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ANALYSIS OF TENSION STRUCTURES

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A method of analysis applicable to the prediction of the response of tension structures is presented. The total potential energy of an assembly of truss-string members is employed as the mathematical model. Solutions to particular problems are generated by direct minimization of the potential energy with respect to the displacement degrees of freedom which are not restricted by boundary conditions. The energy search approach provides a natural means to accommodate a change of structural configuration due to slackening of tension members. The governing equations are based on the deformed geometry of the structure; this permits the prediction of large nodal displacements and post-buckled configurations. An example is presented to demonstrate the effectiveness of the analysis.

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SECTION I
INTRODUCTION

The tension structure has been recognized as an efficient and practical configuration for achieving various structural and architectural objectives. The primary goal in the design of a tension structure is to select and arrange the structural components so that loads are carried primarily in tension and the number of compression members is held to the minimum necessary to maintain stability. A design achieved by extensive use of high strength tension members with relatively small cross sections can offer advantages over a more conventional one. The potential for significant weight reduction is probably the most important, however, since, more often than not, weight is a limiting or controlling factor when a tension structure design is an alternate candidate to perform a specific function.

A common example of the use of tension structures is in the design of roof systems to span the large areas enclosed by modern structures such as stadiums and shopping centers (Reference 1). Networks of slender cables provide the strength necessary to support the tremendous loads involved and at the same time afford the architect the opportunity for imaginative creation. Space activities provide further applications for the tension structure concept since weight is a principal factor in determining feasibility; for example, it is intuitively clear that a space antenna with diameter of the order of 2000 feet must be designed with weight in mind. The concept of inflatable compression members (Reference 2) gives the tension structure a further advantage; for instance, a space antenna can be ''Folded'' into a relatively small, lightweight bundle in the deflated state, transported into space, and then erected simply by inflating the compression members. Potential use for the tension structure also exists underwater where tension forces can be supplied by buoyancy.

The fact that tension structures have a wide range of application implies a need for a dependable analysis capability. To date, most attempts to analyze the response of general tension structures using conventional truss formulations or various specialized analytical and numerical techniques have not met with a great deal of success. Two characteristics common to tension structures are responsible for most of the difficulties:

1. Geometric Nonlinearity - The response of many tension structures is inherently nonlinear, (e.g., planar networks subjected to normal loading).

2. Change of Configuration - Tension members, which exhibit no stiffness to compression, may, under certain loading conditions, "go slack."

The problem of geometric nonlinearity has been treated, with varying degrees of success, by several authors. In Reference 3, nonlinearity is taken into account by applying the load incrementally. The response at each stage of the loading is first computed based on linearizing assumptions and then corrected subsequently by iteration. The energy search approach developed in Reference 4 is extremely well suited to the nonlinear analysis of space trusses as evidenced by the comprehensive stability problems treated by this method in Reference 5. This method is modified in the present work for application to more general tension structures.

The second characteristic of tension structures mentioned above presents a more fundamental problem than do large displacements. The configuration of a tension structure under load may be different from the unloaded state due to tension members dropping out of service as the structure deforms. A solution algorithm, if it is to cope with this situation, must have the capability not only to detect when a tension member goes slack but also to remove mathematically the relaxed member from the structure and to return it to service when the occasion demands. To the author's knowledge, the literature contains no procedure which can, for a general case, predict with certainty the variations in the configuration pattern of a tension structure during loading as well as the final displacement and stress response.

A common assumption in the analysis of tension structures is that the structure is initially prestressed sufficiently to preclude any change in configuration under application of loads (References 3, 6, 7, and 8) Algorithms based on this assumption are limited to rather special cases and, in fact, can produce erroneous results since no means is provided to determine whether the selected prestress is adequate. Another approach is to employ a conventional truss analysis, disregarding the fact that some members have no stiffness to compression. The resulting member forces are examined to determine whether the force in any of the string members is negative - if so, these members are removed, the new structure is then analyzed. Hopefully, the process will result in a stable, fully-stressed configuration; there is no guarantee of this, however.

It is demonstrated below that the conceptual and analytical difficulties associated with variable structural configuration in tension structures is easily resolved by direct minimization of the total potential energy. The formulation presented is essentially the same as that developed by the second author of Reference 4 except for three refinements; (1) the addition of prestress by either temperature change or initial stress (2) the assumed displacement state of a truss member is assumed to be proportional to the first buckling eigenmode instead of a polynominal, and (3) the capability of handling change of configuration is provided.

