RESPONSE OF ELASTIC STRUCTURES TO DETERMINISTIC AND RANDOM EXCITATION

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FOREWORD

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A report such as this often is not the product of a single individual. For this document, the author wishes to acknowledge with thanks the effort of several individuals.

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ABSTRACT

This document discusses the fundamental classical theory governing the response of linear distributed elastic structures to deterministic and to random excitation. A review is made of the basic dynamics theory for discrete and distributed systems when the excitation is deterministic. This review is considered necessary to easily understand the subject material. Integral expressions are then derived for the mean square value and correlation functions for the response of an arbitrary linear elastic structure subjected to stationary random loading. These derived results are then applied to illustrative structural problems. In this way, the association between the parameters in the theoretical expressions and the physical properties of structural systems is demonstrated.

A principal objective is to explore the value and limitations of using classical theory as a tool for predicting structural vibrations in typical flight vehicles. The theoretical results for distributed structures subjected to stationary random excitation are noted to yield complicated analytical expressions even for uniform beams. The direct extension of the shown theoretical results to include typical flight structures, although technically accurate, is not considered practical. However, the derivation procedures and results of this report can be used as a basis for forming statistical parametric techniques for approximating the response behavior of distributed systems to random excitation. In the concluding discussion, several existing techniques reflecting compromises in theoretical rigor are discussed and subject areas for future study are noted.

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ILLUSTRATIONS

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SYMBOLS

A_o	magnitude of a d. c. signal
a_o, A	amplitude of the sinusoidal forcing function
B_r	half-power bandwidth in cycles per second
c	viscous damping coefficient for the single degree-of-freedom system
c(x)	viscous damping coefficient assumed to vary as a function of x
C_o	constant in the transient solution for the single degree-of-freedom system
$\overline{C_i}$	generalized damping in the ith mode
D(x)	spatial differential operator for the flexural vibration of a simple beam
E [ ]	Young's modulus
E [ ]	the expected value of [ ] , i. e., the ensemble average
E_M[ ]	the expected value of [ ] formed by averaging over the mass of the structure
f	frequency of the excitation in cycles/sec.
f(t)	forcing function applied to the mechanical oscillator
f(x, t)	the excitation or forcing function acting on the structure
$\overline{f(x,t)f(x',t)}$	time averaged spatial correlation of external forces acting on the structure at positions x and x'
F_o	peak magnitude of the sinusoidal forcing function
$\overline{F_i}$	generalized force in the ith mode
g	structural damping coefficient
G(f)	power spectral density defined in terms of positive cyclical frequencies (f)
G(ω)	power spectral density defined in terms of positive angular frequencies (ω)
G_o	magnitude of a uniform spectral density
h(τ)	the response of any system to a unit impulse (weighting function)

t	real time
T	time duration of sample record
T_p	period of a sine wave
W	loading per unit length for the simply supported beam problem
x	displacement measured from the static equilibrium position for the single degree-of-freedom system or distance along the length of a structure
$\dot{x}$	velocity for the single degree-of-freedom system
$\ddot{x}$	acceleration for the single degree-of-freedom system
$y(x,t)$	the lateral displacement of the structure from its static equilibrium position
$\dot{y}(x,t)$	the lateral velocity of the structure given by $(\dot{\ }) = \frac{d}{dt}(\ )$
$\ddot{y}(x,t)$	the lateral acceleration of the structure given by $(\ddot{\ }) = \frac{d^2}{dt^2}(\ )$
$\overline{y(x,t)y(x',t)}$	time averaged spatial correlation between the response at x and x'
$\overline{y^2(x)}$	mean square response at x
$Z(\omega)$	mechanical impedance (force to displacement ratio)
$\mathcal{F}[\ ]$	Fourier transform of
$\mathcal{L}[\ ]$	Laplace transform of
$\mathcal{M}(x)$	bending moment of the beam
$(\ )^*$	complex conjugate of ( )
$\overline{(\ )}$	time averaged value of ( )
$\overline{(\ )^2}$	the time averaged mean square value of ( )
Δt	time duration for the mechanical oscillator to decay to 1/e its steady state value.
δ	logarithmic decrement

SYMBOLS (continued)

$H(\omega)$	general notation denoting a frequency response function
I	area moment of inertia of the beam cross section about the bending axis
k	spring constant for the single degree-of-freedom system
K_i	ith modal stiffness
$\overline{K_i}$	generalized stiffness in the ith mode
$L_{jk}(\omega)$	spectral density of the external loading weighted by generalized masses and modal frequencies
m	mass of the single degree-of-freedom system and also used to denote the mass per unit legnth for a beam
$m(x)$	mass per unit dimension
M	total mass of the structure
M_i	ith modal mass
$\overline{M_i}$	generalized mass in the ith mode
$p(x)$	first order probability density function
$p(x_1x_2)$	joint probability density function in the variables x_1 and x_2
$q_i(t)$	ith generalized coordinate
Q	quality factor, i. e., the symbol denoting a measure of damping
$R_x(\tau)$	autocorrelation function for stationary random process
$R_f(x, x', \omega)$	spatial correlation density of the loading at points x and x' at frequency ω
$s(x)$	stress at position x due to bending
$S(f)$	power spectral density defined in terms of both positive and negative cyclical frequencies (f)
$S(\omega)$	power spectral density defined in terms of both positive and negative angular frequencies (ω)
S_o	magnitude of a uniform spectral density

SYMBOLS (continued)

ζ	damping factor
ζ_i	damping factor for the ith mode defined as $\zeta_i = c_i/c_c$ where c_i is the viscous damping coefficient for the ith mode of the structure and c_c is the critical damping coefficient = $2m\,\omega_i$
λ_i	part of the argument for the characteristic equations of the beam problems
μ	arithmetic mean
ν	parameter defining the Rayleigh probability density function
σ	standard deviation
σ^2	variance
τ	period of the damped harmonic oscillation
ϕ_1	phase angle between the response and the excitation for the single degree-of-freedom system
$\phi_i(x)$	ith normal mode of the physical system
Ψ^2	mean square value
$\omega, \omega_o, \omega_o'$	frequency of the forcing function in radians/sec.
ω_i	natural frequency associated with the ith mode of the system(radians/sec.)
ω_n	natural frequency for a single degree-of-freedom system

RESPONSE OF ELASTIC STRUCTURES TO DETERMINISTIC AND RANDOM EXCITATION

INTRODUCTION

With the advent of jet and rocket powered vehicles, it became necessary to consider statistical concepts in predicting the dynamic response characteristics of elastic structures. This consideration is required because the dynamic environment common to high speed jet and rocket powered vehicles is, in general, not deterministic. Hence, it must be characterized by probabilistic statements.

As with deterministic excitation, the typical structural response problems are those associated with structural failure and electronic equipment malfunction. Consequently, problems of structural failure are concerned with peak stress levels, peak displacements and fatigue damage. Problems common to equipment are concerned with maximum excursion (bottoming), time duration above a specified vibration level, and peak acceleration levels. Intimately related to these problems are those of prescribing environmental test specifications for equipment attached to the vehicle structure as well as for the structure, per se. As contrasted with deterministic excitation, no equation can be written which will define solutions to the above problems as explicit functions of time or frequency. One only can estimate, within confidence limits, the response behavior.

A cursory glance at the literature reveals no lack of empirical and/or theoretical documentation pertinent to random excitation and structural response. However, without command of the basic theory of structural response to random excitation, this cataloged information remains but a collection of loosely related data, and its usefulness is severely limited. Basic theory, therefore, can serve to establish an organized perspective of the engineering literature which, in turn, promotes an understanding of the technical concepts contained in the literature.

This report is intended to clarify and illustrate the fundamental classical theory for problems considering the response of distributed elastic structures to random excitation. As such, attention is given to basic properties of random processes and linear, time-invariant, elastic structures. For completeness, this document begins with a summary of the response properties for the single-degree-of-freedom system to sinusoidal excitation and concludes with a treatment for the response of distributed elastic structures to stationary random excitation. For the discussion of the theory, this report is subdivided into four principal sections:

1. Response Characteristics of a Single-Degree-Of-Freedom System to Deterministic Excitation
2. Response Characteristics of Multi-Degree-Of-Freedom Systems to Deterministic Excitation
3. Response Characteristics of a Single-Degree-Of-Freedom System to Random Excitation
4. Response of Continuous Elastic Structures to Random Excitation

Section 1, considers one of the simplest mechanical systems (the mechanical oscillator) and a well-defined excitation (the sinusoid) to develop basic definitions for the dynamics of mechanical systems. Section 2 treats the distributed elastic structure subjected to deterministic excitation and uses the concepts of orthogonality, generalized coordinates, and modal solutions. Section 3 considers basic random process properties such as autocorrelation and covariance and provides the mean square response and spectral density relations for the mechanical oscillator. Section 4 considers the modal approach, Parseval's theorem, and Fourier transforms to derive expressions describing the response properties of a distributed structure subjected to stationary random excitation. Equations are given for spatial correlation functions, mean square response, and a "weighted" generalized force.

The first three sections, therefore, supply background information so that the formulation of Section 4 can proceed with understanding. In addition, the response equations in the various sections are used in selected example problems so the reader can associate the parameters given in the analytical expressions with the physical properties of the structural problems. In this way, an understanding of the theory is bolstered and the computational difficulties of practically applying classical theory are shown.

Some of the material presented here may be found in one form or another in other documents. This report, however, organizes the necessary theory into a single document and

(1) presents a complete discussion of the basic theory appropriate to the response of structures,

(2) clarifies this theory by examining illustrative example problems,

(3) illustrates, with practical examples, the usefulness and limitations of these theoretical techniques.

For example, the mean square response of an arbitrary distributed elastic structure to stationary random excitation is given by an integral equation derived in Section 4. To compute the mean square response, the requirement is but to interpret the physical properties of the problem and substitute these values properly in the integral equation. Even for an uncomplicated structure such as a simply supported beam, the example problems show that the calculations for the mean square response are non-routine tasks. In solving these example problems, several useful analytical techniques are illustrated in evaluating some of the theoretical expressions:

- the use of the delta function in treating point loadings
- treating distributed correlated loadings as sets of discrete correlated point loadings. This is analogous to a lumped parameter model for continuous correlated loading
- citing a procedure for calculating the effect of mass loading on the response of a single dimension, continuous elastic structure.

The concluding section (Section 5) summarizes in brief the theory presented in this report for calculating the mean square response of a distributed elastic structure subjected to stationary random excitation. And, perhaps most important, discussion is given to other references which also illustrate the difficulties and limitations of this theory in practical application. Several procedures are mentioned which reflect compromises in theoretical rigor, and topic areas for future studies are suggested.

1. RESPONSE CHARACTERISTICS OF A SINGLE DEGREE-OF FREEDOM SYSTEM TO DETERMINISTIC EXCITATION

1.1 INTRODUCTORY REMARKS

Although the single degree-of-freedom system is not a complicated mechanical model, many analytical tools and basic dynamics definitions are easily illustrated by a study of such a system. This section reviews some of the more significant dynamic response properties for the mechanical oscillator and serves as a prelude for considering multi-degree-of-freedom systems.

1.2 EQUATIONS OF MOTION

The equations of motion for a single degree-of-freedom system (i.e., a mechanical oscillator) excited by a sinusoidal forcing function applied to the mass is given by:

$$m\ddot{y} + c\dot{y} + ky = f(t) = F_o \sin \omega t \tag{1.1}$$

where

- m = mass of the system (lb-sec.^{2}/in.)
- c = viscous damping coefficient (lb-sec.^{2}/in.)
- k = spring constant (lb./in.)
- y = displacement measured from static equilibrium for the mechanical oscillator (in.)
- $\dot{y}$ = time derivative of displacement or the velocity (in./sec.)
- $\ddot{y}$ = second time derivative of displacement or the acceleration (in./sec.^{2})
- $f(t)$ = sinusoidal forcing function or $F_o \sin \omega t$
- ω = excitation frequency of the forcing function (radians/sec.)
- F_o = peak magnitude of the forcing function (lb.)

In alternate form, Eq. (1.1) can be written as

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \frac{f(t)}{m} \tag{1.2}$$

where

$\omega_n = \sqrt{k/m}$ = natural frequency of the undamped oscillation in radians per second

$f_n = \frac{\omega_n}{2\pi}$ = natural frequency of the undamped oscillation in cycles per second

$\zeta = c/c_c$ = damping factor

$c_c = 2m\omega_n$ = critical damping coefficient

The equation of motion for a mass excited, structurally damped single degree-of-freedom system may be written as

$$m\ddot{y} + k(1 + ig)\,y = F_o e^{i\omega t} \tag{1.3}$$

where

g = structural damping coefficient

i = complex operator of $\sqrt{-1}$

Note that structural damping is defined as proportional to the displacement and in phase with the velocity, and may be interpreted as a complex spring. The appropriateness of Eq. (1.3) as a realistic model for structural damping still is the subject of much controversy in structural mechanics.

The complete solution to Eq. (1.2) for a lightly viscous damped system may be written as

$$y = C_o e^{-\zeta\omega_n t} \sin\left(\sqrt{1 - \zeta^2}\,\omega_n t + \phi_1\right) + \frac{F_o \sin(\omega t - \phi)}{k\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\,\frac{\omega}{\omega_n}\right)^2}} \tag{1.4}$$

The first term of Eq. (1.4) represents the transient term yielded by the complementary solution, whereas the second term of Eq. (1.4) represents the steady state term given by the particular solution.

A solution to the homogeneous part of Eq. (1.3) is given as

$$y = e^{at}\left[C_1 e^{ibt} + C_2 e^{-ibt}\right]$$

where

$$a = -\frac{\omega_n}{\sqrt{2}}\left(\sqrt{1+g^2} - 1\right)^{1/2} \qquad (1.5)$$

$$b = \frac{\omega_n}{\sqrt{2}}\left(\sqrt{1+g^2} + 1\right)^{1/2}$$

This solution is not a trivial mathematical task and, in general, is interpreted for harmonic conditions with light damping near ω_n.

1.3 DAMPING

As shown by Figure 1, the transient response for a lightly (viscous) damped mechanical oscillator is a harmonic motion which decays in amplitude within the boundaries of the $e^{-\zeta\omega_n t}$ envelope.

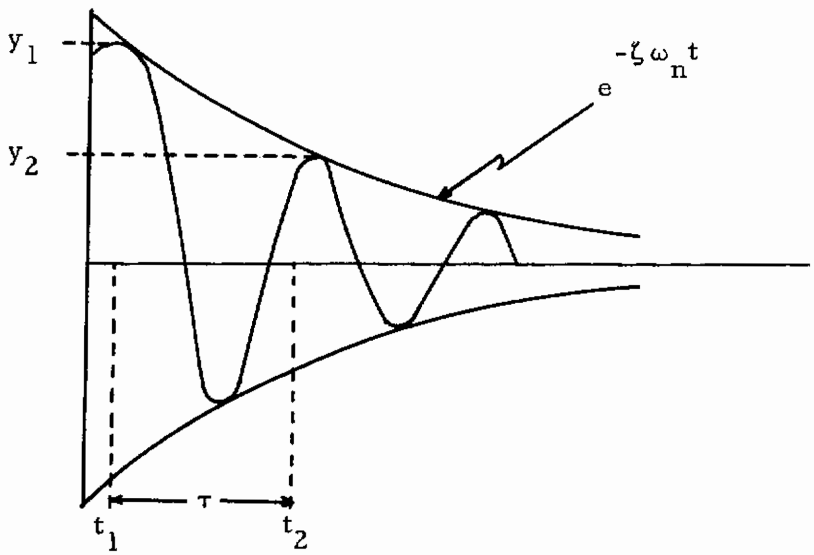


Figure 1. Transient Decay of a Viscous Damped Single Degree-of-Freedom System

This damped harmonic oscillation is given by the transient form of Eq. (1.4) and is noted to be

$$\omega_{damped} = \omega_d = \omega_n \sqrt{1 - \zeta^2} \tag{1.6}$$

For structural damping, the damped harmonic oscillation is given by 'b' term of Eq. (1.5) and is noted to be of the frequency

$$\omega_{structural\ damped} = \omega_{sd} = \omega_n \left\{ \frac{\sqrt{1 + g^2} + 1}{2} \right\}^{\frac{1}{2}} \tag{1.7}$$

By comparing Eqs. (1.6) and (1.7), it is noted that viscous damping tends to decrease the undamped natural frequency whereas structural damping tends to increase the undamped natural frequency.

From the definition of the logarithmic decrement, Figure 1 typically is used to obtain a measure of damping as

$$\delta = \ln \frac{y_1}{y_2} = \zeta \omega_n \tau \tag{1.8}$$

where δ is the logarithmic decrement, y_1 and y_2 are the two successive peak amplitudes, and τ is the period of the damped oscillation. In terms of the damping factors, the logarithmic decrement appears as

$$\delta = \frac{2\pi\zeta}{\sqrt{1 - \zeta^2}} = \frac{2\pi g}{1 + \sqrt{1 + g^2}} \tag{1.9}$$

For small values of damping, therefore, the equivalence between viscous and structural damping becomes simply

$$2\zeta = g \tag{1.10}$$

1.4 FREQUENCY RESPONSE FUNCTIONS

With increasing time, the transient contribution decays to zero and the resulting motion of the mechanical oscillator is described by the steady state response of Eq. (1.4). The steady state condition permits the definitions of frequency response functions. These functions are complex quantities and relate both the magnitude and phase of a steady state response (output) to a steady state excitation (input). A complete listing of frequency response functions for the mechanical oscillator is given in Reference 1, (Table 2, p. 111). One such function, called the magnification factor, represents the magnitude by which the zero frequency deflection y_o (input) must be multiplied to determine the response amplitude y (output). The magnification factor and associated phase angle are defined as

$$\left|H(\omega)\right| = \left|\frac{y}{y_o}\right| = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$$

$$\tan\phi = \frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \tag{1.11}$$

Equation (1.11) may be sketched [see Reference 2 (p. 66)] for various values of damping as shown in Figure 2. Although dependent on the amount of damping, it is noted that the response is nearly in phase with the excitation when $\omega << \omega_n$ and lags the excitation by nearly 180^o when $\omega >> \omega_n$. The phase change is most drastic in the neighborhood of $\omega = \omega_n$. The phase factor specified in Eq. (1.11) is for a mass excited single degree-of-freedom system. This factor is identical to the phase factor for the ratio of the relative motion to the base excitation for a base excited mechanical oscillator.

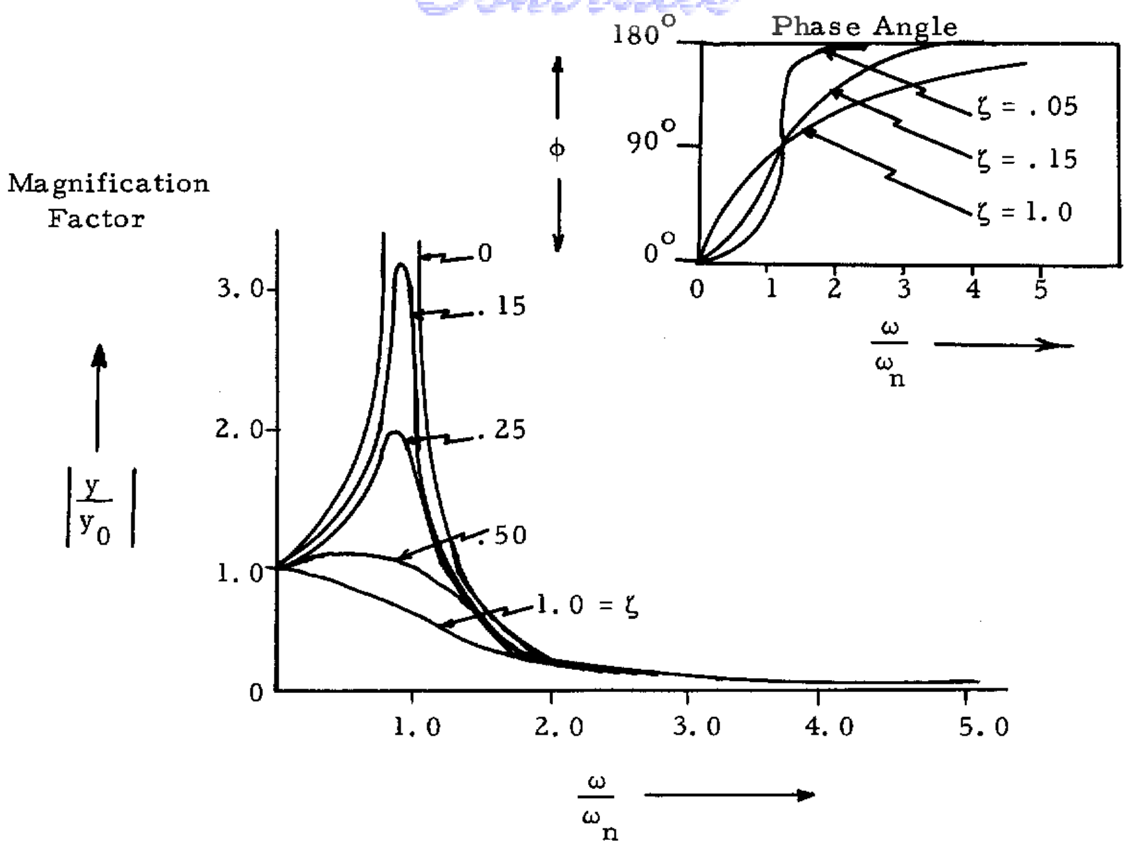


Figure 2. Magnification Factor and Associated Phase Angle For the Vibration of a Viscous Damped Mechanical Oscillator

Note, however, that the phase factor for the ratio of the mass motion to the base excitation for a base excited mechanical oscillator is given by

$$\tan \phi = \frac{2\zeta\left(\frac{\omega}{\omega_n}\right)\left(\frac{\omega}{\omega_n}\right)^2}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + \left(2\zeta\,\frac{\omega}{\omega_n}\right)^2} \tag{1.12}$$

rather than by Eq. (1.11). The phase angle at $\omega = \omega_n$ for Eq. (1.12) is equal to 90° only for extremely small values of $\zeta(\zeta << .10)$ whereas the phase angle for Eq. (1.11) equals 90° for all values of ζ. Before indulging in a usual laboratory practice of locating the natural frequency by observing a 90° phase shift, attention must be given to a theoretical model and appropriately instrument the test specimen in order to monitor the correct phase relationships.

The peak amplitude of the magnification factor can be defined in terms of the quality factor which is written as

$$Q = \frac{1}{2\zeta} = \frac{\sqrt{km}}{c} \tag{1.13}$$

This term is borrowed from electrical engineering and can be defined as the ratio of the reactance to the resistance of an inductor. With resistance proportional to damping, a large Q thus is indicative of the quality or lack of resistance in the inductor. For small values of damping ($Q \gtrsim 10$), the peak of the magnification factor occurs approximately at $\omega = \omega_n$ and the curve is approximately symmetrical for small variations in ω about $\omega = \omega_n$. The amplitude falls off to approximately .707 times the peak value at the frequencies $\omega_n(1 \pm \zeta)$. These frequency values are called the half power or 3 db points and the frequency difference between the half power points is called the bandwidth of the system. Thus, in terms of the bandwidth (B_r) and the natural frequency (f_n)

$$Q = \frac{\omega_n}{\Delta\omega} = \frac{f_n}{B_r} \tag{1.14}$$

where

$$B_r = f_{n+3db} - f_{n-3db} \tag{1.15}$$

It is of interest to note the time duration for the response of a mechanical oscillator to sinusoidal excitation to decay to 1/e times its steady state value. Using the definition of the logarithmic decrement [Eq. (1.8)], this decay time (Δt) is given as

$$\Delta t = \frac{Q}{\pi f_n} \tag{1.16}$$

Equation (1.16) defines Q in terms of the number of cycles for the oscillator to decay to $1/\epsilon$ times its steady state value. In terms of this decay time, the half-power bandwidth is given as

$$B_r = \frac{1}{\pi \Delta t} \qquad (1.17)$$

Three important frequencies are noted to be defined for the single degree-of-freedom system

$$(1) \qquad \omega_n = \sqrt{\frac{k}{m}} \qquad (1.18)$$

$$(2) \qquad \omega_d = \omega_n \sqrt{1 - \zeta^2} \qquad (1.19)$$

$$(3) \qquad \omega_r = \omega_n \sqrt{1 - 2\zeta^2} \qquad (1.20)$$

Equation (1. 18) defines the undamped natural frequency. Equation (1. 19) yields the damped natural frequency in terms of viscous damping and is defined by the logarithmic decrement. Equation (1. 20) provides the frequency for the peak response of the system. The peak response frequency is found by differentiating the steady state part of Eq. (1. 4) with respect to ω and setting the resulting expression equal to zero.

Various computational techniques are available to calculate the frequency response functions for linear systems. All such techniques essentially solve for the steady state solution to the differential equation describing the mechanical system. One particularly useful procedure is illustrated on pages 57, 63, and 65 of Reference 3.

It is of theoretical interest to consider the computation of the frequency response function by using Laplace transforms and noting the response of the system to a unit impulse. The Laplace transform and the corresponding inversion integral of the function y(t) is defined as

$$\mathcal{L}\left[y(t)\right] = \int_0^{\infty} y(t)\, e^{-st}\, dt \tag{1.21}$$

$$y(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \phi(s)\, e^{st}\, ds \tag{1.22}$$

In the above definitions, s is the complex number denoted as $s = \alpha + i\beta$. For harmonic conditions, $\alpha = 0$ and $\beta = \omega$. Under these conditions, the Laplace transform reduces by definition to the Fourier transform. To use Eq. (1.21) to compute a frequency response function, one takes the Laplace transform of the original differential equation where the forcing function $\left[f(t)\right]$ is the unit impulse. One then substitutes $s = i\omega$ in the transformed equation and solves algebraically for the desired transfer function.

By way of illustration, the Laplace transform of Eq. (1.1) appears as

$$\mathcal{L}\left[m\ddot{y} + c\dot{y} + ky\right] = \mathcal{L}\left[f(t)\right] \tag{1.23}$$

Assuming zero initial conditions $\left[\text{i.e., } x(0) = \dot{x}(0) = 0\right]$ and defining $f(t)$ as the unit impulse, Eq. (1.23) becomes

$$(ms^2 + cs + k)y(s) = 1 \tag{1.24}$$

as

$$\mathcal{L}\left[h(\tau)\right] = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau = 1 \tag{1.25}$$

Strictly speaking, the Laplace transform of the weighting function $\left[h(\tau)\right]$ defines the transfer function of a linear system whereas the Fourier transform of the weighting function defines the frequency response function of the linear system. By requiring the Laplace operator $\left[s\right]$ to be equal to $i\omega$, Eq. (1.25) reduces to the Fourier transform of $h(\tau)$. The righthand side of Eq. (1.24) can be written as a constant with the units of force per unit spring constant. This constant is interpreted as the static deflection y_0. Equating $s = i\omega$ in Eq. (1.24) and solving for the ratio of y to y_0 yields

$$\frac{y}{y_0} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + i\,\frac{c\omega}{k}} \tag{1.26}$$

The denominator of Eq. (1.26) may be expressed in polar form as

$$1 - \left(\frac{\omega}{\omega_n}\right)^2 + i\,\frac{c\omega}{k} = \sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(\frac{c\omega}{k}\right)^2} \cdot e^{i\phi} \tag{1.27}$$

where

$$\phi = \tan^{-1} \frac{\frac{c\omega}{k}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \tag{1.28}$$

These results are noted to be equivalent to the expressions of Eq. (1.11).

A linear time-invariant system has the property that the response is the real part of $H(\omega)\,e^{i\omega t}$ when the excitation is the real part (i.e., the cosine term) of $e^{i\omega t}$. Assuming the forcing function $f(t)$ in Eq. (1.1) to be $F_0 \cos \omega t$, then $f(t)$ can be denoted as $R\left[F_0\, e^{i\omega t}\right]$ where $R[\;]$ denotes the real part of the quantity within the brackets. Using this notation, the steady state solution to Eq. (1.1) can be written as

$$y = R\left[Y(\omega)\, e^{i\omega t}\right] = R\left[\frac{F_0 e^{i(\omega t - \phi)}}{m\omega_n^2 \sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\,\frac{\omega}{\omega_n}\right]^2}}\right] \tag{1.29}$$

where

$$Y(\omega) = \frac{F_0}{Z(\omega)} = H(\omega)\, F_0 \tag{1.30}$$

and

$$Z(\omega) = m\omega_n^2 \sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\,\frac{\omega}{\omega_n}\right]^2} \cdot e^{i\phi} \tag{1.31}$$

The quantity $Z(\omega)$, which is a force to displacement ratio, is called a <u>mechanical impedance</u> of the system. This definition is noted to depart from the classical force to velocity ratio as the definition of the mechanical impedance. However, no ambiguity need arise because of this departure from convention. The reciprocal of the mechanical impedance is called the frequency response function $H(\omega)$. The magnification factor of Eq. (1.11) is readily formed from Eq. (1.29) by multiplying both the numerator and denominator by k, noting that $F_0/k = y_0$, then dividing both sides of the equation by y_0.

Given that both the applied force and system response are real quantities, an alternate expression for the steady state displacement response is

$$y(t) = \frac{1}{2}\left[Y(\omega)\, e^{i\omega t} + Y^*(\omega)\, e^{-i\omega t}\right] \tag{1.32}$$

where $Y^*(\omega)$ denotes the complex conjugate of $Y(\omega)$. If the excitation contains j harmonic components, the steady state solution may be written as

$$y(t) = R\left[\sum_j Y_j(\omega)\, e^{i\omega_j t}\right] = \frac{1}{2}\left[\sum_j Y_j(\omega)\, e^{i\omega_j t} + \sum_j Y_j^*(\omega)\, e^{-i\omega_j t}\right] \tag{1.33}$$

1.5 FOURIER TRANSFORMS AND THE CONVOLUTION INTEGRAL

If the excitation contains many closely spaced harmonic components approaching a continuous spectrum, the steady state solution may be expressed using Fourier transforms as

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)\, e^{i\omega t}\, d\omega \tag{1.34}$$

Alternatively, Eq. (1.34) may be given as

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y^*(\omega)\, e^{-i\omega t}\, d\omega \tag{1.35}$$

$Y(\omega)$ is noted to be the Fourier transform of the displacement response and the asterisk superscript denotes the conjugate of $Y(\omega)$. In terms of the properties of the mechanical system, $Y(\omega)$ appears as

$$Y(\omega) = H(\omega)\ F(\omega) \tag{1.36}$$

$$H(\omega) = \frac{1}{Z(\omega)} \tag{1.37}$$

$$F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\ f(t)\ dt \tag{1.38}$$

The displacement in time can be obtained by performing the inverse of Eq. (1.36) which appears as

$$y(t) = Y^{-1}(\omega) = \left[H(\omega) \cdot F(\omega)\right]^{-1} \tag{1.39}$$

where the negative one superscript denotes the inverse. Equation (1.39) is noted to contain the product of two Fourier transform functions. An operational property common to linear transforms is that the inverse of a product of two transform functions yields the convolution integral. For Fourier transforms, the operational property is symbolized as

$$\left[H(\omega)\ F(\omega)\right] \xrightarrow{-1} \int_0^t h(t-\tau)\ f(\tau)\ d\tau = \int_0^t h(\tau)\ f(t-\tau)\ d\tau \tag{1.40}$$

where $h(\tau)$ denotes the system response in τ to a unit impulse and $f(\tau)$ denotes the forcing function acting on the system. Eq. (1.40) thus defines a transformation from a product in the frequency domain into a convolution integral in the time domain. Other discussions pertinent to the derivation of the convolution integral are found in Reference 3, page 58 and Reference 4, page 17.

In form of the Fourier inversion integral given in Eq. (1.34), the displacement response in time can be alternately written as

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\, F(\omega)\, e^{i\omega t}\, d\omega \tag{1.41}$$

It is instructive to interpret the response given by Eq. (1.40) with that of Eq. (1.41). The convolution integral, shown as Eq. (1.40), represents the response as a linear superposition in the time domain of free vibration solutions. The Fourier transform solution, shown as Eq. (1.41), represents the response as a linear superposition in the frequency domain of steady state responses to simple harmonic excitations.

