MEMBRANE ELEMENTS FOR POLYGONAL CYLINDRICAL SHELLS REINFORCED BY RINGS AND STIFFENERS

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FOREWORD

This report was prepared by the Aeronautics and Space Laboratory, University of Liege, Belgium, under Contract AF61(052)-892, Project No. 1467, "Structural Analysis Methods". Task No. 146705, "Automatic Computer Methods of Analysis for Flight Vehicle Structures." The work was administered under the direction of the Air Force Flight Dynamics Laboratory by Mr. James R. Johnson, Project Engineer, and through the European Office of Aerospace Research (OAR), United States Air Force. Major William C. Whicher and subsequently Major Richard T. Boverie were the Project Officers for the European Office of Aerospace Research.

The work reported herein was conducted during the period January 1968 through June 1968. This report was released by the author for publication in November 1968.

This document is the final report of the investigation and concludes the work on Contract AF61(052)-892. Professor B. Fraeijs de Veubeke, Director of the Aeronautics and Space Laboratory, was the Technical Director and Principal Investigator for this study. The findings of the total effort and contributions of the various personnel are incorporated in this and the following list of previously published reports for this contract.

1 - G. Sander, "Upper and Lower Bounds in Spar Matrix Analysis", Report SA-3, Laboratoire de Techniques Aeronautiques et Spatiales, Universite de Liege, Belgium, 1965 (AD 632 929)

2 - B.M. Fraeijs de Veubeke, "Bending and Stretching of Plates. Special Models for Upper and Lower Bounds", Proc. of the Conf. on Matrix Methods in Structural Mechanics, WPAFB, Ohio, Oct. (1965). AFFDL-TR-66-80 (AD 646 300) pp 836-885.

3 - G. Sander and B. Fraeijs de Veubeke, "Upper and Lower Bounds to Structural Deflections by Dual Analysis in Finite Elements. Annual Summary I, AFFDL-TR-66-199 (AD 812 876).

4 - B.M. Fraeijs de Veubeke, "Basis of a Well Conditioned Force Program for Equilibrium Models Via the Southwell Slab Analogies", AFFDL-TR-67-10 (AD 652-239).

5 - B. Fraeijs de Veubeke, "A Conforming Finite Element for Plate Bending," Int. J. Solids Structures, 1968, Vol. 4, pp. 95-108.

6 - B. Fraeijs de Veubeke and G. Sander, "An Equilibrium Model for Plate Bending", Int. J. Solids Structures, 1968, Vol. 4, pp. 447-468.

7 - G. Sander, P. Beckers and H.D. Nguyen, "Digital Computation of Stresses and Deflections in a Box Beam. A Performance Comparison Between Finite Element Models and Idealization Patterns", AFFDL-TR-69- 4 (AD 664 930)

8 - B. Fraeijs de Veubeke and Nguyen Dang Hung, "Natural Strains and Stresses for Trapezoidal Structures Analysis, AFFDL-TR-69- 18

This report has been reviewed and is approved.

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ABSTRACT

The report is concerned with the development of membrane elements for cylindrical, fuselage-type structures. The proposed models include the out of plane bending modes of the reinforcing rings, an important feature in some applications to real situations. The performance of model 4 is tested on a circular cantilevered cylinder, loaded by a radial concentrated force on the end ring. Deflections and stresses are compared with previous analyses and test results. The conclusions are that, while good deflections are already obtained with simple models, more elaborate models are required to avoid stress output interpretations. Model 5 is satisfactory from this point of view.

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LIST OF ABREVIATIONS AND SYMBOLS

u , v	cartesian displacement components in plane of membrane element
x , y	cartesian coordinates
ϵ_x , ϵ_y , γ_{xy}	cartesian strains
σ_x , σ_y , τ_{xy}	cartesian stresses
E	Young's modulus
G	shear modulus
ν	Poisson's ratio
η	dummy integration variable

Appendix

$\xi = x/a$ $\eta = y/b$	dimensionless cartesian coordinates
ψ , ϕ	slope measures, homogeneous to displacements
α	column matrix of displacement parameters
q	column matrix of local displacements
T	transformation matrix defined by equation (A.4)
ϵ	column matrix of strains
σ	column matrix of surface tractions
N	connection matrix defined by equation (A.6)
M	surface tractions-strain matrix displayed in (A.8) for isotropic sheet in (A.12) for reinforced sheet
K	stiffness matrix
S	surface tractions output matrix, defined in equation (A.17)

I. INTRODUCTION.

The success of the finite element method is due, in part to its considerable flexibility in dealing with complex geometries and local accidents such as cutouts or reinforcements.

