NONLINEAR EFFECTS OF LARGE DEFLECTIONS AND MATERIAL DAMPING ON THE STEADY STATE VIBRATIONS OF BEAMS

SHUH-TWU CHOW
P. R. SETHNA

FOREWORD

This report was written at the University of Minnesota, Department of Aeronautics and Engineering Mechanics, Minneapolis, Minnesota 55455, under USAF Contract No. AF 33(615)-1055, Project No. 7351, "Metallic Materials," Task No. 735106, "Behavior of Metals." The work was administered under the direction of Air Force Materials Laboratory, Research and Technology Division, with Mr. J. P. Henderson, MAMD, acting as project engineer.

This report was written during the period April, 1964, to April, 1965.

Our thanks go to Miss Marlene Iverson and Mrs. Nancy Buretta for the preparation of this manuscript.

Manuscript of this report was released by the authors April 1965 for publication as an RTD Technical Report.

This technical report has been reviewed and is approved.

W. J. TRAPP
Chief, Strength and Dynamics Branch
Metals and Ceramics Division

ABSTRACT

Nonlinear forced oscillations of slender beams are studied. The analysis takes into account both the nonlinear effects arising from large deflections of the beam and those arising from nonlinear material behavior. A hysteretic stress-strain law of the Davidenkov type is used in the analysis. Detailed results are given for large amplitude oscillations of beams with hinged ends. Theoretical results for a simply-supported beam are compared with experimental results.

TABLE OF CONTENTS

TABLE OF CONTENTS (CONT'D)

ILLUSTRATIONS

SYMBOLS

ℓ	length of a beam
h	depth of a rectangular beam
b	width of a rectangular beam
c	$\dfrac{I}{A\,\ell^2 \eta}$
e	$\dfrac{E}{k'\,G}\left(\dfrac{I}{A\,\ell^2\,\eta}\right)$
u; w	components of the displacement of the median line of a beam
k'	a shear distortion numerical coefficient -- modifying factor
t	time
a_1; a_n	amplitude
$\bar{u}$	$u/\ell \;\; (= u_1\,\eta + u_2\,\eta^2 + u_3\,\eta^3 + \;. . .)$
$\bar{w}$	$w/\ell \;\; (= w_1\,\eta + w\;\eta^2 + w_3\,\eta^3 + . . .)$
$\tilde{w}$	amplitude of the support motion $(f(t) = \tilde{w}\cos\Omega_n t)$
$\bar{w}_o$	$\tilde{w}/\ell = w_o\,\eta^2$
E	Young's modulus
G	shear modulus
K	transverse load per unit length of the beam
$\bar{K}$	dimensionless load defined by (68)
N	resultant axial force $(= \int \sigma_{xx}\, dA)$
M	resultant bending moment $(= \int z\,\sigma_{xx}\, dA)$
Q	resultant shear force $(= \int \sigma_{xz}\, dA)$
A	cross-sectional area

I	area moment of inertia of the cross section
J	mass moment of inertia per unit length of the beam ($J = \rho I$)
I_1	$2 \int_{o}^{h/2} b\, z\, dz$
I_3	$2 \int_{o}^{h/2} b\, z^3\, dz$
B_m; B_n	Fourier's coefficients of the dimensionless load $\overline{K}$
ϵ_o; ϵ_1; ϵ_2	amplitude of the relative elongation
ϵ	relative elongation in the OX direction at any point ($= \epsilon_{xx}$)
$\overline{\epsilon}$	relative elongation of the median line
θ	angle of rotation of the cross section of a beam
γ	shear angle between axes OX and OZ ($= \epsilon_{xz}$)
α	angle between the deformed median line and the OX axis
σ_{xz}; σ; σ_{xx}	stress components
ρ	mass per unit volume ($= \Gamma/g$)
ν	a material damping parameter
ξ	x/ℓ
Ω_n	excitation frequency ($\Omega_n^2 = \omega_n^2 + \Delta_{1n}\eta + \Delta_{2n}\eta^2 + \ldots$)
τ	$\Omega_n t$ (dimensionless time)
ω_n	linear frequency of the n^{th} normal flexural mode
β_n	$\left(\frac{\rho A \ell^4}{E I} \omega_n^2\right)^{\frac{1}{4}}$

SYMBOLS (CONT'D)

τ_o; $\bar{\tau}_o$	dimensionless time defined by equation (109)
ψ_1	phase angle $(\psi_1 = \psi_{o1} + \psi_{11}\eta + \psi_{21}\eta^2 + \ldots.)$
κ	$\dfrac{I_3 \nu}{2 I \ell}$
η	an arbitrary dimensionless parameter
δ_{mn}	Kronecker delta

SECTION 1

INTRODUCTION

The study of nonlinear vibration of elastic beams and plates has had a long history. The most widely known large deflection theory of plates was established by von Karman in 1910 [1]*. An account of the derivation of the von Karman's equations is given by Timoshenko et al., [2]. Biot [3a, 3b] formulated the theory of large deflection of plates directly from the nonlinear theory of elasticity. Novozhilov [4] used a similar approach. He made the distinction between the case of strong bending and the case of intermediate bending. In the former case, no restriction is placed on the magnitude of the rotation of the cross section; while in the latter case, which is the same as von Karman's theory, the angle of rotation is restricted to be small as compared with unity. Based upon von Karman's theory, a number of problems of rectangular plate were solved by Levy [5a, 5b] and Way [6a], and problems of circular plates by Way [6b] and Stippes and Hausrath [7]. The dynamical equations corresponding to von Karman's theory were studied by Herrmann [8] and more recently by Tadjbakhsh and Saibel [9]. Eringen [10a, 10b], on the other hand, formulated the dynamical equations for beams and later also for membranes which correspond to Novozhilov's strong bending case. The nonlinear free vibrations of string has been studied by Carrier [11a, 11b] and using methods similar to that of Carrier, the nonlinear free vibration of beams having immovable hinged ends has

*Numbers in brackets refer to listed references beginning on page 87.

been treated by Eringen [10a]. Both Carrier and Eringen use a perturbation method. The same perturbation technique was extended in the study of free vibration of rectangular plates with hinged immovable edges by Chu and Herrmann [12]. As for the forced vibration cases, approximate solutions for the nonlinear response of rectangular plates were obtained by Kirchman and Greenspon [13] with the use of the static load-deflection relationship. The nonlinear response of a beam with immovable hinged ends due to an excitation sinusoidally distributed in the space variable was treated by Mettler [14a]. By assuming the resultant axial force to be independent of the spatial coordinate, he was able to obtain a Duffing type response showing a hard spring effect. A similar treatment was also given by Kauderer [15]. The bending equation used by Mettler and Kauderer were the same one as originated by Kirchhoff [16] which are essentially the same as the one of the von Karman's theory. By assuming a first order approximation solution, Yamaki [17] also obtained a hard spring Duffing type response for rectangular and circular plates. The response for circular plates was also studied by Nowinski [18]. In considering the effect on bending due to axial force, an approximate solution in terms of elliptic functions was obtained by Woinowsky-Krieger [19]. McDonald [20] has obtained results in terms of elliptic functions for the case of immovable hinged bars with small motion. Experimentally, the nonlinear hard spring Duffing type response were shown to exist by Lee [21], Tobias [22], Lassiter et al., [23] and Smith et al., [24].

In all the above work the nonlinear effects arise from large deformations. Another source of nonlinearity arises from the nonlinear behavior of the material of the beam.

Resonant structural vibrations can be controlled by the effective use of material damping. The problems of practical interest are those connected with the vibration of metallic structures. As has been shown by Lazan [25], Lazan and Goodman [26], and also others, most metallic materials under cyclic loading exhibit a relationship between stress and strain that is not elastic even at stresses well below the yield point. The stress-strain relationship during loading and unloading are different. The area enclosed by the loading and unloading branches of the stress-strain curve (hysteresis loop) serves as the indication of amount of energy dissipated. This area of the loop is independent of the frequency of cyclic strain. Due to the above reasons, the classical viscoelastic Voight models cannot adequately describe the behavior of metallic materials. In view of this, Mindlin et al., [27] proposed a semi-empirical method. In this work the viscous coefficients are taken inversely proportional to the frequency of the excitation, so as to compensate the frequency effect when the time derivative of strain is taken. Recently Sethna [28], in dealing with transient vibrations of a beam, also adopted the same concept but to a nonlinear visco-elastic stress-strain law. A completely empirical hysteretic stress-strain law has been proposed by Davidenkov [29]. A pair of equations, one for loading, and one for unloading are used. Pisarenko [30], as well as Panovko [31], used Davidenkov's stress-strain law to study the small vibrations of cantilever beams with results that compare very well with experimental results.

It thus appears that there is a large amount of literature concerned with nonlinear effects on beam vibrations where the nonlinear effects arise from large deflections of the beam. In the works concerned with these problems the material is always assumed to be elastic. Then there is a smaller body of literature concerned with nonlinear effects that arise from nonlinear behavior of the material of the beam and in these works the deflections of the beams are assumed small.

The present work is concerned primarily with those problems where the nonlinear effects of large deformations occur along with the nonlinear effects due to nonlinear material behavior.

In surveying the literature on large deflection problems for elastic beams it appears that most of the investigations are concerned with slender pinned beams with immovable ends and a hard spring Duffing type response under sinusoidal excitation is predicted. Relatively little attention is given to problems with other end conditions and in particular nothing is known about the large oscillation of an elastic beam that is truly simply supported, i.e., with one end free to move in the direction of the axis of the undeflected beam. A secondary objective of this work is to investigate this problem.

With these two objectives in mind and for simplicity of presentation this dissertation is divided into two parts. Part one deals with the elastic problem involving large deformations and part two deals with the large deformation problem with nonlinear material behavior. In this latter part Davidenkov's law is used. In both parts, two particular cases are treated. One is the immovable hinged ends case, and the other is the simply supported case. In both cases, steady state solutions are sought. The motions treated belong to Novozhilov's strong bending type, and the equations of motion are similar to those of Eringen's. Both rotatory inertia and shear deformation effects are included in the analysis. The excitations are either in the form of a periodic force uniformly distributed or in the form of a periodic motion exerted at the supports and the nonlinear response of the beam is studied.

The mathematical problems are nonlinear boundary value problems. The method of analysis used here is the perturbation method.

The stability of the nonlinear steady state solutions are also investigated in an appendix* and here an adaptation of the asymptotic method of Krylov, Bogoliubov, and Mitropolsky is used [32, 33].

* The appendix referred to here is the one in a doctoral thesis by the first author at the University of Minnesota, with the same title as that of this report.

SECTION 2
STATEMENT OF THE PROBLEM

Consider a homogeneous rod of uniform cross section, with a longitudinal plane of symmetry. Let the right-handed Cartesian coordinate system Oxyz be chosen as shown in figure 1 with Oxz as the plane of symmetry and Ox passing along the median line. The length of the rod is ℓ

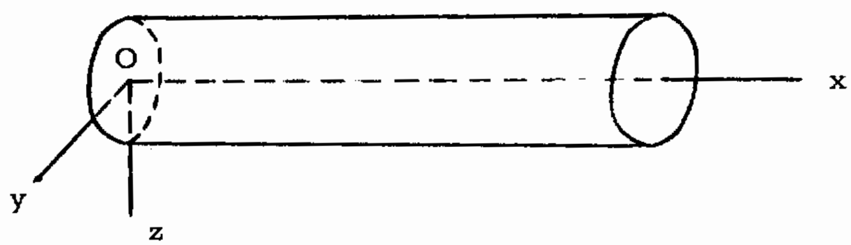


Fig. 1 - Beam in the undeformed state

2.1 DEFORMATION

The following three assumptions will be made:

(a) The rod is free of load in the y-direction and we will consider all quantities independent of y, and that there is no displacement in the y-direction.

(b) Plane cross sections remain plane, though they are not necessarily perpendicular to the median line during the deformation.*

(c) Relative elongation in the z-direction is neglected.

---

*Prescott in deriving Timoshenko beam equation, made use of the same assumption. See [34].

Now examine a segment of a length Δx taken from the beam as shown in figure 2. During the deformation, if the cross section which was originally at x and perpendicular to the median line makes an angle θ with Oz, and if the shear angle is denoted by γ, then α, the angle which the median line makes with x-axis, is:

$$\alpha = \theta + \gamma \tag{1}$$

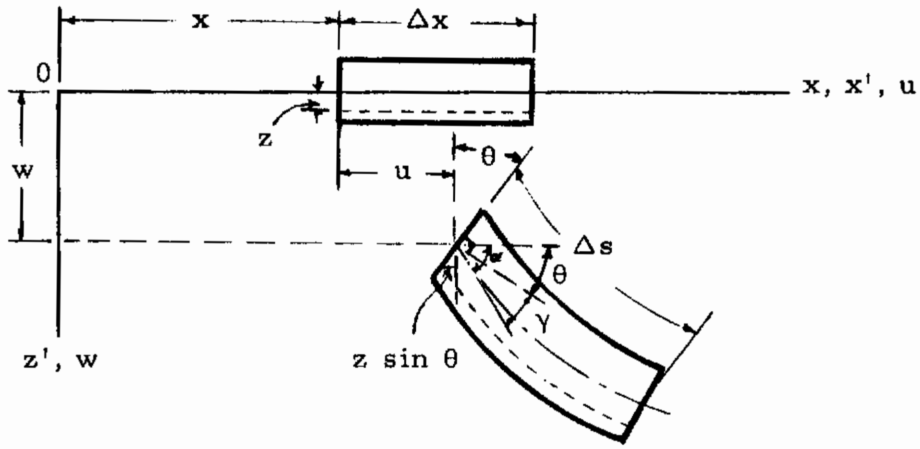


Fig. 2 - Displacement of the beam element

Let a material point with coordinates (x, z) in the undeformed state have new coordinates (x', z') in the deformed state. Then with u(x, y) and w(x, y) as displacements of the median line in the x and z directions respectively, we have

$$\begin{aligned} x' &= x + u - z \sin\theta \\ z' &= w + z \cos\theta \end{aligned} \tag{2}$$

The length of a line element (dx, o) in the deformed state is given by

$$\begin{aligned} ds &= (dx'^2 + dz'^2)^{\frac{1}{2}} \\ &= \left\{ \left[w,_x + (z\cos\theta),_x \right]^2 + \left[1 + u,_x - (z\sin\theta),_x \right]^2 \right\}^{\frac{1}{2}} dx \end{aligned} \tag{3}$$

which is obtained by using (2) and by taking limit as Δx approaches zero. A subscript after a comma will mean differentiation.

The relative elongation in the x-direction is defined as

$$\epsilon = \frac{ds}{dx} - 1 \tag{4}$$

Equations (3) and (4) yield

$$\epsilon = \bar{\epsilon} - z\,\theta,_x \tag{5}$$

with

$$\bar{\epsilon} = \left[w,_x{}^2 + (1 + u,_x)^2 \right]^{\frac{1}{2}} - 1 \tag{6}$$

where $\bar{\epsilon}$ is the relative elongation of the median line.

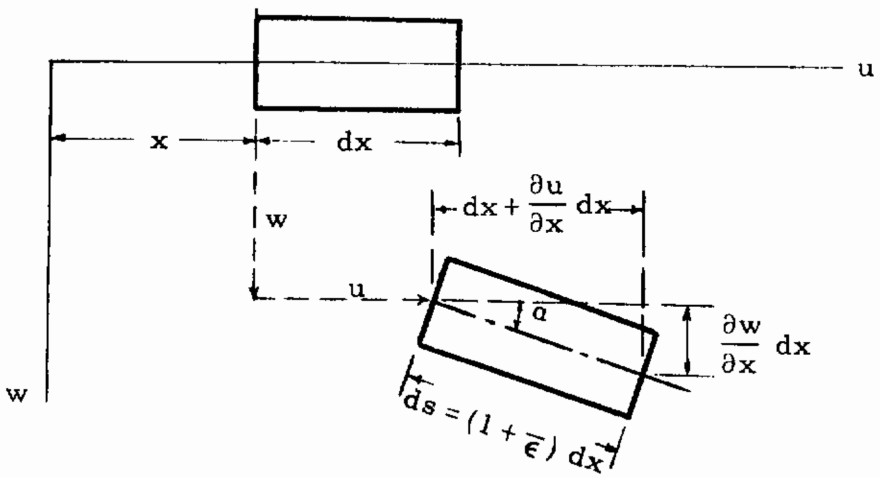


Fig. 3 - Geometry of the deflection curve

From the geometry of the deflection curve as shown in figure 3, we have,

$$\tan\alpha = \frac{w,_x}{1 + u,_x}$$

$$\sin\alpha = \frac{w,_x}{1 + \bar{\epsilon}}$$

$$\cos\alpha = \frac{1 + u,_x}{1 + \bar{\epsilon}} \tag{7}$$

where $\bar{\epsilon}$ is as defined in (6)

2.2 STRESS-STRAIN RELATIONS

If σ_{xx} is the normal stress, and σ_{xz} is the shear stress, then by Hooke's law

$$\sigma_{xx} = E\,\epsilon = E\,\bar{\epsilon} - Ez\,\theta,_x$$

$$\sigma_{xz} = G\,\gamma \qquad (8)$$

Now let

$$N = \int \sigma_{xx}\,dA = EA\bar{\epsilon}$$

$$M = \int \sigma_{xx}\,z dA = -EI\theta,_x$$

$$Q = \int \sigma_{xz}\,dA = k'GA\gamma \qquad (9)$$

In (9), A is the cross-sectional area, I is the area moment of inertia of cross section, k' is the shear-deflection coefficient which is a modifying factor as used by Timoshenko [35] and others, the value of which depends on the shape of the cross section. For instance, for a rectangular cross section, k' = 0.833.

2.3 EQUATIONS OF MOTION

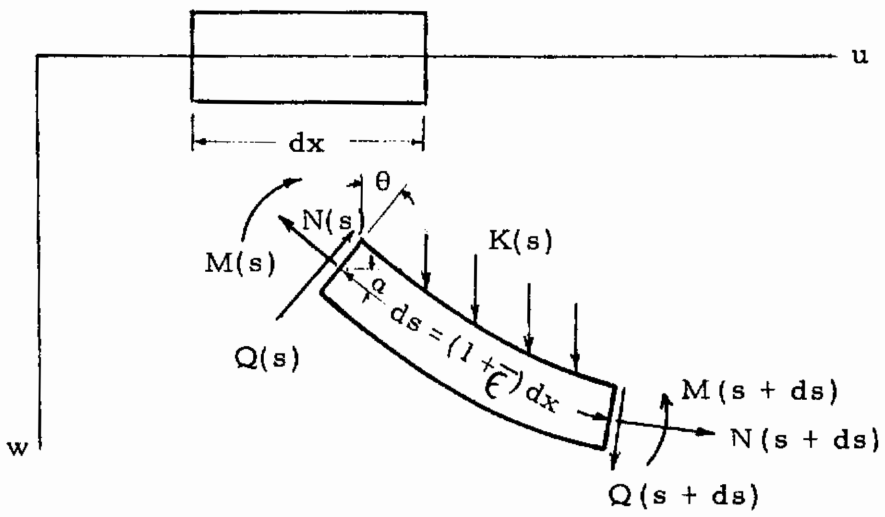


Fig. 4 - Loading diagram

A free body diagram of an element of the beam is shown in figure 4. The element is bounded by the top and bottom surfaces of the beam and on the two sides is bounded by cross sections of beam that are plane and which were normal to the center line before deformation. The shear forces (Q) are parallel to these plane elements but the normal forces (N) are along the center line and are not necessarily normal to these planes. It will be assumed that the Cartesian axis system, Oxyz (reference frame) can have an acceleration $\frac{d^2 f(t)}{dt^2}$ in the z direction. Then from figure 4 the equations of motion are:

$$(N \cos \alpha),_x - (Q \sin \theta),_x = \rho A u,_{tt} \tag{10}$$

$$(N \sin \alpha),_x + (Q \cos \theta),_x + K(x,t) = \rho A (w + f(t)),_{tt} \tag{11}$$

$$M,_x - Q(1 + \bar{\epsilon}) \cos (\alpha - \theta) = -J\theta,_{tt} \tag{12}$$

where K (x, t) is the vertical load during vibration, ρ is the mass density, J is the mass moment of inertia per unit length.

There are eight unknowns occurring in equations (10), (11), and (12), namely N, Q, M, α, θ, u, w, and $\bar{\epsilon}$. But in addition to (10), (11), and (12), we have one equation, (6), another one from (7), three more from (9). Thus, there are enough equations to take care of the unknowns.

We will proceed to reduce the above eight equations to three equations and express them in terms of u, w, and θ. Using equations (7), (6), and (9), equation (12) gives

$$Q = \frac{J \theta,_{tt} - EI \theta,_{xx}}{\left[w,_x \sin\theta + (1 + u,_x) \cos \theta\right]} \tag{13}$$

From equations (1) and (9), we have

$$\alpha = \theta + \gamma = \theta + \frac{Q}{k' AG} \tag{14}$$

From equations (7) and (9), we have

$$N = EA\bar{\epsilon} = EA\left(\frac{1 + u,_x}{\cos \alpha} - 1\right) \tag{15}$$

Substituting equations (7), (13), (15), into (10) and (11), we obtain

$$-\rho A u,_{tt} + EA\left\{(1 + u,_x) - \frac{1 + u,_x}{\left[w,_x{}^2 + (1 + u,_x)^2\right]^{\frac{1}{2}}}\right\},_x - \left\{\frac{J\,\theta,_{tt} - EI\,\theta,_{xx}}{w,_x + (1 + u,_x)\cot\theta}\right\},_x = 0 \tag{16}$$

$$-\rho A w,_{tt} + EA\left\{w,_x - \frac{w,_x}{\left[w,_x{}^2 + (1 + u,_x)^2\right]^{\frac{1}{2}}}\right\},_x + \left\{\frac{J\,\theta,_{tt} - EI\,\theta,_{xx}}{w,_x \tan\theta + (1 + u,_x)}\right\},_x + K(x, t) - \rho A\, f(t),_{tt} = 0 \tag{17}$$

Substituting equations (14) and (13) into the first of equation (7), we obtain

$$\tan\left\{\theta + \frac{J\,\theta,_{tt} - EI\,\theta,_{xx}}{k'\,AG\left[w,_x \sin\theta + (1 + u,_x)\cos\theta\right]}\right\} = \frac{w,_x}{1 + u,_x} \tag{18}$$

Thus, we have three equations, (16), (17), and (18) in terms of three unknowns, w, u, and θ. Up to now, we have not put any restriction on the magnitude of displacement (u, w) and rotation (θ). As long as the strains are within the elastic limit, these three equations can give results for values of u, w, and θ of any magnitude.

2.4 COMPARISON OF THE EQUATIONS WITH OTHER KNOWN THEORIES

Either equations (19), (11), and (12) or equations (16), (17), and (18) can be reduced to equations corresponding to more simplified theories.

2.4.1 Eringen's Equations of Motion

By neglecting the shear deformation, i.e., by letting $\theta = \alpha$, and by dropping the external loads, equations (10), (11), and (12) can be reduced to Eringen's equations [10a].

2.4.2 Equations According to von Karman's Finite Deformation Theory

If we drop shear deformation, i.e., put $\theta = \alpha$, omit rotary inertia term, i.e., put $J = 0$, omit the term $f(t)$, and consider $\bar{\epsilon}$ as very small compared with unity, i.e., put $1 + \bar{\epsilon} \approx 1$, from equations (10), (11), (12), and (15), we obtain

$$-\rho A\, u,_{tt} + EA\Big[(1 + u,_x) - \cos\theta\Big],_x - (Q\sin\theta),_x = 0 \qquad (19)$$

$$-\rho A\, w,_{tt} + EA\Big[(1 + u,_x)\tan\theta - \sin\theta\Big],_x + (Q\cos\theta),_x + K(x,t) = 0 \qquad (20)$$

$$-EI\,\theta,_{xx} - Q = 0 \qquad (21)$$

The second equation of (7) becomes

$$\sin\theta = w,_x \qquad (22)$$

We further assume that θ is sufficiently small so that we can neglect all quantities which have the same order of magnitude as θ^3.

