DEVELOPMENT OF ADVANCED STRUCTURAL OPTIMIZATION PROGRAMS AND THEIR APPLICATION TO LARGE ORDER SYSTEMS

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The methods of matrix structural analysis have been coupled with some of the techniques of operations to provide integrated computer programs capable of performing the fully automated design of minimum weight structures subjected to a multiplicity of loading conditions. The present paper discusses some methods used in the development of such programs. Details of the analysis methods including the discrete elements contained in the program library are presented along with a description of the search procedure used to find the minimum weight. Two new programs and their possible range of practical applications are discussed and the paper is concluded with some brief remarks on prospects for future developmental research.

INTRODUCTION

The feasibility of developing practical techniques for structural optimization has been studied with increasing vigor in recent years. The motivation for this increased activity has been the realization that the well-developed methods of matrix structural analysis, when coupled with operations research methodology, can provide highly automated procedures for optimum structural design with wider scope than had heretofore been possible. Such efforts have led to a considerable number of computer programs of varying degrees of complexity and sophistication. In determining the degree of complexity at which the various researchers have aimed, many factors have been taken into account. Firstly, there has been the question of assigning the absolute importance to the hierarchy of variable design parameters which may be considered. One possible arrangement in order of importance is as follows: (1) member sizes, (2) variable dimensions, (3) variable material properties, (4) types of structures.

As is to be anticipated, the inclusion into a structural optimization program of additional classes of design parameters, as delineated above, will almost inevitably increase the complexity of the computational procedures and hence the computational expense.

This then introduces the second criterion which has hitherto been a controlling factor in the development of optimization programs – the question of economics. It is clear that, given infinite amounts of time and money, any optimization-search program, no matter how inefficient and cumbersome, could eventually generate an optimum structure. Clearly, however, techniques are mandatory which can bring about a saving of weight greater than their computational expense.

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In the work performed at Bell Aerosystems (Reference 1), a careful review of the relative costs of introducing the various design variables led to the belief that the major effort should initially be concentrated into the development of programs of maximum efficiency in which member sizes were the only variables. That is, the configuration and material properties are assumed to be fixed. Factors which influenced this decision included the practical nature of the aerospace structural problems in which the envelope and layout of a system is dictated by other than structural considerations. Consequently, in the technical description of the redesign procedures, the discussions are only concerned with this class of problem. Some comments pertinent to the extension of the techniques to include configurational variables are provided in later sections.

The state of the art has now been advanced to the extent that efficient techniques are indeed available and have been demonstrated in pilot program form. With the knowledge gained from the development of these programs, the further development of large scale optimization programs is now being undertaken.

We must first clarify what is meant by large scale capabilities. Experience has been recorded in the analysis of structures encompassing more than 1000 degrees of freedom. Yet, it is fair to say that such experience does not represent a routine analysis capability or, more importantly, confidence that the numerical results are free from error. On the other hand, at Bell Aerosystems, extensive experience has been accumulated in the routine analysis of structures of 500 degrees of freedom. Computational costs at this level do not often exceed $500 and there is considerable assurance based on numerical evidence, that error-producing difficulties can be circumvented.

In terms of actual structures, what can be accomplished with use of 500 degrees of freedom? This question will be answered in this paper with specific reference to the displacement method of matrix structural analysis, wherein each displacement component at a juncture point of discrete elements (i.e., at each node point) represents a degree of freedom and an equation to be formed and solved. In a three-dimensional truss, three degrees of freedom are specified at each node point and in a space frame six degrees of freedom are generally required. The scope available to these types of structures, with the limits of 500 degrees of freedom, can be readily visualized.

The scope available to airframe-type structures is perhaps less evident. In order to give more meaning to these important classes of conditions, therefore, some recent experience in the design of an actual aircraft is cited. This aircraft, shown in Figure 1, is the X-22A ducted fan VTOL. A comprehensive review of the structure of this aircraft would disclose a minimum idealization requirement of from 3000 to 5000 degrees of freedom for the complete airframe — a requirement well beyond the existing capabilities of advanced structural analysis techniques. Hence, even these procedures must reduce to independent examination of components such as the wing, tail, fuselage, and so forth.

During the design phase, nearly all portions of the X-22A were analyzed with use of discrete element techniques. A good example is furnished by the idealization of the fin shown in Figure 2. The idealization consists of 136 elements and results in 141 degrees of freedom. This is much less than routine capacity, but it is to be recognized that inclusion of any part of the fuselage immediately introduces many more degrees of freedom. Furthermore, the multitudinous number of design conditions for this type of component, as compared with the fuselage, makes imperative the rapid execution of an analysis for each condition.

A second and entirely different problem appears in Figure 3. By the use of truncated cone elements of the form shown in Figure 3b, the complete tank can be analyzed in a single operation. There is only a single loading — pressurization — and the regions of interest from the standpoint of stress analysis are quite localized, being at the juncture point of the

