Damped Response of Visco-Elastic Thick Cylinders of Infinite Extent

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ABSTRACT

Harmonic responses of viscoelastic thick circular cylinders of infinite extent, subjected to harmonic radial and tangential boundary stresses are considered. In development of an analytical solution two dimensional elastodynamic theory is employed and the viscoelastic material for the medium is allowed by assuming complex elastic moduli. The solution provides stresses and displacements at any point in the medium in terms of boundary stresses. The resonant frequencies for different circumferential flexural (lobar) modes and their corresponding thickness modes are computed and satisfactorily compared with an available solution. The present solution is not limited to thin shells, and it equally treats thick cylinders with any values of hysteretic damping. Also, several design charts for estimation of resonant frequencies for a wide range of thickness ratio are developed.

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INTRODUCTION

The trend towards dissipating vibratory energy in cylindrical structures when subjected to circumferential flexural vibrations requires application of viscoelastic materials with high strength. Although many cylindrical structures can be analyzed using the theory of thin shells, thicker cylinders with hysteretic damping have to be studied using the general theory of elasticity with complex moduli. The first investigators to study the vibrations of an infinitely long traction-free hollow cylinder were Greenspon (1957), and Gazis (1958). Armenakas et all (1969), in particular, considered the transmission of elastic energy by means of elastic waves, and formulated the eigenvalue problem for stress free cylindrical surfaces. He presented tables of natural frequencies for different ratios of mean radius/thickness and for different numbers of circumferential wave numbers. McNiven, Shah and Sackman (1966) considered the axisymetric vibrations of hollow cylinder utilizing "Three Modes Theory". Gladwell and Vijay (1975) studied the three dimensional vibrations of a finite length circular cylinder with traction free boundaries, using a finite element approach. Svardh (1984) investigated wave propagation in a semi-infinite, hollow, elastic circular cylinder with traction-free lateral surface initially at rest and subjected to transient end loadings. Hutchinson (1980) developed a series solution of the general three dimensional equation of linear elasto-dynamic problem. Hutchinson and El-Azhari (1986) extended Hutchinson's work in solid cylinders to include free hollow cylinders with finite length. Singal and Williams (1988) studied free vibrations of thick circular cylindrical shells and rings using the energy method and obtained a frequency equation to provide resonant frequencies for breathing and beam type modes. They also conducted experimental investigations to assess the validity of their analysis.

The present study involve the development of an analytical solution to the harmonic response of infinitely long cylindrical structure with internal damping subjected to flexural vibrations around the circumference.

GOVERNING ELASTO-DYNAMIC EQUATIONS

For the isotropic homogeneous elastic medium shown in Figure 1,

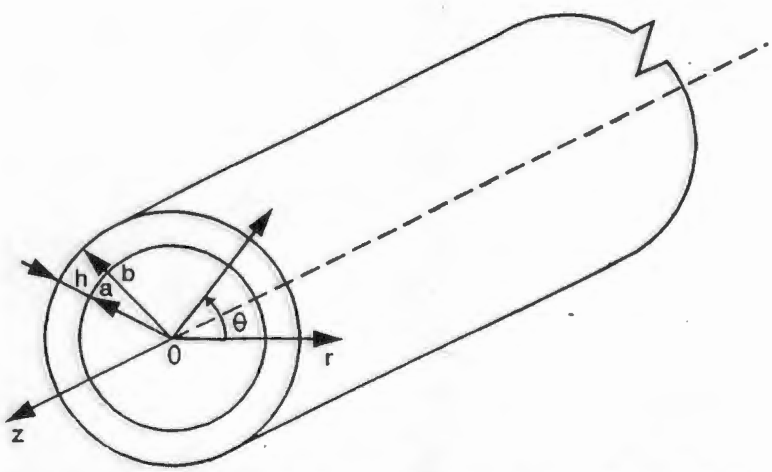

Figure 1. Reference coordinates and dimensions

the governing equation of motion in terms of harmonic radial and tangential displacements amplitudes u and v are:

$$-\rho p^2 u r = (\lambda+2G)\, r \frac{\partial}{\partial r}\epsilon - 2G \frac{\partial}{\partial \theta}\omega_z \tag{1-a}$$

and

$$-\rho p^2 v = (\lambda+2G)\, \frac{1}{r} \frac{\partial}{\partial \theta}\epsilon + 2G \frac{\partial}{\partial r}\omega_z \tag{1-b}$$

where:

$$\epsilon = \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{2-a}$$

and

$$\omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta}\right] \tag{2-b}$$

ϵ and ω_z	are the volumetric strain and elastic rotation about z axis.
u and v	are radial and tangential displacement amplitudes.
p	is the frequency of the harmonic excitation.
G and λ	are shear modulus and Lame's elastic constant.
ρ	is the density of the medium.