Accordingly, (22) becomes

$$\theta = w,_x \qquad (23)$$

Substituting (23) into (21)

$$Q = -EI\, w,_{xxx} \qquad (24)$$

Substituting (24), (23) into (19) and (20), ar etaining only first and second order terms*, we obtain

*In deriving (25), the term $(Q\sin\theta),$ is dropped and in derivating (26), $\tan\theta$ is approximated by $(\theta + \theta^3/3)$, $\sin\theta$ by $(\theta - \theta^3/3!)$ and $(Q\cos\theta),_x$ by $Q,_x$.

$$\rho A\, u,_{tt} - EA\,(u,_x + \tfrac{1}{2}\, w,_x{}^2),_x = 0 \qquad (25)$$

$$\rho A\, w,_{tt} - EA\,(u,_x\, w,_x + \tfrac{1}{2}\, w,_x{}^3),_x + EI\, w,_{xxxx} = K\,(x, t) \qquad (26)$$

Equations (25) and (26) are equations which correspond to von Karman's theory of plates and are also the equations used by Kauderer [15].

2.4.3 Linear Theory and the Timoshenko's Beam Equations

First linearizing the second of equation (7), we obtain

$$\alpha = w,_x$$

Using equation (14), it becomes

$$\theta + \frac{Q}{k'\,AG} = w,_x$$

or

$$Q = k'\,(w,_x - \theta)\,AG \qquad (27)$$

Now linearizing equation (15),

$$N = EA\, u,_x \qquad (28)$$

Finally by using equations (27) and (28), and by linearizing equations (10), (11), and (12), we obtain

$$-\rho A\, u,_{tt} + EA\, u,_{xx} = 0 \qquad (29)$$

$$-\rho A\, w,_{tt} + Q,_x = 0 \qquad (30)$$

$$\rho I\, \theta,_{tt} - EI\, \theta,_{xx} - Q = 0 \qquad (31)$$

We have dropped the external load term and the term f(t) in equation (30), and we have let $J = \rho I$ in equation (31). Equation (29) is the wave equation for the longitudinal waves.

Substituting equation (27) in equations (31) and (30), we obtain

$$-\rho I\, \theta,_{tt} + EI\, \theta,_{xx} + k' (w,_{x} - \theta)\, AG = 0 \qquad (32)$$

$$\rho A\, w,_{tt} - k' (w,_{x} - \theta),_{x}\, AG = 0 \qquad (33)$$

Now eliminating θ in equation (32) by using equation (33), we obtain

$$\rho A\, w,_{tt} + EI\, w,_{xxxx} - (\rho I + \frac{EI\rho}{k'G})\, w,_{xxtt} + \frac{\rho^2 I}{k'G}\, w,_{tttt} = 0 \qquad (34)$$

Equation (34), or equations (32) and (33) are called Timoshenko's beam equations.*

*See Reference [35] p. 331, equations (ℓ) (m) or equation (129).

SECTION 3

SOLUTION OF THE PROBLEM

3.1 EQUATIONS OF MOTION IN DIMENSIONLESS FORM

Equations (16), (17), and (18) will be put into dimensionless form. First the following notation is introduced

$$\frac{\beta_n^4}{\omega_n^2} = \frac{\rho A \ell^4}{E I} \quad , n = 1, 2, 3, \ldots . \tag{35}$$

where ω_n is the natural frequency of flexural vibration of the n^{th} normal mode, and β_n is the corresponding characteristic number, both for the linear case*.

We shall be interested in the steady state solutions of the problem under conditions of sinusoidal excitation. It is not difficult to show that the interesting results occur when the excitation frequency is in the neighborhood of one of the linear natural frequencies of flexural vibrations of the system. It will therefore be assumed that the frequency of excitation Ω_n is in the neighborhood of the n^{th} linear frequency ω_n. The symbol Ω_n will be used to make the time dimensionless, thus:

$$\tau = \Omega_n t \tag{36}$$

Instead of excitation by an external load, the periodic motions of the system can also be caused by a sinusoidal motion of the support. Thus we let

$$f(t) = \tilde{w} \cos \Omega_n t$$

or

$$\ddot{f}(t) = \Omega_n^2 \tilde{w} \cos \Omega_n t \tag{37}$$

*See reference [35], p. 325 (116).

Now let us introduce the following dimensionless quantities

$$\bar{u} = \frac{u}{\ell}\,,\quad \bar{w} = \frac{w}{\ell}\,,\quad \xi = \frac{x}{\ell}\,,\quad \bar{w}_o = \frac{\tilde{w}}{\ell} \tag{38}$$

where ℓ is the length of the beam.

Substituting (35), (36), (37), and (38) into (16) and (17), and multiplying both equations by ℓ/AE, we obtain the equations in dimensionless form:

$$-\frac{I}{A\ell^2}\,\frac{\beta_n^{\,4}\,\Omega_n^{\,2}}{\omega_n^{\,2}}\,\bar{u}_{,\tau\tau} + \left\{(1+\bar{u}_{,\xi}) - \frac{1+\bar{u}_{,\xi}}{\left[\bar{w}_{,\xi}^{\,2} + (1+\bar{u}_{,\xi})^2\right]^{\frac{1}{2}}}\right\}_{,\xi}$$

$$-\frac{I}{A\ell^2}\left\{\frac{\frac{J\ell^2}{EI}\,\Omega_n^{\,2}\,\theta_{,\tau\tau} - \theta_{,\xi\xi}}{\bar{w}_{,\xi} + (1+\bar{u}_{,\xi})\cot\theta}\right\}_{,\xi} = 0 \tag{39}$$

$$-\frac{I}{A\ell^2}\,\frac{\beta_n^{\,4}\,\Omega_n^{\,2}}{\omega_n^{\,2}}\,\bar{w}_{,\tau\tau} + \left\{\bar{w}_{,\xi} - \frac{\bar{w}_{,\xi}}{\left[\bar{w}_{,\xi}^{\,2} + (1+\bar{u}_{,\xi})^2\right]^{\frac{1}{2}}}\right\}_{,\xi}$$

$$+\frac{I}{A\ell^2}\left\{\frac{\left(\frac{J\ell^2}{EI}\,\Omega_n^{\,2}\,\theta_{,\tau\tau} - \theta_{,\xi\xi}\right)}{\bar{w}_{,\xi}\tan\theta + (1+\bar{u}_{,\xi})}\right\}_{,\xi} + \frac{K\ell}{AE} - \frac{I}{A\ell^2}\,\frac{\beta_n^{\,4}\,\Omega_n^{\,2}}{\omega_n^{\,2}}\,\bar{w}_o\cos\tau = 0 \tag{40}$$

Similarly, equation (18) becomes

$$(1+\bar{u}_{,\xi})\tan\left[\theta + \frac{J\Omega_n^{\,2}\,\theta_{,\tau\tau} - \frac{EI}{\ell^2}\,\theta_{,\xi\xi}}{k'GA\left[\bar{w}_{,\xi}\,\sin\theta + (1+\bar{u}_{,\xi})\cos\theta\right]}\right] = \bar{w}_{,\xi} \tag{41}$$

3.2 PERTURBATION SERIES

To solve equations (39), (40), and (41), we shall develop a perturbation procedure. First we assume the following series:

$$\bar{w} = w_1 \eta + w_2 \eta^2 + w_3 \eta^3 + \ldots.$$

$$\bar{u} = u_1 \eta + u_2 \eta^2 + u_3 \eta^3 + \ldots.$$

$$\Omega_n^2 = \omega_n^2 + \Delta_{1n} \eta + \Delta_{2n} \eta^2 + \ldots.$$

$$\theta = \theta_1 \eta + \theta_2 \eta^2 + \theta_3 \eta^3 + \ldots. \tag{42}$$

where η is a small dimensionless parameter, $0 < \eta \ll 1$ depending on the slenderness of the beam as discussed below. The third of equations (42) is written in a manner such that Ω_n stays in the neighborhood of the n^{th} linear natural frequency of flexural vibration.

In equations (39) and (40), there appears a quantity $I/A\ell^2$. If we let $I = Ar^2$, where r is the radius of gyration of the cross section, then $I/A\ell^2 = r^2/\ell^2$, which is the inverse of the square of the slenderness ratio. In the following, we shall consider only slender beams. Thus we let

$$\frac{I}{A\ell^2} = c\eta \tag{43}$$

where c is a dimensionless constant of zero order in η. We also let

$$\frac{E}{k' G} \frac{I}{A\ell^2} = e\eta \tag{44}$$

To express θ in terms of w and u, equation (41) can be used. By using series expansions and by using equations (42), (43), and (44), both sides of equation (41) are expressed in power series in η, and then by equating coefficients of like powers of η, we obtain

$$\theta_1 = w_{1,\xi}$$

$$\theta_2 = w_{2,\xi} - u_{1,\xi}\, w_{1,\xi} + e w_{1,\xi\xi\xi}$$

$$\theta_3 = w_{3,\xi} - u_{1,\xi}(w_{2,\xi} - u_{1,\xi}\, w_{1,\xi} + e w_{1,\xi\xi\xi}) - \frac{w_{1,\xi}^{\;3}}{3} - w_{1,\xi}\, u_{2,\xi}$$

$$- e c\, \beta_n^{\;4}\, w_{1,\xi tt} \frac{J}{\rho I} + e w_{2,\xi\xi\xi} - e(u_{1,\xi} w_{1,\xi})_{,\xi\xi} + e^2\, w_{1,\xi\xi\xi\xi\xi} \qquad (45)$$

From equation (6) and first of equation (9):

$$\frac{N}{EA} = \bar{\epsilon} = \left[\bar{w},_{\xi}^{\;2} + (1 + \bar{u},_{\xi})^2\right]^{\frac{1}{2}} - 1 \qquad (46)$$

Using approximations, we have

$$\frac{N}{EA} = \bar{\epsilon} = u_{1,\xi}\,\eta + \left(u_{2,\xi} + \tfrac{1}{2}\, w_{1,\xi}^{\;2}\right)\eta^2$$

$$+ \left(u_{3,\xi} + w_{1,\xi}\, w_{2,\xi} - \tfrac{1}{2}\, u_{1,\xi}\, w_{1,\xi}^{\;2}\right)\eta^3 + \ldots \qquad (47)$$

From equation (13),

$$\frac{Q}{EA} = \frac{-\dfrac{I}{A\ell^2}\left(\theta,_{\xi\xi} - \dfrac{I}{A\ell^2}\,\dfrac{J}{\rho I}\,\dfrac{\beta_n^{\;4}\,\Omega_n^{\;2}}{\omega_n^{\;2}}\,\theta,_{\tau\tau}\right)}{(1 + \bar{u},_{\xi})\cos\theta + \bar{w},_{\xi}\,\sin\theta}$$

Using series expansions and equations (42), (43), and (45), we have

$$\frac{Q}{EA} = \eta^2\,(-c\, w_{1,\xi\xi\xi}) + \eta^3\,(2c\, u_{1,\xi}\, w_{1,\xi\xi\xi} - c\, w_{2,\xi\xi\xi} + c\, u_{1,\xi\xi\xi}\, w_{1,\xi}$$

$$- c\, e\, w_{1,\xi\xi\xi\xi\xi} + \frac{J}{\rho I}\, c^2\, \beta_n^{\;4}\, w_{1,\xi\tau\tau}) + \ldots \qquad (48)$$

Finally, using the series expansions of sine, cosine and $(1+x)^{-\frac{1}{2}}$ and substituting equations (42), (43), (44), and (45) into equations (39) and (40), we obtain

$$-c\eta \frac{\beta_n^4}{\omega_n^2}(\omega_n^2 + \Delta_{1n}\eta + \ldots)(u_{1,\tau\tau}\eta + u_{2,\tau\tau}\eta^2 + \ldots.)$$
$$+\Big[(1+u_{1,\xi}\eta + u_{2,\xi}\eta^2 + \ldots)(+u_{1,\xi}\eta + u_{2,\xi}\eta^2 + \tfrac{1}{2}w_{1,\xi}^2\eta^2$$
$$-u_{1,\xi}^2\eta^2 + \ldots)\Big]_{,\xi} + c\eta(w_{1,\xi\xi\xi}\, w_{1,\xi}\eta^2 + \ldots)_{,\xi} = 0 \qquad (49)$$

$$-c\eta \frac{\beta_n^4}{\omega_n^2}(\omega_n^2 + \Delta_{1n}\eta + \ldots)(w_{1,\tau\tau}\eta + w_{2,\tau\tau}\eta^2 + \ldots - \bar{w}_o \cos\tau)$$
$$+\Big[(w_{1,\xi}\eta + w_{2,\xi}\eta^2 + \ldots)(u_{1,\xi}\eta + u_{2,\xi}\eta^2 + \tfrac{1}{2}w_{1,\xi}^2\eta^2 - u_{1,\xi}^2\eta^2$$
$$+\ldots)\Big]_{,\xi} + c\eta\Big[(c\eta^2\beta_n^4 w_{1,\xi\tau\tau}\frac{J}{\rho I} - w_{1,\xi\xi\xi}\eta - w_{2,\xi\xi\xi}\eta^2$$
$$+u_{1,\xi\xi\xi}\,w_{1,\xi}\eta^2 + u_{1,\xi}\,w_{1,\xi\xi\xi}\eta^2 - e w_{1,\xi\xi\xi\xi\xi}\eta^2 + \ldots)(1 - \frac{w_{1,\xi}^2}{2!}\eta^2$$
$$+\ldots)(1 - u_{1,\xi}\eta - u_{2,\xi}\eta^2 - \tfrac{1}{2}w_{1,\xi}^2\eta^2 + u_{1,\xi}^2\eta^2 + \ldots)\Big]_{,\xi} + \frac{K\ell}{EA} = 0 \qquad (50)$$

Equating coefficients of like powers of η, we get

$$\eta^1:\ u_{1,\xi\xi} = 0^{*} \qquad (51)$$

$$\eta^2:\ \beta_n^4 w_{1,\tau\tau} + w_{1,\xi\xi\xi\xi} = 0 \qquad (52)$$

*By using the boundary conditions, we will show later that $u_1 \equiv 0$. Consequently, for brevity, we will drop all u_1 terms henceforth.

$$\eta^2:\ (u_{2,\xi} + \tfrac{1}{2} w_{1,\xi}^{\ 2})_{,\xi} = 0 \tag{53}$$

$$\eta^3:\ \beta_n^{\ 4} w_{2,\tau\tau} + w_{2,\xi\xi\xi\xi} + \frac{\beta_n^{\ 4}\Delta_{1n}}{\omega_n^{\ 2}} w_{1,\tau\tau} - \frac{1}{c}(w_{1,\xi}\, u_{2,\xi} + \tfrac{1}{2} w_{1,\xi}^{\ 3})_{,\xi} + e\, w_{1,\xi\xi\xi\xi\xi\xi} - c\,\beta_n^{\ 4} w_{1,\xi\xi\tau\tau}\frac{J}{\rho I} = \frac{K_o \ell^3}{EI} + w_o\,\beta_n^{\ 4}\cos\tau \tag{54*}$$

$$\eta^3:\ -c\,\beta_n^{\ 4} u_{2,\tau\tau} + (u_{3,\xi} + w_{1,\xi}\, w_{2,\xi})_{,\xi} + c(w_{1,\xi}\, w_{1,\xi\xi\xi})_{,\xi} = 0 \tag{55}$$

$$\begin{aligned}\eta^4:\ &\beta_n^{\ 4} w_{3,\tau\tau} + w_{3,\xi\xi\xi\xi} + \frac{\beta_n^{\ 4}}{\omega_n^{\ 2}}(\Delta_{1n} w_{2,\tau\tau} + \Delta_{2n} w_{1,\tau\tau}) \\ &- \frac{1}{c}(u_{2,\xi}\, w_{2,\xi} + \frac{3}{2} w_{1,\xi}^{\ 2} w_{2,\xi} + w_{1,\xi}\, u_{3,\xi})_{,\xi} + (e w_{2,\xi\xi\xi} \\ &+ e^2 w_{1,\xi\xi\xi\xi\xi} - \frac{1}{3} w_{1,\xi}^{\ 3} - w_{1,\xi}\, u_{2,\xi} - \frac{J}{\rho I}\, 2\,ec\,\beta_n^{\ 4} w_{1,\xi\tau\tau})_{,\xi\xi\xi} \\ &- \Big[c\,\beta_n^{\ 4} w_{2,\xi\tau\tau}\frac{J}{\rho I} + \frac{c\,\beta_n^{\ 4}\Delta_{1n}}{\omega_n^{\ 2}} w_{1,\xi\tau\tau}\frac{J}{\rho I} \\ &+ w_{1,\xi\xi\xi}(u_{2,\xi} + \tfrac{1}{2} w_{1,\xi}^{\ 2})\Big]_{,\xi} = \frac{\Delta_{1n}}{\omega_n^{\ 2}}\beta_n^{\ 4} w_o \cos\tau\end{aligned} \tag{56}$$

*where $K = K_o\eta^2$, and $\bar{w}_o = w_o\eta^2$ are substituted

The external load term $\left(\frac{K_o \ell^3}{EI}\right)$ and the term due to the sinusoidal motion of the supports $\left(w_o \beta_n^4 \cos \tau\right)$ are made to appear in equation (54). In this way, we are starting the perturbation with the solution of equation (52) as the generating solution. Thus, the generating solution is a solution of the free vibration problem.

The above equations are applicable for all types of conventional boundary conditions.* Note that since the axial displacement u is also considered in the present theory, a distinction must be made between the conventional simply supported case and the both ends hinged case. In the former case, one end of the beam is allowed to move axially; while in the latter case, both ends are immovable. These two cases will be worked out in detail. The technique used for these two cases can also be applied to other cases, although the analysis for other cases is much more complicated.

3.3 BOTH ENDS HINGED CASE

The boundary conditions are

$$\text{At } \xi = 0;\ 1, \quad w = 0, \quad u = 0 \quad \text{and } M = 0 \tag{57}$$

Since $M = -E\,I\,\theta_{,x}$ we have at $\xi = 0;\ 1,\ \theta_{,x} = 0$

If the series expansions for w, u and θ of equation (42) are used, then conditions (57) will be satisfied if

$$\text{at } \xi = 0,\ 1 \qquad \begin{aligned} w_i &= 0 \\ u_i &= 0 \qquad i = 1, 2, 3 \\ \theta_{i,\xi} &= 0 \end{aligned} \tag{58}$$

*For instance, fixed ends, free ends, etc.

From (51) and (52)

$$u_{1,\xi\xi} = 0 \tag{59}$$

$$\beta_n^{4} w_{1,\tau\tau} + w_{1,\xi\xi\xi\xi} = 0 \tag{60}$$

Equation (59), together with the boundary conditions that at $\xi = 0, 1$, $u_1 = 0$, gives the following solution

$$u_1 \equiv 0 \tag{61}$$

Solving equation (60) under the boundary conditions (58), we have

$$\beta_n = n\pi$$

$$w_1 = \sum_{m=1}^{\infty} a_m \sin m\pi\xi \cos \frac{m^2}{n^2}\tau \tag{62}$$

where n is any positive integer.

Since

$$\theta_1 = w_{1,\xi}$$

$$\therefore \ \theta_1 = \sum_{m=1}^{\infty} a_m \, m\,\pi \cos m\pi\xi \cos \frac{m^2}{n^2}\tau \tag{63}$$

Using the solution (62) for w_1, and taking into consideration of the boundary conditions that at $\xi = o, 1$, $u_2 = o$, equation (53) yields the following solution:

$$u_{2,\xi} + \frac{1}{2} w_{1,\xi}^{\;2} = \frac{1}{4} \sum_{m=1}^{\infty} a_m^{\;2} m^2 \pi^2 \cdot \cos^2 \frac{m^2}{n^2}\tau \tag{64}$$

Equations (61) and (62) give the solution in the first approximation. This solution is the same as that for the free vibration of the linearized problem. In order to proceed to the next approximation, it is advisable to take

$$w_1 = a_n \sin n\pi\xi \cos\tau, \; n = 1, 2, 3, \ldots \tag{65}$$

$$\tau = \Omega_n t$$

i.e., in equation (62) we let

$$a_m = 0, \; m \neq n$$

$$a_m = a_n, \; m = n$$

and consequently equation (64) becomes

$$u_{2,\xi} + \frac{1}{2} w_{1,\xi}^{\;2} = \frac{1}{4} (a_n n \pi)^2 \cos^2 \tau \tag{66}$$

A justification for taking w_1 in the above form is given in Appendix A.* We turn to equation (54)

$$(n\pi)^4 w_{2,\tau\tau} + w_{2,\xi\xi\xi\xi} + \frac{(n\pi)^4 \Delta_{1n}}{\omega_n^2} w_{1,\tau\tau} - \frac{1}{c} \left(u_{2,\xi} w_{1,\xi} + \frac{1}{2} w_{1,\xi}^{\;3}\right)_{,\xi}$$

$$+ e\, w_{1,\xi\xi\xi\xi\xi\xi} - c(n\pi)^4 w_{1,\xi\xi\tau\tau} \frac{J}{\rho I} = \frac{K_o \ell^3}{EI} + w_o n^4 \pi^4 \cos\tau \tag{67}$$

Express the loading as a Fourier's sine series

$$\bar{K} = \frac{K_o \ell^3}{EI} + w_o n^4 \pi^4 \cos\tau = \left(\sum_{m=1}^{\infty} B_m \sin m\pi\xi \right) \cos \Omega_n t \tag{68}$$

* Refer to note on page 5.

Using equations (65), (66) and (68), equation (67) becomes

$$(n\pi)^4 w_{2,\tau\tau} + w_{2\xi\xi\xi\xi} - \sum_{m=1}^{\infty} B_m \sin m\pi\xi \cos\tau$$

$$-(n\pi)^4 a_n \frac{\Delta_{1n}}{\omega_1^2} \sin n\pi\xi \cos\tau + \frac{1}{16c} a_n^3 n^4 \pi^4 \sin n\pi\xi$$

$$(3\cos\tau + \cos 3\tau) - (e + c\frac{J}{\rho I})(n\pi)^6 a_n \sin n\pi\xi \cos\tau = 0 \quad (69)$$

In the right side of (69), the terms with the factor $\sin n\pi\xi$ $\cos\tau$ are the secular terms, so called because they would produce the solution for "w_2" of the form $\tau A_n \sin\tau \sin n\pi\xi$, which then would destroy the periodicity of the solution. Just as in the theory of oscillation for nonlinear systems [30], in the process of eliminating these secular terms, we will be able to find the response relation of the system. Following Pisarenko [30] we let equation (69) be multiplied through by $\sin n\pi\xi \cos\tau$, and then integrate the resulting equation throughout the whole length of the beam (ξ from 0 to 1) and over one cycle (τ from 0 to 2π).

