Contrails

LINEAR AND NONLINEAR ANALYSIS OF SHELLS OF REVOLUTION
WITH ASYMMETRICAL STIFFNESS PROPERTIES

James A. Stricklin* and Jose C. DeAndrade**
Texas A&M University
College Station, Texas

Frederick J. Stebbins*** and Anthony J. Cwiertny, Jr.***
NASA Manned Spacecraft Center
Houston, Texas

Considered in this paper is an extension of the direct stiffness method of structural analysis to the linear and nonlinear analysis of shells of revolution under arbitrary static loading with variable thickness properties in the circumferential direction as well as the meridional direction. The thickness variation in the circumferential direction yields a 8N x 8N element stiffness matrix where N is the number of harmonics. The resulting computer program is used to conduct a linear, nonlinear, and stability analysis of the Apollo aft heat shield.

*Professor, Department of Aerospace Engineering
**Research Assistant, Department of Aerospace Engineering
***Engineer, Structures and Mechanics Division

NOMENCLATURE

A = 8 x 8 matrix relating coefficients to nodal degrees of freedom

C_1 = $E_s t/(1- \nu_{s\theta} \nu_{\theta s})$

C_2 = $E_\theta t/(1- \nu_{s\theta} \nu_{\theta s})$

D_1 = $E_s t^3/ 12(1- \nu_{s\theta} \nu_{\theta s})$

D_2 = $E_\theta t^3/ 12(1- \nu_{s\theta} \nu_{\theta s})$

E = Young's modulus

G = Shear modulus

G_1 = Gt

G_2 = $Gt^3/12$

K = Shell stiffness matrix

k^{mn} = 8 x 8 stiffness matrix relating coupling between m and n harmonics

L^{mn} = 8 x 8 matrix relating the coupling between harmonics m and n

m, n = harmonic numbers

Q = generalized nodal forces

q = nodal displacements and rotations

r = cylindrical coordinate

t = shell thickness

U = internal energy

$u, v, w,$ = displacements in meridional, circumferential and normal directions

z = distance normal to shell midsurface

α = coefficients in expansion for displacements

ϵ = midsurface strains

θ = circumferential angle

χ = changes in curvature

ν = Poisson's ratio

SECTION I

INTRODUCTION

The matrix displacement method of structural analysis has been successfully applied to the linear elastic analysis, (References 1 through 7) linear dynamic analysis, (References 8 and 9) plastic analysis (Reference 10), and the nonlinear static analysis (Reference 11) of shells of revolution with constant thickness in the circumferential direction. Jones and Strome (Reference 12) present a survey of the research in this area prior to 1965. The objective of the present paper is to extend the matrix displacement method to the analysis of shells of revolution with variable stiffness properties in the circumferential direction. The primary difference between this research and previously published results is that all the Fourier harmonics in the circumferential direction are now coupled. This yields an element stiffness matrix of order 8N x 8N where N is the number of harmonics.

The research presented here uses the displacement function of Grafton and Strome (Reference 3) and the curved element of Stricklin, Navaratna and Pian (Reference 6). The displacement function does not explicitly include the rigid body motion of the element but, as first pointed out in Reference 6, this is not necessary. Further, an eigenvalue analysis (Reference 13) and experience (References 6, 7, and 11) with this curved element indicate that rigid body motion is adequately represented. However, for other curved elements the authors agree with the conclusion reached by Schmit, Bogner, and Fox (Reference 14) that higher order displacement functions should be used. There is now an abundance of evidence which shows that the higher order displacement functions are well worth the additional computational effort required in obtaining the element stiffness matrix. They not only better represent rigid body motion but, stated simply, converge much faster than the elements based on lower order displacement functions. The method of Pian (Reference 15) may be used to incorporate these concepts into the shell of revolution computer program without changing the size of the element stiffness matrix. However, it is the authors' opinion that they are not needed for the shell of revolution.

Turner et al (Reference 16) were the first to include large deflections in the matrix displacement method of structural analysis. In their formulation they proposed that the nonlinear problem be solved as a sequence of linear problems with the element stiffness matrix being reevaluated at each loading. A simular procedure has been used by many investigators since the publication of the original paper. Martin (Reference 17) presents a discussion and numerous references on the subject.

1233

It has been pointed out by Marcal (Reference 18) that the use of an initial stress matrix does not yield the correct nonlinear equations of equilibrium. This is due to the fact that in the variation of the strain energy to obtain the equations of equilibrium a variation of the initial stress should be included. In Reference 18 it is stated that this omission leads to a considerable overestimation of the buckling loads.

Another approach to the solution of the nonlinear problem is given by Schmit, Bogner and Fox (Reference 14). They solved the nonlinear problem by seeking the minimum of the total potential energy, and as such, do not encounter the approximations inherent with the use of the initial stress matrix. Still another approach is given by Oden, (Reference 19) but to the authors' knowledge it is not currently being used in the solution of nonlinear problems.

The approach used here for the formulation and solution of the nonlinear equations of equilibrium is the same as presented in Reference 11. The equations are formulated by applying Castigliano's theorem to the expression for the strain energy in terms of the nodal displacement. This formulation yields the correct nonlinear equations of equilibrium. The resulting equations are solved using increments of loading combined with iteration at each loading.

The research presented here was developed concurrently with that presented in Reference 11. However, to prevent duplication only the material which differs from that presented in Reference 11 is presented in the following Sections.

SECTION II

FORMULATION

The shell of revolution is idealized as an assemblage of curved elements connected at their nodal circles or nodes. The element curvature in the meridional direction is described by the slope between the axis of revolution and a tangent to the shell element in the meridional direction. The slope is represented by a second order polynomial function of the meridional distance along the element. The assumed shell element is required to have the same slopes at the nodes as the actual shell and the length of the element, along the meridian, is obtained by assuming an arc of a circle passing through the element nodes at the specified slopes.