Differentiating equations (1-a) and (1-b) with respect to r and θ and adding them together yields:

$$-\rho p^2 r \epsilon = (\lambda+2G)\left[r \frac{\partial^2}{\partial r^2}\epsilon + \frac{\partial}{\partial r}\epsilon + \frac{1}{r}\frac{\partial^2}{\partial \theta^2}\epsilon\right] \tag{3-a}$$

Differentiating equations (1-a) and (1-b) with respect to θ and r, after arranging the results, yields:

$$-\rho p^2 r \omega_z = G\left[\frac{1}{r}\frac{\partial^2}{\partial \theta^2}\omega_z + \frac{\partial}{\partial r}\omega_z + r\frac{\partial^2}{\partial r^2}\omega_z\right] \tag{3-b}$$

Introducing two parameters β and μ such that

$$\beta^2 = \frac{\rho p^2}{\lambda+2G} \tag{4-a}$$

$$\mu^2 = \frac{\rho p^2}{G} \tag{4-b}$$

Substituting β^2 and μ^2 in equation (3-a) and (3-b) they become:

$$r^2 \frac{\partial^2}{\partial r^2}\epsilon + r\frac{\partial}{\partial r}\epsilon + r^2\beta^2\epsilon = -\frac{\partial^2}{\partial \theta^2}\epsilon \tag{5-a}$$

$$r^2\frac{\partial^2}{\partial r^2}\omega_z+r\frac{\partial}{\partial r}\omega_z+r^2\mu^2\omega_z=-\frac{\partial^2}{\partial\theta^2}\omega_z \tag{5-b}$$

Considering the boundary conditions, the solution to these equations are:

$$\epsilon(r,\theta)=\sum_{n=0}^{\infty}\epsilon_n(r,\theta) \tag{6-a}$$

$$\omega_z(r,\theta)=\sum_{n=0}^{\infty}\omega_{zn}(r,\theta) \tag{6-b}$$

where:

$$\epsilon_n(r,\theta)=\beta^2[A_nJ_n(\beta r)+B_nY_n(\beta r)]\cos(n\theta) \tag{7-a}$$

$$\omega_{zn}(r,\theta)=\mu^2[C_nJ_n(\mu r)+D_nY_n(\mu r)]\sin(n\theta) \tag{7-b}$$

J_n and Y_n are first and second kinds of Bessel functions of n^{th} order.

MODAL DISPLACEMENT AND STRESS COMPONENTS

Substituting equations (7) into equations (1), modal displacement components will become:

$$\begin{aligned}u_n(r_1\theta)=&-\beta[A_nJ_n(\beta r)+B_nY_n(\beta r)]\cos(n\theta)\\&+(2n/r)[C_nJ_n(\mu r)+D_nY_n(\mu r)]\cos(n\theta)\end{aligned} \tag{8-a}$$

$$\begin{aligned}v_n(r_1\theta)=&\,n/r[A_nJ_n(\beta r)+B_nY_n(\beta r)]\sin(n\theta)+\\&-2\mu[C_nJ_n(\mu r)+D_nY_n(\mu r)]\sin(n\theta)\end{aligned} \tag{8-b}$$

Amplitude of stresses on the plane normal to the radial axis in the elastic medium, in terms of volumetric strain ϵ and elastic rotation ω_z are:

$$\sigma_{rr}=\lambda\epsilon+2G\frac{\partial}{\partial r}u \tag{9-a}$$

$$\tau_{r\theta}=2G[\frac{\partial}{\partial r}v-\omega_z] \tag{9-b}$$

substituting from equations (8) and (7) into (9), component of stresses will be presented as:

$$\sigma_{rrn}(r,\theta)=[A_nE_n(r)+B_nF_n(r)+C_nG_n(r)+D_nH_n(r)]\cos(n\theta) \tag{10-a}$$

$$\tau_{r\theta n}(r,\theta)=[A_nE_n*(r)+B_nF_n*(r)+C_nG_n*(r)+D_nH_n*(r)]\sin(n\theta) \tag{10-b}$$