For a given structure a first approximation to the state of stress and strain is always possible by using very simple elements. But, as soon as more exact solutions are wanted, there arises the problem of convergence of the approximate fields of displacements and stresses to the exact fields. This can in principle be obtained by reducing the sizes of elements and increasing simultaneously their number. Another procedure to improve solutions is to devise more sophisticated elements.

The two approaches were tested on various types of structures [4, 9], plates in bending, box beams and a multispar sweptback wing. The measure of convergence on displacements was obtained by the dual analysis principle [1, 2, 3], whereby upper and lower bounds are obtained if the element models are endowed with the special properties of being either "conforming" or "stress-diffusing". This experience tends to prove that, on the basis of an equal number of final generalized coordinates, the results obtained with more sophisticated elements are the better ones. They simplify the interpretation of stress outputs of the displacement models and make it less sensitive to the geometrical pattern of idealization. Also, in addition to a good bracketting of displacements, they yield a good agreement between the stress fields obtained from conforming elements on one side and stress-diffusing elements on the other side.

On the basis of this experience a reliable finite element analysis seems possible without recourse to the dual principle, which represents an appreciable economy in the case of large problems. There are however limits to the use of ever more complicated elements because, in covering each a larger part of the structure, they prevent the correct representation of finer details and critical stress gradients.

The present research was prompted by the requirement for solving fuselage type structures. It became soon apparent that new membrane type elements of the right degree of complexity were to be developped for an economical and reasonably correct analysis. Their performance was then tested on a regular cylindrical shell problem involving rings and stiffeners and compared to previous analyzes by J. TURNER, H.C. MARTIN and R.C. WEIKEL [8] and P. DENKE [13] and J. BARLOW [14].

2. DISCRETE ELEMENTS FOR FUSELAGE ANALYSIS.

This section is devoted to the development of elements suitable for thin cylindrical shells of arbitrary cross section, stiffened by transversal ring frames and longitudinal stringers. Such a geometry is typical for fuselage structures. It is represented on figure (2.I) in the ideal case of a circular cross section as analyzed and tested by J. TURNER, H.C. MARTIN and R.C. WEIKEL [8]. In more practical applications the fuselage shell can present additional connection members like bulkheads and floors.

The analysis can be based on the following assumptions

a) the skin is in a membrane state of stress so that the stiffness normal to the generators is introduced by the ring frames only.

b) The cross section is replaced by a closed polygonal arc, as illustrated on figure (2.I). Then, only flat membranes, or beam elements, are to be used.

c) The bending stiffness of the ring frames out of their plane is neglected by comparison with the in plane stiffness of the membrane. The use of such an assumption depends not only on the relative stiffnesses in question but also on the loading modes. If the assumption is used, the conformity requirement in the axial direction between rings and membranes disappears and previously developped general purpose membrane elements can be used. This procedure was followed in the aforementioned analysis [8]. In our case the out of plane bending stiffness of rings will be taken into account. Hence, the axial conformity problem arises and, because beam bending necessitates at least a third degree polynomial representation, a new membrane element must be devised with the same axial degrees of freedom in deformation.
Such an element is represented on figure (2,2). The displacement in the axial direction has to be of the form

$$u = f_1(x) + y\, f_2(x) + y^2\, f_3(x) + y^3\, f_4(x) \qquad \text{(I)}$$

d) The axial stringers bordering the membrane panels have only axial and torsional stiffness. The analysis would be considerably complicated by taking their bending stiffness into account. In such a case they would have to be treated as beams curved by shearing actions as well as by bending moments. Otherwise, their rigid connexions to the ring frames would result in zero shearing deformations and zero shear stresses at the corners of the membrane panels. This would be unrealistic and detrimental to their essential role of shear transmission elements. Assuming therefore that the bending stiffness

of the bordering stringers is neglected, the conformity requirement between two panels which are not in the same plane reduces to the singlevaluedness of the axial displacement u along their common edge. The displacement components w, normal to each plane, induce no strain energy and adjust themselves to prevent any gap due to non conforming v components. The v components are those in the plane of each panel and parallel to a side of the polygonal ring frames.
The freedom in the v components along the axial edges of the panels will be used to advantage later.

e) The combined assumptions of a membrane state of stress in the panels and a zero bending stiffness in the bordering stringers leads to the conclusion that, in an exact solution, the σ_y stresses of the panels must vanish along their common edge. This observation suggests two possible formulations for the panel models.