The displacements of the shell element are represented by three components in the normal, meridional, and circumferential directions. These displacement components are expressed by polynomials in the meridional direction, s and a Fourier series in the circumferential angle, θ, with both cosine and sine terms being retained. For the displacement component in the normal direction a third order polynomial in the meridional distance is used and linear expressions are used for the meridional and circumferential components. These displacements are

$$
\begin{aligned}
w = & \sum_{i=0}^{IA} (\alpha_1^i + \alpha_2^i s + \alpha_3^i s^2 + \alpha_4^i s^3) \cos i\theta \\
& + \sum_{i=0}^{IB} (\alpha_1^{-i} + \alpha_2^{-i} s + \alpha_3^{-i} s^2 + \alpha_4^{-i} s^3) \sin i\theta \\
u = & \sum_{i=0}^{IA} (\alpha_5^i + \alpha_6^i s) \cos i\theta + \sum_{i=0}^{IB} (\alpha_5^{-i} + \alpha_6^{-i} s) \sin i\theta \\
v = & \sum_{i=0}^{IA} (\alpha_7^i + \alpha_8^i s) \sin i\theta + \sum_{i=0}^{IB} (\alpha_7^{-i} + \alpha_8^{-i} s) \cos i\theta
\end{aligned}
\tag{1}
$$

where

w, u and v = the displacement in the normal, meridional, and circumferential directions respectively.

s = meridional distance along the element

θ = circumferential angle

α^i, α^{-i} = generalized displacement coefficients

IA + 1 and IB + 1 are the number of A and B harmonics respectively.

For any A or B harmonic the 8 α coefficients are related to the nodal displacements and rotations by the expression

$$\boldsymbol{\alpha} = \mathbf{A}\,\mathbf{q} \tag{2}$$

where the generalized degrees of freedom for the element correspond to the radial, circumferential, and axial displacements and a rotation at each node. The matrix $\mathbf{A}$ is given in Reference 11.

The strain-displacements are those given by Novozhilov (Reference 20) with the assumption that only those nonlinearities due to moderate rotations about the shell coordinate axes are important.

$$\epsilon_s = \hat{e}_s + \frac{1}{2}\hat{e}_{13}^2$$

$$\epsilon_\theta = \hat{e}_\theta + \frac{1}{2}\hat{e}_{23}^2 \tag{3}$$

$$\epsilon_{s\theta} = \hat{e}_{s\theta} + \frac{1}{2}\hat{e}_{13}\,\hat{e}_{23}$$

The expressions for the strains $\hat{e}_s$, $\hat{e}_\theta$, and $\hat{e}_{s\theta}$ and rotations $\hat{e}_{13}$, $\hat{e}_{23}$ are also given in Reference 11.

The internal energy expression is given in Ambartsumian (Reference 21) and is valid for orthotropic shells

$$U = \frac{1}{2}\iint(C_1\,\epsilon_s^2 + C_2\,\epsilon_\theta^2 + 2\nu_{s\theta}C_1\epsilon_s\epsilon_\theta$$

$$+ G_1\,\epsilon_{s\theta}^2 + D_1\,X_s^2 + D_2\,X_\theta^2 \tag{4}$$

$$+ 2\nu_{s\theta}D_1 X_s X_\theta + G_2 X_{s\theta}^2)\,r\,ds\,d\theta$$

Substituting the expressions for the midsurface strains, the internal energy may be separated into two parts

$$U = U_L + U_{NL} \tag{5}$$

where U_L is the usual expression for the internal energy based on linear theory and U_{NL} contains all the dependence of U on $\hat{e}_{13}$ and $\hat{e}_{23}$.

Using the expression for U_L the element stiffness matrix may be evaluated as described in the next Section III. Assembling the element stiffness matrices the equations of equilibrium may be written as

$$\mathbf{K\,q} = \mathbf{Q} - \frac{\partial U_{NL}}{\partial \mathbf{q}} \qquad (6)$$

where $\mathbf{K}$ = shell stiffness matrix

$\mathbf{Q}$ = generalized forces, which are here assumed to be those based on linear theory.

$\dfrac{\partial U_{NL}}{\partial \mathbf{q}}$ = additional generalized forces due to nonlinearities.

The column matrix $\mathbf{q}$ is arranged with all the degrees of freedom together at each node. Since at each node the number of degrees of freedom is equal to four times the number of harmonics the total number of equations represented by Equation 6 is considerable. For example, the maximum permissible case of 17 harmonics and 51 nodes yields 3468 equations. The stiffness matrix $\mathbf{K}$ does, however, have a narrow band width and may be effectively solved for prescribed values of the right side by Choleski's method (Reference 22).

The solution of the nonlinear equations of equilibrium given by Equation 6 is obtained by increasing the loads in increments, combined with iteration at each value of the loading. For the first load the initial solution is obtained by solving the linear equations with the contribution due to nonlinearities being zero. For loads other than the first the initial guess is obtained by extrapolating from previous solutions. The treatment of the nonlinear terms is the same as given in Reference 11.

STIFFNESS MATRIX

The element stiffness matrix is evaluated in two steps. First the internal energy based on linear theory, U_L, is evaluated in terms of the generalized coefficients, α, and then transformed to a quadratic form in terms of the nodal displacements through Equation 2.

Substituting the assumed displacement functions (Equation 1) into the strain energy expression based on linear theory, U_L, the strain energy for the element may be written as

$$U_L = \frac{1}{2}\left\lfloor \alpha^0 \alpha^1 \cdots \alpha^{IA} \vdots \overline{\alpha}^0 \overline{\alpha}^1 \cdots \overline{\alpha}^{IB} \right\rfloor
\begin{bmatrix}
L^{00} & L^{01} & \cdots & L^{0IA} & L^{0\overline{0}} & L^{0\overline{1}} & \cdots & L^{0\overline{IB}} \\
L^{10} & L^{11} & \cdots & L^{1IA} & L^{1\overline{0}} & L^{1\overline{1}} & \cdots & L^{1\overline{IB}} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
L^{IA0} & L^{IA1} & \cdots & L^{IAIA} & L^{IA\overline{0}} & L^{IA\overline{1}} & \cdots & L^{IA\overline{IB}} \\
L^{\overline{0}0} & L^{\overline{0}1} & \cdots & L^{\overline{0}IA} & L^{\overline{0}\overline{0}} & L^{\overline{0}\overline{1}} & \cdots & L^{\overline{0}\overline{IB}} \\
L^{\overline{1}0} & L^{\overline{1}1} & \cdots & L^{\overline{1}IA} & L^{\overline{1}\overline{0}} & L^{\overline{1}\overline{1}} & \cdots & L^{\overline{1}\overline{IB}} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
L^{\overline{IB}0} & L^{\overline{IB}1} & \cdots & L^{\overline{IB}IA} & L^{\overline{IB}\overline{0}} & L^{\overline{IB}\overline{1}} & \cdots & L^{\overline{IB}\overline{IB}}
\end{bmatrix}
\begin{Bmatrix}
\alpha^0 \\ \alpha^1 \\ \vdots \\ \alpha^{IA} \\ \overline{\alpha}^0 \\ \overline{\alpha}^1 \\ \vdots \\ \overline{\alpha}^{IB}
\end{Bmatrix}
\quad (7)$$

where α^m, α^{-n} = eight coefficients corresponding to the mth and nth B harmonic respectively.