Note that by using integration by parts and by applying the boundary condition of (58), we have*

$$\int_0^{2\pi}\int_0^1 \left[(n\pi)^4 w_{2,\tau\tau} + w_{2,\xi\xi\xi\xi}\right] \sin n\pi\xi \cos\tau \, d\xi \, d\tau = 0 \qquad (70)$$

Taking into the consideration equation (70), we have

*See Reference [30] Chapter II, Section 10, p. 54.

$$\int_0^{2\pi} \int_0^1 \Bigg[\sum_{m=1}^{\infty} B_m \sin m\pi\xi \cos\tau + (n\pi)^4 a_n \frac{\Delta_{1n}}{\omega_n^2} \sin n\pi\xi \cos\tau$$

$$- \frac{1}{16c} a_n^3 (n\pi)^4 \sin n\pi\xi \,(3\cos\tau + \cos 3\tau)$$

$$+ \left(e + c\frac{J}{\rho I}\right)(n\pi)^6 a_n \sin n\pi\xi \cos\tau \Bigg] \sin n\pi\xi \cos\tau = 0 \qquad (71)$$

Carrying out the integration, we obtain

$$B_n + (n\pi)^4 a_n \frac{\Delta_{1n}}{\omega_n^2} - \frac{3}{16\,c} a_n^3 (\pi n)^4 + \left(e + c\frac{J}{\rho I}\right)(n\pi)^6 a_n = 0$$

or

$$\frac{\Delta_{1n}}{\omega_n^2} = \frac{3}{16\,c} a_n^2 - \left(e + c\frac{J}{\rho I}\right)(n\pi)^2 - \frac{B_n}{(n\pi)^4 a_n} \qquad (72)$$

The response equation can thus be written as

$$\frac{\Omega_n^2}{\omega_n^2} = 1 + \frac{\Delta_{1n}}{\omega_n^2}\eta = 1 - \frac{I}{A\ell^2}\left(\frac{J}{\rho I} + \frac{E}{k'G}\right)(n\pi)^2$$

$$+ \left[\frac{3\,A\ell^2}{16I}(a_n\eta)^2 - \frac{B_n \eta^2}{(n\pi)^4 \eta a_n}\right] \qquad (73)$$

As might be expected, the linear correction terms are due to the rotatory inertia and the shear deformation,* while the nonlinear

*It agrees with equation (140) of reference [35] , p. 335

term $\frac{3\,A\ell^2}{16\,I}\,(\eta a_n)^2$ is due to the axial force, the magnitude of which is proportional to the slenderness constant $\frac{1}{\eta c} = \frac{A\ell^2}{I} = \frac{\ell^2}{r^2}$. When the term containing the external force (B_n) is omitted in equation (73), the resulting response equation represents the stem of the response curve. Since the term $\frac{3\,A\ell^2}{16\,I}\,(a_n\eta)^2$ is proportional to the square of the amplitude and is always positive, the stem of the response curve bends toward the right (figure 10). This is so called hard spring effect. In fact, the pattern of equation (73) is similar to that of the Duffing's equation of nonlinear oscillation theory.*

Another physical phenomenon associated with nonlinear spring effect is the jump phenomenon.** Here jump refers to the phenomenon of a sudden and abrupt increase or decrease of the amplitude when the excitation frequency changes only slightly. In the present case, it happens in the region of $\overline{dfbc}$ of figure 12. In figure 12, as excitation frequency increases, the amplitude increases from point a up to point b. Further increase of the frequency will cause a drop in the amplitude to point c. On the other hand, if the frequency is gradually reduced from point c toward the left, the amplitude will go along the right branch of the response curve up to point f. Further decrease of the frequency will cause a sudden jump of the amplitude from point f to point d. The portion $\overline{f\,e\,b}$ of the right branch of the response curve can be shown to be unstable. (See Appendix B).***

---

*Reference [36] p. 85 equation (25) and p. 88 figure 23

**Reference [32] p. 245 and figure 80.

***Refer to note on page 5.

Let us examine the case when there is no external load and the beam is excited by the sinusoidal motion of the supports only, i.e.,

$$\bar{K} = f(t)_{,tt} = w_o (n\pi)^4 \cos \Omega_n t$$

Expressing w_o in terms of a sine series,* we have

$$\bar{K} = 4 n^4 \pi^3 \bar{w}_o \cos \Omega_n t \sum_{m=1,3,\ldots}^{\infty} \frac{\sin m\pi\xi}{m} \qquad (74)$$

Using equation (74), the response equation (73) becomes

$$\frac{\Omega_n^2}{\omega_n^2} = 1 - \frac{I}{A\ell^2} (n\pi)^2 \left(\frac{J}{\rho I} + \frac{E}{k'G} \right) + \left[\frac{9\ell^2}{4h^2} (\eta a_n)^2 - \frac{4 w_o \eta^2}{n\pi\eta a_n} \right] \qquad (75)$$

for odd n, and

$$\frac{\Omega_n^2}{\omega_n^2} = 1 - \frac{I}{A\ell^2} (n\pi)^2 \left(\frac{J}{\rho I} + \frac{E}{k'G} \right) + \frac{9\ell^2}{4h^2} (\eta a_n)^2 \qquad (76)$$

for even n.

The difference of the response for odd n or even n in the linear case will be discussed in Appendix C.** To solve for u_2, using equations (65) and (66), and the boundary condition that at $\xi = 0$, $u_2 = 0$, thus we have

$$u_{2,\xi} = \frac{1}{4} a_n^2 n^2 \pi^2 \cos^2 \tau (1 - 2 \cos^2 n\pi\xi)$$

*Use is made of the Fourier sine series formula, valid for $f(\xi) = -\bar{w}_o$, $-1 < \xi < 0$, and $f(\xi) = \bar{w}_o$, $0 < \xi < 1$ (here only the right half, $0 < \xi < 1$ is needed). For reference, see Churchill "Fourier Series and Boundary Value Problems," McGraw-Hill, 1st ed. p. 64 Prob. 3.

**Refer to note on page 5.

By integration

$$u_2 = -\frac{1}{16} a_n^2 n\pi \sin 2n\pi\xi \, (1 + \cos 2\tau) \tag{77}$$

In view of equation (72), equation (69) becomes

$$n^4 \pi^4 w_{2,\tau\tau} + w_{2,\xi\xi\xi\xi} = \sum_{m=1, m\neq n}^{\infty} B_m \sin m\pi\xi \cos \tau$$

$$- \frac{1}{16c} a_n^3 n^4 \pi^4 \sin n\pi\xi \cos 3\tau$$

Solution of which is

$$w_2 = \sum_{m=1, m\neq n}^{\infty} \frac{B_m \sin m\pi\xi \cos \tau}{(m^4 - n^4)\pi^4} + \frac{a_n^3}{128c} \sin n\pi\xi \cos 3\tau \tag{78}$$

Thus $\bar{u}$ and $\bar{w}$ can be written as

$$\bar{u} = u_1 \eta + u_2 \eta^2 \tag{79}$$

$$= \frac{-1}{16} (a_n \eta)^2 n \pi \sin 2n\pi\xi \, (\cos 2\Omega_n t + 1)$$

$$w_o = w_1 \eta + w_2 \eta^2 = (\eta a_n) \sin n\pi\xi \cos \Omega_n t$$

$$+ \sum_{m=1, m\neq n}^{\infty} \frac{(\eta^2 B_m) \sin m\pi\xi}{(m^4 - n^4)\pi^4} \cos \Omega_n t$$

$$+ \frac{(\eta a_n)^3 A\ell^2}{128 I} \sin n\pi\xi \cos 3\Omega_n t \tag{80}$$

Once Ω_n is known, the amplitude (ηa_n) can be determined by equation (73), and the solution of $\bar{u}$ and $\bar{w}$ can be obtained by equations (79) and (80). Equation (79) shows that period of $\bar{u}\left(\frac{2\pi}{2\Omega_n}\right)$ is only half of that of the external excitation or of $\bar{w}$. It also indicates that when $(1 + \cos 2\Omega_n t) = 0$, $\bar{u} = 0$, and thus $\bar{u} = 0$, when $\bar{w} = 0$, and an examination of equation (79) shows that the nodal points $(\bar{u} \equiv 0)$ are at $\xi = \frac{1}{2}$ which is the middle of the beam and at $\xi = \frac{1}{2n}$, $\frac{2}{2n}$, ... Since $\sin 2n\pi\xi$ is anti-symmetrical about the point $\xi = \frac{1}{2}$, the beam thus is always stretching out or compressing in a symmetric way about the middle section.

3.4 SIMPLY SUPPORTED CASE

Assuming that the left end is hinged and the right end is roller-supported, the boundary conditions are

at $\xi = 0$, $u = 0$, $w = 0$, $M = 0$ (i.e., $\theta,_\xi = 0$)

and at $\xi = 1$, $w = 0$, $M = 0$ $(\theta,_\xi = 0)$, and

$\Sigma F_x = N \cos \alpha - Q \sin \theta = 0$ (i.e., no resultant axial force at $\xi = 1$). If series expansions are used, we have

$$\text{at} \quad \xi = 0, \quad u_i = 0, \quad w_i = 0, \quad \theta_{i,\xi} = 0 \qquad i = 1, 2, 3 \ldots \tag{81}$$

$$\text{and at} \quad \xi = 1, \quad w_i = 0, \quad \theta_{i,\xi} = 0 \qquad i = 1, 2, 3 \ldots \tag{82}$$

Substituting for N, Q from equations (47) and (48), we have

$$\Sigma F_x = (N \cos \alpha - Q \sin \theta) = EA\left[\eta u_{1,\xi} + \eta^2 (u_{2,\xi} + \frac{1}{2} w_{1,\xi}{}^2)\right.$$

$$\left. + \eta^3 (u_{3,\xi} + w_{1,\xi} w_{2,\xi} + c\, w_{1,\xi} w_{1,\xi\xi\xi} - u_{1,\xi} w_{1,\xi}{}^2) + \eta^4 \ldots\right.$$

Thus we have, at $\xi = 1$,

$$u_{1,\xi} = 0$$

$$u_{2,\xi} + \frac{1}{2} w_{1,\xi}{}^2 = 0$$

$$u_{3,\xi} + w_{1,\xi} w_{2,\xi} + c\, w_{1,\xi} w_{1,\xi\xi\xi} - u_{1,\xi} w_{1,\xi}{}^2 = 0 \qquad (83)$$

Equations (81), (82), and (83) are the boundary conditions for the simple supported case.

To obtain the solution, we start with equations (51), (52), and (53)

$$u_{1,\xi\xi} = 0$$

$$\beta_n^4 w_{1,\tau\tau} + w_{1,\xi\xi\xi\xi} = 0$$

$$(u_{2,\xi} + \frac{1}{2} w_{1,\xi}{}^2)_{,\xi} = 0$$

Considering also the boundary conditions (81), (82), and (83), we obtain

$$u_1 = 0$$

$$w_1 = a_n \sin n\pi\xi \cos \tau$$

$$\theta_1 = a_n n\pi \cos n\pi\xi \cos \tau$$

$$\beta_n = n\pi, \ \tau = \Omega_n t, \ n = \text{any fixed integer} \qquad (84)$$

Here again instead of a series, we choose w_1 to be a one term solution. The reason of this choice is exactly the same as that in the case of immovable hinged ends.

Solution of equation (53) together with the boundary condition that at $\xi = 1$, $u_{2,\xi} + \frac{1}{2} w_{1,\xi}^{\;2} = 0$ (equation (83)) is

$$u_{2,\xi} + \frac{1}{2} w_{1,\xi}^{\;2} \equiv 0 \text{ (for all } \tau \text{ and } \xi) \tag{85}$$

Thus

$$\begin{aligned} u_{2,\xi} &= -\frac{1}{2} w_{1,\xi}^{\;2} \\ &= -\frac{1}{2} (n\pi a_n \cos n\pi\xi \cos \tau)^2 \end{aligned}$$

and

$$u_2 = -\frac{1}{8} (a_n n\pi)^2 \left(\xi + \frac{1}{2\pi n} \sin 2n\pi\xi\right)(1 + \cos 2\tau) \tag{86}$$

To obtain $\frac{\Delta_{1n}}{\omega_n^{\;2}}$, we again use equation (54). But this time we have

$$\left(u_{2,\xi} w_{1,\xi} + \frac{1}{2} w_{1,\xi}^{\;3}\right) = 0 \text{ (equation (85))}$$

Thus instead of (73), we now have the response equation up to the order η^1 as

$$\frac{\Omega_n^{\;2}}{\omega_n^{\;2}} = 1 - \frac{I}{A\ell^2}\left(\frac{J}{\rho I} + \frac{E}{k'G}\right) n^2 \pi^2 - \frac{\eta^2 B_n}{n^4 \pi^4 \eta a_n} \tag{87}$$

For excitation by support motion only, and for $\Omega_n \approx \omega_1$, (87) becomes

$$\frac{\Omega_1^{\;2}}{\omega_1^{\;2}} = 1 - \frac{I\pi^2}{A\ell^2}\left(\frac{J}{\rho I} + \frac{E}{k'G}\right) - \frac{4\eta^2 w_o}{\pi \eta a_1} \tag{88*}$$

*If the nonlinear ($a_n^{\;2}$) term is dropped and for $n = 1$, equation (75) is reduced to equation (88).

As is remarked in Appendix C,* this formula is no different from the response relation of the linear case, and thus we conclude that the effect of large amplitude on the frequency will be at most of the order η^2.

We now proceed to find the quantities w_2, u_3, and Δ_{2n} from equations (54), (55), and (56).

For the sake of simplicity, let us consider the case when n = 1 and the loading:

$$\overline{K} = \pi^2 w_o \cos\tau = 4\pi^2 w_o \cos\tau \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m} \sin m\pi\xi \qquad (89)$$

Equation (54) becomes

$$\pi^4 w_{2,\tau\tau} + w_{2,\xi\xi\xi\xi} = 4\pi^3 w_o \cos\tau \sum_{m=1,3,\ldots}^{\infty} \frac{\sin m\pi\xi}{m}$$
$$+ \pi^4 \frac{\Delta_{11}}{\omega_1^2} a_1 \sin\pi\xi \cos\tau + \pi^6 (e+c) a_1 \sin\pi\xi \cos\tau \qquad (90)$$

But equation (87) gives

$$4\pi^3 w_o + \pi^4 a_1 \frac{\Delta_{11}}{\omega_1^2} + \pi^6 (e+c) a_1 = 0$$

Thus equation (90) becomes

$$\pi^4 w_{2,\tau\tau} + w_{2,\xi\xi\xi\xi} = 4\pi^3 w_o \cos\tau \sum_{m=3,5,\ldots}^{\infty} \frac{\sin m\pi\xi}{m} \qquad (91)$$

*Refer to note on page 5.

The solution of (91) is

$$w_2 = \frac{4w_o}{\pi}\cos\tau \sum_{m=3,5,\dots}^{\infty} \frac{\sin m\pi\xi}{m(m^4-1)} \tag{92}$$

To find u_3 we use equation (55)

$$u_{3,\xi\xi} = c\beta_n^4 u_{2,\tau\tau} - (w_{2,\xi} w_{1,\xi} + c w_{1,\xi} w_{1,\xi\xi\xi})_{,\xi}$$

Thus

$$u_{3,\xi} = c\beta_n^4 \int^{\xi} u_{2,\tau\tau} d\xi - (w_{2,\xi} w_{1,\xi} + c w_{1,\xi} w_{1,\xi\xi\xi}) + k_1(\tau) \tag{93}$$

After substitution, it becomes

$$u_{3,\xi} = \frac{\pi^6}{2} c a_1^2 \left(\frac{\xi}{2} - \frac{\cos 2\pi\xi}{4\pi^2}\right)\cos 2\tau - a_1\pi \sum_{m=3,5,\dots}^{\infty} \frac{4w_o \cos m\pi\xi \cos \pi\xi}{m^4-1}$$

$$\cos^2\tau + c a_1^2 \pi^4 \cos^2\pi\xi \cos^2\tau + k_1(\tau)$$

By further integration, we have

$$u_3 = \frac{\pi^6}{2} c a_1^2 \left(\frac{\xi^3}{6} - \frac{\sin 2\pi\xi}{8\pi^3}\right)\cos 2\tau - a_1\pi \sum_{m=3,5,\dots}^{\infty} \frac{4w_o\cos^2\tau}{2\pi(m^4-1)}$$

$$\left[\frac{\sin(m-1)\pi\xi}{m-1} + \frac{\sin(m+1)\pi\xi}{m+1}\right] + c a_1^2 \pi^4 \cos^2\tau \left(\frac{\xi}{2} + \frac{\sin 2\pi\xi}{4\pi}\right)$$

$$+ \xi k_1(\tau) + k_2(\tau)$$

Since at $\xi = 0$, $u_3 = 0$ and at $\xi = 1$,

$$u_{3,\xi} + w_{2,\xi}\, w_{1,\xi} + c\, w_{1,\xi}\, w_{1,\xi\xi\xi} = 0,$$

it yields

$$k_2(\tau) = 0$$

and

$$k_1(\tau) = -\frac{\pi^6}{2}\, c\, a_1^2 \left(\frac{1}{2} - \frac{1}{4\pi^2}\right) \cos 2\tau$$

Thus:

$$u_3 = \frac{\pi^6}{2}\, c\, a_1^2 \left[\frac{1 - 2\pi^2}{4\pi^2}\, \xi + \frac{\xi^3}{6} - \frac{\sin 2\pi\xi}{8\pi^3}\right] \cos 2\tau - 2 a_1 \sum_{m=3,5,\ldots}^{\infty} \frac{w_o \cos^2\tau}{m^4 - 1}$$

$$\left[\frac{\sin(m-1)\pi\xi}{m-1} + \frac{\sin(m+1)\pi\xi}{m+1}\right] + c\, a_1^2\, \pi^4 \cos^2\tau \left(\frac{\xi}{2} + \frac{\sin 2\pi\xi}{4\pi}\right) \qquad (94)$$

To obtain $\dfrac{\Delta_{21}}{\omega_1^2}$, we multiply equation (56) by $\dfrac{w_1}{a_1}\, d\xi\, d\tau$ and integrate as before:

$$\int_0^{2\pi}\int_0^1 \Bigg\{ \pi^4 w_{3,\tau\tau} + w_{3,\xi\xi\xi\xi} + \left(e^2 w_{1,\xi\xi\xi\xi\xi} - \frac{w_{1,\xi}^3}{3} - u_{2,\xi} w_{1,\xi} + e\, w_{2,\xi\xi\xi}\right.$$

$$\left. - 2ce\, \frac{J}{\rho I}\, \pi^4 w_{1,\xi\tau\tau}\right)_{,\xi\xi\xi} + \frac{\pi^4}{\omega_1^2}\, \Delta_{21} w_{1,\tau\tau} + \frac{\pi^4}{\omega_1^2}\, \Delta_{11}\, w_{2,\tau\tau}$$

$$- \frac{1}{c}\left(w_{1,\xi}\, u_{3,\xi} + w_{2,\xi}\, u_{2,\xi} + \frac{3}{2}\, w_{1,\xi}^2\, w_{2,\xi}\right)_{,\xi}$$

$$- \frac{\Delta_{11}}{\omega_1^2}\, \pi^4 w_o \cos\tau - \left[\frac{Jc}{\rho I}\, \pi^4 w_{1,\xi\tau\tau}\, \frac{\Delta_{11}}{\omega_1^2} + \frac{cJ}{\rho I}\, \pi^4 w_{2,\xi\tau\tau}\right.$$

$$\left. + w_{1,\xi\xi\xi}\left(w_{1,\xi}^2 + u_{2,\xi}\right)\right]_{,\xi} \Bigg\} \sin\pi\xi \cos\tau\, d\xi\, d\tau = 0 \qquad (95)$$

Carrying out the integration, and substituting the values $\frac{\Delta_{11}}{\omega_1^2}$ from equation (88), and $c\,\eta$ and $e\,\eta$ from equations (43) and (44), we obtain

$$\eta^2 \frac{\Delta_{21}}{\omega_1^2} = \left[\left(\frac{E}{k'G}\right)^2 + \left(\frac{J}{\rho I}\right)^2 + \frac{3JE}{\rho I k'G}\right]\left(\frac{I\pi^2}{A\ell^2}\right)^2 + \frac{15}{32}(a_1\pi\eta)^2 - \frac{\pi^4}{12} a_1^2 \eta^2$$
$$+ \left(\frac{4w_o}{a_1\eta\pi}\right)^2 + 4\left(\frac{J}{\rho I} + \frac{E}{k'G}\right)\frac{I}{A\ell^2}\frac{w_o\pi}{a_1\eta} \tag{96}$$

Thus in the second approximation, the response equation takes the following form:

$$\frac{\Omega_1^2}{\omega_1^2} = 1 + \eta\frac{\Delta_{11}}{\omega_1^2} + \eta^2\frac{\Delta_{21}}{\omega_1^2}$$
$$= 1 - \frac{I\pi^2}{A\ell^2}\left(\frac{J}{\rho I} + \frac{E}{k'G}\right) + \left(\frac{I\pi^2}{A\ell^2}\right)^2\left[\left(\frac{J}{\rho I}\right)^2 + \left(\frac{E}{k'G}\right)^2 + \frac{3EJ}{k'G\rho I}\right]$$
$$\frac{(8\pi^2 - 45)\pi^2}{96}(a_1\eta)^2 + \left(\frac{J}{\rho I} + \frac{E}{k'G}\right)\frac{4I}{A\ell^2}\frac{\pi w_o}{a_1\eta} - \frac{4w_o}{\pi a_1\eta} + \left(\frac{4w_o}{\pi a_1\eta}\right)^2 \tag{97}$$

In equation (97), terms with the factor $\frac{J}{\rho I}$ are due to rotatory inertia ($J = \rho I$, $\frac{J}{\rho I} = 1$, the factor $\frac{J}{\rho I}$ is retained solely for identification purposes); while with $\frac{E}{k'G}$ are due to shear deformation. All these terms are with the slenderness ratio $\left(\frac{I}{A\ell^2}\right)$ as their coefficients. Since beams associated with large deflection are usually of the slender type, for all practical purposes, the shear deformation and rotatory inertia terms can be neglected, and thus equation (97) can be reduced to

$$\frac{\Omega_1^{\,2}}{\omega_1^{\,2}} = 1 - \frac{(8\pi^2 - 45)\pi^2}{96}(a_1\eta)^2 - \left(1 - \frac{4\,w_o}{\pi\, a_1\eta}\right)\frac{4\,w_o}{\pi\, a_1\eta}$$

Equation (98) indicates that the nonlinear effect on frequency due to large amplitude is of the soft spring type (i.e., the response curve bends towards the left). As is seen in figure 16, it also demonstrates the jump phenomena. By increasing the frequency it jumps up from point f (on left branch) to point d (on right branch), and by decreasing the frequency it jumps down from point b to c. The portion b e f of the left branch of the response curve is unstable. (Appendix B)

The solution for $\bar{w}$ and $\bar{u}$ up to the second approximation can be written as

$$\bar{w} = a_1\eta \sin\pi\xi \cos\Omega_1 t + \frac{4\,w_o\eta^2}{\pi}\cos\Omega_1 t \sum_{m=3,5,\ldots}^{\infty} \frac{\sin m\pi\xi}{m(m^4 - 1)} \tag{99}$$

$$\bar{u} = -\frac{1}{8}(\pi a_1\eta)^2\left(\xi + \frac{1}{2\pi}\sin 2\pi\xi\right)(1 + \cos 2\Omega_1 t) \tag{100}$$

For a given frequency Ω_1, the amplitude (ηa_1) is determined by equation (98), and the solution of $\bar{u}$ and $\bar{w}$ can be obtained by equation (99) and (100). Equation (100) shows that period of $\bar{u}\left(\frac{2\pi}{2\Omega_1}\right)$ is only half of that of the external excitation or of $\bar{w}$. It also indicates that when $(1 + \cos 2\Omega_1 t) = 0$, $\bar{u} = 0$, and thus $\bar{u} = 0$ when $\bar{w}$ is zero. The maximum of $|\bar{u}|$ occurs at $\xi = 1$ and $\Omega_1 t = 0$, or $|\bar{u}|_{max} = \left(\frac{\pi a_1\eta}{2}\right)^2$. Examining equation (99), it is seen that the second term, i.e., the term with the summation sign, is very small and converges rapidly. Therefore, it can be neglected.

SECTION 4

MODIFICATION OF THE PROBLEM DUE TO MATERIAL DAMPING

4.1 STRESS-STRAIN LAW WITH HYSTERESIS

Under alternating loads, the deformation of materials do not follow Hooke's law. Part of the deformation energy is converted into the internal friction loss. Due to this process of dissipation of energy, the stress-strain curve forms a loop, which is commonly known as the hysteresis loop. The area of the loop determines the amount of energy dissipated per unit volume of the material during one cycle of oscillation. This property of energy dissipation is called the internal damping, or material damping, and is a characteristic of the material. Material damping serves as an effective agent to reduce the amplitude of vibration.

One relationship that is commonly used for material damping is in the form of viscous damping, i.e., the damping is proportional to the rate of deformation during oscillation. This is the same as saying that damping is proportional to the frequency of the oscillation. According to this assumption, the equations of oscillations are linear, and can be solved often by rigorous mathematical methods. However, experimental work shows that damping in general is only dependent on the amplitude of the oscillation and is independent of the frequency. The use of viscous damping for the vibration analysis, therefore, is not justified.

Based on the fact that the material damping is dependent on the amplitude and is independent of frequency, Davidenkov proposed an empirical stress-strain law. Davidenkov's law was adopted by Pisarenko for cantilever beams (both Euler-Bernoulli and Timoshenko beams), and the analysis agrees quite well with the experimental results.

