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PERIODIC STRUCTURES ON CURVED SURFACES.

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ABSTRACT

The present report extends the theory developed for plane periodic structures to cylindrical structures having an azimuthal periodicity. The main object of this report is obtaining k - v diagrams (where v is the complex azimuthal propagation constant).

Since the cylindrical structures considered in this report have azimuthal periodicity, the fields can be expanded, in accordance with Floquet's theorem, in space harmonics.

Two particular structures are considered: a) the curved corrugated surface and b) the curved periodic slotted conductors. For a) the characteristic equation for v is obtained by equating appropriate energies on the surface of the structure; for b), the characteristic equation is obtained by using the transverse resonance condition.

An approximate solution for v is found for structure a). In this case, a perturbation technique permits obtaining the real and imaginary part of the azimuthal propagation constant for the slow region and for the n = - 1 leaky wave region.

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I. Introduction

The properties for plane periodic structures have been studied extensively by many workers. The dispersion curves for closed or open plane periodic structures are often presented by means of k - β diagram.[1,2,3,4,5,6]

The problem of finding the similar k - ν diagram (when ν is the complex azimuthal propagation constant) of periodic structures on curved surfaces has not received much attention in literature yet. In 1953, Barlow and Cullen[7] pointed out the effects of Bends in their paper about surface waves. A physical interpretation of the complex ν based on an increasing separation between adjacent equiphase planes with increasing distance from the surface was given in their paper. When the radial distance increases a point will be reached at which the separation becomes greater than $\frac{\lambda}{2}$. Then the evanescent character of the field disappears, and the energy is no longer trapped in a surface wave mode but is partly radiated outward. Therefore, ν becomes complex. The condition for the critical radius when the waves are still trapped in the surface in their paper is exactly the condition we use for the unperturbated value of ν, which in turn, gives more rigourous solution for ν by perturbation technique.

In 1955, Elliott[8] solved the problem of leaky azimuthal surface waves on corrugated cylinder for large kb, where kb is from 60 to infinity. For smaller kb, the complex propagation constant ν needs better approximation.

In this report, the perturbation technique is used to find the effect of curvature on propagation constant. The kb can be as small as 5 as long as the large argument approximation for the zero order Bessel function and Hankel function are valid.

The main purposes in this report are the k - Re(ν) and k - Im(ν) curves both in slow wave region and the n = - 1 leaky wave region for the curved periodic structures.

II. Statement of Problems

Two classes of periodic structure on curved surfaces are studied in this report:

1. A two dimensional corrugated surface on a cylinder of radius a as shown in Figure 1.

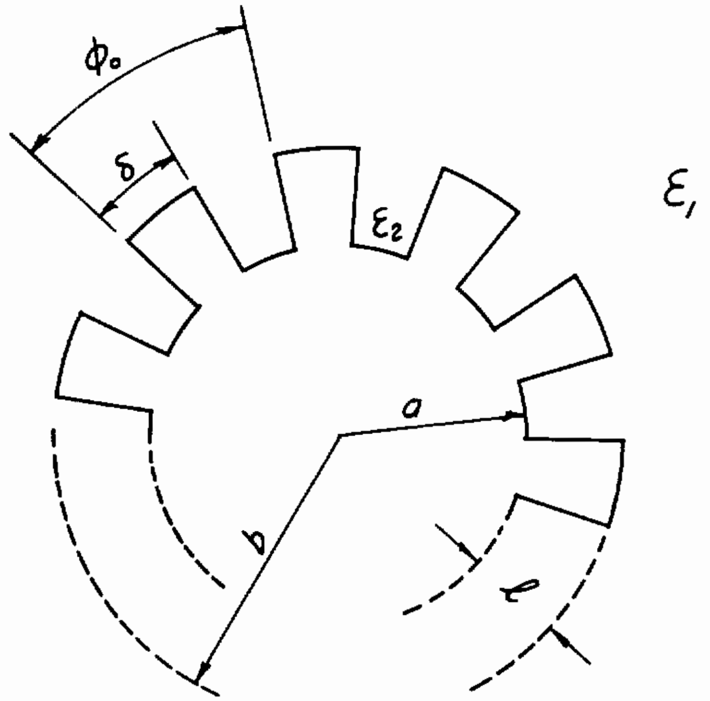

Figure 1. Curved Corrugated Surface.

where ϕ_0 is the periodicity of the curved corrugated surface in radian

δ is the angle of the slot width in radian

ℓ is the depth of the slot

$b = a + \ell$ is the outer radius of the structure

ε_1 is the dielectric constant in outside region where $\rho > b$

ε_2 is the dielectric constant in the slots region where $\rho < b$.

2. A two dimensional periodic slotted conductor over a cylinder of radius a as shown in Figure 2.

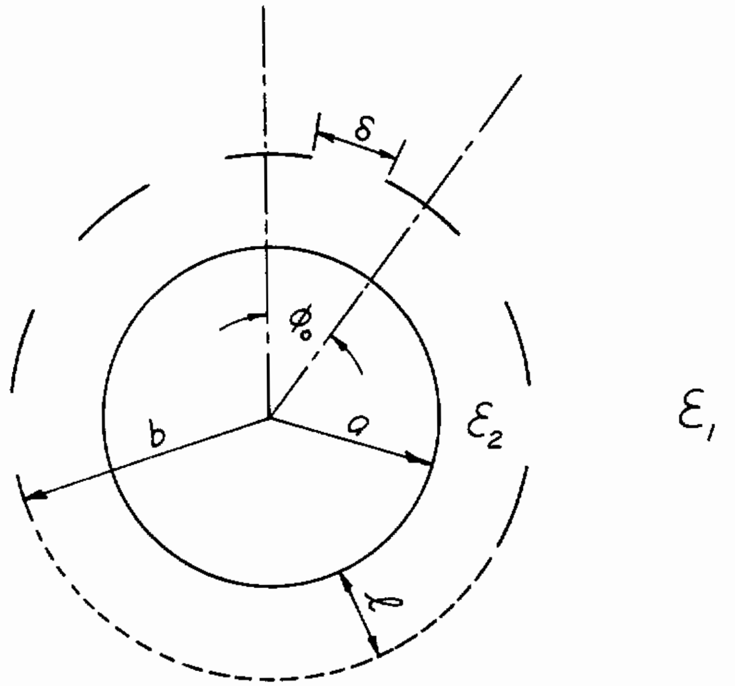

Figure 2. Curved Periodic Slotted Conductor.

where ϕ_0 is the periodicity in radian

δ is the angle of the slot width

ℓ is the distance between the cylinder and the slotted conductor

ε_1 is the dielectric constant in the outside region where $\rho > b$

ε_2 is the dielectric constant in the inside region where $\rho < b$.