SECTION II
FORMULATION

TRUSS-STRING MEMBER

A typical truss-string member in both the undeformed and the deformed states is shown in Figure 1. The initial position of the element is described by the coordinates of the nodes,

$$\vec{L} = \vec{r}^{(2)} - \vec{r}^{(1)}$$

and the initial length is

$$L = |\vec{L}| = \left\{ \sum_{i=1}^{3} \left[x_i^{(2)} - x_i^{(1)} \right]^2 \right\}^{1/2} \qquad (1)$$

Under loading the element nodes undergo displacements $u^{(j)}$ measured with respect to the reference coordinate system. The distance between the nodes in the deformed position is defined by

$$\vec{S} = \vec{r}^{(2)} + \vec{u}^{(2)} - \vec{r}^{(1)} - \vec{u}^{(1)}$$

and the corresponding length is

$$S = \left|\vec{S}\right| = \left\{ \sum_{i=1}^{3} \left[(x_i + u_i)^{(2)} - (x_i + u_i)^{(1)} \right]^2 \right\}^{1/2} \tag{2}$$

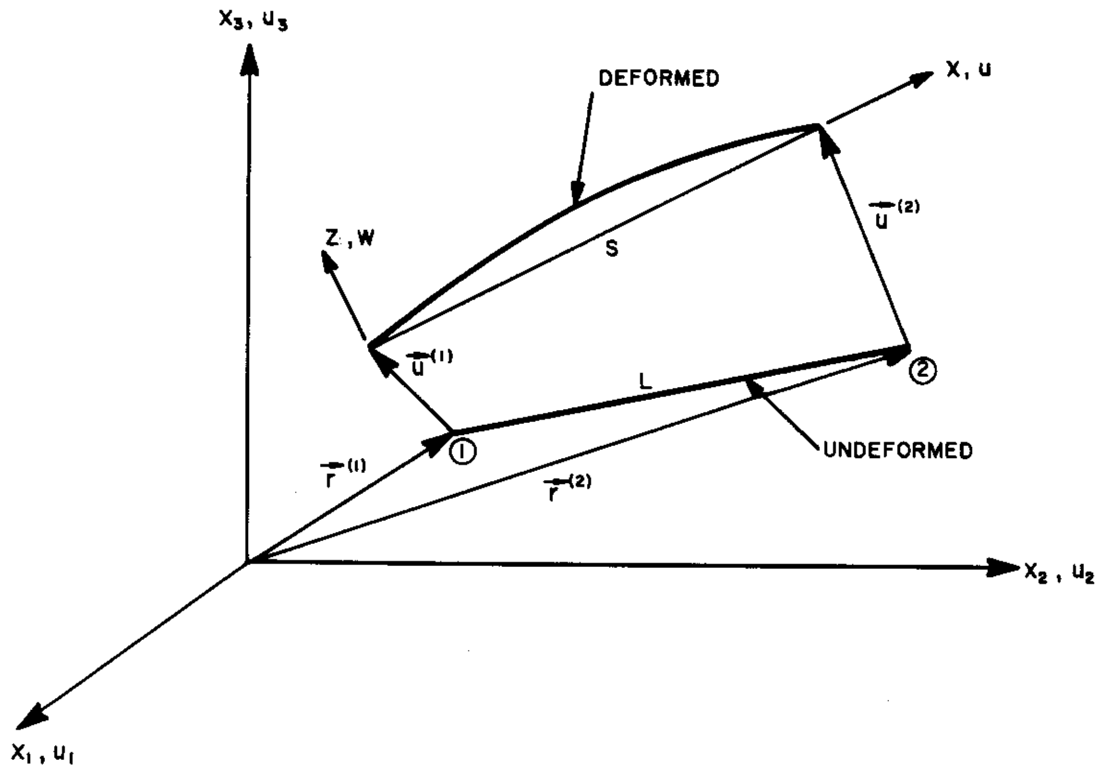

Figure 1. Truss-String Member

STRAIN DISPLACEMENT

The strain of an individual truss-string member is expressed in terms of displacements, (u,w), measured in a local coordinate system, (x,z), with origin at one of the endpoints, (Figure 1). The x-axis is defined by the line between the displaced positions of the nodes of the member; the z-axis is normal to the x-axis and lies in the plane of bending. The transverse displacement, w, is not considered a primary deformation as in the case of a beam (loading is assumed to be applied only at the nodes of a truss structure), but is included to describe buckling of a truss member. The strain-displacement relation for a typical truss-string member is written in the form

$$\epsilon = u_x + \frac{1}{2} w_x^2 - \zeta w_{xx} \tag{3}$$

where ζ is measured from the neutral axis and the subscript x denotes differentiation.

