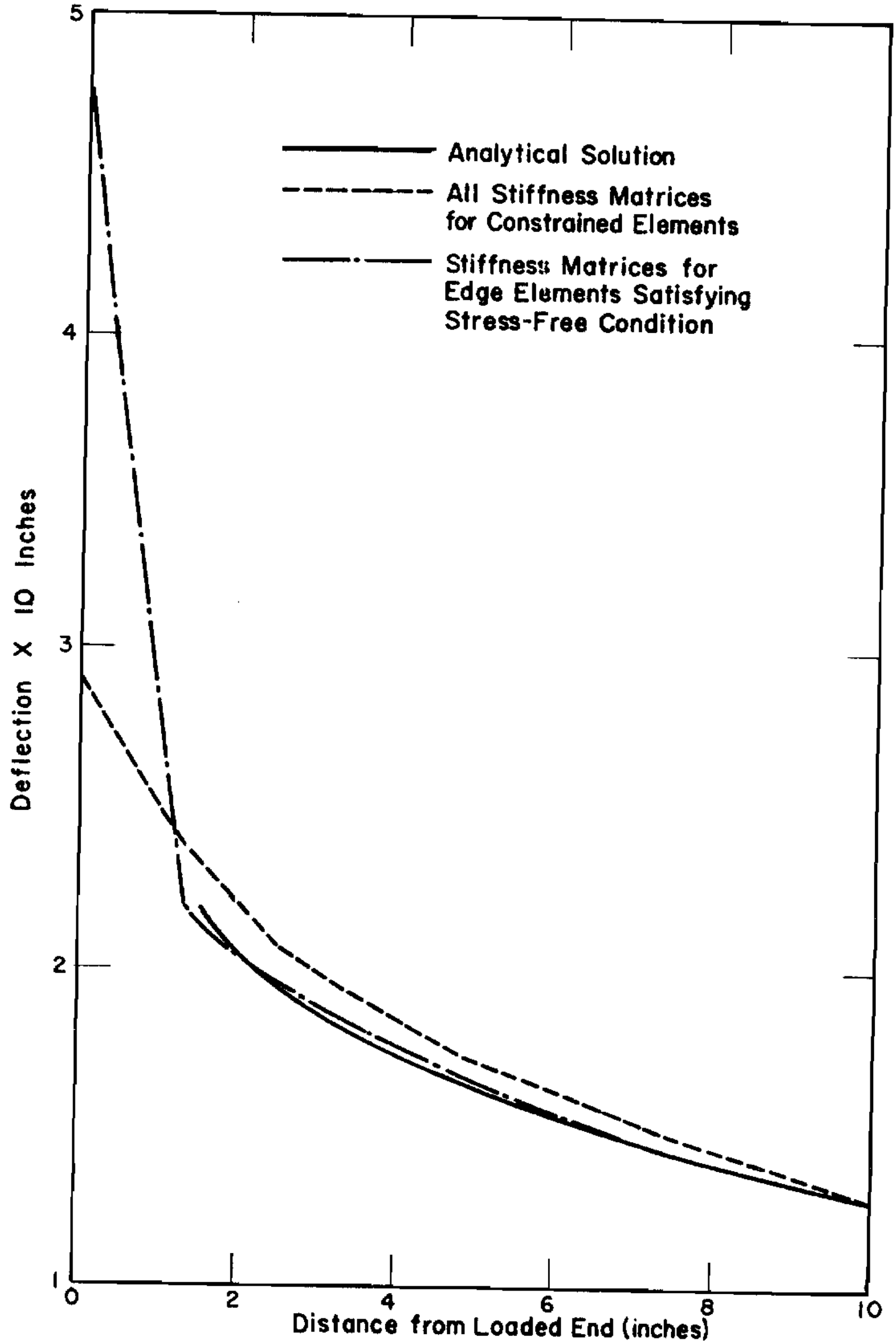

Figure 3. Deflection of Panel Centerline in Direction of Applied Load

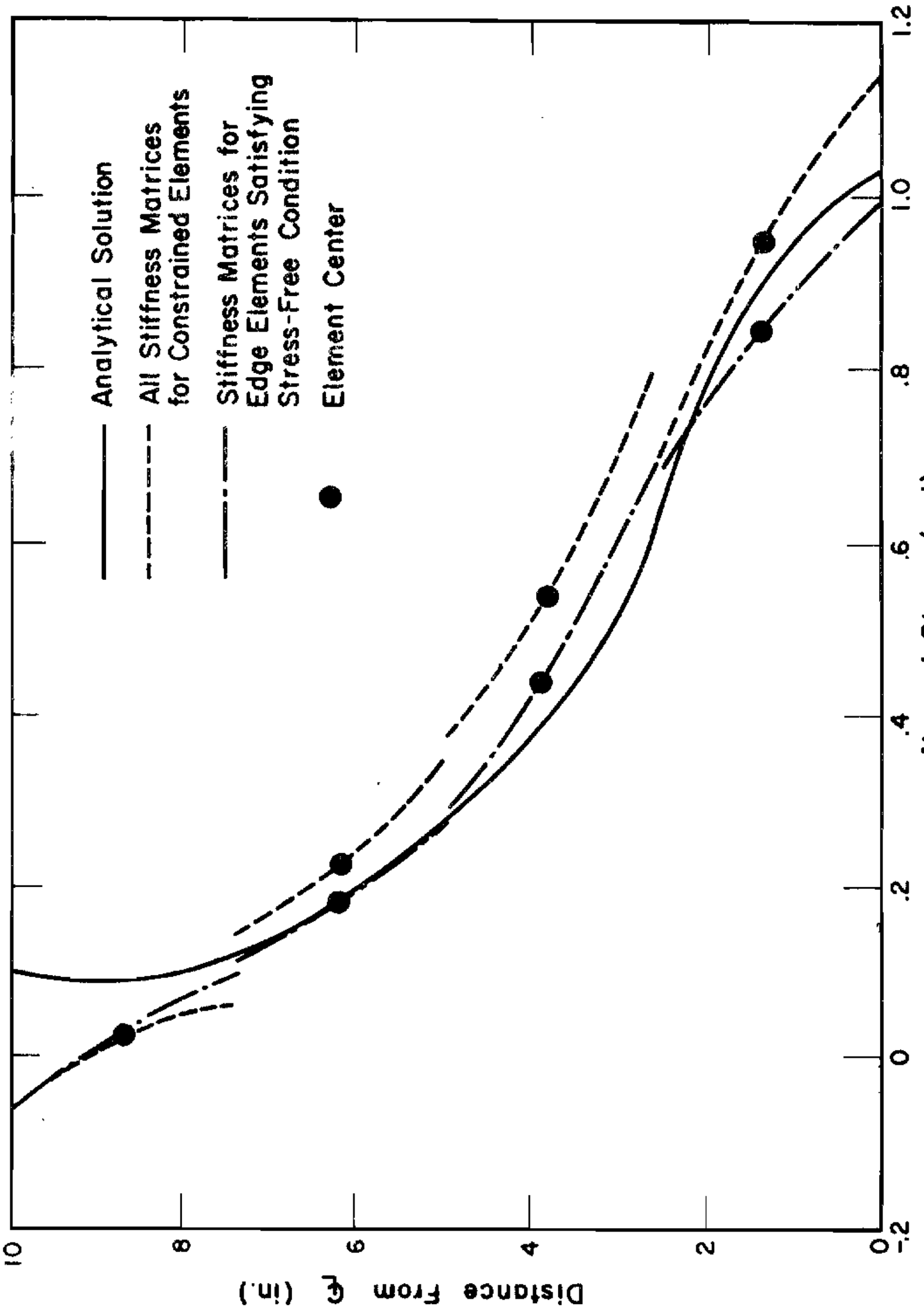

Figure 4. Normal Stress Across Quarter Panel at Section 5 Inches from Loaded End

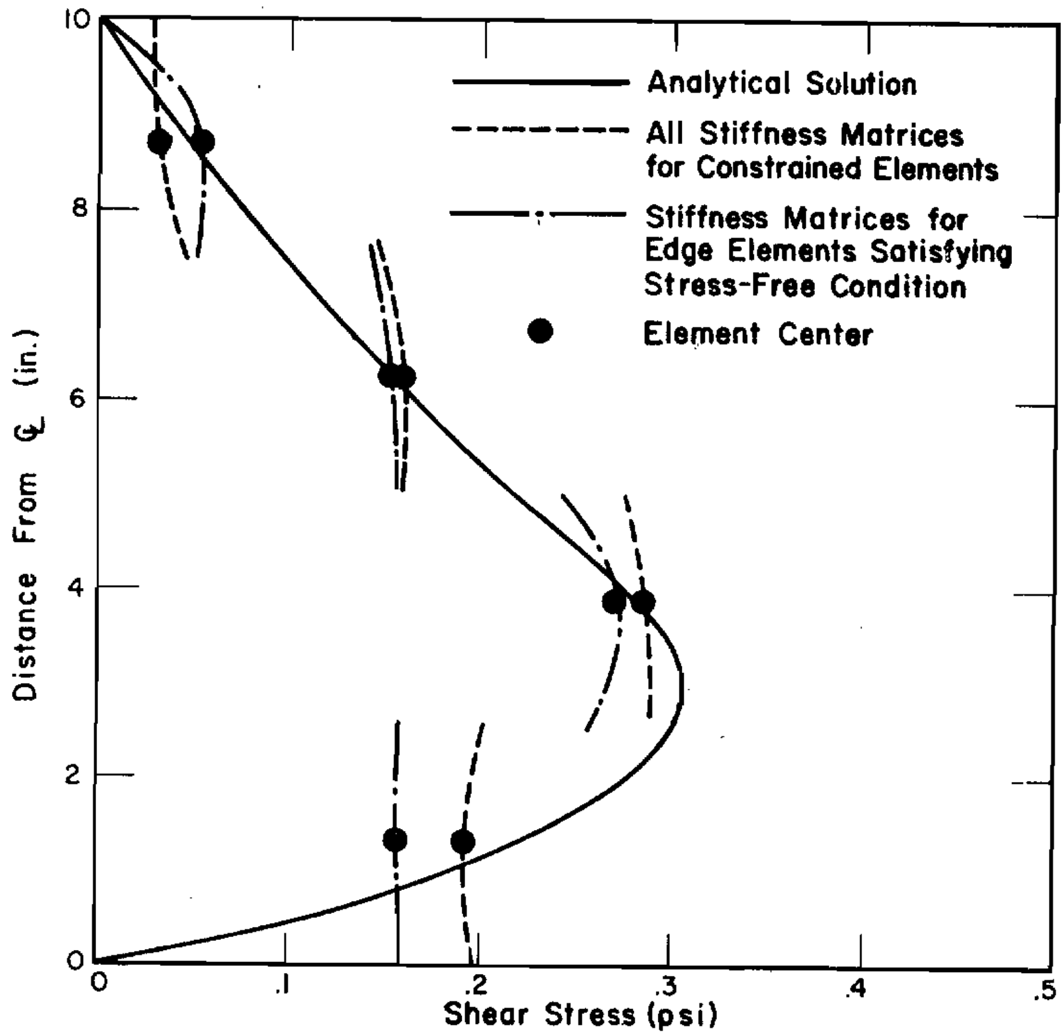

Figure 5. Shear Stress Across Quarter Panel at Section 5 Inches from Loaded End

By inspection of Figure 5 one can discover immediately that the shear-stress distribution violates the condition of symmetry at the centerline. Thus, to make further improvement in the finite-element analysis for this problem the stiffness matrices for all elements along the centerline of the panel should be derived based on shear-stress free conditions along the lower edges.

BENDING OF RECTANGULAR PLATE ELEMENT

The next structural element under consideration is a rectangular plate of dimension a x b, under transverse shear force and bending and twisting moments. Figure 6 shows the twelve generalized forces at the four corners