In the above equations, E_n, F_n, G_n, H_n, E_n*, F_n*, G_n*, and H_n* are functions of Bessel functions, where:

$$E_n(r)=\beta^2\lambda J_n(r\beta)-2G\beta^2 J''(r\beta) \tag{11-a}$$

$$F_n(r)=\beta^2\lambda Y_n(r\beta)-2G\beta^2 Y_n''(r\beta) \tag{11-b}$$

$$G_n(r)=4Gn\mu/rJ_n'(r\mu)-4Gn/r^2 J_n(r\mu) \tag{11-c}$$

$$H_n(r)=4Gn\mu/rY_n(r\mu)-4Gn/r^2 Y_n(r\mu) \tag{11-d}$$

$$E_n^*(r)=2G\beta n/rJ_n(r\beta)-2Gn/r^2 J_n(r\beta) \tag{11-e}$$

$$F_n^*(r)=2G\beta n/rY_n'(r\beta)-2Gn/r^2 Y_n(r\beta) \tag{11-f}$$

$$G_n^*(r)=-4G\mu^2 J''(r\mu)-2G\mu^2 J_n(r\mu) \tag{11-g}$$

$$H_n^*(r)=-4G\mu^2 Y''(r\mu)-2G\mu^2 Y_n(r\mu) \tag{11-h}$$

Functions $J_n'(x)$, $J_n''(x)$, $Y_n'(x)$ and $Y_n''(x)$ are first and second derivative of $J_n(x)$ and $Y_n(x)$ with respect to x.

MODAL HARMONIC RESPONSE

Considering the boundary stresses in inner and outer surfaces these stresses can be presented as:

$$\begin{bmatrix} E_n(a) & F_n(a) & G_n(a) & H_n(a) \\ E_n^*(a) & F_n^*(a) & G_n^*(a) & H_n^*(a) \\ E_n(b) & F_n(b) & G_n(b) & H_n(b) \\ E_n^*(b) & F_n^*(b) & G_n^*(b) & H_n^*(b) \end{bmatrix} \begin{Bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{Bmatrix} = \begin{Bmatrix} \sigma_{rrn}(a) \\ \tau_{r\theta n}(a) \\ \sigma_{rrn}(b) \\ \tau_{r\theta n}(b) \end{Bmatrix} \tag{12-a}$$

or

$$T_n a_n = \sigma_n \tag{12-b}$$

where:

σ_n is a vector containing radial and shear stresses on the inner and outer surface of the medium.

a_n is a vector containing arbitrary constants.

T_n is a square matrix containing the coefficients in terms of Bessel functions.

To provide displacement and stress components at any point in the cylinder, equations (8) and (10) can be arranged in the following matrix equation.

$$\begin{Bmatrix} U_n \ (r,\theta)/\cos(n\theta) \\ V_n \ (r,\theta)/\sin(n\theta) \\ \sigma_{rrn} \ (r,\theta)/\cos(n\theta) \\ \tau_{r\theta n} \ (r,\theta)/\sin(n\theta) \end{Bmatrix} = \begin{bmatrix} -\beta J_n(\beta r) & -\beta Y_n(\beta r) & \frac{2n}{r} J_n(\mu r) & \frac{2n}{r} Y_n(\mu r) \\ \frac{n}{r} J_n(\beta r) & \frac{n}{r} Y_n(\beta r) & -2\mu J_n(\mu r) & -2\mu Y_n(\mu r) \\ E_n(r) & F_n(r) & G_n(r) & H_n(r) \\ E_n^*(r) & F_n^*(r) & G_n^*(r) & H_n^*(r) \end{bmatrix} \begin{Bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{Bmatrix} \tag{13-a}$$

The above matrix equation can be abbreviated as:

$$\boldsymbol{R}_n(r) = \boldsymbol{S}_n(r)\,\boldsymbol{a}_n \tag{13-b}$$

where:

$R_n(r)$ is a vector containing components of radial and tangential displacement and stress.

$S_n(r)$ is a coefficient square matrix.

Arranging equation (12-b) and (13-b) they result in

$$\boldsymbol{D}_n(r)\,\boldsymbol{R}_n(r) = \boldsymbol{\sigma}_n \tag{14}$$

where:

$$\boldsymbol{D}_n(r) = \boldsymbol{T}_n\,\boldsymbol{S}_n^{-1}(r) \tag{15}$$

For given lobar boundary stress components, Equation (14) can provide the displacement and stress components at any point in the medium.

RESULTS AND DISCUSSION

The frequency response for different lobar modes of vibrations can be computed for any cross sectional geometry of the elastic or viscoelastic cylinders. Figure 2 illustrates the lobar vibration forms for the first three modes. Computations were conducted to determine the resonant responses of the first three lobar modes ($n = 2$ to 4) and five of their corresponding thickness modes ($m = 1$ to 5). Results presented in Figure 3 provide the non-dimensionalized resonant frequency (frequency factor) versus thickness ratio for the lobar modes ($n = 2,3$ and 4) of elastic cylinder. Figure 3 demonstrate coupling between the different thickness modes, at particular thickness ratios. These results are computed for Poisson's ratio $= 0.33$.