Davidenkov's stress-strain law consists of two expressions. One expression is for the ascending branch of the hysteresis loop (loading); the other for the descending branch of the hysteresis loop (unloading). In functional form, it is expressed by

$$\begin{aligned} \overrightarrow{\sigma} &= E\left\{\epsilon - \frac{\nu}{n}\left[(\epsilon_o + \epsilon)^n - 2^{n-1}\epsilon_o^{\,n}\right]\right\} \\ \overleftarrow{\sigma} &= E\left\{\epsilon + \frac{\nu}{n}\left[(\epsilon_o - \epsilon)^n - 2^{n-1}\epsilon_o^{\,n}\right]\right\} \end{aligned} \tag{101}$$

or taking the derivatives, we obtain

$$\begin{aligned} \frac{d\overrightarrow{\sigma}}{d\epsilon} &= E\left[1 - \nu(\epsilon_o + \epsilon)^{n-1}\right] \\ \frac{d\overleftarrow{\sigma}}{d\epsilon} &= E\left[1 - \nu(\epsilon_o - \epsilon)^{n-1}\right] \end{aligned} \tag{102}$$

where arrows towards the right indicate the ascending branch, i.e., loading or $d\epsilon/dt > 0$ (t:time), while arrows towards the left indicate the descending branch, i.e., unloading or $d\epsilon/dt < 0$, and

E is the Young's modulus

ϵ_o is the amplitude of the strain (absolute value)

ϵ is the strain at any instant

σ is the corresponding stress

ν, n are parameters for the hysteresis loop.

A sketch of the hysteresis loop of equation (101) is shown in figure 5.

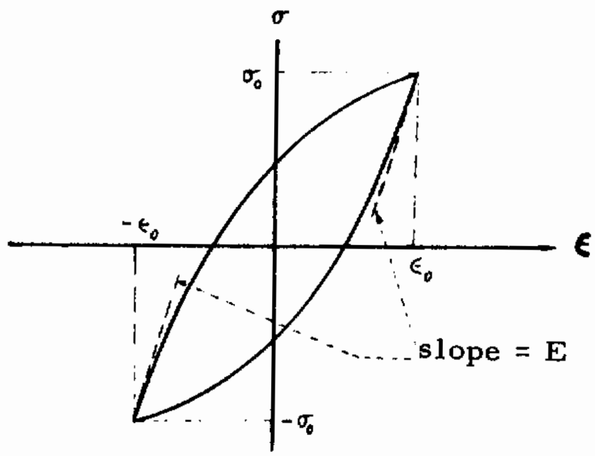


Fig. 5 - Hysteresis Loop -- Davidenkov's Stress-Strain Law

The following are the properties of the Davidenkov's stress-strain law:

(a) The damping energy, i.e., the area of the loop, is independent of the frequency but is dependent on the strain amplitude ϵ_o.

(b) For the same amplitude of strain, different materials have different loop areas. The loop area then depends on the two parameters ν and n. These two parameters also determine the shape of the hysteresis loop. However, as shown by Pisarenko [30], the shape of the hysteresis loop has little effect on the damping properties and therefore if we take n to be the number 2 or 3 and vary ν accordingly, the hysteresis loop thus obtained usually gives satisfactory results for damped vibration problems.

(c) The following conditions of symmetry are satisfied:

$$\left.\frac{d\overrightarrow{\sigma}}{d\epsilon}\right|_{\epsilon = -\epsilon_o} = \left.\frac{d\overleftarrow{\sigma}}{d\epsilon}\right|_{\epsilon = +\epsilon_o}$$

$$\left.\frac{d\overleftarrow{\sigma}}{d\epsilon}\right|_{\epsilon = +\epsilon_o} = \left.\frac{d\overrightarrow{\sigma}}{d\epsilon}\right|_{\epsilon = -\epsilon_o}$$

(d) The starting points of loading and unloading are always with a slope equal to E, the Young's modulus. Thus

$$\left.\frac{d\overrightarrow{\sigma}}{d\epsilon}\right|_{\epsilon = -\epsilon_o} = E$$

$$\left.\frac{d\overleftarrow{\sigma}}{d\epsilon}\right|_{\epsilon = +\epsilon_o} = E$$

Davidenkov's law, as expressed by equation (101) is only for the case when the strain is symmetric about the origin. For the case in which the hysteresis loop is not symmetrical about the origin, i.e., the extreme values are not equal, $|\epsilon_1| \neq |\epsilon_2|$; $|\sigma_1| \neq |\sigma_2|$, (see figure 6), we can adopt

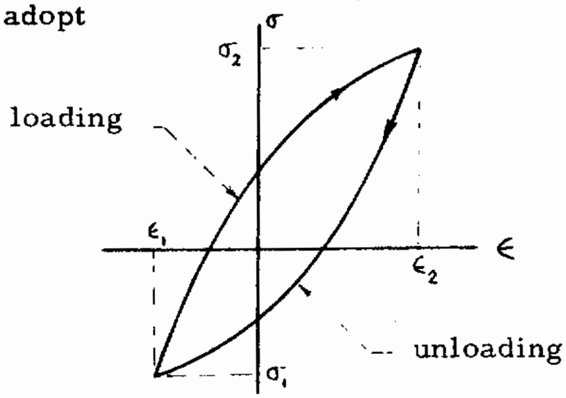


Fig. 6 - Unsymmetrical Hysteresis Loop

the same functional relationship of Davidenkov's law and make the assumption that the loop is symmetrical about the point $\left((\epsilon_1 + \epsilon_2)/2, (\sigma_1 + \sigma_2)/2\right)$. The stress-strain law then takes the following modified form:

$$\overrightarrow{\sigma} = E\left\{\epsilon - \frac{\nu}{n}\left[(\epsilon - \epsilon_1)^n - \frac{1}{2}(\epsilon_2 - \epsilon_1)^n\right]\right\}$$

$$\overleftarrow{\sigma} = E\left\{\epsilon + \frac{\nu}{n}\left[(\epsilon_2 - \epsilon)^n - \frac{1}{2}(\epsilon_2 - \epsilon_1)^n\right]\right\} \qquad (103)$$

This modified form of equation (103) also possesses the properties (a) - (d) of Davidenkov's law, but now the amplitude of the strain is defined as the average of the extreme values, i.e., $\epsilon_o = \epsilon_2 - \epsilon_1/2$. For most metals, the parameter n can be taken as 2 [30]. If n = 2, equations (103) become

$$\overset{\rightarrow}{\sigma} = E\left\{\epsilon - \frac{\nu}{2}\left[\epsilon^2 - 2\epsilon_1\epsilon + \frac{1}{2}\left(\epsilon_1^{\,2} - \epsilon_2^{\,2}\right) + \epsilon_1\epsilon_2\right]\right\}$$

$$\overset{\leftarrow}{\sigma} = E\left\{\epsilon + \frac{\nu}{2}\left[\epsilon^2 - 2\epsilon_2\epsilon - \frac{1}{2}\left(\epsilon_1^{\,2} - \epsilon_2^{\,2}\right) + \epsilon_1\epsilon_2\right]\right\} \qquad (104)$$

In the following development, the Davidenkov's stress-strain law in its modified form expressed by equations (103) or (104) will be used.

4.2 NATURE OF STRAIN CYCLES IN BEAMS AND STRESS IN TERMS OF BEAM DISPLACEMENTS

Before deriving the equations for the motion of beams with hysteretic material properties, it is necessary first to get a physical understanding of the strain cycle occurring at any point in the beam.

For the sake of simplicity, the excitation frequency will be assumed to be near the lowest linear flexural mode frequency, and just as in the undamped case, only two cases, one with both ends hinged and the other with simply supported ends, will be considered.

4.2.1 Both Ends Hinged Case

Since the hysteretic damping effects are very small in comparison with the elastic effects (for metallic materials) it will be assumed that the basic strain cycle is similar to the one for an elastic beam.

Section 3.3 gives the results in the first approximation for an elastic beam with hinged immovable ends as follows:

$$w = \ell \bar{w} \approx \ell w_1 \eta = \ell a_1 \eta \sin \pi\xi \cos \tau$$

$$u_1 = 0, \; u_{2,\xi} + \frac{1}{2} w_{1,\xi}^{\;2} = \frac{1}{4} a_1^{\;2} \pi^2 \cos^2 \tau \tag{105}$$

The axial strain of the middle plane is expressed by equation (47):

$$\bar{\epsilon} = u_{1,\xi} \eta + (u_{2,\xi} + \frac{1}{2} w_{1,\xi}^{\;2}) \eta^2 + \ldots \tag{47}$$

The axial strain at any point is obtained from equation (5):

$$\epsilon = \bar{\epsilon} - z \theta_{,x} \tag{5}$$

Substituting equation (105) into (47) and (5), we have

$$\epsilon = \frac{1}{4} (a_1 \eta \pi \cos \tau)^2 + \frac{z}{\ell} a_1 \eta \pi^2 \sin \pi\xi \cos \tau \tag{106}$$

where the first term is due to axial tension, while the second term is due to bending.

Differentiating ϵ with respect to τ,

$$\frac{d\epsilon}{d\tau} = - a_1 \eta \pi^2 \sin \tau \left(\frac{1}{2} a_1 \eta \cos \tau + \frac{z}{\ell} \sin \pi\xi\right) \tag{107}$$

Positive values of $d\epsilon/d\tau$ corresponds to the loading branch of the hysteresis loop (figure 6); while negative values of $d\epsilon/d\tau$ corresponds to the unloading branch. The number of strain cycles that occur at any point during one cycle of beam motion is determined by the number of changes in sign that occurs in the strain rate during that interval of time at that point. The hysteretic stress-strain law for each strain cycle is also dependent on the maximum and minimum value of ϵ of that cycle (ϵ_1, ϵ_2 of equation (104)). For these two reasons, the stationary values of equation (106) will now be examined. The stationary values of ϵ is determined by the zeros of equation (107), i.e., at $\tau = 0, \pi, 2\pi, \ldots$.

and when

$$\frac{1}{2}\, a_1 \eta \cos\tau + \frac{z}{\ell}\, \sin \pi\xi = 0 \tag{108}$$

i.e., when

$$\tau_o ; \bar{\tau}_o = \cos^{-1}\left(\frac{-2\, z \sin \pi\,\xi}{a_1 \eta \ell}\right) \tag{109}$$

for

$$z > 0,\ \frac{\pi}{2} < \tau_o < \pi;\ \text{and}\ \ \pi < \bar{\tau}_o < \frac{3\pi}{2}$$

Since $|\cos\tau| \leq 1$, no solution of equation (109) is possible if

$$|2\, z \sin \pi\xi| > a_1 \eta \ell$$

There are two types of cyclic variations of strain and they depend on the solution of equation (109). These two types are described below.

(a) Case when $|2\, z \sin \pi\xi| \leq a_1 \eta \ell$

For this case, for the interval $0 \leq \tau \leq 2\pi$, there are two roots of equation (109). These roots shall be called τ_o and $\bar{\tau}_o$.

For $z > 0$, we have: $\frac{\pi}{2} \leq \tau_o \leq \pi$, and $\pi \leq \bar{\tau}_o \leq \frac{3\pi}{2}$. Based on the knowledge of all the zeros of equation (107) and the signs of $d\epsilon/d\tau$, it is possible to determine the time dependence of the strain from equation (106), and the maximum and minimum values of strain. Thus, at $\tau = 0$, $\frac{d\epsilon}{d\tau} = 0$, and since for $\tau \approx 0^-$, $\frac{d\epsilon}{d\tau} = +$, i.e., ϵ is increasing, and $\tau \approx 0^+$, $\frac{d\epsilon}{d\tau} = -$, i.e., ϵ is decreasing, ϵ must be a maximum at $\tau = 0$. On the other hand, at $\tau = \tau_o$, again $\frac{d\epsilon}{d\tau} = 0$, but for $0 < \tau < \tau_o$, $\frac{d\epsilon}{d\tau} = -$ and hence ϵ is decreasing, and for $\tau_o < \tau < \pi$, $\frac{d\epsilon}{d\tau} = +$ and hence ϵ is increasing, ϵ must be a minimum at $\tau = \tau_o$. By the similar reasoning ϵ also must be a maximum at $\tau = \pi$; and a minimum at $\tau = \bar{\tau}_o$. In summary, for $z > 0$, we have

τ	0		τ_o		π		$\bar{\tau}_o$		2π
$\frac{d\epsilon}{d\tau}$	0	- (decreases)	0	+ (increases)	0	- (decreases)	0	+ (increases)	0
state of ϵ	max.		min.		max.		min.		max.

For a given set of values of z and ξ, the strain cyclic variation is shown in figure 7(a). It is seen that there are two cyclic variations of ϵ during one cycle of excitation ($0 \leq \tau \leq 2\pi$).

(b) Case when $|2z \sin \pi\xi| > a_1 \epsilon \ell$

In this case there are no solutions for equation (109). It is observed from equation (107) that the following is true. For $z > 0$

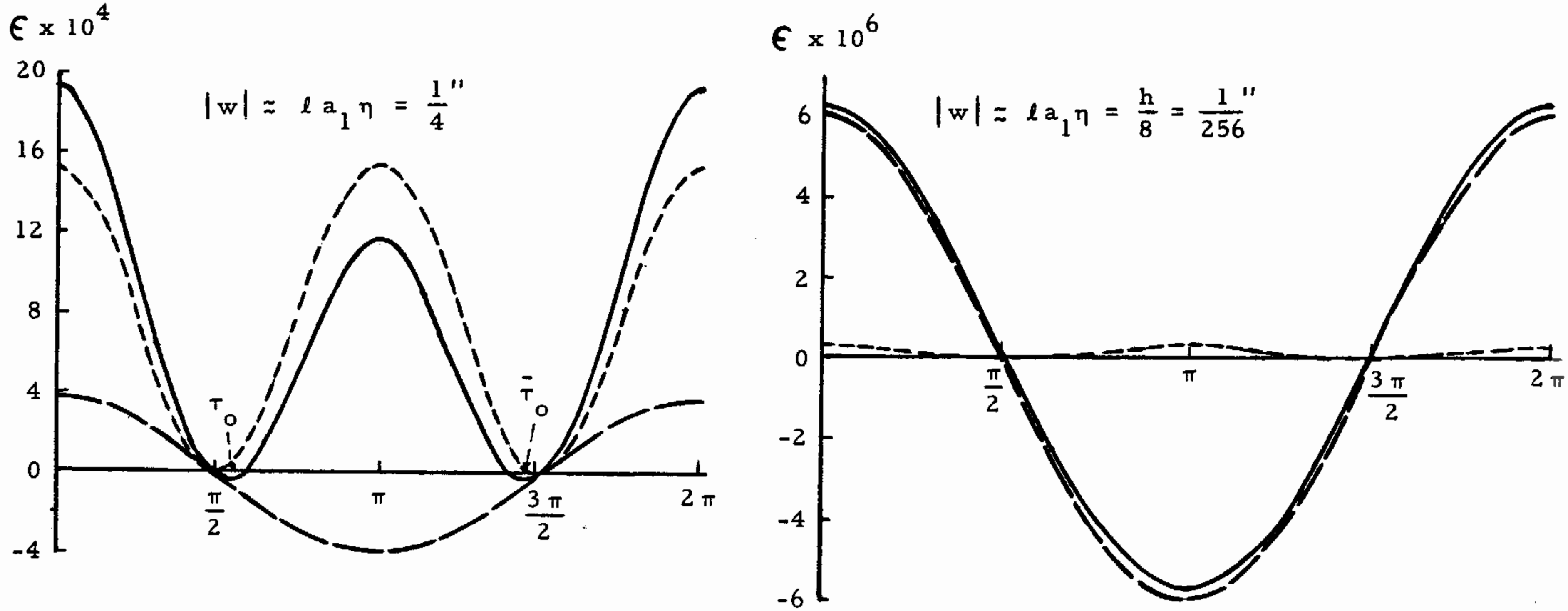
τ	0		π		2π
$\frac{d\epsilon}{d\tau}$	0	- (decreases)	0	+ (increases)	0
state of ϵ	max.		min.		max.

Therefore, there will be only one cycle of variation of ϵ during the interval of $0 \leq \tau \leq 2\pi$. The time variation of ϵ for this case is shown in figure 7(b).

At a given instant (i.e., τ = a constant) equation (108) represents a curve in the z - ξ plane. Sketches of the curve are shown in figures 8(a) and (b). A series of curves at different instants of time for the case when $a_1\eta\ell > h$ are shown in figure 8(a). Starting from outside (at $\tau = 0$), the curve moves toward the center of the beam, and coincides with the center line of the beam (ξ axis) at $\tau = \pi/2$. From $\tau = \pi/2$ to $\tau = \pi$ the curve moves outward in the other direction. From $\tau = \pi$ to $\tau = 2\pi$, the motion of the curve is reversed. Figure 8(b) shows the curves for the case $a_1\eta\ell < h$. The curve again moves with time inward at the side for $z < 0$ during the interval $0 \leq \tau \leq \pi/2$, coincides with the ξ axis at $\tau = \pi/2$, moves outward at the side for $z > 0$ during the interval $\pi/2 \leq \tau \leq \pi$, and then reverses the motion during the interval $\pi \leq \tau \leq 2\pi$. The difference between the cases represented by figure 8(b) and figure 8(a) is that in figure 8(b), the curve starts inside some part of the beam at either $\tau = 0$ or $\tau = \pi$, and there is a portion of the beam (two shaded areas) that is always outside the curve; while in figure 8(a), at either $\tau = 0$, or $\tau = \pi$, the curve is completely outside of the beam. This difference is significant in that the portion of the beam which is always outside the curve (the two shaded areas) is the area for which $|2z \sin \pi\xi| > a_1\eta\ell$, and thus, for which no solutions (τ_o, $\bar{\tau}_o$) of equation (109) exist. Consequently the strain cycle for this portion of the beam can be represented by figure 7(b). For the other portion of the beam (area which is not shaded in figure 8(b), and the entire beam of the case shown in figure 8(a), the strain has two cycles which is represented by figure 7(a).

$$\epsilon = \left(\frac{a_1 \eta \pi}{2} \cos \tau \right)^2 + \frac{z}{\ell} a_1 \eta \pi^2 \sin \pi \xi \cos \tau, \quad \ell = 10'', \quad h = \frac{1}{32}'', \quad \xi = \frac{1}{2}, \quad z = \frac{h}{2}$$

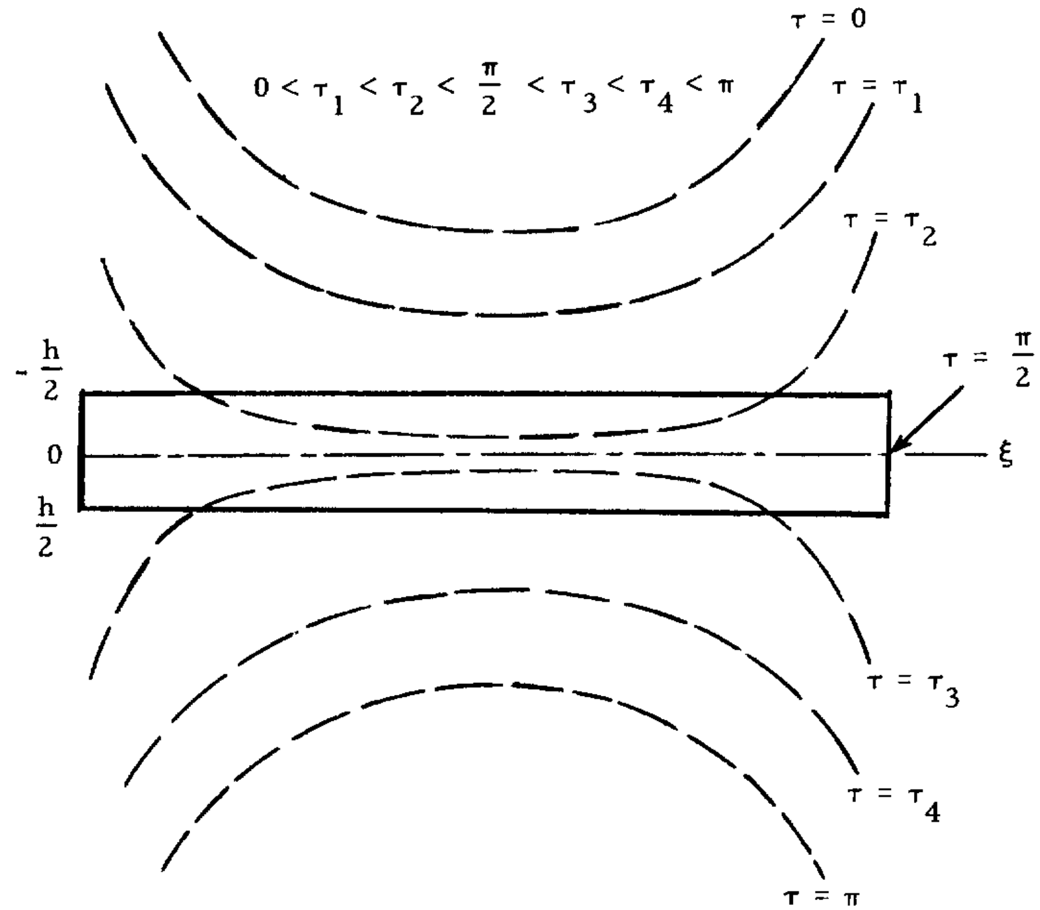
——— Resultant, — — — Due to Bending, - - - - - Due to Axial Tension



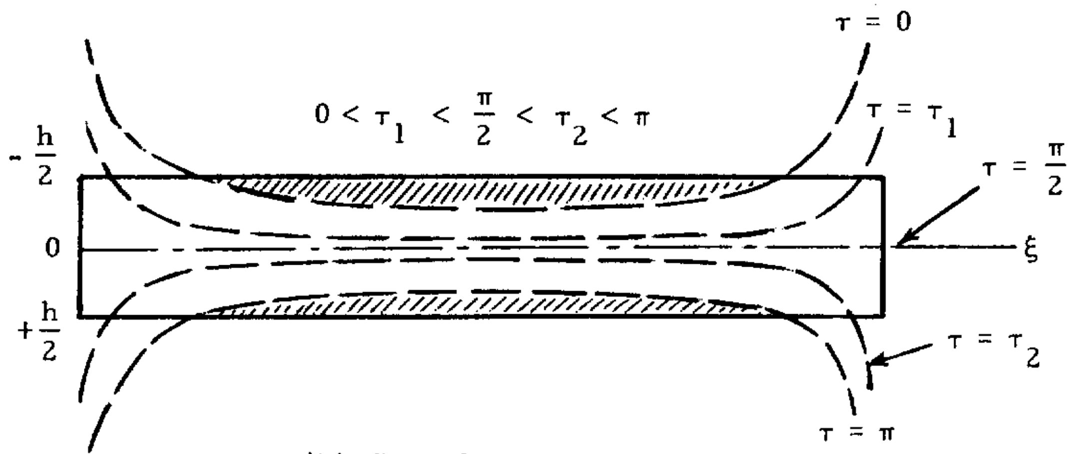
(a) $|2z \sin \pi\xi| \leq a_1 \eta \ell$

(b) $|2z \sin \pi\xi| > a_1 \ell \eta$

Figure 7 - Periodic Variation of Strains for Immovable Hinged Ends Case

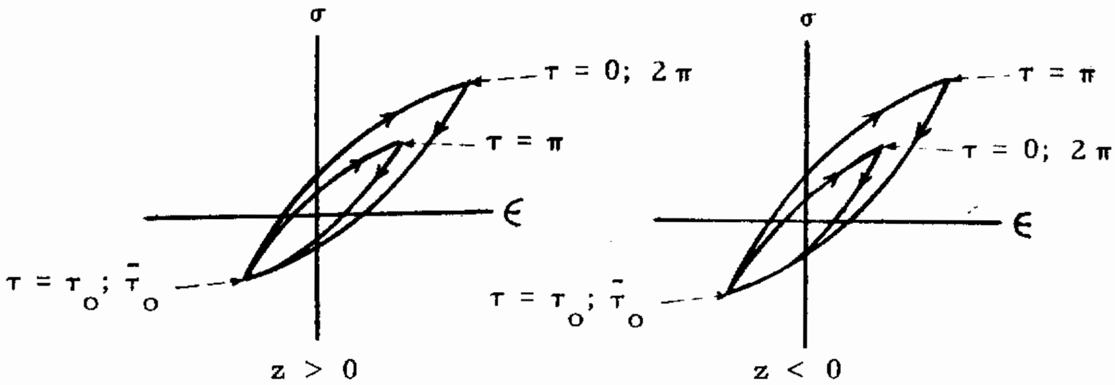


(a) Case for $a_1 \eta \ell > h$

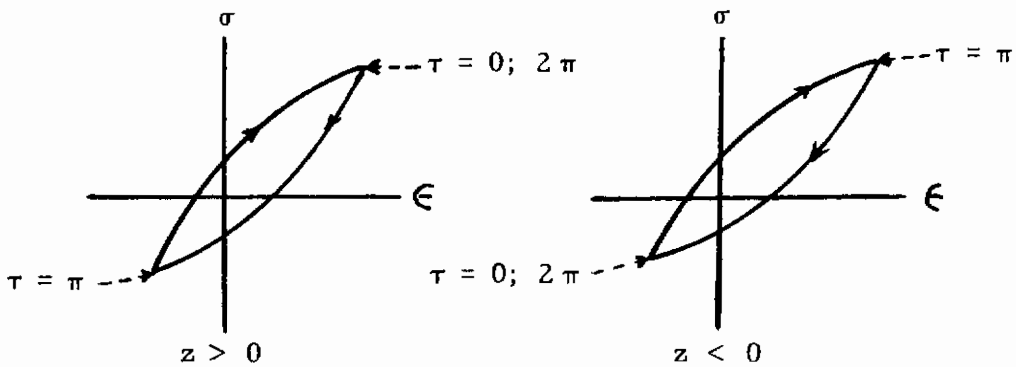


(b) Case for $a_1 \eta \ell < h$

Figure 8 - Sketches of the Curve $\frac{a_1 \eta}{2} \cos \tau + \frac{z}{\ell} \sin \pi \xi = 0$



(a) Two loop case -- $|2z \sin \pi \xi| < a_1 \eta \ell$, corresponding to Figure 8(a) and unshaded area of Figure 8(b).