L^{mn} = 8 x 8 matrix indicating the coupling between the mth and nth A harmonics with a bar over m or n indicating a B harmonic.

The individual terms of the 8 x 8 L^{mn} matrices are given by

$$L^{mn}_{ij} = \frac{\partial^2 U_L}{\partial \alpha^m_i \partial \alpha^n_j} \quad (8)$$

Using Equation 8 all the elements of the 8 x 8 matrices representing coupling between A-A, A-B, and B-B harmonics may be obtained from the expressions for the upper half of one L^{mn} matrix. An example of one of these 36 terms is

$$L^{mn}_{17} = C^{mn}_2 \, n \int \frac{\cos\phi}{r}\,ds - \nu_{s\theta} C^{mn} \, n \int \phi'\,ds + D^{mn}_2 \, m^2 n \int \frac{\cos\phi}{r^3}\,ds$$
$$+ 4G^{\overline{m}\overline{n}}_2 \, m \int \frac{\sin^2\phi\cos\phi}{r^3}\,ds + 2G^{\overline{m}\overline{n}} \, m \int \frac{\phi'\sin^2\phi}{r^2}\,ds \quad (9)$$

where $C^{mn}_2 = \int_0^2 C_2 \cos m\theta \cos n\theta \, d\theta$ and $\overline{m}$ or $\overline{n}$ indicate that the cosine is replaced by a sine function. This integral and the integral for D and G are evaluated by expanding t and t^3 in a Fourier series in the circumferential direction A maximum number of 49 Fourier coefficients is permitted in the expansion.

The other terms for the upper half of the 8 x 8 L^{mn} matrix are given in Reference 23.

The following rules yield the complete L matrix.

1. The terms below the diagonal are obtained by interchanging m and n in the expressions for the terms above the diagonal. i.e. L_{ji}^{mn} (m,n) = L_{ij}^{mn}(n,m). As required, this yields a symmetrical matrix for m = n. This interchange does not apply to the C's, D's or G's.

2. If the superscript m or n of L_{ij}^{mn} has a bar over it, use a negative m or n in the expressions for L_{ij}^{mn} and change the unbarred m or n over the C's, D's and G's to barred m or n or vice-versa.

The components of the element stiffness matrix representing the coupling between any barred or unbarred m and n harmonics are given by

$$k^{mn} = A^T L^{mn} A$$

(10)

where k^{mn} is an 8 by 8 matrix.

Once the element stiffness matrix is obtained, the structural stiffness matrix may be assembled. The assemblage rules are the same as used for beam elements except the element stiffness matrix is much larger.
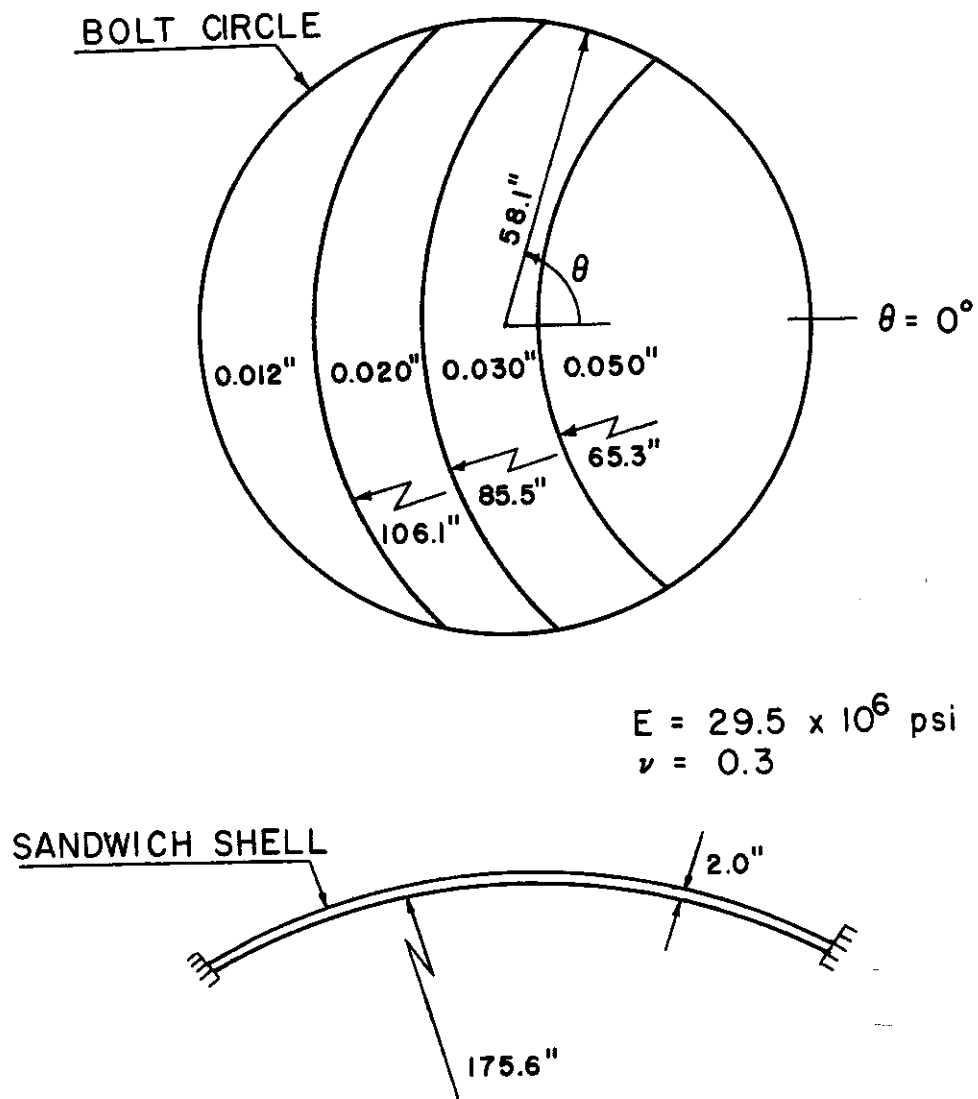
BOLT CIRCLE

$\theta = 0°$

0.012" 0.020" 0.030" 0.050"

58.1"

θ

65.3"

85.5"

106.1"

$E = 29.5 \times 10^6$ psi
$\nu = 0.3$

SANDWICH SHELL

2.0"

175.6"

Figure 1. Scalloped Apollo Aft Heat Shield

SECTION III

APPLICATIONS

The computer code has been in operation for quite some time and has been used to analyze several different types of problems. The primary emphasis has been on the linear, nonlinear, and stability analyses of the Apollo aft heat shield, which is shown in Figure 1. The heat shield is of honeycomb construction with scalloped face sheets. The effects of transverse shear deformations were not included in the analysis. A discussion of transverse shear effects is given in a later section.