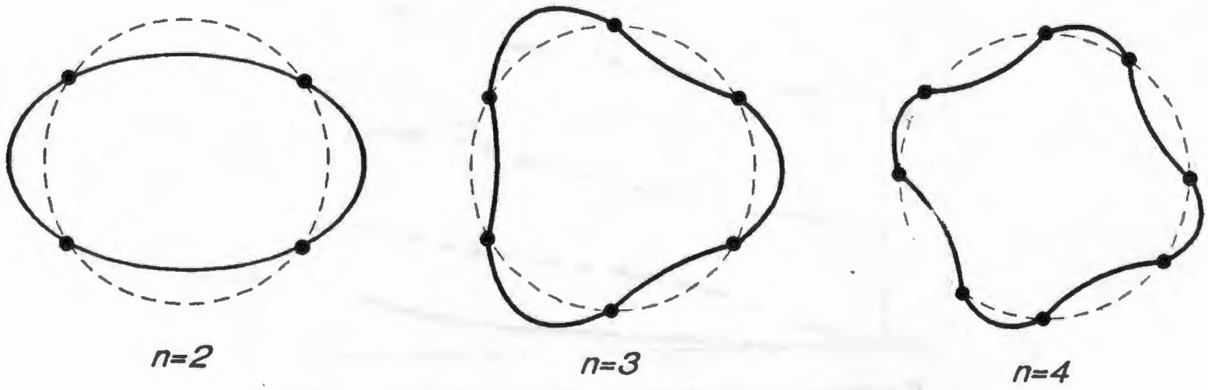

Figure 2. Lobar Vibration Forms

To verify the validity of the present results, computed resonant frequency were compared with Armenakas et al (1969) results. The comparison of the results for different thickness ratios indicates satisfactory agreement between them. It is believed that the present results are more accurate than Armenakas' results. This is due to the fact that in his computation, only a few terms in expansion of the Bessel functions are assumed, however, the present results are obtained by utilizing higher accuracy for the Bessel functions of complex arguments.

Table 1. Comparison of present resonant frequency factors with Armenakas (1969) natural frequency factors for different thickness ratios

n,m	h/a = 0.1276		h/a = 0.1739		h/a = 0.1978	
	Present	Armenakas	Present	Armenakas	Present	Armenakas
2,1	0.0101	0.010	0.0175	0.0176	0.0230	0.0223
2,2	0.2408	0.2414	0.1800	0.1763	0.3600	0.3604
3,1	0.0240	0.0270	0.0480	0.0490	0.6078	0.6152
3,2	0.3360	0.3411	0.4440	0.4530	0.5040	0.5084
4,1	0.0530	0.523	0.0910	0.0914	0.1140	0.1145
4,2	0.4440	0.4442	0.5880	0.5895	0.6600	0.6611

Frequency responses of maximum radial displacment for a cylinder having thickness ratio of 0.5 and poisson's ratio of 0.25, subjected to harmonic radial stress from inside, for three damping factors of $\eta = 0.0$, 0.05 and 0.1 are presented in Figure 4.