(b) Single loop case -- $|2z \sin \pi \xi| > a_1 \eta \ell$, corresponding to shaded area of Figure 8(b).

Figure 9 - Hysteresis Loop of Beam Stresses

Based on the observations on figure 7 and 8, and the above discussions on the two types of cyclic variation of strain, stress-strain curves of the hysteretic type can be formed as shown in figure 9(a) and (b).

The functional form of the stress-strain law can be formulated by using equation (104). For the two cycle case, for $z > o$ and the part of the cycle $o\tau_o$, figure 9(a) as compared with figure 6, we have,

$$\epsilon_1 = \epsilon(\tau_o), (\text{min.}) \text{ and } \epsilon_2 = \epsilon(0), (\text{max.})$$

while for $z > o$, and part of the cycle $\pi\bar{\tau}_o$:

$$\epsilon_1 = \epsilon(\bar{\tau}_o), (\text{min.}), \text{ and } \epsilon_2 = \epsilon(\pi), (\text{max.})$$

Note that $\epsilon(\tau_o) = \epsilon(\bar{\tau}_o)$. From equation (5), we have

$$\epsilon(0) = \bar{\epsilon}(0) - z\,\theta,_x(0)$$

$$\epsilon(\pi) = \bar{\epsilon}(\pi) - z\,\theta,_x(\pi)$$

from (84), (105) and (47), it can be shown that

$$\bar{\epsilon}(0) = \bar{\epsilon}(\pi), \text{ and } \theta,_x(\pi) = -\theta,_x(0).$$

For brevity, let us denote $\bar{\epsilon}(0)$ by $\bar{\epsilon}_m$, and $\theta,_x(0)$ by $\theta,_{xm}$, then we have

$$\epsilon(0) = \bar{\epsilon}_m - z\,\theta,_{xm}$$

$$\epsilon(\pi) = \bar{\epsilon}_m + z\,\theta,_{xm} \qquad (110)$$

To obtain the minimum value $\epsilon(\tau_o)$, substituting equation (109) into equation (106), thus we have

$$\epsilon(\tau_o) = -\left(\frac{z}{\ell}\,\pi \sin \pi\,\xi\right)^2 \qquad (111)$$

This indicates that $\epsilon(\tau_o)$ is proportional to $(z/\ell)^2$, and if (z/ℓ) is small, $\epsilon(\tau_o)$ can be neglected.

Using equations (104), (110), and neglecting the $\epsilon(\tau_o) = \epsilon_1$ terms, we have, in terms of $\bar{\epsilon}$ and $\theta,_x$, the following stress-strain law with two loops

$$\left.\begin{matrix} \vec{\sigma}_{o\tau_o} \\ \vec{\sigma}_{\pi\tau_o} \end{matrix}\right\} = E\left\{(\bar{\epsilon} - z\,\theta,_x) + \frac{\nu}{2}\left[-(\bar{\epsilon}^2 - \frac{1}{2}\bar{\epsilon}_m^{\,2}) + z(2\bar{\epsilon}\theta,_x \mp \bar{\epsilon}_m\,\theta,_{xm}) - z^2(\theta,_x^{\,2} - \frac{1}{2}\theta,_{xm}^{\,2})\right]\right\} \tag{112}$$

$$\left.\begin{matrix} \overleftarrow{\sigma}_{o\tau_o} \\ \overleftarrow{\sigma}_{\pi\tau_o} \end{matrix}\right\} = E\left\{(\bar{\epsilon} - z\,\theta,_x) + \frac{\nu}{2}\left[(\bar{\epsilon}^2 + \frac{1}{2}\bar{\epsilon}_m^{\,2} - 2\bar{\epsilon}\bar{\epsilon}_m) - z(2\bar{\epsilon}\theta,_x \pm \bar{\epsilon}_m\theta,_{xm} - 2\bar{\epsilon}_m\,\theta,_x \mp \bar{\epsilon}\theta,_{xm}) + z^2(\theta,_x^{\,2} + \frac{1}{2}\theta,_{xm}^{\,2} \mp 2\,\theta,_x\,\theta,_{xm})\right]\right\} \tag{113}$$

Now we turn our attention to the single loop case (figure 9(b)). As figure 9(b) is compared with figure 6, we have

For $z > 0$: $\epsilon_1 = \epsilon(\pi)$, min.
$\epsilon_2 = \epsilon(0)$, max.

and for $z < 0$: $\epsilon_1 = \epsilon(0)$, min.
$\epsilon_2 = \epsilon(\pi)$, max.

However, due to symmetry, we have

$$\epsilon(z)_{\text{at } \tau = 0} \equiv \epsilon(-z)_{\text{at } \tau = \pi}$$

Thus, for the part of the cycle $o\pi$, we have

$$\epsilon_2 = \bar{\epsilon}_m - |z| \; \theta,_{xm} \qquad \text{for all } z$$

$$\epsilon_1 = \bar{\epsilon}_m + |z| \; \theta,_{xm} \tag{114}$$

Substituting equation (114) into equation (104), the stress-strain law of the single loop type is obtained:

$$\overset{\rightleftarrows}{\sigma} = E \left\{ (\bar{\epsilon} - z\,\theta,_x) + \frac{\nu}{2} \left[\mp (\bar{\epsilon} - \bar{\epsilon}_m)^2 \pm 2z\,(\bar{\epsilon} - \bar{\epsilon}_m)\,\theta,_x \right.\right.$$

$$\left.\left. + \; 2|z|\;\theta,_{xm}\,(\bar{\epsilon} - \bar{\epsilon}_m) \mp (\theta,_x{}^2 - \theta,_{xm}{}^2)\; z^2 - 2\theta,_x\,\theta,_{xm}\, z\,|z| \right] \right\} \tag{115}$$

4.2.2 <u>Simply Supported Case</u>

The first approximate solution for the simply supported case, as given in Section 3.4, shows that

$$w = \ell \bar{w} \simeq \ell w_1 \eta = \ell a_1 \eta \sin \pi\xi \cos \tau$$

$$u_1 = 0, \; u_{2,\xi} + \frac{1}{2} w_{1,\xi}{}^2 = 0 \tag{116}$$

Substituting equation (116) into equations (47); (5), we have,

$$\epsilon = \frac{z}{\ell} \pi^2 a_1 \eta \sin \pi\xi \cos \tau$$

$$\frac{d\epsilon}{d\tau} = -\frac{z}{\ell} \pi^2 a_1 \eta \sin \pi\xi \sin \tau \tag{117}$$

Equation (117) indicates that the strain varies linearly across the depth of the beam and the neutral axis is at the center. It also forms a single loop from 0 to π and back from π to 2π, and the loop is similar to the one shown in figure 9(b). The functional form also must be the same. In other words, for the simply supported case with one end axially free to move, the stress-strain law is expressed by equation (115).

Note that the main parts (elastic part) of the stress-strain law of equation (112), (113), and (115) are all the same as expressed by the term $E(\bar{\epsilon} - z\theta,_x)$. The only difference is the second order effect of hysteresis which is represented by the terms with the coefficient ν.

4.3 AXIAL FORCE AND BENDING MOMENT TAKING MATERIAL DAMPING INTO ACCOUNT

The axial tension and bending moment are defined respectively:

$$N = \int_A \sigma \, dA$$
$$M = \int_A \sigma z \, dA \qquad (118)$$

The integration is over the entire cross section of the beam. Again the immovable hinged ends case and simply supported case shall be considered separately.

4.3.1 Both Ends Hinged Case

In Section 4.2.1, it is shown that the addition of the axial tension causes the beam to have at some points two cycles of strain and some points one cycle of strain during one cycle of beam motion. The criterion of determining which part of the beam has two cycles and which part has one cycle is, as explained before, obtained by examining the roots of equation (109); while the criterion of deciding which branch of the hysteresis loop is applicable is to examine the signs of the loading rate ($d\epsilon/d\tau$) of equation (107). In both equations, (109) and (107), there are three variables -- τ, ξ and z, and thus at any given instant τ, but at different locations of beam (different ξ and z), stress-strain relationships may be different. Consequently, the integration of equation (118) will not only give complicated expressions for N and M but also take different forms at different part (ξ) of the beam. In general, therefore, the analytical expressions for the damped motion of the two hinged ends case are very complicated. For the sake of simplicity, only two limiting cases will be considered. One limiting case will be for beam amplitude of vibration large in comparison with beam depth; while the other limiting case is the one for beam amplitude small in comparison with beam depth but still large enough to produce nonlinear effects (figure 7(b)). It may be expected that the general case (intermediate case) will have a behavior intermediate between these two limiting cases.

4.3.1.1 When Amplitude is Large Compared with Depth of the Beam*

If the amplitude is sufficiently large so that in equation (109)

$$\cos\tau = \frac{-2z}{\ell a_1 \eta} \sin \pi\xi \approx 0 \text{ (i.e. } \ell a_1 \eta >> 2z \sin \pi\xi)$$

then $\quad \tau_o \approx \frac{\pi}{2}, \bar{\tau}_o \approx \frac{3}{2}\pi$ (figure 7(a)).

*In the following, for brevity, this case will be referred to simply as the very large amplitude case

If the approximations $\tau_o = \frac{\pi}{2}$ and $\bar{\tau}_o = \frac{3\pi}{2}$ are used, the unloading and the loading cycles of the beam can be written as

$$
\begin{aligned}
&0 < \tau < \frac{\pi}{2} && \sigma = \overleftarrow{\sigma}_{0\tau_o} \\
&\frac{\pi}{2} < \tau < \pi && \sigma = \overrightarrow{\sigma}_{\pi\tau_o} \\
&\pi < \tau < \frac{3\pi}{2} && \sigma = \overleftarrow{\sigma}_{\pi\tau_o} \\
&\frac{3\pi}{2} < \tau < 2\pi && \sigma = \overrightarrow{\sigma}_{0\tau_o}
\end{aligned}
\tag{119}
$$

Using the appropriate formula of equations (112) and (113) and substituting (119) into equation (118), we obtain the following results.

$$
\left.\begin{matrix} N\left(\frac{\pi}{2} - \pi\right) \\ N\left(\frac{3\pi}{2} - 2\pi\right) \end{matrix}\right\} = EA\bar{\epsilon} - \frac{\nu}{2} EA\left(\bar{\epsilon}^2 - \frac{1}{2}\bar{\epsilon}_m^2\right) - \frac{\nu}{2} EI\left(\theta,_x^2 - \frac{1}{2}\theta,_{xm}^2\right)
$$

$$
\left.\begin{matrix} N\left(0 - \frac{\pi}{2}\right) \\ N\left(\pi - \frac{3\pi}{2}\right) \end{matrix}\right\} = EA\bar{\epsilon} + \frac{\nu}{2} EA\left(\bar{\epsilon}^2 + \frac{1}{2}\bar{\epsilon}_m^2 - 2\bar{\epsilon}\bar{\epsilon}_m\right) + \frac{\nu}{2} EI \left(\theta,_x^2 + \frac{1}{2}\theta,_{xm}^2 \mp 2\theta,_x\,\theta,_{xm}\right)
\tag{120}
$$

$$
\left.\begin{matrix} M\left(\frac{\pi}{2} - \pi\right) \\ M\left(\frac{3\pi}{2} - 2\pi\right) \end{matrix}\right\} = -EI\theta,_x + \frac{\nu}{2} EI\left(2\bar{\epsilon}\theta,_x \pm \bar{\epsilon}_m\,\theta,_{xm}\right)
$$

$$
\left.\begin{matrix} M\left(0 - \frac{\pi}{2}\right) \\ M\left(\pi - \frac{3\pi}{2}\right) \end{matrix}\right\} = -EI\theta,_x - \frac{\nu}{2} EI\left(2\bar{\epsilon}\theta,_x \pm \bar{\epsilon}_m\,\theta,_{xm} - 2\bar{\epsilon}_m\,\theta,_x \mp \bar{\epsilon}\theta,_{xm}\right)
\tag{121}
$$

where

$$A = \int dA = bh$$

$$I = \frac{1}{12} bh^3$$

4.3.1.2 When Amplitude Is Small Compared with Depth of the Beam*

If the amplitude is small so that we make the approximation $2z \sin\pi\xi > \ell a_1 \eta$, then there is no solution for equation (109), and the stress-strain law takes the simpler form of equation (115). Strictly speaking, however, the relationship of $2z \sin\pi\xi \geq \ell a_1 \eta$ can not hold for the whole span of the beam, since $\sin\pi\xi$ approaches zero toward the end of the beam (at $\xi = 0$ and 1), while $\ell a_1 \eta$ is never zero, no matter how small it may be. The approximation is taken in the sense that $\ell a_1 \eta$ is sufficiently small so that at any instant only a very small portion of the entire beam behaves with the hysteretic behavior of equations (112) and (113) rather than with the hysteretic behavior of equation (115), and the substitution of equation (115) for equations (112) and (113) for this small portion of the beam does not alter the damping behavior too much. Physically speaking, this is the case when the shaded areas of the beam shown in figure 8(b) become predominant; while the unshaded area becomes negligibly small.

*In the following, for brevity, this case will be referred to simply as the small amplitude case.

Based on this approximation, and the loading cycle as shown in figure 9(b), the substitution of equation (115) into equation (118) gives

$$\overleftrightarrow{N} = EA\bar{\epsilon} + EI_1 \nu (\bar{\epsilon} - \bar{\epsilon}_m)(\theta,_{xm} \pm \theta,_x)$$

$$\overleftrightarrow{M} = -EI\theta,_x - EI_3 \nu \theta,_x \theta,_{xm} \mp \frac{E\nu}{2} \left[I_1 (\bar{\epsilon} - \bar{\epsilon}_m)^2 + I_3 (\theta,_x^2 - \theta,_{xm}^2) \right] \qquad (122)$$

where upper signs refer to $\pi \leq \tau \leq 2\pi$, lower signs refer to $0 \leq \tau \leq \pi$, and

$$I_1 = 2 \int_0^{\frac{h}{2}} bzdz = \frac{1}{4} bh^2$$

$$I_3 = 2 \int_0^{\frac{h}{2}} bz^3 dz = \frac{1}{32} bh^4 \qquad (123)$$

4.3.2 Simply Supported Case

As stated in Section 4.2.2, for the simply supported case the stress-strain law follows the same form of equation (115), which in turn gives the axial tension N and bending moment M the same expressions as shown by equation (122).

4.4 EQUATIONS OF MOTION WITH MATERIAL DAMPING

As observed in the analysis of Part I, for the type of slender beams considered, the influence of the shear deformation and rotatory inertia is small. In the following analysis the shear deformation and rotatory inertia will be neglected for simplicity.

Since shear deformation is neglected, we have

$$\gamma = 0, \text{ and } \theta = \alpha \tag{124}$$

and equation (7) becomes

$$\tan\theta = \frac{w,_x}{1 + u,_x}$$

$$\sin\theta = \frac{w,_x}{1 + \bar{\epsilon}}$$

$$\cos\theta = \frac{1 + u,_x}{1 + \bar{\epsilon}} \tag{125}$$

When both shear deformation and rotatory inertia are neglected, equation (12) is reduced to $Q = M,_x/(1 + \bar{\epsilon})$. By using this expression and equation (125), Q and $\bar{\epsilon}$ are eliminated from equations (10) and (11), and thus we have

$$(N\cos\theta),_x - \frac{1}{2}\left(\frac{M,_x \sin 2\theta}{1 + u,_x}\right),_x = \rho A u,_{tt} \tag{126}$$

$$(N\sin\theta),_x + \left(\frac{M,_x \cos^2\theta}{1 + u,_x}\right),_x + K(x,t) = \rho A\,(w,_{tt} - \Omega_1^2\,\tilde{w}\cos\Omega_1 t) \tag{127}$$

Since there will be a phase difference between the external excitation and the beam motion, a phase angle ψ_1 is introduced as follows

$$\tau = \Omega_1 t + \psi_1\ , \quad \text{or} \quad \Omega_1 t = \tau - \psi_1 \tag{128}$$

and we let ψ_1 be expressed by a perturbation series:

$$\psi_1 = \psi_{01} + \psi_{11}\eta + \psi_{21}\eta^2 + \ldots \tag{129}$$

Then

$$\cos \Omega_1 t = \cos (\tau - \psi_1)$$

$$= \cos (\tau - \psi_{01}) + \eta \psi_{11} \sin (\tau - \psi_{01})$$

$$- \eta^2 \left[\frac{\psi_{11}}{2} \cos (\tau - \psi_{01}) + \psi_{21} \sin (\tau - \psi_{01}) \right] + \ldots \quad (130)$$

Now values of N and M are substituted in equations (126) and (127) and the resulting equations are transformed into dimensionless form. Perturbation equations are obtained by techniques similar to those used in Part I for the undamped case. Among the difference in the resulting equations are those due to the introduction of the phase angle ψ_1, and those due to the fact that the expressions for N and M at different intervals of time (τ) are different now. Consequently the equations of motion take different forms during different time (τ) intervals. After some computation, we have

$$u_{1,\xi\xi} = 0 \quad (131)$$

$$\beta_1^4 w_{1,\tau\tau} + w_{1,\xi\xi\xi\xi} = 0 \quad (132)$$

$$\left(u_{2,\xi} + \frac{1}{2} w_{1,\xi}^2 \right)_{,\xi} = 0 \quad (133)$$

$$c \beta_1^4 u_{2,\tau\tau} = (u_{3,\xi} + w_{1,\xi} w_{2,\xi} + c w_{1,\xi} w_{1,\xi\xi\xi})_{,\xi} \quad (134)$$

where the notation is the same as in Part I.

Equations (131) (132), (133) and (134) are valid for all intervals of time ($0 \leq \tau \leq 2\pi$), and for all boundary conditions. In fact, they are identical to the corresponding equations for undamped motion, (equations (51) (52), (53) and (55)). The second order and third order equations for w (i.e., w_2, and w_3), however, are all different and are listed below.

(a) For both ends hinged case with very large amplitude

$$\beta_1^{\,4} w_{2,\tau\tau} + w_{2,\xi\xi\xi\xi} + \beta_1^{\,4} \frac{\Delta_{11}}{\omega_1^{\,2}} w_{1,\tau\tau} - \frac{1}{c}\left(\frac{1}{2} w_{1,\xi}^{\,3} + w_{1,\xi} u_{2,\xi}\right)_{,\xi}$$

$$-\beta_1^{\,4} w_o \cos(\tau - \psi_{01}) + \frac{\nu\eta}{2}\Big[(2w_{1,\xi\xi} \mp w_{1,\xi\xi m})$$

$$(u_{2,\xi} + \frac{1}{2} w_{1,\xi}^{\,2} - u_{2,\xi m} - \frac{1}{2} w_{1,\xi m}^{\,2})\Big]_{,\xi\xi} = 0 \qquad (135)$$

where the upper sign is for $0 \le \tau \le \frac{\pi}{2}$, and the lower sign is for $\pi \le \tau \le \frac{3\pi}{2}$ and

$$\beta_1^{\,4} w_{2,\tau\tau} + w_{2,\xi\xi\xi\xi} + \beta_1^{\,4} \frac{\Delta_{11}}{\omega_1^{\,2}} w_{1,\tau\tau} - \frac{1}{c}\left(\frac{1}{2} w_{1,\xi}^{\,3} + w_{1,\xi} u_{2,\xi}\right)_{,\xi}$$

$$-\beta_1^{\,4} w_o \cos(\tau - \psi_{01}) - \frac{\nu\eta}{2}\Big[2w_{1,\xi\xi}(u_{2,\xi} + \frac{1}{2} w_{1,\xi}^{\,2})$$

$$\pm w_{1,\xi\xi m}(u_{2,\xi} + \frac{1}{2} w_{1,\xi}^{\,2})_m\Big]_{,\xi\xi} = 0 \qquad (136)$$

where the upper sign is for $\frac{\pi}{2} \le \tau \le \pi$, and the lower sign is for $\frac{3\pi}{2} \le \tau \le 2\pi$.

(b) For both ends hinged case with small amplitude and also for a simply supported case

$$\beta_1^4 w_{2,\tau\tau} + w_{2,\xi\xi\xi\xi} + \beta_1^4 \frac{\Delta_{11}}{\omega_1^2} w_{1,\tau\tau} - \frac{1}{c}\left(\frac{1}{2} w_{1,\xi}^3 + w_{1,\xi} u_{2,\xi}\right)_{,\xi}$$

$$+ 2K \left(w_{1,\xi\xi} w_{1,\xi\xi m}\right)_{,\xi\xi} \mp K \left(w_{1,\xi\xi}^2 - w_{1,\xi\xi m}^2\right)_{,\xi\xi}$$

$$- w_o \beta_1^4 \cos(\tau - \psi_{01}) = 0 \qquad (137)$$

$$\beta_1^4 w_{3,\tau\tau} + w_{3,\xi\xi\xi\xi} + \beta_1^4 \frac{\Delta_{11}}{\omega_1^2} w_{2,\tau\tau} + \beta_1^4 \frac{\Delta_{21}}{\omega_1^2} w_{1,\tau\tau}$$

$$- \frac{1}{c}\left(w_{1,\xi}^2 w_{2,\xi} + u_{3,\xi} w_{1,\xi}\right)_{,\xi} + \left(w_{1,\xi} w_{1,\xi\xi}^2\right)_{,\xi}$$

$$\mp 2K \left(w_{1,\xi\xi} w_{2,\xi\xi} - w_{1,\xi\xi m} w_{2,\xi\xi m}\right)_{,\xi\xi} + 2K \left(w_{2,\xi\xi} w_{1,\xi\xi m}\right.$$

$$\left. + w_{1,\xi\xi} w_{2,\xi\xi m}\right)_{,\xi\xi} - \frac{\Delta_{11}}{\omega_1^2} \beta_1^4 w_o \cos(\tau - \psi_{01})$$

$$- \beta_1^4 w_o \psi_{11} \sin(\tau - \psi_{01}) = 0 \qquad (138)$$

where $K = \frac{I_3 \nu}{2I\ell} = \frac{3 h \nu}{16\ell}$, and the upper signs are for $0 \leq \tau \leq \pi$, and the lower signs for $\pi \leq \tau \leq 2\pi$.