III. The Characteristic Equations for the Complex Propagation Constant V_n

1. Curved Corrugated Surface

Since the structure is cylindrical, the fields satisfy the two-dimensional wave equation in cylindrical coordinates,

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + k^2 \right) H_z = 0 \qquad \text{(III - 1)}$$

where H_z is the z-component magnetic field if we confine ourselves in solving a TM type wave with respect to the direction of propagation, which is most common in the study of plane slow wave structure. H_z can be replaced by E_z if we study a TE type wave.

The boundary conditions at $\rho = b$;

(1) Tangential E - field is continuous over the slot.

$$E_{\phi_1} = E_{\phi_2} \qquad \text{(III - 2)}$$

(2) Tangential H - field is continuous over the slot.

(3) The total power from inside the slot must be equal to the power flowing outside per period,

$$\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} E_{\phi_2} \times H_z \, d\phi = \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} E_{\phi_1} \times H_{z_1} d\phi \qquad \text{(III - 3)}$$

where E_{ϕ_1}, H_{z_1} are fields in the outside region where $\rho > b$.

In the outside region $\rho > b$, because of the periodicity of the structure, we can apply the Floquet's theorem. The fields can be written in terms of azimuthal space harmonics;

$$H_{z_1} = \sum_n AnH_{Vn}^{(2)}(k_1\rho)\ e^{-jVn\phi}$$

$$E_{\phi_1} = \frac{-k_1}{j\omega\varepsilon_1} \sum_n AnH_{Vn}^{(2)'}(k_1\rho)\ e^{-jVn\phi} \qquad \text{(III - 4)}$$

where

$$Vn = V + \frac{2n\pi}{\phi_0}, \quad n = 0, \pm 1, \pm 2 \cdots\cdots$$

Vn is the azimuthal propagation constant for the n^{th} space harmonic with a fundamental complex propagation constant V.

An is the magnitude of the n^{th} space harmonics

$H_{Vn}^{(2)}(k_1\rho)$ is the Hankel function of second kind.

In the inside region $\rho < b$, each slot can be considered as a section of a wedge-shaped region shorted at $\rho = a$ and open at $\rho = b$ as shown in Figure 3.

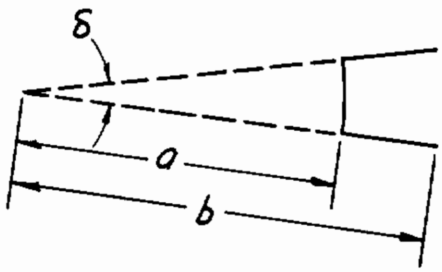

Figure 3. One slot of the curved corrugated surfaces.

The fields in this region inside the slot have to satisfy the wave equation with the following boundary conditions;

$$E_{\phi_2} = 0 \qquad \text{at } \rho = a$$

$$E_{\phi_2} = E_{\phi_1} \qquad \text{at } \rho = b$$

Then the fields are in the form of

$$H_{z_2} = C\left(H_p^{(2)'}(k_2a)\, J_p(k_2\rho) - J_p'(k_2a)\, H_p^{(2)}(k_2\rho)\right) \cos p\phi$$

$$E_{\phi_2} = \frac{-k_2}{j\omega\varepsilon_2}\, C\left(H_p^{(2)'}(k_2\rho)\, J_p'(k_2\rho) - J_p'(k_2a)\, H_p^{(2)'}(k_2\rho)\right) \cos p\phi \qquad \text{(III - 5)}$$

where

$$p = \frac{m\pi}{\delta} \qquad m = 0, 1, 2, \cdots\cdots$$

C is the unknown constant

$J_p(k_{2_\rho})$ is the Bessel function of first kind.

In order to evaluate the unknown constants An and C, we have to apply the boundary conditions (III - 2) and (III - 3) at $\rho = b$. Assume the slot field at $\rho = b$ to be

$$E_{\phi_1} = E_{\phi_2} = E_0 \cos p\phi \qquad \text{at } \rho = b \qquad \text{(III - 6)}$$

one can equate equations (III - 5) and (III - 6) and get

$$H_{z_2} = \frac{E_0}{jZ_2}\; \frac{H_p^{(2)'}(k_2a)\, J_p(k_2\rho) - J_p'(k_2a)\, H_p^{(2)}(k_2\rho)}{H_p^{(2)'}(k_2a)\, J_p'(k_2b) - J_p'(k_2a)\, H_p^{(2)'}(k_2b)} \cos p\phi$$

$$E_{\phi_2} = E_0\; \frac{H_p^{(2)'}(k_2a)\, J_p'(k_2\rho) - J_p'(k_2a)\, H_p^{(2)'}(k_2\rho)}{H_p^{(2)'}(k_2a)\, J_p'(k_2b) - J_p'(k_2a)\, H_p^{(2)'}(k_2b)} \cos p\phi$$

where

$$Z_2 = \sqrt{\frac{\mu}{\varepsilon}}$$

Similarly, one can find the expression for An by equating equations (III - 4) and (III - 6) and due to the orthogonality of the space harmonics,

$$A_n = \frac{E_0}{2jZ_1 H_{Vn}^{(2)'}(k_1 b)} \frac{\delta}{\phi_0} \left[\frac{\sin (Vn + p)\frac{\delta}{2}}{(Vn + p)\frac{\delta}{2}} + \frac{\sin (Vn - p)\frac{\delta}{2}}{(Vn - p)\frac{\delta}{2}} \right]$$

The fields can then be obtained by substituting An into equation (III - 4). By applying the third boundary condition (III - 3),

$$\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} E_0 \cos p\phi \, H_{z_2} d\phi = \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} E_0 \cos p\phi \cdot H_{z_1} d\phi \qquad \text{at } \rho = b$$

The characteristic equation for Vn can then be obtained,

$$\frac{Z_1}{Z_2} \cdot \frac{H_p^{(2)'}(k_2 a)\, J_p(k_2 b) - J_p'(k_2 a)\, H_p^{(2)}(k_2 b)}{H_p^{(2)'}(k_2 a)\, J_p'(k_2 b) - J_p'(k_2 a)\, H_p^{(2)'}(k_2 b)}$$

$$= \frac{\delta}{\phi_0} \sum_n \frac{H_{Vn}^{(2)}(k_1 b)}{H_{Vn}^{(2)'}(k_1 b)} \frac{1}{4} \left[\frac{\sin (Vn + p)\frac{\delta}{2}}{(Vn + p)\frac{\delta}{2}} + \frac{\sin (Vn - p)\frac{\delta}{2}}{(Vn - p)\frac{\delta}{2}} \right]^2$$

(III - 10)

As a special case, consider the case when $p = \frac{m\pi}{z} = 0$, only the fundamental mode m = 0 existing inside the slot. Under the condition that the width of the slot is narrow such that $\delta b < \frac{\lambda}{2}$, this special case holds. Then equation (III - 10) reduces to

$$\frac{Z_1}{Z_2}\ \frac{H_0^{(2)'}(k_2a)\ J_0(k_2b) - J_0'(k_2a)\ H_0^{(2)}(k_2b)}{H_0^{(2)'}(k_2a)\ J_0'(k_2b) - J_0'(k_2a)\ H_0^{(2)'}(k_2b)}$$

$$= \frac{\delta}{\phi_0} \sum_n \frac{H_{Vn}^{(2)}(k_1b)}{H_{Vn}^{(2)'}(k_1b)} \left(\frac{\sin Vn \frac{\delta}{2}}{Vn \frac{\delta}{2}} \right)^2 \qquad \text{(III - 11)}$$

we shall come to study this equation later on.