ASSESSMENT OF ACCURACY

The following studies have been conducted to demonstrate the accuracy of the method.

1.   Comparison With Finite Difference Solution.

The linear analysis of the aft heat shield was conducted under a uniform pressure loading covering a circle with a 40-inch radius and with the center of the circle at 15 degrees from the apex. The results were compared with results using the method developed in Reference 24. The results for the deflections and stress resultants are in reasonable agreement. However, the shear resultants do not agree especially near the clamped base where the method of Reference 24 yields a divergent solution.

2.   Thickness Representation in Circumferential Direction.

To determine the number of Fourier harmonics needed for the representation of the thickness, analyses were conducted on the aft heat shield using different numbers of terms in the expansion. Due to symmetry, only cosine terms were used in the expansion. Results were obtained with 3, 6, 9 and 25 Fourier harmonics. The results show less than a 10% change in going from 3 to 9 harmonics and no change between 9 and 25. Since 49 harmonics are permitted in the expansion for the thickness, it is concluded that drastic changes in the thickness may be accurately represented.

3.   Representation of Loads.

The number of harmonics needed in Equation 1 depends on the complexity of the loading. Consequently, analyses were conducted using different numbers of Fourier's harmonics in

the expressions for the displacements. For a circular pressure loading distributed over a circle with a 20-inch radius centered at 15 degrees from the apex it was found that 6 and 9 harmonics yielded the same results. For another shell with concentrated forces at 90-degree increments around the circumference it was found that going from 8 to 13 A harmonics gave a 10% increase in the maximum stress resultant. It cannot be said that the solution has converged but it is believed that additional harmonics will give little change. The maximum number of 17 harmonics is considered adequate for most physical problems.

4. Stability Analyses.

Since the solution of the nonlinear equations is obtained by an iterational procedure, the possibility of a numerical instability exists for loads in the neighborhood of the buckling load. To check the ability of the computer code to predict buckling pressures, application was made to the analysis of shallow caps under uniform loading and results compared with known values. It was found that the iterational procedure consistently failed to converge for loads below the actual buckling load, but above most experimental data. The procedure used here to determine the buckling load is to plot the load-deflection curve and extrapolate to the buckling load. For a structure such as the Apollo aft heat shield this gives a rather smooth curve as illustrated in Figure 2, and, consequently the buckling load may be accurately determined.

It is interesting to note that if the omission of the rigid body modes in the displacement functions were important, the buckling load would be overestimated, which is not the case.

RESULTS

During "splashdown" the aft heat shield of the Apollo spacecraft encounters rather high hydrodynamic forces. The actual distribution consists of a high pressure near the edge of the wetted area with decreasing values near the center. The wetted area is, of course, increasing with time.

The idealization used here assumes the pressure to be a constant over a circular wetted area and, more important, neglects dynamic forces. The results reported are for the cases when the center of impact is at 10 degrees from the apex and the pressure loading has a radius of 10, 20, or 40 inches.

For localized pressure loadings such as a wetted area with a 10- or 20- inch radius, it was found that the effects of nonlinearities are quite small until the pressure approaches

the buckling pressure. However, for a pressure loading over a large area, such as the 40-inch radius, nonlinearities become important at rather low values of the pressure. This is illustrated in Figure 2 which presents a plot of the maximum displacement versus loading for a wetted area with a 40-inch radius. Figures 3 and 4 present the displacement and meridional stress in the upper face sheet for this case at a pressure of 80 psi. It is noted that linear theory underestimates the maximum stresses by about 20%.

Table 1 presents the values of the buckling pressures for the three different loadings. Also presented in this table are the results obtained by Gallagher et al (Reference 25) based on using the in plane forces from a linear analysis in the stability analysis.

TABLE I

BUCKLING PRESSURES (PSI) FOR SCALLOPED HEAT SHIELD

Wetted Area Radius	10	20	40
Present Research (Nonlinear)	1100	300	100
Gallagher (Linear)	1179	396	210