Although the Fourier transform and convolution integral differ in analytical format, the solutions obtained by both procedures must be identical. As shown in Reference 4, page 20, the two methods are related mathematically as

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\, e^{i\omega t}\, d\omega \tag{1.42}$$

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt \tag{1.43}$$

That is, the complex frequency response $H(\omega)$ and the unit impulse response $h(t)$ are Fourier transform pairs.

1.6 EXAMPLE PROBLEM

The example problem included here is intended to illustrate in detail the application of classic vibration theory and transform theory in obtaining a forced response solution for a mechanical oscillator.

Problem 1.1. The forced displacement response of a mechanical oscillator to a periodic exciting force consisting of three harmonic components.

Three approaches are illustrated in obtaining the solution to this problem: (1) classic assault on the differential equation, (2) the use of Fourier transforms and (3) the use of Laplace transforms and the convolution integral. The equation of motion for the mechanical oscillator is given by Eq. (1.2) and appears as

$$\ddot{y} + 2\zeta\,\omega_n\dot{y} + \omega_n^2 y = \frac{1}{m}\,f(t) \tag{1.44}$$

where

$$f(t) = f_0(\sin\omega_1 t + \sin\omega_2 t + \sin\omega_3 t) \tag{1.45}$$

The coefficients of the oscillator are assumed to be constants. The sinusoidal components of the forcing function are noted to be of the same amplitude (f_0) and phase, but differing in frequency. For the resulting excitation to be periodic, the frequency ratios (ω_2/ω_1) and ω_3/ω_1) need be rational numbers.

Classic Approach:

The steady state solution to Eq. (1.44) is given in form by Eq. (1.27) and may be expressed as

$$y(t) = \mathrm{Im}\left[\sum_{j=1}^{3} Y_j(\omega)\, e^{i\omega_j t}\right] \tag{1.46}$$

In expanded form, Eq. (1.46) becomes

$$y(t) = \frac{f_0}{m\omega_n^2}\left[\sum_{j=1}^{3} \frac{\sin(\omega_j t - \phi_j)}{\sqrt{\left[1-\left(\frac{\omega_j}{\omega_n}\right)^2\right]^2 + \left[2\zeta\,\frac{\omega_j}{\omega_n}\right]^2}}\right] \tag{1.47}$$

where

$$\phi = \tan^{-1}\frac{2\zeta\,\frac{\omega_j}{\omega_n}}{1-\left(\frac{\omega_j}{\omega_n}\right)^2} \tag{1.48}$$

Fourier Transforms and the Convolution Integral:

Denoting the Fourier transform by capital letters, Eq. (1.44) appears in transformed format as

$$(-\omega_j^2 + i2\zeta\ \omega_n\omega_j + \omega_n^2)\ Y(\omega_j) = \frac{1}{m}\ F(\omega_j) \tag{1.49}$$

Solving Eq. (1.49) for the transform of the displacement yields

$$Y(\omega_j) = \frac{1}{m\omega_n^2}\ H(\omega_j)\ F(\omega_j) \tag{1.50}$$

where

$$H(\omega_j) = \frac{1}{1-\left(\frac{\omega_j}{\omega_n}\right)^2 + i\ 2\zeta\frac{\omega_j}{\omega_n}} \tag{1.51}$$

$$F(\omega_j) = \frac{f_0}{2i}\left[\delta(\omega+\omega_j) - \delta(\omega-\omega_j)\right] \tag{1.52}$$

Eq. (1.51) is the frequency response function in ω_j while the delta (δ) notation in Eq. (1.52) is the classic delta function, with $i = \sqrt{-1}$.

The displacement response is obtained from Eq. (1.50) and is written as

$$y(t) = \sum_{j=1}^{3} Y^{-1}(\omega_j) = \frac{1}{m\omega_n^2} \sum_{j=1}^{3} \left[H(\omega_j) \cdot F(\omega_j) \right]^{-1} \tag{1.53}$$

where the negative superscript denotes the inverse transform. Considering only the jth term, Eq. (1.53) can be written according to Eq. (1.46) as

$$y_j(t) = \int_0^t h(\tau)\, f_j(t - \tau)\, d\tau \tag{1.54}$$

In terms of Laplace transforms, the response of the single degree-of-freedom system to a unit impulse may be expressed as

$$h(t) = \mathcal{L}^{-1}\left[h(\tau)\right] \tag{1.55}$$

where

$$\mathcal{L}\left[h(\tau)\right] = \frac{1}{m(s^2 + 2\zeta\,\omega_n s + \omega_n^2)} \tag{1.56}$$

Equation (1.55) is evaluated by making use of the following transformation

$$\frac{1}{(s-\alpha)(s-\beta)} \Longrightarrow \frac{e^{\beta t} - e^{\alpha t}}{\alpha - \beta} \tag{1.57}$$

where

$$\alpha = \left(-\zeta + i\sqrt{1 - \zeta^2}\right) \omega_n \tag{1.58}$$

$$\beta = \left(-\zeta - i\sqrt{1 - \zeta^2}\right) \omega_n \tag{1.59}$$

The Eqs. of (1.57) symbolically define a transformation from the complex domain (s) into the time domain (t). Substituting (1.58) and (1.59) into (1.57) yields

$$mh(t) = \frac{1}{\omega_n \sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \left\{ \sin \sqrt{1 - \zeta^2}\ \omega_n t \right\} \tag{1.60}$$

Equation (1.60) agrees with the results given by Crandall and Mark (Reference 3, page 64) who obtain the impulse response using an alternate approach.

Substituting the jth term of Eq. (1.45) and (1.60) into Eq. (1.54) yields

$$y_j(t) = \frac{1}{m\omega_d} \int_0^t e^{-\zeta\omega_n \tau} \sin \omega_j (t - \tau) \sin \omega_d \tau \, d\tau \tag{1.61}$$

where ω_d is the damped natural frequency and defined by Eq. (1.19).

Ignoring the transient terms after performing the integration of Eq. (1.61) provides

$$y_j(t) = \frac{f_0}{m\omega_n^2} \cdot \frac{\sin(\omega_j t - \phi_j)}{\sqrt{\left[1 - \left(\frac{\omega_j}{\omega_n}\right)^2\right]^2 + \left[2\zeta \frac{\omega_j}{\omega_n}\right]^2}} \tag{1.62}$$

where ϕ_j is the phase angle and defined by Eq. (1.48). The total displacement therefore is formed by summing the j terms as

$$y(t) = \sum_{j=1}^{3} y_j(t) \tag{1.63}$$

Laplace Transforms:

Assuming zero initial conditions ($y = \dot{y} = 0$), Eq. (1.44) appears in Laplace transform format as

$$\mathcal{L}\left[y(t)\right] = \frac{\frac{1}{m}\mathcal{L}\left[f(t)\right]}{s^2 + 2\zeta\,\omega_n s + \omega_n^2} \tag{1.64}$$

For the forcing function defined by Eq. (1.45), $f(t)$ may be written as

$$\mathcal{L}\left[f(t)\right] = f_0 \sum_{j=1}^{3} \frac{\omega_j}{s^2 + \omega_j^2} \tag{1.65}$$

In terms of the inverse transform, the displacement response is expressed as

$$y(t) = \mathcal{L}^{-1}\left[y(s)\right] \tag{1.66}$$

where

$$y(s) = \frac{f_0}{m} = \sum_{j=1}^{3} \frac{1}{s^2 + 2\zeta\,\omega_n s + \omega_n^2} \cdot \frac{\omega_j}{s^2 + \omega_j^2} \tag{1.67}$$

The transformation from the s domain into the time domain is represented symbolically as

$$\frac{1}{(s^2 + \lambda^2)(\left[s + \alpha\right]^2 + \beta^2)} \Longrightarrow$$

$$\frac{\frac{1}{\lambda}\sin(\lambda t + \phi_1) + \frac{1}{\beta}e^{-\alpha t}\sin(\beta t + \phi_2)}{\sqrt{4\alpha^2\lambda^2 + (\alpha^2 + \beta^2 - \lambda^2)^2}} \tag{1.68}$$

where

$$\phi_1 = \tan^{-1} \frac{2\alpha\lambda}{\alpha^2 + \beta^2 - \lambda^2} \tag{1.69}$$

$$\phi_2 = \tan^{-1} \frac{2\alpha\beta}{\alpha^2 - \beta^2 + \lambda^2} \tag{1.70}$$

Ignoring the transient motion, Eq. (1.68) reduces to the steady state response as shown by Eq. (1.47), and Eq. (1.48).

Rather than use the complete transformation defined by Eq. (1.68), the steady state response can be obtained directly from Eq. (1.67) by using Heaviside's expansion theorem and suppressing the transient roots. This approach yields the displacement response as

$$y(t) = \frac{f_0}{m\,\omega_n^2} \sum_{j=1}^{3} \frac{\sin \omega_j t}{1 - \left(\frac{\omega_j}{\omega_n}\right)^2 + i\,2\zeta \frac{\omega_j}{\omega_n}} \tag{1.71}$$

In polar form, Eq. (1.71) compares identically with Eqs. (1.47) and (1.48).

2. RESPONSE CHARACTERISTICS OF CONTINUOUS SYSTEMS TO DETERMINISTIC EXCITATION

2.1 INTRODUCTORY REMARKS

Modal procedures are often applied in calculating the dynamic response properties of continuous elastic structures. Since the latter part of this report assumes the reader is familiar with modal properties, it is appropriate to review modal theory and to illustrate a use of this theory in solving several beam problems. More detailed discussions of modal theory are available from Reference 5 (Chapter 9), Reference 6, and Reference 7 (Chapter 7).

Consider an arbitrary physical system which is assumed to be a uniform linear, lightly damped, continuous elastic structure excited by a forcing function dependent upon space and time. From the physics of the problem, the system may be defined as a partial differential equation of the form:

$$m(x)\ddot{y}(x, t) + c(x)\dot{y}(x, t) + D(x)\,y(x, t) = f(x, t) \qquad (2.1)$$

where

$m(x)$	denotes mass per unit length
$c(x)$	denotes a viscous damping coefficient assumed to vary as a function of x
$D(x)$	denotes a spatial differential operator whose properties are defined by the specific structure
x	denotes the distance along the length of the structure
$y(x, t)$	denotes the lateral displacement of the structure from its static equilibrium position
$\dot{y}(x, t)$	denotes the lateral velocity of the structure defined as the first time derivative of the displacement
$\ddot{y}(x, t)$	denotes the lateral acceleration of the structure defined as the second time derivative of the displacement
$f(x, t)$	denotes the forcing function acting on the structure and is assumed to be expressable as the product of the space (x) and time (t) variable.

It is desired to solve for the steady-state displacement response of Eq. (2.1) using modal theory.

While not detracting from the generality of this discussion, the assumption of uniformity defines the coefficients of Eq. (2.1) as constants and thereby decreases the complexity of the mathematics in the example problems. Structures with nonuniform properties can be represented as partial differential equations with variable coefficients but it is, in general, computationally more convenient to represent such systems by lumped parameter models. When this is done, the integral equations associated with the analysis of continuous elastic structures are replaced by summations, and matrix techniques can be applied to obtain approximate solutions.

2.2 SEPARATION OF VARIABLES

Classically, the solution to a partial differential equation similar in form to Eq. (2.1) can often be obtained by the method of separation of variables. While this method is not universally applicable, many of the problems in structural dynamics can be solved by this approach. The procedure is to first assume a solution consisting of the functional products of the space and time variables. Upon substitution, this assumed solution reduces the original <u>partial</u> differential equation to two <u>ordinary</u> differential equations, one in terms of the space variable and the other in terms of the time variable. Solving the space variable equation subject to the boundary conditions of the structure yields the resonant frequencies and associated mode shapes for the structure. Solving the time variable equation subject to the initial conditions of the structure provides a solution in terms of generalized coordinates. These generalized coordinates are mathematical definitions and may or may not have a convenient physical interpretation.

The modal technique is a separation of variables approach in which the space function represents the normal modes of the mechanical system and the time function represents generalized coordinates. Symbolically, the assumed modal solution to Eq. (2.1) may be written as

$$y(x,\ t) = \sum_{j} \phi_j(x)\, q_j(t) \tag{2.2}$$

where $\sum_{j}$ denotes the summation $j = 1, 2, 3 \ldots \infty$

$\phi_j(x)$ denotes the ith normal mode of the physical system

$q_j(t)$ denotes the jth generalized coordinate

Note that Eq. (2.2) defines a modal transformation between the generalized coordinates and the physical coordinates of the structure. In matrix format, this corresponds to premultiplying the generalized coordinate vector by the modal transformation matrix to obtain the displacement vectors in the coordinates of the structure.

Theoretically, a continuous elastic structure will have an infinite number of normal modes and an infinite number of degrees-of-freedom or generalized coordinates. In calculating the displacement response, the summation of Eq. (2.2) usually is truncated as it can be shown that the contributions of the higher modes to the total displacement response are negligible. For the response in terms of acceleration, however, the truncation rationale is somewhat more subtle as contrasted to the approach used for the displacement response.

Before considering a formal treatment of Eq. (2.1), it is well to review a most useful theoretical property associated with the mode shapes of an elastic structure. This has to do with the orthogonality of the elastic modes and suggests the analytical procedure to use in obtaining a solution to Eq. (2.1).

2.3 ORTHOGONALITY CONDITIONS

The orthogonality property of mode shapes is of concern here and is guaranteed by the rather well-known Sturm-Liouville conditions. Rigorously stated from Ref. 7, p. 233, the Sturm-Liouville theorem appropriate to a fourth-order differential equation (characteristic of beams and plates) is

GIVEN: the differential equation of the form

$$\frac{d^2\left[r(x)\phi''(x)\right]}{dx^2} + \left[q(x) + \lambda\, p(x)\right]\phi(x) = 0 \tag{2.3}$$

where: $p(x)$, $q(x)$, and $r(x)$ are continuous functions over the interval (a, b), and

$$(\ )' = \frac{d}{dx}\left(\ \right)$$

IF: (1) $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the values of the parameter for which there exist solutions of this differential equation satisfying at both a and b boundary conditions of the form

$$\begin{aligned} c_1\phi(x) &= d_1(r\phi''(x))' \\ c_2\phi'(x) &= d_2(r\phi''(x)) \end{aligned} \tag{2.4}$$

(2) $\phi_1(x), \phi_2(x), \phi_3(x) \ldots$ are the solutions corresponding to $\lambda_1, \lambda_2, \lambda_3, \ldots$

THEN: the set $\left\{\phi_j(x)\right\}$ is orthogonal with respect to the weight function $p(x)$ over the interval (a, b).

Note that the 'weight function' $p(x)$ is an arbitrary function defined from the form of Eq. (2.3) and may or may not have any relationship to the physical weight of a system. For structural problems, $p(x)$ equals the mass per unit length of the structure $m(x)$. Equation (2.4) defines the usual boundary conditions for a beam where $\phi(x)$ denotes displacement, $\phi'(x)$ denotes the slope, $\phi''(x)$ is proportional to the bending moment, and $\phi'''(x)$ is proportional to shear.

Rather than dwell upon the rigor of the S-L conditions, it suffices to say that the orthogonality property of the normal modes and the resonant frequencies corresponding to each mode are provided by the S-L theorem. Expressed in terms of structural modes, the orthogonality condition can be expressed as

$$\int_0^{\ell} m(x)\,\phi_j(x)\,\phi_k(x)\,dx = \begin{cases} 0 & , \quad j \neq k \\ \text{constant} & , \quad j = k \end{cases} \tag{2.5}$$

Mathematically, Eq. (2.5) states the normal modes are orthogonal with respect to the weighting function $m(x)$. Less formally, Eq. (2.5) shows that the integral over the length of the structure (with the integrand being the product of any two mode shapes and the mass distribution) is equal to zero if the mode shapes are different and is equal to a constant if the mode shapes are the same. Since the scale of $\phi_j(x)$ is arbitrary, the modal self terms ($j=k$) usually are scaled such that Eq. (2.5) reduces to either the total mass of the body or to unity. For the first condition, the mode shapes are said to be normalized to the total mass of the body. For the second condition, the mode shapes are said to be orthonormally scaled.

The solution to Eq. (2.3) yields the mode shapes for a given physical system. In the literature, these mode shapes also are called characteristic functions or eigenvectors. As will be seen in the example problems, the computation of the mode shapes is associated with a frequency equation (transcendental in form and functionally dependent upon λ) which yields the resonant frequencies for each mode of the physical system. This transcendental equation sometimes is called the characteristic equation of the solution. Therefore, the resonant frequencies which satisfy the characteristic equation are called the characteristic values or eigenvalues.

Physically, normal modes of a structure can be interpreted as the free vibrations of the system in the absence of damping and all external forces. Hence, if a body is distorted into one of its normal mode shapes $\phi_j(x)$, then released, the body will vibrate for all time in the $\phi_j(x)$ mode at the modal frequency ω_j. This is equivalent to the response of a modal spring-mass system in which the modal mass and modal stiffness are dependent only upon the geometry, the mass, and the stiffness distributions of the physical structure. Also, it is noted that the mode shape and the associated natural frequency are independent of amplitude.

As will be demonstrated in the beam modal solution, Eq. (2.5) provides the analytical tactics for solving Eq. (2.1). In other words, Eq. (2.1) is operated upon to make use of the orthogonality condition in writing simplified equations for the distributed elastic system.

2.4 BEAM MODAL SOLUTION

In keeping with the separation of variables approach, Eq. (2.2) is substituted into Eq. (2.1) to obtain

$$m(x) \sum_j \phi_j(x)\, \ddot{q}_j(t) + c(x) \sum_j \phi_j(x)\, \dot{q}_j(t) + EI \sum_j \frac{\partial^4 \phi_j(x)}{\partial x^4}\, q_j(t) = f(x,t) \qquad (2.6)$$

From basic mechanical vibration theory, it may be shown that for an undamped conservative elastic system at the jth natural frequency,

$$m(x)\omega_j^2\, \phi_j(x) q_j(t) = EI \frac{\partial^4 \phi_j(x)}{\partial x^4}\, q_j(t) = D(x) q_j(t) \qquad (2.7)$$

Equation (2.7) defines the jth undamped modal oscillator in the generalized coordinates for a Bernoulli-Euler beam.

By comparing Eq. (2.7) with the form of the response equation for a simple spring-mass system, the jth modal mass and stiffness are noted to be

$$M_j = m(x)\,\phi_j(x)$$
$$K_j = EI\,\frac{\partial^4 \phi_j(x)}{\partial x^4} \qquad (2.8)$$

Equation (2.7) is used to calculate the mode shapes for the elastic structure. Assuming the beam to be of constant mass and flexibility, Eq. (2.7) may be expressed in operator form as

$$(D^4 - \lambda_j^4)\,\phi_j(x) = 0 \qquad (2.9)$$

where

$$\lambda_j^4 = \frac{m}{EI}\,\omega_j^2 \qquad (2.10)$$

The solution to Eq. (2.9) yields the mode shape $\phi_j(x)$ and is given as

$$\phi_j(x) = C\cos\lambda_j x + D\sin\lambda_j x + E\cosh\lambda_j x + F\sinh\lambda_j x \qquad (2.11)$$

Applying the boundary conditions of the beam structure yields the coefficients for Eq. (2.11) as well as the appropriate frequency equation expressed in terms of $\lambda_j \ell$. In concert with Eq. (2.10), the solution to the frequency equation gives the modal frequency (ω_i) associated with the jth mode ($\phi_j[x]$) of the beam.

Substituting Eq. (2.7) into Eq. (2.6) yields

$$m(x)\sum_j \phi_j(x)\ddot{q}_j(t) + c(x)\sum_j \phi_j(x)\dot{q}_j(t) + m(x)\sum_j \omega_j^2\phi_j(x)q_j(t) = f(x,t) \qquad (2.12)$$

To make use of the orthogonality property of the modes, Eq. (2. 12) is multiplied by $\phi_k(x)$ and integrated over the complete length of the beam. Eq. (2. 12) then appears as

$$\sum_j \int_0^{\ell} \phi_j(x)\phi_k(x)m(x)dx\,\ddot{q}_j(t) + \sum_j \int_0^{\ell} \phi_j(x)\,\phi_k(x)c(x)dx\,\dot{q}_j(t)$$
$$+ \sum_j \omega_j^2 \int_0^{\ell} \phi_j(x)\phi_k(x)m(x)\,dx\,q_i(t) = \sum_j \int_0^{\ell} \phi_j(x)f(x,t)\,dx \tag{2.13}$$

In less pretentious form, Eq. (2. 13) may be expressed for the jth mode as

$$\overline{M}_j\,\ddot{q}_j(t) + \overline{C}_j\,\dot{q}_j(t) + \overline{K}_j\,q_j(t) = \overline{F}_j \tag{2.14}$$

where

$\overline{M}_j$ denotes the generalized mass in the jth mode defined for the beam as

$$\int_0^{\ell} \phi_j(x)\phi_k(x)m(x)\,dx \tag{2.15}$$

$\overline{C}_j$ denotes the generalized viscous damping in the jth mode defined for the beam as

$$\int_0^{\ell} \phi_j(x)\,\phi_k(x)c(x)\,dx \tag{2.16}$$

$\overline{K}_j$ denotes the generalized stiffness in the jth mode defined as

$$\omega_j^2 \overline{M}_j \tag{2.17}$$

ω_j denotes the jth modal frequency expressed in radians per second

$\overline{F}_j$ denotes the generalized force in the jth mode defined as

$$\int_0^{\ell} \phi_j(x) f(x, t) dx \tag{2.18}$$

The generalized ma ss, generalized damping, generalized stiffness and generalized force terms are definitions introduced to conveniently reference the equations of a multi-mode system to the equation of a single degree-of-freedom system. Assuming $c(x) = 2\zeta_j \omega_j m(x)$ reduces the generalized damping [Eq. (2.16)] to a form so that the orthogonality property can be used. As a consequence, Eq. (2.14) may be written as

$$\ddot{q}_j(t) + 2\zeta_j \omega_j \dot{q}_j(t) + \omega_j^2 q_j(t) = \frac{\overline{F}_j}{\overline{M}_j} \tag{2.19}$$

Equation (2. 19) represents an uncoupled second-order linear differential equation in q_j with constant coefficients; and the solution is readily found in most introductory texts in differential equations. The utility of the damping assumption is clear by using the orthogonality property; the computational problem now is reduced to solving n <u>independent</u> second-order differential equations in lieu of n <u>coupled</u> second-order differential equations.

To conclude the general modal discussion, it is appropriate to review the preceeding derivation. First of all, Eq. (2.2) represents the steady-state displacement response to Eq. (2.1). This response consists of contributions from an infinite number of mechanical oscillators (each oscillator responding at its modal frequency) weighted by the mode shapes of the distributed structure. These mode shapes are given by Eq. (2.11) and are dependent upon the physical properties and the boundary conditions of the structure. The magnitude of the contributions from the generalized coordinates is given by the solution to Eq. (2.19) and is dependent upon the initial conditions of the problem, the values for the generalized mass, modal damping, and generalized force. As a matter of computational sequence, the mode shapes and resonant frequencies are calculated first. As guaranteed by the Sturm-Liouville conditions, the mode shapes are orthogonal with respect to the mass per unit length $\left[m(x)\right]$ of the structure. Then, the generalized mass and generalized force must be calculated to establish the terms of Eq. (2.19). Equation (2.19), in turn, is solved to obtain explicit expressions in the generalized coordinates $q_j(t)$ at each of the modal frequencies. Having the expressions for the mode shapes and generalized coordinates at each resonant frequency, the total response is obtained by summing as specified by Eq. (2.2).

2.5 EXAMPLE PROBLEMS

In the illustrative problems which follow, simple beam theory is assumed and damping, in general, is ignored. The first assumption specifies the spatial operator D(x) while the second assumption eases the numerical effort. If desired, damping can be approximately accounted for by substituting the complex modal frequency $\omega_j^2(1 + ig)$ for ω_j^2 into the undamped solution. The "i" coefficient of the structural damping term (g) refers to the imaginary operator $\sqrt{-1}$.

Problem 2.1: The forced lateral response of a uniform cantilevered beam sinusoidally excited at the base.

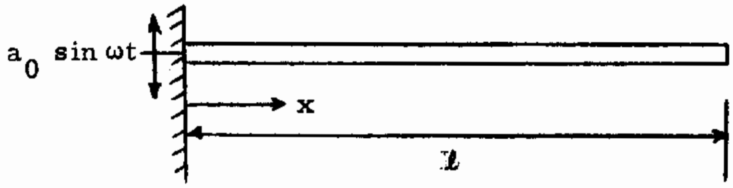


Figure 3. Cantilever Beam Excited at the Base with a Sinusoidal Forcing Function

The physical system is depicted in Figure 3 and is defined analytically as

$$EI \frac{\partial^4 y(x, t)}{\partial x^4} + m \frac{\partial^2 y(x, t)}{\partial t^2} = m\omega^2 a_0 \sin \omega t = F_0 \sin \omega t \qquad (2.20)$$

where

E =	Youngs modulus	$= lb/in^2$
I =	cross-section area moment of inertia about the bending axis	$= in^4$

m	=	beam mass per unit length	= lb-sec^2/in^2
a_0	=	maximum amplitude of the sinusoidal excitation	= in
ω	=	frequency of the excitation	= radian/sec
y	=	lateral displacement at the beam from its static equilibrium position	= in
x	=	distance along the length of the beam	= in
t	=	real time	= sec

The physical properties of the beam (I and m) are assumed as constants. It is desired to calculate the displacement response $y(x,t)$ of the cantilevered beam excited at the root by a sinusoidally varying displacement of amplitude a_o. Analytically, it is required to solve Eq. (2.20) for $y(x,t)$ subject to the boundary conditions for the cantilevered beam.

According to modal theory, the solution to Eq. (2.20) is

$$y(x, t) = \sum_j \phi_j(x)\, q_j(t) \tag{2.21}$$

The jth mode shape is given by

$$\omega_j(x) = C \cos\lambda_j x + D \sin\lambda_j x + E \cosh\lambda_j x + F \sinh\lambda_j x \tag{2.22}$$

where

$$(\lambda_j \ell)^4 = \frac{m\ell^4}{EI}\, \omega_j^{\,2} \tag{2.23}$$

The jth generalized coordinate is given by the solution to

$$\ddot{q}_j(t) + \omega_j^2\, q_j(t) = \frac{\overline{F}_j}{\overline{M}_j} \tag{2.24}$$

where

$$\overline{F}_j = \int_0^{\ell} \phi_j(x)\, f(x,t)\, dx \tag{2.25}$$

$$\overline{M}_j = \int_0^{\ell} m\, \phi_j(x)\, \phi_k(x)\, dx \tag{2.26}$$

Eq. (2.25) defines the generalized force ($\overline{F}_j$) and Eq. (2.26) defines the generalized mass ($\overline{M}_j$).

Mode Shapes:

The mode shapes for the cantilevered mass are given by Eq. (2.22) subject to the following boundary conditions

$$x = 0 \begin{cases} y(0,t) = 0 \\ \dfrac{\partial y(0,t)}{\partial x} = 0 \end{cases} \qquad \left. \begin{aligned} EI\,\frac{\partial^2 y(\ell,t)}{\partial x^2} &= 0 \\ EI\,\frac{\partial^3 y(\ell,t)}{\partial x^3} &= 0 \end{aligned} \right\} x = \ell \tag{2.27}$$

Expression (2.27) states the relative deflection $\left[y(0,t)\right]$ and the slope $\left[\partial y(0,t)/\partial x\right]$ are both zero at the root of the beam; and at the free end, the bending moment $\left[EI\ \partial^2 y(\ell,t)/\partial x^2\right]$ and shear load $\left[EI\ \partial^3 y(\ell,t)/\partial x^3\right]$ are both equal to zero. Imposing the statements of (2.27) on Eq. (2.22) yields

$$\phi_j(0) = 0 = C + E \tag{2.28}$$

$$\frac{\partial \phi_j(0)}{\partial x} = 0 = D + F \tag{2.29}$$

$$\frac{\partial^2 \phi_j(\ell)}{\partial x^2} = 0 = -C \cos\lambda_j\ell - D \sin\lambda_j\ell + E \cosh\lambda_j\ell + F \sinh\lambda_j\ell \tag{2.30}$$

$$\frac{\partial^3 \phi_j(\ell)}{\partial x^3} = 0 = C \sin\lambda_j\ell - D \cos\lambda_j\ell + E \sinh\lambda_j\ell + F \cosh\lambda_j\ell \tag{2.31}$$

Substituting Eqs. (2.28) and (2.29) into Eqs. (2.30) and (2.31) gives

$$\begin{bmatrix} (\cos\lambda_j\ell + \cosh\lambda_j\ell) & (\sin\lambda_j\ell + \sinh\lambda_j\ell) \\ (-\sin\lambda_j\ell + \sinh\lambda_j\ell) & (\cos\lambda_j\ell + \cosh\lambda_j\ell) \end{bmatrix} \begin{Bmatrix} E \\ F \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2.32}$$

In demanding a nontrivial solution to Eq. (2.32), the determinant of the 2 x 2 matrix is set equal to zero and the resulting frequency equation is

$$\cos\lambda_j\ell \cdot \cosh\lambda_j\ell + 1 = 0 \tag{2.33}$$

Eq. (2.33) is a transcendental equation in $\lambda_j\ell$ and the solutions (in concert with Eq. [2.23]) yield the modal frequencies of the beam. These solutions are obtained by solving directly for the arguments of the frequency equation and also are found in standard vibration texts (see Reference 2, pg. 217, or Reference 8). The solutions for the first six modes are given as

j	$\lambda_j \ell$
1	1.875
2	4.694
3	7,855
4	10.996
5	14.137
6	17.279

(2.34)

Substituting Eqs. (2.25) and (2.29) into Eq. (2.22) allows the mode shape to be written as

$$\phi_j(x) = E\left\{(-\cos\lambda_j x + \cosh\lambda_j x) + \frac{F}{E}(-\sin\lambda_j x + \sinh\lambda_j x)\right\} \tag{2.35}$$

F/E is obtained from Eq. (2.32) and appears as

$$\frac{F}{E} \equiv \alpha_j = \frac{\cos\lambda_j \ell + \cosh\lambda_j \ell}{\sin\lambda_j \ell + \sinh\lambda_j \ell} \tag{2.36}$$

Arbitrarily setting E equal to one yields the mode shape for the uniform cantilever beam as

$$\phi_j(x) = \cosh\lambda_j x - \cos\lambda_j x - \alpha_j(\sinh\lambda_j x - \sin\lambda_j x) \tag{2.37}$$

The modal frequencies corresponding to the mode shapes of (2.37) are given by Eq. (2.23) where the $\lambda_j \ell$ values are specified by (2.34).

Generalized Coordinates:

The generalized coordinates are given by the solutions to Eq. (2.24). For the cantilever beam, the generalized mass is given as

$$\overline{M}_j = m\int_0^{\ell} \phi_j(x)\,\phi_k(x)\,dx = \begin{cases} 0, & j \neq k \\ m\ell, & j \neq k \end{cases} \tag{2.38}$$

In form, Eq. (2.38) conforms to the orthogonality conditions defined by Eq. (2.5).

The generalized force ($\overline{F}_j$) appears as

$$\overline{F}_j = \int_0^{\ell} \phi_j(x) f(x,t)\,dx = \int_0^{\ell} \phi_j(x) F_0 \sin\omega t\,dx = 2F_0 \frac{\alpha_j}{\lambda_j} \sin\omega t \tag{2.39}$$

where

$$f(x,t) = F_0 \sin\omega t = m\omega^2 a_0 \sin\omega t \tag{2.40}$$

For the jth mode, therefore, Eq. (2.24) resolves into

$$\ddot{q}_j(t) + \omega_j^2 q_j(t) = \frac{2a_0}{\ell}\,\omega^2 \frac{\alpha_j}{\lambda_j} \sin\omega t \tag{2.41}$$

The steady state solution to (2.41) is written as

$$q_j(t) = 2a_0 \frac{\alpha_j}{\lambda_j \ell}\left(\frac{\omega}{\omega_j}\right)^2 \cdot \frac{1}{1-\left(\frac{\omega}{\omega_j}\right)^2} \sin\omega t \tag{2.42}$$

Displacement Response:

The derived displacement response is formed by substituting Eqs. (2.37) and (2.42) into Eq. (2.21) and appears as

$$y(x,t) = \sum_j \phi_j(x)\, q_j(t)$$

$$= 2a_0 \sum_{j=1,2,3,\ldots}^{\infty} \left(\frac{\omega}{\omega_j}\right)^2 \cdot \frac{1}{1-\left(\frac{\omega}{\omega_j}\right)^2} \cdot \frac{\alpha_j}{\lambda_j \ell}\, \phi_j(x) \sin \omega t \qquad (2.43)$$

α_j is defined by Eq. (2.36); $\lambda_j \ell$ is given by Eq. (2.23); and $\phi_j(x)$ is specified by Eq. (2.37).