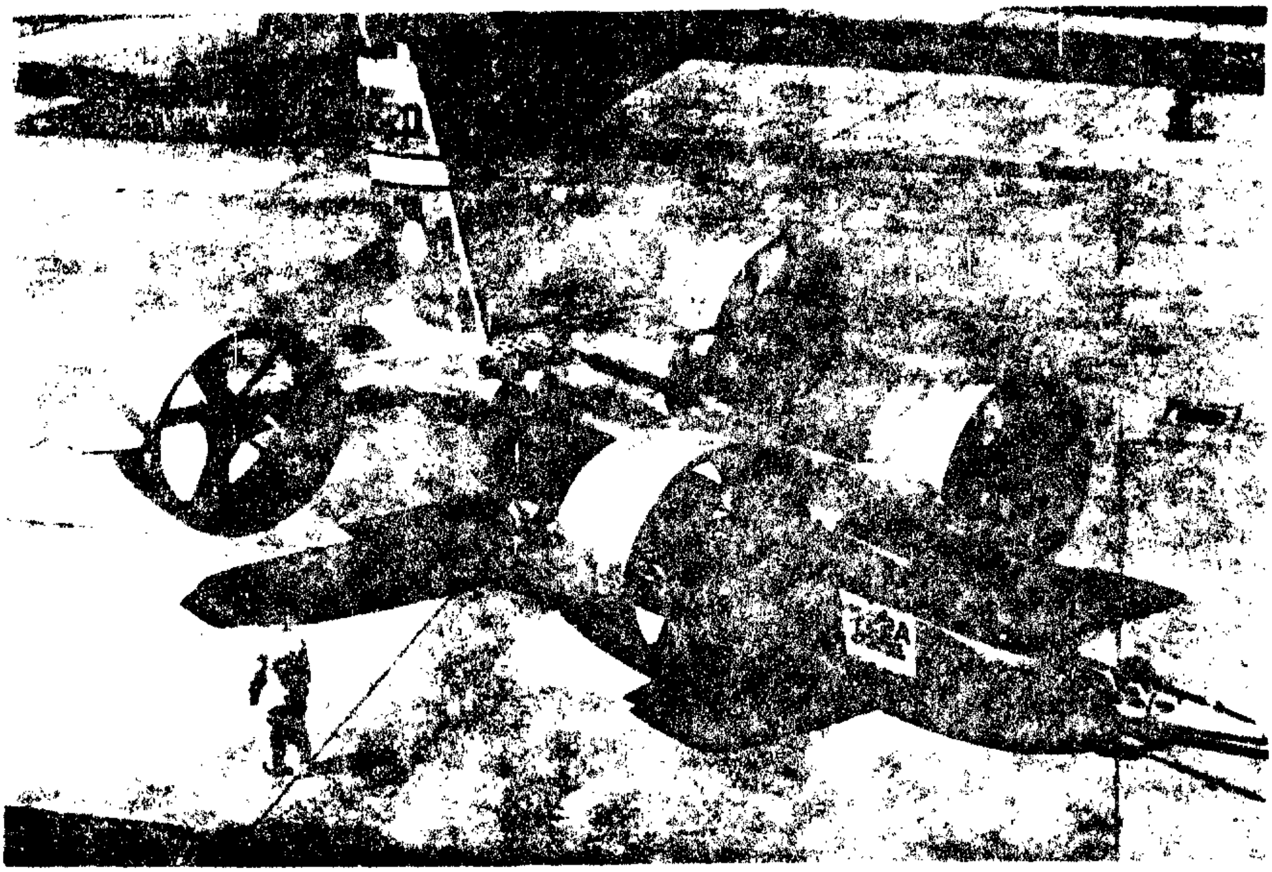


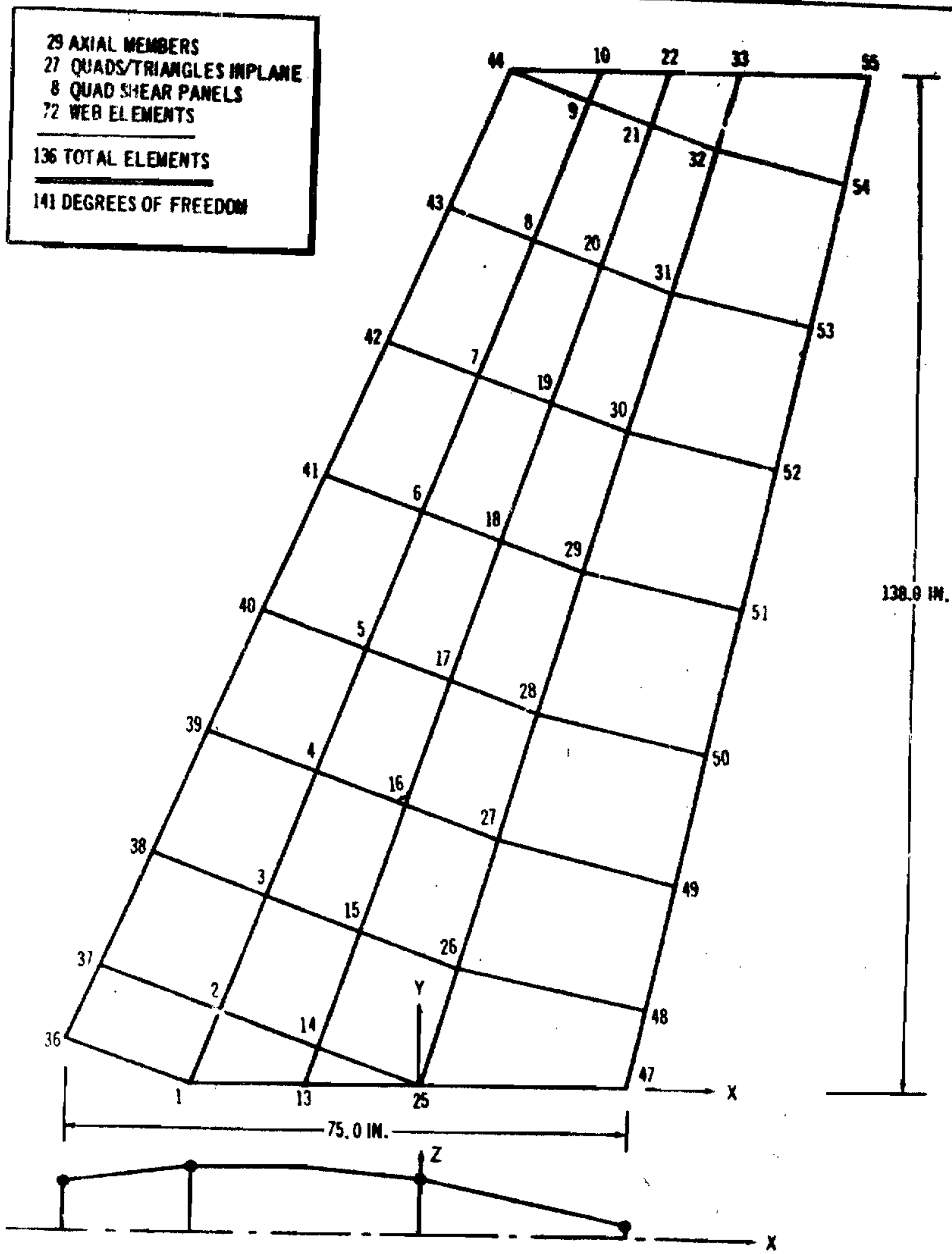
Figure 1. X-22A VTOL Research Airplane

ead and shell and at the flanged opening at the end. The total of 350 degrees of freedom is epresentative of this type of problem.

With the foregoing necessarily brief review of the features of a large scale analysis apability, attention can be turned to the envisioned requirements of a computer based, mini-um weight design capability of corresponding scope. Clearly, the latter capability must ossess the following attributes (in addition to large capacity) if it is to be responsive to the emands of an actual design organization: (1) Versatility (2) Efficiency (3) Maximum Auto-ation. These attributes, which have been referred to briefly in the context of the foregoing scussion, are examined more closely in the following.

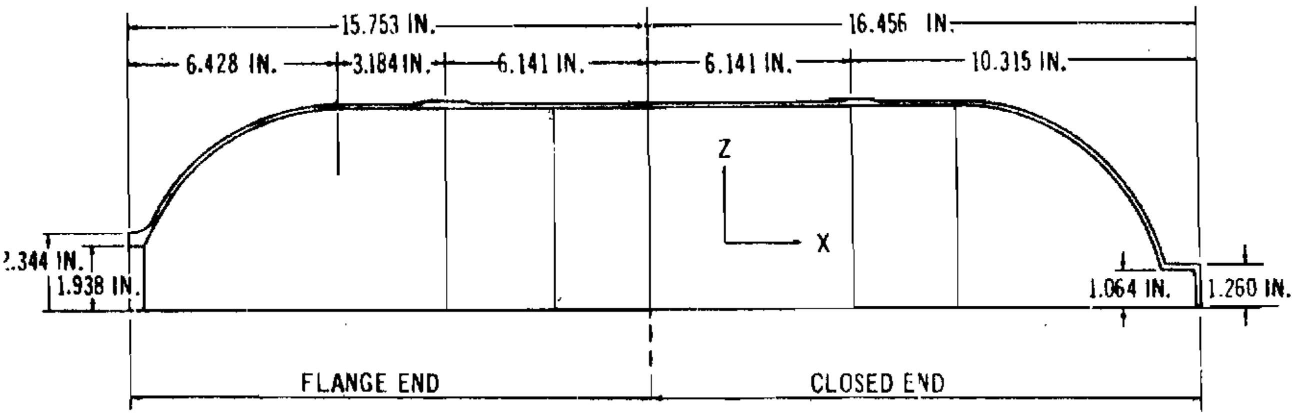
Very few structural design organizations are limited in their attention to simply one type structural analysis problem (e.g., trusses). Aerospace industrial organizations, in partic-ar, are faced with a variety of design configurations. It is not unusual for these to extend — a single company — from underwater craft to space vehicles, involving the airframe, nks, and rocket engines. The only unified means for dealing with these problems is through e of the discrete element techniques of matrix structural analysis. These techniques are a bject in themselves, and hence this paper will only outline the fundamental concepts and dicate specifically the extent to which they are implemented at a practicing organization.