A comparison of the results reveals that the stability analysis based on linear theory agrees quite well with the nonlinear stability analysis in cases where nonlinearities are unimportant for loads approaching the buckling loads (10- and 20-in. radii). However, when the nonlinearities are important for loads below the buckling load (40-inch radius) the linear stability analysis yields unconservative results.
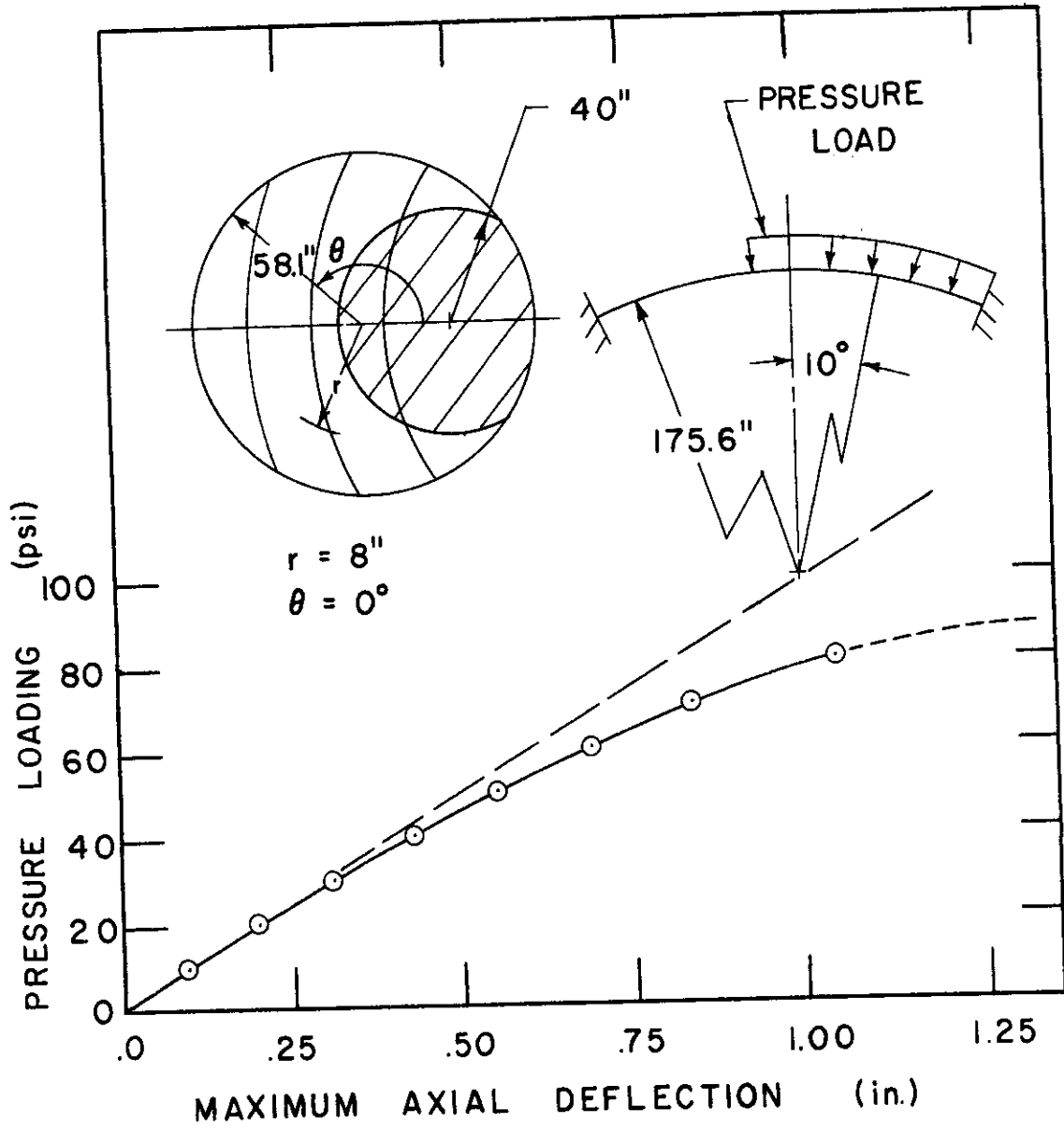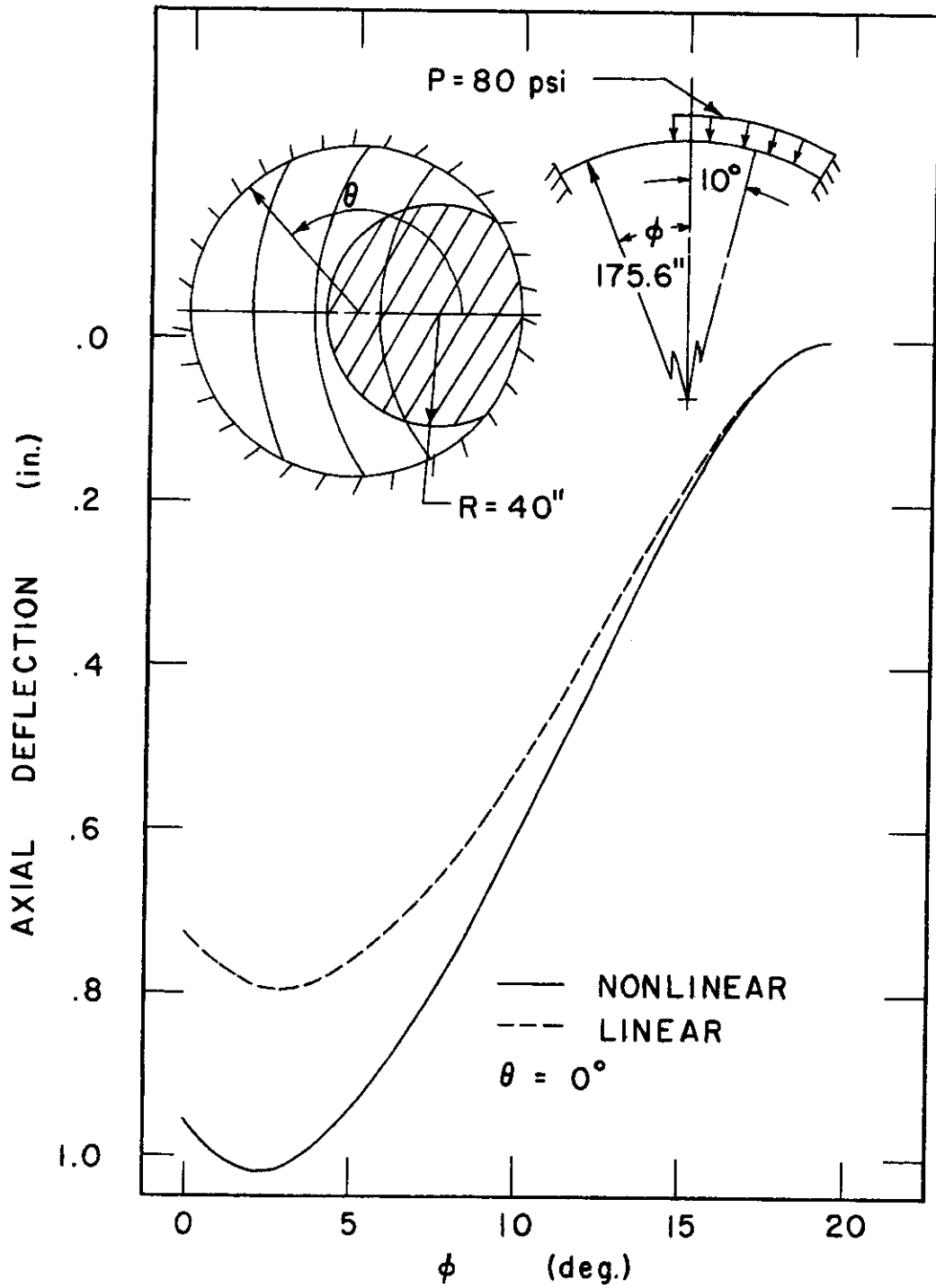
Figure 2. Maximum Axial Deflection Versus Loading
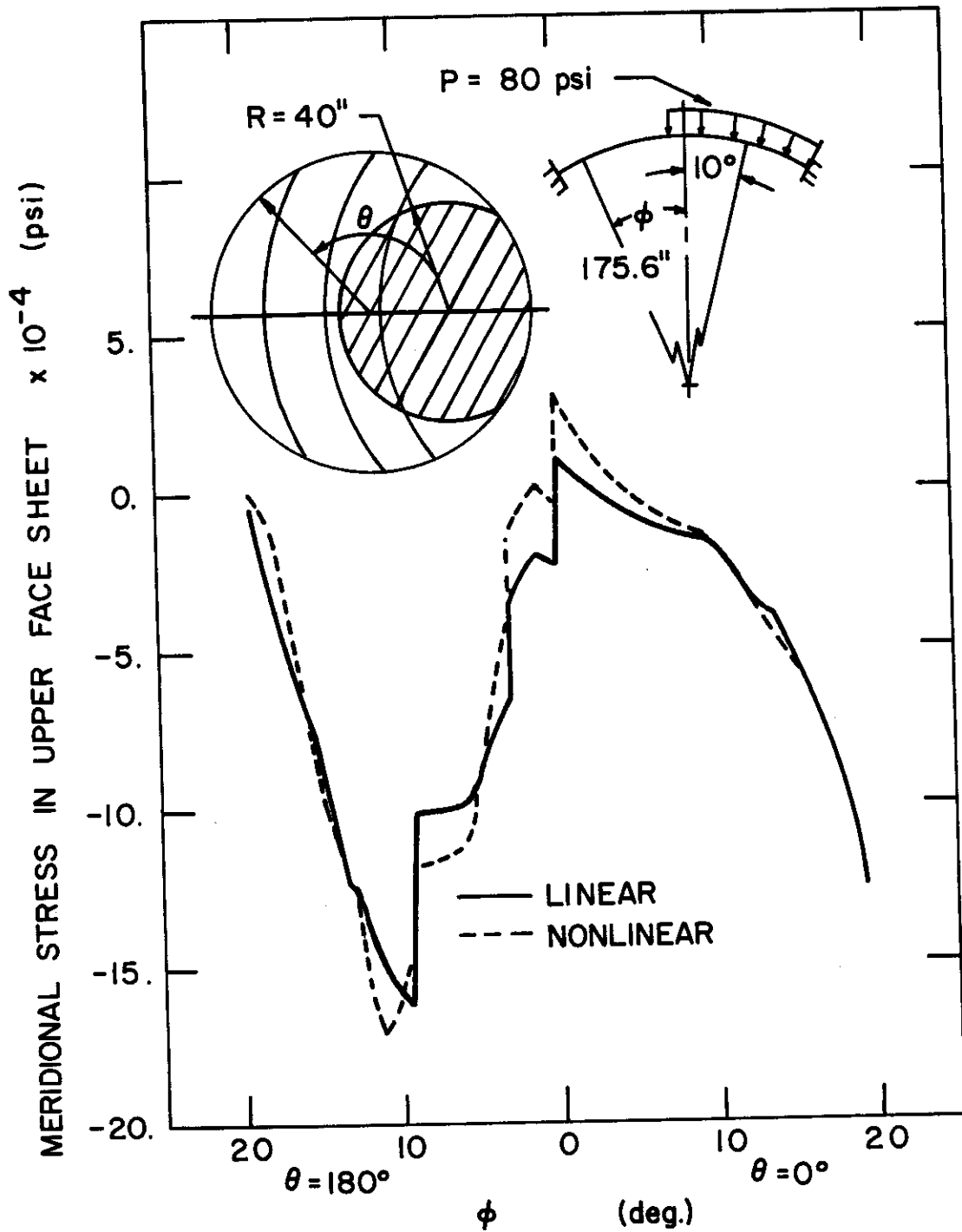
Figure 3. Axial Deflection