2. Curved Periodic Slotted Conductors

From the structure as shown in Figure 2, solving the cylindrical wave equation with the following boundary conditions;

(1) Tangential E - field vanishes on the conductor at $\rho = a$.

(2) Tangential E - field is continuous over the slots and vanishes on the strips of conductor at $\rho = b$.

(3) Tangential H - field is continuous over the slots at $\rho = b$.

One can write down the expressions for E and H fields in region one and two in terms of space harmonics;

$$H_{z_1} = \sum_n An\ H_{Vn}^{(2)}(k_1\rho)\ e^{-jVn\phi}$$

$$E_{\phi_1} = \frac{-k_1}{j\omega\varepsilon_1} \sum_n An\ H_{Vn}^{(2)'}(k_1\rho)\ e^{-jVn\phi} \quad , \qquad \rho > b$$

and

$$H_{z_2} = \sum_n Cn\left[H_{Vn}^{(2)'}(k_2a)\ J_{Vn}(k_2\rho) - J_{Vn}'(k_2a)\ H_{Vn}^{(2)}(k_2\rho)\right]\ e^{-jVn\phi}$$

$$E_{\phi_2} = \frac{-k_1}{j\omega\varepsilon_2} \sum_n Cn\left[H_{Vn}^{(2)'}(k_2a)\ J_{Vn}'(k_2\rho) - J_{Vn}'(k_2a)\ H_{Vn}^{(2)'}(k_2\rho)\right]\ e^{-jVn\phi} \ ,$$

$$a < \rho < b$$

Instead of matching the boundary condition to evaluate the unknown coefficients An and Cn, one can use the network approach.[9] Considering the case when only one mode propagating transversely in the ρ - direction, one can represent the field problem for a unit cell of the periodic structure (in this case the unit cell is shown as in Figure 4) by a transverse transmission line of characteristic impedance Y_1 and Y_2 with a shunt susceptance B which is due to the discontinuity of the slot at $\rho = b$. The network representation is shown in Figure 5.

By the so called transverse resonance condition at $\rho = b$, $Y_2 + j\,B = Y_1$ we have the characteristic equation for Vn,

$$\sum_n \frac{j\omega\varepsilon_2}{k_1} \frac{H_{Vn}^{(2)'}(k_2a)\, J_{Vn}(k_2b) - J_{Vn}'(k_2a)\, H_{Vn}^{(2)}(k_2b)}{H_{Vn}^{(2)'}(k_2a)\, J_{Vn}'(k_2b) - J_{Vn}'(k_2a)\, H_{Vn}^{(2)'}(k_2b)} + jB$$

$$= \sum_n \frac{j\omega\varepsilon_1}{k_1} \frac{H_{Vn}^{(2)}(k_1b)}{H_{Vn}^{(2)'}(k_1b)} \qquad \text{(III - 12)}$$

where the susceptance B can be found by solving a diffraction problem of the sectional diaphragm as shown in Figure 4, although it might have a little difficulty for the closed form expression. As long as ϕ_0 is small and radius b is reasonably large, we can approximate the sectional diaphragm by a two plate capacitive diaphragm. Then the susceptance due to the diaphragm at $\rho = b$, B, is given approximately by

$$B = \frac{2kb\phi_0\, Y_0}{\pi}\, \ell n\, \mathrm{Csc}\, \frac{\pi\delta}{2\phi_0} \qquad \text{(III - 13)}$$

where

$$Y_0 = \sqrt{\frac{\varepsilon_0}{\mu}}$$

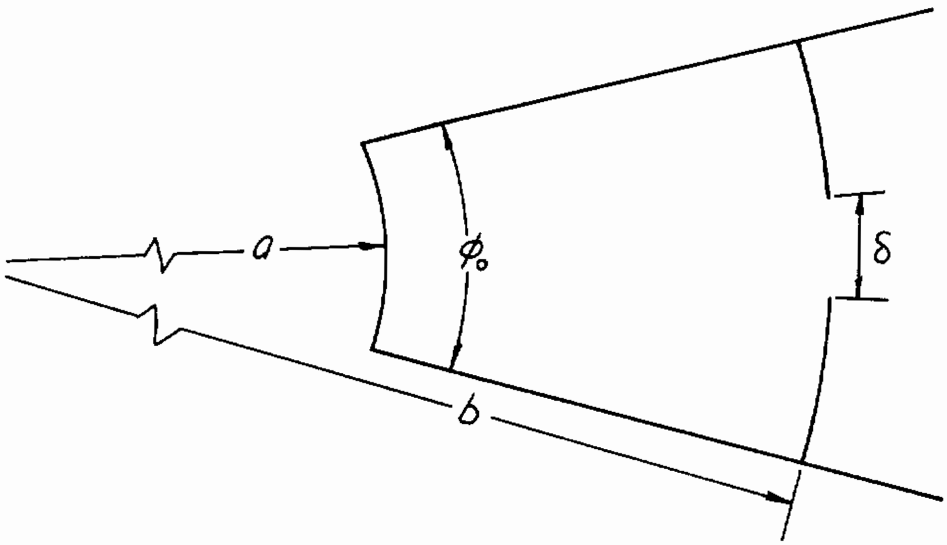

Figure 4. The unit cell of the curved periodic slotted conductor.

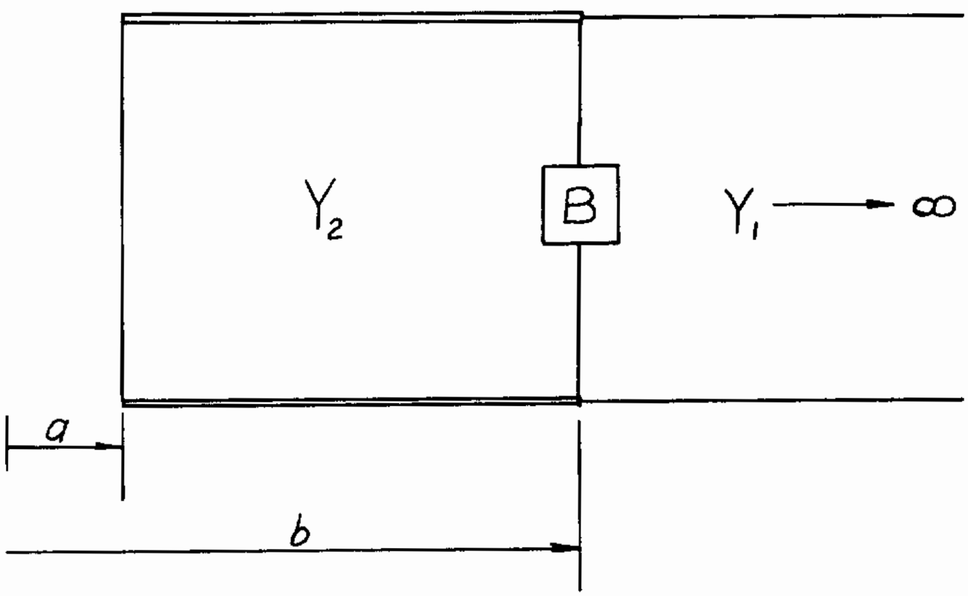

Figure 5. The transverse transmission line for the unit cell in Figure 4.