Damping:

It is noted that damping is introduced into the solution by the formulation of the differential equation in the generalized coordinate $\left[q_j(t)\right]$. This differential equation is given by Eq. (2.19) and appears as

$$\ddot{q}_j(t) + 2\zeta_j \omega_j \dot{q}_j(t) + \omega_j^2 q_j(t) = \frac{\overline{F}_j}{\overline{M}_j} \qquad (2.44)$$

The steady state solution to Eq. (2.44) is given as

$$q_j(t) = \frac{1}{\omega_j^2 - \omega^2 + i\, 2\zeta_j \omega_j \omega} \cdot \frac{\overline{F}_j(x)}{\overline{M}_j} \quad \sin \omega t \qquad (2.45)$$

where

$$\overline{F}_j = \overline{F}_j(x) \sin \omega t \qquad (2.46)$$

The absolute magnitude of the displacement in the generalized coordinate may be written as

$$\left|q_j\right| = \frac{1}{\omega_j^2\left\{\left(1-\left[\frac{\omega}{\omega_j}\right]^2\right)^2 + \left(2\zeta_j \frac{\omega}{\omega_j}\right)^2\right\}^{\frac{1}{2}}} \cdot \frac{\overline{F}_j(x)}{\overline{M}_j} \qquad (2.47)$$

$\overline{F}_j(x)$ is the spatial function of the generalized force and is resolved for the cantilevered beam as (see Eq. (2. 39)),

$$\overline{F}_j(x) = 2m\,\omega^2 a_0 \frac{\alpha_j}{\lambda_j} \tag{2. 48}$$

The steady state solution in the generalized coordinates with no damping may be expressed as

$$q_j(t) = \frac{1}{\omega_j^2 - \omega^2} \frac{\overline{F}_j(x)}{\overline{M}_j} \sin \omega t \tag{2. 49}$$

Equation (2. 49) evaluated for the base excited cantilevered beam is given by Eq. (2. 42). Accounting for structural damping by substituting $\omega_j^2(1+ig)$ for ω_j^2 in Eq. (2. 49) yields

$$\left|q_j\right| \simeq \frac{1}{\left|\omega_j^2(1+ig) - \omega^2\right|} \cdot \frac{\overline{F}_j(x)}{\overline{M}_j} \tag{2. 50}$$

or

$$\left|q_j\right| \simeq \frac{1}{\omega_j^2\left\{\left(1 - \left[\frac{\omega}{\omega_j}\right]^2\right)^2 + g^2\right\}^{\frac{1}{2}}} \cdot \frac{\overline{F}_j(x)}{\overline{M}_j} \tag{2. 51}$$

Noting the definition that $2\zeta = g$ (see Eq. (1. 10)), Eqs. (2. 47) and (2. 51) are seen to be identical at resonance. For frequency values other than at resonance, Eq. (2. 41) and Eq. (2. 51) are noted to differ in magnitude for the radical ($\sqrt{\;}$) in the denominator. However, this difference is numerically negligible for small values of damping.

The displacement response for the base excited cantilevered beam including damping can be written as

$$y(x,t) = \sum_j \phi_j(x)\, q_j(t) = \sum_{j=1,2,3,\ldots}^{\infty} \frac{\phi_j(x)}{\omega_j^2 - \omega^2 + i2\zeta_j\, \omega_j \omega} \; \frac{\overline{F}_j(x)}{\overline{M}_j} \sin \omega t \qquad (2.52)$$

The absolute magnitude of the response may be written as

$$\left| y(x) \right| = \sum_{i=1,2,3\ldots}^{\infty} \frac{1}{\omega_j^2 \left\{ \left(1 - \left[\frac{\omega}{\omega_j} \right]^2 \right)^2 + \left(2\zeta_j \frac{\omega}{\omega_j} \right)^2 \right\}^{\frac{1}{2}}} \phi_j(x) \frac{\overline{F}_j(x)}{\overline{M}_j} \qquad (2.53)$$

In a form compatible with Eq. (2. 43), Eq. (2. 53) may be expressed as

$$\left| y(x) \right| = 2a_0 \sum_{j=1,2,3\ldots}^{\infty} \frac{\left(\frac{\omega}{\omega_j} \right)^2}{\left\{ \left(1 - \left[\frac{\omega}{\omega_j} \right]^2 \right)^2 + \left(2\zeta_j \frac{\omega}{\omega_j} \right)^2 \right\}^{\frac{1}{2}}} \cdot \frac{\alpha_j}{\lambda_j \ell} \phi_j(x) \qquad (2.54)$$

<u>Problem 2. 2:</u> The forced lateral response of a simply supported beam excited by a uniform loading distributed over the length of the beam

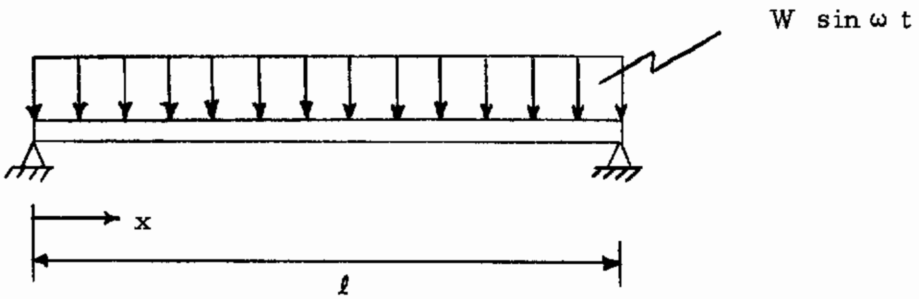


Figure 4. Simply Supported Beam with Loading Acting Over the Beam Length

The physical system is shown in Figure 4 and is represented analytically as

$$EI \frac{\partial^4 y(x,t)}{\partial x^4} + m \frac{\partial^2 y(x,t)}{\partial t^2} = W \sin \omega t \qquad (2.55)$$

The dimensions of the symbols used in Eq. (2.55) are the same as those for the cantilevered beam with W being the magnitude of the loading per unit length.

The solution in modal theory for the problem is outlined by Eqs. (2.21) through (2.26).

Mode Shapes:

The mode shapes for the simply supported beam are given by Eq. (2.22) subject to the following boundary conditions

$$x = 0 \begin{cases} x(0,t) = 0 \\ EI \dfrac{\partial^2 y}{\partial x^2}(0,t) = 0 \end{cases} \qquad \begin{rcases} y(\ell,t) = 0 \\ EI \dfrac{\partial y^2(\ell,t)}{\partial x^2} = 0 \end{rcases} x = \ell \tag{2.56}$$

Equation (2.56) states that the deflection (y) and the moment $\left(EI \dfrac{\partial^2 y}{\partial x^2}\right)$ at the ends of the beam are zero. Imposing the conditions of Eq. (2.56) on Eq. (2.22) yields

$$\phi_j(0) = 0 = C + E \tag{2.57}$$

$$\frac{\partial^2 \phi_j(0)}{\partial x^2} = 0 = -C + E \tag{2.58}$$

$$\phi_j(\ell) = 0 = C\cos\lambda_j\ell + D\sin\lambda_j\ell + E\cosh\lambda_j\ell + F\sinh\lambda_j\ell \tag{2.59}$$

$$\frac{\partial^2 \phi_j(\ell)}{\partial x^2} = 0 = -C\cos\lambda_j\ell - D\sin\lambda_j\ell + E\cosh\lambda_j\ell + F\sinh\lambda_j\ell \tag{2.60}$$

Equations (2.57) and (2.58) yield $C = E = 0$. Adding Equations (2.59) and (2.60) provide

$$2F \sinh \lambda_j \ell = 0 \tag{2.61}$$

Since $\sinh \lambda_j \ell$ is zero only at $\ell = 0$, the value of F (a constant coefficient) satisfying Eq. (2.61) is $F = 0$.

Subtracting Eq. (2.59) from Eq. (2.60) yields

$$2D \sin \lambda_j \ell = 0 \tag{2.62}$$

Since $D = 0$ would yield a trivial solution for $\phi_j(x)$, the solution to Eq. (2.62) must satisfy

$$\sin \lambda_j \ell = 0 \tag{2.63}$$

Eq. (2.63) is the frequency equation for the simply supported uniform beam and the solutions are given as

$$\lambda_j \ell = j\pi, \quad j = 1, 2, 3 \cdots \tag{2.64}$$

Statement (2.64) together with Eq. (2.23) define the modal frequencies where the mode shapes are given by

$$\phi_j(x) = \sin \lambda_j x = \sin \frac{j\pi}{\ell} x \tag{2.65}$$

Generalized Coordinates:

The expression for the generalized coordinate $(q_j[t])$ is given by Eq. (2.49) and appears as

$$q_j(t) = \frac{1}{\omega_j^2 - \omega^2} \frac{\bar{F}_j}{\bar{M}_j} \tag{2.66}$$

For the uniform simply supported beam, the generalized mass is

$$\overline{M}_j = m\int_0^{\ell} \phi_j(x)\,\phi_k(x)\,dx = m\int_0^{\ell} \sin\lambda_j x \cdot \sin\lambda_k x\,dx = \begin{cases} 0 & ; \quad j \neq k \\ \dfrac{m\ell}{2} & ; \quad j = k \end{cases} \tag{2.67}$$

Likewise, the generalized force is

$$\overline{F}_j = \int_0^{\ell} \phi_j(x)\, f(x,t)\,dx = W\int_0^{\ell} \sin\ \lambda_j x \sin\omega t\,dx \tag{2.68}$$

Carrying out the integration of Eq. (2.68) yields

$$\overline{F}_j = \frac{2W}{\lambda_j} \sin\omega t \qquad j = 1, 3, 5 \cdots \tag{2.69}$$

Substituting Eqs. (2.67) and (2.69) into Eq. (2.66) provides

$$q_j(t) = \frac{4W}{m} \cdot \frac{1}{\lambda_j \ell} \cdot \frac{1}{\omega_j^2 - \omega^2} \sin\omega t \tag{2.70}$$

System Response:

Using Eqs. (2.65) and (2.70),the desired displacement response for the uniform simply supported beam may be written as

$$y(x,t) = \sum_j \phi_j(x)\, q_j(t) = \frac{4W}{m} \sum_{j=1,3,5\ldots}^{\infty} \frac{1}{\lambda_j \ell} \cdot \frac{1}{\omega_j^2 - \omega^2} \sin\lambda_j x \sin\omega t \tag{2.71}$$

Using Eqs. (2.23) and (2.64), the displacement response given by Eq. (2.71) may be restated as

$$y(x,t) = \frac{4W\ell^4}{EI\pi^5} \sum_{j=1,3,5\ldots}^{\infty} \frac{1}{j^5\left[1 - \left(\dfrac{\omega}{\omega_j}\right)^2\right]} \sin\frac{j\pi x}{\ell} \sin\omega t \tag{2.72}$$

From an inspection of Figure 4 the loading is seen to be symmetric about the midspan of the beam and, consequently, can excite only odd numbered elastic modes. The formal solutions substantiate this statement as the system response is noted to depend only on the odd numbered modes.

By inspection of Eq. (2.71) and (2.72), the following terms can be defined as a dynamic modal magnification factor for the simply supported beam

$$(DMF)_j = \frac{4W}{m} \cdot \frac{1}{\lambda_j \ell} \cdot \frac{1}{\omega_j^2 - \omega^2} = \frac{4W\ell^4}{EI(j\pi)^5} \cdot \frac{1}{\omega_j^2 - \omega^2} \tag{2.73}$$

As expected for the beam without damping, Eq. (2.73) is unbounded ($\rightarrow \infty$) at resonance ($\omega = \omega_j$). Finite response values will be obtained at resonance, however, if damping is included in the solution.

<u>Problem 2.3:</u> The forced lateral response of a simply supported beam excited by a uniform loading distributed over a partial length of the beam

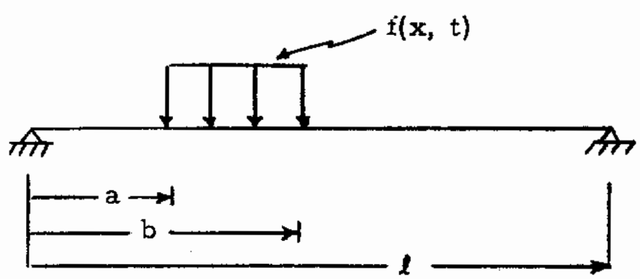


Figure 5. Simply Supported Elastic Beam Subjected to a Partially Distributed Uniform Loading

This problem is intended to illustrate the calculation of the generalized force for a partially distributed uniform loading. Ignoring damping, the partial differential equation defining this problem is given as

$$EI \frac{\partial^4 y(x,t)}{\partial x^4} + m\ddot{y}(x,t) = f(x,t) \tag{2.74}$$

where

$$f(x,t) = W \sin \omega t \tag{2.75}$$

W is a constant for this problem with the dimensions of force per unit length and defined over the (b-a) length of the beam. The beam is assumed to have uniform physical properties and is noted to have simply supported boundary conditions. As with Problem 2.2, the displacement response in modal theory is outlined by Eqs. (2.21) through (2.26).

Mode Shapes:

The mode shapes for the simply supported beam are calculated in Problem 2.2 and are expressed here as

$$\phi_j(x) = \sqrt{2} \sin \lambda_j x = \sqrt{2} \sin \frac{j\pi x}{\ell} \tag{2.76}$$

The resonant frequencies associated with the elastic modes of Eq. (2.76) are given by

$$(\lambda_j \ell)^4 = \frac{m\ell^4}{EI} \omega_j^2 \tag{2.77}$$

where

$$\lambda_j \ell = j\pi \quad , \quad j = 1, 2, 3 \cdots \tag{2.78}$$

Equation (2.78) is obtained from the solution to the frequency equation (see Eq. (2.63)) for this beam. It is noted that the amplitude of the mode shapes is assumed equal to $\sqrt{2}$ instead of unity as in Eq. (2.65) This is done to resolve the generalized mass to the mass of the beam.

Generalized Coordinates:

As with Problem 2.2, the jth generalized coordinate for the undamped beam is given as

$$q_j(t) = \frac{1}{\omega_j^2 - \omega^2} \frac{\overline{F}_j}{\overline{M}_j} \tag{2.79}$$

The generalized mass is

$$\overline{M}_j = 2m \int_0^{\ell} \sin \lambda_j x \sin \lambda_k x \, dx = \begin{cases} 0 \;, & j \neq k \\ m\ell \;, & j = k \end{cases} \tag{2.80}$$

Similarly, the generalized force is

$$\overline{F}_j = W\sqrt{2} \int_a^b \sin \lambda_j x \sin \omega t \, dx \tag{2.81}$$

Carrying out the integration of (2.81) yields

$$\overline{F}_j = \frac{2\sqrt{2}\ W}{\lambda_j} \left[\sin (b+a) \frac{\lambda_j}{2} \cdot \sin(b-a) \frac{\lambda_j}{2} \right] \sin \omega t \tag{2.82}$$

In alternate form, Eq. (2.82) may be stated as

$$\overline{F}_j = \frac{2\sqrt{2}\ W\ell}{j\pi} \left[\sin(b+a) \frac{j\pi}{2\ell} \cdot \sin(b-a) \frac{j\pi}{2\ell} \right] \sin \omega t \tag{2.83}$$

Substituting Eqs. (2.80) and (2.82) into (2.79) yields the jth generalized coordinate as

$$q_j(t) = \frac{1}{\omega_j^2 - \omega^2} \frac{2\sqrt{2}\ W}{m\lambda_j \ell} \left[\sin(b+a) \frac{\lambda_j}{2} \cdot \sin(b-a) \frac{\lambda_j}{2} \right] \sin \omega t \tag{2.84}$$

Displacement Response:

The displacement response, therefore, can be written as

$$y(x, t) = \sum_j \phi_j(x)\, q_j(t) = \frac{4W}{m} \sum_{j=1,2,3,\ldots}^{\infty} \frac{\sin \lambda_j x}{\lambda_j (\omega_j^2 - \omega^2)} \left(\sin (b+a) \frac{\lambda_j}{2} \cdot \sin (b-a) \frac{\lambda_j}{2} \right) \sin \omega t \tag{2.85}$$

<u>Problem 2.4:</u> The forced lateral response of a simply supported beam excited by a point loading

This problem is essentially the same as Problem 2.3 except that the beam loading is a concentrated force of magnitude (P) acting on the simply supported beam at position a. The mode shapes are given by Eq. (2.76) and the generalized force is found by taking the limit ($b \rightarrow a$) of Eq. (2.82) and is expressed as

$$\begin{aligned} \overline{F}_j &= \lim_{b \to a} \frac{2\sqrt{2}\; W}{\lambda_j} \left[\sin (b+a) \frac{\lambda_j}{2} \cdot \sin (b-a) \frac{\lambda_j}{2} \right] \\ &= \sqrt{2}\; W \cdot \frac{2}{\lambda_j} \left[(b-a) \frac{\lambda_j}{2} \cdot \sin \lambda_j a \right] \end{aligned} \tag{2.86}$$

Hence,

$$\overline{F}_j = \sqrt{2}\; W (b-a) \sin \lambda_j a = \sqrt{2}\; P \sin \lambda_j a \tag{2.87}$$

where $W(b-a)$ denotes the total load (P) acting on the beam between positions a and b.

The steady state displacement response appears as

$$y(x, t) = \frac{2P}{m\ell} \sum_{j=1,2,3,\ldots}^{\infty} \frac{1}{\omega_j^2 \left(1 - \left[\frac{\omega}{\omega_j} \right]^2 \right)} \sin \lambda_j a \sin \lambda_j x \sin \omega t \tag{2.88}$$

Using Eq. (2.77), the displacement response can be expressed as

$$y(x, t) = \frac{2P\ell^3}{EI\pi^4} \sum_{j=1, 2, 3, \ldots}^{\infty} \frac{1}{j^4} \cdot \frac{1}{1 - \left[\frac{\omega}{\omega_j}\right]^2} \sin \frac{j\pi a}{\ell} \sin \frac{j\pi x}{\ell} \sin \omega t \quad (2.89)$$

In both Eq. (2.88) and Eq. (2.89), the summations vary incrementally as 1, 2, 3, . . . , ∞. These equations imply that, in general, all modes contribute to the total response. If the load (P) were placed at the center of the beam ($a = \ell/2$), the displacement response reduces to

$$y(x, t) = \frac{2P\ell^3}{EI\pi^4} \sum_{j=1, 3, 5, \ldots}^{\infty} \frac{1}{j^4} \cdot \frac{1}{1 - \left[\frac{\omega}{\omega_j}\right]^2} \sin \frac{j\pi x}{\ell} \sin \omega t \quad (2.90)$$

As with Problem 2.3, the response consists of contributions only from the symmetrical modes (j = 1, 3, 5, . . .) of the structure.

Problem 2.5: The forced lateral response of a simply supported beam excited by a point loading moving over the length of the beam with a constant velocity v.

This problem is depicted by the sketch of Figure 6.

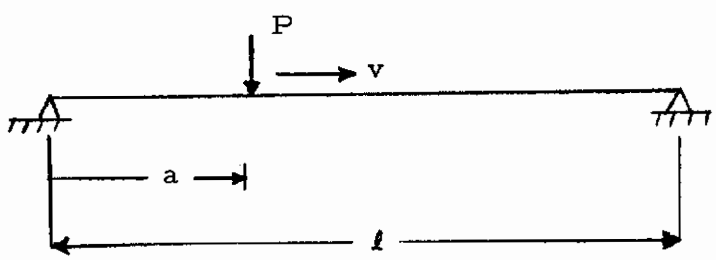


Figure 6. Simply Supported Elastic Beam with a Point Loading Moving Over the Length of the Beam with Velocity v

The solution to this problem rests with evaluating the generalized force as the mode shapes are given by Eq. (2.76). The spatial part of the generalized force for a point loading at position 'a' is given by Eq. (2.87) as

$$\overline{F}_j(x) = \int_0^{\ell} \phi_j(x)\, f(x)\, dx = \sqrt{2}\; P \sin \frac{j\pi a}{\ell} \tag{2.91}$$

For this problem

$$a = vt \tag{2.92}$$

so

$$\sin \frac{j\pi a}{\ell} = \sin \beta_0 t \tag{2.93}$$

where

$$\beta_j = \frac{j\pi v}{\ell} \tag{2.94}$$

The solution for the jth generalized coordinate is given by Eq. (2.79) and appears here as

$$q_j = \frac{1}{\omega_j^2 - \beta_j^2} \sqrt{2} \;\frac{P}{m\ell} \;\sin \beta_j t \tag{2.95}$$

The displacement response becomes

$$y(x,t) = \sum_j \phi_j(x)\, q_j(t) = \sum_{j=1,3,5,\ldots}^{\infty} \frac{1}{\omega_j^2 \left[1 - \left(\frac{\beta_j}{\omega_j}\right)^2\right]} \cdot \frac{2P}{m\ell} \cdot \sin \frac{j\pi x}{\ell} \sin \beta_j t \tag{2.96}$$

In alternate form, Eq. (2.96) may be written as

$$y(x,t) = \frac{2P\ell^3}{EI\pi^4} \sum_{j=1,2,3,\ldots}^{\infty} \frac{1}{j^4} \cdot \frac{1}{1 - \left(\frac{\beta_j}{\omega_j}\right)^2} \;\sin \frac{j\pi x}{\ell} \;\;\sin \beta_j t \tag{2.97}$$

Problem 2.6: Simply supported elastic beam with a point loading at position 'a' and decaying exponentially in time.

Given the mode shapes of Eq. (2.76), this solution also rests with evaluating the expression for the generalized force. The generalized force is written as

$$\overline{F}_j = \int_0^{\ell} \phi_j(x)\, f(x,t)\, dx = \sqrt{2}\; P e^{-\alpha t} \sin\frac{j\pi a}{\ell} \tag{2.98}$$

The oscillator equation in the generalized coordinates is given by Eq. (2.24) and may be written as

$$\ddot{q}_j + \omega_j^2 q_j = \sqrt{2}\; P e^{-\alpha t} \sin\frac{j\pi a}{\ell} \tag{2.99}$$

The jth generalized coordinate is the solution to Eq. (2.99) and appears as

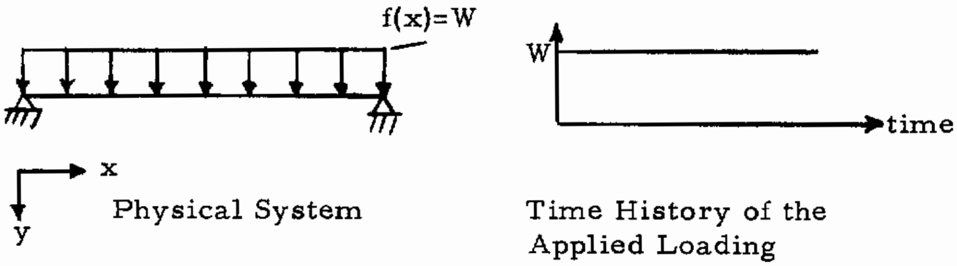
$$q_j = \frac{1}{\omega_j^2 + \alpha^2} \sqrt{2}\; P e^{-\alpha t} \sin\frac{j\pi a}{\ell} \tag{2.100}$$

The displacement response becomes

$$y(x,t) = \sum_j \phi_j(x)\, q_j(t) = \frac{2P\ell^3}{EI\pi^4} \sum_{j=1,2,3,\ldots}^{\infty} \frac{1}{j^4} \cdot \frac{e^{-\alpha t}}{1 + \left(\frac{\alpha}{\omega_j}\right)^2} \sin\frac{j\pi a}{\ell} \sin\frac{j\pi x}{\ell} \tag{2.101}$$

As contrasted with the other displacement solutions for the simply supported beam, this response is noted to decay exponentially in time as opposed to varying sinusoidally in time. The denominator $j^4\left[1 + (\alpha/\omega_j)^2\right]$ of the amplitude factor is noted to be finite as α is defined as a positive number for the physical conditions of this problem. Hence, for the classic conditions of resonance ($\alpha = \omega_j$), the displacement response is bounded even though structural damping is ignored. It is concluded, therefore, that the structure cannot be excited into resonance by the given exponential loading.

Problem 2.7: The forced lateral response of a simply supported beam excited by a uniformly distributed loading applied (in time) as a step function



Physical System

Time History of the Applied Loading

Figure 7. Simply Supported Elastic Beam with a Uniform Loading Applied as a Step Function

The physical system is a uniform elastic beam with simply supported boundary conditions. The loading of magnitude W pounds per unit length is uniformly distributed over the length of the beam and is applied in time as a step function. The beam is assumed to be initially at rest prior to the application of the loading. It is desired to calculate the displacement response.

Using modal theory, the solution may be written as

$$y(x, t) = \sum_j \phi_j(x)\, q_j(t) \tag{2.102}$$

Imposing the boundary conditions on Eq. (2.11), yields the mode shapes $\phi_j(x)$ of the frequency equation as [see problem 2.2]

$$\phi_j(x) = \sin \lambda_j x \tag{2.103}$$

$$\sin \lambda_j \ell = j\pi \quad ; \quad j = 1, 2, 3, \ldots \tag{2.104}$$

The modal frequency corresponding to the jth mode shape is

$$(\lambda_j \ell)^4 = \frac{m\ell^4}{EI} \omega_j^2 \qquad (2.105)$$

The jth generalized coordinate corresponding to the jth mode shape is given by Eq. (2.24) and appears as

$$\ddot{q}_j(t) + \omega_j^2 q_j(t) = \frac{\overline{F}_j}{\overline{M}_j} \qquad (2.106)$$

This is noted as a linear differential equation for an undamped oscillator with the generalized mass given as

$$\overline{M}_j = \frac{m\ell}{2} \qquad (2.107)$$

and the generalized force defined as

$$\overline{F}_j = W \int_0^{\ell} \sin \lambda_j x \, dx \cdot f(t) = \frac{2W}{\lambda_j} f(t) \ , \ j = 1, 3, 5, \ldots \qquad (2.108)$$

Due to the uniform spatial distribution of the loading, the even numbered modes cannot be excited and hence cannot contribute to the displacement response of the system.

Substituting Eqs. (2.107) and (2.108) into Eq. (2.106) yields

$$\ddot{q}_j(t) + \omega_j^2 q_j(t) = \frac{4W}{m} \cdot \frac{1}{\lambda_j \ell} \cdot f(t) \qquad (2.109)$$

f(t) is defined as a step function and it becomes convenient to solve Eq. (2.109) using Laplace transforms. Given the initial conditions

$$y(x, 0) = \dot{y}(x, 0) = 0 \qquad (2.110)$$

Equation (2.109) appears in transformed form as

$$\mathscr{L}\left(q_j[t]\right) = \frac{4W}{m} \cdot \frac{1}{\lambda_j \ell} \cdot \frac{1}{s(s^2 + \omega_j^2)} \tag{2.111}$$

Taking the inverse of Eq. (2.111) yields

$$q_j(t) = \frac{4W}{m\lambda_j \ell} \cdot \frac{1}{\omega_j^2} \left[1 - \cos \omega_j t\right] \tag{2.112}$$

Substituting Eqs. (2.108) and (2.112) into Eq. (2.102) gives the displacement response as

$$y(x, t) = \frac{4W}{m} \sum_{j=1, 3, 5, \dots}^{\infty} \frac{\sin \lambda_j x}{\lambda_j \ell} \cdot \frac{1}{\omega_j^2} \left[1 - \cos \omega_j t\right] \tag{2.113}$$

3. RESPONSE CHARACTERISTICS OF A SINGLE DEGREE-OF-FREEDOM SYSTEM TO RANDOM EXCITATION

3.1 DESCRIPTION OF RANDOM EXCITATION

The response of any linear mechanical system to random excitation cannot be defined explicitly as a function to time; i.e., no equation for the response time history can be written as is possible for a deterministic forcing function. The response characteristics must be defined in terms of the statistical properties of a random process. The stochastic properties of a Gaussian process are described completely by the first two statistical moments. Moreover, any realizable linear operation on a Gaussian random process yields another Gaussian random process. For random processes other than Gaussian, more than the first two statistical moments are required to define the stochastic properties; and, linear operations usually change the random process. In contrast to a Gaussian process, the mathematics for non-Gaussian processes are generally much more complicated and complex.

For many practical problems, the random loadings acting on a structure can be described as a Gaussian process. If the structure is a linear time-invariant mechanical system, then the resulting response also is a Gaussian random process. It is helpful, therefore, to review those basic statistical quantities that characterize the structural response of linear, time invariant mechanical systems to Gaussian random excitation.

3.1.1 Random Process Properties

A simple observed time history record of any random physical phenomenon constitutes only one possible outcome from an infinitely large number of time history records which might have occurred. This collection of all possible records (called the ensemble) which might have occured forms a random process which can be used to describe the random phenomenon of interest. A random process is illustrated in Figure 8. For example, each time history in Figure 8 could represent the acceleration or stress time history measured at a distinct point on a flight vehicle structure.

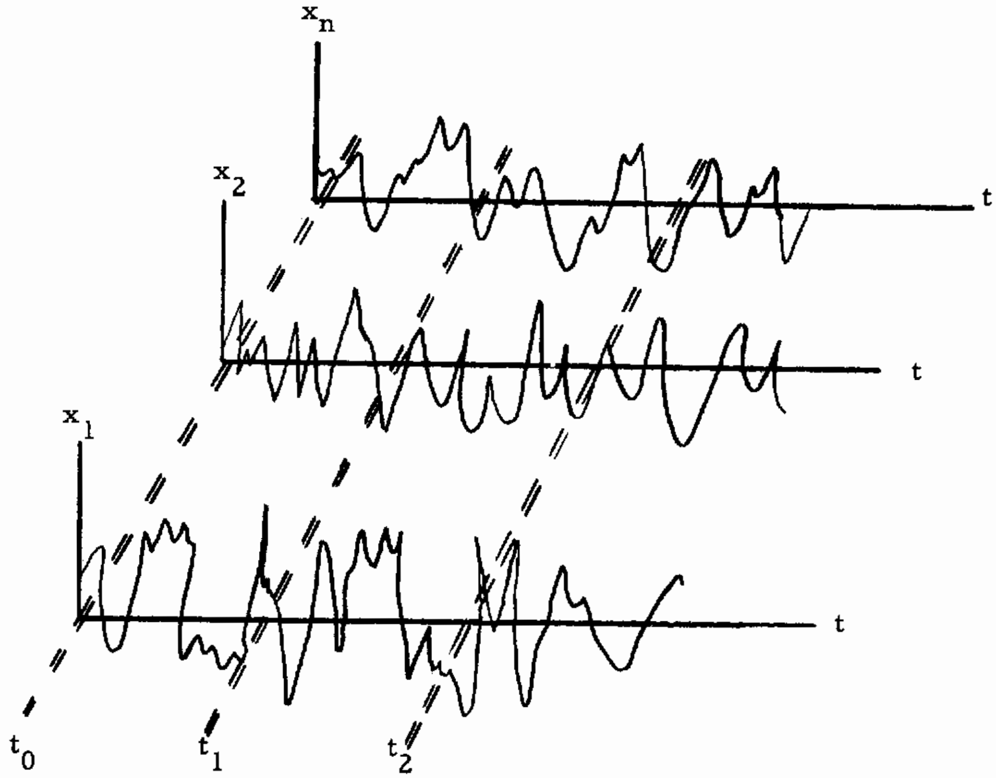


Figure 8 Schematic Representation of a Random Process

The collection of records would then represent the time histories measured at that same point for separate flights performed under identical conditions.

The statistical properties of the random process are computed by averaging over the ensemble at any instant of time, or by averaging over time on individual sample records. If the statistical properties found by averaging over the ensemble are invariant with respect to translations in time, then the random process is said to be stationary. If the random process is stationary and the ensembled averaged properties are numerically equal to the properties found by time averaging each individual record, then the random process is said to be ergodic. Ergodicity, therefore, allows one to use the time averaged properties of an individual record to describe the properties of the entire random process. More extensive discussions of these concepts are available from Reference 5, Section 6. In all discussions to follow, it will be assumed that the random processes of interest are stationary and ergodic so that all properties of the random processes can be described in terms of time averages of individual sample records.