The first term on the righthand side of Equation 3 describes the primary deformation which occurs as the nodes of the element displace relative to one another. The remaining terms, containing the transverse displacement, w, represent the effect of secondary deformation induced by buckling of the member (since string members have no stiffness to compression, it is understood that these terms are included only for truss members).

ELEMENT STRAIN ENERGY

The strain energy of a prestressed truss-string member is

$$U = \frac{1}{2} \int_V (\epsilon + \epsilon_p) \sigma \, dV$$

where ϵ is the strain of deformation (Equation 3), ϵ_p is the strain due to prestress, σ is the stress in the element, and V is the volume. Prestressing can be effected in various ways; for example, a temperature increment: $\epsilon_p = -\alpha \Delta T$, or an initial tension: $\epsilon_p = T_p/AE$. The stress induced in an element by the combined effects of deformation and prestress is

$$\sigma = E(\epsilon + \epsilon_p)$$

The element strain energy is then

$$U = \frac{E}{2} \int_V (\epsilon + \epsilon_p)^2 \, dV \tag{4}$$

Upon substitution of Equation 3 into Equation 4 and integration over the cross section, the strain energy is given in terms of the local displacements of the element

$$U = \frac{AE}{2} \int_0^S \left\{ (u_x + \frac{1}{2} w_x^2 + \epsilon_p)^2 + \frac{I}{A} w_{xx}^2 \right\} dx \tag{5}$$

where S is the distance between the element nodes in the deformed state, (Equation 2).

DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

The following expression is obtained after taking the first variation of Equation 5 and integrating by parts

$$\delta U = 0 = -AE\int_0^S \left\{ \frac{d}{dx}\left(u_x + \frac{1}{2} w_x^2 + \epsilon_p \right) \right\} \delta u \; dx \qquad (a)$$

$$+AE\int_0^S \left\{ \frac{I}{A} w_{xxxx} - \frac{d}{dx}\left[\left(u_x + \frac{1}{2} w_x^2 + \epsilon_p \right) w_x \right] \right\} \delta w \; dx \qquad (b)$$

$$+AE \left\{ u_x + \frac{1}{2} w_x^2 + \epsilon_p \right\} \delta u \Big|_0^S \qquad (c) \quad (6)$$

$$-AE \left\{ \frac{I}{A} w_{xxx} - \left(u_x + \frac{1}{2} w_x^2 + \epsilon_p \right) w_x \right\} \delta w \Big|_0^S \qquad (d)$$

$$+EI \left\{ w_{xx} \right\} \delta w_x \Big|_0^S \qquad (e)$$

Each of the contributions (a through e) in Equation 6 must individually be zero in order that the total variation of the strain energy functional be zero.

DIFFERENTIAL EQUATIONS

It follows from the fundamental lemma of the calculus of variations that the expressions in the braces of terms (a) and (b) of Equation 6 must vanish,

$$\frac{d}{dx}\left(u_x + \frac{1}{2} w_x^2 + \epsilon_p \right) = 0 , \qquad (7)$$

$$\frac{I}{A} w_{xxxx} - \frac{d}{dx}\left[\left(u_x + \frac{1}{2} w_x^2 + \epsilon_p \right) w_x \right] = 0. \qquad (8)$$

Integrating Equation 7,

$$u_x + \frac{1}{2} w_x^2 + \epsilon_p = K . \qquad (9)$$

Since K is constant with respect to x, Equation 8 can be rewritten in the form

$$w_{xxxx} - \frac{AK}{I} w_{xx} = 0. \qquad (10)$$

NATURAL BOUNDARY CONDITIONS

It is assumed that the truss-string member has pinned ends; therefore, the variational quantity δw_x is arbitrary at the ends of the member and the term in braces of Equation 6(e) is zero. The natural boundary conditions are then

$$w_{xx}\Big|_{x=0} = 0, \qquad w_{xx}\Big|_{x=S} = 0. \tag{11}$$

IMPOSED BOUNDARY CONDITIONS

It is apparent from Figure 1 that the imposed boundary conditions are

$$u\Big|_{x=0} = 0, \qquad u\Big|_{x=S} = S - L, \tag{12}$$

$$w\Big|_{x=0} = 0, \qquad w\Big|_{x=S} = 0. \tag{13}$$

The variational quantity δu must vanish at the ends of the member; therefore, Equation 6(c) and Equation 9, indicate that the force in the member is a constant given by