One of the major impediments to the introduction of computer based analysis methods has en the view that such techniques add unnecessary cost to the product. This view will also pede practical application of optimization techniques unless it can clearly be shown that



X-22A Fin Idealization

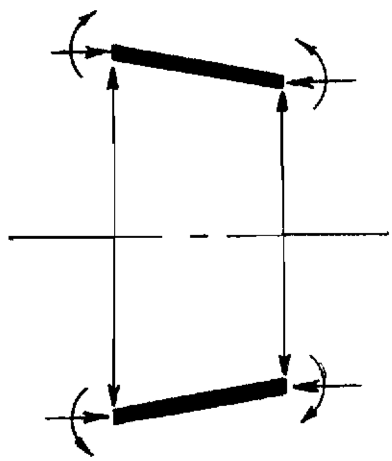
Figure 2. Discrete Element Idealization



Number of Elements 108

a. LEM Fuel Tank

Number of Degrees of Freedom 350



b. Truncated Cone Element

Figure 3. Idealization for Shell Analysis

cost savings as well as an improved product will result. Direct improvements in the efficiency of the optimization technique are an obvious route to the desired objective but one should also seek to supplant, through directed redesign, some of the present costs of analysis.

Maximum automation has a close tie to desired efficiency, since this is also a means to reduce cost. It has been found through long experience with computer techniques, however, that automation is essential to eliminate the human errors which preclude any large scale operations from being a working proposition without almost complete automation.

In the following section of the paper, the general concepts of optimization are dealt with, followed by a discussion of the matrix analysis procedure employed. A detailed description of the redesign techniques follows and the paper is concluded by a section on the specific programs under development with applications and future prospects.

GENERAL CONCEPTS

A given structural design can be viewed as a point in an n-dimensional hyperspace wherein each dimension represents a design variable. If we restrict our attention initially to designs under the circumstances of fixed configuration and material properties, only the cross-sectional dimensions of the elements act as design variables. For any given weight of a structure, there exists an infinity of possible combinations of the design variables which produce this weight. All these possible structures form a surface (or hypersurface) in the design space. If only one variable is permitted for each structural element (e.g., cross-sectional area, skin thickness, etc.), each weight is a linear function of the design parameters and corresponds to a plane in three-dimensional space (Figure 4a). The introduction of additional configurational variables would merely increase the dimensionality of the design space and would also generate a nonlinear weight function.

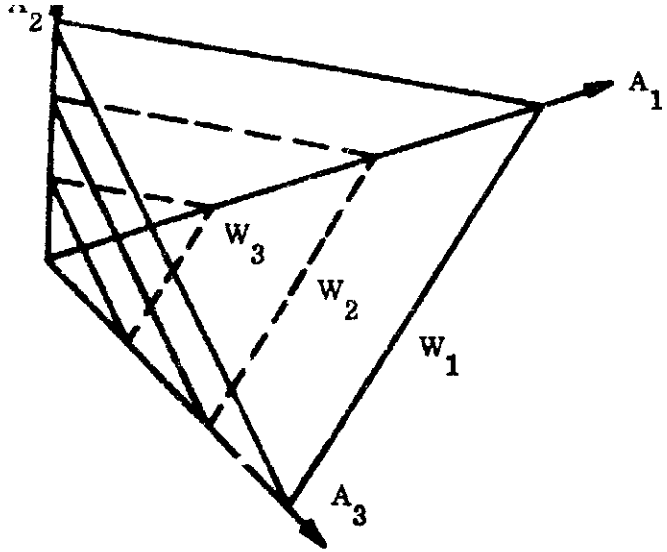
The idea of a "constraint" surface can now be introduced. If a limitation is provided on any characteristic of a structural system (e.g., the maximum positive stress in an element), all possible designs satisfying this condition lie on a surface in the design space. Since stresses and deflections are, in general, nonlinear functions of the design parameters, this "constraint" surface is a nonlinear function of the design parameters. Figure 4b presents the form of a typical constraint surface, which is generally concave when viewed from above. The term "above" refers to the space on the side of the surface distant from the origin. If a design point lies in the space above a constraint surface, the characteristic stress or deflection will lie below its critical value. This is referred to as "free-point" space.

For every element stress and nodal displacement, upper and lower limits can be specified so that a large number of constraint surface exists. These surfaces will, in general, intermesh in a complex fashion. When viewed from free-point space, the uppermost (dominant) constraints form a composite constraint surface, as shown in Figure 4c, in which each "patch" represents a segment of an individual surface. The composite surface is also of concave form. In addition to these main constraints, there exist also "side" constraints. These are limitations that members must have positive areas and appear simply as planes parallel to coordinate directions.

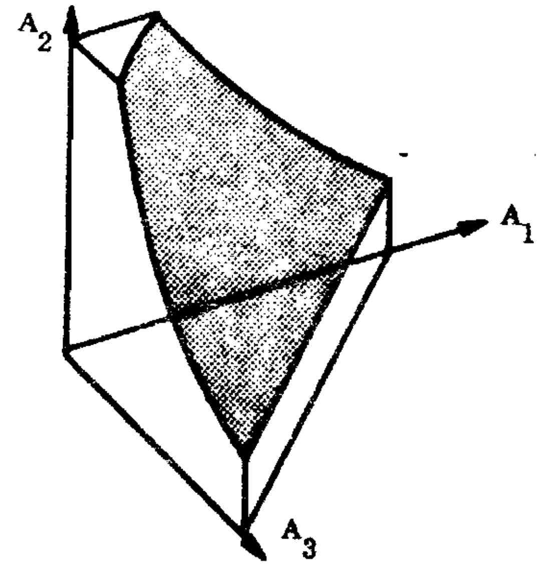
Now the problem may be reduced to physical concepts. Any design process must begin with an initial guess as to the member sizes. From this point of departure, the process can be viewed as a trip through the associated n-dimensional hyperspace, the trip being concluded when a design point of lowest weight is reached. This condition is defined such that it is impossible to travel further in any direction from the lowest weight point without exceeding initially imposed limits on stresses, deflections, areas, etc. Physically, this may be seen as the point at which a weight surface touches the composite constraint surface at one point, but does not intersect it (Figure 4d).

When at a given design point, the prevailing stresses and deflections must be examined in order that the adherence of the related design to the limiting values of these quantities can be assessed. That is, at each design point, at the outset or immediately after the travel from a previous point (after changes in the design parameters), an analysis must be performed. The analysis operations are completely independent of the operations employed to achieve a directed change (a redesign) from the previous design point.

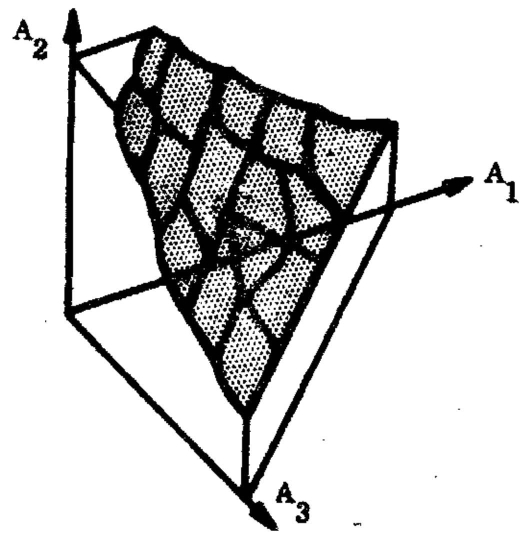
The redesign process devised herein consists of three independent modes of travel: (1) An initial step (2) Steepest descent (3) Side-step. Once the initial step has been taken, modes (2) and (3) are successively and repeatedly applied. The concept of proceeding to the minimum weight design solely by use of the latter two modes of travel has been characterized as the "Method of Alternate Steps" (Reference 2).