3.1.2 Probability Density Functions

Consider a sample record $x(t)$ from a stationary random process. The probability that $x(t)$ will take on a value in a narrow interval between x and $x+\Delta x$ is given by

$$\text{Prob}\left[x \leq x(t) \leq x+\Delta x\right] \approx p(x)\,\Delta x \qquad (3.1)$$

where $p(x)$ is called the first-order probability density function for $x(t)$. More precisely,

$$p(x) = \lim_{\Delta x \to 0} \frac{\text{Prob}\left[x \leq x(t) \leq x+\Delta x\right]}{\Delta x} \qquad (3.2)$$

The first-order probability density function for a stationary random process defines the probability of a sample record taking on values within any defined range. That is,

$$\text{Prob}\left[a \leq x \leq b\right] = \int_a^b p(x)\,dx \qquad (3.3)$$

In other words, the area under the probability density plot between two values a and b defines the probability that $x(t)$ will take on a value in that range at any instant of time. Clearly, the entire area under the probability density plot must equal unity since the probability of any value occurring is a certainty. That is,

$$\text{Prob}\left[-\infty \leq x \leq \infty\right] = \int_{-\infty}^{\infty} p(x)\, dx = 1 \qquad (3.4)$$

Note that the mean value μ, the mean square value Ψ^2, and the variance σ^2 of a random process are related to the probability density function as follows.

$$\mu = \int_{-\infty}^{\infty} x\, p(x)\, dx \qquad (3.5)$$

$$\Psi^2 = \int_{-\infty}^{\infty} x^2\, p(x)\, dx \qquad (3.6)$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2\, p(x)\, dx = \Psi^2 - \mu^2 \qquad (3.7)$$

However, for stationary ergodic random processes, these quantities may actually be computed by time averaging a single sample record as follows.

$$\mu = \overline{x} = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)\, dt \qquad (3.8)$$

$$\Psi^2 = \overline{x^2} = \lim_{T \to \infty} \frac{1}{T} \int_0^T x^2(t)\, dt \qquad (3.9)$$

$$\sigma^2 = \overline{(x - \mu)^2} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left[x(t) - \mu\right]^2 dt \qquad (3.10)$$

The positive square root of the mean square value, given by Ψ or $\sqrt{\overline{x^2}}$, defines the root mean square (rms) value for the random process.

The first-order probability density function for a random process may theoretically be of many different forms. However, because of the practical implications of the central limit theorem in statistics, a specific probability density function called the normal (Gaussian) probability density function is usually assumed for random physical phenomenon such as the pressure fluctuations created by jet exhaust gas mixing and aerodynamic boundary layer noise. The normal density function is defined in Eq. (3.11) and illustrated in Figure 9.

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-(x-\mu)^2/2\sigma^2\right] \tag{3.11}$$

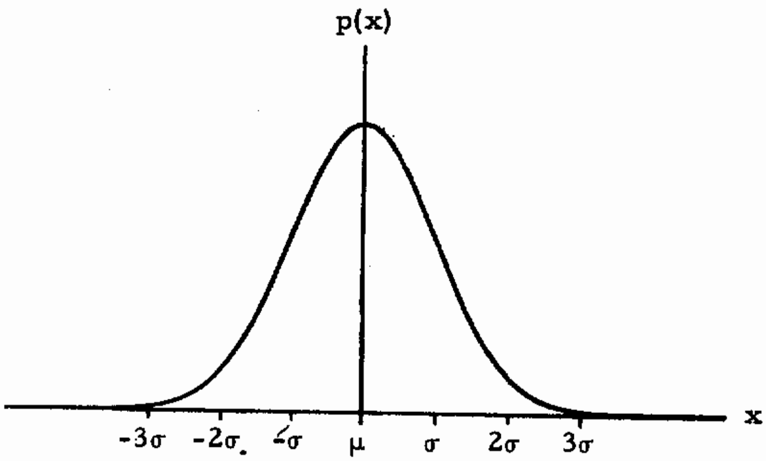


Figure 9 Normal Probability Density Function

The integral of Eq. (3.11) from the lower limit to some upper value of x, say a, is commonly called the distribution function or cumulative distribution function for x(t). The normal distribution function is defined in Eq. (3.12) and illustrated in Figure 10.

$$p(x)\ \mathrm{Prob}\left[x \le a\right] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{a} \exp\left[-(x-\mu)^2/2\sigma^2\right] dx \qquad (3.12)$$

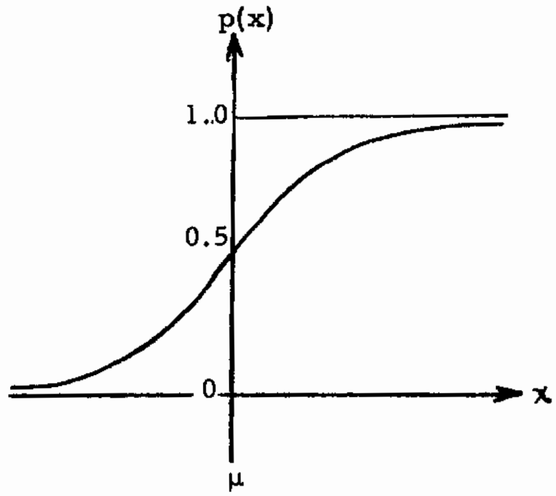


Figure 10 Normal Probability Distribution Function

As stated earlier, but repeated here for emphasis, an important property of stationary random processes with normal first-order probability density functions (Gaussian random processes) is as follows. All linear operations on a Gaussian random process produce another Gaussian random process. This is not necessarily true for random processes other than Gaussian. Hence, if the excitation to a linear structure is stationary and Gaussian, the response of the structure will also be stationary and Gaussian.

The probability density functions discussed thus far are first-order probability density functions. That is, the density functions define the instantaneous probability characteristics of a single stationary random process. Given two stationary random processes with sample records x(t) and y(t), a second-order probability density function which yields the joint instantaneous probability characteristics for the two processes can be defined as

$$p(x, y) = \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \frac{\text{Prob}\left[x \leq x(t) \leq x+\Delta x \; ; \; y \leq y(t) \leq y+\Delta y \right]}{\Delta x \Delta y}$$

The same general relationship given by Eqs. (3.3) and (3.4) still applies, except two variables must now be considered jointly. That is,

$$\text{Prob}\left[a \leq x \leq b \; ; \; c \leq y \leq d \right] = \int_a^b \int_c^d p(xy) \, dx \, dy \qquad (3.13)$$

$$\text{Prob}\left[-\infty \leq x \leq \infty \; ; \; -\infty \leq y \leq \infty \right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(xy) \, dx \, dy = 1 \qquad (3.14)$$

For the special case where the two random processes are statistically independent, the following important relationship is true.

$$p(xy) = p(x) \, p(y) \qquad (3.15)$$

The extension of the concept of joint probability density functions to more than two variables may be accomplished in a similar manner.

The assumption of statistical independence greatly simplifies probabilistic analyses and allows for a linear superposition principle for random variables. For example, suppose $x(t)$ is a stationary function of time defined as

$$x(t) = x_1(t) + x_2(t) + x_3(t) \ldots \tag{3.16}$$

Then the time or temporal average of $\overline{x^2}$ can be shown to equal

$$\overline{x^2} = \overline{x_1^2} + \overline{x_2^2} + \overline{x_3^2} + \ldots \tag{3.17}$$

as the values of all of the cross terms $(x_1x_2,\ x_1x_3,\ x_2x_3, \ldots)$ equal zero. Similar statements can be made for the output response spectrum of a linear system excited by independent random excitations.

3.1.3 Autocorrelation and Autocovariance

Consider a sample record $x(t)$ from a stationary random process. The dependence of the value of $x(t)$ at some future time based upon an observed value τ seconds before is given by the autocorrelation defined as

$$R(\tau) = E\left[x(t)\,x(t+\tau)\right] = \lim_{T\to\infty} \int_0^T x(t)\,x(t+\tau)\,dt \tag{3.18}$$

The autocovariance function is given by

$$\rho(\tau) = E\left[\left\{x(t) - \mu\right\}\left\{x(t+\tau) - \mu\right\}\right] = \lim_{T\to\infty} \int_0^T \left[x(t) - \mu\right]\left[x(t+\tau) - \mu\right] dt \tag{3.19}$$

The covariance is an average of the product of the deviation from the means at two instants of time. When $\mu = 0$, the autocovariance is identical to the autocorrelation function. A normalized form of the co-variance obtained by dividing the covariance by the variance σ^2 defines a function which is bounded by the values of -1 and +1. A necessary but not sufficient condition for $x(t)$ and $x(t+\tau)$ to be statistically independent is that the correlation function must equal zero. For Gaussian random processes, being uncorrelated is also a sufficient condition for being statistically independent. All other values are referred to as being either positively or negatively correlated. Note that when $\tau = 0$, $R(0)$ is the mean square value of $x(t)$ and $\rho(0)$ is the variance of $x(t)$. That is,

$$R(0) = \overline{x^2} \tag{3.20}$$

$$\rho(0) = \overline{(x-\mu)^2} \tag{3.21}$$

3.1.4 Spectral Density Functions

For a sample record $x(t)$ from a stationary random process, the frequency composition of $x(t)$ may be conveniently described by the Fourier transform of the autocorrelation function. In terms of transform pairs, the relationships between the power spectral density function $S(f)$ and the autocorrelation function $R(\tau)$ is written as

$$\begin{aligned} S(f) &= \int_{-\infty}^{\infty} R(\tau)\, e^{-i2\pi f\tau}\, d\tau \\ R(\tau) &= \int_{-\infty}^{\infty} S(f)\, e^{i2\pi f\tau}\, df \end{aligned} \tag{3.22}$$

In terms of circular frequencies ω, the transform pairs may be expressed as

$$\begin{aligned} S(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau \\ R(\tau) &= \int_{-\infty}^{\infty} S(\omega)\, e^{i\omega\tau}\, d\omega \end{aligned} \tag{3.23}$$

$S(f)$ has the units of mean square value per cps whereas $S(\omega)$ has the units of mean square value per radian per second. Note that the power spectral density function is a non-negative real-valued function which is defined for both positive and negative frequencies. A physically realizable power spectral density function defined for only non-negative frequencies is given by

$$G(f) = \begin{cases} 2\,S(f) & \text{for} \quad f > 0 \\ 0 & \text{for} \quad f < 0 \end{cases} \qquad (3.24)$$

$$G(\omega) = \begin{cases} 2\,S(\omega) & \text{for} \quad \omega > 0 \\ 0 & \text{for} \quad \omega < 0 \end{cases} \qquad (3.25)$$

The mean square value for a sample record $x(t)$ is related to the power spectral density function as

$$\overline{x^2} = \int_{-\infty}^{\infty} S(f)\,df = \int_{-\infty}^{\infty} S(\omega)\,d\omega = \int_{0}^{\infty} G(f)\,df = \int_{0}^{\infty} G(\omega)\,d\omega \qquad (3.26)$$

From Eq. (3.26), the following relationships are obtained

$$S(f) = 2\pi\,S(\omega) \qquad (3.27)$$

$$G(f) = 4\pi\,S(\omega) \qquad (3.28)$$

The 2π term of Eq. (3.27) is formed simply by a change in the frequency units. The 4π term of Eq. (3.28) consists of 2π due to the change in frequency units and a factor of 2 due to a consideration of positive frequencies only (see Eqs. (3.24) and (3.25)).

Adapted to a physically realizable spectral density function, the transform definitions given by Eq. (3.22) yield

$$R(0) \equiv \Psi^2 = \overline{x^2} = \int_0^\infty G(f)\, df \tag{3.29}$$

$$G(0) = \int_{-\infty}^{\infty} R(\tau)\, d\tau \tag{3.30}$$

For a record $x(t)$ from a stationary and ergodic random process, Eq. (3.29) defines the mean square value as the magnitude of the autocorrelation function at $\tau = 0 \left[R(0)\right]$; and is formed by integrating the spectral density function over the complete frequency range. Equation (3.30) states that the magnitude of the spectral density function at zero frequency (i.e., the mean value of the signal) is given by integrating the autocorrelation function over its complete range.

Since the autocorrelation function and spectral density function are related as transform pairs, both yield essentially identical information but in somewhat different formats. By way of illustration, consider Figure 11 which depicts sketchss of the time history and the associated spectral density and autocorrelation functions for four distinct signals: (1) direct current (d.c.), (2) sine wave, (3) narrow band random noise, and (4) bandlimited white noise.

The spectral density of a discrete frequency component is denoted as a Dirac delta function of infinite magnitude but whose area equals the mean square value of the signal. Note the spectral densities in Figure 11 for the d.c. signal and the sine wave. A mean value component (such as the d.c. signal of magnitude A_0) appears as a delta function at zero frequency in the power spectrum whereas, in the autocorrelation function, the same component appears as a shift in the ordinate by an amount equal to the square of the mean value. A sinusoidal component appears as a delta function at the periodic frequency (T_P) of the sine wave whereas, in the autocorrelation function, the sinusoidal component appears as a nonconvergent cosine function with the same period as the sine wave. The autocorrelation function for the sine wave at τ equal to zero yields $A^2/2$ which is the mean square value of a sine wave with peak amplitude A.

The time history of narrow band random noise has the appearance of a sinusoid at frequency f_0 with slowly varying random amplitude and random phase. The dashed line bounds the amplitude of the signal and is defined as an envelope function. The autocorrelation function appears as a cosine function with frequency ω_d decaying in amplitude as $e^{-2\pi f_n \zeta \tau}$. This type of response is typical of the output of a lightly damped mechanical oscillator subjected to bandlimited white noise whose frequency band includes and is wide compared to the resonant frequency of the mechanical system. For these conditions, the autocorrelation function appears as

$$R(\tau) \simeq \frac{\pi}{2} \frac{S_0}{\zeta \omega_n^3} e^{-\zeta \omega_n \tau} \left(\cos \omega_d \tau + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d \tau \right) \qquad (3.31)$$

The mean square value, therefore, becomes [see Eq. (2.50) of Reference 3].

$$R(0) = \Psi^2 = E\left[x(t)^2 \right] \simeq \frac{\pi}{2} \frac{S_0}{\zeta \omega_n^3} \qquad (3.32)$$

Note that the spectral density functions for the sine wave and narrow band random noise are shown with zero mean values [i.e., $G(0) = 0$]. The corresponding autocorrelation functions are observed to be anti-symmetric with respect to the τ axis and it is apparent that the integral over τ equals zero as required by Eq. (3.30). This integral equation, however, does not demand that all autocorrelation functions for signals with zero mean value be anti-symmetric about τ. Equation (3.30) requires simply that the sum of the areas of the autocorrelation functions above and below the τ axis equals zero for a signal with zero mean value. Should either the sine wave or narrow band random noise have a mean value, the autocorrelation functions would then be anti-symmetric about an axis parallel to the τ axis and intersecting $R(\tau)$ at a magnitude equal to the square of the mean value.

The spectral density of bandlimited white noise is noted as a constant of magnitude G_0 over the entire frequency band B_f. The autocorrelation function is peaked near $\tau = 0$ and decays rapidly to small values of $R(\tau)$ with increasing τ. In terms of the center frequency of the bandwidth (f_c), the bandwidth of the noise (B_f), and the magnitude of the spectral density (G_0), the autocorrelation function is

$$R(\tau) = B_f G_0 \left(\frac{\sin \pi B_f \tau}{\pi B_f \tau} \right) \cos 2\pi f_c \tau \tag{3.33}$$

The mean square value appears as

$$\Psi^2 = R(0) = \int_0^f G_0 df = B_f G_0 \tag{3.34}$$

Thus, the autocorrelation function implies bandlimited white noise is correlated only for small values of τ and is a maximum where $\tau = 0$.

Should the time history be white noise, the spectral density is a constant of magnitude G_0 extending over an infinite bandwidth. The autocorrelation function is a Dirac delta function whose area equals Ψ^2. The mean square value for such a physically unrealizable process is infinite since the area under the spectrum also is infinite. In comparison with the d. c. signal, the spectral density of white noise appears equivalent in form to the d. c. autocorrelation function whereas the autocorrelation function of white noise appears equivalent in form to the d. c. spectral density.

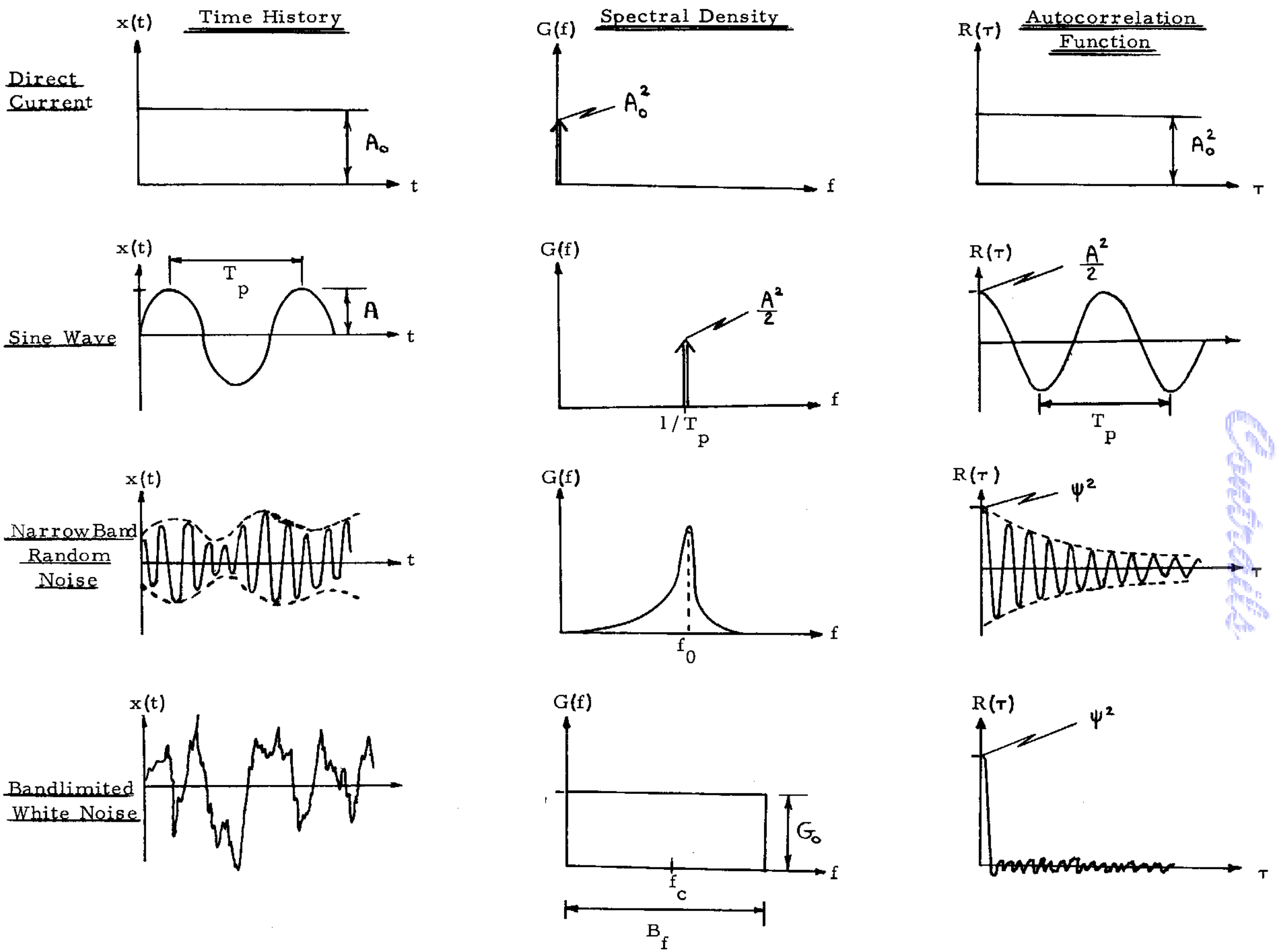


Figure 11. Sketches of the Time History and the Associated Spectral Density and Autocorrelation Functions for Various Signals

3.2 MEAN SQUARE RESPONSE AND SPECTRAL DENSITY RELATIONS FOR STATIONARY RANDOM EXCITATION

Let the response (output) be denoted by $y(t)$ and the excitation (input) be denoted by $x(t)$. The relationship between $y(t)$ and $x(t)$ is conveniently given by the convolution integral as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t - \tau)\, d\tau \tag{3.35}$$

Allowing capital letters to denote the Fourier transform (F. T.), the F. T. of the response $\left[y(t)\right]$ appears as

$$Y(\omega) = H(\omega)\, X(\omega) \tag{3.36}$$

where

$$Y(\omega) = \mathcal{F}\left[y(t)\right] \tag{3.37}$$

$$H(\omega) = \mathcal{F}\left[h(\tau)\right] \tag{3.38}$$

$$X(\omega) = \mathcal{F}\left[x(t)\right] \tag{3.39}$$

The mean square value of the response of a linear mechanical system to stationary random excitation may be written as

$$\overline{y^2} = \frac{1}{T} \int_0^T y^2(t)\, dt \tag{3.40}$$

Equation (3. 40) denotes a time averaging of a single record from the random process and is taken to be representative of the random process due to the ergodicity assumption.

In alternate form, the mean square response can be expressed in terms of the spectral density as (see Eq. (3. 26))

$$\overline{y^2} = R(0) = \int_{-\infty}^{\infty} S_y(\omega)\, d\omega \tag{3.41}$$

In terms of the F. T. of the response, the response spectral density is

$$S_y(\omega) = E\left[Y(\omega)\, Y^*(\omega)\right] \tag{3.42}$$

Substituting Eq. (3. 36) into Eq. (3. 42) provides

$$S_y(\omega) = \left[H(\omega)\, H^*(\omega)\right] E\left[X(\omega)\, X^*(\omega)\right] = \left|H(\omega)\right|^2 S_x(\omega) \tag{3.43}$$

where

$$H(\omega)\, H^*(\omega) = \left|H(\omega)\right|^2 \tag{3.44}$$

$$E\left[X(\omega)\, X^*(\omega)\right] = S_x(\omega) \tag{3.45}$$

Equation (3. 43) is the widely quoted equation stating the response spectral density equals the product of the square of the absolute value of the frequency response function times the input spectral density.

Another quantity of interest is the cross-spectrum and is defined as

$$S_{yx}(\omega) = E\left[Y(\omega)\, X^*(\omega)\right] \tag{3.46}$$

Substituting Eq. (3. 36) into Eq. (3. 46) yields

$$S_{yx}(\omega) = H(\omega)\, S_x(\omega) \tag{3.47}$$

The cross-spectrum finds application in directly calculating frequency response function and coherence functions [see Reference 1]

For the single degree-of-freedom system excited by a sinusoidal forcing function, the mean square response may be written as

$$\overline{y^2} = E\left[Y(\omega)\, Y^*(\omega)\right] = \frac{\overline{F_0^2}}{Z(\omega)\, Z^*(\omega)} = \frac{1}{2}\, \frac{F_0^2}{\left|Z(\omega)\right|^2} \tag{3.48}$$

where $Y(\omega)$ is given by Eq. (1.30). If the excitation contains more than one harmonic component, the mean square response appears as

$$\overline{y^2} = \sum_j E\left[Y_j(\omega)\, Y_j^*(\omega)\right] = \sum_j \frac{1}{2}\, Y_j(\omega)\, Y_j^*(\omega) \tag{3.49}$$

where again

$$Y_j(\omega) = \frac{F_0}{Z_j(\omega)} \tag{3.50}$$

If the excitation contains more closely spaced harmonic components, the mean square response is given as

$$\overline{y^2} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{T}\, Y(\omega)\, Y^*(\omega)\, \frac{\sin(\omega - \omega')T}{(\omega - \omega')}\, d\omega\, d\omega' \tag{3.51}$$

which reduces to

$$\overline{y^2} = \frac{1}{2\pi T} \int_0^{\infty} Y(\omega)\, Y^*(\omega)\, d\omega \tag{3.52}$$

The time interval (T) is retained in this expression as the record length over which the averaging takes place (see Eq. (3.40)).

Implicit in the equation for the mean square response given by Eq. (3.52) is the following definition for transform pairs

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} y(t)\, e^{-i\omega t}\, dt \\ y(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega)\, e^{i\omega t}\, d\omega \end{aligned} \tag{3.53}$$

Note that the expression of Eq. (3.53) differ in the use of the $1/2\pi$ term from the definitions given by Eqs. (3.22) and (3.23).

Comparing the terms of Eqs. (3.41) and (3.52), the integrand of Eq. (3.52) may be considered as the spectral density of the response and is written as

$$S_y(\omega) = \frac{1}{4\pi T} Y(\omega) Y^*(\omega) \tag{3.54}$$

In terms of a physically realizable spectra, Eq. (3.54) appears as

$$G_y(\omega) = \frac{1}{2\pi T} Y(\omega) Y^*(\omega) \tag{3.55}$$

Similarly, the power spectral density function of the input excitation $\left[x(t)\right]$ is written as

$$S_x(\omega) = \frac{1}{4\pi T} F(\omega) F^*(\omega) \tag{3.56}$$

where $F(\omega)$ is the Fourier transform of the forcing function and $F^*(\omega)$ is the complex conjugate of $F(\omega)$. Substituting Eq. (3.36) into Eq. (3.54) provides the relationship between the input and output spectral densities for a linear time invariant system and is given by

$$S_y(\omega) = \frac{1}{4\pi T} \frac{F(\omega) F^*(\omega)}{Z(\omega) Z^*(\omega)} = \frac{1}{\left|Z(\omega)\right|^2} S_x(\omega) = \left|H(\omega)\right|^2 S_x(\omega) \tag{3.57}$$

Hence, as also stated by Eq. (3.43), the output spectral density is equal to the input spectral density multiplied by the square of the absolute magnitude of the frequency response function.

3.3 EXAMPLE PROBLEMS

These problems are abridged from Reference 1, Chapter 7, and are inserted here to illustrate the application of the equations discussed in Section 3.

Problem 3.1: The power spectrum and the rms amplitude for the displacement response of a mass excited mechanical oscillator to a periodic forcing function consisting of three harmonic components.

The parametric values for the mechanical oscillator are assumed to be

$$\begin{aligned} m &= 0.025 \text{ lb-sec}^2/\text{inch} \\ c &= 0.25 \text{ lb-sec/inch} \\ k &= 1000 \text{ lb/inch} \end{aligned} \tag{3.58}$$

The steady state forcing function consists of three harmonic components with equal magnitudes and is expressed as

$$x(t) = 17(\sin 30\pi t + \sin 60\pi t + \sin 90\pi t) \tag{3.59}$$

Using the values of Eq. (3.58) with the definitions of Eq. (1.2) yields

$$\begin{aligned} f_n &= \frac{1}{2\pi}\sqrt{\frac{k}{m}} &&= 31.8 \text{ cps} \\ c_c &= 2\sqrt{km} &&= 10 \text{ lb-sec/inch} \\ \zeta &= \frac{c}{c_c} &&= 0.025 \end{aligned} \tag{3.60}$$

The magnitude of the appropriate frequency response function (i.e., the ratio of the displacement to the exciting force) is given by the steady-state solution of Eq. (1.4), and may be written as

$$\left|\frac{y}{x}\right| = \left|H_{y,x}(f)\right| = \left|H_{y,x}(\omega)\right| = \frac{1}{k\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}} \tag{3.61}$$

Note that the only difference between the frequency response functions of Eq. (3.61) and the magnification factor of Eq. (1.11) is the term $1/k$. This term normalizes the displacement/force frequency response function to the displacement frequency response function as, by definition, y_0 equals the applied loading divided by the spring constant.

From Section 2.2 of Reference 1, the periodic input excitation $x(t)$ may be described by the discrete power spectrum

$$G_x(f) = \sum_{j=0}^{3} \overline{x_j^2} = 144\left[\delta(f-15) + \delta(f-30) + \delta(f-45)\right] \text{ lbs.}^2/\text{cps.} \tag{3.62}$$

where the subscript of the power spectrum denotes the forcing function, the (f) denotes a functional dependence on frequency, and the δ's denote delta functions for the 15, 30, and 45 cps. frequencies.

Expressed in the terminology of realizable spectra, Eq. (3.57) appears as

$$G_y(f) = \left|H_{y,x}(f)\right|^2 G_x(f) \tag{3.63}$$

Hence, the displacement response power spectrum may be written as

$$G_y(f) = \frac{144}{10^6} \frac{\left[\delta(f-15) + \delta(f-30) + \delta(f-45)\right]}{\left[1-\left(\frac{f}{31.8}\right)^2\right]^2 + \left[\frac{0.05f}{31.8}\right]^2} \text{ inches}^2/\text{cps}. \tag{3.64}$$

Equation (3.58) appears in pictorial form as Figure 12.

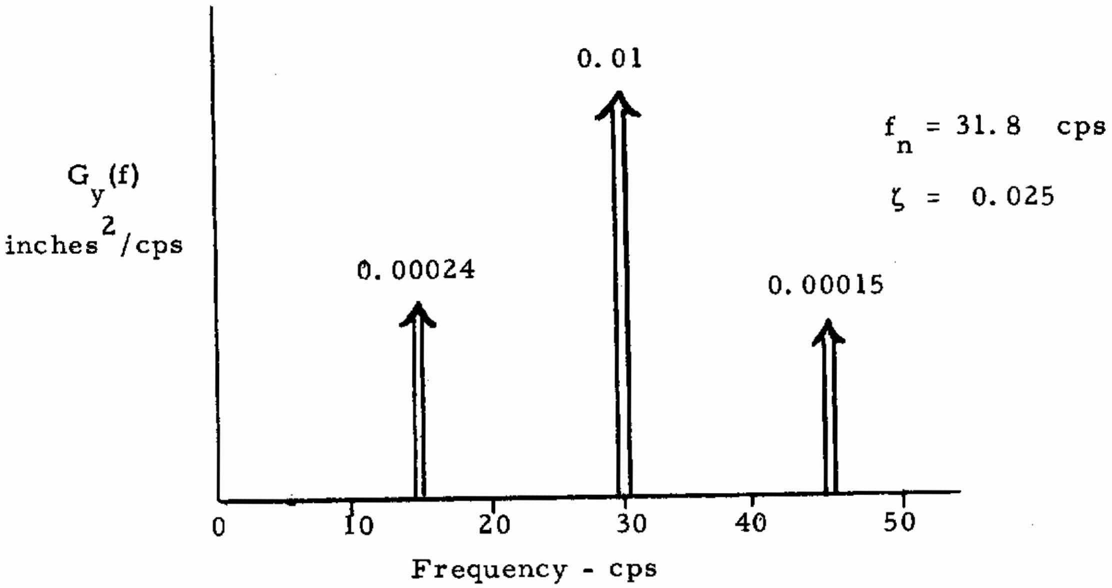


Figure 12 Discrete Response Power Spectral Density Function for the Displacement Response of a Mechanical Oscillator to Periodic Force Excitation

The mean square value for the displacement response is given in integral form by Eq. (3.41). This equation is expressed in summation form as

$$\overline{y^2} = 144 \sum_{j=1}^{3} \frac{10^{-6}}{\left[1-\left(\frac{15j}{31.8}\right)^2\right]^2 + \left[\frac{0.75j}{31.8}\right]^2} = 0.010 \text{ inches}^2 \tag{3.65}$$

Problem 3.2: The power spectrum and rms amplitude for the displacement response of a specified mass excited mechanical oscillator to bandlimited white noise.

The mechanical oscillator is assumed to be identical to that of Problem 3.1. The exciting force is assumed to have a uniform spectral density of magnitude 10 lb^2/cps over the frequency range from 0 cps $\gg f_n$.

The displacement response spectral density is given by Eq. (3.63) and appears as

$$G_y(f) = \frac{10}{10^6} \frac{1}{\left[1 - \left(\frac{f}{31.8}\right)^2\right]^2 + \left[\frac{0.05\,f}{31.8}\right]^2} \text{ inches}^2/\text{cps.} \tag{3.66}$$

In graphical form, Eq. (3.66) appears as shown in Figure 13.