IV. The Approximate Vn for the Curved Corrugated Surfaces

Equations (III - 11) and (III - 12) are the characteristic equations of Vn. Just as the k - β diagram which is widely used in the plane periodic structure, so there are k - V curves for the curved periodic structures. Depending on the frequency range the structure is operating, the k - V curves will be different for a particular structure. Here we only consider the case of a curved corrugated surface of radius b in slow wave region and n = - 1 leaky wave region.

1. In the slow wave region

Equation (III - 11) can be simplified if kb is large enough that we can use the large argument asymptotic form for the Bessel function and Hankel function of zero order.

$$J_0(k_2b) = \sqrt{\frac{2}{\pi k_2 b}} \cos\left(k_2b - \frac{\pi}{4}\right)$$

$$J_0'(k_2b) = -\sqrt{\frac{2}{\pi k_2 b}} \sin\left(k_2b - \frac{\pi}{4}\right)$$

$$H_0^{(2)}(k_2b) = \sqrt{\frac{2}{\pi k_2 b}}\ e^{-j(k_2b - \frac{\pi}{4})}$$

$$H_0^{(2)'}(k_2b) = -j\sqrt{\frac{2}{\pi k_2 b}}\ e^{-j(k_2b - \frac{\pi}{4})}$$

Then equation (III - 11) becomes the following after some algebraic manipulation

$$\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \cot k_2 \ell = \frac{\delta}{\phi_0} \sum_n \left(- \frac{H_{Vn}^{(2)}(k_2 b)}{H_{Vn}^{(2)'}(k_1 b)} \right) \left(\frac{\sin Vn \frac{\delta}{2}}{Vn \frac{\delta}{2}} \right)^2$$

$$= \frac{\delta}{\phi_0} \sum_n Bn \qquad \text{(IV - 1)}$$

where

$$Bn = - \frac{H_{Vn}^{(2)}(k_1 b)}{H_{Vn}^{(2)'}(k_1 b)} \left(\frac{\sin Vn \frac{\delta}{2}}{Vn \frac{\delta}{2}} \right)^2$$

Since we are considering the case in slow wave region, we have to use the Debye's asymptotic form applicable for $k_1 b < |Vn|$ for Hankel function

$$H_{Vn}^{(2)}(k_1 b) = \sqrt{\frac{2}{\Pi k_1 b \sin \alpha n}} \; e^{-k_1 b(\sinh \alpha n - \alpha n \cosh \alpha n) + j \frac{\pi}{4}}$$

$$Vn = k_1 b \cosh \alpha n$$

And the derivative of Hankel function is

$$H_{Vn}^{(2)'}(k_1 b) = A \, H_{Vn}^{(2)}(k_1 b)$$

where

$$A = - \sinh \alpha n + \frac{1}{2 k_1 b \sin^2 \alpha n}$$

then Bn can be written as

$$Bn = \frac{1}{\sinh \alpha n - \dfrac{1}{2k_1 b \sinh^2 \alpha n}} \left(\frac{\sin Vn \frac{\delta}{2}}{Vn \frac{\delta}{2}} \right)^2 \qquad \text{(IV - 2)}$$

This simplified transcendental equation for Vn is still difficult to solve. A perturbation technique is used to find the approximate Vn. The justification of employing the perturbation technique will become clear later on.

Assume we can write Vn in the form of

$$Vn = V + \frac{2n\pi}{\phi_0}$$

$$= V_0 + dV + \frac{2n\pi}{\phi_0}$$

$$= Vn_0 + dV$$

where $Vn_0 = V_0 + \frac{2n\pi}{\phi_0}$ is real

dv is a small complex value which is independent of n.

let $dv = g_1 - jg_2$

Under the condition that $|Vn_0| >> |dv|$, one can expand equation (IV - 1) about the value Vn_0.

$$\sqrt{\frac{\epsilon_2}{\epsilon_1}} \cot k_2 \ell = \frac{\delta}{\phi_0} \sum_n Bn \Big|_{Vn_0} + \frac{\delta}{\phi_0} Bn' \Big|_{Vn_0} dV \qquad \text{(IV - 3)}$$

$$+ \frac{\delta}{2\phi_0} \sum_n Bn'' \Big|_{Vn_0} (dV)^2 + - - - - -$$

where

$$Bn' = \frac{dBn}{dVn}$$

$$Bn'' = \frac{d^2Bn}{dVn^2}$$

Substituting $dv = g_1 - jg_2$ into equation (IV - 3) and separating the real and imaginary parts, one has the following equations.

$$\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\ \frac{\phi_0}{\delta} \cot k_2\ell = \sum_n \left[Bn \Big|_{Vn_0} + g_1 Bn' \Big|_{Vn_0} + \frac{1}{2}(g_1{}^2 - g_2{}^2)\ Bn'' \Big|_{Vn_0} \right] \qquad \text{(IV - 4)}$$

$$0 = \sum_n Bn' \Big|_{Vn_0} + g_1^1 \sum_n Bn'' \Big|_{Vn_0} \qquad \text{(IV - 5)}$$

From equation (IV - 5) the real part of dv, g_1, can be found

$$g_1 = \frac{- \sum_n Bn' \Big|_{Vn_0}}{\sum_n Bn'' \Big|_{Vn_0}} \qquad \text{(IV - 6)}$$

And the imaginary part of dV, g_2, can be obtained by rearranging equation (IV - 4) in the following way,

$$\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\ \frac{\phi_0}{\delta} \cot k_2\ell = \sum_n \left\{ \frac{1}{\sinh \alpha n} \left(\frac{\sin Vn \frac{\delta}{2}}{Vn \frac{\delta}{2}} \right)^2 + \right.$$

$$\left. \left(\frac{1}{-A} - \frac{1}{\sinh \alpha n} \right) \left(\frac{\sin Vn \frac{\delta}{2}}{Vn \frac{\delta}{2}} \right) \Big|_{Vn_0} + g_1 Bn' \Big|_{Vn_0} + \frac{1}{2}(g_1{}^2 - g_2{}^2)\ Bn'' \Big|_{Vn_0} \right\} \qquad \text{(IV - 7)}$$