1245

Figure 4. Meridional Stress in Upper Face Sheet

SECTION IV

TRANSVERSE SHEAR EFFECTS

As pointed out previously the effects of transverse shear deformations were neglected in the computer code. The purpose of this section is to present a formulation which automatically includes transverse shear deformations and to present results for a highly idealized loading. The material presented in this Section is taken from Reference 26.

It is assumed that the shell of revolution has constant thickness properties in the circumferential direction and is loaded axisymmetrically. The displacement functions are written for a laminate at a distance z from the midsurface of the shell.

$$w = a_1 + a_2 s + a_3 s^2 + a_4 s^3$$

$$u = a_5 + a_6 s + (a_7 + a_8 s) z$$

where the a coefficients are related to two displacements and two rotations, $\frac{\partial w}{\partial s}$, $\frac{\partial u}{\partial z}$, at each node. With this formulation the rotations $\frac{\partial w}{\partial s}$ may be eliminated from the element stiffness matrix. However, most of a computer code was already available for the treatment of four unknowns at each node so $\frac{\partial w}{\partial s}$ was retained as a degree of freedom.

The element stiffness matrix is obtained by substituting into the strain energy expression, given by three-dimensional elasticity theory in body of revolution coordinates, and integrating over the length and around the circumference of the elements. The strain energy expression depends on the displacements and their first derivatives only. This is the reason the degree of freedom, $\frac{\partial w}{\partial s}$, may be eliminated from the element stiffness matrix.