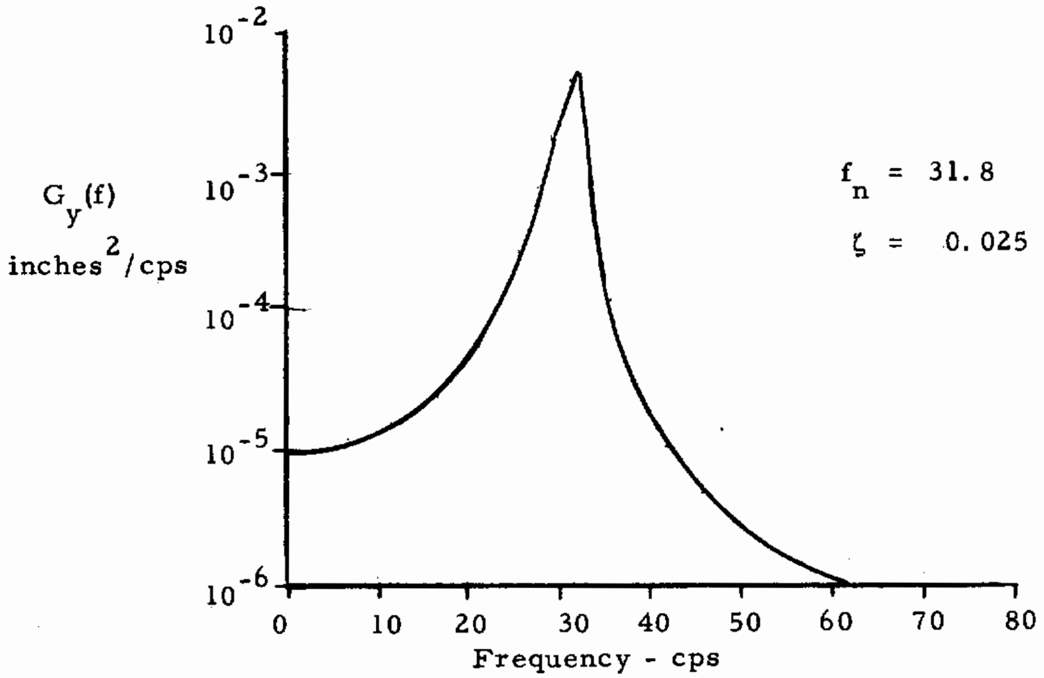


Figure 13 Displacement Response Spectral Density Function For a Mass Excited Mechanical Oscillator. The excitation is a bandlimited white noise.

For a uniform power spectrum G_x, the mean square response is given by Eq. (3.26) which reduces to

$$\overline{y^2} = \int_0^\infty G_y(f)\, df = \frac{\pi f_n (1 + 4\zeta^2) G_x}{4\zeta} \qquad (3.67)$$

Note that G_x must have the dimensions of inches^2/cps. if Eq. (3.67) is to yield the mean square displacement response. As stated for Problem 3.2, G_x is given in terms of lbs.^{2}/cps. Since $f = kx$ for this linear system,

$$G_x(\text{inches}^2/\text{cps.}) = \frac{1}{k^2}\, G_x\,(\text{lbs.}^2/\text{cps.}) \qquad (3.68)$$

Therefore, the rms amplitude for the displacement response becomes

$$y_{rms} = \sqrt{\overline{y^2}} = 0.10 \text{ inches} \qquad (3.69)$$

Note that the integral of Eq. (3.67) represents the area under the displacement response spectral density. Hence, the rms value for this problem also can be obtained by graphically integrating under the curve of Figure 13.

It is seen that the response power spectrum has a sharp peak at a frequency of about $f = f_n = 31.8$ cps. The power spectral density of the peak is about 400 times (26 db) greater than the power spectral density of the excitation. The magnitude of this peak is a function only of the damping ratio for the structure.

For the general case of an excitation with a uniform power spectrum, the frequency and magnitude of the peak in the response power spectrum is obtained by taking the first derivative of Eq. (3.63) with respect to frequency, and setting the derivative equal to zero. The following result is obtained.

$$f_r = f_n \sqrt{1 - 2\zeta^2} \qquad \text{for} \quad \zeta < \frac{1}{\sqrt{2}} \tag{3.70}$$

$$G_y(f_r) = \frac{G_x(f_d)}{4\zeta^2(1 - \zeta^2)} \qquad \text{for} \quad \zeta < \frac{1}{\sqrt{2}} \tag{3.71}$$

The frequency f_r where the peak occurs is the resonant frequency of the structure. The expressions in Eqs. (3.70) and (3.71) apply only if the damping ratio ζ is less than $1/\sqrt{2}$ because the response power spectrum will not show a peak (the structure will not have a clearly defined resonant frequency) if ζ is larger than $1/\sqrt{2}$.

Referring to Eq. (3.70), note that the resonant frequency f_r approaches the undamped natural frequency f_n as the damping ratio ζ becomes small. For the case of a small damping ratio, say $\zeta < 0.1$, the frequency and magnitude of the response power spectrum peak are given as

$$f_r \approx f_n \tag{3.72}$$

$$G_y(f_r) \approx \frac{G_x(f_n)}{4\zeta^2} \tag{3.73}$$

For $\zeta << 1$ and $G_x(f)$ = constant, the response power spectrum falls to one-half its peak value when $f \approx f_n(1 \pm \zeta)$. Noting that $f_r \approx f_n$, this gives a bandwidth for the resonance between half-power point frequencies of

$$B_r \approx 2\zeta f_r \tag{3.74}$$

In general, actual flight vehicle structures have relatively small equivalent viscous damping coefficients. Typical damping ratios for assembled structures range from $\zeta = 0.01$ to $\zeta = 0.05$. Hence, the value of $\zeta = 0.025$ used in the numerical example is appropriate, and the resulting sharp peak in the response power spectrum is generally typical of actual flight vehicle vibration data. It should be emphasized that the simple structure considered in the numerical example is a single degree-of-freedom system with only one resonant frequency. Thus, only one peak appears in the response power spectrum. For real structures with many degrees-of-freedom and many different resonant frequencies, a number of sharp peaks may appear in the response power spectrum.

<u>Problem 3.3:</u> The power spectrum and the rms amplitude for the displacement response of a mass excited mechanical oscillator subjected to a forcing function consisting of three harmonic components <u>and</u> bandlimited white noise.

The mechanical oscillator for this problem is assumed to be the same as in Problems 3.1 and 3.2. Moreover, the excitation for this problem is assumed to be those used in both Problems 3.1 and 3.2. Hence, the displacement response power spectrum may be expressed as

$$G_y(f) = \left| H_{y,x}(f) \right|^2 \left\{ G_x(f)_{\text{sinusoids}} + G_x(f)_{\text{white noise}} \right\} \quad (3.75)$$

where

$$\left| H_{y,x}(f) \right|^2 = \frac{1}{10^6 \left\{ \left[1 - \left(\frac{f}{31.8} \right)^2 \right]^2 + \left[\frac{0.05 f}{31.8} \right]^2 \right\}} \quad (3.76)$$

$$G_x(f)_{\text{sinusoids}} = 144 \left[\delta(f-15) + \delta(f-30) + \delta(f-45) \right] \text{lbs.}^2/\text{cps.} \quad (3.77)$$

$$G_x(f)_{\text{white noise}} = 10 \text{ lbs.}^2/\text{cps.} \quad (3.78)$$

In graphical form, the displacement response spectrum consists of the superposition of Figures 12 and 13 and is shown as Figure 14.

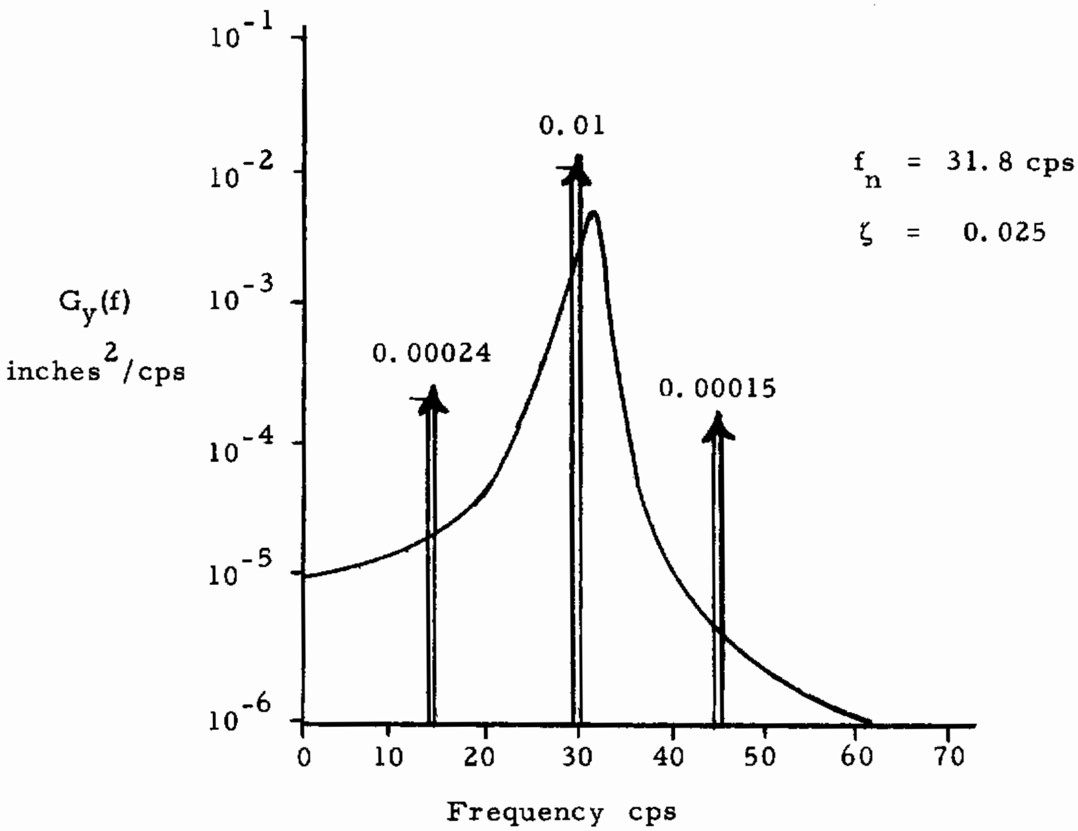


Figure 14. Displacement Response Power Spectral Density for a Mass Excited Mechanical Oscillator. The excitation consists of three harmonic forces and bandlimited white noise.

Note that the periodic components of Figure 14 are denoted as delta functions in the spectral density plot. The magnitudes of the periodic components are mean square values and represent the area of each of the delta functions at the specified frequencies.

Theoretically, one always can detect a harmonic component in noise as the spectral density of a harmonic function is infinite at the harmonic frequency and zero at all other frequencies. Hence, as mentioned in the discussion of Section 3, one way to detect a sinusoidal component in noise is to compute a spectral density plot and observe the peaks in the frequency spectrum. Other practical methods are (1) to calculate the autocorrelation function, or (2) to cross-correlate the output with a signal from a reference oscillator. For a more detailed discussion on this topic, the reader is directed to Section 2.1 of Reference 7.

The mean square response can be obtained by integrating to compute the area under the curve of Figure 14. In an alternate manner, one can add the mean square values for Problems 3.1 and 3.2 to calculate the mean square magnitude for this problem. Adding the mean square values yield

$$\overline{y^2} = .020 \text{ inches}^2 \tag{3.79}$$

Hence, the rms amplitude is

$$y_{rms} = \sqrt{\overline{y^2}} = .447 \text{ inches} \tag{3.80}$$

4. RESPONSE OF CONTINUOUS ELASTIC STRUCTURES TO RANDOM EXCITATION

In this section, attention is given to calculating the response of distributed, linear elastic structures to stationary random excitation. Mathematically, this task essentially reduces to examining the solution to partial differential equations where the nonhomogeneous term must be described by stochastic quantities. The structures treated here are assumed to be homogeneous and uniform; thus, the coefficients of the partial differential equations are constants.

4.1 MODAL APPROACH AND FOURIER TRANSFORM APPLICATION

Consider any elastic structure whose normal modes are defined by $\phi_j(x)$ and ω_j. The deflection at any point x can be expressed in terms of the normal modes and generalized coordinates $q_j(t)$ as

$$y(x,t) = \sum_j \phi_j(x)\, q_j(t) \tag{4.1}$$

Damping is, in general, an unknown quantity; however, it is assumed to be small and defined as viscous. In order not to introduce coupling between modes, its distribution over the structure is given as

$$\int_0^{\ell} c(x)\, \phi_j(x)\, \phi_k(x)\, dx \tag{4.2}$$

Assuming $c(x) = 2\zeta_j \omega_j\ m(x)$, the equation of motion in generalized coordinates for the jth mode is

$$\ddot{q}_j(t) + 2\zeta_j\, \omega_j \dot{q}_j(t) + \omega_j^2\, q_j(t) = \frac{\overline{F}_j}{\overline{M}_j} \tag{4.3}$$

where $\overline{M}_j$ is called the generalized mass (see Eq. (2.15)) and $\overline{F}_j$ is called the generalized force (see Eq. (2.18)).

When f(x,t) is random, Eq. (4.3) must be solved in a statistical sense. For this, the method of Fourier transforms (F. T.) the concept of power spectral density, and correlation between quantities are the analytical tools appropriate for use.

Letting the capital letters denote the F. T., Eq. (4.3) may be written as

$$(-\omega^2 + i2\zeta_j\,\omega_j\omega + \omega_j^2)Q_j(\omega) = \frac{1}{\overline{M}_j}\int_0^{\ell} F(x,\omega)\ \phi_j(x)\ dx \tag{4.4}$$

Solving explicitly for the F. T. of the generalized coordinate (q) yields

$$Q_j(\omega) = \frac{H_j(\omega)}{\overline{M}_j\omega_j^2}\int_0^{\ell} F(x,\omega)\phi_j(x)\ dx \tag{4.5}$$

where

$$H_j(\omega) = \frac{1}{1 - \left(\frac{\omega}{\omega_j}\right)^2 + i\,2\,\zeta_j\,\frac{\omega}{\omega_j}} \tag{4.6}$$

The absolute value of (4.6) may be interpreted as a modal magnification factor defined by Eq. (1.11). Substituting Eq. (4.5) into the F. T. of Eq. (4.1) yields

$$Y(x,\omega) = \sum_j \phi_j(x)\ \frac{H_j(\omega)}{\overline{M}_j\ \omega_j^2}\int_0^{\ell} F(x,\omega)\phi_j(x)\ dx \tag{4.7}$$

The displacement response can now be obtained by taking the inverse of Eq. (4.7)

4.2 CORRELATION CONSIDERATIONS AND MEAN SQUARE RESPONSE

The correlation between the response at points x and x' can be expressed as

$$\overline{y(x)\,y(x')} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} y(x,t)\,y(x',t)\,dt \tag{4.8}$$

Parseval's theorem for integrals can be expressed as

$$\int_{-\infty}^{\infty} y^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|Y(\omega)\right|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)\,Y^*(\omega)\,d\omega \tag{4.9}$$

where the integral is assumed finite (page 65, Reference 9). Applying Parseval's theorem to Eq. (4.8) yields

$$\overline{y(x)\,y(x')} = \frac{1}{2}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2\pi T}\; Y^*(x,\omega)\,Y(x',\omega)\,d\omega \tag{4.10}$$

Substituting Eq. (4.7) into Eq. (4.10) provides

$$\overline{y(x)\,y(x')} = \frac{1}{2}\sum_j \sum_k \phi_j(x)\,\phi_k(x')\int_{-\infty}^{\infty} \frac{H_j^*(\omega)\,H_k(\omega)}{\overline{M}_j\overline{M}_k\,\omega_j^2\,\omega_k^2}\;\cdot$$

$$\int_0^{\ell}\!\!\int_0^{\ell} \lim_{T\to\infty} \frac{1}{2\pi T}\; F^*(x,\omega)\,F(x',\omega)\,\phi_j(x)\,\phi_k(x')\,dx\,dx'\,d\omega \tag{4.11}$$

To interpret this equation, examine the applied force without consideration of the structure and define the spatial correlation of the applied forces at x and x' as

$$\overline{f(x)\,f(x')} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x,t)\,f(x',t)\,dt \tag{4.12}$$

Applying Parseval's theorem to Eq. (4.12) yields

$$\overline{f(x)\, f(x')} = \frac{1}{2}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2\pi T} F^{*}(x,\omega)\, F(x',\omega)\, d\omega \tag{4.13}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} G_f(x,x',\omega)\, d\omega$$

where

$$G_f(x,x',\omega) = \lim_{T\to\infty} \frac{1}{2\pi T} F^{*}(x,\omega)\, F(x',\omega) \tag{4.14}$$

Equation (4.14) defines a spatial correlation density for the applied forces at frequency ω. The quantity $G_f(x,x',\omega)$ can be estimated electronically by multiplying the outputs obtained by simultaneously passing $f(x,t)$ and $f(x',t)$ through identical narrow band filters whose central frequency varies slowly over the desired frequency range. Substituting Eq. (4.14) into Eq. (4.11) produces

$$\overline{y(x)\, y(x')} = \frac{1}{2}\sum_j\sum_k \phi_j(x)\,\phi_k(x')\int_{-\infty}^{\infty} \frac{H_j^{*}(\omega)\, H_k(\omega)}{\overline{M}_j\overline{M}_k\,\omega_j^2\,\omega_k^2}\,\cdot$$

$$\int_0^{\ell}\int_0^{\ell} G_f(x,x',\omega)\,\phi_j(x)\,\phi_k(x')\,dx\,dx'\,d\omega \tag{4.15}$$

As a convenience for writing equations, define

$$L_{jk}(\omega) = \frac{1}{\overline{M}_j\overline{M}_k\,\omega_j^2\,\omega_k^2}\int_0^{\ell}\int_0^{\ell} G_f(x,x',\omega)\,\phi_j(x)\,\phi_k(x')\,dx\,dx' \tag{4.16}$$

Making use of the definition of (4.16), Eq. (4.13) may be rewritten as

$$\overline{y(x)\, y(x')} = \frac{1}{2}\sum_j\sum_k \phi_j(x)\,\phi_k(x')\int_{-\infty}^{\infty} L_{jk}(\omega)\, H_j^{*}(\omega)\, H_k(\omega)\, d\omega \tag{4.17}$$

Since the integrand in Eq. (4. 17) is an even function of ω, the 1/2 may be deleted and the lower limit of the integrand changed from $-\infty$ to 0.

The mean square response at x is found by letting $x' = x$ in Eq. (4. 17), and may be written as

$$\overline{y^2(x)} = \sum_j \sum_k \phi_j(x)\, \phi_k(x) \int_0^\infty L_{jk}(\omega)\, H_j^*(\omega)\, H_k(\omega)\, d\omega \tag{4. 18}$$

In terms of even and odd functions of j and k, Eq. (4.1 8) can be written as

$$\begin{aligned} \overline{y^2(x)} &= \sum_j \phi_j^2(x) \int_0^\infty H_j(\omega)^2\, L_{jj}(\omega)\, d\omega \\ &\quad + \sum_j \sum_k \phi_j(x)\, \phi_k(x) \int_0 L_{jk}(\omega)\, H_j(\omega)\, H_k(\omega)\, d\omega \end{aligned} \tag{4. 19}$$

The first summation of Eq. (4. 19) is the arithmetic sum of independent contributions from all of the normal modes whereas the second summation $(j \neq k)$ is the contribution to the total response due to coupling between the motions of the various modes.

In terms of a physically realizable spectra, the mean square response can be expressed as

$$\overline{y^2(x)} = \int_0^\infty G_y(x, \omega)\, d\omega \tag{4. 20}$$

where the response spectral density for the distributed system is

$$G_y(x, \omega) = \sum_j \sum_k \phi_j(x)\, \phi_k(x)\, L_{jk}(\omega)\, H_j^*(\omega)\, H_k(\omega) \tag{4. 21}$$

4.3 DISCUSSION OF RESULTS

This discussion considers the use of structural modes, generalized coordinates, and Fourier transforms to derive the correlation between the response at points x and x' and to derive the mean square response of a distributed structure to random excitation. These results are given as Eqs. (4.17) and (4.18). Both the response correlation and the mean square response are seen to consist of the summation of the normal modes, the frequency response functions and their conjugates, and a factor defined as $L_{jk}(\omega)$.

It is not unusual to express Eq. (4.17) and (4.18) in terms of spectral densities. For example, the correlation expression of (4.15) may be stated as

$$\overline{y(x)\, y(x')} = \int_0^\infty G_y(x, x', \omega)\, d\omega \tag{4.22}$$

where

$$G_y(x, x', \omega) = \sum_j \sum_k \phi_j(x)\, \phi_k(x') \; \frac{H_j^*(\omega)\, H_k(\omega)}{\overline{M}_j\, \overline{M}_k\, \omega_j^2\, \omega_k^2} \; \mathcal{G}^f_{jk}(\omega) \tag{4.23}$$

and

$$\mathcal{G}^f_{jk}(\omega) = \int_0^\ell \int_0^\ell G_f(x, x', \omega)\, \phi_j(x)\, \phi_k(x')\, dx\, dx' \tag{4.24}$$

Equation (4.23) defines a realizable cross-spectral density of the response for a distributed elastic structure and relates the response spectral density to the spectral density of the generalized force. In comparison with the expressions for the input-output spectral density for a single degree-of-freedom system (see Eq. (3.57)), Eq. (4.23) is similar in form but requires a modal summation to account for the many modes of the distributed structure.

Equation (4.24) defines a relationship between the spectral density of the generalized force $\left[\mathcal{G}^f_{jk}(\omega)\right]$ and the realizable cross spectra of the external loading $\left[G_f(x, x', \omega)\right]$. The external force spectral density, in turn, is the Fourier transform of the time average of the external forces at positions x and x' (see Eq. (4.12)).

The response cross-spectra is noted to vary (for j = k) as the reciprocal of the fourth power of the modal frequencies. Hence, it may be expected that the response at positions x and x' become less correlated with increasing frequencies. It is apparent, however, that the magnitude of the correlation also is dependent upon the mode shapes, the frequency response of the modes, and the cross-spectra of the excitation. For complex structures such as shells, for example, the mode shapes are not necessarily simple functions so that the computations implied by Eq. (4.23) may be tedious indeed. For realistic aircraft and missile structures, the mode shapes may be impossible to measure accurately let alone express in functional form.

The power spectra of the generalized force can be written as

$$\mathcal{G}^{f}_{jk}(\omega) = G_f(x_0, \omega)\,\ell^2 \left[\frac{1}{\ell^2}\int_0^{\ell}\int_0^{\ell} G_0(x, x', \omega)\,\phi_j(x)\,\phi_k(x')\,dx\,dx'\right] \qquad (4.25)$$

where

$$G_0(x, x', \omega) = \frac{G_f(x, x', \omega)}{G_f(x_0, \omega)} \qquad (4.26)$$

and

$$G_f(x_0, \omega) = \lim_{T\to\infty} \frac{1}{2\pi T}\; F^*(x_0, \omega)\, F(x_0, \omega) \qquad (4.27)$$

Equation (4.26) is a normalized correlation coefficient in a narrow band of frequency ω whereas Eq. (4.27) is the spectral density of the loading in a narrow band at frequency ω and spatial position x_0. $G_0(x, x', \omega)$ describes a coefficient normalized by dividing by a mean square loading at an arbitrarily selected reference position x_0. This reference position (x_0) usually is a point on the structure of maximum loading so that the coefficient defined by (4.26) has a maximum value of unity. The advantage of the form of Eq. (4.25) is that the expression within the square brackets is nondimensional and has values ranging from 0 to 1. It is a measure of the interaction between the excitation force and mode shapes and has been termed "joint acceptance" by Powell in Reference 6.

In terms of the spectral density of the applied loading, the L_{jk} term of Eq. (416) appears as

$$L_{jk}(\omega) = \frac{1}{\overline{M}_j \, \overline{M}_k \, \omega_j^2 \, \omega_k^2} \, \mathcal{G}^f_{jk}(\omega) \tag{4.28}$$

Thus, the $L_{jk}(\omega)$ term may be considered as a weighted density function of the generalized loading and provides a measure of the effectiveness of a particular forcing field in exciting a given mode of vibration.

In terms of modal impedances, the cross-spectral density of the response may be written as

$$G_y(x, x', \omega) = \sum_j \sum_k \phi_j(x)\, \phi_k(x') \, \frac{\mathcal{G}^f_{jk}(\omega)}{Z_j^*(\omega)\, Z_k(\omega)} \tag{4.29}$$

where

$$Z_k(\omega) = \frac{H_k(\omega)}{\overline{M}_k \, \omega_k^2} \tag{4.30}$$

$$Z_j^*(\omega) = \frac{H_j^*(\omega)}{\overline{M}_j \, \omega_j^2} \tag{4.31}$$

Note that the modal impedance of Eq. (4.30) is similar in form to the impedance defined for the mechanical oscillator in Eq. (1.31). Comparing these two equations shows the magnification factors to be identical and the mass of the mechanical oscillator is equivalent to the generalized mass for the particular mode.

The equations presented here theoretically allow one to compute the correlation between the response at points x and x' and the mean square response at any position x for any distributed elastic structure to stationary random excitation. Consider those quantities which are necessary for these computations:

(1) External Loading

- power spectral density at some convenient reference point (x_0) defined as equal to $G_f(x_0, \omega)$
- spatial correlation coefficient of the loading normalized as shown in Eq. (4.26) and defined as $G_0(x, x', \omega)$

or, alternatively,

- spatial correlation density at frequency ω defined as $G_f(x, x', \omega)$

(2) Mode shapes of the structure $\phi_j(x)$, $\phi_k(x')$

(3) Mechanical impedance of the structure

- natural frequencies of the normal modes
- damping of the normal modes

It is not an easy task to either empirically measure or analytically compute these quantities for realistic aircraft and missile structures. Moreover, if all of these quantities could be accurately determined, the correlation and mean square expressions shown in the text would not be computationally easy to evaluate. Although the equations for the distributed structure may prove impractical for realistic structures, it is instructive to consider the application of the theoretical expressions to simpler structural systems.

4.4 EXAMPLE PROBLEMS

Problem 4.1 : The mean square response at any point of a simply supported uniform beam acted upon by two concentrated periodic loads.

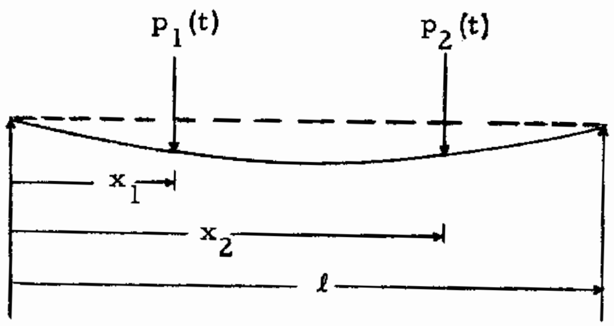


Figure 15. Simply Supported Uniform Beam Acted Upon by Forces $p_1(t)$ and $p_2(t)$ located at Positions x_1 and x_2.

As shown in Figure 15, the periodic loads are defined as $p_1(t)$ and $p_2(t)$, and, are applied at positions x_1 and x_2 respectively. The mean square response is given by Eq. (4.18) and appears as

$$\overline{y^2(x)} = \sum_j \sum_k \phi_j(x)\, \phi_k(x) \int_0^\infty L_{jk}(\omega)\, H_j^*(\omega) \quad H_k(\omega) \; d\omega \tag{4.32}$$

It is required but to evaluate Eq. (4.32) for the simply supported beam and the prescribed loading conditions of this problem.

The mode shapes may be written as

$$\phi_j(x) = \sqrt{2}\ \sin \frac{j\pi x}{\ell} \tag{4.33}$$

$$\phi_k(x) = \sqrt{2}\ \sin \frac{k\pi x}{\ell} \tag{4.34}$$

The product of the modal magnification factor and its conjugate appears in expanded form, as

$$H_j^*(\omega)\ H_k(\omega) = \frac{1}{1-\left(\frac{\omega}{\omega_j}\right)^2 - i\,2\zeta_j \frac{\omega}{\omega_j}} \cdot \frac{1}{1-\left(\frac{\omega}{\omega_k}\right)^2 + i2\zeta_k \frac{\omega}{\omega_k}} \tag{4.35}$$

The quantity L_{jk} is given by Eq. (4.16) and appears as

$$L_{jk}(\omega) = \frac{1}{\overline{M}_j\ \overline{M}_k\ \omega_j^2\ \omega_k^2} \int_0^\ell \int_0^\ell G_f(x, x', \omega)\ \phi_j(x)\ \phi_k(x)\ dx\ dx' \tag{4.36}$$

By selecting the coefficient $\sqrt{2}$ for the mode shapes, the generalized mass (see Eq. (2.80)) reduces to the mass of the beam. Therefore,

$$\overline{M}_j = \overline{M}_k = m\ell = M \tag{4.37}$$

The associated modal frequencies for the simple supported beam are given as

$$\omega_j = j^2\ \pi^2 \sqrt{\frac{EI}{M\ell^3}} \tag{4.38}$$

$$\omega_k = k^2\ \pi^2 \sqrt{\frac{EI}{M\ell^3}} \tag{4.39}$$

It is required now to evaluate the spatial correlation density $G_f(x, x', \omega)$ so that $L_{jk}(\omega)$ can be expressed in expanded form. The loading on the beam can be written in terms of delta functions as

$$f(x,t) = p_1(t)\ \ \delta(x-x_1) + p_2(t)\ \delta(x-x_2) \tag{4.40}$$

Using the definition of Eq. (4. 13), the spatial correlation density for the loading resolves into

$$G_f(x,x',\omega) = \lim_{T\to\infty} \frac{1}{2\pi T} F^*(x,\omega)\ F(x',\omega)$$

$$= \lim_{T\to\infty} \frac{1}{2\pi T} \left\{P_1^*(\omega)\ \delta(x-x_1) + P_2^*(\omega)\ \delta(x-x_2)\right\} \cdot \left\{P_1(\omega)\ \delta(x'-x_1) + P_2(\omega)\ \delta(x'-x_2)\right\} \qquad (4.41)$$

Substituting (4. 41) into (4. 37) yields

$$L_{jk}(\omega) = \frac{1}{\overline{M}_j \overline{M}_k \omega_j^2 \omega_k^2} \left\{\lim_{T\to\infty} \frac{1}{2\pi T} \left(P_1^* P_1\ \phi_j(x_1)\phi_k(x_1) + P_1^* P_2\ \phi_j(x_1)\ \phi_k(x_2) + P_2^* P_1\ \phi_j(x_2)\phi_k(x_1) + P_2^* P_2\ \phi_j(x_2)\phi_k(x_2)\right)\right\} \qquad (4.42)$$

Note the delta functions define the integrals of Eq. (4. 37) equal to zero except where $x = x_1$ and $x' = x_2$. Moreover, the correlation densities of the applied forces may be written as

$$\lim_{T\to\infty} \frac{1}{2\pi T} P_1^*\ P_1 = \text{autocorrelation density} = G_f(x_1, x_1, \omega) \qquad (4.43)$$

$$\lim_{T\to\infty} \frac{1}{2\pi T} P_1^* P_2 = \text{spatial cross-correlation density of the loading at points } x_1 \text{ and } x_2 = G_f(x_1, x_2, \omega) \qquad (4.44)$$

Substituting Eq. (4.43) and (4.44) into (4.42) provides

$$L_{jk}(\omega) = \frac{1}{\overline{M}_j \overline{M}_k \omega_j^2 \omega_k^2} \Big\{ G_f(x_1, x_1, \omega)\phi_j(x_1)\phi_k(x_1) + G_f(x_1, x_2, \omega)\phi_j(x_1)\phi_k(x_2) \tag{4.45}$$

$$+ G_f(x_1, x_2, \omega)\phi_j(x_2)\phi_k(x_1) + G_f(x_2, x_2, \omega)\phi_j(x_2)\phi_k(x_2) \Big\}$$

In the form of Eq. (4.28), L_{jk} may be stated as

$$L_{jk} = \frac{\ell^6}{j^4 k^4 \pi^8 (EI)^2} \mathcal{G}_{jk}^f(\omega) \tag{4.46}$$

where

$$\frac{1}{\overline{M}_j \overline{M}_k \omega_j^2 \omega_k^2} = \frac{\ell^6}{j^4 k^4 \pi^8 (EI)^2} \tag{4.47}$$

and

$$\mathcal{G}_{jk}^f(\omega) = G_f(x_1, x_1, \omega)\phi_j(x_1)\phi_k(x_1) + G_f(x_1, x_2, \omega)\phi_j(x_1)\phi_k(x_2) \tag{4.48}$$

$$+ G_f(x_1, x_2, \omega)\phi_j(x_2)\phi_k(x_1) + G_f(x_2, x_2, \omega)\phi_j(x_2)\phi_k(x_2)$$

In terms of generalized spectral densities, Eq. (4.48) appears as

$$\mathcal{G}_{jk}^f(\omega) = \mathcal{G}_{jk}^f(x_1, x_1, \omega) + \mathcal{G}_{jk}^f(x_1, x_2, \omega) + \mathcal{G}_{jk}^f(x_2, x_1, \omega) + \mathcal{G}_{jk}^f(x_2, x_2, \omega) \tag{4.49}$$

The first and last terms of (4.49) describe generalized ordinary spectral densities whereas the second and third terms describe generalized cross-spectral densities.

