

## EXCITATION OF DISTRIBUTED SYSTEMS

by

H. H. Bleich  
Institute of Flight Structures  
Columbia University  
New York, New York

### ABSTRACT

A vibration analysis of very complex distributed systems by exact methods is usually impracticable, and numerical methods using the large computers now available are indicated. The use of numerical methods is roughly equivalent to using a lumped system, but in problems of acoustical fatigue the relatively high frequencies involved may require a rather large number of masses.

The purpose of the presentation is a discussion of the appropriate number of masses (or subdivisions of the structure) necessary to obtain a sufficient approximation of the structural response. The matter will be considered for longitudinal and bending vibrations (including shear effects) of linear elastic and viscoelastic members.

### I. INTRODUCTION

The forced vibrations of a beam or plate of simple geometry, Figure 1, can be determined from the differential equations of the continuous system without difficulty, even if the material exhibits linear viscoelastic behavior. On the other hand, if the geometry is complicated due to many irregular spaced supports and/or variation in mass, Figure 2, a lumped mass approach or equivalent numerical methods may be advantageous because the treatment of the continuous system becomes too unwieldy.

To be able to replace a continuous system by a lumped one, it is necessary to have criteria to decide how fine a mesh of masses must be selected. This depends not solely on the system but also on the range of frequencies of interest, as may be seen from Figures 3 and 4. Figure 3 shows a simple beam under an oscillating load of low frequency (below the first natural frequency) when the response has no nodes. At high frequencies,

on the other hand, Figure 4, many nodes will occur. In the case of Figure 3, a replacement of the beam by as few as 3 masses might give reasonable results, but many more are required for the case of high frequency, Figure 4.

One can conclude, therefore, that the number of lumped masses required is frequency dependent, and that the high frequencies of interest in acoustic fatigue problems may require a large number of mass points, much larger than those required in flutter analysis in aircraft design.

To obtain rational criteria, the case of longitudinal vibration of elastic and viscoelastic bars is considered in detail as an introduction. The results are thereafter generalized for the more interesting case of bending vibrations, allowing for shear effects.

## II. LONGITUDINAL VIBRATIONS OF BARS

Consider the bar of unit area shown in Figure 5. The coordinate  $x$  defines the original position of an element, the displacement of which is  $u$ . Let the stress be  $\sigma$  (positive if tension), while the strain is  $\epsilon \equiv u'$ , and the equation of motion of an element of thickness  $dx$  becomes

$$\rho \ddot{u} = \sigma' \quad \text{Equation (1)}$$

For a linear viscoelastic material the relation between stress  $\sigma$  and strain  $\epsilon$  will be of the form

$$L(\epsilon) = S(\sigma) \quad \text{Equation (2)}$$

where  $L$  and  $S$  are in general differential operators; for a Maxwell Body

$$L = E \frac{\partial}{\partial t} \quad \text{and} \quad S = \frac{\partial}{\partial t} + \frac{1}{\tau}$$

where  $\tau$  is the relaxation time and  $E$  an elastic constant, while for an elastic material

$$L = E \quad \text{and} \quad S = 1 \quad \text{Equation (3a)}$$

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\* Primes and dots indicate differentiation with respect to the coordinate  $x$  and time  $t$ , respectively.

# Contrails

Taking a derivative of Equation (2) with respect to  $x$  and noting  $\epsilon \equiv u'$  gives

$$L(u'') = S(\sigma')$$

Performing the operation  $S$  on Equation (1) results in

$$\rho S(\ddot{u}) = S(\sigma')$$

and therefore

$$\rho S(u) = L(u'') \quad \text{Equation (4)}$$

which is the general differential equation for vibrations and wave propagation. Boundary conditions for  $u$  can easily be found for any particular problem.

Consider now cases of harmonic vibrations such that the displacement  $u(x,t)$  may be written

$$u(x,t) = U(x) e^{i\Omega t} \quad \text{Equation (5)}$$

The operators (Eq. 3) for the Maxwell Body become

$$L = E i \Omega \quad \text{and} \quad S = i \Omega + \frac{1}{\tau} \quad \text{Equation (6)}$$

and the differential equation (4) may be written

$$\alpha^2 U + U'' = 0 \quad \text{Equation (7)}$$

where

$$c^2 = \frac{E}{\rho} \quad \text{and} \quad \alpha^2 = \frac{\Omega^2}{c^2} \left(1 - \frac{1}{\Omega \tau}\right) \quad \text{Equation (8)}$$

The general solution of Equation (7) is

$$U = C_1 e^{i\alpha x} + C_2 e^{-i\alpha x} \quad \text{Equation (9)}$$

such that the time dependent displacement  $u$  becomes

$$u = C_1 e^{i\Omega \left(t + \frac{\alpha}{\Omega} x\right)} + C_2 e^{i\Omega \left(t - \frac{\alpha}{\Omega} x\right)} \quad \text{Equation (10)}$$

The two terms represent the familiar waves progressing in the direction of the positive and negative  $x$ -axis, respectively.