$$F = AEK \tag{14}$$

DISPLACEMENT MODES

Since the member can have transverse displacement only for $K < 0$ (external loading is assumed to be applied at the nodes of a structure), the general solution to Equation 10 is

$$w = B_1 + B_2 x + B_3 \sin\sqrt{-\frac{AK}{I}}\,x + B_4 \cos\sqrt{-\frac{AK}{I}}\,x \tag{15}$$

subject to the boundary conditions given in Equations 11 and 13. An attempt to solve for the four constants in Equation 15 leads to the following relations,

$$B_1 = B_2 = B_4 = 0,$$

$$B_3 \sin\sqrt{-\frac{AK}{I}}\,S = 0.$$

If the trivial solution is disregarded, then K must be treated as an eigenvalue

$$K_{cr} = -\left(\frac{k\pi}{S}\right)^2 \frac{I}{A}, \qquad k = 1, 2, \cdots \tag{16}$$

The critical force (buckling load) in the element is (Equation 14),

$$F_{cr} = -\left(\frac{k\pi}{S}\right)^2 EI, \quad k = 1, 2, \cdots \tag{17}$$

The eigenmodes corresponding to the critical values of K are obtained upon substitution of Equation 16 into Equation 15

$$w_k = B_k \sin \frac{k\pi x}{S}, \quad k = 1, 2, \cdots$$

The assumed transverse displacement of a typical truss member is taken to be proportional to the first buckling eigenmode,

$$w = C \sin \frac{\pi x}{S}. \tag{18}$$

The constant C (midspan displacement) is treated as a generalized coordinate for a truss member which is permitted to buckle; C is taken to be identically zero otherwise.

The constant K is determined by integrating Equation 9 over the length,

$$\int_0^S K\, dx = \int_0^S (u_x + \frac{1}{2} w_x^2 + \epsilon_p)\, dx$$

or

$$K = 1 - \frac{L}{S} + \epsilon_p + \left(\frac{\pi C}{2S}\right)^2. \tag{19}$$

Equation 19 incorporates the imposed boundary conditions given by Equation 12. This determination of K insures that axial equilibrium is satisfied on the average over the entire span of the member. Note that K is indirectly a nonlinear function of the nodal displacements, measured with respect to a reference coordinate system, through S (Equation 2).

STRAIN ENERGY

The strain energy for a truss-string member is obtained in terms of the nodal displacements and the buckling amplitudes upon substituting Equations 9, 19 and 18 into Equation 5 and performing the indicated integration

$$U = \frac{AE}{2}\left[SK^2 + \frac{1}{2}\frac{I}{A}\frac{\pi^4 C^2}{S^3}\right] \tag{20}$$

where S is given by Equation 2 and K by Equation 19.

TOTAL POTENTIAL ENERGY

The total potential energy of an assembly of N truss-string members is

$$\Pi_p = \sum_{i=1}^{N} U^{(i)} - W \qquad (21)$$

where $U^{(i)}$ is the strain energy of the i^{th} member (Equation 20) and W is the external work done by forces applied to the nodes of the structure.

SECTION III
ENERGY SEARCH

The mathematical model selected for the analysis of general truss and tension structures is the total potential energy (Equation 21) for an assembly of one-dimensional, truss-string members. In general, the potential energy is a function of n — undetermined displacement parameters:

1. The nodal displacements of the structure, measured with respect to a reference coordinate system, which are not determined by imposed boundary conditions,

2. The constants describing the local transverse displacement state of the truss members in the structure which are permitted to buckle (Equation 18).

Each of these independent degrees of freedom is assigned a distinct number from 1 to n and a corresponding position in an n-component vector $\vec{q}$. The potential energy is then written as a function of $\vec{q}$,

$$\Pi_p = \Pi_p(\vec{q})$$

The n-components of the independent degree-of-freedom vector together with the potential energy itself are viewed as coordinates in an (n+1)-dimensional space. The potential energy expression defines a hypersurface in this space. Mathematical programming methods can be used to search the space directly to locate points corresponding to local minima of the energy. These points represent displacement states which satisfy the equilibrium equatiotions. The particular search technique used to generate the solutions to the examples presented below is given in Reference 9.