Case 1: Both loads are the same frequency (ω_0) and differ from one another by the phase angle α. The loading conditions can be expressed as

$$p_1(t) = p_1 \sin \omega_0 t; \; p_2(t) = p_2 \sin(\omega_0 t - \alpha) \tag{4.50}$$

According to the definition of Eq. (4.12), the value of the spatial correlation functions for the external sinusoidal loadings can be calculated as

$$\begin{aligned} \overline{p_1(t)\, p_2(t)} &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} p_1 p_2 \sin \omega_o t \sin(\omega_o t - \alpha)\, dt \\ &= \frac{p_1 p_2}{2} \cos \alpha \end{aligned} \tag{4.51}$$

and

$$\overline{p_1^2(t)} = \frac{p_1^2}{2} \; ; \qquad \overline{p_2^2(t)} = \frac{p_2^2}{2} \tag{4.52}$$

Statement (4.51) describes a spatial cross-correlation value for the loadings at positions x_1 and x_2 whereas (4.52) describes ordinary spatial correlation values (a mean square value, in this case). The spectral densities corresponding to these correlation functions are formed by taking the Fourier transforms of the correlation functions. For a sinusoid at frequency (ω_0), Eqs. (4.51) and (4.52) can be expressed in terms of spectral densities as

$$G_f(x_1, x_2, \omega_o) = \frac{p_1 p_2}{2} \cos\alpha \ \delta(\omega - \omega_o) \tag{4.53}$$

$$G_f(x_1, x_1, \omega_o) = \frac{p_1^2}{2} \ \delta(\omega - \omega_o) \tag{4.54}$$

$$G_f(x_2, x_2, \omega_o) = \frac{p_2^2}{2} \ \delta(\omega - \omega_o) \tag{4.55}$$

Making use of the expressions for the mode shapes (Eqs. (4.33) and (4.34)), the modal magnification factor (Eq. (4.35)), the weighted generalized force term $\left[L_{jk}(\omega)\right]$ and the spectral densities (Eqs. (4.53), (4.54), and (4.55)), the frequency interval of Eq. (4.32) can be expressed as

$$\begin{aligned}
\int_0^\infty L_{jk}(\omega)\, H_j^*(\omega)\, H_k(\omega)\, d\omega = \\
\frac{2}{M^2 \omega_j^2 \omega_k^2} \cdot \Bigg\{ & \frac{p_1^2}{2} \sin\frac{j\pi x_1}{\ell} \sin\frac{k\pi x_1}{\ell} \int_0^\infty H_j^*(\omega)\, H_k(\omega)\, \delta(\omega-\omega_o)\, d\omega \\
& + \frac{p_1 p_2}{2} \cos\alpha \sin\frac{j\pi x_1}{\ell} \sin\frac{k\pi x_2}{\ell} \int_0^\infty H_j^*(\omega)\, H_k(\omega)\, \delta(\omega-\omega_o)\, d\omega \\
& + \frac{p_1 p_2}{2} \cos\alpha \sin\frac{j\pi x_2}{\ell} \sin\frac{k\pi x_1}{\ell} \int_0^\infty H_j^*(\omega)\, H_k(\omega)\, \delta(\omega-\omega_o)\, d\omega \\
& + \frac{p_2^2}{2} \sin\frac{j\pi x_2}{\ell} \sin\frac{k\pi x_2}{\ell} \int_0^\infty H_j^*(\omega)\, H_k(\omega)\, \delta(\omega-\omega_o)\, d\omega \Bigg\}
\end{aligned} \tag{4.56}$$

The integral $\int_0^\infty H_j^*(\omega) H_k(\omega)\,\delta(\omega-\omega_o)\,d\omega$ is common to every term and, by making use of the property of the delta function, is equal to $H_j^*(\omega_o)\,H_k(\omega_o)$. Therefore, Eq. (4.56) becomes

$$\int_0^\infty L_{jk}(\omega)\,H_j^*(\omega)\,H_k(\omega)\,d\omega =$$

$$\frac{2H_j^*(\omega_o)\,H_k(\omega_o)}{M^2\,\omega_j^2\,\omega_k^2}\left\{\frac{p_1^2}{2}\sin\frac{j\pi x_1}{\ell}\sin\frac{k\pi x_1}{\ell}+\frac{p_1 p_2}{2}\cos\alpha\left(\sin\frac{j\pi x_1}{\ell}\sin\frac{k\pi x_2}{\ell}\right.\right.$$

$$\left.\left.+\sin\frac{j\pi x_2}{\ell}\sin\frac{k\pi x_1}{\ell}\right)+\frac{p_2^2}{2}\sin\frac{j\pi x_2}{\ell}\sin\frac{k\pi x_2}{\ell}\right\} \tag{4.57}$$

As in Eq. (4.32), the mean square response at any point x is obtained by multiplying Eq. (4.57) by $\phi_j(x)\phi_k(x)$ and summing over all j and k. In alternate form, Eq. (4.32) can be rewritten as

$$\overline{y^2}(x) = \sum_j \sum_k a_{jk} \tag{4.58}$$

where

$$a_{jk} = \frac{2\phi_j(x)\,\phi_k(x)\left[H_j^*(\omega_o)\right]\left[H_k(\omega_o)\right]}{M^2\,\omega_j^2\,\omega_k^2}\left\{\frac{p_1^2}{2}\sin\frac{j\pi x_1}{\ell}\sin\frac{k\pi x_1}{\ell}\right.$$

$$+\frac{p_1 p_2}{2}\cos\alpha\left(\sin\frac{j\pi x_1}{\ell}\sin\frac{k\pi x_2}{\ell}+\sin\frac{j\pi x_2}{\ell}\sin\frac{k\pi x_1}{\ell}\right) \tag{4.59}$$

$$\left.+\frac{p_2^2}{2}\sin\frac{j\pi x_2}{\ell}\sin\frac{k\pi x_2}{\ell}\right\}$$

When $j = k$, the mean square terms appear as

$$a_{jj} = \frac{2\phi_j^2(x)\left|H_j(\omega_o)\right|^2}{M^2\omega_j^4}\left\{\frac{p_1^2}{2}\sin^2\frac{j\pi x_1}{\ell} + p_1 p_2 \cos\alpha\left(\sin\frac{j\pi x_1}{\ell}\sin\frac{j\pi x_2}{\ell}\right) + \frac{p_2^2}{2}\sin^2\frac{j\pi x_2}{\ell}\right\} \tag{4.60}$$

where

$$\left|H_j(\omega_o)\right|^2 = \frac{1}{\left[1-\left(\frac{\omega_o}{\omega_j}\right)^2\right]^2 + \left[2\zeta_j\frac{\omega_o}{\omega_j}\right]^2} \tag{4.61}$$

Since $a_{jk} = a_{kj}$, the sum of the odd terms $a_{jk} + a_{kj}$ equals

$$\left\{\text{factor}\right\}\left\{H_j(\omega_o)H_k(\omega_o) + H_j(\omega_o)H_k^*(\omega_o)\right\} = \left\{\text{factor}\right\} 2\,\text{Real}\left[H_j^*(\omega_o)\right]\left[H_k(\omega_o)\right] \tag{4.62}$$

From Eq. (4.61), the factor term of Eq. (4.62) is

$$\left\{\text{factor}\right\} = \frac{4\phi_j(x)\phi_k(x)}{M^2\omega_j^2\omega_k^2}\left\{\frac{p_1^2}{2}\sin\frac{j\pi x_1}{\ell}\sin\frac{k\pi x_1}{\ell} + \frac{p_1 p_2}{2}\cos\alpha\left(\sin\frac{j\pi x_1}{\ell}\sin\frac{k\pi x_2}{\ell} + \sin\frac{j\pi x_2}{\ell}\sin\frac{k\pi x_1}{\ell}\right) + \frac{p_2^2}{2}\sin\frac{j\pi x_2}{\ell}\sin\frac{k\pi x_2}{\ell}\right\} \tag{4.63}$$

and the real part of the product of the modal magnification factor and its conjugate is

$$\text{Real}\left[H_j^*(\omega_o)\right]\left[H_k(\omega_o)\right] = \frac{\left\{1-\left(\frac{\omega_o}{\omega_j}\right)^2\right\}\left\{1-\left(\frac{\omega_o}{\omega_k}\right)^2\right\} + 4\zeta_j\zeta_k\,\frac{\omega_o}{\omega_j}\cdot\frac{\omega_o}{\omega_k}}{\left\{\left[1-\left(\frac{\omega_o}{\omega_j}\right)^2\right]^2+\left[2\zeta_j\,\frac{\omega_o}{\omega_j}\right]^2\right\}\left\{\left[1-\left(\frac{\omega_o}{\omega_k}\right)^2\right]^2+\left[2\zeta_k\,\frac{\omega_o}{\omega_k}\right]^2\right\}} \tag{4.64}$$

Expanded to include the first four modes, Eq. (4.58) yields the mean square response of x as

$$\overline{y^2(x)} = \sum_{j=1}^{4}\sum_{k=1}^{4} a_{jk} = a_{11} + 2a_{12} + a_{22} + 2(a_{13}+a_{23}) + a_{33} + 2(a_{14}+a_{24}+a_{34}) + a_{44} \tag{4.65}$$

The first four terms of the mean square response (both j and k ranging from 1 to 2) appear as

$$a_{11} = \frac{4\sin^2\frac{\pi x}{\ell}\left|H_1(\omega_o)\right|^2}{M^2\,\omega_1^4}\left\{\frac{p_1^2}{2}\sin^2\frac{\pi x_1}{\ell} + p_1p_2\cos\alpha\left(\sin\frac{\pi x_1}{\ell}\sin\frac{\pi x_2}{\ell}\right) + \frac{p_2^2}{2}\sin^2\frac{\pi x_2}{\ell}\right\} \tag{4.66}$$

where

$$\left|H_1(\omega_o)\right|^2 = \frac{1}{\left[1-\left(\frac{\omega_o}{\omega_1}\right)^2\right]^2 + \left[2\zeta_1\,\frac{\omega_o}{\omega_1}\right]^2} \tag{4.67}$$

$$a_{12} = a_{21} = \frac{4 \sin\frac{\pi x}{\ell} \sin\frac{2\pi x}{\ell} \, \mathrm{Real}\left[H_1^*(\omega_o)\right]\left[H_2(\omega_o)\right]}{M^2 \, \omega_1^2 \, \omega_2^2} \left\{ \frac{p_1^2}{2} \sin\frac{\pi x_1}{\ell} \sin\frac{2\pi x_1}{\ell} \right. \tag{4.68}$$

$$\left. + \frac{p_1 p_2}{2} \cos\alpha \left(\sin\frac{\pi x_1}{\ell} \sin\frac{2\pi x_2}{\ell} + \sin\frac{\pi x_2}{\ell} \sin\frac{2\pi x_1}{\ell} \right) + \frac{p_2^2}{2} \sin\frac{\pi x_2}{\ell} \sin\frac{2\pi x_2}{\ell} \right\}$$

where

$$\mathrm{Real}\left[H_1^*(\omega_o)\right]\left[H_2(\omega_o)\right] = \frac{\left\{1 - \left(\frac{\omega_o}{\omega_1}\right)^2\right\}\left\{1 - \left(\frac{\omega_o}{\omega_2}\right)^2\right\} + 4\zeta_1 \zeta_2 \frac{\omega_o}{\omega_1} \cdot \frac{\omega_o}{\omega_2}}{\left\{\left[1 - \left(\frac{\omega_o}{\omega_1}\right)^2\right]^2 + \left[2\zeta_1 \frac{\omega_o}{\omega_1}\right]^2\right\}\left\{\left[1 - \left(\frac{\omega_o}{\omega_2}\right)^2\right]^2 + \left[2\zeta_2 \frac{\omega_o}{\omega_2}\right]^2\right\}} \tag{4.69}$$

$$a_{22} = \frac{4 \sin^2 \frac{2\pi x}{\ell} \left|H_2(\omega_o)\right|^2}{M^2 \, \omega_2^4} \left\{ \frac{p_1^2}{2} \sin^2 \frac{2\pi x_1}{\ell} + p_1 p_2 \cos\alpha \left(\sin\frac{2\pi x_1}{\ell} \sin\frac{2\pi x_2}{\ell} \right) \right. \tag{4.70}$$

$$\left. + \frac{p_2^2}{2} \sin^2 \frac{2\pi x_2}{\ell} \right\}$$

where

$$\left|H_2(\omega_o)\right|^2 = \frac{1}{\left[1 - \left(\frac{\omega_o}{\omega_2}\right)^2\right]^2 + \left[2\zeta_2 \frac{\omega_o}{\omega_2}\right]^2} \tag{4.71}$$

By considering the expressions of Eqs. (4.59) and (4.61), it is seen that the parameters affecting the mean square response are

- the excitation frequency (ω_o)
- the modal frequencies of the structure (ω_j , ω_k)
- the magnitude of the modal damping (ζ_j , ζ_k)
- the mass of the beam (M)
- the magnitudes of the external loading (p_1 , p_2)
- the value of the phase angle of the applied loading (α)
- the position of the applied loads (x_1 , x_2)
- the location of the measured response (x)

Although these parameters can be readily listed, their relative effects on the mean square response are not immediately obvious from the form of the response equations. However, by inspecting these equations in more detail, useful judgments can be formed.

Consider for example, the maximum magnitude of the mean square response in a single mode. From Eq. (4.66), the mean square response is formed from the effects of four quantities:

(1) a coefficient varing inversely as the product of the square of the mass of the structure and the fourth power of the modal frequency; i. e.,

$$\frac{4}{M^2 \omega_1^4} \tag{4.72}$$

(2) a sine squared term whose argument contains the spatial position of the response; i. e.,

$$\sin^2 \frac{j\pi x}{\ell} \tag{4.73}$$

This expression has a maximum of unity when x is located at the midpoint of the beam.

(3) a modal magnification factor whose maximum value is found when the excitation frequency coincides with the modal frequency of the structure; i. e.,

$$\left| H_1(\omega_o) \right|^2 = \frac{1}{(2\zeta)^2} = 25 \tag{4.74}$$

The numerical value of (4.74) is for $\zeta = .1$ which is a damping typical of practical structures.

(4) a generalized loading expression containing the magnitude of the applied forces, the phase angle between the loads, a sine function whose arguments contain the spatial position of the external loads. This expression is written as

$$\frac{p_1^2}{2}\left\{\sin^2 \frac{\pi x_1}{\ell} + 2\,\frac{p_2}{p_1}\cos\alpha\left(\sin\frac{\pi x_1}{\ell}\sin\frac{\pi x_2}{\ell}\right)+\left(\frac{p_2}{p_1}\right)^2 \sin^2 \frac{\pi x_2}{\ell}\right\} \tag{4.75}$$

For Eq. (4.75) to be a maximum, x_1 and x_2 must be defined at the midspan of the beam ($x_1 = x_2 = \ell/2$) and the loads must be in phase ($\alpha = 0$). Then, Eq. (4.75) reduces to

$$\frac{p_1^2}{2}\left[1 + 2\left(\frac{p_2}{p_1}\right)+\left(\frac{p_2}{p_1}\right)^2\right] = \frac{1}{2}\left[p_1 + p_2\right]^2 \tag{4.76}$$

Equation (4.76) shows a conservative estimate of the magnitude of this loading is one-half of the square of the sum of the applied loads. Should the loads be phased in quadrature ($\alpha = 90^o$), the loading magnitude reduces to the sum of the mean square values of the applied loads; i. e.,

$$\frac{p_1^2}{2} + \frac{p_2^2}{2} \tag{4.77}$$

Hence, the phase angle governs the loading expression in the ratio

$$\frac{1+\left(\frac{p_2}{p_1}\right)^2}{\left(1+\frac{p_2}{p_1}\right)^2} \tag{4.78}$$

This ratio ranges from 1 when $p_2/p_1 = 0$ to 0.5 when $p_2/p_1 = 1$. Beyond $p_2/p_1 = 1$, Eq. (4.78) increases and asymptotically approaches unity for very large values of p_2/p_1. This behavior points out that the phase angle can vary the magnitude of the mean square response in a single mode by a factor of two. Taking into account the preceding statements, an approximate value for the maximum mean square response in a single mode is

$$a_{11} \simeq \frac{50}{M^2 \omega_1^4} (p_1 + p_2)^2 \tag{4.79}$$

Compare now the a_{22} expression of Eq. (4.70) with the a_{11} expression of Eq. (4.66). The loading expression corresponding to Eq. (4.75) does not change. The position of maximum response, however, is at the quarter-span of the beam instead of the midspan as with a_{11}. This occurs as the argument of the sine squared term containing the spatial position of the response is $2\pi x/\ell$ instead of $\pi x/\ell$. The maximum value of the magnification factor occurs when $\omega_o = \omega_2$ and may be written as

$$\left|H_2(\omega_o)\right|^2 = \left(\frac{\omega_2}{\omega_1}\right)^2 \left|H_1(\omega_o)\right|^2 \tag{4.80}$$

Making use of Eqs. (4.80) and (4.70), the a_{22} mean square displacement response reduces to

$$a_{22} = \left(\frac{\omega_1}{\omega_2}\right)^2 a_{11} \tag{4.81}$$

Thus, compared to the fundamental mode contribution (a_{11}), the mean square displacement response contributions for all of the even terms ($j = k$) vary in the ratio of the frequency squared.

If the structure is at resonance ($\omega_o = \omega_1$), the $j = 1$, $k = 2$ cross term (see Eq. (4.69)) reduces to

$$\text{Real}\left[H_1^*(\omega_o)\right]\left[H_2(\omega_o)\right] = \frac{1}{1 + \dfrac{\left[1 - \left(\dfrac{\omega_o}{\omega_2}\right)^2\right]^2}{2\zeta_2 \dfrac{\omega_o}{\omega_2}}} \tag{4.82}$$

Assuming the modal damping is very small, Eq. (4.82) approaches zero. Hence, all odd terms containing a resonant frequency as a subscript are zero. This implies that, for small values of modal damping, no cross coupling exists between the resonant frequency and any of the modal frequencies.

If the external loads are positioned at x_1 and x_2 equal to ℓ/j where j is a given integer, all the terms of Eq. (4.61) containing the particular j value as a subscript equal zero. These conditions simply state that the jth mode of the structure cannot be excited if the external loads are applied at the nodes of that mode. Should the loads be of equal magnitude and positioned symmetrically about the midspan of the beam, the loading expressions of Eq. (4.61) reduces to

$$\frac{p_1^2}{2} \sin^2 \frac{j\pi x_1}{\ell} \; (1 + \cos \alpha) \tag{4.83}$$

Thus, as implied by Eq. (4.79), the phase angle can vary the mean square response by a factor of two.

Case 2: Conditions of Case 1 with a damped cosine spatial correlation.

The spatial part of the correlation function may be expressed as

$$e^{-\gamma\xi} \cos \frac{2\pi\xi}{\lambda} \tag{4.84}$$

where

$\xi = x_2 - x_1$ = spatial separation

λ = wave length of the damped cosine wave

γ = damping or decay factor for the cosine wave

Pictorially, this spatial correlation function appears as shown in Figure 16.

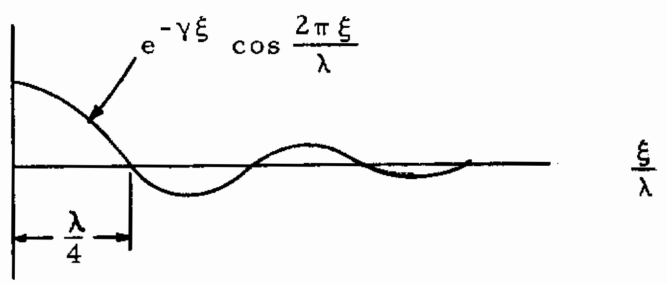


Figure 16. Damped Cosine Spatial Correlation Function

The only difference between the problems defined in Case 1 and Case 2 is the spatial correlation density which, for this problem, is given as

$$G(x_1, x_2, \omega_o) = \frac{P_1 P_2}{2} \cos\alpha \cos\frac{2\pi\xi}{\lambda} \cdot e^{-\gamma\xi} \delta(\omega - \omega_o) \tag{4.85}$$

The mean square response can be represented symbolically as

$$\overline{y^2}(x) = \sum_j \sum_k a_{jk} \tag{4.86}$$

where

$$a_{jk} = \frac{2\phi_j(x)\,\phi_k(x)\left[H_j^*(\omega_o)\right]\left[H_k(\omega_o)\right]}{M^2\,\omega_j^2\,\omega_k^2}\left\{\frac{p_1^2}{2}\sin\frac{j\pi x_1}{\ell}\sin\frac{k\pi x_1}{\ell}\right.$$

$$+\;p_1p_2\cos\alpha\cos\frac{2\pi\xi}{\lambda}\,e^{-\gamma\xi}\left(\sin\frac{j\pi x_1}{\ell}\sin\frac{k\pi x_2}{\ell}+\sin\frac{j\pi x_2}{\ell}\sin\frac{k\pi x_2}{\ell}\right) \tag{4.87}$$

$$\left.+\;\frac{p_2^2}{2}\sin\frac{j\pi x_2}{\ell}\sin\frac{k\pi x_2}{\ell}\right\}$$

For the j = k terms, Eq. (4.87) yields

$$a_{jj} = \frac{2\phi_j^2(x)\left|H_j(\omega_o)\right|^2}{M^2\,\omega_j^4}\left\{\frac{p_1^2}{2}\sin^2\frac{j\pi x_1}{\ell}+p_1p_2\cos\alpha\cos\frac{2\pi\xi}{\lambda}\,e^{-\gamma\xi}\left(\sin\frac{j\pi x_1}{\ell}\sin\frac{j\pi x_2}{\ell}\right.\right. \tag{4.88}$$

$$\left.\left.+\sin\frac{j\pi x_2}{\ell}\sin\frac{k\pi x_2}{\ell}\right)+\frac{p_2^2}{2}\sin^2\frac{j\pi x_2}{\ell}\right\}$$

The absolute value of the square of the modal magnification factor $\left[\left|H_j(\omega_o)\right|^2\right]$ is given by Eq. (4.61) whereas the product of a modal magnification factor and its conjugate $\left[H_j^*(\omega_o)\right]\left[H_j(\omega_o)\right]$ is given by Eq. (4.35).

In addition to those parameters listed in Case 1 which affect the mean square response, three additional quantities are introduced by the correlation function of Case 2.

- the rate of decay of the damped cosine term (γ)
- the spatial separation of the two loadings $(\xi = x_2 - x_1)$
- the wave length of the damped cosine term (λ)

In contrast to altering the phase angle (α) between the applied loads, these loads now can become uncorrelated if they are positioned at

$$\xi = n\frac{\lambda}{4}\quad,\quad n = 1, 3, 5, \ldots \tag{4.89}$$

Equation (4.89) defines the zero value crossings in the damped cosine plot of Figure 16. The correlation function becomes negligibly small if the exponent of the decay term is very large. This condition can be described by the sketch shown in Figure 17.

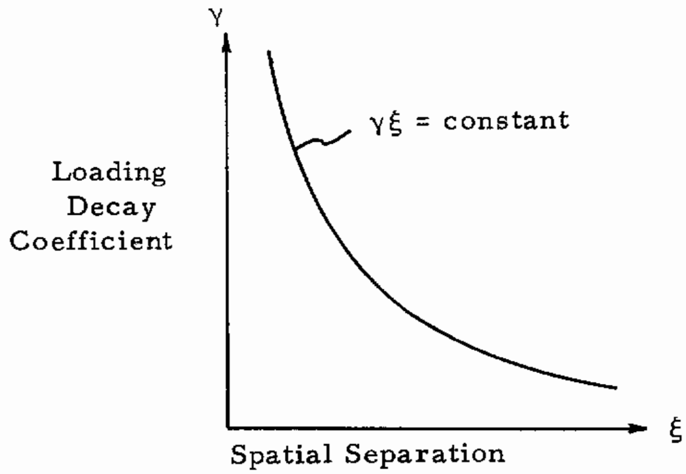


Figure 17. Sketch of the Damping Coefficient (γ) versus the Spatial Separation (ξ) for the Decay Term of the Damped Cosine Function

This sketch shows the relationship between γ and ξ is hyperbolic. In the limits, zero correlation occurs either with rapid decay (γ is very large) or by widely separating the loads (ξ is very large). For given positions of the applied loads (ξ is then a constant), the curve yields the minimum γ to insure zero correlation between the loads. Alternatively, given a fixed value of decay, the hyperbolic curve yields the spatial separation beyond which the loads are uncorrelated.

Case 3: Conditions of Case 1 except that each load is applied at a different frequency.

In contrast to Eq. (4.50), these loading conditions can be written as

$$p_1(t) = p_1 \sin \omega_o t \quad ; \quad p_2 = p_2 \sin \omega_o' t \tag{4.90}$$

The value for the spatial cross-correlation functions may be calculated as

$$\overline{p_1(t)\, p_2(t)} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} p_1 p_2 \sin \omega_o t \sin \omega_o' t \; dt$$

$$= p_1 p_2 \lim_{T \to \infty} \frac{1}{2T} \left\{ \frac{\sin (\omega_o - \omega_o')T}{\omega_o - \omega_o'} - \frac{\sin (\omega_o + \omega_o')T}{\omega_o + \omega_o'} \right\} = 0 \tag{4.91}$$

The autocorrelation functions yield mean square values of

$$\overline{p_1^2(t)} = \frac{p_1^2}{2} \quad ; \quad \overline{p_2^2} = \frac{p_2^2}{2} \tag{4.92}$$

Note the correlation densities are Fourier transforms of the correlation functions; that is, in symbolic form

$$G_f(x, x', \omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} \; \overline{p(x, t)\, p(x', t+\tau)} \; d\tau \tag{4.93}$$

$$G_f(x, x, \omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} \; \overline{p(x, t)\, p(x, t+\tau)} \; d\tau \tag{4.94}$$

Interpreting Eqs. (4.93) and (4.94) for the loading conditions of this problem yield the following spectral densities for the correlation functions

$$G_f(x_1, x_2, \omega_o, \omega_o') = 0 \tag{4.95}$$

$$G_f(x_1, x_1, \omega_o) = \frac{p_1^2}{2} \delta(\omega - \omega_o) \tag{4.96}$$

$$G_f(x_2, x_2, \omega_o') = \frac{p_2^2}{2} \delta(\omega - \omega_o') \tag{4.97}$$

The frequency terms of Eq. (4.32) now can be written as

$$\int_0^\infty L_{jk}(\omega) H_j^*(\omega) H_k(\omega) d\omega =$$

$$\frac{2}{M^2 \omega_k^2 \omega_j^2} \left\{ \frac{p_1^2}{2} \sin \frac{j\pi x_1}{\ell} \sin \frac{k\pi x_1}{\ell} \int_0^\infty H_j^*(\omega) H_k(\omega) \delta(\omega - \omega_o) d\omega \right. \tag{4.98}$$

$$\left. + \frac{p_2^2}{2} \sin \frac{j\pi x_2}{\ell} \sin \frac{k\pi x_2}{\ell} \int_0^\infty H_j^*(\omega) H_k(\omega) \delta(\omega - \omega_o') d\omega \right\}$$

The mean square response is given by Eq. (4.58) which is repeated here as

$$\overline{y^2(x)} = \sum_j \sum_k \phi_j(x) \phi_k(x) \int_0^\infty H_j^*(\omega) H_k(\omega) L_{jk}(\omega) d\omega = \sum_j \sum_k a_{jk} \tag{4.99}$$

where

$$a_{jk} = \frac{\phi_j(x)\,\phi_k(x)}{M^2 \omega_j^2 \omega_k^2} \left\{ \left[H_j^*(\omega_o)\right]\left[H_k(\omega_o)\right] p_1^2 \sin\frac{j\pi x_1}{\ell} \sin\frac{k\pi x_1}{\ell} + \left[H_j^*(\omega_o')\right]\left[H_k(\omega_o')\right] p_2^2 \sin\frac{j\pi x_2}{\ell} \sin\frac{k\pi x_2}{\ell} \right\} \tag{4.100}$$

For the j = k terms, Eq. (4.100) provides

$$a_{jj} = \frac{\phi_j^2(x)}{M^2 \omega_j^4} \left\{ \left|H_j(\omega_o)\right|^2 p_1^2 \sin^2\frac{j\pi x_1}{\ell} + \left|H_j(\omega_o')\right|^2 p_2^2 \sin^2\frac{j\pi x_2}{\ell} \right\} \tag{4.101}$$

The absolute value of the square of the modal magnification factor $\left[\left|H_j(\omega_o)^2\right|\right]$ is given by Eq. (4.61) whereas the product of a modal magnification factor and its conjugate $\left[H_j^*(\omega_o)\,H_k(\omega_o)\right]$ is given by Eq. (4.35).

As contrasted with the a_{jk} and a_{jj} coefficients in both Case 1 and Case 2, the mean square coefficients for this problem (see Eqs. (4.100) and (4.101)) contain no coupling due to correlation between the loads. However, as with both of the previous problems, the mean square response may contain contributions due to the coupling of the external loads by the structural modes. This modal coupling is evidenced in the $j \neq k$ terms of Eq. (4.100) and the magnitude is noted to be dependent upon the product of the mode shapes, the mass squared of the structure, the product of the square of each of the modal frequencies, the position of the applied loads, and the product of a modal magnification factor and its conjugate. The magnification factor term will contain quantities identical in form to Eq. (4.64) so that the value of the modal damping (ζ_j and ζ_k) and the ratio of the driving frequencies to the modal frequencies (ω_o/ω_j, ω_o/ω_k, ω_o'/ω_j, ω_o'/ω_k) also are important to the magnitude of the a_{jk} coefficients.

Problem 4.2: The mean square response at any point x of a simply supported elastic beam subjected to uniformly distributed white noise.

Pictorially, the problem is depicted as shown in Figure 18.

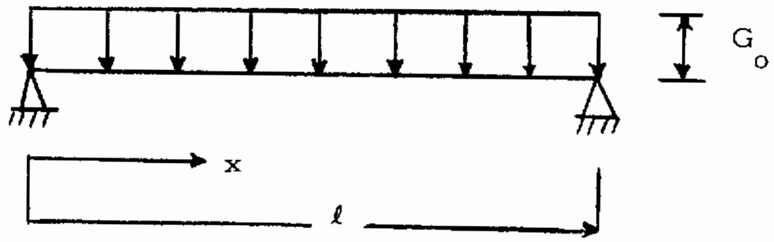


Figure 18. Simply Supported Elastic Beam Excited by Uniformly Distributed White Noise of Density G_o

The beam is assumed as homogeneous, to obey Hooke's Law, and its deformation is given by small deflection theory. The dynamic properties of the beam are described by Bernoulli-Euler theory. Note that the units associated with the magnitude of the spectral density (G_o) are force per unit length squared per radian per second.