$$\{Q\} = \{F_1, F_2, \cdots - F_{12}\} \tag{24}$$

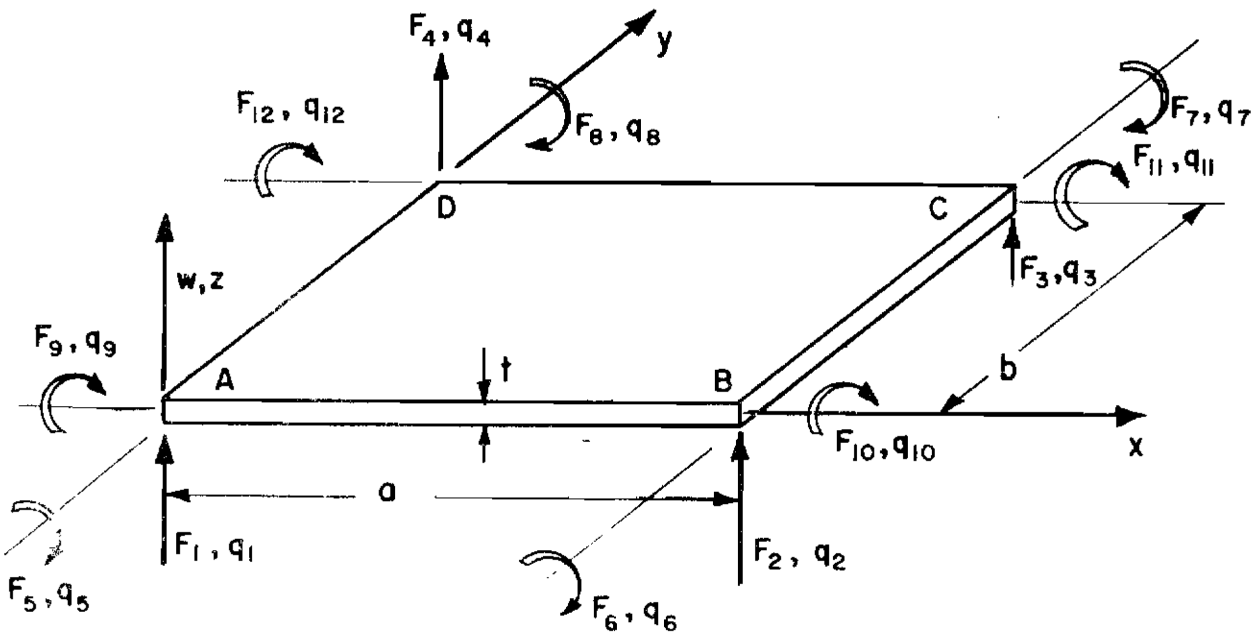

Figure 6. Generalized Forces and Displacements of a Rectangular Plate Element Under Bending

and the corresponding generalized displacements

$$\{q\} = \{ w_A, w_B, w_C, w_D, -w_{x_A}, -w_{x_B}, -w_{x_C}, -w_{x_D}, -w_{y_A}, -w_{y_B}, -w_{y_C}, -w_{y_D} \} \tag{25}$$

where w is the deflection of the plate and the subscripts x and y represent derivatives with respect to x and y, respectively. It is seen that the positive w, negative $\partial w/\partial x$ and negative $\partial w/\partial y$ are considered as the positive directions of the boundary forces.

The stiffness matrices for such elements are obtained again by using various numbers of terms in the stress functions. For this problem, although there are five stress resultants and couples (Figure 7), i.e., the transverse shear forces Q_x and Q_y, the bending moments M_x and M_y and the twisting moment M_{xy}, under simple plate theory, only the last three enter into the expression for the strain energy. The transverse shear forces, of course, appear in the boundary forces. They are, however, related to the moment distribution by the following equations of equilibrium (Reference 10)

$$Q_x = \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y}$$

$$Q_y = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} \tag{26}$$

The third equation of equilibrium is

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 0 \tag{27}$$

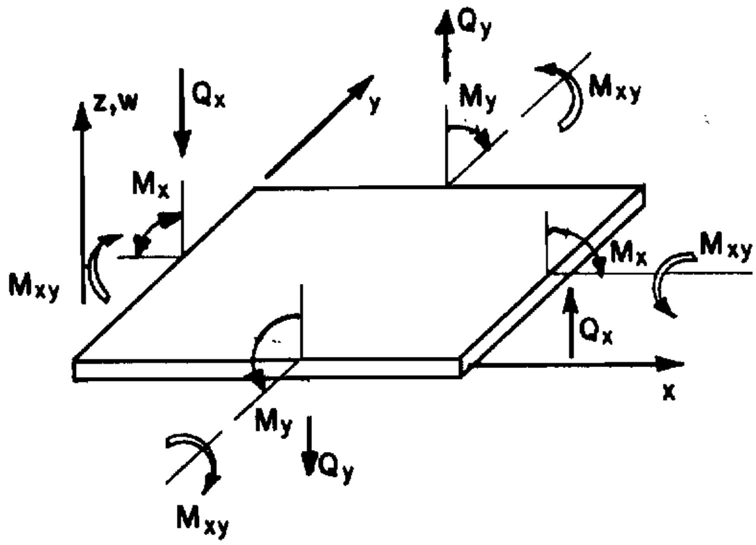

Figure 7. Stress Resultants and Couples in a Plate

which leads to the following equilibrium condition for bending and twisting moments

$$\frac{\partial^2 M_x}{\partial x^2} + 2\,\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} = 0 \tag{28}$$

To satisfy these equations, the following expressions for the bending and twisting moments are assumed.