The first is to take a "u" displacement field of the type of equation (I) and for "v" any function that conforms at the edge with the corresponding displacement components of the rings. Such a model is of the conforming displacement type and will yield the lower bound property for displacements [3, 4]. It will present non zero σ_y stresses at the axial edges, which is a form of local lack of equilibrium characteristic of the displacement approach. The simplest model of this type is defined by

$$v = V(x) \qquad (2)$$

The transverse fibers are inextensible

$$\varepsilon_y = 0 \qquad (3)$$

and the strain energy reduces to

$$\frac{1}{2} \int_{Vol} \left(\frac{E}{1-\nu^2} \left(\frac{\partial u}{\partial x}\right)^2 + G \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \right) dVol \qquad (4)$$

This model will be called model I. It is only compatible with <u>inextensible</u> polygonal ring frames. The closure of the ring results in a homogeneous algebraic relation between the displacement components in the plane of the ring, expressing its inextensibility, which is not easily or economically

introduced in the computer program.
Preference was therefore given to models which allow ε_y, the direct strain in a beam of the polygonal ring, to be at least a non zero constant.
In model 4 the assumptions are in consequence represented by (I) for the u component but

$$v = g_1(x) + y\, g_2(x) \qquad (5)$$

for the other component. They allow a correct representation of the fundamental deformation modes and a good rate of convergence can be expected as far as displacements are concerned. However, experience with stress representations indicates that an additional deformation mode is important, in particular with respect to the shearing stress. The following physical considerations are pertinent to the problem : in the case of elongated elements $\partial u/\partial y$ is nearly constant in the y direction because of the small influence of the warping of the transverse cross section. In fact the behavior of the element is very similar to that of a spar, as shown on figure (2.3). The degree in y of τ_{xy} is governed in major part by the degree in y of $\partial v/\partial x$; it leads to improve the previous model by replacing assumption (5) by

$$v = g_1(x) + y\, g_2(x) + y^2\, g_3(x) \qquad (6)$$

Models built according to assumptions (I) and (6) are referred to as models 5.
It is now necessary to formulate precisely the functions $f_i(x)$ and $g_i(x)$ appearing in those pure displacement models. One should attempt to limitate the number of generalized displacements to a minimum while representing the fundamental deformation modes. The elements are loaded in the y direction by the ring members alone, so that the shear force between two rings is constant. The global bending moment will have linear variation in x and these global characteristics will be approximately true for each panel. Hence we shall take, as for a beam bent under constant shear load

$$g_1(x) = B_1 + B_2\, x + B_3\, x^2 + B_4\, x^3 \qquad (7)$$

$$f_2(x) = A_4 + A_5\, x + A_6\, x^2 \qquad (8)$$

Furthermore each panel will be subjected to a traction (or compression) between rings, which can vary linearly because of an approximately constant shear feed from the other panels. Hence the assumption

$$f_1(x) = A_1 + A_2\, x + A_3\, x^2 \tag{9}$$

The functions $f_3(x)$, $f_4(x)$ represent cross-sections warping effects which need only vary linearly between rings, whence

$$f_3(x) = A_7 + A_8\, x \tag{10}$$

$$f_4(x) = A_9 + A_{10}\, x \tag{11}$$

For $g_2(x)$ the assumption

$$g_2(x) = B_5 + B_6\, x + B_7\, x^2 + B_8\, x^3 \tag{12}$$

that was retained, could be reduced to a first degree polynomial, because it is also related to a deformation mode of secundary importance.
The more complete form was adopted because it represented a first step towards the adaptation of the model to incorporate a bending stiffness of the bordering stringers. For model 5 the function $g_3(x)$ needs only be linear

$$g_3(x) = B_9 + B_{10}\, x \tag{13}$$

A second possibility for a rational development of a panel element is to take the stress component σ_y into consideration. Because of the change in plane orientation between panels and the neglection of the bending stiffness of bordering stringers, this stress should vanish at the level of the axial edges. At the limit of panel heights, when the number of sides of the polygonal ring is increased, the stress σ_y should vanish completely and this agrees with the equilibrium possibilities offered by a curved cylindrical membrane reinforced only by axial filaments. The assumption