The mean square response is noted as

$$\overline{y^2(x)} = \sum_j \sum_k \phi_j(x)\,\phi_k(x) \int_0^\infty L_{jk}(\omega)\,H_j^*(\omega)\,H_k(\omega)\,d\omega \tag{4.102}$$

where

$$L_{jk}(\omega) = \frac{1}{\overline{M}_j \overline{M}_k \omega_j^2 \omega_k^2} \int_0^\ell \int_0^\ell G_f(x, x', \omega)\,\phi_j(x)\,\phi_k(x')\,dx\,dx' \tag{4.103}$$

The values for the mode shapes $\phi_j(x)$ and $\phi_k(x)$, the generalized masses M_j and M_k, and the modal frequencies ω_j and ω_k are given for the simply supported beam in Problem 4.1. For white noise, the spatial correlation densities are expressed as

$$G_f(x, x', \omega) = G_o(\omega)\,\delta(x - x') \tag{4.104}$$

Equation (4.104) states that white noise is completely uncorrelated for any two positions ($x \neq x'$) on the beam and the ordinary spectral density at any position (x) is of magnitude G_o over all frequencies.

Due to the property of the delta function,

$$\int_0^{\ell} G_o(\omega)\,\phi_j(x)\,\delta(x - x')\,dx = G_o(\omega)\,\phi_j(x') \tag{4.105}$$

Thus, it follows that

$$\int_0^{\ell}\int_0^{\ell} G_f(x, x', \omega)\,\phi_j(x)\,\phi_k(x')\,dx\,dx' = 2\,G_o \int_0^{\ell}\int_0^{\ell} \sin\frac{j\pi x}{\ell}\,\sin\frac{k\pi x'}{\ell}\,dx\,dx' \tag{4.106}$$

Due to the orthogonality property of sine functions, the integral of Eq. (4.106) has nonzero values only when $j = k$. Hence, Eq. (4.106) reduces to

$$2\,G_o \int_0^{\ell}\int_0^{\ell} \sin^2\frac{j\pi x}{\ell}\,dx\,dx = G_o\,\ell^2 \tag{4.107}$$

The L_{jk} terms of Eq. (4.103) can now be written as

$$L_{jj} = \frac{1}{M_j^2\,\omega_j^4}\,G_o\,\ell^2 = \frac{G_o}{m^2\,\omega_j^4} \tag{4.108}$$

The mean square response then becomes

$$\overline{y^2(x)} = \phi_j^2(x)\int_0^{\infty} L_{jj}(\omega)\,H_j^*(\omega)\,H_j(\omega)\,d\omega \tag{4.109}$$

In expanded form, the integrand of Eq. (4.109) appears as

$$\frac{G_0}{m^2 \omega_j^4} \int_0^\infty \frac{1}{\left[1 - \left(\frac{\omega}{\omega_j}\right)^2\right]^2 + \left[2\zeta_j \frac{\omega}{\omega_j}\right]^2} d\omega = \frac{\pi G_0}{2} \frac{1}{2\zeta_j} \cdot \frac{1}{m^2 \omega_j^3} \tag{4.110}$$

Note that the $1/2\zeta_j$ term appearing in (4.110) is the Q value for the jth mode (Q_j). Thus, the mean square displacement response can be written as

$$\overline{y^2(x)} = \frac{\pi G_0}{m^2} \left\{ \frac{Q_1}{\omega_1^3} \sin^2 \frac{\pi x}{\ell} + \frac{Q_2}{\omega_2^3} \sin^2 \frac{2\pi x}{\ell} + \frac{Q_3}{\omega_3^3} \sin^2 \frac{3\pi x}{\ell} + \ldots \right\} \tag{4.111}$$

Equation (4.111) suggests a procedure for calculating the mean square response of a distributed elastic structure to white noise. The frequency response function at position (x) is assumed to have well defined modal peaks and the magnitude of the mode shape corresponding to the modal peaks is assumed known. The Q values corresponding to each of the modal peaks then can be obtained using the half power points as implied by Eq. (1.14). Knowing the mass density of the responding structure (m), the magnitude of the spectral density of the noise excitation (G_0), the mean square displacement response can be calculated by

$$\overline{y^2(x)} = \frac{\pi G_0}{2m^2} \sum_j \frac{Q_j}{\omega_j^3} \phi_j^2(x) \tag{4.112}$$

It is of interest to compare the expression for the mean square response at any point (Eq. (4.111)) to a reference value obtained by averaging the mean square response over the total mass of the structure. Powell (Reference 6, page 194) calculates a similar reference value but averages the mean square response over the length or surface area of the structure. Averaging the mean square response with respect to the mass allows use of the orthogonality properties associated with the generalized mass (see Eq. (2.5)). For this problem, the averaged response spectra is given as

$$E_M\left[G_y(x,\omega)\right] = \frac{1}{M}\int_0^{\ell} G_y(x,\omega)\, m(x)\, dx \tag{4.113}$$

where the response spectra is

$$G_y(x,\omega) = \sum_j \phi_j^2(x)\, L_{jj}(\omega)\, \left|H_j(\omega)\right|^2 \tag{4.114}$$

In expanded form, Eq. (4.113) appears as

$$E_M\left[G_y(x,\omega)\right] = \frac{\left|H_1(\omega)\right|^2 L_{11}(\omega)}{M}\int_0^{\ell} \phi_1^2(x)\, m(x)\, dx + \frac{\left|H_2(\omega)\right|^2 L_{22}(\omega)}{M}\int_0^{\ell} \phi_2^2(x)\, m(x)\, dx \ldots \tag{4.115}$$

For the simply supported beam, the integrals of Eq. (4.115) reduce to the total mass of the structure so that

$$E_M\left[G_y(x,\omega)\right] = \left|H_1(\omega)\right|^2 L_{11}(\omega) + \left|H_2(\omega)\right|^2 L_{22}(\omega) + \ldots \tag{4.116}$$

The mass averaged mean square response value is denoted as

$$\overline{y_M^2} = E_M\left[\overline{y^2}(x)\right] = \int_0^{\infty} E_M\left[G_y(x,\omega)\right] d\omega \tag{4.117}$$

For this problem, Eq. (4.117) is written as

$$\overline{y_M^2} = L_{11}\int_0^{\infty} \left|H_1(\omega)\right|^2 d\omega + L_{22}\int_0^{\infty} \left|H_2(\omega)\right|^2 d\omega + \ldots \tag{4.118}$$

which can be resolved to yield

$$\overline{y_M^2} = \frac{\pi G_0}{2m^2}\left\{\frac{Q_1}{\omega_1^3} + \frac{Q_2}{\omega_2^3} + \frac{Q_3}{\omega_3^3} + \ldots\right\} \tag{4.119}$$

Equation (4. 119) is noted to differ from Eq. (4. 111) in the absence of the mode shape squared term $\left[\phi_j^2(x)\right]$.

Considering only the response of the lowest mode, the ratio of the mean square response at x to the mass averaged mean square response is

$$\frac{\overline{y^2(x)}}{\overline{y_M^2}} = 2 \sin^2 \frac{\pi x}{\ell} \tag{4.120}$$

Equation (4. 120) is observed to vary from zero to two depending only upon the spatial dimension x. Hence, if this mass averaged value were used as an estimate of the mean square response of the structure, it would be accurate only at positions $x/\ell = 1/4$ and $3/4$. All other positions would be in error ranging with the mean square response at the supports being zero and twice the mass averaged value at the mid-span.

Including all of the modes for the simply supported beam, Eq. (4. 120) appears as

$$\frac{\overline{y^2(x)}}{\overline{y_M^2}} = \frac{2\left\{\frac{Q_1}{\omega_1^3}\sin^2\frac{\pi x}{\ell} + \frac{Q_2}{\omega_2^3}\sin^2\frac{2\pi x}{\ell} + \frac{Q_3}{\omega_3^3}\sin^2\frac{3\pi x}{\ell} + \ldots\right\}}{\left\{\frac{Q_1}{\omega_1^3} + \frac{Q_2}{\omega_2^3} + \frac{Q_3}{\omega_3^3} + \ldots\right\}} \tag{4.121}$$

Since the individual terms in both the numerator and denominator of (4. 121) vary inversely with the cube of the modal frequency, the higher modal frequencies, in general, have a minor effect in contrast with the lower modal frequencies.

Problem 4.3: The mean square response of a base excited cantilevered beam. The excitation is white noise.

Using simple beam theory, the equation of motion for a cantilever beam excited at the base with a sinusoidal forcing function is

$$EI \frac{\partial^4 y(x,t)}{\partial x^4} + c \frac{\partial y(x,t)}{\partial t} + m \frac{\partial^2 y(x,t)}{\partial t^2} = -m\ddot{y}_0 \sin \omega t \tag{4.122}$$

where

y = beam elastic deformation

$\ddot{y}_0$ = base acceleration

By assuming the beam as homogeneous and uniform, the coefficients of Eq. (4.122) are constants. The modal solution is written as

$$y(x,t) = \sum_j \phi_j(x)\, q_j(t) \tag{4.123}$$

Substituting Eq. (4.123) into Eq. (4.122) and imposing orthogonality conditions results in

$$\ddot{q}_j(t) + 2\zeta_j \omega_j \dot{q}_j(t) + \omega_j^2 q_j(t) = -\frac{m\ddot{y}_0}{\overline{M}_j} \int_0^{\ell} \phi_j(x)\, dx \sin \omega t \tag{4.124}$$

The Fourier transform of Eq. (4.124) yields

$$Q_j(\omega) = -\frac{H_j(\omega)}{\overline{M}_j \omega_j^2} \int_0^{\ell} \phi_j(x)\, dx \cdot \ddot{Y}_0(\omega) \tag{4.125}$$

Making use of Eq. (4.123) and (4.125), the Fourier transform of the response appears as

$$Y(x,\omega) = \sum_j \phi_j(x)\, Q_j(\omega) = \sum_j \phi_j(x) \frac{H_j(\omega)}{\omega_j^2 \overline{M}_j} \int_0^{\ell} \phi_j(x)\, m\ddot{Y}_0(\omega)\, dx \tag{4.126}$$

The correlation between the response at points x and x' is given as

$$\overline{y(x)\ y(x')} = \frac{1}{2} \int_{-\infty}^{\infty} \lim_{T \to \infty} \frac{1}{2\pi T} Y^{*}(x, \omega) Y(x', \omega)\, d\omega \tag{4.127}$$

In terms of the spectral density between the response at points x and x', Eq. (4.127) may be expressed as

$$\overline{y(x)\ y(x')} = \int_{0}^{\infty} G_y(x, x', \omega)\, d\omega \tag{4.128}$$

where

$$G_y(x, x', \omega) = \lim_{T \to \infty} \frac{1}{2\pi T} Y^{*}(x, \omega)\ Y(x', \omega) \tag{4.129}$$

From Eq. (4.126), the cross-spectral density of the response is written as

$$G_y(x, x', \omega) = \sum_j \sum_k \phi_j(x) \phi_k(x') \frac{H_j(\omega)\ H_k^{*}(\omega)}{\overline{M}_j \overline{M}_k\ \omega_j^2\ \omega_k^2} \mathcal{G}_{jk}^{f}(\omega) \tag{4.130}$$

where the spectral density of the generalized force is

$$\mathcal{G}_{jk}^{f}(\omega) = \int_0^{\ell} \int_0^{\ell} m(x')\ \phi_k(x')\ m(x)\ \phi_j(x)\ G_{\alpha}(x, x', \omega)\, dx\, dx' \tag{4.131}$$

and the cross-spectral density of the external loading (acceleration) is denoted as

$$G_{\alpha}(x, x', \omega) = \lim_{T \to \infty} \frac{1}{2\pi T}\ \ddot{Y}_0(x, \omega)\ \ddot{Y}_0^{*}(x', \omega) \tag{4.132}$$

Equation (4.132) is noted to define a power spectral density of the base acceleration. For white noise, the excitation is uncorrelated with spatial position so that the Fourier transforms of Eq. (4. 132) are independent of the positions x and x'. Requiring the structure to have a constant mass per unit length (m), (4. 131) now can be written as

$$\mathcal{J}^{f}_{jk}(\omega) = m^2 G_\alpha(\omega) \int_0^\ell \int_0^\ell \phi_k(x') \, \phi_j(x) \; \delta(x-x') \tag{4. 133}$$

where the spectral density is written as

$$G_\alpha(x, x', \omega) = G_0(\alpha) \; \delta(x-x') \tag{4. 134}$$

and $G_0(\alpha)$ has the units for acceleration squared per radian per second. Since

$$\int_0^\ell \phi_j(x) \; \delta(x-x') \, dx = \phi_j(x') \tag{4. 135}$$

Equation (4. 133) then reduces to

$$\mathcal{J}^{f}_{jj}(\omega) = m^2 G_0(\alpha) \; \ell^2 \tag{4. 136}$$

In terms of the L_{jk} parameter, the cross-spectral density of the response is

$$G_y(x, x', \omega) = \sum_j \sum_k \; \phi_j(x) \, \phi_k(x') \; H_j(\omega) \, H_k^*(\omega) \, L_{jk}(\omega) \tag{4. 137}$$

where

$$L_{jk}(\omega) = \frac{1}{\overline{M}_j \, \overline{M}_k \; \omega_j^2 \, \omega_k^2} \; \mathcal{J}^{f}_{jj}(\omega) \tag{4. 138}$$

For this problem, the L_{jk} expression has meaning only when j = k and appears as

$$L_{jj}(\omega) = \frac{(m\ell)^2 G_0(\alpha)}{\overline{M}_j^2 \; \omega_j^4} = \frac{G_0(\alpha)}{\omega_j^4} \tag{4.139}$$

The mean square response at point x is noted as

$$\overline{y^2(x)} = \sum_j \phi_j^2(x) \int_0^\infty L_{jj}(\omega) \left|H_j(\omega)\right|^2 \, d\omega \tag{4.140}$$

where the square of the absolute value of the modal magnification factor $\left[\left|H_j(\omega)\right|^2\right]$ is given by Eq. (4.61). The integrand of (4.140) reduces to

$$\int_0^\infty L_{jj}(\omega) \left|H_j(\omega)\right|^2 d\omega = \frac{\pi G_0(\alpha)}{2} \cdot \frac{1}{\omega_j^3} \cdot \frac{1}{2\zeta_j} = \frac{\pi G_0(\alpha)}{2} \cdot \frac{Q_j}{\omega_j^3} \tag{4.141}$$

and, as shown in (2.37), the mode shape for the cantilever beam is

$$\phi_j(x) = \cosh \lambda_j x - \cos \lambda_j x - \alpha_j (\sinh \lambda_j x - \sin \lambda_j x) \tag{4.142}$$

where

$$\alpha_j = \frac{\cos \lambda_j \ell + \cosh \lambda_j \ell}{\sin \lambda_j \ell + \sinh \lambda_j \ell} \tag{4.143}$$

and

$$(\lambda_j \ell)^4 = \frac{m\ell^4}{EI} \omega_j^2 \tag{4.144}$$

To compute the rms stresses in the beam, the rms bending moment must be determined. The relationship between the rms stress level $\left[s_{rms}(x)\right]$ at position x at the rms bending moment $\left[M_{rms}(x)\right]$ at position x is given as

$$s_{rms}(x) = \frac{h}{I} M_{rms}(x) \tag{4.145}$$

where h is the vertical distance from the neutral axis of the beam and I is the area moment of inertia of the beam cross section about the bending axis. The rms bending moment is written as

$$M_{rms}(x) = \sqrt{\overline{M^2(x)}} \tag{4.146}$$

From simple beam theory, the bending moment at x is

$$M(x) = EI \frac{\partial^2 y(x,t)}{\partial x^2} \tag{4.147}$$

The second order partial expression may be written in terms of the mode shapes and generalized coordinates as

$$M(x) = EI \sum_j \frac{\partial^2 \phi_j(x)}{\partial x^2} \, q_j(t)$$

Following the derivation procedures in Sections 4.1 and 4.2, the mean square bending moment appears as

$$\overline{M^2(x)} = (EI)^2 \sum_j \sum_k \frac{\partial^2 \phi_j(x)}{\partial x^2} \cdot \frac{\partial^2 \phi_k(x)}{\partial x^2} \int_0^{\infty} L_{jk}(\omega)\, H_j(\omega)\, H_k^*(\omega)\, d\omega \tag{4.148}$$

For the conditions of this problem, Eq. (4.148) reduces to

$$\overline{M^2(x)} = \frac{\pi G_0}{2} \left(\frac{EI}{m}\right)^2 \cdot \sum_j \frac{Q_j}{\omega_j^3} \left(\frac{\partial^2 \phi_j(x)}{\partial x^2}\right)^2 \tag{4.149}$$

The partial derivative terms of (4.149) for the cantilever beam are to be found tabulated by Felgar in Ref. 8. For other than homogeneous elastic structures with the simpler boundary conditions and geometry (such as a simply supported rectangular plate or beam), accurate mode shapes - let alone their second derivative - are difficult to obtain in functional form.

Consider now the derivation of an expression for the mean square acceleration response of an arbitrary linear elastic structure to stationary random excitation. In terms of normal modes $\left[\phi_j(x)\right]$ and generalized coordinates $\left[q_j(t)\right]$, the displacement response is expressed by Eq. (4.1)

$$y(x,t) = \sum_j \phi_j(x)\, q_j(t) \tag{4.150}$$

The acceleration is defined as the second time derivative of (4.150) and may be written as

$$\ddot{y}(x,t) = \sum_j \phi_j(x)\, \ddot{q}_j(t) \tag{4.151}$$

where the double dot notation is defined as

$$\ddot{(\ )} = \frac{d^2(\ )}{dt^2} \tag{4.152}$$

Only when the generalized coordinate is a harmonic function in time will

$$\ddot{y}(x,t) = -\omega^2\, y(x,t) \tag{4.153}$$

For the condition where the excitation is random, the simple expression of Eq. (4.153) relating the acceleration and displacement no longer is valid.

The Fourier transform of the displacement response in the jth mode is

$$Y_j(\omega) = \phi_j(x)\, Q_j(\omega) \tag{4.154}$$

For zero initial conditions, the Fourier transform of the acceleration response in the jth mode is, however,

$$\ddot{Y}_j(\omega) = \phi_j(x)\,\ddot{Q}_j(\omega) = \phi_j(x)\,\omega^2\,Q_j(\omega) \tag{4.155}$$

From the derivation in Section 4.1 and Eq. (4.155), the Fourier transform of the acceleration response may be written as

$$\ddot{Y}(x,\omega) = \sum_j \phi_j(x)\left(\frac{\omega}{\omega_j}\right)^2 \frac{H_j(\omega)}{\overline{M}_j}\int_0^{\ell} F(x,\omega)\,\phi_j(x)\,dx \tag{4.156}$$

and, the mean square acceleration response becomes

$$\overline{\ddot{y}^2(x)} = \sum_j \sum_k \phi_j(x)\,\phi_k(x)\int_0^{\infty} \omega^4\,L_{jk}(\omega)\,H_j^*(\omega)\,H_k(\omega)\,d\omega \tag{4.157}$$

In contrast to the mean square displacement response of Eq. (4.18), Eq. (4.157) is noted to contain an ω^4 term in the integrand. This can introduce serious convergence problems when evaluating the integral expression shown in (4.157). For the conditions of this example problem, the integral expression appears as

$$\int_0^{\infty} \omega^4 L_{jj}(\omega)\,\left|H_j(\omega)\right|^2 d\omega = \int_0^{\infty} \frac{G_0(\alpha)\left(\dfrac{\omega}{\omega_j}\right)^4}{\left[1-\left(\dfrac{\omega}{\omega_j}\right)^2\right]^2+\left[2\zeta_j\,\dfrac{\omega}{\omega_j}\right]^2}\,d\omega \qquad (4.158$$

Therefore, the mean square acceleration is of the form

$$\overline{y^2(x)} = \sum_j \phi_j^2(x)\int_0^{\infty}\frac{G_0(\alpha)\left(\dfrac{\omega}{\omega_j}\right)^4}{\left[1-\left(\dfrac{\omega}{\omega_j}\right)^2\right]^2+\left[2\zeta_j\,\dfrac{\omega}{\omega_j}\right]^2}\,d\omega$$

If $G_0(\alpha)$ is bandlimited white noise, the integral of (4.159) can be evaluated; otherwise, (4.159) is bounded without limit.

Problem 4.4: The mean square response of a simply supported elastic beam subjected to a continuous loading spatially correlated over the length of the beam.

It is the intent of this problem to illustrate a procedure for representing a continuous, spatially correlated loading as sets of correlated discrete loadings. This approach is analogous to a lumped parameter model for a distributed structure and is but an approximation of the distributed loading. However, computational advantages are to be gained in that integral functions can be evaluated as algebraic summations as will be illustrated here.

As shown by Eq. (4.18), the mean square displacement response at any point x of a continuous elastic structure subjected to stationary random excitation is

$$\overline{y^2(x)} = \sum_j \sum_k \phi_j(x)\,\phi_k(x) \int_0^\infty L_{jk}(\omega)\, H_j^*(\omega)\, H_k(\omega)\, d\omega \qquad (4.160)$$

Consider now a distributed load $p(x,t)$ over a one dimensional structure such as a Bernoulli-Euler beam. The quantity $L_{jk}(\omega)$ is then

$$L_{jk}(\omega) = \frac{1}{\overline{M}_j \overline{M}_k \omega_j^2 \omega_k^2} \int_0^\ell \int_0^\ell G_f(x, x', \omega)\, \phi_j(x)\, \phi_k(x')\, dx\, dx' \qquad (4.161)$$

where

$$G_f(x, x', \omega) = \int_0^\infty e^{-i\omega t}\; \overline{p(x,t)\, p(x',t)}\; dt \qquad (4.162)$$

Generally, the spatial correlation function of the loading $\left[\overline{p(x,t)\, p(x',t)}\right]$ can be expressed as a function of the spatial difference $|x-x'|$. For many practical problems, the spatial correlation function of the loading tends to decay as a damped cosine function with increasing values of $|x - x'|$.

Often, due to the complexity of the mode shapes $\left[\phi_j(x)\right]$ as well as the spatial correlation function, a direct analytical integration of the double integrals given in Eq. (4.161) is difficult or even impossible. In many applications, it is expedient to evaluate the double integral numerically. One such procedure is to replace the distributed load p(x, t) by a set of concentrated loads as

$$p(x,t) = p_1(t)\ \delta(x-x_1) + p_2(t)\ \delta(x-x_2) + p_3(t)\ \delta(x-x_3) + \ldots \tag{4.163}$$

where $\delta(x-x_i)$ is a delta function which is zero everywhere except at $x = x_i$.

The double integral shown in Eq. (4.161) now appears as

$$\begin{Bmatrix} G_f(x_1,x_1,\omega)\phi_j(x_1)\phi_k(x_1) + G_f(x_1,x_2,\omega)\phi_j(x_1)\phi_k(x_2) + G_f(x_1,x_3)\phi_j(x_1)\phi_k(x_3) + \ldots \\ + G_f(x_2,x_1,\omega)\phi_j(x_2)\phi_k(x_1) + G_f(x_2,x_2,\omega)\phi_j(x_2)\phi_k(x_2) + G_f(x_2,x_3,\omega)\phi_j(x_2)\phi_k(x_3) + \ldots \\ + G_f(x_3,x_1,\omega)\phi_j(x_3)\phi_k(x_1) + G_f(x_3,x_2,\omega)\phi_j(x_3)\phi_k(x_2) + \ldots \end{Bmatrix} \tag{4.164}$$

where $G_f(x_3,x_2,\omega)$, is the correlation density at frequency ω between the loads positioned at x_3 and x_2. The calculations indicated in (4.164) can most conveniently be carried out using a digital computer. In this example, however, a hand computation is illustrated using four concentrated loads and four normal modes.

Assume the spatial correlation of the loading is a triangular function shown by the sketch in Figure 19.

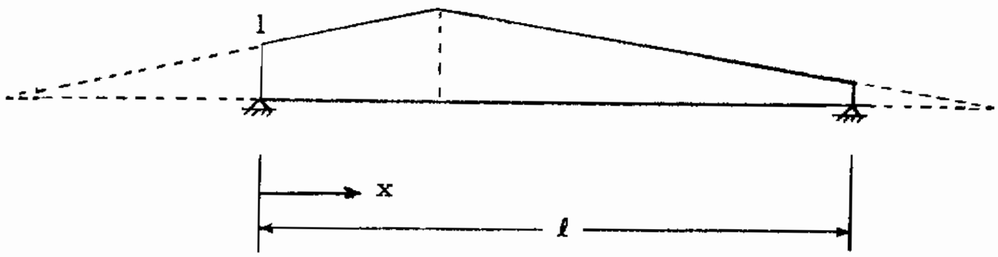


Figure 19. Loading Spatially Correlated as a Ramp Function and Distributed Over the Length of a Simply Supported Elastic Beam

The loading for this problem is to be concentrated or "lumped" at four positions shown in Figure 20.

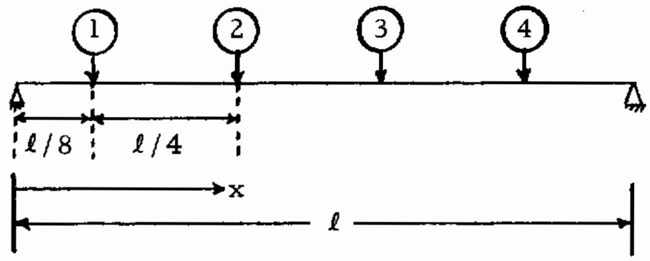


Figure 20. Assumed Discrete Loading Positions for the Distributed Load Acting Over the Length of the Simply Supported Beam.

In functional form, the loading of Figure 20 is denoted as

$$p(x,t) = p_1(t)\,\delta(x-x_1) + p_2(t)\,\delta(x-x_2) + p_3(t)\,\delta(x-x_3) + p_4(t)\,\delta(x-x_4) \tag{4.165}$$

In a matrix format, Eq. (4.161) can be expressed as

$$\overline{M}_j\,\overline{M}_k\,\omega_j^2\omega_k^2\,L_{jk}(\omega) = \left\{\phi_j(x_n)\;.\;.\;.\right\}\left[G(n,m,\omega)\right]\left\{\begin{matrix}\phi_k(x_m)\\ \vdots\end{matrix}\right\} \tag{4.166}$$

where

$$\left\{\phi_j(x_n)\right\} = \left\{\phi_j(x_1),\ \phi_j(x_2),\ \phi_j(x_3),\ \phi_j(x_4)\right\} \tag{4.167}$$

$$\left\{\phi_k(x_m)\right\} = \left\{\begin{matrix}\phi_k(x_1)\\ \phi_k(x_2)\\ \phi_k(x_3)\\ \phi_k(x_4)\end{matrix}\right\} \tag{4.168}$$

$$\left[G(n, m, \omega) \right] = \begin{bmatrix} G(1,1) & G(1,2) & G(1,3) & G(1,4) \\ G(2,1) & G(2,2) & G(2,3) & G(2,4) \\ G(3,1) & G(3,2) & G(3,3) & G(3,4) \\ G(4,1) & G(4,2) & G(4,3) & G(4,4) \end{bmatrix} \qquad (4.169)$$

Equations (4. 167) and (4. 168) define the magnitude of the j and k modes at positions x_1, x_2, x_3 and x_4. Equation (4. 169) describes the spatial correlation of the applied loading and is assumed in this problem to be triangular in shape (see Figure 19) and have the magnitudes shown in (4. 170).

$\left[G(n, m, \omega) \right] =$ (4. 170)

G	m=1	2	3	4
n=1	1	3/4	1/2	1/4
2	3/4	1	3/4	1/2
3	1/2	3/4	1	3/4
4	1/4	1/2	3/4	1

Note that the table shown as Eq. (4. 170) describes a symmetric matrix.

Consider, in detail, the evaluation of the $L_{jk}(\omega)$ expression of Eq. (4. 161) when $j = k = 1$. The loading positions and spatial density function appear as shown in Figure 21. For the assumed ramp function, the magnitudes of the spatial densities are the altitudes of the triangular function at the four spatial positions. It is recalled that the mode shapes for a simply supported beam are

$$\begin{aligned} \phi_j(x) &= \sqrt{2} \sin \frac{j\pi x}{\ell} \\ \phi_k(x) &= \sqrt{2} \sin \frac{k\pi x}{\ell} \end{aligned} \qquad (4.171)$$

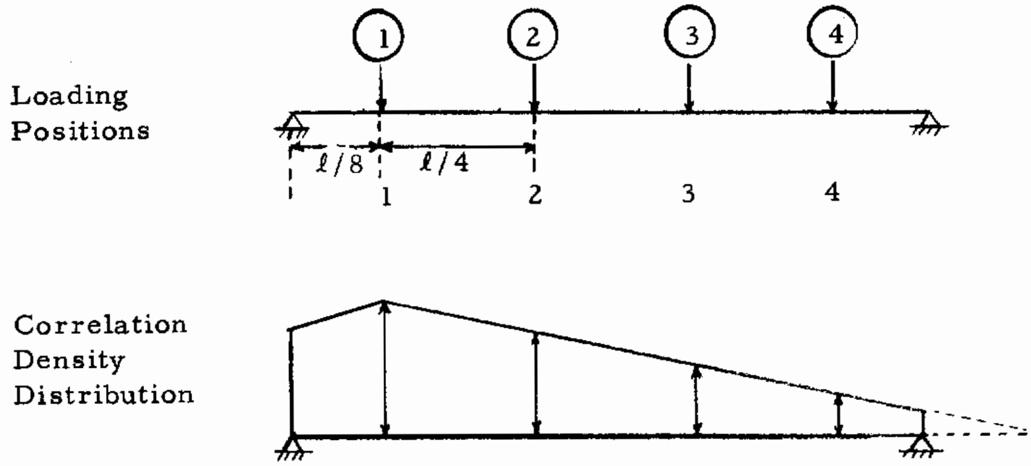


Figure 21. Loading Positions and Spatial Density Distribution for the Simply Supported Beam

For j = k = 1, Eq. (4.166) yields

$$\frac{M^2 \omega_1^4 L_{11}(\omega)}{(\sqrt{2})^2 \left(\frac{\ell}{4}\right)^2} = \left[\begin{array}{llll} 1(.1465) & + 3/4(.357) & + 1/2(.357) & + 1/4(.1465) + \\ 3/4(357) & + 1(.853) & + 3/4(.853) & + 1/2(.357) + \\ 1/2(.357) & + 3/4(.853) & + 1(.853) & + 3/4(.357) + \\ 1/4(.1465) & + 1/2(.357) & + 3/4(.357) & + 1(.1465) \end{array}\right] \tag{4.172}$$

The $\ell/4$ term in (4.172) is the incremental length over which the distributed loading is lumped and the radical is due to the assumed mode shapes of the beam.

Adding the terms of Eq. (4.172) allows $L_{11}(\omega)$ to be written as

$$L_{11}(\omega) = 0.64 \frac{\ell^2}{M^2 \omega_1^4} = \frac{.64}{m^2 \omega_1^4} \tag{4.173}$$

Consider, now, the evaluation of $L_{12}(\omega)$. In expanded form, Eq. (4.166) may be written as

$$\frac{M^2\omega_1^2\omega_2^2}{\left(\frac{\ell}{4}\right)^2} L_{12}(\omega) = \Big[1\phi_1(x_1)\phi_2(x_1) + \frac{3}{4}\phi_1(x_1)\phi_2(x_2) + \frac{1}{2}\phi_1(x_1)\phi_2(x_3) + \frac{1}{4}\phi_1(x_1)\phi_2(x_4)$$
$$+ \frac{3}{4}\phi_1(x_2)\phi_2(x_1) + 1\phi_1(x_2)\phi_2(x_2) + \frac{3}{4}\phi_1(x_2)\phi_2(x_3) + \frac{1}{2}\phi_1(x_2)\phi_2(x_4)$$
$$+ \frac{1}{2}\phi_1(x_3)\phi_2(x_1) + \frac{3}{4}\phi_1(x_3)\phi_1(x_2) + 1\phi_1(x_3)\phi_2(x_3) + \frac{3}{4}\phi_1(x_3)\phi_2(x_4)$$
$$+ \frac{1}{4}\phi_1(x_4)\phi_2(x_1) + \frac{1}{2}\phi_1(x_4)\phi_2(x_2) + \frac{3}{4}\phi_1(x_4)\phi_2(x_3) + 1\phi_1(x_4)\phi_2(x_4)\Big] \tag{4.174}$$

The magnitudes of the first and second mode shapes evaluated at the four loading positions are shown in Figure 22.