The difficulties associated with the numerical prediction of the response of structures which may change configuration, (slackening of tension members), under load are easily resolved with the energy search approach. Part of the potential energy calculation for an assembly of truss-string members involves summing the strain energy contributions from the individual members which comprise the structure, (Equation 21). If, during the search process, a point (displacement vector) in the energy space is generated which causes the force in a string member to be negative, then the contribution of that member to the total potential energy is simply not included in the summation. This has the effect of mathematically removing slack members from the structure. Thus, string members are automatically removed from the structure and brought back into service as the situation requires. In this manner the variable configurations of a loaded tension structure can be predicted as well as the displacement and stress response.

SECTION IV
EXAMPLE

A structure resembling a guyed roof system is shown in Figure 2. Although this is a relatively simple model geometrically, the response includes the rather complex characteristics of snap-through instability phenomena coupled with typical tension structure behavior. The simplicity of the structural model permits the visualization and qualitative appraisal of the results generated by the analysis.

The structure is initially prestressed by prescribing the length of the guy wires to be 0.2 in. shorter than the distance from the top of the columns to the anchor points shown in the figure. The resulting forces in the members in this initial state are

$$F(\text{struts}) = 0$$
$$F(\text{columns}) = -2434.9 \text{ lb.}$$
$$F(\text{ties}) = 608.8 \text{ lb.}$$
$$F(\text{guys}) = 1362.5 \text{ lb.}$$

and the displacements of nodes 1 and 2 are,

$$w_1 = .1790 \text{ in.}$$
$$u_2 = -.0292 \text{ in.}$$
$$v_2 = -.0292 \text{ in.}$$
$$w_2 = .0029 \text{ in.}$$

The displacements of the remaining nodes are determined from the symmetry of the structure. The effect of prescribing short guy wires is to: a) induce prestress forces in the columns, ties and guys, b) displace the tops of the columns away from the center of the structure, and c) displace node 1 downward without inducing forces in the struts.

The load-displacement history for the apex (node 1) of the structure is shown in Figure 3. The portions of the curve with negative slope correspond to unstable equilibrium positions which occur during snap-through. The points at which the dash-dot curve deviate from the solid curve are points of bifurcation with the force in the struts exactly equal to the buckling load. If the struts are not permitted to buckle, the behavior of the structure is represented by the solid branch with snap-through occurring along the dashed curve. When the struts are