# Contrails

It is noted that in a viscoelastic material  $\alpha$  is complex and the waves decay, while for the elastic case  $\alpha$  is real and no decay occurs.

Having found the solution for the case of a continuous bar, we can proceed and replace it by a lumped system, Figure 6, and compare the solutions. Instead of obtaining the equations of motion of the physical system shown in Figure 6, we can use a purely formal approach and replace each derivative in Equation (7) by the appropriate finite difference expression:

$$U' \approx \frac{U_{k+1} - U_k}{h}, \quad U'' \approx \frac{U_{k+1} - 2U_k + U_{k-1}}{h^2} \quad \text{Eq. (11)}$$

where  $h$  is the interval. Instead of the differential equation (7), we obtain thus the finite difference equation

$$U_{k+1} - [2 - \alpha^2 h^2] U_k + U_{k-1} = 0 \quad \text{Equation (12)}$$

where  $\alpha$  is defined by Equation (8). The theory of solution of such equations is well established.\* Equation (12) has constant coefficients and its solution may be written in the form of trigonometric functions

$$U_k = \frac{\sin \beta k}{\cos \beta k} \quad \text{Equation (13)}$$

To express the functions  $U_k$  of  $k$  by  $U(x)$ , let

$$\beta k \equiv \alpha h k = \bar{\alpha} x \quad \text{Equation (13a)}$$

because  $h k = x$ , Figure 6.

Substituting Equation (13) into (12) one obtains a condition for  $\beta$

$$\cos \beta = 1 - \frac{\alpha^2 h^2}{2}$$

and because of (13a)

$$\cos \bar{\alpha} h = 1 - \frac{\alpha^2 h^2}{2} \quad \text{Equation (14)}$$

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\* See Karman and Biot, Math. Methods in Engineering, McGraw Hill Book Company

# Contrails

If the lumped mass solution is a reasonable approximation of the true continuous one,  $\alpha$  found from Equation (14) must differ only little from the actual wave number  $\alpha$  such that we can ask for the error  $\bar{E}$  defined by

$$\bar{\alpha} = \alpha(1 + \bar{E})$$

Substitution in Equation (14) and expansion of the cosine in a power series, permissible if  $\bar{E} \ll 1$  and  $\alpha h \ll 1$ , gives the first approximation for the error

$$\bar{E} \approx \left| \frac{\alpha^2 h^2}{24} \right| \quad \text{Equation (15)}$$

It is possible to interpret this result in a quite elementary manner. If the lumped mass system, Figure 7, is to represent the displacement  $U$ , Equation (9), one may write

$$U = C_1 e^{i\alpha x} + C_2 e^{-i\alpha x} = C \sin(\alpha x + \varphi) \quad \text{Eq. (16)}$$

The period of the sine being  $\frac{2\pi}{\text{Re}(\alpha)}$  it is obviously necessary that the spacing be much smaller than the period. Thus

$$h \ll \frac{2\pi}{\text{Re}(\alpha)} \quad \text{Equation (17)}$$

Computing  $h$  from Equation (15), stipulating  $E \ll 1$  gives

$$h \ll \frac{\sqrt{24}}{|\alpha|}$$

which is equivalent to Equation (17).

If the damping is small, i.e.,  $\Omega\tau \ll 1$ , then  $\alpha$  according to Equation (8) may be approximated by

$$\alpha \approx \frac{\Omega}{c} \quad \text{Equation (18)}$$

and (17) becomes

$$h \ll \frac{2\pi c}{\Omega} = \frac{c}{f} \quad \text{Equation (19)}$$

where  $f$  is the frequency and  $c$  the phase velocity of the wave.