$$\sigma_y = 0 \tag{14}$$

is therefore worth considering and leads to a modification of the strain-energy expression that can be obtained by ignoring completely the stress-strain relation

$$E \frac{\partial v}{\partial y} = \sigma_y - \nu \sigma_x \tag{15}$$

that would otherwise result from the variational derivative of σ_y in Reissner's principle. The other stress-strain relations reduce to

$$E \frac{\partial u}{\partial x} = \sigma_x \tag{16}$$

$$G \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_{xy} \tag{17}$$

and, replaced into the Clapeyron form of the energy

$$\frac{1}{2} \int_{Vol} \left\{ \sigma_x \frac{\partial u}{\partial x} + \tau_{xy} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \right\} dVol \tag{18}$$

yield the required formulation consistent with (14)

$$\frac{1}{2} \int_{Vol} \left\{ E \left(\frac{\partial u}{\partial x}\right)^2 + G \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \right\} dVol \tag{19}$$

It is seen to differ from (4) only by the factor $(1-\nu^2)$ affecting the first term. The same assumptions concerning displacements can now be introduced into (19) instead of (4) in order to derive the stiffness matrices. Because of the simultaneous appearance of stress and displacement assumptions these models are "hybrids". The hybrid model based on (19) and assumptions (1) and (2) will be referred to as model 2.

Finally there is another approach to hybrid models which, in this case, would consist in satisfying exactly (15) instead of ignoring it. In this case, eliminating σ_x between (15) and

$$E \frac{\partial u}{\partial x} = \sigma_x - \nu \sigma_y \tag{20}$$

and introducing (14)

$$\frac{\partial v}{\partial y} = - \nu \frac{\partial u}{\partial x} \tag{21}$$

Integration of this equation yields

$$v = V(x) - \nu \int_o^y \frac{\partial u(x,\eta)}{\partial x} d\eta \qquad (22)$$

Hence any parametric expansion of u would partially influence the parametric expansion of v, in such a manner that the stress assumption (I4) is automatically satisfied by the stress-strain relations. While the parametric expansion in u can be organized for conformity with adjacent members, the influence of u on v will not generally make conformity possible for v. However average compatibility conditions can be set up. The possible generalization of this approach is a subject of further study. The element based on (22) and (I) will be referred to as model 3. Its energy expression, derived from (I4) and (22) is as follows

$$\frac{1}{2} \int_{Vol} \{ E \left(\frac{\partial u}{\partial x}\right)^2 + G \left(\frac{\partial u}{\partial y} + \frac{dV}{dx} - \nu \int_o^y \frac{\partial^2 u(x,\eta)}{\partial x^2} d\eta\right)^2 \} dVol \qquad (23)$$

3. PERFORMANCE OF THE MODELS IN A BENDING MODE.

The relative merits of the five panel models can already be judged in the simplified case where they work as spar webs in a bending mode. Then only terms in odd powers of y will appear in u and in even powers of y in v. Model I will be a displacement model, based on the strain energy expression (4), with

$$u = y(\alpha_1 + \alpha_2 x + \alpha_3 x^2) + y^3(\alpha_{10} + \alpha_{11} x) \qquad (24)$$

$$v = \alpha_4 + \alpha_5 x + \alpha_6 x^2 + \alpha_7 x^3 \qquad (25)$$

Model 4 is identical to model I. Model 2 is also based on assumptions (24) and (25) but uses the modified strain energy (I9). Model 5 is again a displacement type based on (4) and (24) but with a more refined

$$v = \alpha_4 + \alpha_5 x + \alpha_6 x^2 + \alpha_7 x^3 + y^2(\alpha_8 + \alpha_9 x) \qquad (26)$$

Model 3 is based on

$$u = y(\alpha_1 + \alpha_2 x + \alpha_3 x^2) \qquad (27)$$

and a v derived from formula (22)

$$v = \alpha_4 + \alpha_5 x + \alpha_6 x^2 + \alpha_7 x^3 - \frac{1}{2} \nu y^2 (\alpha_2 + 2 \alpha_3 x) \qquad (28)$$

with strain energy (4).
Experience with these models in a box beam analysis was reported earlier [9].
It also included the pure equilibrium model derived from the simplest Airy stress function [3]. Since, for constant web thickness, the strain field of this model is integrable, it can be presented here in the form

$$u = y(\alpha_1 + \alpha_2 x + \alpha_3 x^2) - (2 + \nu)\alpha_3 y^3 \qquad (29)$$