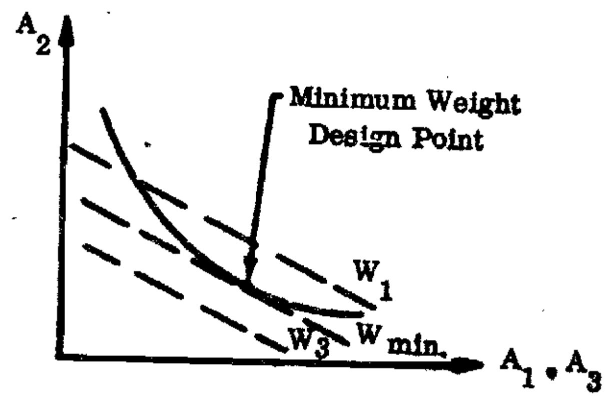
a. Planes of Constant Weight



b. Constraint Surface for One Load Condition



c. Typical Constraint Surface



d. Cross-Section Through Minimum Weight Design Point

Figure 4. Geometric Interpretation of Synthesis Problem

In the initial step, the "first guess" at a design is employed in an analysis and the results of this analysis are applied to the resizing of elements so that in the new design each element is subjected to its maximum allowable stress under at least one load condition. This procedure for revising the design is repeated as often as desired, tending to generate a design in which each element achieves a limiting stress under at least one loading condition.

At the conclusion of the initial step, the steepest descent mode of travel is entered. In the steepest descent mode, the weight of the total structure is reduced in the most rapid possible manner until limitations on stresses, and other parameters are reached. Then, the side step mode, which effects a redesign of the structure at constant weight, is introduced. This mode of travel is so contrived that it results in a new design in which all of the stresses lie below their limiting values. The steepest descent mode is then re-entered, and the process continues until the minimum weight design point is reached.

ANALYSIS PROCEDURE

The analysis portion of the design process furnishes a list of the stresses and displacements caused by the respective load conditions; these are for the purposes of comparison with the specified limiting values. As noted previously, the analysis method utilized herein is the "displacement" or "direct stiffness" method (Reference 3). In matrix displacement analysis, structural systems are idealized as assemblages of discrete elements. Relationships between the element juncture point forces $\mathbf{F}$ and displacements $\boldsymbol{\delta}_{EL}$, in the presence of a temperature change from the stress-free state, are stated in the form (considering, for simplicity, only one load-temperature condition)

$$\mathbf{F} = \mathbf{k}\,\boldsymbol{\delta}_{EL} - \mathbf{F}^{\alpha} \tag{1}$$

where $\mathbf{k}$ is the so-called element stiffness matrix, and $\mathbf{F}^{\alpha}$ are the nodal forces representing full restraint of displacements due to the temperature change (i.e., thermal force matrix)

Upon numerical evaluation, the element relationships are combined, in accordance with the requirements of node point equilibrium and compatible displacements, to yield

$$\mathbf{P} = \mathbf{K}\,\boldsymbol{\delta} - \mathbf{P}^{\alpha} \tag{2}$$

where $\mathbf{P}$ are the node point external loads
$\mathbf{K}$ is the stiffness matrix of the assembled idealization
$\boldsymbol{\delta}$ are the node point displacements, and
$\mathbf{P}^{\alpha}$ are the node point net thermal forces

The matrices $\mathbf{P}$ and $\mathbf{P}^{\alpha}$ are known quantities, and the solution to these equations (after they are modified in recognition of the geometric boundary conditions), by inversion of the $\mathbf{K}$ matrix or by other means, produces values for the displacements, $\boldsymbol{\delta}$.

In order to obtain the stresses acting upon the elements once the node point displacements are known, it would be possible to extract from the total column of displacements $\boldsymbol{\delta}$ the node point displacements of the respective elements and then to use these in the element equations (Equation 1) in determination of the internal forces. Finally, the internal forces can be transformed into values of stress. This is an inefficient procedure in the present context, however. Here, element relationships are immediately formulated for the stresses in terms of the element node point displacements

$$\boldsymbol{\sigma} = \mathbf{S}\,\boldsymbol{\delta} + \boldsymbol{\sigma}^{\alpha} \tag{3}$$

The matrix S is referred to as the element stress matrix while σ^{α} represents the stresses associated with full constraint of the strain due to the specified element temperature change. When considered as an integral portion of an optimization procedure in which an analysis is required at every stage of redesign, it is possible to increase efficiency by recomputing only the minimum number matrices at each stage. For example, the stress matrix S in Equation 3 is entirely independent of the design parameters (e.g. cross-sectional area or plate thickness), it can be formulated at the start of the design process, and will remain unchanged as the process advances through the many changes in the design parameters. On the other hand, although the K and P^{α} matrices must be assembled anew at every stage, the individual element stiffness k and thermal force matrices F^{α} are linearly dependent on the design variables and need to be calculated only once, and stored on tape. At each stage, these elemental matrices are then read from tape and scaled by the current values of the design variables before assembly into the master matrices.

From the general efficiency standpoint, every effort has been made to reduce to a minimum the computational time for the individual matrix operations. Wherever possible, only non-zero elements of large matrices are stored and operated upon. In the case of the master stiffness matrix, for which, due to symmetry, only one-half need be considered, the matrix will be stored and operated upon in variable bandwidth form, which will permit equation solving up to approximately 500th order within the core of an IBM 7090.

As noted earlier, the redesign procedure simply requires from the analysis section a list of stresses and displacements of the structure at the current point in design space. This list which is referred to as the behavior function of the design parameters may be denoted as

$$\left[\frac{\sigma_c}{\delta}\right].$$

The behavior matrix has one column for each external loading condition. The σ_c is simply a list of stresses, one for each element, and is derived from the σ matrix generated in Equation 3. For an axial force member, the reference stress is merely the axial stress, but for plate or other elements where multiple stresses are generated e.g. σ_x, σ_y and τ_{xy}, these are combined using, say, a yield stress criterion to provide a single reference stress for the element. The δ portion of the behavior matrix is the δ matrix obtained by solution of Equation 2.

The key to the versatility of an optimization, or indeed a general analysis program, resides in the type of discrete elements contained within the program. The pilot optimization program developed at Bell contains the five types of elements shown in Figure 5. With these elements, it is possible to design space frames, stiffened plates, and shells in a membrane state of stress. In the Bell General Purpose Structural Analysis Program, approximately twenty-five elements are available in the library storage. For the expanded optimization programs under development at Bell, the precise number and types of elements to be incorporated has not yet been finalized. The analysis capabilities will, however, be equivalent to those of the General Purpose Program and the element library will be open ended to permit addition of new elements as desired.

REDESIGN PROCESS

Initial Step

Any design process must begin with an initial estimate of member sizes. If conventional cut-and-try procedures are applied to the initial estimation of member sizes, the automated design process adds cost to the conventional design procedures. Preferably, the automated design procedures should serve to reduce the cost of conventional design procedures as well as to accomplish structural weight reductions.