III. BENDING VIBRATIONS

As the derivation of Equations (17) and (19) did not specify wave propagation in bars, one can expect that these two conditions apply quite generally, i.e., for bending vibrations. They define the mesh distance  $h$  as function of either the wave number  $\alpha$  or of the phase velocity  $c$  and frequency  $f$ . To obtain more accurate information on the error  $E$  for the case of bending, the cases with and without shear effects were also studied. The respective differential equations for elastic bending are:

$$E r^2 y^{iv} + \rho \ddot{y} = 0 \quad \text{Equation (20)}$$

where  $r = \sqrt{I/A}$  is the radius of gyration. If shear deformations are included, the differential equations are:

$$\begin{aligned} E r^2 \psi'' + k G (y' - \psi) - \rho r^2 \ddot{\psi} &= 0 \\ k G (y'' - \psi') - \rho \ddot{y} &= 0 \end{aligned} \quad \text{Equation (21)}$$

where  $k$  is a coefficient depending on the cross section (for rectangles  $k = 5/6$ ).

Equations (20) and (21) apply also to viscoelastic materials provided  $E$  and  $G$  are considered as appropriate complex quantities.

The solution of (20) and (21) leads to characteristic equations having four roots (wave numbers)

$$\pm \alpha_1 \quad \text{and} \quad \pm \alpha_2 \quad \text{Equation (22)}$$

from which the complete solution is constructed.\*

Considering the lumped mass system, one can again obtain the finite difference equations and solve for the approximate wave numbers  $\alpha$ . One finds that the latter are obtained from equations similar to (14),

$$\cos \bar{\alpha} h = 1 - \frac{h^2}{2} \alpha_{1,2}^2 \quad \text{Equation (23)}$$

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\* In the elastic case  $\alpha_2 = i \alpha_1$ .

such that the error  $\bar{E}$  is again given by Equation (15)

$$\bar{E} \approx \frac{\alpha^2 h^2}{24} \quad \text{Equation (15)}$$

As the largest error is of interest, the larger of the values  $\alpha_1^2$  and  $\alpha_2^2$  must be used in Equation (15).

To facilitate applications, Figure 8 gives a non-dimensional plot of  $\alpha_1^2$  and  $\alpha_2^2$  for shear beams of rectangular section  $k = 5/6$  and also  $\alpha_1^2$  for  $k = 0.1$  (which would apply for an I-beam or sandwich panel). Figure 8 also gives the value of  $\alpha^2$  if shear is neglected, indicating that this is permissible only at very low levels of the parameter  $r\Omega/c_1$ .

#### IV. PERMISSIBLE ERROR $\bar{E}$

We have obtained an expression for the error  $\bar{E}$  in the wave numbers  $\alpha$ . To decide the permissible magnitude of  $\bar{E}$  it is necessary to consider the effect of an error  $\bar{E}$  on the error in the displacements or stresses.

The general expression for any physical quantity is a sum of terms  $C.e^{i\alpha x}$ , and if we specify that the error in each term shall not be more than a certain number per cent,  $100 \Delta$ , we can compute  $\bar{E}$ . In doing so, the length of the element  $L$  enters, because  $0 \leq x \leq L$ . Substituting  $\bar{\alpha} = \alpha(1 + \bar{E})$  into each exponential term

$$e^{i\bar{\alpha}x} = e^{i\alpha x + i\alpha \bar{E}x} = e^{i\alpha x} [1 + \Delta] \quad \text{Equation (24)}$$

If  $\alpha \bar{E} x \ll 1$  the term  $e^{i\alpha \bar{E}x}$  may be expanded, giving

$$\Delta = i\alpha \bar{E}x$$

The largest value of the error  $\Delta$  occurs for  $x = L$ , and its absolute value is

$$\max |\Delta| = \bar{E} |\alpha| L$$

such that

$$\text{permissible } \bar{E} = \frac{\Delta}{L |\alpha|} \quad \text{Equation (25)}$$

where  $\Delta$  is the desired maximum error.

If the overall length  $L$  of the structure is very large, the permissible error  $\bar{E}$  becomes very small and Equation (25) is very severe. If the structure is long, and damping is present, the condition (25) may be relaxed by requiring that the error at any point shall be smaller than a stated percentage,  $100\Delta$ , of the largest value of the term  $C e^{i\alpha x}$  anywhere. (The error will then be less than  $100\Delta\%$  where stresses are large, but may be larger where they are small).