If we let Vn_0 be the solution for

$$\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\ \frac{\phi_0}{\delta} \cot k_2 \ell = \sum_n \frac{1}{\sinh \alpha n} \left(\frac{\sin Vn \frac{\delta}{2}}{Vn \frac{\delta}{2}} \right)^2 \qquad \text{(IV - 8)}$$

then equation (IV - 7) reduces to

$$0 = \sum_n Pn \Big|_{Vn_0} + g_1 \sum_n Bn' \Big|_{Vn_0} + \frac{1}{2} (g_1^2 - g_2^2) \sum_n Bn'' \Big|_{Vn_0} \qquad \text{(IV - 9)}$$

where

$$Pn = \left(\frac{1}{-A} - \frac{1}{\sinh \alpha n} \right) \left(\frac{\sin Vn \frac{\delta}{2}}{Vn \frac{\delta}{2}} \right)^2$$

and the expression for g_2 can be found by substituting equation (IV - 6) into (IV - 9),

$$g_2 = (-g_1) \sqrt{\frac{2 \sum_n Pn \Big|_{Vn_0}}{-g_1 \sum_n Bn' \Big|_{Vn_0}} - 1} \qquad \text{(IV - 10)}$$

So far we have the expressions for g_1 and g_2 under the assumption that $|Vn_0| >> |dV|$, where Vn_0 is the solution of equation (IV - 8).

It seems it is necessary to check the assumption and have some physical interpretation of Vn_0.

Recalling that $\sinh \alpha n = \sqrt{\left(\frac{Vn}{k_1 b} \right)^2 - 1}$,

one can see the similarity between equation (IV - 7) and the well known characteristic equation for βn, the space phase constant of a plane corrugated surface with the periodicity L, width of slot $\overline{W}$ and depth of slot ℓ,

$$\sqrt{\frac{\epsilon_2}{\epsilon_1}}\ \frac{L}{\overline{W}} \cot k_2 \ell = \sum_n \frac{1}{\sqrt{(\frac{\beta n}{k})^2 - 1}} \left(\frac{\sin \beta n \frac{\overline{W}}{2}}{\beta n \frac{\overline{W}}{2}} \right)^2 \qquad \text{(IV - 11)}$$

The only difference in equation (IV - 8) compared with equation (IV - 11) is its dependence on the radius of the structure. When b approaches infinite equation (IV - 8) can be reduced to the plane case. Therefore, Vn_0 , the unperturbated propagation constant, is the solution for a structure of large radius where

$$2k_1 b \sinh^3 \alpha n >> 1$$

The effect of a smaller radius gives the complex dV. So it is safe to use the assumption $|Vn_0| >> |dV|$ under which the perturbation technique was used, when the radius is reasonably large.

2. In the n = - 1 leaky wave region.

The characteristic equation for Vn is still the same form as the slow wave case which can be seen from equation (IV - 1). The differences are the conditions which give a leaky wave in n = - 1 space harmonic. Those conditions can be summarized as follows:

$$|Vn| = |V_0 - \frac{2n\pi}{\phi_0}| > k_1 b \qquad n \neq -1$$

$$|V_{-1}| = |V_0 - \frac{2\pi}{\phi_0}| < k_1 b \qquad \text{(IV - 13)}$$

Therefore, equation (IV - 1) can be rearranged, as follows

$$\sqrt{\frac{\epsilon_2}{\epsilon_1}} \cot k_2 \ell = \frac{\delta}{\phi_0} \sum_{n \neq -1} Dn + \frac{\delta}{\phi_0} D_{-1} \qquad \text{(IV - 14)}$$

where

$$Dn = \frac{-H_{Vn}^{(2)}(k_1 b)}{H_{Vn}^{(2)'}(k_1 b)} \left(\frac{\sin Vn \frac{\delta}{2}}{Vn \frac{\delta}{2}} \right)^2$$

$$D_{-1} = \frac{-H_{V-1}^{(2)}(k_1 b)}{H_{V-1}^{(2)'}(k_1 b)} \left(\frac{\sin V\text{-}1 \frac{\delta}{2}}{V\text{-}1 \frac{\delta}{2}} \right)^2$$

Because of the condition (IV - 13), we have to use Debye's asymptotic form for Hankel function in the region $Vn > k_1 b$ for all the space harmonics other than $n = -1$, and Debye's asymptotic form in the region $V_1 < k_1 b$ for the $n = -1$ space harmonic. Then Dn and D_{-1} become

$$Dn = \frac{1}{\sin \alpha n - \dfrac{1}{2k_1 b \sinh^2 \alpha n}} \left(\frac{\sin Vn \frac{\delta}{2}}{Vn \frac{\delta}{2}} \right)^2$$

$$D_{-1} = \frac{1}{j \sin \alpha_{-1} + \dfrac{1}{2k_1 b \ \sin^2 \alpha_{-1}}} \left(\frac{\sin Vn \frac{\delta}{2}}{Vn \frac{\delta}{2}} \right)^2 \qquad \text{(IV - 15)}$$

Similar to the case of slow wave, if the condition that $|Vn_0| >> |dV|$, we can assume $Vn = Vn_0 + dV$ and use perturbation technique. Let $dV = t_1 + jt_2$ and expand equation (IV - 14) about Vn_0,

$$\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \frac{\phi_0}{\delta} \cot k_2 \ell \cong \left(\sum_{n \neq -1} Dn + D_{-1} \right) \Big|_{Vn_0} + \left(\sum_{n \neq -1} Dn' + D_{-1}' \right) \Big|_{Vn_0} (t_1 + jt_1)$$

(IV - 16)

Use a short hand notation $Dn\big|_{Vn_0} = Dn_0$ $\quad D_{-1}\big|_{Vn_0} = D_{-10}$. Since

$D_{-10} = D_{-10r} + j\, D_{-10i}$ where D_{-10r} is the real part of D_{-10} and D_{-10i}

is the imaginary part of D_{-10}, one can separate the real part and imaginary part of equation (IV - 16).

$$\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\frac{\phi_0}{\delta} \cot k_2 \ell = \sum_{n\neq -1} Dn_0 + d_{-10r} + (D_{-10r}' + \sum_{n\neq -1} Dn_0')\, t_1 - D_{-10i}'\, t_2 \qquad \text{(IV - 17)}$$

$$0 = D_{-10i} + D_{-10i}' + (D_{-10r}' + \sum_{n\neq -1} Dn_o')\, t_2 \qquad \text{(IV - 18)}$$

If we let Vn_0 be the solution of

$$\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\ \frac{\phi_0}{\delta} \cot k_2 \ell = \sum_{n\neq -1} Dn_0 \qquad \text{(IV - 19)}$$

then equations (IV - 17) and (IV - 18) become

$$0 = D_{-10r} + (D_{-10r}' + \sum_{n\neq -1} Dn_o')\, t_1 - D_{-10i}'\, t_2 \qquad \text{(IV - 20)}$$

$$0 = D_{-10i} + D_{-10i}'\, t_1 + (D_{-10r}' + \sum_{n\neq -1} Dn_o')\, t_2 \qquad \text{(IV - 21)}$$