$$\begin{aligned}
M_x &= \beta_1 + \beta_4 y + \beta_6 x + \beta_{10} y^2 + \beta_{12} x^2 + \beta_{14} xy + \beta_{16} y^3 + \beta_{18} x^3 \\
&\quad + \beta_{20} x^2 y + \beta_{22} xy^2 \\
M_y &= \beta_2 + \beta_5 x + \beta_7 y + \beta_{11} x^2 + \beta_{13} y^2 + \beta_{15} xy + \beta_{17} x^3 + \beta_{19} y^3 \\
&\quad + \beta_{21} x^2 y + \beta_{23} xy^2 \\
M_{xy} &= \beta_3 + \beta_8 y + \beta_9 x + (\beta_{12} + \beta_{13}) xy + \frac{1}{2}(3\beta_{18} + \beta_{23}) x^2 y + \frac{1}{2}(3\beta_{19} + \beta_{20}) xy^2
\end{aligned} \tag{29}$$

The corresponding **P** matrix is given in Table 2. The transverse shear forces Q_x and Q_y, obtained by Equation 24, are as follows:

$$\begin{aligned}
Q_x &= \beta_6 - \beta_8 + (\beta_{12} - \beta_{13}) x + \beta_{14} y + \frac{1}{2}(3\beta_{18} - \beta_{23}) x^2 \\
&\quad + (\beta_{20} - 3\beta_{19}) xy + \beta_{22} y^2 \\
Q_y &= \beta_7 - \beta_9 + (\beta_{13} - \beta_{12}) y + \beta_{15} x + \frac{1}{2}(3\beta_{19} - \beta_{20}) \\
&\quad + (\beta_{23} - 3\beta_{18}) xy + \beta_{21} x^2
\end{aligned} \tag{30}$$

Thus, when all cubic terms are included in the moments, a total of 23 β's are required. However, to derive the stiffness matrix for an element with stress-free conditions along the edges, many of the β's may be eliminated.

In deriving the relation between the boundary force matrix and the stress distribution in the interior, it should be recognized that the distribution of boundary twisting moment M_{xy} is statically equivalent to a distribution of boundary shear forces V_x and V_y which are related to Q_x, Q_y and M_{xy} (Reference 10, page 81). For example, along the edges AB and CD,

$$V_y = Q_y - \partial M_{xy} / \partial x \tag{31}$$

while along BC and AD,

$$V_x = Q_x - \partial M_{xy} / \partial y \tag{32}$$

The boundary force matrix is then

$$\mathbf{S} = \left\{ -V_{y_{AB}}, -M_{y_{AB}}, V_{x_{BC}}, M_{x_{BC}}, V_{y_{DC}}, M_{y_{DC}}, -V_{x_{AD}}, -M_{x_{AD}} \right\} \tag{33}$$

and the **R** matrix is shown in Table 2.

It should be remembered that the replacement of twisting moments by equivalent shear forces also introduces concentrated forces at the corners. The magnitude of the concentrated corner force is equal to twice the magnitude of M_{xy} at the corresponding corner. Since the concentrated corner forces are not included in the boundary forces **S**, they should be added to the integrals given by Equation 11 to form the corrected expressions of **T** and $\mathbf{T}^T$ matrices. The increments of the corner forces are linearly related to the β's as follows:

$$\begin{aligned}
\Delta F_1 &= 2\beta_3 \\
\Delta F_2 &= -2\beta_3 - 2a\beta_9 \\
\Delta F_3 &= 2\beta_3 + 2b\beta_8 + 2a\beta_9 + 2ab(\beta_{12} + \beta_{13}) \\
&\quad + a^2 b(3\beta_{18} + \beta_{23}) + ab^2(3\beta_{19} + \beta_{20}) \\
\Delta F_4 &= -2\beta_3 - 2b\beta_8
\end{aligned} \tag{34}$$

To obtain the boundary displacements **u** in terms of the generalized coordinates at the nodes, one can make use of the Lagrangian-Hermite's interpolation formula (Reference 11). Along a straight edge between two coordinates $s_1 = 0$ and $s_2 = \ell$, the lateral displacement w(s) can be written as

$$w(s) = h_1(s)\, w(s_1) + h_2(s)\, w(s_2) + \bar{h}_1(s)\, w'(s_1) + \bar{h}_2(s)\, w'(s_2) \tag{35}$$

where w' represents the derivative of w with respect to s and,

$$\begin{aligned}
h_1(s) &= 1 - 3\left(\frac{s}{\ell}\right)^2 + 2\left(\frac{s}{\ell}\right)^3 \\
h_2(s) &= 3\left(\frac{s}{\ell}\right)^2 - 2\left(\frac{s}{\ell}\right)^3 \\
\bar{h}_1(s) &= \ell\left[\frac{s}{\ell} - 2\left(\frac{s}{\ell}\right)^2 + \left(\frac{s}{\ell}\right)^3\right] \\
\bar{h}_2(s) &= -\ell\left[\left(\frac{s}{\ell}\right)^2 - \left(\frac{s}{\ell}\right)^3\right]
\end{aligned} \tag{36}$$