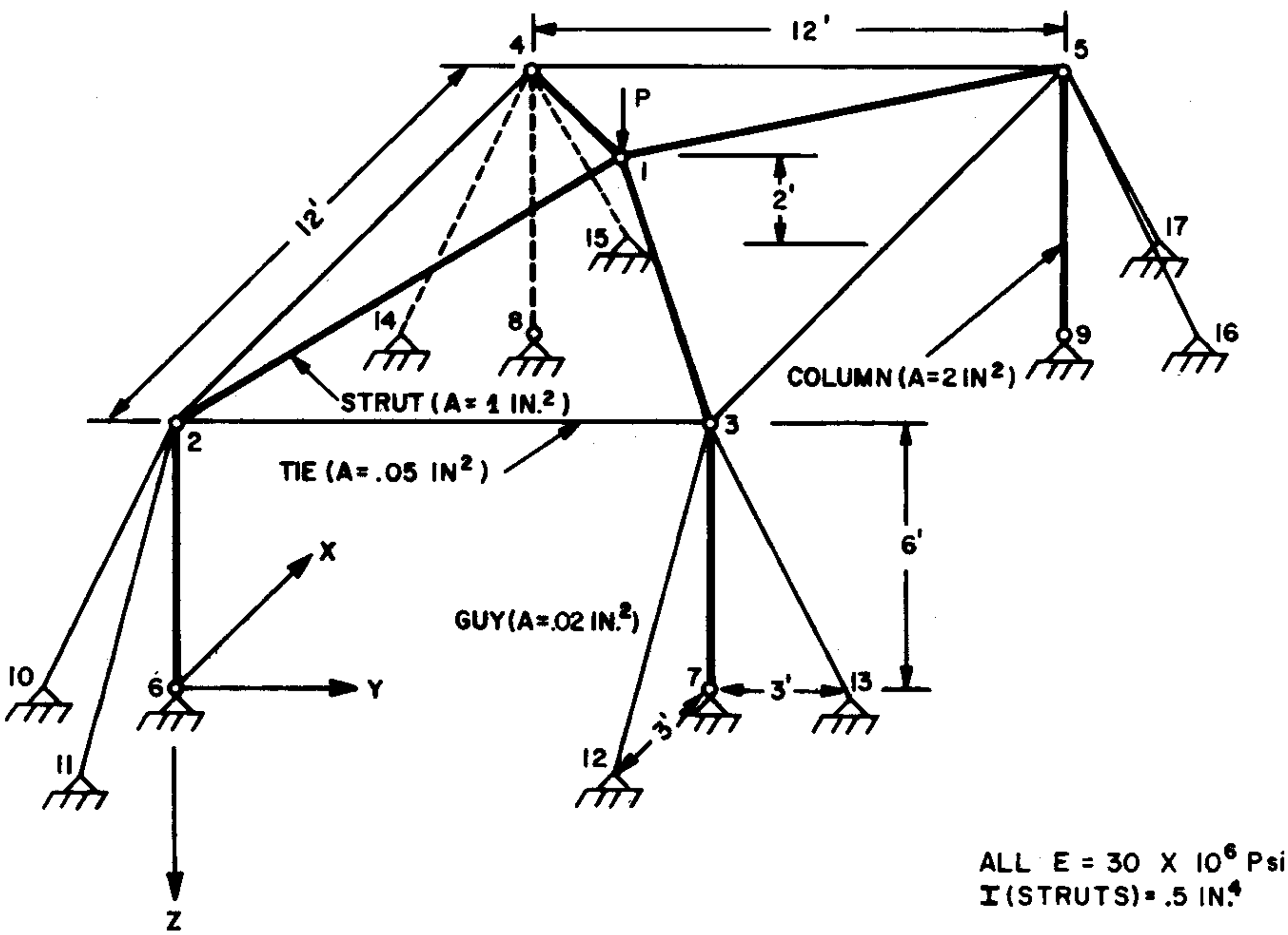

Figure 2. Guyed Roof System

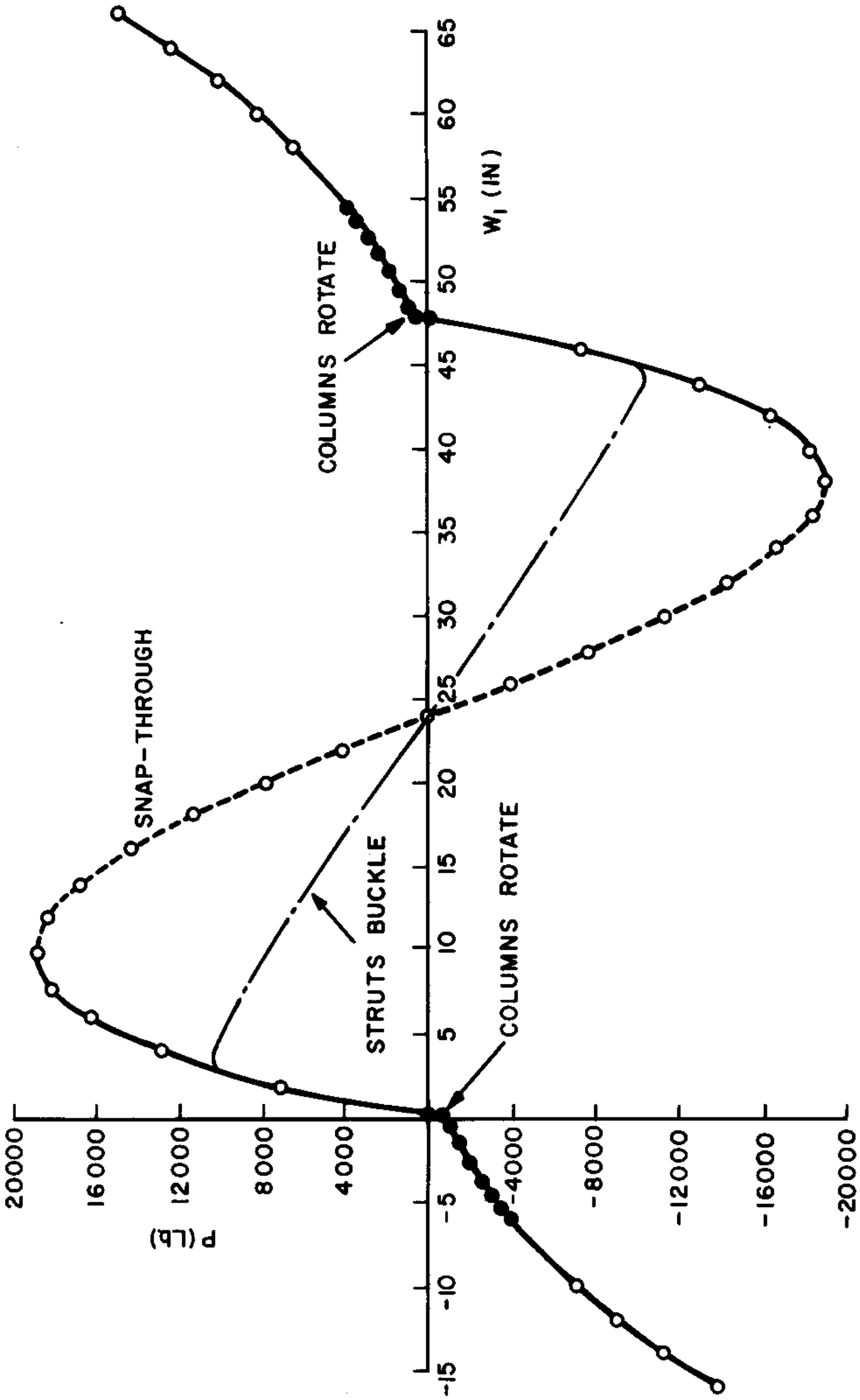

Figure 3. Load - Displacement History

allowed to buckle, general instability results at a considerably reduced load level as indicated by the dash-dot curve.

The sharp bends in the curve in the vicinity of $w_1 = 0$ and $w_1 = 48$ in. indicate the rapid change in stiffness of the structure which takes place as the columns rotate from one side of the vertical to another. Points on the curve to the left of the bend near $w_1 = 0$ and to the right of the bend near $w_1 = 48$ in. correspond to stable equilibrium configurations in which the columns are rotated toward the center of the structure and the horizontal tie wires between the columns are slack. The guy wires are slack between $w_1 \approx 4$ in. and $w_1 \approx 44$ in. The curve does not pass through the origin due to the initial prestressed state of the structure.

The force histories of the guys and ties are presented in Figure 4. It is interesting that all the members of the structure participate simultaneously only in a small portion of the load history, for instance, either the ties or the guys are slack most of the time. It is apparent from this figure why the slackening of the ties affects the smoothness of the load-deflection curve but the slackening of the guys does not: the ties go slack abruptly but the slackening of the guys is a smooth transition.

The load-displacement history shown in Figure 4 was generated by imposing both values of the load and the vertical displacement of the apex. Points obtained from imposed load conditions are represented by solid dots while those generated from prescribed displacement conditions are denoted by circles. The entire dash-dot curve was obtained by imposing the displacement. Either method can be used to generate points on the parts of the plot with positive slope; however, the displacement must be prescribed on the portions with negative slope since the potential energy has no minimum corresponding to these points when the load is imposed.

The symmetry of the structure and loading was not taken into consideration when the input data for the computer program was prepared; therefore, the potential energy of the structure when the struts were not permitted to buckle was a function of 15 degrees of freedom for the imposed load cases and 14 degrees of freedom for the prescribed displacement cases. The search procedure seemed to sense the symmetry of the problem, however, and the majority of the cases converged in three to five moves in the energy space. The total run-time for 55 prescribed load and displacement conditions was approximately three minutes on a GE 635 computer.