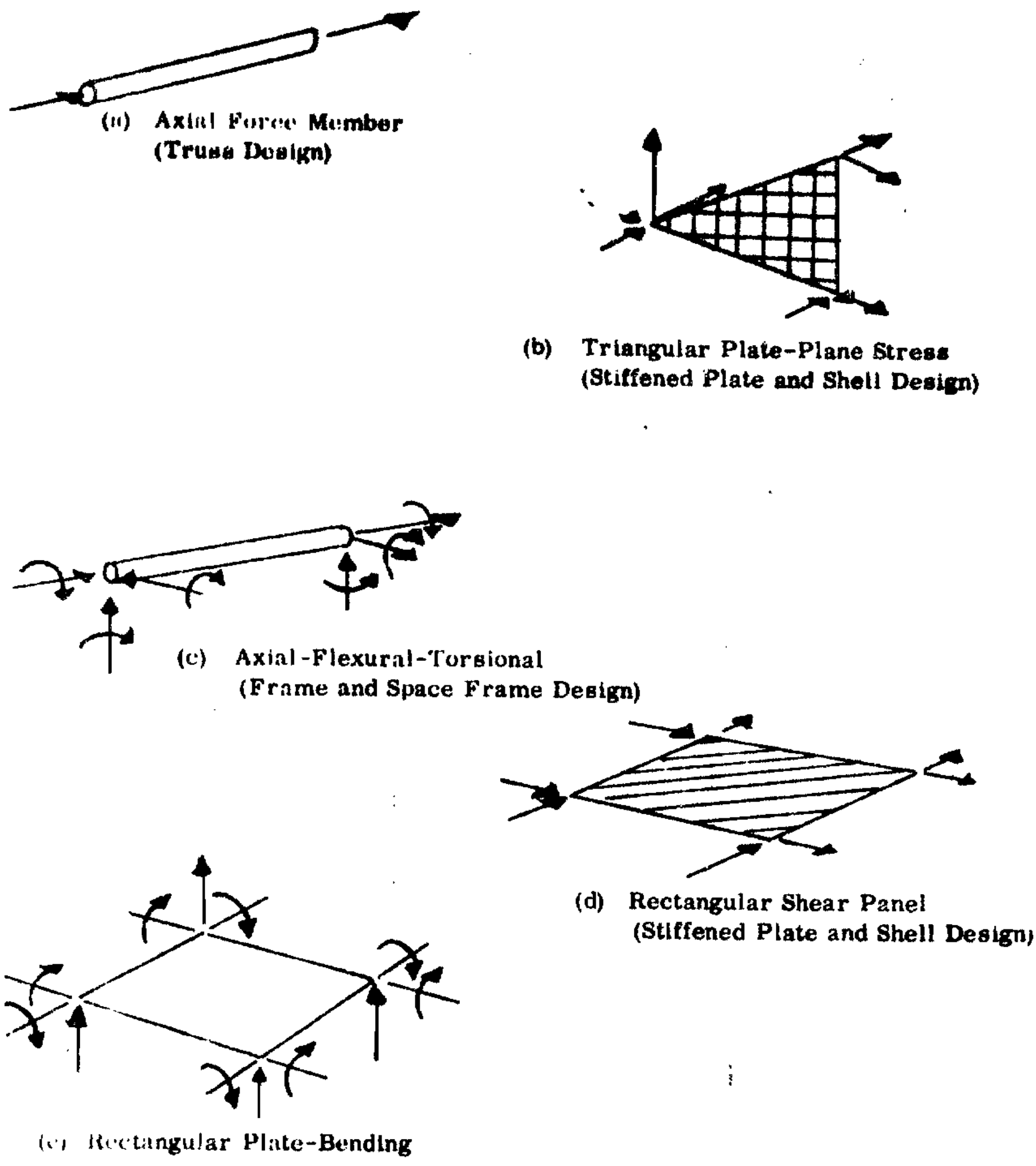


Figure 5. Discrete Elements for Synthesis Program

In view of this consideration, it is suggested that the process begin by designation of the specified lower limits of the respective member sizes as the initial guess as to these same member sizes. This procedure would appear to prejudice computational time, since it would seem that a number of redesign attempts are necessary before realistic member sizes are reached. One purpose of the initial step portion of the redesign process is to obviate this disadvantage. The initial step mode of travel can be described as follows.

Let the first guess as to the design parameters be designated by the superscript "o". Also, let all computations associated with the first guess be designated by the same superscript. On this basis, all steps in the analysis process described previously (Equations 1 to 3) are undertaken, producing a corresponding behavior function of the design parameters $\{\sigma\,\delta\}^o$. The stresses listed in this behavior function must then be compared with the specified upper and lower limits $[\sigma]^u$ and $[\sigma]^L$ respectively.

The inequalities which must be satisfied by the final design can be written as

$$\left[\sigma\right]^L \leq \left[\sigma\right] \leq \left[\sigma\right]^U \qquad (4)$$

If the lower limits on member size are used as the basis for this first computation of the behavior function, $\{\sigma\,\delta\}^o$ usually lies well outside the specified stress limits. These limits are then used in a determination of new member sizes. Each value of σ^o is compared with the corresponding limiting values and the ratios between σ^o and σ^U and σ^L are formed. The largest ratio (which will be greater than 1.0) then dictates the change in the value of the design parameter in the pertinent element, that is, the design parameter is multipled by this ratio. The process is repeated for each row of stresses in $\{\sigma\,\delta\}^o$, resulting in complete change of all design parameters.

The resultant change in design parameters will often bring the design to a point which is not far removed from the minimum weight, but in any case, the procedure is repeated as often as it is considered necessary. Clearly, repeated application of the process until convergence is obtained in successive applications leads to a "fully-stressed" design; that is, one in which each element reaches limiting stress under at least one of the load conditions. A fully-stressed design would not be acceptable if displacement limitations prevail. Less obvious is the possibility that, in the absence of displacement limitations, there is a design which is of less weight than the full-stressed design. This possibility dictates the continuation of the design process under such circumstances.

If any elements of the structure are statically determinate, the design process can be slightly simplified for these elements. Therefore, a test for static determinancy of any or all elements is incorporated into the initial step mode. In a statically determinate element, the load in the member is not a function of its stiffness and hence is independent of its cross-sectional area. Thus, in making the first step from an initial guess, the load in any statically determinate member is unaltered when the area is altered. As described above, an attempt is made in this step to modify the design parameters such that the stress in each member would reach its critical value, if the element load were to remain constant. If, at the second iteration, any element stress lies exactly at its critical value, the load must have been constant, and hence the member is determinate. When only stress limits are present, the determinate member will require no further redesign and can be removed from the synthesis process. If, however, displacement constraints are present, this simplification is not possible.

Steepest Descent Mode of Travel

In the n-dimensional hyperspace representation of the minimum weight design problem, all possible structures of a given weight can be defined by points which lie upon a surface (or hypersurface). Thus, all possible weights of a particular configuration can be considered as a family of such surfaces. With member sizes as variables, the structural weight is a linear function of the design parameters and the weight surfaces are planar (Figure 4a).

In the redesign process, the weight of the structure can be reduced most rapidly if the direction of travel is normal to the weight surfaces. Travel in this direction is the so-called "Steepest Descent Mode". As in any mode of travel from some initial design point, it is necessary to define both direction and distance of travel. Clearly, if the direction can be established, the distance may be determined by some iterative process. The direction of travel is established in the present case as follows.

The weight of the structure is the sum of the weights of the individual elements, that is:

$$W = \lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3 + \cdots \tag{5}$$

where A_1, A_2 are the design variables
λ_1, λ_2 are constants dependent upon material properties, geometry, etc.

For the axial force member, for example, the design variable is the cross-sectional area and $\lambda = \gamma \ell$, where γ is the specific weight of the material and ℓ is the length of the element. Equation 5 is the equation of the weight surface in a coordinate system defined by the variables A_1, A_2, etc. Now, if Equation 5 is differentiated with respect to the individual design variables and the resultant derivatives are normalized, the direction cosines V_{w_i} of the normal to the weight surface are obtained. Algebraically, this can be written as

$$V_{w_i} = \frac{\lambda_i}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \cdots}} \tag{6}$$

The steepest descent process can now be expressed in the matrix form:

$$(D_p)_{p+1} = (D_p)_p - \epsilon \, V_w \tag{7}$$

where D_p is an assembled vector consisting of the individual design variables (e.g. A_1, A_2, ...).
V_w is the vector of the direction cosines, and
ϵ is some distance of travel along the normal between the p^{th} and $(p+1)^{th}$ iteration.