The simultaneous solutions for t_1 and t_2 can be solved.

$$t_1 = \frac{\begin{vmatrix} -D_{-10r} & -D_{-10i}' \\ -D_{-10i} & (D_{-10r}' + \sum_{n\neq -1} Dn\ ') \end{vmatrix}}{\Delta}$$

$$t_2 = \frac{\begin{vmatrix} (D_{-10r}' + \sum\limits_{n\neq -1} Dn_0') & -D_{-10r} \\ D_{-10i}' & -D_{-10i} \end{vmatrix}}{\Delta}$$

where

$$\Delta = \begin{vmatrix} \sum\limits_{n\neq -1} Dn_0' & -D_{-10i}' \\ D_{-10i}' & \sum\limits_{n\neq -1} Dn_0' \end{vmatrix}$$

V. The Numerical Solution and the K - V Diagram for the Curved Corrugated Surfaces.

As an example, we consider numerically a special case where

$$\epsilon_1 = \epsilon_2 = \epsilon_0, \qquad \frac{\delta}{\phi_0} = A = 0.5, \qquad \phi_0 = \frac{\pi}{27}$$

and let $q = \frac{\ell}{b\phi_0}$ be the ratio of the depth of slot to the periodicity in length of the structure.

1. In the slow wave region.

In terms of $Vn\phi_0$ and $kh\phi_0$, equation (IV - 8) can be written as

$$\cot qkb = A \sum_n \frac{1}{\sinh \alpha n} \left(\frac{\sin Vn\ \phi_0 \frac{A}{2}}{Vn\ \phi_0 \frac{A}{2}} \right)^2 \qquad (V - 1)$$

where

$$\sinh \alpha n = \sqrt{\left(\frac{Vn\ \phi_0}{kb\ \phi_0} \right)^2 - 1}$$

The plot of $kb\phi_0$ verse $Vn\phi_0$ has been shown in Figure 6. Similar to the plane case, it has stop band or bands depending on the value of g.

After getting the value of Vn_0 for each kb, the calculation of g_1 and g_2 can be carried out straight forward although it is lengthy. The curves for $Vn_0\phi_0 + g_1\phi_0$ verse $kb\phi_0$ in the slow wave region are also shown in the first triangle bound by two 45° lines $kb\phi_0 = V_0\phi_0$ and $kb\phi_0 = -(V_0\phi_0 - 2\pi)$ in Figure 6. The curves for the imaginary part of dV, $g_2\phi_0$ verse $kb\phi_0$ are shown in Figure 7.

Since $g_2\phi_0$ is positive value, the complex propagation constant V for a slow wave curved periodic structure can be written as