SECTION 5

SOLUTION OF THE PROBLEM

5.1 BOTH ENDS HINGED CASE

5.1.1 Large Amplitude Case

Up to the first three equations (131, 132 and 133), the solution is the same as that of the undamped motion, and thus we have

$$u_1 = 0 \tag{61}$$

$$w_1 = a_1 \sin \pi\xi \cos \tau \tag{65}$$

$$\beta_1 = \pi$$

$$u_{2,\xi} + \frac{1}{2} w_{1,\xi}^2 = \frac{1}{4} a_1^2 \pi^2 \cos^2 \tau \tag{66}$$

$$u_2 = \frac{-1}{8} a_1^2 \pi \sin 2\pi\xi \cos^2 \tau \tag{77}$$

To obtain the periodic solution of w_2 from equations (135) and (136), first we must eliminate the secular terms in equations (135) and (136). With phase shift added, these terms consist of both $\cos \tau \sin \pi\xi$ and $\sin \tau \sin \pi\xi$. These terms can be eliminated by first multiplying equations (135) and (136) with $\cos \tau \sin \pi\xi$, and secondly by $\sin \tau \sin \pi\xi$, and then integrating the resulting two equations throughout the whole beam length $(0 \leq \xi \leq 1)$, and over the period of one cycle $(0 \leq \tau \leq 2\pi)$, which gives:

$$\int_0^1 \sin \pi\xi d\xi \int_0^{2\pi} \left[\beta_1{}^4 w_{2,\tau\tau} + w_{2,\xi\xi\xi\xi} + \beta_1{}^4 \frac{\Delta_{11}}{\omega_1{}^2} w_{1,\tau\tau}\right.$$

$$\left. - \frac{1}{c}\left(\frac{1}{2} w_{1,\xi}{}^3 + w_{1,\xi} u_{2,\xi}\right)_{,\xi} - \beta_1{}^4 w_o \cos(\tau - \psi_{o1})\right] \cos \tau \, d\tau$$

$$+ \frac{\nu\eta}{2} \int_0^1 \sin \pi\xi d\xi \left\{ \left[\int_0^{\frac{\pi}{2}} - \int_{\frac{\pi}{2}}^{\pi} + \int_{\pi}^{\frac{3\pi}{2}} - \int_{\frac{3\pi}{2}}^{2\pi} \right] \left[2w_{1,\xi\xi} \left|u_{2,\xi}\right.\right.\right.$$

$$\left.\left.\left. + \frac{1}{2} w_{1,\xi}{}^2\right| \right]_{,\xi\xi} \cos \tau \, d\tau \right\} - \frac{\nu\eta}{2} \int_0^1 \sin \pi\xi d\xi \left\{ \left[\int_0^{\frac{\pi}{2}}\right.\right.$$

$$\left.\left. + \int_{\pi}^{\frac{3\pi}{2}} \right] \left[w_{1,\xi\xi} \left(u_{2,\xi} + \frac{1}{2} w_{1,\xi}{}^2\right)_m \right]_{,\xi\xi} \cos \tau \, d\tau \right\}$$

$$- \frac{\nu\eta}{2} \int_0^1 \sin \pi\xi d\xi \left\{ \left[\int_0^{\frac{\pi}{2}} - \int_{\pi}^{\frac{3\pi}{2}} \right] \left[w_{1,\xi\xi m} \left|u_{2,\xi}\right.\right.\right.$$

$$\left.\left.\left. + \frac{1}{2} w_{1,\xi}{}^2\right| \right]_{,\xi\xi} \cos \tau \, d\tau \right\} + \frac{\nu\eta}{2} \int_0^1 \sin \pi\xi d\xi \left\{ \left[\int_0^{\frac{\pi}{2}} - \int_{\frac{\pi}{2}}^{\pi} - \int_{\pi}^{\frac{3\pi}{2}}\right.\right.$$

$$\left.\left. + \int_{\frac{3\pi}{2}}^{2\pi} \right] \left[w_{1,\xi\xi m} \left(u_{2,\xi} + \frac{1}{2} w_{1,\xi}{}^2\right)_m \right]_{,\xi\xi} \cos \tau \, d\tau \right\} = 0 \qquad (139)$$

and

$$
\begin{aligned}
&\int_0^1 \sin \pi\xi d\xi \int_0^{2\pi} \Bigg[\beta_1{}^4 w_{2,\tau\tau} + w_{2,\xi\xi\xi} + \beta_1{}^4 \frac{\Delta_{11}}{\omega_1{}^2} w_{1,\tau\tau} \\
&- \frac{1}{c}\left(\frac{1}{2} w_{1,\xi}{}^3 + u_{2,\xi} w_{1,\xi}\right)_{,\xi} - \beta_1{}^4 w_o \cos(\tau - \psi_{o1})\Bigg] \sin \tau d\tau \\
&+ \frac{\nu\eta}{2}\int_0^1 \sin \pi\xi d\xi \Bigg\{\Bigg[\int_0^{\frac{\pi}{2}} - \int_{\frac{\pi}{2}}^{\pi} + \int_{\pi}^{\frac{3\pi}{2}} - \int_{\frac{3\pi}{2}}^{2\pi}\Bigg]\Bigg[2w_{1,\xi\xi} \Big| u_{2,\xi} \\
&+ \frac{1}{2} w_{1,\xi}{}^2 \Big| \Bigg]_{,\xi\xi} \sin \tau d\tau\Bigg\} - \frac{\nu\eta}{2}\int_0^1 \sin \pi\xi d\xi \Bigg\{\Bigg[\int_0^{\frac{\pi}{2}} \\
&- \int_{\pi}^{\frac{3\pi}{2}}\Bigg]\Bigg[w_{1,\xi\xi} \Big| u_{2,\xi} + \frac{1}{2} w_{1,\xi}{}^2 \Big|_m \Bigg]_{,\xi\xi} \sin \tau d\tau\Bigg\} \\
&- \frac{\nu\eta}{2}\int_0^1 \sin \pi\xi d\xi \Bigg\{\Bigg[\int_0^{\frac{\pi}{2}} - \int_{\pi}^{\frac{3\pi}{2}}\Bigg]\Bigg[w_{1,\xi\xi m} \Big| u_{2,\xi} \\
&+ \frac{1}{2} w_{1,\xi}{}^2 \Big| \Bigg]_{,\xi\xi} \sin \tau d\tau\Bigg\} + \frac{\nu\eta}{2}\int_0^1 \sin \pi\xi d\xi \Bigg\{\Bigg[\int_0^{\frac{\pi}{2}} - \int_{\frac{\pi}{2}}^{\pi} - \int_{\pi}^{\frac{3\pi}{2}} \\
&+ \int_{\frac{3\pi}{2}}^{2\pi}\Bigg]\Bigg[w_{1,\xi\xi m} \Big| u_{2,\xi} + \frac{1}{2} w_{1,\xi}{}^2 \Big|_m \Bigg]_{,\xi\xi} \sin \tau d\tau\Bigg\} = 0 \qquad (140)
\end{aligned}
$$

where, for the sake of brevity, integrations with same integrand but different integration limits are grouped together. Using the relationships that

$$\int_0^1 \sin \pi\xi d\xi \int_0^{2\pi} \left(\beta_1^4 w_{2,\tau\tau} + w_{2,\xi\xi\xi\xi} \right) \begin{matrix}\cos \tau \\ \sin \tau\end{matrix} \, d\tau \equiv 0,$$

and substituting all the known quantities into equations (139) and (140), and carrying out the integration, we obtain

$$-\frac{a_1 \pi^5 \Delta_{11}}{2\omega_1^2} + \frac{3a_1^3 \pi^5}{32c} - 2\pi^4 w_o \cos \psi_{o1} - \nu\eta a_1^3 \pi^6 \frac{3\pi - 8}{48} = 0 \qquad (141)$$

$$-2\pi^4 w_o \sin \psi_{o1} - \nu\eta \frac{a_1^3 \pi^6}{24} = 0 \qquad (142)$$

Solving these two equations for Δ_{11} and ψ_{o1}, we obtain

$$\psi_{o1} = \sin^{-1} \left(- \frac{\nu\eta a_1^3 \pi^2}{48 \, w_o} \right) \qquad (143)$$

and

$$\frac{\Delta_{11}}{\omega_1^2} = \frac{9\ell^2}{4h^2} a_1^2 \eta - \frac{(3\pi-8)\pi}{24} \nu\eta a_1^2 \mp \frac{4}{\pi a_1} \left[w_o^2 - \left(\frac{\nu\eta \pi^2 a_1^3}{48} \right)^2 \right]^{\frac{1}{2}} \qquad (144)$$

The response equation is

$$\frac{\Omega_1^2}{\omega_1^2} = 1 + \frac{9\ell^2}{4h^2} a_1^2 \eta^2 - \frac{(3\pi-8)\pi}{24} \nu\eta^2 a_1^2 \mp \frac{4}{\pi a_1 \eta} \left[w_o^2 \eta^4 - \left(\frac{\nu\pi^2 a_1^3 \eta^3}{48} \right)^2 \right]^{\frac{1}{2}} \qquad (145)$$

Note that if $\nu = 0$, equation (145) is reduced to the response equation for the undamped case, i.e., equation (76), and that because of the damping, the response curves are closed at the top now. To solve for w_2 we observe that equations (135) and (136) both can be put into the following form:

$$\pi^4 w_{2,\tau\tau} + w_{2,\xi\xi\xi\xi} = F(\tau,\xi) \tag{146}$$

By expressing $F(\tau,\xi)$ into a double Fouries's series for the intervals $(0 \le \xi \le 1)$ and $0 \le \tau \le 2\pi$, these two equations can be combined into one equation. Thus

$$\pi^4 w_{2,\tau\tau} + w_{2,\xi\xi\xi\xi} = 4\pi^3 w_o \cos(\tau - \psi_{o1}) \sum_{m=3,5,\ldots}^{\infty} \frac{\sin m\pi\xi}{m}$$

$$- \frac{1}{16c} a_1^3 \pi^4 \sin \pi\xi \cos 3\tau + \frac{\nu\eta}{4} a_1^3 \pi^5 \sin \pi\xi \Bigg[2 \sum_{m=3,5,\ldots}^{\infty}$$

$$(-1)^{\frac{m+1}{2}} \frac{m^2 - 3}{m(m^2-4)} \cos m\tau + \sum_{m=5,7,\ldots}^{\infty} \frac{(m^2+1)(m^4 - 7m^2 + 18) \sin m\tau}{m(m^2-4)(m^2-1)(m^2-9)}$$

$$+ \frac{22}{15} \sin 3\tau \Bigg] \tag{147}$$

$$0 \le \tau \le 2\pi, \quad 0 \le \xi \le 1$$

where the secular terms are being removed by the use of equations (141) and (142). Solution of equation (147) is

$$w_2 = \frac{4}{\pi} w_o \cos(\tau - \psi_{o1}) \sum_{m = 3, 5, \ldots}^{\infty} \frac{\sin m\pi\xi}{m(m^4 - 1)} + a_1^3 \frac{\sin \pi\xi \cos 3\tau}{128c}$$

$$+ \frac{\nu\eta}{4} a_1^3 \pi \sin \pi\xi \Bigg[2 \sum_{m = 3, 5, \ldots}^{\infty} \frac{(m^2 - 3) \cos m\tau}{m(m^2 - 4)(1 - m^2)}$$

$$- \sum_{m = 3, 5, \ldots}^{\infty} \frac{(m^2 + 1)(m^4 - 7m^2 + 18) \sin m\tau}{m(m^2 - 4)(m^2 - 1)^2 (m^2 - 9)} - \frac{22}{120} \sin 3\tau \Bigg] \tag{148}$$

If all the terms higher than m = 3 are neglected, equation (148) becomes

$$w_2 = \frac{1}{60\pi} w_o \cos(\tau - \psi_{o1}) \sin 3\pi\xi + a_1^3 \frac{\cos 3\tau}{128c} \sin \pi\xi$$

$$- \frac{\nu\eta}{40} a_1^3 \pi \sin \pi\xi \left(\cos 3\tau + \frac{11}{6} \sin 3\tau\right) \tag{149}$$

Equations (148) or (149) satisfies the boundary conditions of (58).

The solutions of $\overline{w}$ and $\overline{u}$, up to the second approximation can be written as

$$\overline{w} = a_1\eta \sin \pi\xi \cos(\tau - \psi) + \frac{1}{60\pi} w_o \eta^2 \sin 3\pi\xi \cos(\tau - \psi)$$

$$+ \frac{a_1^3 \eta^3 A \ell^2}{128 I} \sin \pi\xi \cos 3\tau - \frac{\nu\pi}{40} a_1^3 \eta^3 \sin \pi\xi \left(\cos 3\tau + \frac{11}{6} \sin 3\tau\right) \tag{150}$$

$$\overline{u} = - \frac{\pi}{16} (a_1\eta)^2 \sin 2\pi\xi (1 + \cos 2\tau) \tag{151}$$

Compared with the undamped solution, i.e., (79) and (80), it is seen that the expression of $\overline{u}$ remains the same while the expression of $\overline{w}$ is changed. The change is in the form of the phase angle ψ and some additional terms with the damping parameter ν.

5.1.2 Small Amplitude Case

Again we can use the undamped solutions for u_1, u_2, and w_1.

$$u_1 = 0 \tag{61}$$

$$w_1 = a_1 \sin \pi\xi \cos \tau \tag{65}$$

$$\beta_1 = \pi$$

$$u_{2,\xi} + \frac{1}{2} w_{1,\xi}^{\;2} = \frac{1}{4} a_1^{\;2} \pi^2 \cos^2 \tau \tag{66}$$

$$u_2 = \frac{-1}{8} a_1^{\;2} \pi \sin 2\pi\xi \cos^2 \tau \tag{77}$$

To eliminate secular terms in equation (137), and at the same time to seek the solution of $\Delta_{11}/\omega_1^{\;2}$ and ψ_{o1}, the same technique used in Section 5.1.1 will be applied again. First equation (137) is multiplied by $\sin \pi\xi \cos \tau$ and $\sin \pi\xi \sin \tau$ and then the two resulting equations are integrated over τ $(0 - 2\pi)$ and ξ $(0 - 1)$. Thus we have

$$\int_o^1 \sin \pi\xi d\xi \left\{ \int_o^{2\pi} \left[\beta_1^{\;4} w_{2,\tau\tau} + w_{2,\xi\xi\xi\xi} + \beta_1^{\;4} \frac{\Delta_{11}}{\omega_1^{\;2}} w_{1,\tau\tau} \right.\right.$$

$$- \frac{1}{c} \left(\frac{1}{2} w_{1,\xi}^{\;3} + w_{1,\xi}\, u_{2,\xi}\right)_{,\xi} + 2K \left(w_{1,\xi\xi}\, w_{1,\xi\xi m}\right)_{,\xi\xi}$$

$$- w_o \beta_1^4 \cos(\tau - \psi_{o1}) \Big] \cos \tau d\tau - K \left[\int_o^{\pi} - \int_{\pi}^{2\pi} \right] (w_{1,\xi\xi}^2$$

$$- w_{1,\xi\xi m}^2)_{,\xi\xi} \cos \tau d\tau \Big\} = 0 \tag{152}$$

and

$$\int_o^1 \sin \pi\xi d\xi \Big\{ \cdot \int_o^{2\pi} \Big[\beta_1^4 w_{2,\tau\tau} + w_{2,\xi\xi\xi\xi} + \beta_1^4 \frac{\Delta_{11}}{\omega_1^2} w_{1,\tau\tau} - \frac{1}{c} (\frac{1}{2} w_{1,\xi}^3$$

$$+ w_{1,\xi} u_{2,\xi})_{,\xi} + 2K (w_{1,\xi\xi} w_{1,\xi\xi m})_{,\xi\xi} - w_o \beta_1^4 \cos(\tau - \psi_{o1}) \Big] \sin \tau d\tau$$

$$- K \left[\int_o^{\pi} - \int_{\pi}^{2\pi} \right] (w_{1,\xi\xi}^2 - w_{1,\xi\xi m}^2)_{,\xi\xi} \sin \tau d\tau \Big\} = 0 \tag{153}$$

The evaluation of the integrals gives the following two equations

$$- \frac{a_1 \pi^5 \Delta_{11}}{2\omega_1^2} + \frac{3\pi^5}{c32} a_1^3 - \frac{8K}{3} a_1^2 \pi^6 - 2w_o \pi^4 \cos \psi_{o1} = 0 \tag{154}$$

$$- \frac{32}{9} K a_1^2 \pi^5 - 2\pi^4 w_o \sin \psi_{o1} = 0 \tag{155}$$

Solving for Δ_{11}/ω_1^2 and $\sin \psi_{o1}$, and letting

$$K = \frac{3h\nu}{16\ell}, \quad c\eta = \frac{I}{A\ell^2} = \frac{h^2}{12\ell^2}$$

We have

$$\psi_{o1} = \sin^{-1} \left(- \frac{\nu \pi h a_1^2}{3 w_o \ell} \right) \tag{156}$$

$$\eta \frac{\Delta_{11}}{\omega_1^2} = \frac{9\ell^2}{4h^2}(a_1\eta)^2 - \frac{h}{\ell}\nu a_1\eta\pi \mp \frac{4}{\pi a_1\eta}\left[w_o^2\eta^4 - \left(\frac{\nu h}{3\ell}a_1^2\eta^2\pi\right)^2\right]^{\frac{1}{2}} \tag{157}$$

The response equation takes the following form

$$\frac{\Omega_1^2}{\omega_1^2} = 1 + \frac{9\ell^2}{4h^2}(a_1\eta)^2 - \frac{h}{\ell}\nu a_1\eta\pi \mp \frac{4}{\pi a_1\eta}\left[w_o^2\eta^4 - \left(\frac{\nu h}{3\ell}\pi a_1^2\eta^2\right)^2\right]^{\frac{1}{2}} \tag{158}$$

Again we see that if there is no damping, i.e., $\nu = 0$, equation (158) is reduced to the response equation of the undamped case, i.e., (76). The addition of the damping, however, makes the response curves closed at top (figure 12).

To solve equation (137) for w_2, first all the known quantities of equation (137) are expressed in a double Fourier's series for the intervals $0 \leq \tau \leq 1$, $0 \leq \quad \leq 2\pi$

Thus

$$\pi^4 w_{2,\tau\tau} + w_{2,\xi\xi\xi\xi} = - \sum_{m=3,5,\ldots}^{\infty} \frac{16K\,m}{m^2-4} a_1^2\pi^5 \sin m\pi\xi \cos\tau + 4 w_o\pi^3$$

$$\cos(\tau - \psi_{o1}) \sum_{m=3,5,\ldots}^{\infty} \frac{\sin m\pi\xi}{m} - \frac{1}{16c} a_1^3\pi^4 \sin\pi\xi \cos 3\tau$$

$$+ \sum_{m=3,5,\ldots}^{\infty} \frac{16K \sin m\tau}{m(m^2-4)} a_1^2\pi^5 \cos 2\pi\xi$$

$$- \frac{64}{3} \kappa a_1^2 \pi^4 \sin \tau \sum_{m=3,5,\ldots}^{\infty} \frac{m \sin m\pi\xi}{m^2 - 4}, \quad 0 \le \tau \le 2\pi, \quad 0 \le \xi \le 1 \qquad (159)$$

where the secular terms are being removed by the use of (154) and (155).

Solution of (159) is

$$w_2 = - \sum_{m=3,5,\ldots}^{\infty} a_1^2 \pi \kappa \frac{16 m \sin m\pi\xi \cos \tau}{(m^4 - 1)(m^2 - 4)}$$

$$+ \frac{4}{\pi} w_o \sum_{m=3,5,\ldots}^{\infty} \frac{\sin m\pi\xi}{m(m^4 - 1)} \cos(\tau - \psi_{o1})$$

$$+ \frac{1}{128c} a_1^3 \sin \pi\xi \cos 3\tau$$

$$- \sum_{m=3,5,\ldots}^{\infty} 16 \kappa a_1^2 \pi \frac{\sin m\tau \cos 2\pi\xi}{m(m^2 - 4)(m^4 - 16)}$$

$$- \frac{64}{3} \kappa a_1^3 \sin \tau \sum_{m=3,5,\ldots}^{\infty} \frac{m \sin m\pi\xi}{(m^2 - 4)(m^4 - 1)} \qquad (160)$$

In equation (160) if all terms higher than m = 3 are neglected, the equation can be reduced to:

$$w_2 = - \frac{3\pi}{25} \kappa a_1^2 \sin 3\pi\xi \cos \tau + \frac{w_o}{60\pi} \sin 3\pi\xi \cos(\tau - \psi_{o1})$$

$$+ \frac{\sin \pi\xi}{128c} a_1^3 \cos 3\tau - \frac{16\pi}{975} K a_1^2 \sin 3\tau \cos 2\pi\xi$$

$$- \frac{4}{25} K a_1^2 \sin \tau \sin 3\pi\xi \tag{161}$$

Now the solution of $\overline{w}$ and $\overline{u}$, up to the second approximation, can be written as

$$\overline{w} = a_1 \eta \sin \pi\xi \cos (\tau - \psi) + w_o \eta^2 \frac{\sin 3\pi}{60\pi} \cos (\tau - \psi)$$

$$+ \frac{A\ell^2}{128I} a_1^3 \eta^3 \sin \pi\xi \cos 3\tau - \frac{\nu h}{\ell} a_1^2 \eta^2 \left(\frac{\pi}{325} \cos 2\pi\xi \sin 3\tau \right.$$

$$\left. - \frac{9\pi}{400} \sin 3\pi\xi \cos \tau + \frac{3}{100} \sin 3\pi\xi \sin \tau \right) \tag{162}$$

$$\overline{u} = - \frac{\pi}{16} a_1^2 \eta^2 \sin 2\pi\xi \, (1 + \cos 2\tau) \tag{163}$$

Compared with the undamped solution (79) and (80), it is seen that the expression of $\overline{u}$ remains the same, while $\overline{w}$ is changed. The change is in the form of the phase angle ψ and some additional terms with the damping parameter ν.

5.2 SIMPLY SUPPORTED CASE

The solution of u_1, w_1, u_2, are the same as that for the undamped case, i.e.,

$$u_1 = 0$$

$$w_1 = a_1 \sin \pi\xi \cos \tag{84}$$

$$u_{2,\xi} + \frac{1}{2} w_{1,\xi}^2 = 0 \tag{85}$$

$$u_2 = - \frac{1}{8} a_1^2 \pi^2 \left(\xi + \frac{1}{2\pi} \sin 2\pi\xi\right) (1 + \cos 2\tau) \tag{86}$$

To obtain Δ_{11}/ω_1^2, ψ_{o1} and w_2, again equation (137) will be used. In fact, all the results for Δ_{11}/ω_1^2, ψ_{o1}, and w_2 of Section 5.1.2 can be directly applied here. But this time, we have

$$u_{2,\xi} + \frac{1}{2} w_{1,\xi}^2 = 0$$

Thus instead of equation (157), we have

$$\eta \frac{\Delta_{11}}{\omega_1^2} = -\frac{h}{\ell} \nu a_1 \eta \pi \mp \frac{4}{\pi a_1 \eta} \left[w_o^2 \eta^4 - \left(\frac{\nu h}{3\ell} a_1^2 \eta^2 \pi \right)^2 \right]^{\frac{1}{2}} \tag{164}$$

The solution for the phase angle, however, is the same as (156),

$$\psi_{o1} = \sin^{-1} \left(- \frac{\nu \pi h a_1^2}{3 w_o \ell} \right) \tag{165}$$

Similarly solution of w_2 (161) becomes:

$$w_2 = -\frac{3K}{25} a_1^2 \pi \sin 3\pi\xi \cos \tau + \frac{1}{60\pi} w_o \sin 3\pi\xi \cos (\tau - \psi_{o1}) \tag{166}$$

$$- \frac{16\pi}{975} K a_1^2 \cos 2\pi\xi \sin 3\tau - \frac{4K}{25} a_1^2 \sin 3\pi\xi \sin \tau$$

To obtain the second order approximate solution of response relationship (Δ_{21}, ψ_{11}), first equation (134) is solved for u_3. Thus

$$u_3 = \frac{c}{2} a_1^2 \pi^6 \left(\frac{1}{6} \pi^3 - \frac{1}{8\pi^3} \sin 2\pi\xi \right) \cos 2\tau - \frac{c}{8} (2\pi^2 - 1) a_1^2 \pi^4 \xi \cos 2\tau$$

$$+\left[\frac{9K}{50} a_1^3 \pi^2 \cos^2\tau - a_1 \overline{w}_o \frac{\cos\tau}{40} \cos(\tau - \psi_{o1})\right] \left(\frac{\sin 2\pi\xi}{2} + \frac{\sin 4\pi\xi}{4}\right)$$

$$+\frac{16K}{975} a_1^3 \pi^2 \cos\tau \cos 3\tau \left(\cos\pi\xi + \frac{\cos 3\pi\xi}{3}\right) - \frac{9\pi}{50} K a_1^3 \cos\tau \sin\tau$$

$$+\frac{64K}{2925} a_1^3 \pi^2 \sin 3\tau \cos\tau - \frac{6\pi}{25} K a_1^3 \cos\tau \sin\tau \left(\frac{\cos 2\pi\xi}{2} + \frac{\cos 4\pi\xi}{4}\right)$$

$$+ c a_1^2 \pi^4 \cos^2\tau \left(\frac{\xi}{2} + \frac{1}{4\pi} \sin 2\pi\xi\right) \qquad (167)$$

In solving u_3, the boundary conditions (81) and (83) are used.