It is possible to travel along the normal to the weight surface until some limitation is reached. These limitations can be stresses and deflections, which are referred to as main constraints, or on the design parameters themselves (side constraints). It is, however, not possible to determine directly the distance measured from any point along the steepest descent path to the nearest constraint. Thus, as indicated previously, an iterative method is employed.

A distance ϵ is chosen arbitrarily and the design parameters are modified as indicated by Equation 7. If the new design does not violate the constraints, this distance is doubled. This process is repeated until a design is reached which violates on a main constraint (side constraints are ignored at this stage). Upon violation, the incremental distance ϵ is halved and the direction of travel reversed. In all subsequent iterations, the distance is always halved and the direction reversed after each transition between a violated and nonviolated condition. Thus, the process is directed to and converges upon the constraint surface. No difficulty will be encountered if the initial choice of a design violates constraints; the steepest descent mode will immediately "ascend" toward the constraint surface.

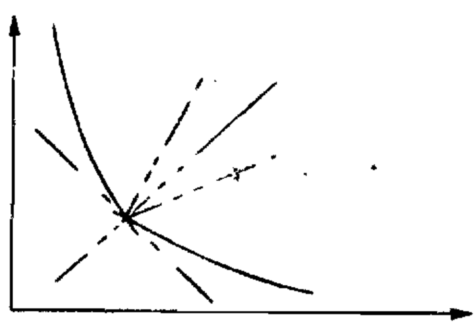
It should be emphasized that each "travel" in the steepest descent mode or any other mode is simply a change in the design parameters and should strictly occasion a recomputation of the stiffness matrix (Equation 2) and redetermination of the behavior function of the design parameters, these being required for comparison of the results with the limiting stresses and displacements in order to determine if a constraint has been reached.

Upon reaching the main constraint surface, the design variables are tested against their side constraints. If any values of the design variables violate the side constraints, they are restored to their lower limiting values and a steep descent mode (with the limited design variables suppressed) is re-entered. This direction of travel prevents further violation of the side constraints associated with the limited design variables. When a main constraint which does not violate side constraints is reached, the side-step mode is entered.

Side-Step — Optimum Vector Mode

The method developed herein for the mode of travel from the constraint surface following completion of a steep descent is designated as a "side-step" in the direction of the "optimum vector".

It is clear from the definition of the optimum structure that if one constraint condition predominates, the lowest weight plane will be tangential to the constraint surface so that the respective normals to the weight and constraint surfaces will coincide. This consideration provides a basis for the selection of a direction of travel in a constant weight plane. It must be emphasized, however, that the minimum weight design will most often lie at the intersection of many constraints, that is, the minimum weight design is bounded on many stresses and deflections. In this case, the individual normals to the constraint surface do not coincide with the normal to the weight surface, but the average normal will tend to coincide (see sketch).



An algebraic development of this procedure is as follows. This first operation consists of the determination of the vector $V_c = \frac{\partial q_c}{\partial A_i}$ of the direction cosines of the normal to the current constraint surface. (Assuming, for the moment, that only one constraint is active.) Each element of this vector represents the partial derivative of the bounded behavior parameter q_c with respect to a particular design parameter A_i. The constraint surface in question is, of course, a representation of the limiting value of the pertinent behavior parameter q_c which may be either a stress σ or a displacement δ.

The relation between total external loading and nodal displacements is given by Equation 2.

$$P + P^{\alpha} = K\,\delta \qquad (2)$$

If the design variable of the i^{th} element of the structure is altered by a small amount ∇A_i, the stiffness matrix, and the thermal force matrix of the modified structure become

$$K' = K + \nabla A_i\, k_i \qquad (8a)$$

$$P^{\alpha'} = P^{\alpha} + \nabla A_i\, F_i^{\alpha} \qquad (8b)$$

where k_i and F_i^{α} are expanded stiffness and transformation matrices for the i^{th} element for a unit value of A_i. The displacements and stresses are also altered by small quantities

$$\delta' = \delta + \nabla\delta \qquad (9a)$$

$$\sigma' = \sigma + \nabla\sigma \qquad (9b)$$

Equation 2 may now be written as

$$P + P^{\alpha'} = K'\,\delta'$$

or

$$P + P^{\alpha} + \nabla A_i\, F_i^{\alpha} = K\,\delta + \nabla A_i\, k_i\,\delta + K\,\nabla\delta + \nabla A_i\, k_i \nabla\delta \qquad (10)$$

Neglecting second order terms, and with $K\,\delta = P + P^{\alpha}$, Equation 10 becomes

$$\nabla A_i\, F_i^{\alpha} = \nabla A_i\, k_i\,\delta + K\,\nabla\delta \qquad (11)$$

thus

$$\nabla\delta = -\nabla A_i\, K^{-1}\left(k_i\,\delta - F_i^{\alpha}\right)$$

or

$$\frac{\partial\delta}{\partial A_i} = -K^{-1}\left(k_i\,\delta - F_i^{\alpha}\right) \qquad (12)$$

Also, from Equation 3

$$\frac{\partial\sigma}{\partial A_i} = S\,\frac{\partial\delta}{\partial A_i} \qquad (13)$$

$\frac{\partial\delta}{\partial A_i}$ and $\frac{\partial\sigma}{\partial A_i}$ are assembled to form a behavior matrix derivative, from which the relevant element $\frac{\partial q_c}{\partial A_i}$ can be selected. The entire process, which involves the evaluation of Equations 12 and 13, is repeated for the design parameters, in turn, to generate the complete vector.

It should be noted that this method is particularly attractive since it only requires the reformation of the simple matrices k_i and F_i^{α} and the matrix multiplications of Equations 12 and 13 for each parameter A_i. The matrices K^{-1} and δ are only computed once and used repeatedly. Other methods of determining the vector V_c all involve a complete re-analysis of a modified structure for every design parameter.

The second necessary item of information is the direction cosine vector of the normal to the weight surfaces, V_w . This vector will have been developed during the steepest descent mode, as described by Equations 5 and 6.

With V_c and V_w both evaluated, the desirable direction of travel in the constant weight plane represented by the vector V_D can be determined. This direction is orthogonal to V_w . By the right triangle of vectors shown in Figure 6:

$$R \; V_C = V_D + V_W \tag{14}$$

where R is the length of the resultant vector in the V_c direction. V_w is a unit vector due to its normalization, so that V_c , being a hypotenuse, has a length greater than unity). Thus

$$V_D = R \; V_C - V_W \tag{15}$$

Also, since V_D and V_w are orthogonal

$$V_W^T \; V_D = 0 \tag{16}$$

then, substituting 15 into 16

$$R \; V_W^T \; V_C - V_W^T \; V_W = 0 \tag{17}$$

but, since V_w is a normalized vector; i.e., $V_w^T \; V_w = 1$, Equation 17 becomes

$$R \, V_W^T \; V_C - 1 = 0 \tag{18}$$

or

$$R = \frac{1}{V_W^T \; V_C}$$

Substituting Equation 18 into Equation 15,

$$V_D = \frac{V_C}{V_W V_C} - V_W \tag{19}$$

Equation 19 defines the direction of travel within a constant weight plane. After the new direction of travel orthogonal to the line, AF is chosen (i.e., after V_D is computed from Equation 19), a step of arbitrary magnitude (AB) is taken (see Figure 7). This and the following steps in the side-step, in aggregate, are taken in a manner similar to the steepest descent mode of travel. If a violation occurs as a result of the travel ($\overline{AB}$), this distance is halved until a free point is reached. Certain provisions have been incorporated to permit selection of a new direction of travel when new bounds are encountered, but this is discussed at a later stage.