The slope of w with respect to the normal direction can be expressed simply as,

$$\frac{\partial w}{\partial n} = \left(1 - \frac{s}{\ell}\right)\left(\frac{\partial w}{\partial n}\right)_{s_1} + \frac{s}{\ell}\left(\frac{\partial w}{\partial n}\right)_{s_2} \tag{37}$$

It should be noted that the displacement function given by Equation 35 and 36 is identical to that given by Schmit (Reference 5) while the variation of slope given by Equation 37 is different from Schmit's expression. The boundary displacement matrix contains

$$\mathbf{u} = \left\{ w_{AB},\ -w_{y_{AB}},\ w_{BC},\ -w_{x_{BC}},\ w_{DC},\ -w_{y_{DC}},\ w_{AD},\ -w_{x_{AD}} \right\} \tag{38}$$

It is related to the twelve generalized coordinates **q** by the **L** matrix shown in Table 2.

The stress-strain relation matrix **N** is

$$\mathbf{N} = \frac{12}{Eh^3}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \tag{39}$$

Again with **P**, **N**, **R** and **L** given the element stiffness matrix can be determined.

Element stiffness matrices of a completely constrained square plate in bending have been evaluated using different number of β's for the stress function. The resulting matrices again tend to become more stiff when more β's are used as can be seen by comparing the diagonal elements of the 12 x 12 matrices obtained by using different number of β's as shown in Table 3. Element stiffness matrices have also been evaluated for square plates which have one or two edges under bending-moment free condition.

These element stiffness matrices have been used to calculate the deflection of a square plate under a central concentrated load. Both clamped and simply-supported boundary conditions are considered. For the case of a clamped plate the stiffness matrices for all elements are those derived for completely constrained edges. For the case of simply-supported plate, the stiffness matrices for the elements along the edges are derived for the corresponding boundary-stress conditions. In order to find out the improvement of result due to the use of the special stiffness matrices for the edge elements, the simply-supported case has also been analyzed by using the stiffness matrix for completely constrained edges for all elements. Table 4 presents a comparison of the computed central deflections and the exact solutions given in the Timoshenko test (Reference 10). Also listed in the table are the results given by Tocher and Kapur in Reference 4. The stiffness matrices used to obtain the results in that reference do not satisfy the slope continuity along adjacent edges of the plate elements.

Table 4 indicates numerical convergence for all different types of matrices used. However, even when the stiffness matrices are carefully derived by satisfying slope continuity at the edges there is still a lack of monotonic convergence in some cases. Furthermore, it appears that when the stiffness matrices are derived using fewer β's the convergence approaches from "above" but when they are derived using up to quadratic and cubic terms in the stress function, the convergence approaches from "below". Thus, it would be difficult to establish whether a given computed result by finite element approach is the upper bound or the lower bound. Finally, it appears to be a general tendency that when the stiffness matrices are derived by using more β terms in the stress function the solutions will converge more rapidly to the exact solution.

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TABLE 1

MATRICES USED FOR EVALUATION OF STIFFNESS
MATRIX OF RECTANGULAR PANELS IN PLANE-STRESS CONDITIONS

(1) $\boldsymbol{\sigma} = \{\sigma_x, \sigma_y, \tau\} = \mathbf{P}\ \{\beta_1, \beta_2, \ldots, \beta_{18}\}$

where $\mathbf{P}$ is as follows:

$$\begin{bmatrix} 1 & y & 0 & 0 & 0 & x & 0 & y^2 & 0 & x^2 & xy & 0 & xy^2 & y^3 & 0 & x^2y & x^3/3 & 0 \\ 0 & 0 & 1 & x & 0 & 0 & y & 0 & x^2 & y^2 & 0 & xy & 0 & 0 & x^3 & y^3/3 & xy^2 & x^2y \\ 0 & 0 & 0 & 0 & 1 & -y & -x & 0 & 0 & -2xy & -y^2/2 & -x^2/2 & -y^3/3 & 0 & 0 & -xy^2 & -x^2y & -x^3/3 \end{bmatrix}$$