Selecting the direction of positive  $x$  such that  $e^{i\alpha x}$  decreases with increasing  $x$ , the largest value of  $|e^{i\alpha x}|$  in the range  $0 \leq x \leq L$  occurs for  $x = 0$ , and is unity. Instead of Equation (24) we have then

$$e^{i\bar{\alpha}x} = e^{i\alpha x} + \Delta \quad \text{Equation (26)}$$

Substituting  $\bar{\alpha} = \alpha(1 + \bar{E})$  and expanding one finds

$$\Delta = e^{i\alpha x} (i\alpha \bar{E}x) \quad \text{Equation (27)}$$

The absolute value of  $\Delta$  is [noting  $\alpha = R(\alpha) + i \text{Im}(\alpha)$ ]

$$|\Delta| = e^{-x \text{Im}(\alpha)} \bar{E} \alpha x \quad \text{Equation (28)}$$

and the maximum value of  $|\Delta|$  absolute occurs for  $x = 1/\text{Im}(\alpha)$  if this value  $x < L$ . Thus

$$\max |\Delta| = \frac{\bar{E} |\alpha|}{e \text{Im}(\alpha)} \quad \text{Equation (29)}$$

The permissible error  $\bar{E}$  is therefore in terms of the prescribed error  $\Delta$

$$\bar{E} = \frac{\Delta e \text{Im}(\alpha)}{|\alpha|} \quad \text{Equation (30)}$$

Equation (30) is to be used only if it gives a larger value than Equation (25).



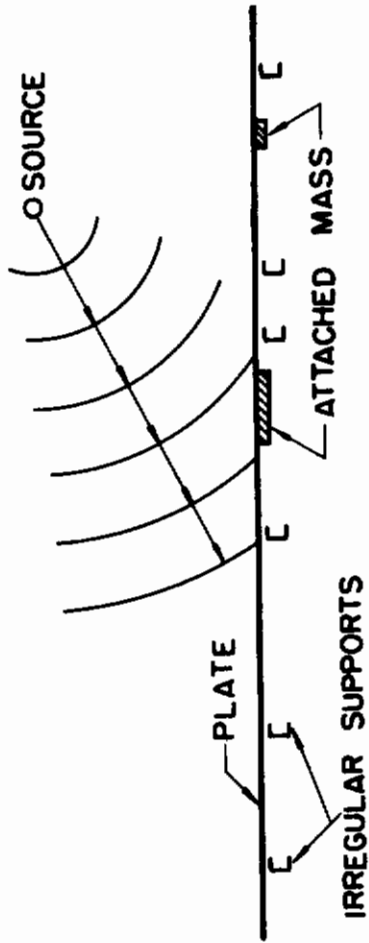


FIG.2



FIG.1



FIG.3 (LOW FREQUENCY)

FIG.4 (HIGH FREQUENCY)