After all the known quantities are substituted into equation (138), and the same technique as used in obtaining Δ_{11}/ω_1^2 and ψ_{o1} is applied to the resulting equation, we get two secular equations for Δ_{21}/ω_1^2 and ψ_{11}.

$$-\frac{8\pi^2 - 45}{192} a_1^3 \pi^7 + \frac{I_3\nu}{I\ell} \left(\frac{4\pi}{75} a_1 w_o \sin\psi_{o1} + \frac{32(997\pi^2 - 936)}{9 \times 13000} \frac{I_3\nu}{I\ell} a_1^3 \pi^5\right.$$

$$-\frac{2\pi^5}{25} a_1 w_o$$

$$\left.\cos\psi_{o1}\right) - \frac{a_1}{2}\pi^5 \frac{\Delta_{21}}{\omega_1^2} - 2\pi^4 w_o \cos\psi_{o1} \frac{\Delta_{11}}{\omega_1^2} + 2\pi^4 w_o \psi_{11} \sin\psi_{o1} = 0 \qquad (168)$$

and

$$\frac{I_3\nu}{I\ell} \left(\frac{72\nu I_3}{1251\ell} a_1^3 \pi^6 - \frac{8w_o}{135} a_1 \pi^4 \cos\psi_{o1} - \frac{w_o}{25} a_1 \pi^5 \sin\psi_{o1}\right)$$

$$-2\pi^4 w_o \sin\psi_{o1} \frac{\Delta_{11}}{\omega_1^2} - 2\pi^4 w_o \psi_{11} \cos\Delta_{o1} = 0 \qquad (169)$$

Solving (168) and (169) for Δ_{21} and ψ_{11}, we obtain

$$\frac{\Delta_{21}}{\omega_1^2} = \left(\frac{8\pi^2 - 45}{96}\right)\pi^2 a_1^2 - \frac{4 w_o}{\pi a_1 \cos\psi_{o1}} \frac{\Delta_{11}}{\omega_1^2} + \frac{997\pi^2 - 936}{13000}\left(\frac{h\nu a}{\ell}\right)^2$$

$$+ \frac{3h\nu}{8\ell}\left(\frac{-8 w_o}{675\pi}\sin\psi_{o1} - \frac{2 w_o}{25\cos\psi_{o1}} - \frac{2 w_o}{25}\cos\psi_{o1}\right.$$

$$\left. + \frac{54 h\nu}{125\ell} a_1^2 \pi \tan\psi_{o1}\right) \qquad (170)$$

$$\psi_{11} = -\frac{\Delta_{11}}{\omega_1^2}\tan\psi_{o1} - \frac{3\nu h}{8 w_o \ell \cos\psi_{o1}}\left(\frac{4 w_o}{135} a_1 \cos\psi_{o1}\right.$$

$$\left. + \frac{w_o}{50} a_1 \pi \sin\psi_{o1}\right) + \frac{81}{2000 w_o \cos\psi_{o1}} a_1^3 \pi^2 \left(\frac{\nu h}{\ell}\right)^2 \qquad (171)$$

Terms inside the parentheses of equations (170) and (171) are small when compared with other terms, and hence for simplicity they shall be neglected. Also the term Δ_{11}/ω_1^2 is substituted for the use of equation (164). Thus equations (170) and (171) become

$$\frac{\Delta_{21}}{\omega_1^2} = \frac{(8\pi^2 - 45)\pi^2}{96} a_1^2 \eta^2 + \left(\frac{2 w_o}{\pi a_1}\right)^2 \left(4 \pm \frac{3\nu h \pi^2 a_1^2}{\sqrt{(3 w_o \ell)^2 - (\nu \pi h a_1^2)^2}}\right)$$

$$+ \frac{997\pi^2 - 936}{13000}\left(\frac{h\nu a}{\ell}\right)^2 \qquad (172)$$

and

$$\psi_{11} = -\frac{\nu h a}{3\ell}\left(4 \pm \frac{3\nu h\pi^2 a_1^2}{\sqrt{(3w_o\ell)^2 - (\nu\pi h a_1^2)^2}}\right) \tag{173}$$

Thus the response equation, up to terms of second order, becomes

$$\frac{\Omega_1^2}{\omega_1^2} = 1 + \frac{\Delta_{11}}{\omega_1^2}\eta + \frac{\Delta_{21}}{\omega_1^2}\eta^2$$

$$= 1 - \frac{(8\pi^2-45)\pi^2}{96}a_1^2\eta^2 - \frac{w_o\eta^2}{\pi a_1\eta}\left[1 \mp \frac{12\,\ell w_o^2\eta^4}{\pi a_1\eta\sqrt{(3w_o\eta^2\ell)^2 - (\nu\pi h a_1^2\eta^2)^2}}\right]$$

$$\left[\frac{\nu h(a_1\eta\pi)^2}{w_o\eta^2\ell} \pm 4\sqrt{1 - \left(\frac{\nu\pi h a_1^2\eta^2}{3w_o\ell\eta^2}\right)^2}\right]$$

$$+\frac{997\pi^2-936}{13000}\left(\frac{h\nu a_1\eta}{\ell}\right)^2 \tag{174}$$

The phase angle is

$$\psi = \psi_{o1} + \eta\psi_{11}$$

$$= \sin^{-1}\left(-\frac{\nu\pi h a_1^2\eta^2}{3w_o\eta^2\ell}\right) - \frac{4\nu h a_1\eta}{3\ell}$$

$$\mp \frac{\frac{1757}{2000}\left|\nu\pi h a_1^2\eta^2\right|^2}{\ell\sqrt{\left|3w_o\eta^2\ell\right|^2 - \left|\nu\pi h a_1^2\eta^2\right|^2}} \tag{175}$$

Again we see that if $\nu = 0$, equation (174) is reduced to equation (98), i.e., the response equation of the undamped case, but due to the addition of damping, the response curves are closed at top now (figures 14, 15, 16).

Up to the second approximation, the solutions of $\bar{w}$ and $\bar{u}$ are:

$$\begin{aligned}\bar{w} = {} & a_1\eta \sin \pi\xi \cos(\tau - \psi) + w_o\eta^2 \frac{\sin 3\pi\xi}{60\pi} \cos(\tau - \psi) \\ & - \frac{3\nu h\pi}{100\ell} a_1^2\eta^2 \sin 3\pi\xi \left(\frac{3}{4}\cos\tau + \sin\tau\right) \\ & - \frac{\nu h\pi}{325\ell} a_1^2\eta^2 \cos 2\pi\xi \sin 3\tau\end{aligned} \tag{176}$$

and

$$\bar{u} = -\frac{1}{8}(a_1\eta\pi)^2 \left(\xi + \frac{1}{2\pi}\sin 2\pi\xi\right)(1 + \cos 2\tau) \tag{177}$$

Again it is seen that due to the effect of damping, the expression of $\bar{w}$ is changed. The change is in the form of the phase angle ψ and some additional terms with the damping parameter ν. The expression for $\bar{u}$, however, remains the same.

SECTION 6

NUMERICAL RESULTS AND EXPERIMENTAL VERIFICATION

Calculations for two sample beams are made. One is of the very slender type $\left(\frac{\ell}{r} = \left(\frac{A\ell^2}{I}\right)^{\frac{1}{2}} = 1110\right)$, the other is of the moderately slender type $\left(\frac{\ell}{r} = \left(\frac{A\ell^2}{I}\right)^{\frac{1}{2}} = 69.3\right)$. Response curves are shown in figure 10 through figure 17.

Steel beams are used. The physical constants for steel are: $E = 30 \times 10^6$ psi, $G = 12 \times 10^6$ psi, density $\Gamma = 0.283$ lb/in.^{3} For a rectangular beam, the shear distortion constant $k' = 0.833$, the damping parameter $\nu = 18.6$ [30]. Except the case of figure 17, all the other cases are calculated for the external excitation in the form of support accelerations.

The dimensions are:

Beam (a) -- very slender type

length	ℓ =	10 inch
depth	h =	$\frac{1}{32}$ inch
width	b =	$\frac{1}{2}$ inch

$$\frac{I}{A\ell^2} = 0.813 \times 10^{-6}$$

The first natural linear frequency is

$$\frac{\omega_1}{2\pi} = \frac{\pi}{2}\left(\frac{gEI}{\Gamma A\ell^4}\right)^{\frac{1}{2}} = 28.5 \text{ cps}$$

Note: Although the dimensions indicate a strip type plate rather than a beam, tests show that the natural frequency is equal to the value (28.5 cps) calculated from the beam formula.

Beam (b) -- moderately slender type

$$\ell = 10 \text{ inch}$$

$$h = \frac{1}{2} \text{ inch}$$

$$b = \frac{1}{2} \text{ inch}$$

$$\frac{I}{A\ell^2} = 2.085 \times 10^{-4}$$

$$\frac{\omega_1}{2\pi} = 458.5 \text{ cps}$$

Note that the values of $\frac{I}{A\ell^2}$ for both beams are small. The correction due to shear deformation and rotatory inertia can be neglected.

Figure 10 shows response curves of Beam (a) for the immovable hinged ends case. The hard spring effect is very strong (equation (75)).

Figure 11 shows the response curve of Beam (b) with immovable hinged ends. The peak amplitude is twice as high as the depth of the beam. The hard spring effect due to large amplitude is so overwhelmingly strong that the influence of damping is relatively insignificant. However, the damping does make the top closed. In calculating the response curves, equation (75) (undamped) and equations (145) and (158) (damped) are used.

Figure 12 shows the response of Beam (b) with immovable hinged ends. But this time the peak amplitude is limited to 1/8 inch (depth of beam = 1/2 inch). Equation (75) (undamped) and equation (158) (damped) are used.

Figure 13 shows the response of Beam (b) while simply supported. The soft spring effect can be seen (on the stem curve) but not significant.

For the response curves of figures 13, 14, 15 and 16, equations (98) (undamped) and (174) (damped) are used.

If instead of the support acceleration, the beam is excited by a periodic uniform loading, the right side of equation (56) is zero, and consequently the last term of the response equation (98) is dropped. The difference is shown up in figure 17, where by equivalent uniform loading means that if only the linear theory is considered, both loadings would give exactly the same response relationship.

Tests have been conducted on the simply supported very slender beam. Both the soft spring effect and jump phenomena are demonstrated. Some of the test results are plotted in figures 14, 15, and 16.

The specific damping of a material can be expressed by*

$$D = J\sigma^m \text{ (in-lb per cubic inch per cycle)}$$

where σ is the amplitude of the stress. But D is equal to the area of the hysteresis loop, i.e.,

$$D = \int_{\epsilon_1}^{\epsilon_2} (\vec{\sigma} - \overleftarrow{\sigma})\, d\epsilon \quad \text{(Figure 6)}$$

By the use of equation (104), the integration gives

$$D = \frac{4}{3}\, \nu E \epsilon_0^3 \quad (\epsilon_0 \approx \sigma/E)$$

For any value of ϵ_0, ν is determined by equating these two expressions of D. For example, for Sandvik steel we have:* $m = 2.3$ and $J = 8.9 \times 10^{-12}$ If $\epsilon_0 = 3.08 \times 10^{-3}$, $\nu = 2.03$. For comparison, response curves for $\nu = 2.03$ and $\nu = 18.6$ are both shown in Figure 16.

*See [26], equation 36.9, page 36-7 and Table 36.5, page 36-35.

SECTION 7

CONCLUSIONS

(a) The nonlinear effects due to large amplitude on the response depends on support conditions. With immovable hinged ends the response is of hard spring type; and for the case of simply-supported ends the response is of the soft spring type.

(b) Nonlinear material damping in addition to limiting the amplitude at resonance to a finite value, modifies the shape of the response curve. It makes those response curves of the soft spring type more soft; and those of the hard spring type less hard.

(c) The effect of shear distortion and rotatory inertia on the linear or nonlinear response curves is related to the slenderness parameter $\frac{I}{A\ell^2}$. For beams with small $\frac{I}{A\ell^2}$ values, the correction due to shear distortion and rotatory inertia is usually negligible.

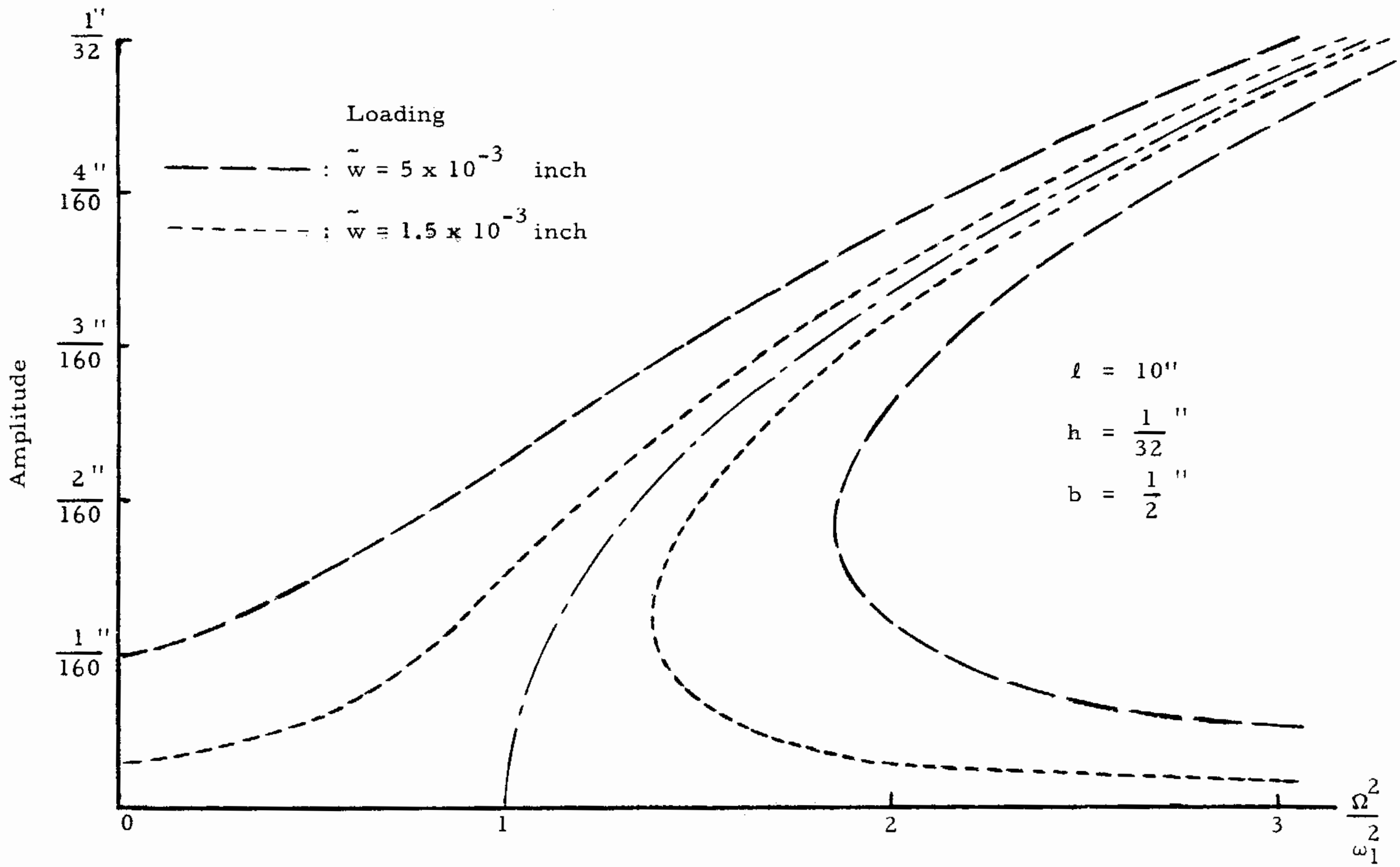


Figure 10 - Response Curve -- Very Slender Beam with Immovable Hinged Ends, Undamped Case Only

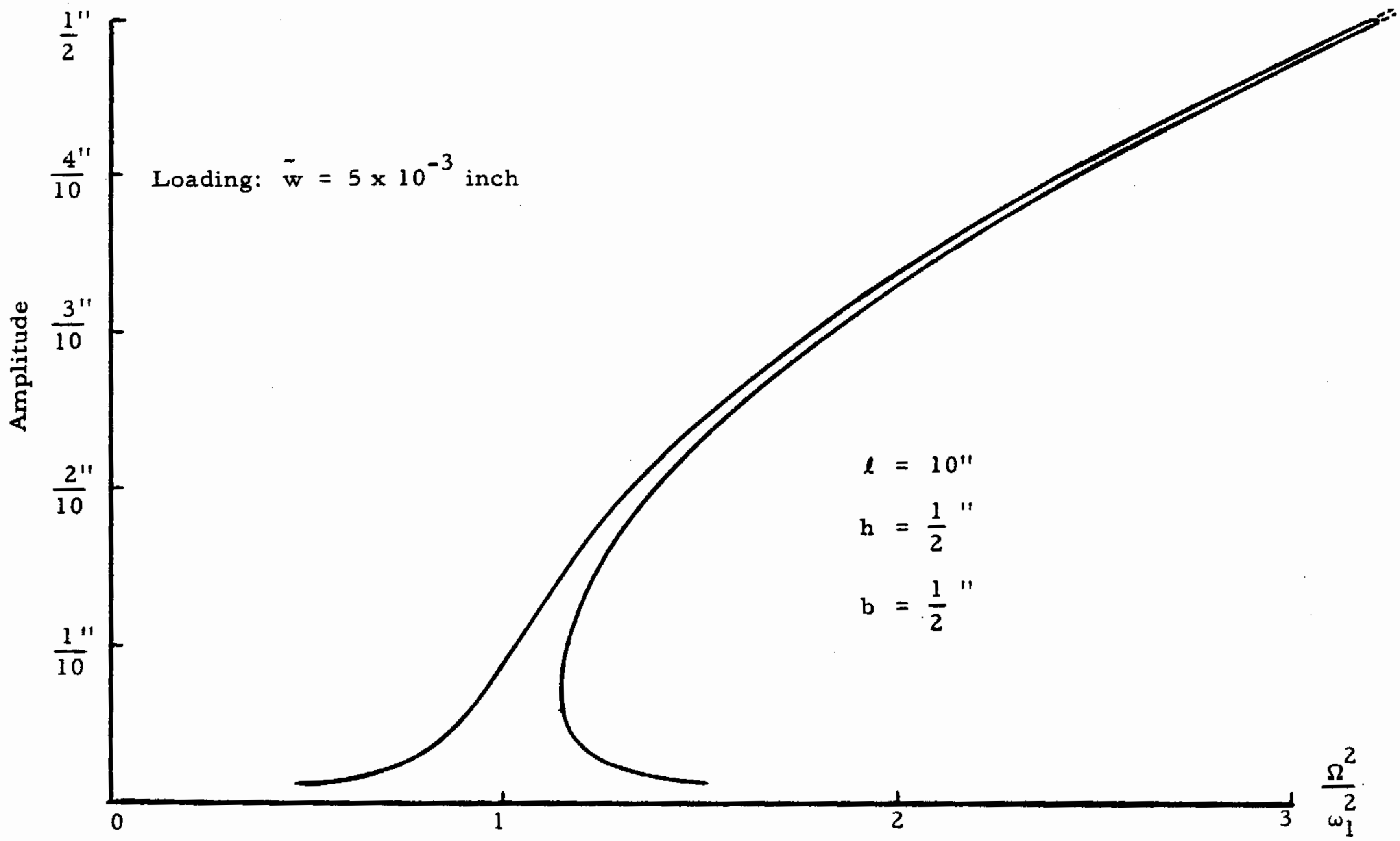


Figure 11 - Response Curve -- Moderately Slender Beam with Immovable Hinged Ends, with Amplitude Large Compared with Depth of the Beam

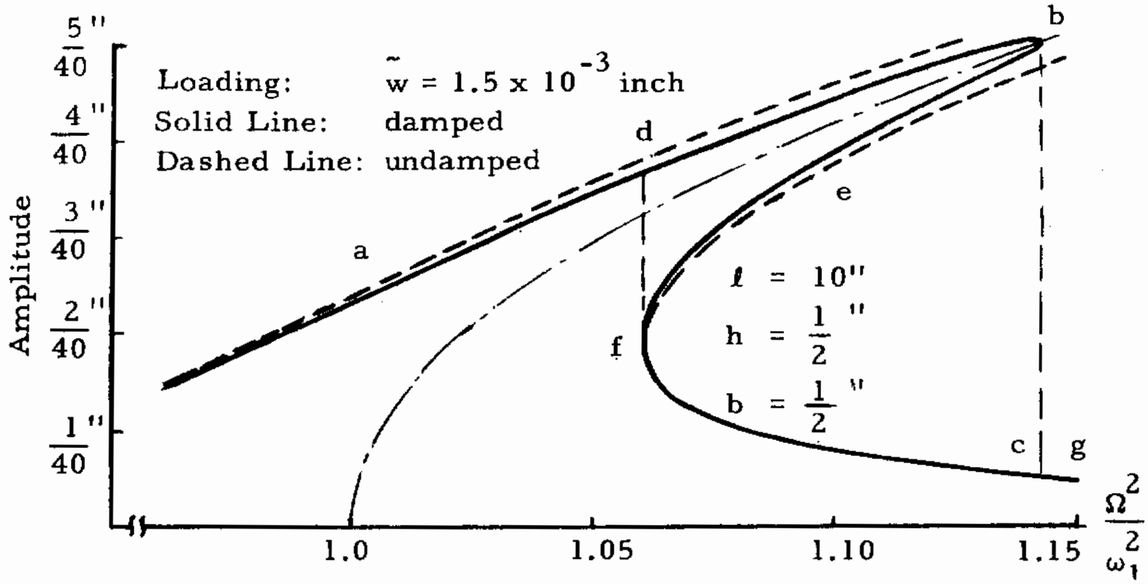


Figure 12 - Response Curve -- Moderately Slender Beam with Immovable Hinged Ends, with Amplitude Comparable to the Depth of the Beam