The analysis was conducted for a shallow honeycomb cap clamped at the base. A uniform pressure of 10 psi was applied over a circle with a 15.3-inch radius centered at the apex. The results for the axial displacement and the meridional stresses in the upper and lower face are shown in Figures 5 and 6. It is noted that no appreciable differences are obtained even for very low values of the transverse shear modulus.
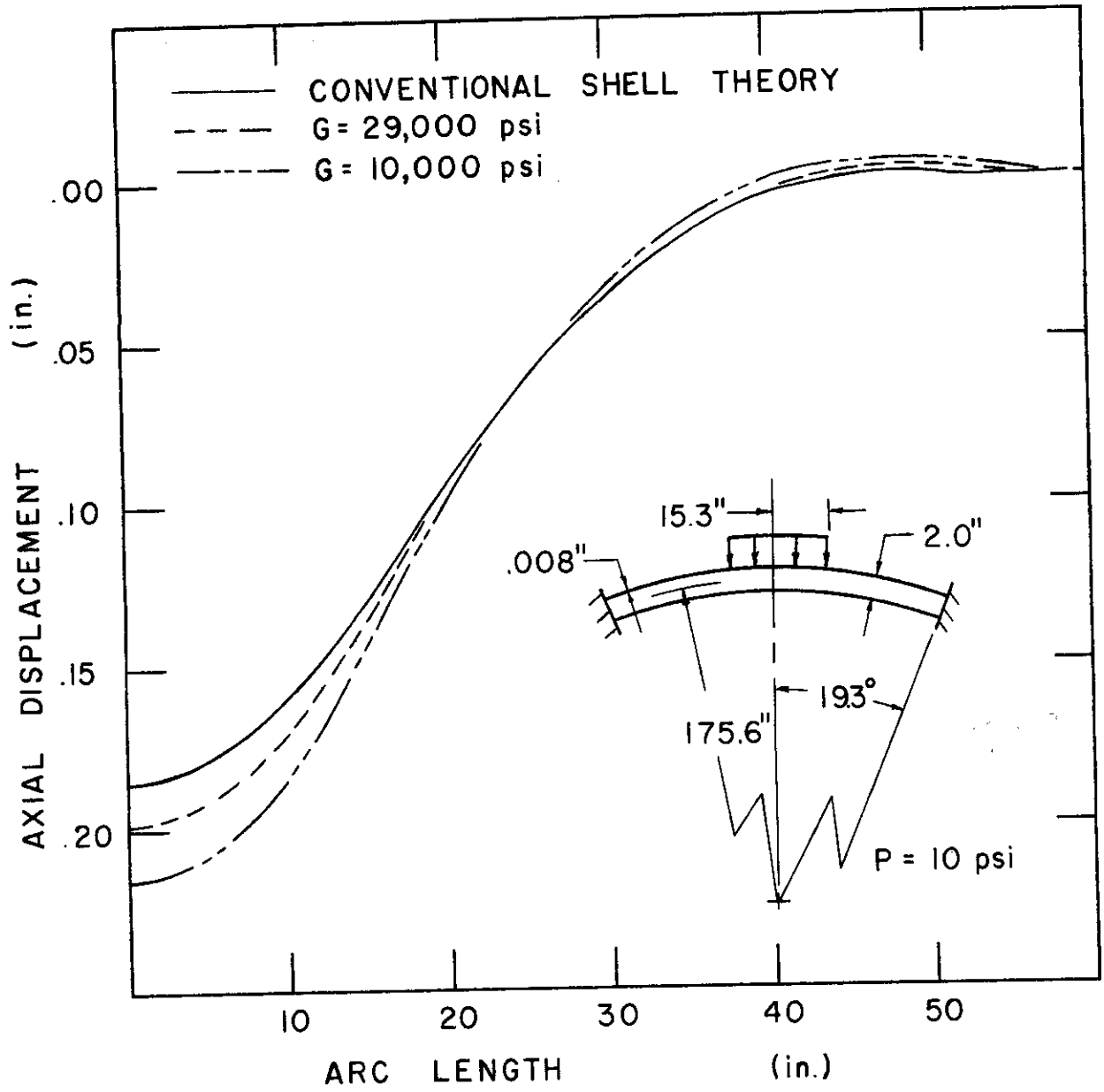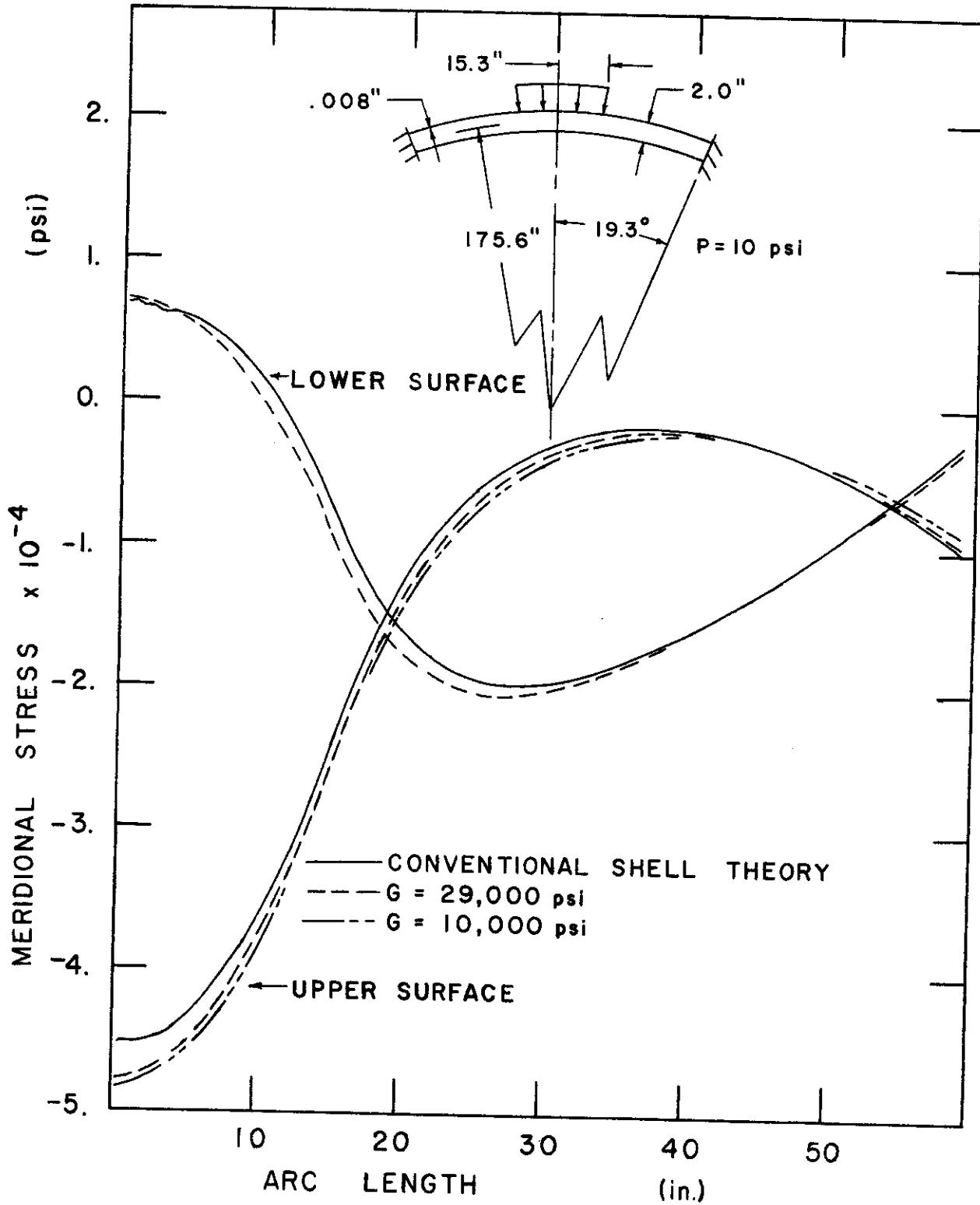
Figure 5. Axial Displacement Versus Arc Length

Figure 6. Meridional Stress Versus Arc Length

REFERENCES

1. Meyer, R. R. and Harmon, M. B., "Conical Segment Method for Analyzing Open Crown Shells of Revolution for Edge Loads," AIAA Journal, Vol. 1, No. 4, pp. 886-891, April 1963.