Second Mode = $\sin \frac{2\pi x}{\ell}$ First Mode = $\sin \frac{\pi x}{\ell}$

Figure 22. Magnitude of the First and Second Modes for the Simply Supported Beam

Given that the density function is symmetric, the summation of the product of an odd and even mode totals zero. In numerical form, Eq. (4.174) appears as

$$L_{12}(\omega) = \frac{8}{m^2\omega_1^2\omega_2^2}\Big[\ .271 + \frac{3}{4}(.653) + \frac{1}{2}(.653) + \frac{1}{4}(.271)$$
$$+ \frac{3}{4}(.271) + .653) + \frac{3}{4}(.653) + \frac{1}{2}(.271)$$
$$- \frac{1}{2}(.271) - \frac{3}{4}(.653) - .653 - \frac{3}{4}(.271)$$
$$- \frac{1}{4}(.271) - \frac{1}{2}(.653) - \frac{3}{4}(.653) - .271\ \Big] = 0 \tag{4.175}$$

From the form evidenced in Eq. (4.174), $L_{21}(\omega) = L_{12}(\omega)$. The remaining fifteen terms can be evaluated in a similar manner and appear as Eq. (4.176).

$\frac{\omega_j^2 \omega_k^2}{m^2} L_{ij}(\omega)$	k = 1	k = 2	k = 3	k = 4
j = 1	.64	0	.042	0
j = 2	0	.0117	0	.00138
j = 3	.042	0	.00134	0
j = 4	0	.00138	0	.00049

(4.176)

These terms are noted to be in agreement with the conclusions of Reference 11, Equations 41, 42, and 43. That is, for the uniform beam under stationary and homogeneous random excitation, the response is statistically correlated in two odd modes or two even modes and is uncorrelated (zero) between an odd mode and an even mode.

The mean square response may now be written as

$$\overline{y^2(x)} = \sum_j \sum_k \phi_j(x)\,\phi_k(x)\,L_{jk} \int_0^\infty H_j^*(\omega) H_k(\omega)\, d\omega \tag{4.177}$$

For the diagonal terms only, Eq. (4.177) reduces to

$$\overline{y^2(x)} = \sum_j \phi_j^2(x)\,L_{jj} \int_0^\infty \left| H_j(\omega) \right|^2 d\omega \tag{4.178}$$

so that

$$\overline{y^2(x)} = \sum_j \phi_j^2(x)\,L_{jj}\,\frac{\pi\omega_j}{4\zeta_j} \tag{4.179}$$

Using the numerical values from Eq. (4. 176) and assuming a constant damping ratio for all modes, the mean square displacement response of Eq. (4. 179) yields

$$\overline{y^2(x)} = \frac{.64\pi}{4\zeta m^2 \omega_1^3} \left\{ \phi_1^2(x) + .1432\phi_2^2(x) + .0252\phi_3^2(x) + .0122\phi_4^2(x) \right\} \tag{4. 180}$$

Problem 4. 5: Modification of the response of a continuous elastic structure due to mass loading.

Pictorially, this problem is represented in Figure 23.

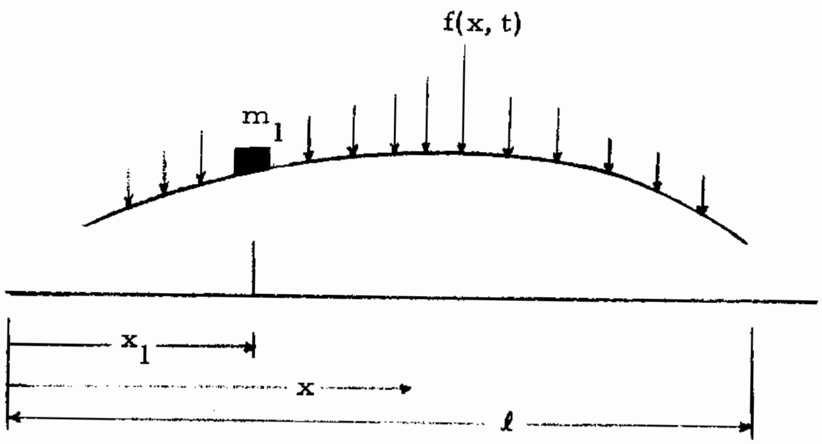


Figure 23. Representation of the Primary Elastic Structure and the Attached Mass

Figure 23 depicts an arbitrary uniform elastic structure acted upon by the forcing function f(x, t). At position x_1, mass m_1 is attached to the primary structure. It is intended to derive the expressions needed to calculate the effect of the attached mass on the displacement response of the structure.

Expressed in terms of the normal modes, the displacement response is given as

$$y(x,t) = \sum_j \phi_j(x)\, q_j(t) \tag{4. 181}$$

The mass at x_1 introduces an inertial force $-m_1\ddot{y}(x_1, t)$ on the structure so that the equation for the generalized force becomes

$$\ddot{q}_j(t) + 2\zeta_j\,\omega_j\,\dot{q}_j(t) + \omega_j^2\,q_j(t) = \frac{1}{\overline{M}_j}\int_0^{\ell} f(x,t)\,\phi_j(x)\,dx - \frac{1}{\overline{M}_j}\,\phi_j(x_1)\,m_1 \sum_k \phi_k(x_1)\,\ddot{q}_k(t) \tag{4.182}$$

where

$$\frac{m_1}{\overline{M}_j}\,\phi_j(x_1)\sum_k \phi_k(x_1)\,\ddot{q}_k(t) = \frac{m_1}{\overline{M}_j}\,\phi_j(x_1)\left\{\phi_1(x_1)\,\ddot{q}_1(t) + \phi_2(x_1)\,\ddot{q}_2(t) + \ldots\right\} \tag{4.183}$$

Note that Eq. (4.183) has the effect of coupling the generalized coordinates via terms of generalized accelerations. To rigorously calculate the mass loading effect, therefore, one must solve these coupled equations for each of the generalized coordinates, then substitute into Eq. (4.181) to calculate the effects of the mass at x_1 on the displacement response. This approach is done most effectively using a digital computer.

If only the first mode is considered in the mass coupling term, Eq. (4181) reduces to

$$\left\{1 + \frac{m_1\phi_1^2(x_1)}{\overline{M}_1}\right\}\ddot{q}_1(t) + 2\zeta_1\omega_1\dot{q}_1(t) + \omega_1^2\,q_1(t) = \frac{1}{\overline{M}_1}\int_0^{\ell} f(x,t)\,\phi_1(x)\,dx \tag{4.184}$$

This is equivalent to changing the first natural frequency from ω_1 to

$$\frac{\omega_1}{\sqrt{1 + \dfrac{m_1 \phi_1^2(x_1)}{\overline{M}_1}}} = \omega_1 \tag{4.185}$$

The damping factor ζ_1 is also changed from ζ_1 to

$$\frac{\zeta_1}{\sqrt{1 + \dfrac{m_1 \phi_1^2(x_1)}{\overline{M}_1}}} = \zeta_1' \tag{4.186}$$

and the generalized mass is increased to

$$\overline{M}_1 \left\{ 1 + \frac{m_1 \phi_1^2(x_1)}{\overline{M}_1} \right\} = \overline{M}'_1 \tag{4.187}$$

With these modifications, the structure loaded with m_1, has the equation

$$\ddot{q}_1(t) + 2\zeta_1 \, \omega_1' \, \dot{q}_1(t) + \omega_1'^{\,2} \, q_1(t) = \frac{1}{\overline{M}'_1} \int_0^{\ell} f(x,t) \, \phi_1(x) \, dx \tag{4.188}$$

and the random vibration problem can be treated in the usual sense with the primed coefficients. Considering the complexity of the general problem, these assumptions may well provide an acceptable approximate solution if the convergence for the series with higher modes is rapid.

Case 1: Moment of inertia effects in addition to mass loading.

Aside from the change in mass, damping, and frequency of the previous case, further modifications are introduced due to the generalized force associated with J_1 which is

$$\frac{1}{\overline{M}_j}\int_0^{\ell} \mathcal{M}(x,t)\,\frac{d}{dx}\left[\ddot{y}\right]\frac{d}{dx}\left[\phi_j(x)\right]\,\delta(x-x_1)\,dx$$

$$= \frac{J_1}{\overline{M}_j}\,\frac{d}{dx}\left[\phi_1(x_1)\right]\sum_k \frac{d}{dx}\left[\phi_k(x_1)\right]q_k(t) \tag{4.189}$$

Again using a single mode approximation, Eq.(4.189) becomes

$$\frac{J_1}{\overline{M}_j}\,\frac{d}{dx}\left\{\left[\phi_1(x_1)\right]\right\}^2\;q_1(t) \tag{4.190}$$

Using the same format as with mass loading, the primed coefficients for mass loading and rotary inertia effects appear as

$$\omega_1' = \frac{\omega_1}{\sqrt{1+\frac{m_1}{\overline{M}_1}\phi_1^2(x_1)+\frac{J_1}{\overline{M}_1}\;\frac{d}{dx}\left\{\left[\phi_1(x_1)\right]\right\}^2}} \tag{4.191}$$

$$\zeta_1' = \frac{\zeta_1}{\sqrt{1+\frac{m_1}{\overline{M}_1}\phi_1^2(x_1)+\frac{J_1}{\overline{M}_1}\;\frac{d}{dx}\left\{\left[\phi_1(x_1)\right]\right\}^2}} \tag{4.192}$$

$$M_1' = M_1 \left\{ 1 + \frac{m_1 \phi_1^2(x_1)}{\overline{M}_1} + \frac{J_1}{\overline{M}_1} \frac{d}{dx}\left[\phi_1(x_1)\right] \right\}^2 \tag{4.93}$$

The solutions presented here are functional in form and illustrate the procedure to be followed in calculating the displacement response effects for any specific problem.

For mass loading alone, the modification factor is noted as

$$\left\{ 1 + \frac{m_1}{\overline{M}_1} \phi_1^2(x_1) \right\} \tag{4.94}$$

Assuming the modal mass $\overline{M}_1$ is normalized to equal the mass of the structure, the magnitude of the modification factor is dependent upon the product of the mass ratio and the square of the mode shape magnitude at the attachment location.

For other discussion on the mass loading problem, the reader is directed to Section 9. 3 of Reference 5. If the attached mass is small compared to the mass of the structure, the modification factor will be approximately unity providing the mode shape magnitude ≤ 1. On the other hand, attaching a large mass at a nodal position $\left[\phi_1(x_1) = 0\right]$ is seen to produce no effect on the displacement response. Similar comments can be made for Case 1.

5. CONCLUDING REMARKS

5.1 SUMMARY OF THE THEORY

The conclusions of this report are appropriate to linear, time invariant, homogeneous elastic structures. Small deflection theory is assumed and two classes of forcing functions are considered: (1) steady-state harmonic conditions and (2) stationary random excitation.

Although basic dynamics theory (Section 1) and random process theory (Section 3) are discussed, prime interest is focused upon classical theory for computing the response properties of distributed elastic structures to both harmonic and random loadings. It is found that the partial differential equations defining the dynamics of many structures are of a form compatible with the Sturm-Liouville conditions. This fact guarantees the orthogonality of the structural modes and may be defined for a one-dimensional structure as

$$\int_0^{\ell} m(x)\,\phi_j(x)\,\phi_k(x)\,dx = \begin{cases} 0 & , \quad i \neq j \\ \text{constant} & , \quad j = k \end{cases} \tag{5.1}$$

As shown in Section 2, the orthogonality condition is used to enormously simplify the mathematics by uncoupling the modal equations for the elastic system.

The displacement response for distributed elastic structures to deterministic excitation is assumed to be expressable as a summation of the product of the normal modes of the structure and the generalized coordinates. This is written as

$$y(x,\ t) = \sum_j \phi_j(x)\, q_j(t) \qquad (5,\ 2)$$

The generalized coordinates are definitions with convenient mathematical properties and need not correspond to the physical coordinates of the structure. Indeed, in classical mechanics, momenta often is used as generalized coordinates. The example problems of Section 2 illustrate response calculations for uniform beams subjected to distributed and discrete deterministic loadings.

For the distributed structure subjected to stationary random loadings, it is improper to speak of calculating the instantaneous displacement time history. As explained in Section 3, the response properties must be expressed by quantities which have statistical meaning. One such expression is the mean square response at any point x which appears as

$$\overline{y^2(x)} = \sum_j \sum_k \phi_j(x)\,\phi_k(x) \int_0^\infty L_{jk}(\omega)\, H_j^*(\omega)\, H_k(\omega)\, d\omega \tag{5.3}$$

where

$$L_{jk}(\omega) = \frac{1}{M_j M_k \omega_j^2 \omega_k^2} \int_0^\ell \int_0^\ell G_f(x, x', \omega)\, \phi_j(x)\, \phi_k(x')\, dx\, dx' \tag{5.4}$$

and

$$H_j^*(\omega)\, H_k(\omega) = \frac{1}{1 - \left(\frac{\omega}{\omega_j}\right)^2 - i2\zeta_j \left(\frac{\omega}{\omega_j}\right)} \quad \frac{1}{1 - \left(\frac{\omega}{\omega_k}\right)^2 + i2\zeta_k \left(\frac{\omega}{\omega_k}\right)} \tag{5.5}$$

The ϕ's represent the mode shapes, the $H(\omega)$'s are the modal magnification functions, and the $L_{jk}(\omega)$ term denotes the spectral properties of a generalized force. $L_{jk}(\omega)$ is closely related to the joint acceptance function and is thus a measure of the effectiveness of a forcing field in exciting a given mode of vibration.

5.2 PROBLEMS IN THE APPLICATION OF THEORY

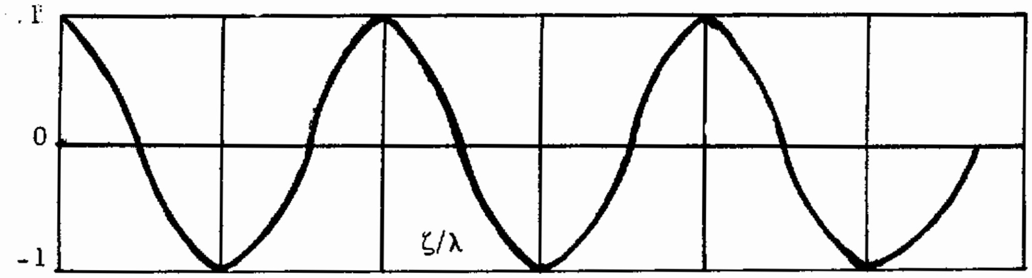
As shown in the illustrative problems of Section 4, one prime difficulty in calculating the mean square displacement response is the evaluation of $L_{jk}(\omega)$. This expression requires the mode shapes of the structure, and for other than idealized structures, the mode shapes may be extremely difficult or impossible to obtain either analytically or experimentally. Even for idealized structures, these mode shapes may not appear in simple analytical form as is noted by comparing the mode shapes for a simply supported beam to those for a cantilevered beam. The simply supported beam has sinusoidal mode shapes whereas the cantilevered beam has mode shapes defined by both trigonometric and hyperbolic functions.

In computing the mean square displacement response, it is seen that the higher order modes contribute very little to the total response as compared to the lower order modes [note the $\omega_j^2 \omega_k^2$ term in the denominator of Eq.(5.4)]. However, for computing other properties such as mean square stresses, the derivatives of the mode shapes are required rather than the mode shapes, per se. For computing the mean square acceleration, the higher order modal quantities are important in that the response spectral density is a function of the fourth power of frequency (see Eq. (4.157)).

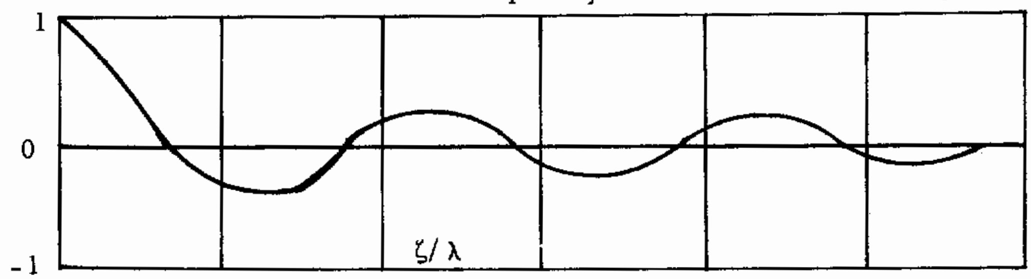
Another nontrivial problem is to obtain by calculation, measurement, or even assumption, the spatial correlation density $G_f(x, x', \omega)$ for the loadings at position x and x' at the frequency ω. The spatial correlation functions for four acoustic pressure fields are obtained from Reference 12, page 63 and are shown here as Figure 24. These spatial correlation functions are observed to vary from a simple cosine expression for acoustic pressure of discrete frequency and fixed incidence to a relatively complicated exponential cosine term for acoustic pressure in the near field of a jet engine. Upon substituting the expressions for the mode shapes and the spatial correlation density function into Eq. (5.4), the resulting integral expression for $L_{jk}(\omega)$ may be impractical to evaluate. References 12, 13, and 14 illustrate the complexities of evaluating such an integral.

By expanding the correlation function as a Fourier series, Mayer (Reference 12) computes the generalized loading for simply supported beams and plates excited by the pressure fields of Figure 24. Reference 13 suggests representing a correlated distributed loading as sets of concentrated correlated

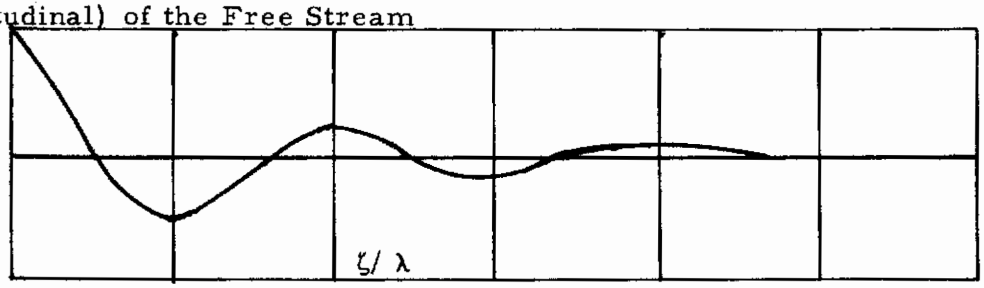
a) Acoustic Pressures of Discrete Frequency and Fixed Incidence



b) Acoustic Pressures of Discrete Frequency and Random Incidence



c) Pressure Fluctuations in a Subsonic Boundary Layer in the Direction (Longitudinal) of the Free Stream



d) Acoustic Pressures in the Near Field of a Jet Engine

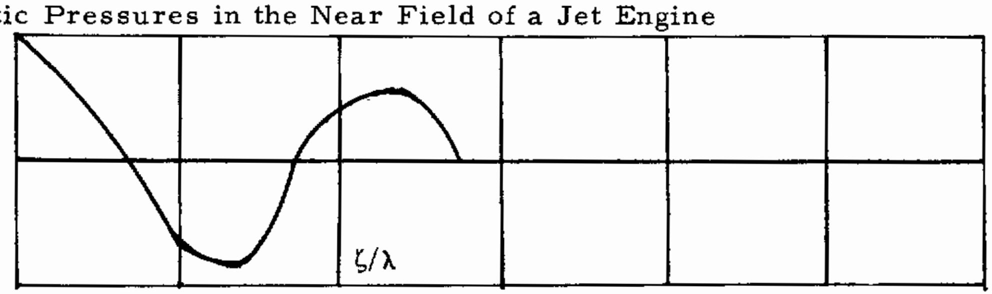


Figure 24. Spatial Correlation Functions for Four Typical Pressure Fields (Reference 12.)

forces with the discrete loadings defined by use of delta functions. This procedure is illustrated in Problem 4.4 of this report.

Reference 14 discusses the calculations for the joint acceptance squared for flat rectangular panels with the sides simply supported and then with the sides fully clamped. The excitation consists of a sinusoidal pressure wave propagating over the surface of the panel. The normalized pressure amplitudes are defined by an exponential which decays with increasing spatial correlation. From the discussion of Reference 15, this spatial correlation length is shown from experimental measurements to be the distance from maximum correlation (zero separation) to the first zero crossing, and not the distance between the first two zero crossings. Figure 25 (Reference 14, page 54) gives the numerical results for a rectangular panel simply supported on all four sides. The joint acceptance squared terms are for the integer mode numbers $m = 1$, $n = 1, 2, 3, 4,$ and 5; the decay constant $c_0 = \infty$, 20, and 1 where ℓ is the panel length and c equals the velocity of sound propagation. Note also it is assumed that small motion theory applies, that the modes are widely separated in frequency, and that the damping is small.

The curves of Figure 25 show that as the spatial correlation decreases, the panel mode shapes lose their wavelength selectivity and the joint acceptance becomes uniform for all modes. This behavior is due both to wavelength relationships that imbalance the force distribution and to the decaying spatial correlation which results in the noncancellation of harmonics as evidenced by the asymptotic approach to a $1/\omega$ decrease in $j^2\omega$.

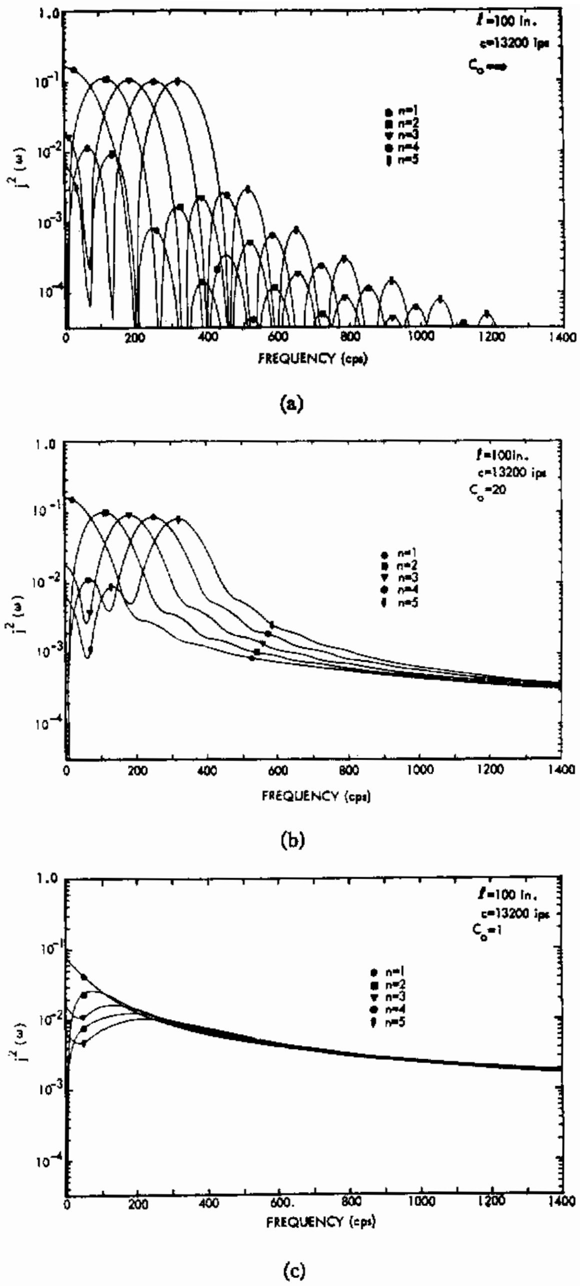


Figure 25 Joint Acceptance Squared for a Rectangular Panel with Sides Simply Supported (Reference 14.)

5.3 OMISSIONS IN THE THEORY

The analytical effort yields important parametric results which, in general, can be accurately applied only to idealized structural elements such as homogeneous plates and beams. Experiments can be readily conducted to verify the theoretical results for such simple structures (see Reference 16). However, it is not at all clear how to extrapolate the results for these component structures to predict the response behavior of typical aircraft and missile construction. Two well-known deviations between simple panel experiments and typical vehicle structure are (1) the vibration characteristics of a stiffened cylindrical fuselage cannot be represented as a single panel, and (2) many correlation patterns in the noise field are difficult and even impossible to reproduce by a single speaker or siren.

Moreover, the analytical models often neglect the effect of the surrounding medium in deriving the equations for structures excited by random pressure fields. If the medium is moving in phase with the acceleration of the structure, a slight increase in the effective mass of the structure is expected. If the medium is moving in phase with the velocity, a damping force is created in which the structure acts as a radiator of acoustic energy. As implied by the work of Reference 17, the acoustic radiation effects may not always be ignorable. Another consideration is that internal air resonances for air trapped in compartments of the vehicle may couple with the response of the structure. Still another task is to analytically include the internal damping of the material itself as well as the structural damping due to friction at the joints.

5.4 COMPROMISES WITH THE THEORY

It becomes apparent that an analytical model which includes the effects of the surrounding medium, the various damping mechanisms, and details of complex structures would appear extremely complex and may be impossible to solve. To obtain answers for engineering use, it is necessary to sacrifice detail for tractability in creating a simplified method for predicting a type of average vibration for a complex structure. Two such suggested techniques are (1) a modal density-energy approach, and (2) the use of dynamically similar scaled models.

The modal density-energy concepts are discussed in detail in References 17, 18, and 19. This approach uses statistical ideas and concepts from room acoustics to gain an approximation for the multi-mode response of an elastic structure subjected to reverberant acoustic fields. It is assumed that the energy contained in a structure vibrating at some mean square velocity is uniformly distributed and that the structural modes do not overlap. The latter assumption implies that the vibrations of the various modes are statistically independent and energies do not interact but may be summed directly. Terms such as modal density, average modal energy, mechanical resistance and absorption coefficients are synonomous with this approach and must be evaluated empirically for other than the simpler elastic structures. Although not fully developed, this technique holds promise as a useful tool.

The use of scaled models as a prediction tool is discussed in References 20 and 21. In general, physical size is the main scaling parameter so that the test model can be constructed by model builders after studying the drawings for the full scale vehicle. However, scaling the actual pressure fields is no small task, and carrying out the scaled experiments is expensive in time and dollars. Other problems involve the control of damping and the physical scaling of such items as rivet holes and honeycomb construction. Reference 20 shows acceptable agreement for several frequency response functions and power spectra for both full scale and test models of the SNARK missile fuselage. Reference 21 provides a general discussion of scaling laws and presents SATURN and SCOUT data for models and full-scale vehicles which bolster the argument for using scaled models.

Reference 22 describes an experimental program to study the modal behavior of typical aircraft structures excited by acoustic pressures. The structure considered is an aft section of the Caravelle. The excitation is provided by a Rolls Royce Avon engine. By considering the stress spectrum for the various panels, the fuselage frames,and stringers as well as the correlation spectrum between panels, the modal behavior of the Caravelle structure is determined.

As an example of typical data from Reference 22, consider Figure 26 which shows the power spectra and correlation spectrum for panels 2 and 10. Panels 2 and 10 are adjacent panels across a frame and show a very low correlation at the major panel resonance of 715 cps. It is concluded that the panels are vibrating independently of one another and that the intermediate frame can be considered to be a rigid boundary. If the correlation were 0.9 or greater, then the panels would be considered to be vibrating in phase as part of a gross mode of the structure. For a more complete discussion of the correlation function and its application to structural response, the reader is urged to read References 9, 10, and 23.

Stress Spectrum at
Panel Position 2

Stress Spectrum at
Panel Position 10

Correlation Spectrum between
Panels 2 and 10

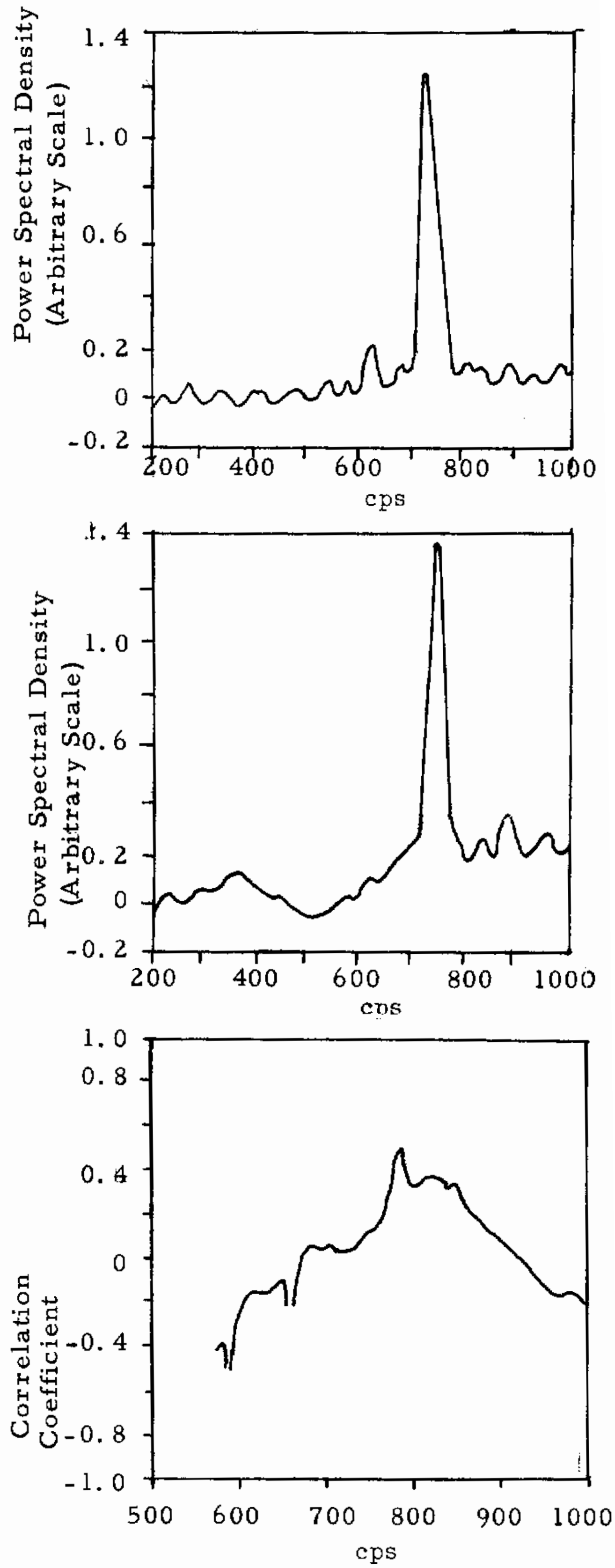


Figure 26 Stress Spectra and the Correlation Spectrum for Two Typical Aircraft Panels (Reference 22.)

5.5 FUTURE CONSIDERATIONS

Although much useful analytical and experimental work has been done to develop a tool for predicting structural response to random excitation, no technique can accurately predict the life of the structure at the drawing board stage. Prototype and proof-testing still are essential and perhaps always will be. It appears that a linear analytical model which accounts for most physical effects will be overly complex to be of practical use to the engineer. For nonlinear effects, this complexity is compounded. Hence, the desired technique is one which must compromise mathematical rigor and incorporate experimental results to approximately predict the structural response and the related stress levels and fatigue life. Such a technique has yet to be developed.

More analytical and experimental work must be done to quantitatively predict the behavior of structures in a combined environment, the thermal-vibro acoustics environment being one example. Additional effort should be given to empirical correlation studies in order to parametrically analyze actual flight data and then to extrapolate results from structural component testing to predict properties for actual vehicles. In this area, the use of structural models such as the passive element electrical analog may be of considerable use. Attention also should be given to parametrically isolating the factors which significantly affect structural behavior in a nonstationary random pressure field in a manner similar to that shown in Reference 24.

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13. ABSTRACT

This document discusses the fundamental classical theory governing the response of linear distributed elastic structures to deterministic and to random excitation. A review is made of the basic dynamics theory for discrete and distributed systems when the excitation is deterministic. This review is considered necessary to easily understand the subject material. Integral expressions are then derived for the mean square value and correlation functions for the response of an arbitrary linear elastic structure subjected to stationary random loading. These derived results are then applied to illustrative structural problems. In this way, the association between the parameters in the theoretical expressions and the physical properties of structural systems is demonstrated.

A principal objective is to explore the value and limitations of using classical theory as a tool for predicting structural vibrations in typical flight vehicles. The direct extension of the shown theoretical results to include typical flight structures although technical accurate, is not considered practical. However, the derivation procedures and results of this report can be used as a basis for forming statistical parametric techniques for approximating the response behaviro of distributed systems to random excitation. In the concluding discussion, several existing techniques reflecting compromises in theoretical rigor are discussed and subject areas for future study are noted.

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