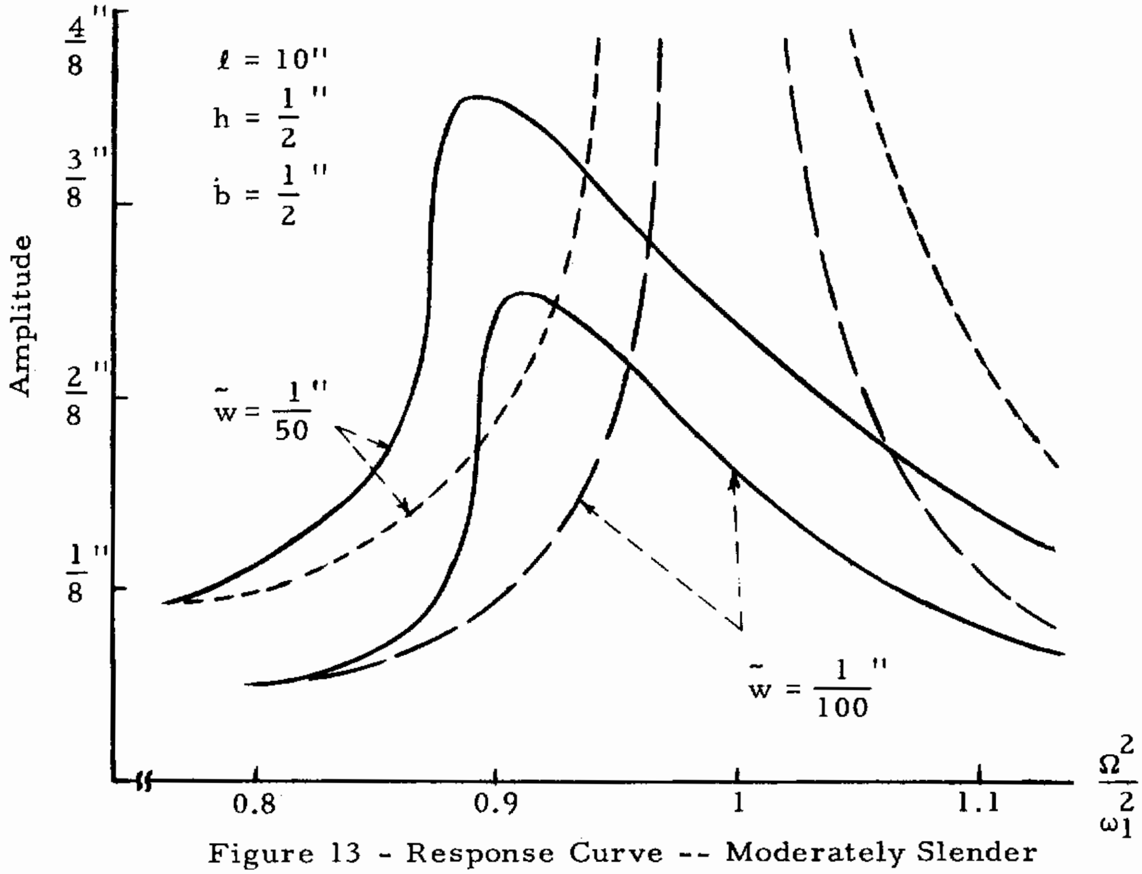


Figure 13 - Response Curve -- Moderately Slender Simply Supported Beam

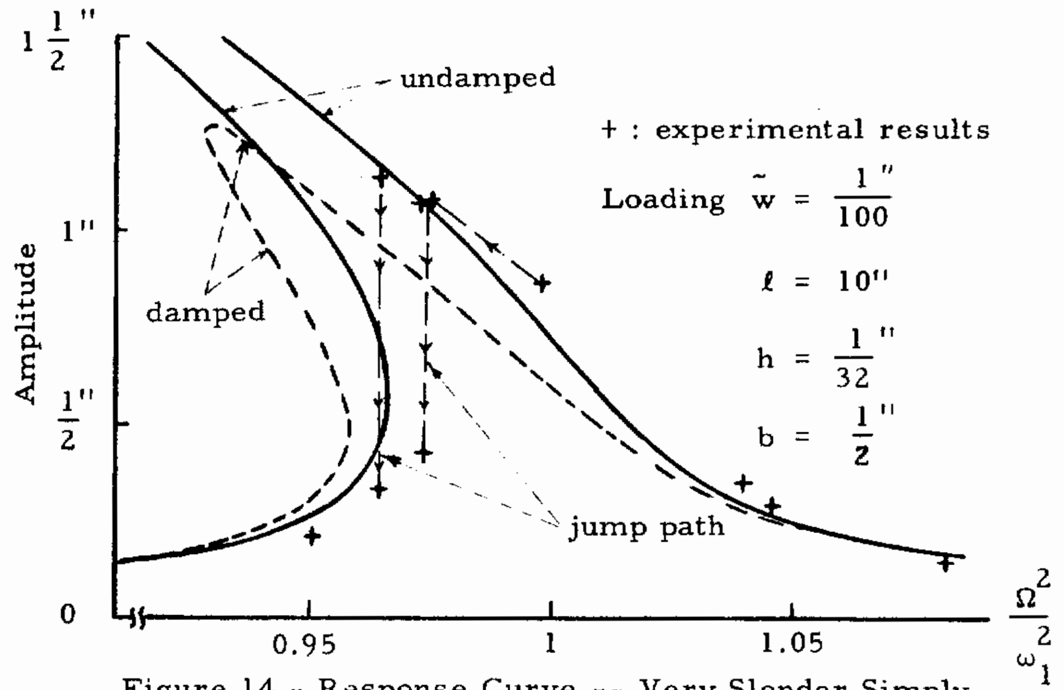


Figure 14 - Response Curve -- Very Slender Simply Supported Beam (with Experimental Results)

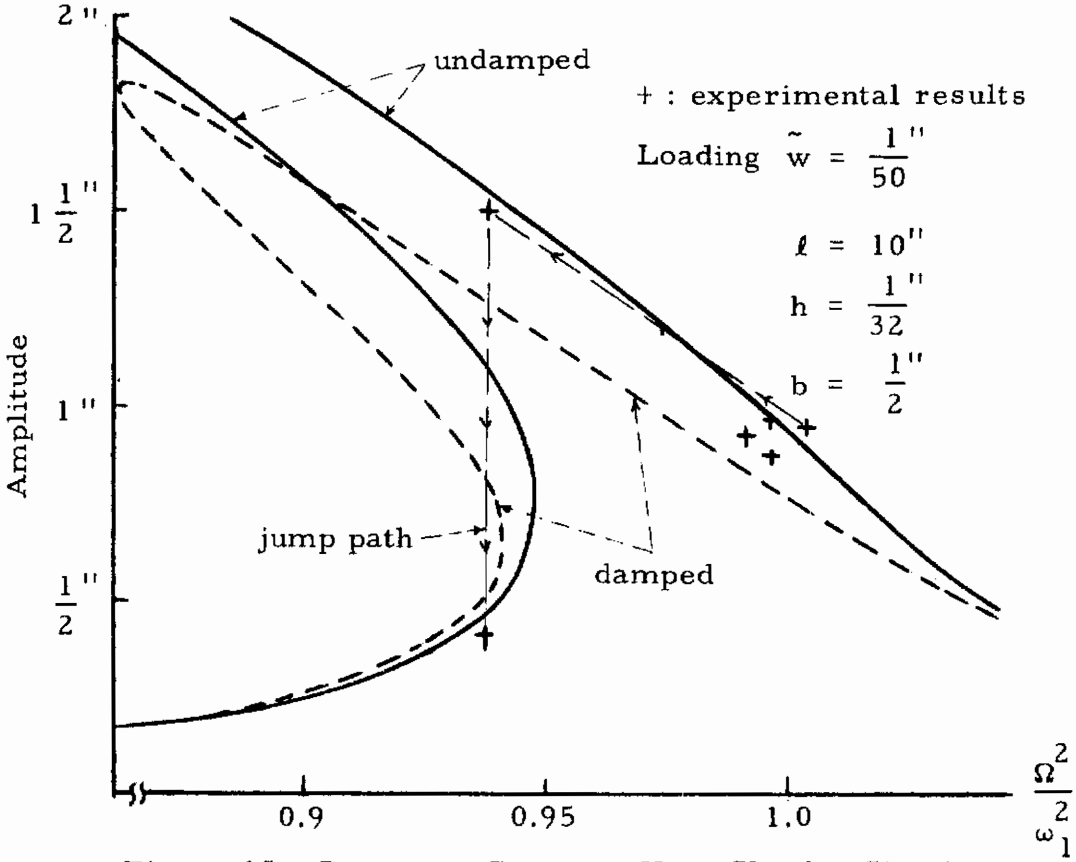


Figure 15 - Response Curve -- Very Slender Simply Supported Beam (with Experimental Results)

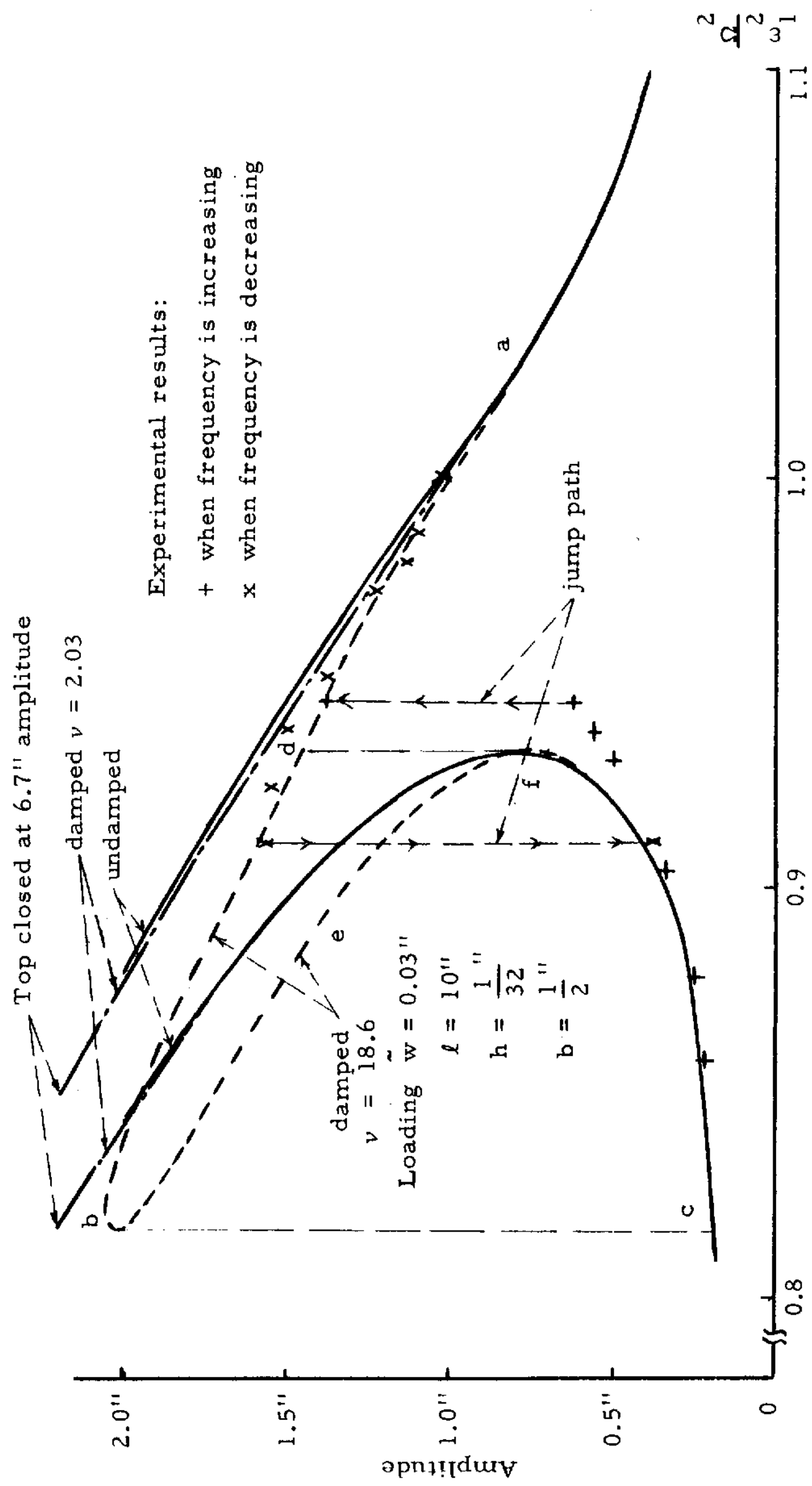


Figure 16 - Response Curve -- Very Slender Simply Supported Beam (with Experimental Results)

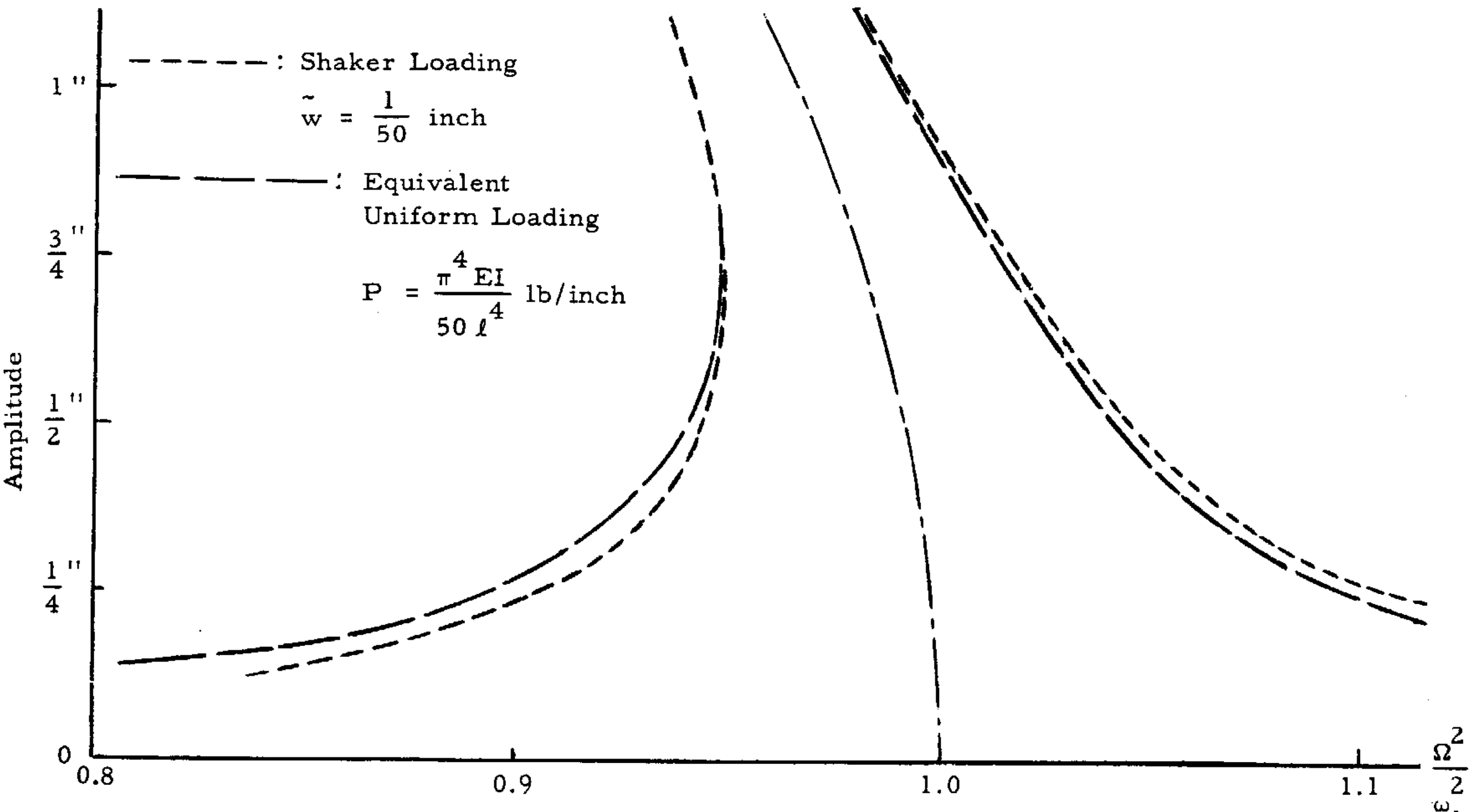


Figure 17 - Comparison of Response Curve of a Very Slender Simply Supported Beam Due to (a) Shaker Loading, (b) Equivalent Uniform Loading

REFERENCES

1. Von Kármán, Encyklopädie der Math. Wissenschaften, Vol. IV p. 349, 1910.

2. Timoshenko and Woinowsky - Krieger, "Theory of Plates and Shells," McGraw-Hill, 1959.

3. Biot, M. A., (a) "Theory of Elasticity with Large Displacements and Rotations," Proc. Fifth International Congress of Appl. Mech., 1938; (b) "Elastizitätstheorie Zweiter Ordnung Mit Anwendungen," ZAMM Vol. 20, 1940; (c) "Linear Thermodynamics and the Mechanics of Solids," Proc. 3rd U.S. National Congress of Appl. Mech., 1958.

4. Novozhilov, "Foundations of the Nonlinear Theory of Elasticity," Graylock, 1953.

5. Levy, S., (a) "Bending of Rectangular Plates with Large Deflections," NACA Report No. 737, 1942. (b) "Large Deflection Theory for Rectangular Plates," Proc. of Symposia in Appl. Math. Am. Math. Society, New York, Vol. 1, 1949.

6. Way, S., (a) "Uniform Loaded, Clamped, Rectangular Plates with Large Deflection," Proc. 5th Interm. Congress Appl. Mech. 1938. (b) Way, S., "Bending of Circular Plate with Large Deflection," Trans. ASME 56, p. 627, 1934.

7. Stippes, M. and Hausrath, A. H., "Large Deflection of Circular Plates," J. Appl. Mech. 19, p. 287, 1952.

8. Herrmann, G.,"Influence of Large Amplitudes on Flexual Motions of Elastic Plates," NACA Tech. Note 3578, 1955.

9. Tadjbakhsh, I. and Saibel, E., "On the Large Elastic Deflections of Plates," ZAMM 40, p. 259, 1960.

10. Eringen, A. C., (a) "On the Nonlinear Vibration of Elastic Bars," Q. of Appl. Math. 9, p. 361, January 1952. (b) "On the Nonlinear Vibration of Circular Membrane," Proc. 1st U.S. National Congress of Appl. Mech., p. 139, 1951. (c) "On the Nonlinear Oscillations of Viscoelastic Plates," J. Appl. Mech. 22, p. 563, 1955.

11. Carrier, G. F., (a) "On the Nonlinear Vibration Problem of the Elastic String," Q. of Appl. Math. 3, p. 157, 1945 (b) "A Note on the Vibrating String," Q. of Appl. Math. 7, p. 98, 1949.

12. Chu, H. N. and Herrmann, G., "Influence of Large Amplitudes on Free Flexural Vibrations of Rectangular Plates," J. of Appl. Mech. 23, p. 532, 1956.

13. Kirchman, E. J., and Greenspon, J. E., "Nonlinear Response of Aircraft Panels in Acoustic Noise," J. Acous. Soc. Am. 29 p. 854, 1957.

14. Mettler, E., (a) "On the Stability Problem of Forced Vibrations of Elastic Bodies," (in German) ZAMM 31, 1951. (b) "Uber die Stabilität erzwungener Schwingringen elastischer Körper," Ing-Arch 13, 1942. (c) "A General Theory of the Stability of Forced Vibrations of Elastic Bodies," (in German) Ing-Arch 17, 1949.

15. Kauderer, H., "Nonlinear Mechanics," (in German) Springer-Verlag, 1958.

16. Kirchhoff, G. R., "Vorlesungen über Mechanik 4, Anfl. Leipzig 1897 p. 440 Eq. (16).

17. Yamaki, N., "Influence of Large Amplitudes on Flexural Vibrations of Elastic Plates," ZAMM 41, p. 501, 1961.

18. Nowinski, J., "Nonlinear Transverse Vibrations of Circular Elastic Plates Built-in at the Boundary," Proc. 4th U.S. National Congress of Appl. Mech. p.325, 1962.

19. Woinowsky - Krieger, S., "The Effect of an Axial Force on the Vibration of Hinged Bars," JAM 17, 1950.

20. McDonald, Jr., P. H., "Nonlinear Dynamic Coupling in a Beam Vibration," JAM 22, 1955.

21. Lee, E. W., "Nonlinear Forced Vibration of a Stretched String," British J. of Appl. Physics 8, 1957.

22. Tobias, S. A., "Nonlinear Forced Vibrations of Circular Disc," Eng. 186, 1958.

23. Lassiter, Hess and Hubbard, "An Experimental Study of the Response of Simple Panels to Intense Acoustic Loading," J. Aero. Sci. 24, p. 19, 1957.

24. Smith, Jr., P. W., et al, "Nonlinear Response of a Simple Clamped Panel," J. Acous. Soc. Am. 33, 11, p. 1476, November 1961.

25. Lazan, B. J., "Energy Dissipation Mechanisms in Structures with Particular Reference to Material Damping," Structural Damping Edited by Ruzicka, ASME 1959.

26. Lazan, B. J. and Goodman, L. E., "Material and Interface Damping," Ch. 36 Vol. II, Shock and Vibration Handbook, McGraw-Hill, 1961.

27. Mindlin, R. D., Stubner, F. W., and Cooper, H. L., "Response of Damped Elastic Systems to Transient Disturbance," Proc. Soc. Experimental Stress Analysis, Vol. 5 No. 2, 1948.

28. Sethna, P. R., "Free Vibrations of Beams with Nonlinear Viscoelastic Material Properties," Proc. 4th U.S. National Congress of Appl. Mech., 1962.

29. Davidenkov, N. N., "Energy Dissipation in Vibration," Journal of the Tech. Physics Vol. 8 No. 6 (in Russian) 1938.

30. Pisarenko, G. S., "Vibrations of Elastic Systems Taking Account of Energy Dissipation in the Material." A Translation from Russian by A. R. Robinson as a WADD Technical Report WADD TR 60-582, 1962.

31. Panovko, Ya. G., "Calculation of the External and Internal Inelastic Resisting Forces in Problems of the Applied Theory of Elastic Vibration," (in Russian) Prob's of Dynamics and Dynamic Strength No. 1, Riga, Akas, Nauk. Latv. SSR, 1953.

32. Bogaliubov and Mitropolsky, "Asymptotic Methods in the Theory of Nonlinear Oscillation," A Translation from Russian by Hindustan Publishing Corp. (India) 1961.

33. Krylov and Bogaliubov, "Introduction to Nonlinear Mechanics," Translation by Lefschetz, Princeton Press, 1947.

34. Prescott, J., "Vibration of Thin Rod," Phil. Mag. 33, 1942.

35. Timoshenko and Young, "Vibration Problems in Engineering," D. Van Nostrand 3rd. Ed.

36. Stoker, J. J., "Nonlinear Vibrations," Interscience Publisher, 1950.

37. Malkin, "Theory of Stability of Motion," Translated from Russian by AEC, AEC-tr-3352.

38. Cesari, L., "Asymptotic Behavior and Stability Problems in Ordinary Differential Equations," Academic Press, New York, 1963.

39. Clebsch, A., "Theorie der Elasticität der fester Körper," Leipzig, 1862.

40. Todhunter, J. and Pearson, K., "A History of the Theory of Elasticity," Vol. II Part I and II, University Press, Cambridge, England.

41. Poincare, H., (a) "Les Méthodes nouvelles de la mécanique céleste T. 3 Dover, 1957. (b) J. des Mathe (3) 7, 1881 also Ceuvres T. 1, Gauthier-Villars, Paris 1928.

42. Lord Rayleigh, "The Theory of Sound," Dover, 2 Vol's.

43. Love, A. E. H., "Mathematical Theory of Elasticity," Dover Ed.

44. Myklestad, "The Concept of Complex Damping," JAM Vol. 19, p. 284, 1952.

45. Moulton, F. R., "Differential Equations," Dover 1958.

46. Sethna, P. R., "Steady State Undamped Vibrations of a Class of Nonlinear Discrete Systems," JAM 27 March 1960.

47. Chow, S. T. and Sethna, P. R., "Steady State Damped Vibrations of a Class of Nonlinear Discrete Systems," WADC Tech. Rep. 59-543, 1960.

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* University of Minnesota, Dept. of Aeronautics and Engineering Mechanics, Minneapolis, Minnesota 55455	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	2b. GROUP

3. REPORT TITLE

NONLINEAR EFFECTS OF LARGE DEFLECTIONS AND MATERIAL DAMPING ON THE STEADY STATE VIBRATIONS OF BEAMS

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Summary Report - April 1964 to April 1965

5. AUTHOR(S) *(Last name, first name, initial)*

Chow, Shuh-Twu and Sethna, P. R.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
July 1965	90	47

8a. CONTRACT OR GRANT NO. AF 33(615)-1055	9a. ORIGINATOR'S REPORT NUMBER(S) AFML-TR-65-167
b. PROJECT NO. 7351	
c. TASK NO. 735106	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	

10. AVAILABILITY/LIMITATION NOTICES Qualified users may obtain copies of this report from the Defense Documentation Center, Alexandria, Va. The distribution of this report is limited because it contains technology identifiable with items on the Mutual Defense Assistance Control List excluded from export under U. S. Export Control Act of 1949, as implemented by AFR 400-10.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Metals and Ceramics Division (MAMD), Air Force Materials Laboratory, Wright-Patterson AFB, Ohio 45433

13. ABSTRACT

Nonlinear forced oscillations of slender beams are studied. The analysis takes into account both the nonlinear effects arising from large deflections of the beam and those arising from nonlinear material behavior. A hysteretic stress-strain law of the Davidenkov type is used in the analysis. Detailed results are given for large amplitude oscillations of beams with hinged ends. Theoretical results for a simply-supported beam are compared with experimental results.

DD FORM 1 JAN 64 1473

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through
__________________________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through
__________________________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through
__________________________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.