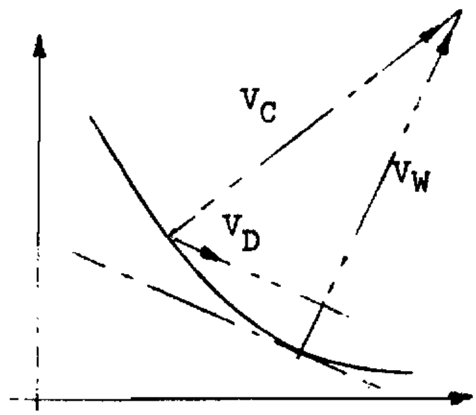


Figure 6. Triangles of Vectors for Side-Step Mode

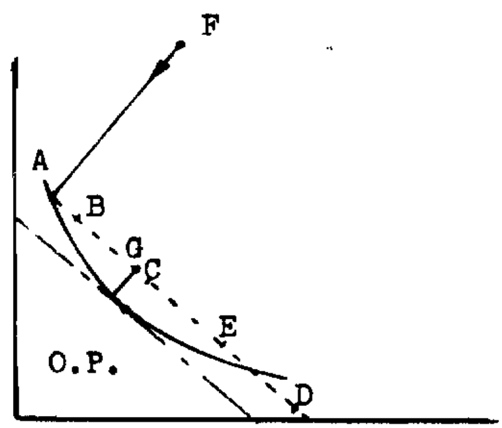


Figure 7. Travel in Side-Step Mode

If, as in Figure 7, the point B is free (not bounded), the initial increment is doubled and step ($\overline{BC}$) is taken. This process is repeated until a violation D occurs. Since an exact knowledge of the displacement boundary is not required, it can be assumed that it lies at the midpoint (E) of the last step ($\overline{CD}$). Then, a free point (G), halfway between A and E is chosen as a suitable point from which to initiate a steepest descent.

If the size of the initial arbitrary increment in the above side-step procedure is ϵ, then the distance ($\overline{AD}$) traveled after i iterations, when a violation occurs, is

$$(\overline{AD}) = (2^i - 1)\,\epsilon \tag{20}$$

and the size of the last increment is

$$(\overline{CD}) = (2^{i-1})\epsilon \tag{21}$$

Thus, the distance $(\overline{DG})$ through which the design point is returned from D is

$$\begin{aligned}(\overline{DG}) &= (\overline{AD}) - (\overline{AG}) = (\overline{AD}) - \left(\frac{\overline{AE}}{2}\right) \\ &= (\overline{AD}) - \frac{1}{2}\left[(\overline{AD}) - (\overline{DE})\right] \\ &= (\overline{AD}) - \frac{1}{2}\left[(\overline{AD}) - \frac{(\overline{CD})}{2}\right] \\ &= \frac{\epsilon}{2}\left[(5)(2)^{i-2} - 1\right]\end{aligned} \tag{22}$$

To prevent any element from becoming disproportionately large, as may occur in early stages of a side-step mode, an additional limitation is introduced to ensure that no element is increased in size by more than a given factor (approx. 1.5 or 2.0) during any one travel in the side-step mode.

If multiple constraints occur at the termination of steepest descent, the vector of the normal to each constraint surface is computed and the vector $\mathbf{V}_C$ is taken as the average of all the normal vectors, that is,

$$\mathbf{V}_C = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{V}_C)_i \tag{23}$$

It is possible that during the side-step process, when the current point is near the optimum, that the next step reaches new constraints while the design point is still bounded on the former constraints. In Figure 8, the solid lines represent constraint surfaces, with the dotted lines defining the tolerance within which boundedness is considered to exist. If a design lies at some point B, within the boundary zone of one constraint only, then the direction of travel may be such that the design moves into the shaded zone A, in which the design is bounded on two constraints. If this occurs, a new averaged normal vector, which includes a contribution from the latest constraint, is generated and the side-step mode restarted. This will direct travel away from both above constraints.

In the event that the bound point at the end of the steepest descent is bounded on side constraints as well as main constraints, a test is incorporated to prevent travel which would tend to violate these side constraints. Travel in these directions is set to zero and a new orthogonal travel vector $\mathbf{V}_D$ is computed.

PAST AND FUTURE PROSPECTS

As part of past developmental studies at Bell, a working optimization program of moderate capacity has been generated. This program, which contains in its library the elements detailed in a previous section, has been used to optimize a considerable range of structures. The capacity of the program, in its present form, is approximately 50 variable elements or 50 degrees of freedom. As part of the checkout and use of the program, details of optimizations performed on two structures are given here.

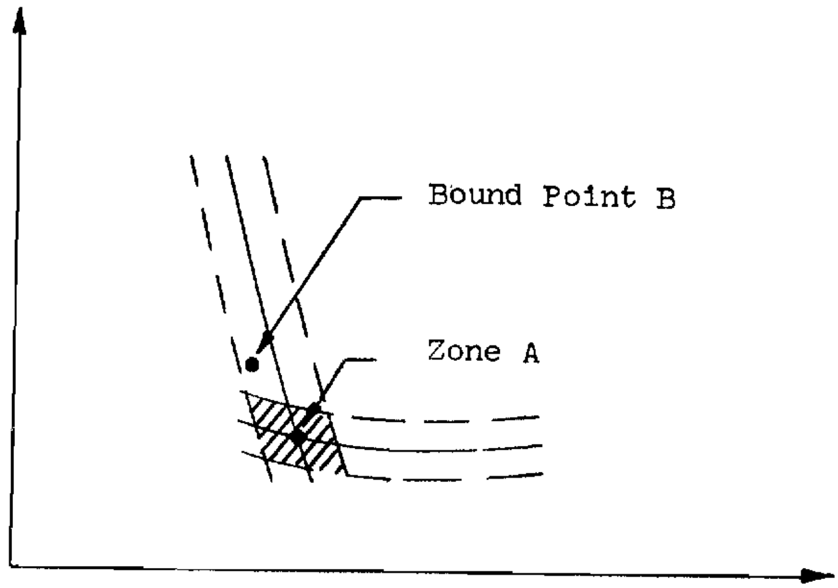


Figure 8. Design Point Within Boundary Zones

In the first example, a plane frame (Figure 9) containing 46 elements and possessing 29 degrees of freedom was subjected to 10 loading conditions. Dimensions, etc. of the frame along with a typical loading case are shown in Figure 9. For this structure, which was actually the most complex system in a large series of optimizations used to check out the workability of the computer program, the computational time was only of the order of six minutes.

In the second example, the twenty-five member space frame (Figure 10) optimized by Fox and Schmit (Reference 4) was examined. This frame has 10 node points and is subjected to six loading conditions, and both stress and displacement limitations were included. In defining the stress limits, the effect of local Euler buckling was taken into account. Since all members are of circular tubular cross-section, they will provide the greatest resistance to buckling when the radius is at its maximum allowable value. r_{max}. The Euler critical stress is then given by

$$\sigma_{cr} = \frac{\pi^2 E I}{A \ell^2} \tag{24}$$

where A is the cross sectional area and ℓ is the length of the strut. Substituting expressions for A and I, Equation 25 reduces to

$$\sigma_{cr} = \frac{\pi^2}{2} \left(\frac{r^2}{\ell^2} \right) E \tag{25}$$

Since both r and ℓ are fixed, the critical stress may be defined independently of the actual variable cross sectional area. Using the Bell program, the final weight obtained was 547.2 lb. which compares well with the figure of 570.4 lb. obtained by Fox and Schmit. The difference in values is principally due to variations in accuracy criteria.