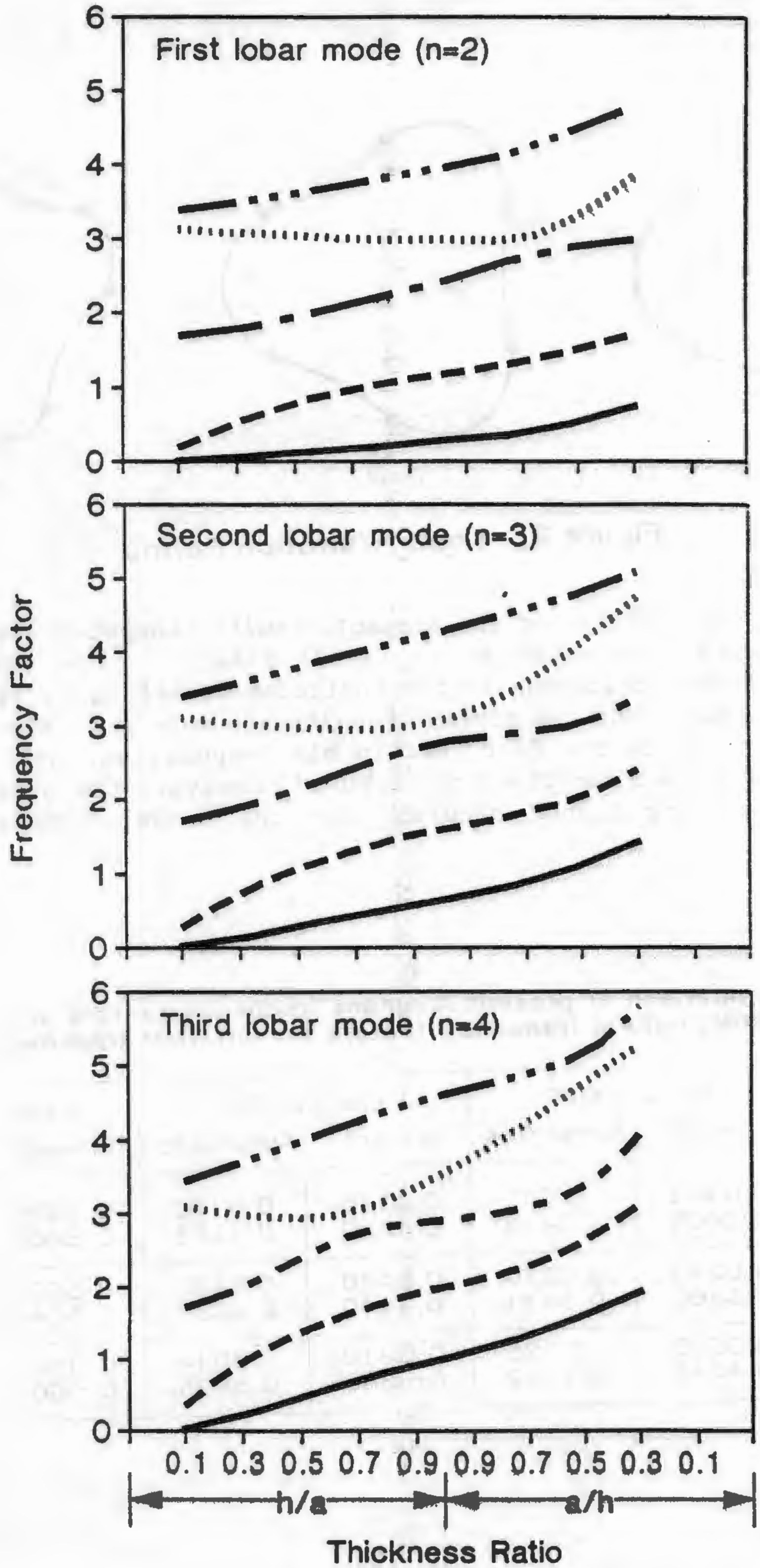

Figure 3. First five resonant frequency factors for different thickness ratio

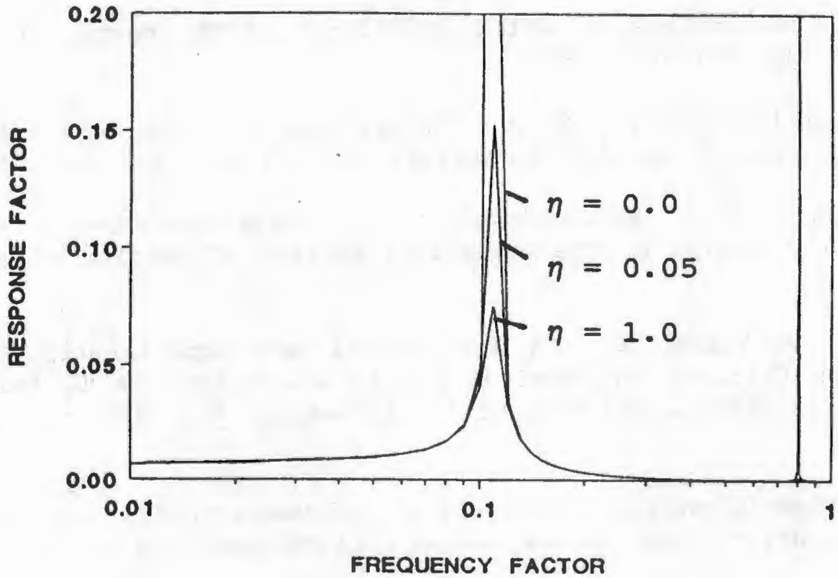

Figure 4. Frequency response of a point with maximum radial displacement on a cylinder having thickness ratio of h/a = 0.5 and Poisson's ratio of 0.25. Excited by a harmonic internal radial stress with an amplitude of 10^4 psi for three different damping factors.

CONCLUSIONS

Harmonic lobar vibrations of thick viscoelastic cylinders were considered and a general solution based on two dimensional wave propagation was developed. Design charts for estimation of the Non-dimensional resonant frequencies were provided and results were compared with available data and satisfactory agreement was established.

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