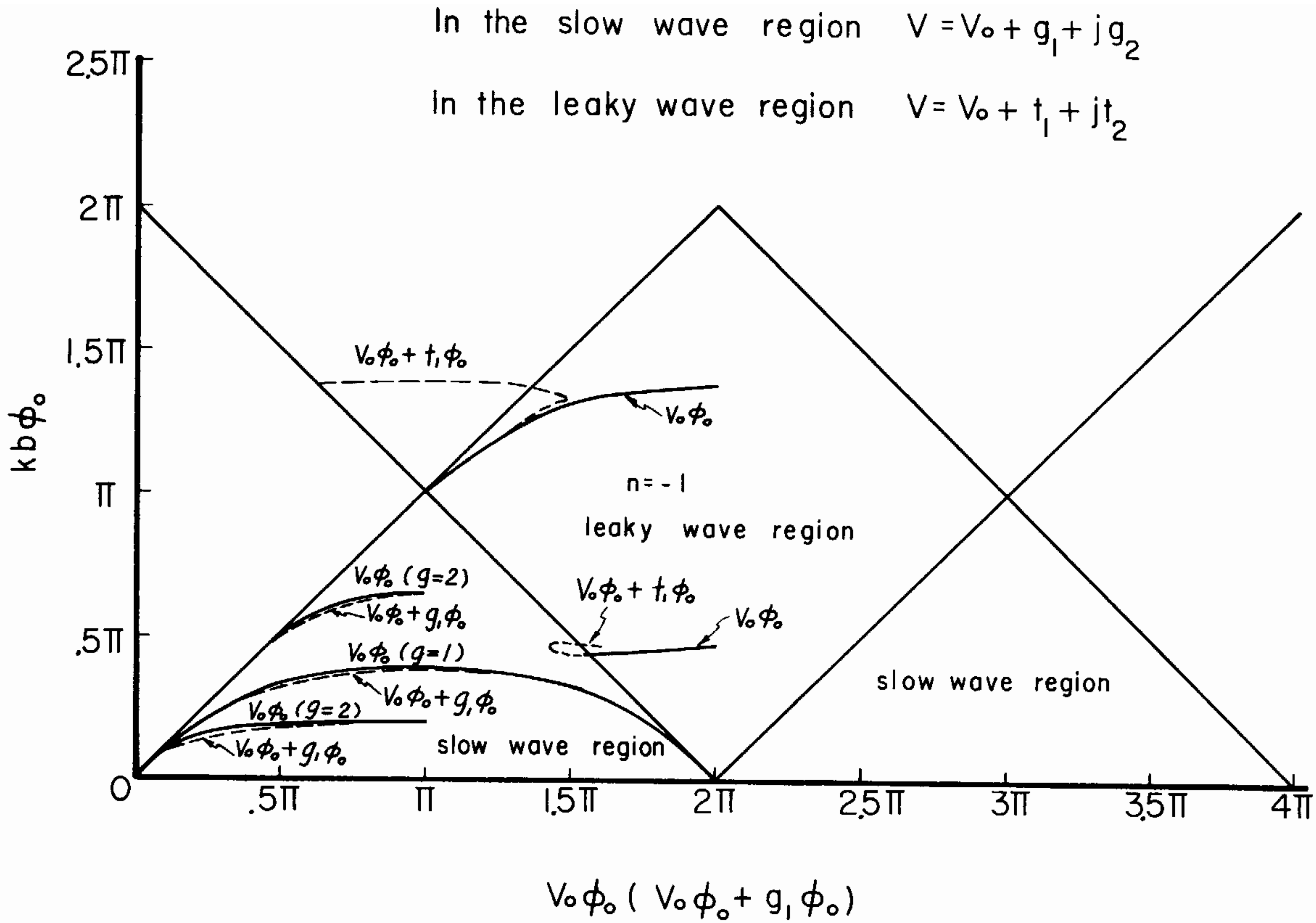

Fig. 6 The curves for $V_o\phi_o$ and $V_o\phi_o + g_1\phi_o$ verse $kb\phi_o$

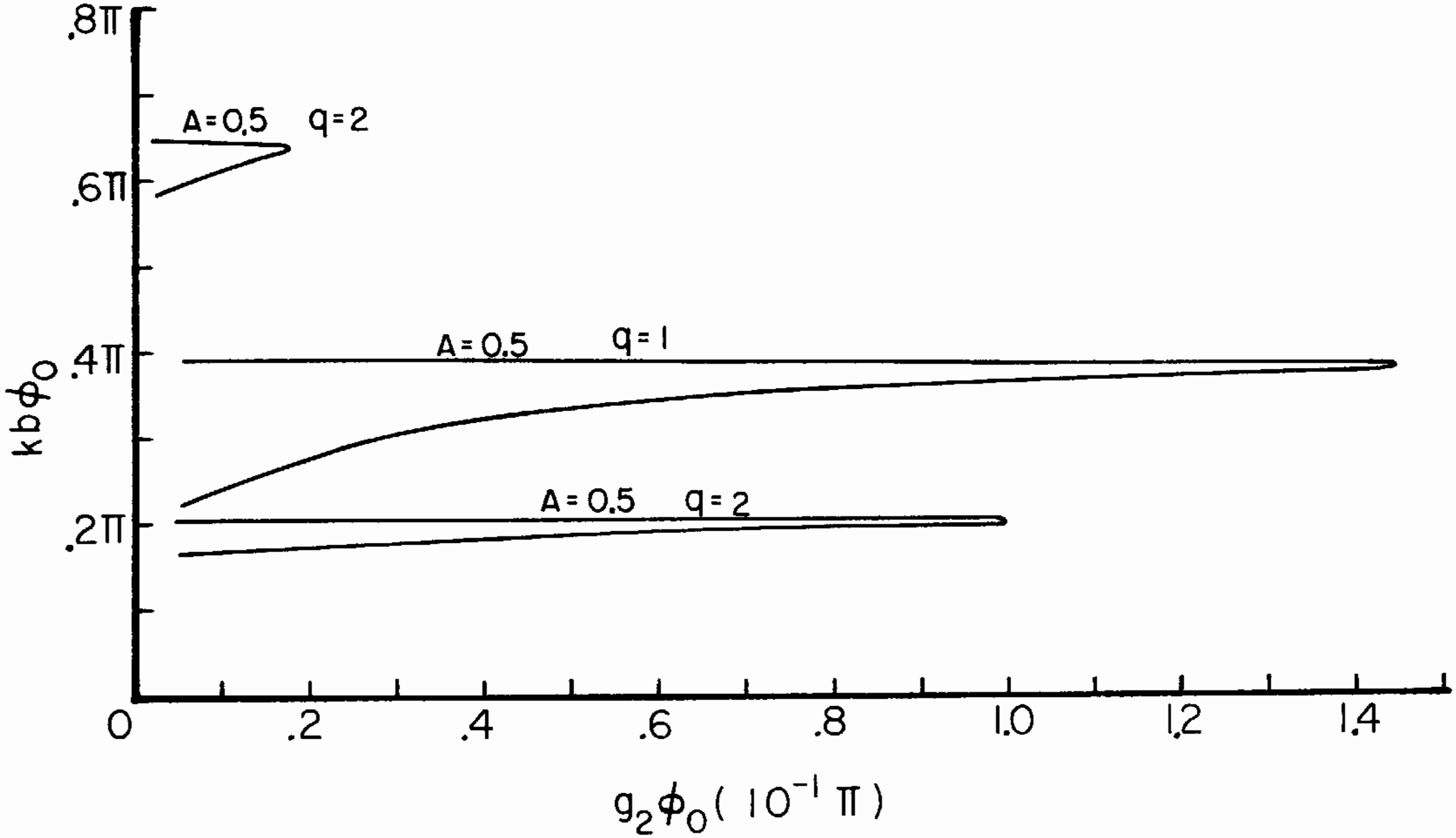

Fig. 7 The attenuation constant $g_2\phi_0$ verse $kb\phi_0$

$$V = V_0 + g_1 - jg_2$$

This is easily recognized to be the characteristics of leaky wave by recalling

$$e^{-jV\phi} = e^{-j(V_0 + g_1)\ \phi_0 - g_2\phi}$$

In addition to a phase constant (Vn + g_1) propagating along the ϕ - direction, there is an attenuation constant g_2 along the ϕ - direction indicating the leaking of energy as the waves travel, which is the effect of the curvature of the structure.

2. In the n = - 1 leaky wave region.

The plot of $V_0\phi_0$ verse $kb\phi_0$ for equation (IV - 19) is shown in Figure 6 in n = - 1 leaky wave region. Since only the n = - 1 leaky wave is allowed to radiate the region must be confined in the square where the condition (IV - 13) is satisfied as shown in Figure 6.

From the curves in Figure 6, we realize there are two branches of curve for g = 1, which gives a single curve only in slow wave region. The lower branch is in the frequency range where $kb\phi_0$ is around 0.4π, while the upper branch is in higher frequency range from π to 1.4π. When the operating frequency is lower than $kb\phi_0 = \pi$, only the lower branch can be seen.

The values for t_1 and t_2 can be calculated from equation (IV - 22) and (IV - 23). The curves for $V_0\phi_0 + t_1\phi_0$ have been shown in Figure 6. The curve for $- t_2\phi_0$ verse $kb\phi_0$ for the lower branch is shown in Figure 8 and that for the upper branch is shown in Figure 9. They appear very interesting. More time and effort of research work are needed to investigate the real properties of them.