(2) $\mathbf{S} = \{-\tau_{AB}, -\sigma_{y_{AB}}, \sigma_{x_{BC}}, \tau_{BC}, \tau_{DC}, \sigma_{y_{DC}}, -\sigma_{x_{AD}}, -\tau_{AD}\} = \mathbf{R}\ \{\beta_1, \beta_2, \ldots, \beta_{18}\}$

where $\mathbf{R}$ is as follows:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & x & 0 & 0 & 0 & 0 & x^2/2 & 0 & 0 & 0 & 0 & 0 & x^3/3 \\ 0 & 0 & -1 & -x & 0 & 0 & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 & -x^3 & 0 & 0 & 0 \\ 1 & y & 0 & 0 & 0 & a & 0 & y^2 & 0 & a^2 & ay & 0 & ay^2 & y^3 & 0 & a^2y & a^3/3 & 0 \\ 0 & 0 & 0 & 0 & 1 & -y & -a & 0 & 0 & -2ay & -y^2/2 & -a^2/2 & -y^3/2 & 0 & 0 & -ay^2 & -a^2/y & -a^3/3 \\ 0 & 0 & 0 & 0 & 1 & -b & -x & 0 & 0 & -2bx & -b^2/2 & -x^2/2 & -b^3/3 & 0 & 0 & -b^2x & -bx^2 & -x^3/3 \\ 0 & 0 & 1 & x & 0 & 0 & b & 0 & x^2 & b^2 & 0 & bx & 0 & 0 & x^3 & b^3/3 & b^2x & bx^2 \\ -1 & -y & 0 & 0 & 0 & 0 & 0 & -y^2 & 0 & 0 & 0 & 0 & 0 & -y^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & y & 0 & 0 & 0 & 0 & y^2/2 & 0 & y^3/3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

TABLE 1 (Con't)

(3) $u = \{ u_{AB}, v_{AB}, u_{BC}, v_{BC}, u_{DC}, v_{DC}, u_{AD}, v_{AD} \} = L \{ q_1, q_2, \ldots, q_8 \}$

where L is given by:

$$
\begin{bmatrix}
1-x/a & x/a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1-x/a & x/a & 0 & 0 \\
0 & 1-y/b & y/b & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1-y/b & y/b & 0 \\
0 & 0 & x/a & 1-x/a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x/a & 1-x/a \\
1-y/b & 0 & 0 & y/b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1-y/b & 0 & 0 & y/b
\end{bmatrix}
$$

TABLE 2

MATRICES USED FOR EVALUATION OF STIFFNESS MATRIX OF RECTANGULAR PLATES IN BENDING

(1) $\sigma = \{M_x, M_y, M_{xy}\} = P\,\{\beta_1, \beta_2, \ldots, \beta_{23}\}$

where P is as follows:

$$P = \begin{bmatrix} 1 & 0 & 0 & y & 0 & x & 0 & 0 & 0 & y^2 & 0 & x^2 & 0 & xy & 0 & y^3 & 0 & x^3 & 0 & x^2y & 0 & xy^2 & 0 \\ 0 & 1 & 0 & 0 & x & 0 & y & 0 & 0 & 0 & x^2 & 0 & y^2 & 0 & xy & 0 & x^2 & 0 & y^3 & 0 & x^2y & 0 & xy^2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & y & x & 0 & 0 & xy & xy & 0 & 0 & 0 & 0 & \frac{3}{2}x^2y & \frac{3}{2}xy^2 & \frac{1}{2}xy^2 & 0 & 0 & \frac{1}{2}x^2y \end{bmatrix}$$

(2) $S = \{-V_{y_{AB}}, -M_{y_{AB}}, V_{x_{BC}}, M_{x_{BC}}, V_{y_{DC}}, M_{y_{DC}}, -V_{x_{AD}}, -M_{x_{AD}}\} = R\,\{\beta_1, \ldots, \beta_{23}\}$

where R is as follows:

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 0 & -x^2 & 0 & 0 \\ 0 & -1 & 0 & 0 & -x & 0 & 0 & 0 & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 & -x^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & -2a & y & 0 & 0 & 0 & 0 & -6ay & 0 & 0 & y^2 & -a^2 \\ 1 & 0 & 0 & y & 0 & a & 0 & 0 & 0 & y^2 & 0 & a^2 & 0 & ay & 0 & y^3 & 0 & a^3 & 0 & a^2y & 0 & ay^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & -2b & 0 & 0 & x & 0 & 0 & -6bx & 0 & -b^2 & x^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & x & 0 & b & 0 & 0 & 0 & x^2 & 0 & b^2 & 0 & bx & 0 & x^3 & 0 & b^3 & 0 & bx^2 & 0 & b^2x \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y^2 & 0 \\ -1 & 0 & 0 & -y & 0 & 0 & 0 & 0 & 0 & -y^2 & 0 & 0 & 0 & 0 & 0 & -y^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