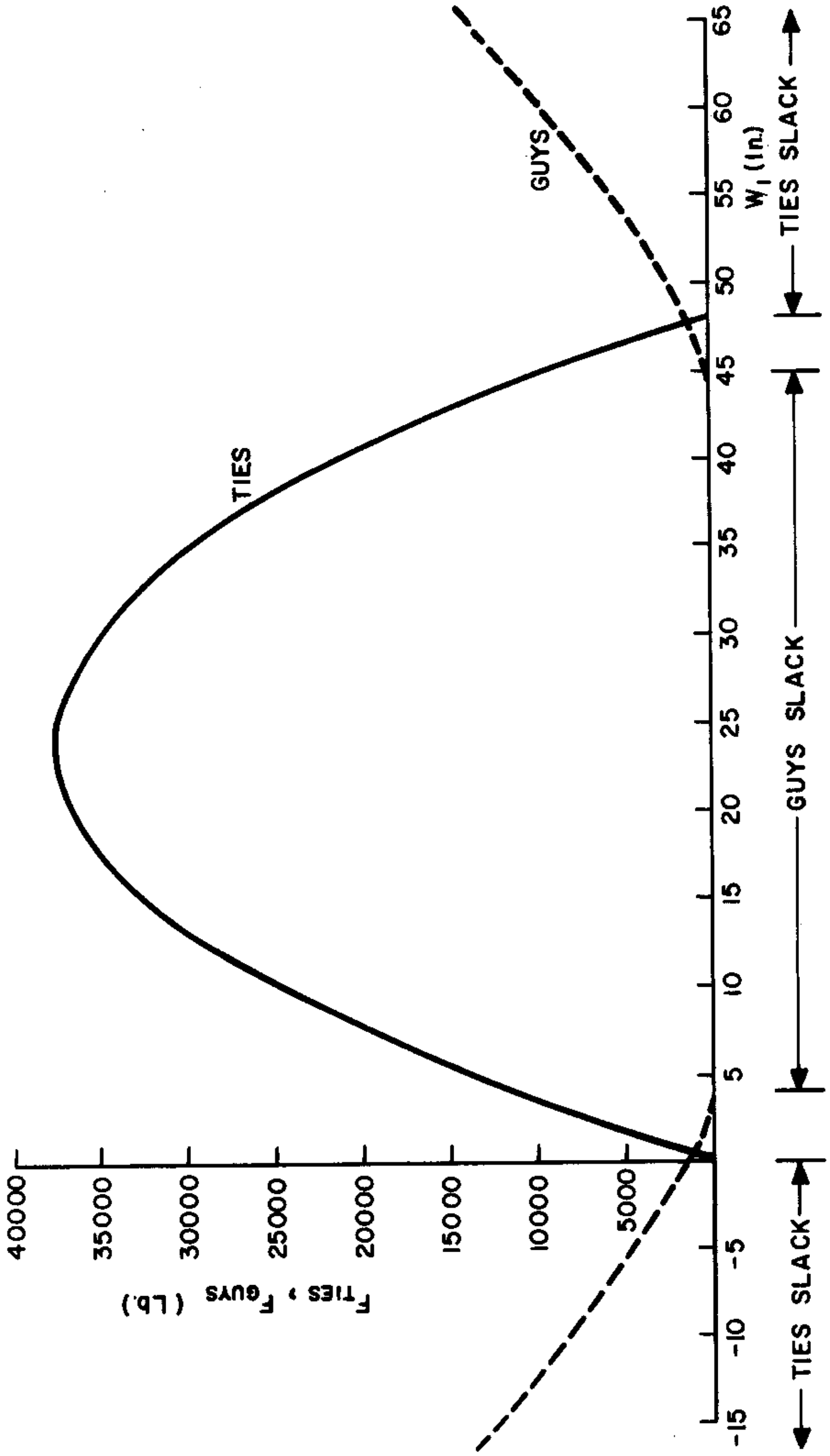

Figure 4. Force Histories of the Ties and Guys

For the cases in which the struts were permitted to buckle the potential energy was a function of 18 degrees of freedom (14 nodal displacements since the apex displacement was imposed plus four midspan amplitudes of the struts). The total run-time required to generate the dash-dot curve in Figure 3 (26 imposed displacement conditions) was approximately three minutes.

SECTION V
SUMMARY

An analysis method applicable to the prediction of the response of tension structures has been formulated. The mathematical approach employs the total potential energy of an assembly of truss-string members. Solutions to particular problems are generated by direct minimization of the potential energy with respect to the midspan amplitudes of the truss members which are permitted to buckle and the nodal displacement degrees of freedom which are not restricted by boundary conditions.

The governing equations are based on the deformed geometry of the structure. This permits the prediction of large nodal displacements and post-buckled configurations resulting from gross instability (instability of the structure as a whole). Provision for buckling of individual members is included so that gross instability resulting from the accumulation of local buckling effects can be detected (local buckling occurs when the force in a truss member exceeds a critical value).

The conceptual difficulties associated with tension members dropping out of service are easily resolved using the energy search approach. The calculation of the total potential energy of a tension structure involves summing the individual contributions from the elements which comprise the system. This provides a natural means to accommodate an altered structural configuration at any point in the search process by simply not including the contribution of slack members in the summation. Therefore, the point in the energy space corresponding to the minimum value of the potential energy of a structure provides a description of the final structural configuration as well as the displacement and force state of the structure.

The example treated demonstrates that the analysis capability presented can be a useful tool for predicting the response of complex tension structures.

SECTION VI

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