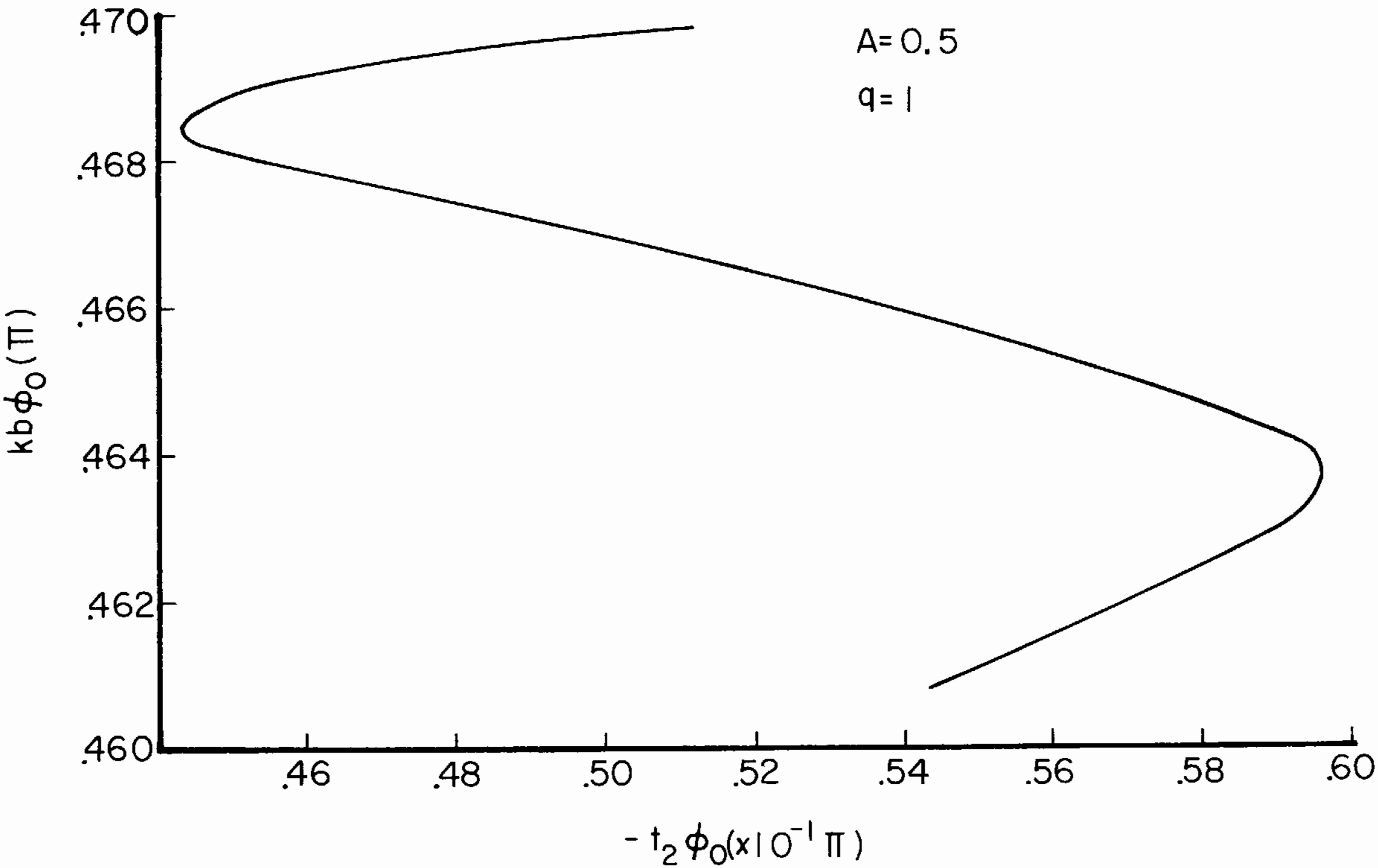

Fig. 8 Attenuation constant $-t_2\phi_0$ verse $kb\phi_0$ for the lower branch in $n=-1$ leaky wave region

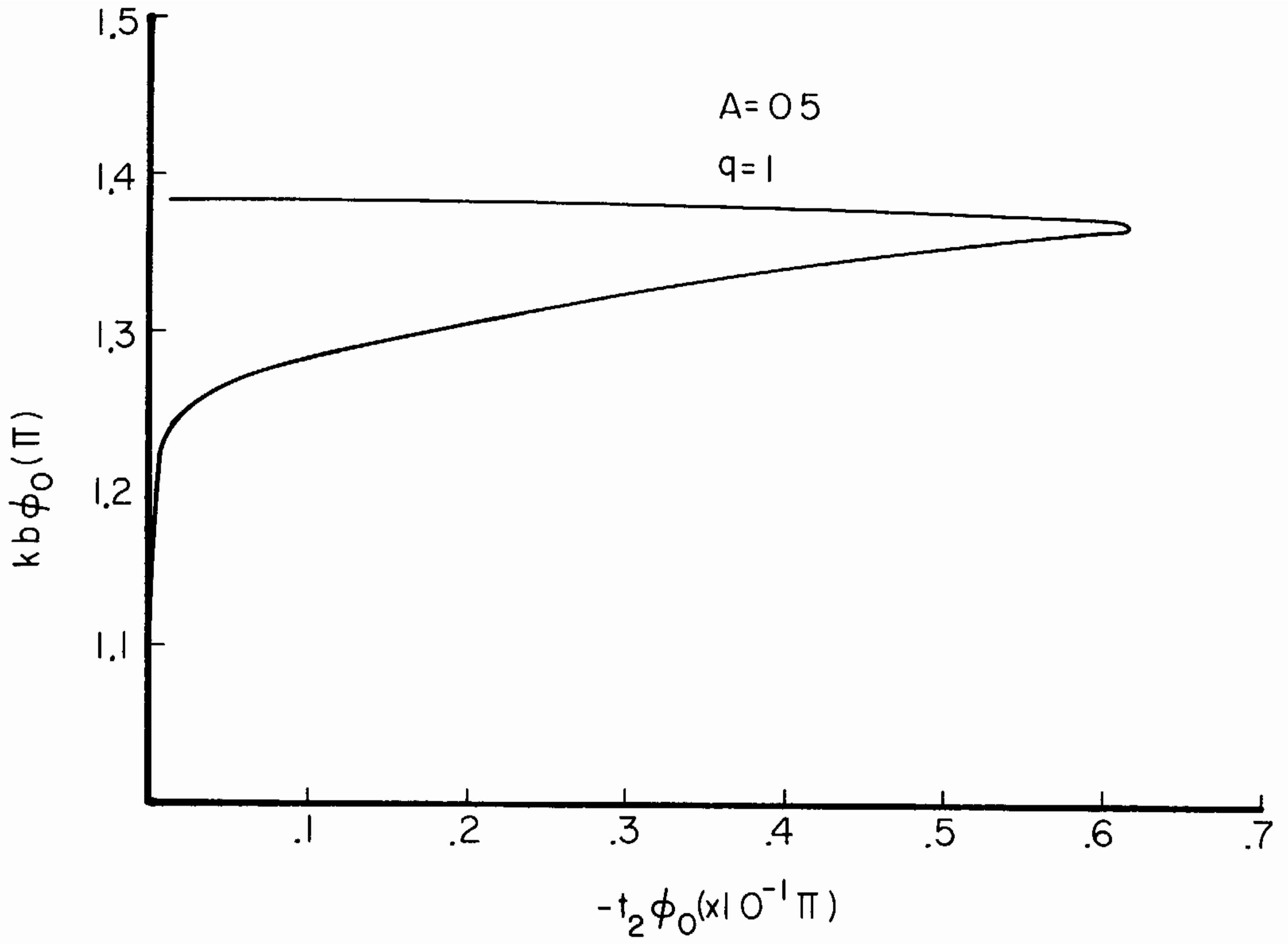

Fig 9 The attenuation constant $-t_2\phi_o$ verse $kb\phi_o$ for the upper branch in the n=-1 leaky wave region

VI. Conclusion

The k - v diagram for the curved periodic structures has been studied. Interesting points appear in the k - ReV and k - ImV curves both in the slow wave region and the n = - 1 leaky wave region, which need more research work to investigate. A graphical method has been used to find the solutions of equations (IV - 8) and (IV - 19), which might not be as accurate as using the computer to find the roots of equations (IV - 8) and (IV - 19). A more careful calculation for k - ReV and k - ImV for the leaky wave case will be very interesting.

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