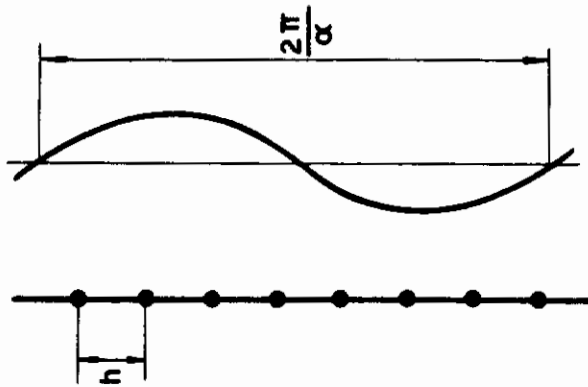


FIG. 7

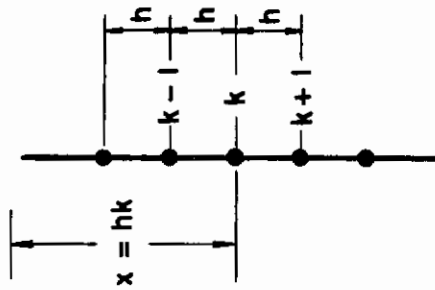


FIG. 6

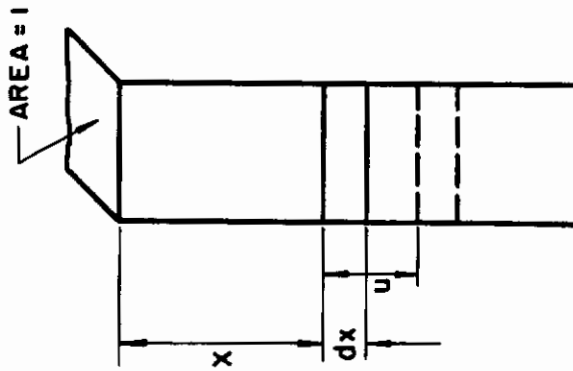


FIG. 5

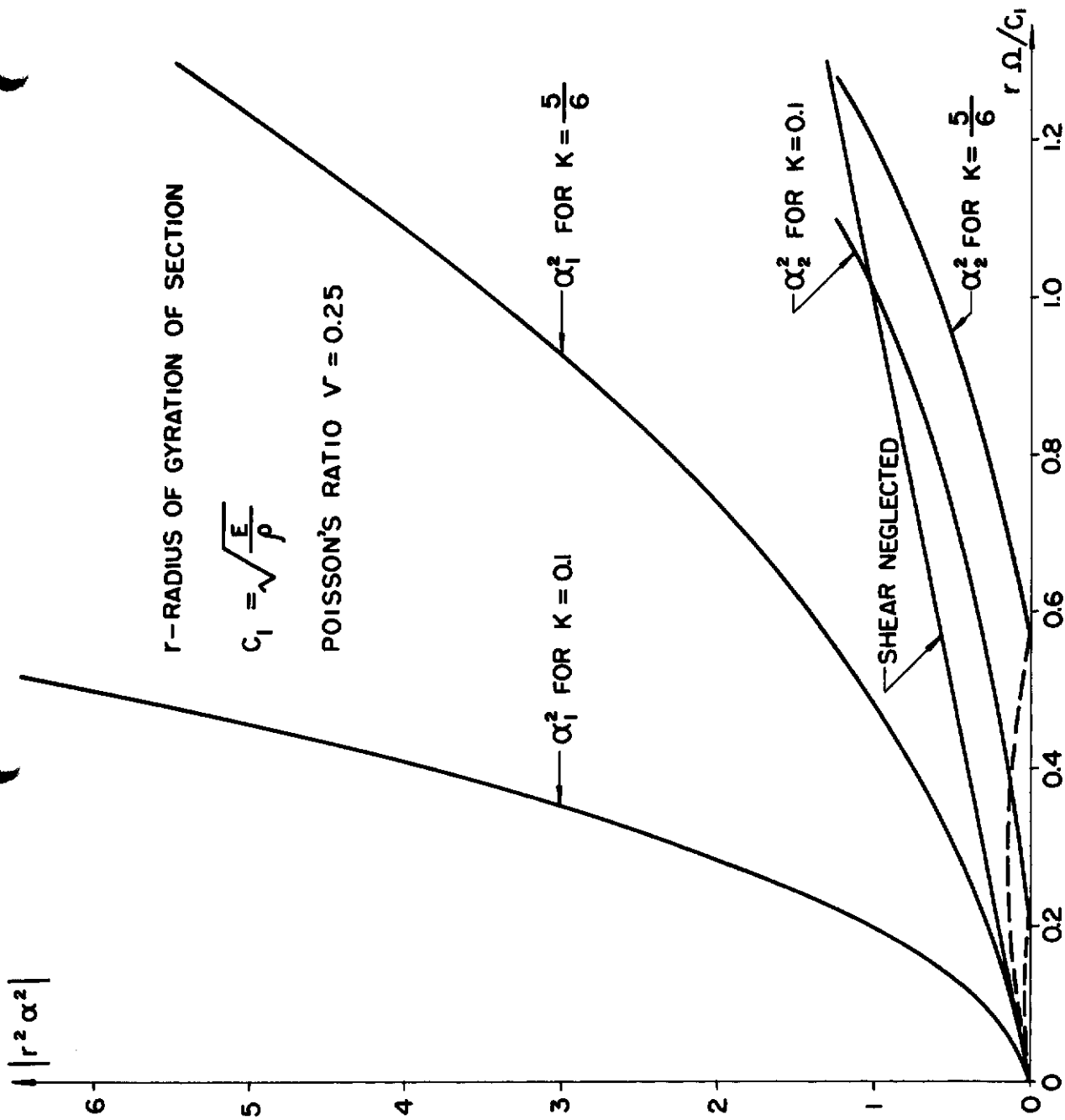


FIG. 8 SHEAR BEAM.  $\alpha$  AS FUNCTION OF  $\Omega$