TABLE 2 (Con't)

(3) $u = \{ w_{AB}, -w_{y_{AB}}, w_{BC}, -w_{x_{BC}}, w_{DC}, -w_{y_{DC}}, w_{AD}, -w_{x_{AD}} \}$

$= L \{ q_1 \ldots q_{12} \}$

where L is given by

$$
\begin{bmatrix}
1-3\xi^2+2\xi^3 & 3\xi^2-2\xi^3 & 0 & 0 & a(-\xi+2\xi^2-\xi^3) & a(\xi^2-\xi^3) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\xi & \xi & 0 & 0 \\
0 & 1-3\eta^2+2\eta^3 & 3\eta^2-2\eta^3 & 0 & 0 & 0 & 0 & 0 & 0 & b(-\eta+2\eta^2-\eta^3) & b(\eta^2-\eta^3) & 0 \\
0 & 0 & 0 & 0 & 0 & 1-\eta & \eta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3\xi^2-2\xi^3 & 1-3\xi^2+2\xi^3 & 0 & 0 & a(\xi^2-\xi^3) & a(-\xi+2\xi^2-\xi^3) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi & 1-\xi \\
1-3\eta^2+2\eta^3 & 0 & 0 & 3\eta^2-2\eta^3 & 0 & 0 & 0 & 0 & b(-\eta+2\eta^2-\eta^3) & 0 & 0 & b(\eta^2-\eta^3) \\
0 & 0 & 0 & 0 & 1-\eta & 0 & 0 & \eta & 0 & 0 & 0 & 0
\end{bmatrix}
$$

where $\xi = x/a$

$\eta = y/b$

TABLE 3

DIAGONAL ELEMENTS OF STIFFNESS MATRICES OF A COMPLETELY CONSTRAINED SQUARE PLATE IN BENDING

Plate thickness = t

Plate span = a

Modulus = E

Poisson's ratio = 0.3

	Up to All Linear terms (9 β's)	Up to All Quadratic terms (15 β's)	Up to all Cubic terms (23 β's)
$k_{11} = k_{22} = k_{33} = k_{44}$	$0.67766 \frac{Et^3}{a^2}$	$1.02051 \frac{Et^3}{a^2}$	$1.05744 \frac{Et^3}{a^2}$
$k_{55} = k_{66} = k_{77} = k_{88} = k_{99} = k_{1010} = k_{1111} = k_{1212}$	$0.12210 \frac{Et^3}{a^2}$	$0.14866 \frac{Et^3}{a^2}$	$0.15752 \frac{Et^3}{a^2}$

TABLE 4

COMPUTED COEFFICIENTS α FOR CENTRAL DEFLECTION OF A SQUARE PLATE DUE TO A CONCENTRATED LOAD FOR SEVERAL MESH-SIZE AND DIFFERENT STIFFNESS MATRICES

$$w_{max} = \alpha P a^2 / D$$

Simply Supported Plate

Mesh Size	Total No. of Nodes	Stiffness Matrices Compatible with Boundary Stress Conditions — Stress Function			All Stiffness Matrices for Re-strained Edges	Solution by Tocher and Kapur (Ref. 3)
		up to β_9	up to β_{15}	up to β_{23}	Stress Function up to β_{23}	
2x2	9	.02188	.01279	.01129	.01022	.01378
4x4	25	.01191	.01142	.01133	.01125	.01233
8x8	81	.01169	.01156	.01153	.01151	.01183
12x12	169	.01164	.01157	.01159	.01159	.01172
16x16	289	--	--	--	--	.01167

Exact (Timoshenko) .01160

Clamped Plate

Mesh Size	Total No. of Nodes	All Stiffness Matrices for Restrained Edges — Stress Function			Solution by Tocher and Kapur (Ref. 3)
		up to β_9	up to β_{15}	up to β_{23}	
2x2	9	.008446	.005608	.005410	.005919
4x4	25	.005806	.005375	.005276	.006134
8x8	81	.005689	.005556	.005514	.005803
12x12	169	.005650	.005587	.005572	.005710
16x16	289	.005633	.005593	.005601	.005672

Exact (Timoshenko) .00560

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