2. Lu, Z. A., Penzien, J., and Popov, E. P., Finite Element Solution for Thin Shells of Revolution, NASA CR-37, July 1964.

3. Grafton, P. E. and Strome, D. R., "Analysis of Axisymmetric Shells by the Direct Stiffness Method," AIAA Journal, Vol. 1, No. 10, pp. 2343-2347, October 1963.

4. Percy, J. H., Pian, T. H. H., Klein, S., and Navaratna, D. R., "Application of Matrix Displacement Method to Linear Elastic Analysis of Shells of Revolution," AIAA Journal, Vol. 3, No. 11, pp. 2138-2145, November 1965.

5. Jones, R. E. and Strome, D. R., "Direct Stiffness Method Analysis of Shells of Revolution Utilizing Curved Elements," AIAA Journal, Vol. 4, No. 9, pp. 1519-1525, September 1966.

6. Stricklin, J. A., Navaratna, D. R., and Pian, T. H. H., "Improvements on the Analysis of Shells of Revolution by the Matrix Displacement Method," AIAA Journal, Vol. 4, No. 11, pp. 2069-2072, November 1966.

7. Witmer, E. A., et. al., SABOR 4: An Improved Discrete-Element Analysis and Program for the Linear-Elastic Static Analysis of Meridionally-Curved, Varible-Thickness, Branched, Thin Shells of Revolution Subjected to General External and Thermal Loads, Aeroelastic and Structures Research Laboratory, TR 146-4, Massachusetts Institute of Technology, Cambridge, Massachusetts.

8. Klein, S. and Sylvester, R. J., "The Linear Elastic Dynamic Analysis of Shells of Revolution by the Matrix Displacement Method," Proceedings of Conference on Matrix Methods in Structural Mechanics, AFFDL-TR-66-80, Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio, 1966.

9. Popov, E. P. and Penzien, J. "Seventh Progress Report Submitted to NASA under Research Grant NsG-274", Dec. 1965.

10. Khajasteh-Bekht, M. Analysis of Elastic-Plastic Shells of Revolution Under Axisymmetric Loading by the Finite Element Method, Structural Engineering Laboratory Report SESM 67-8, University of California, Berkeley, California.

11. Stricklin, J. A., Haisler, W. E., MacDougall, H. R., and Stebbins, F. J., "Nonlinear Analysis of Shells of Revolution by the Matrix Displacement Method," AIAA Paper No. 67-177, January 1968.

12. Jones, R. E. and Strome, D. R., "A Survey of the Analysis of Shells by the Displacement Method", Proceeding of Conference on Matrix Methods in Structural Mechanics, AFFDL-TR-66-80, Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio, 1966.

13. Haisler, W. E. and Stricklin, J. A., "Rigid-Body Displacements of Curved Elements in the Analysis of Shells by the Matrix-Displacement Method," AIAA Journal, Vol. 5, No. 8, pp. 1525-1527, August 1967.

REFERENCES (CONT)

14. Schmit, L. A., Bogner, F. K., and Fox, R. L., "Finite Deflection Structural Analysis Using Plate and Shell Discrete Elements," AIAA Journal, Vol. 6, No. 5, pp. 781-791, May 1968.

15. Pian, T. H. H., "Derivation of Element Stiffness Matrices", AIAA Journal, Vol. 2, No. 3, March 1964.

16. Turner, M. J. et al, "Large Deflections of Structures Subjected to Heating and External Loads," Journal of Aerospace Sciences, Vol. 27, pp. 97-106, 1960.

17. Martin, H. C., "On the Derivation of Stiffness Matrices for the Analysis of Large Deflections and Stability Problems," Proceedings of the Conference on Matrix Methods in Structural Mechanics, AFFDL-TR-66-80, Air Force Flight Dynamics Lab., Wright-Patterson Air Force Base, Ohio, 1966.

18. Marcal, P. V., "The Effects of Initial Displacements on Problems of Large Deflection and Stability", Report AR PA E54, Division of Engineering, Brown University, Providence, R. I.

19. Oden, T., "Numerical Formulation of Nonlinear Elasticity Problems," Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 93, No. ST3, pp. 235-255, June 1967.

20. Novozhilov, V. V., Foundations of the Nonlinear Theory of Elasticity, Graylock Press, Rochester, New York, 1953.

21. Ambartsumian, S. A., Theory of Anisotropic Shells, NASA TT F-118, p. 53, May 1964.

22. Przemieniecki, J. S., Theory of Matrix Structural Analysis, McGraw-Hill Book Company, New York, 1968.

23. Haisler, W. E., Nonlinear Elastic Analysis of Arbitrarily Loaded Shells of Revolution by the Matrix Displacement Method, M. S. Thesis, Texas A&M University, College Station, Texas, August 1968.

24. Cappelli, A. P., Nishimoto, T. S., and Radkowski, P. P., "The Analysis of Shells of Revolution Having Arbitrary Stiffness Distribution," AIAA/ASME 8th Structures, Structural Dynamics, and Materials Conference, American Institute of Aeronautics and Astronautics, New York, 1967, pp. 732-749.

25. Gallagher, R. H., Gellatly, R. A., and Batt, J. R., Structural and Dynamic Analysis of the Apollo Aft Heat Shield, Report No. D 7218-933005, Bell Aerosystems Company, December 1965.

26. Abshier, J. C., "Effects of Transverse Shear Deformations in the Analysis of Shells of Revolution by the Matrix Displacement Method," paper presented at AIAA Southwest Student Paper Competition, Forth Worth, Texas, April 1969.