Using this program as a basis, two further computer programs are under current development at Bell. The initial program was written in conventional FORTRAN IV with the allocation

46 elements
29 degrees of freedom

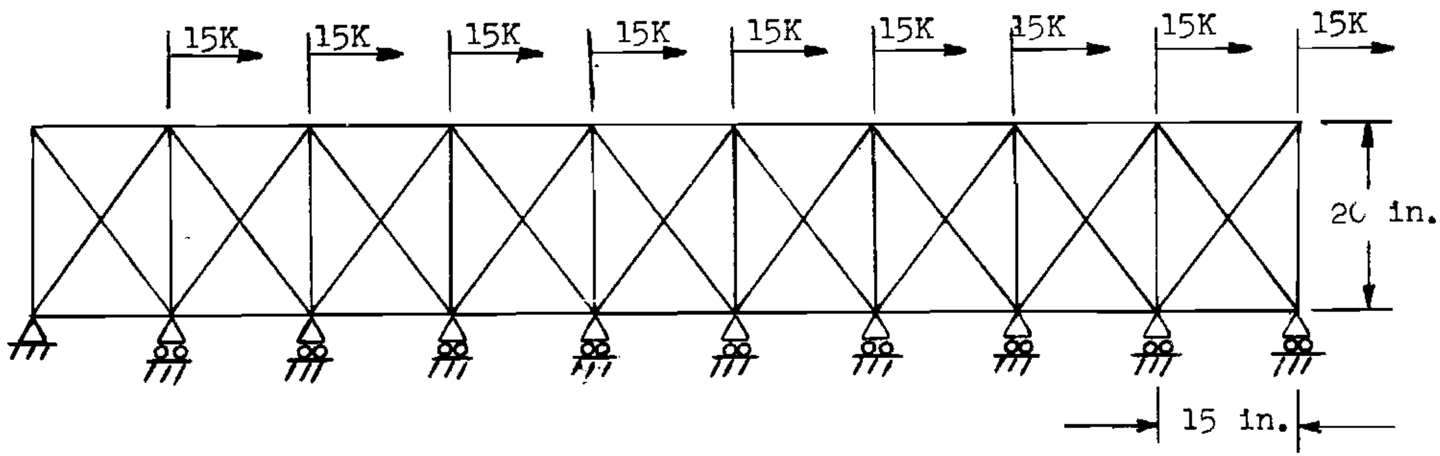


Figure 9. Plane Frame Structure

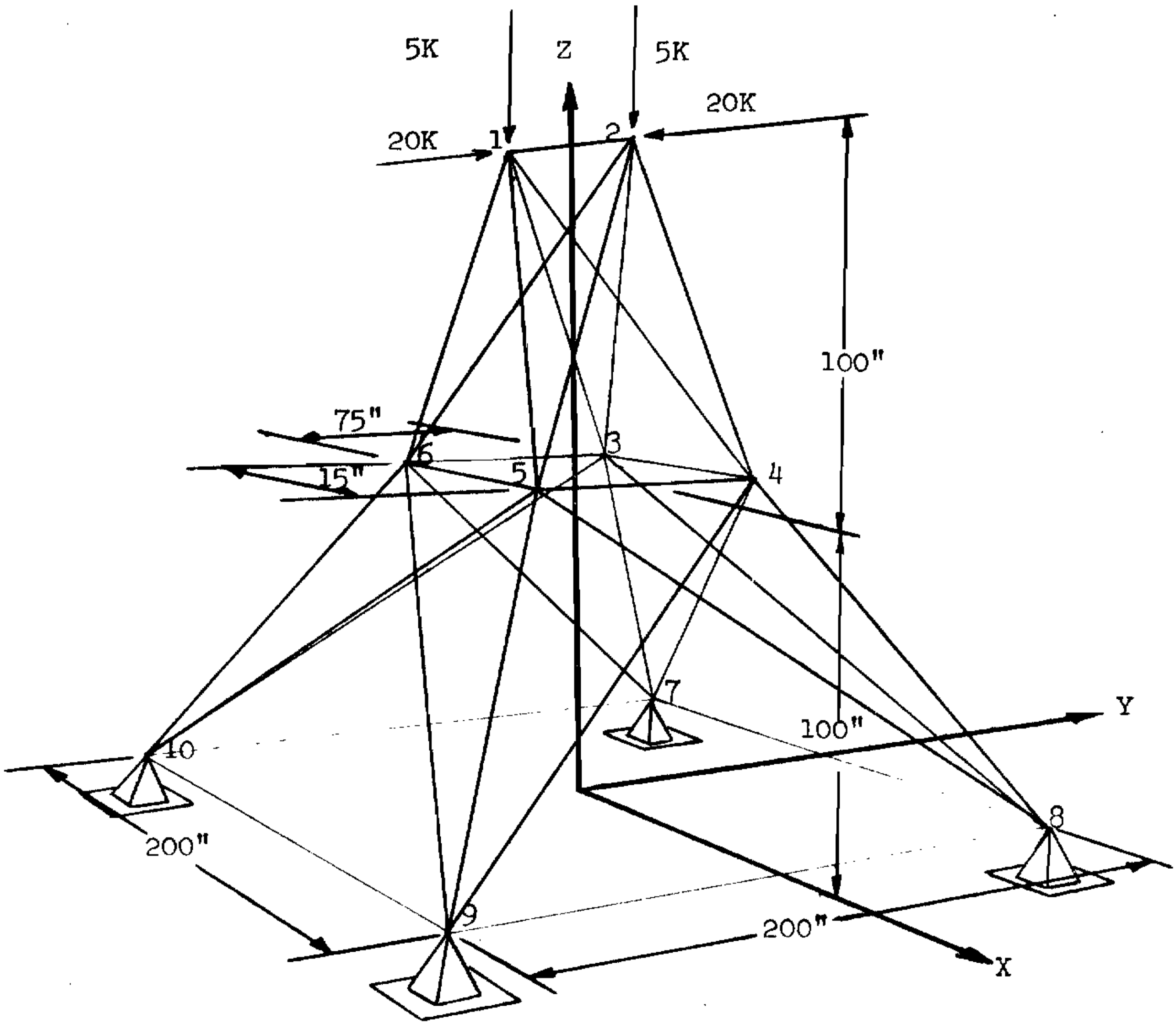


Figure 10. Twenty-five Member Space Frame

in dimension statements of large blocks of storage of fixed size, reducing the flexibility and capacity of the program to the aforementioned level. Accordingly, the original program is being redeveloped to increase the total capacity and to provide a much greater degree of flexibility of operation. The program will still operate effectively entirely within the computer core storage and will have a high operating speed. The anticipated capacity will be approximately 90 variables and over 100 total degrees of freedom. The element library will be extended to included relationships for wing type structures and it is intended that this program will be applied to the optimization of actual airframe structures, either wings or tail units.

In parallel to this moderate capacity high speed program, the development of a program capable of handling approximately 450 variables is in work. Due to the very much increased capacity, very extensive use of storage devices is necessary with a consequent reduction in operating efficiency. This program will possess the open ended element library mentioned previously and will be capable of including such special effects as instability. Although one version of the original optimization programs considers instability, this has not been included in the intermediate capacity high-speed program. In addition, the large program will be capable of handling a limited number of configurational variables using a modification of the alternate step techniques discussed in the present paper.

While the practicability of the method of alternate steps has been demonstrated and its extension into non-linear problems is in hand, the possibility of developing alternate methods is not excluded. Among the new techniques under current review is the Cutting Plane Method. This method which, in effect, generates a series of linear approximations to the nonlinear constraint surfaces and also, if necessary, linearizes the merit function, would appear to present an interesting avenue of further research in the field.

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