$$v = \alpha_4 + \alpha_5 x - \frac{1}{2} \alpha_2 x^2 - \frac{1}{3} \alpha_3 x^3 - \frac{1}{2} \nu y^2 (\alpha_2 + 2 \alpha_3 x) \qquad (30)$$

of a displacement model which resembles model 3. Its major difference is that it satisfies internal equilibrium with zero body loads.
From the displacements point of view all those models were satisfactory. But, for a correct stress representation, it was necessary to use the more sophisticated model 5. In the others, only average stresses, in particular the average shear stress, behaved satisfactorily . The cantilevered box beam analyzed is sketched on figures (3.I) and (3.2); it is submitted to symmetrical bending by two equal tip loads. The side panels work as spar webs with some variable shear introduced by the cover sheets. The shear stress fields obtained are compared on the figures for model 5 webs (six consecutive elements along the span) and equilibrium elements (twelve consecutive elements along the span).
Because of the symmetry, only the upper half of the web is shown. The agreement between the two solutions is seen to be excellent except in the neighborhood of the built in section. The explanation of the discrepancy in this region is as follows : the displacement model can represent accurately the built in condition which is a particular case of the parametric displacement field of model 5; in the case of the equilibrium model only average values of the displacements can be set equal to zero.
The simpler displacement models produce a constant shear field in a section, equal to the average shear of the preceding diagrams.
The conclusion is that if a detailed stress picture in the web is required, model 5 or an equilibrium model is needed. If the average shear stress is considered sufficient information, the simpler displacement models can be used.
Similar conclusions will hold for the results of the fuselage analysis presented below.

4. RING STIFFENED CIRCULAR CYLINDER ANALYSIS.

Figure (4.I) represents a cantilevered circular cylinder with ring stiffeners as investigated experimentally by KUHN, DUBERG and GRIFFITH [10] and analyzed theoretically by HOFF [11], JENSEN [12] and DENKE [13]. It was also analyzed by the Direct Stiffness Method by TURNER, MARTIN and WEIKEL [8] and their structural idealization is shown on figure (4.2). It consists in a polygonal approximation to the circular shape; their finite elements consist of bars and triangular panels with linear displacement fields. Our idealization is similar but uses model 4 for the panels and is thus able to take into consideration the rigidity of rings out of their plane.

The analysis has been simplified by consideration of the symmetry in the structural geometry and loading with respect to the xy plane. There are 28 active nodes, numbered on figure (4.2) and at each node 3 translational and 3 rotational components. In the plane of symmetry the nodes experience only two translations and one rotation. At the built in section all nodal degrees of freedom are suppressed. The total number of degrees of freedom, which indicates the size of the problem, is then I44. Numerical data from measurements are based on $E = 10.6 \times 10^6$ psi and $\nu = 0.33$.

Figure (4.3) shows the deformation of the rings in their own plane. The agreement with reference 8 is so close that the differences cannot be depicted. It is a confirmation of the fact that, from a displacement point of view, the sensitivity to idealization patterns is generally small. The bending moments in the rings in their own plane are shown on figures (4.4) and (4.5) and compared to the measurements of reference IO and the numerical results of reference 8. The agreement is excellent. The results for ring torsional moments and out of plane bending moments could not be compared. They are represented respectively on figures (4.6) and (4.7). Another interesting result is the distribution of the radial and circumferential reactions in the built in section, they are generalized loads computed at the fixed nodes. It is seen in figure (4.8) how they add up to balance the applied load. The same figure also shows the vectorial addition of the total shear loads in the adjacent panels.

Turning now to the stress fields in the panels, the shortcoming of model 4 is apparent as regards the τ_{xy} stresses in a panel. The function v in equation (5) does not allow a good shear distribution output and model 5, based on equation (6), would probably give better results.

Figure (4.9) shows the discontinuities in the shear stress diagrams at the level of the built in section that results from the constraints in the y direction in equation (4). As in the case of the triangular elements with linear variation used

by TURNER, it is necessary to use an averaging process.
Figure (4.9) shows the regularity of the curve based on the average shear stress, which becomes comparable to the same average obtained in reference 8. Another way around this difficulty of stress output interpretation is to calculate the stress resultants which are the nodal forces at the element level through the equation $g = Kq$, where K is the element stiffness matrix and q the calculated displacements of the nodes of the element. Those stress resultants can then be used to determine a more satisfactory stress picture. This procedure was found to give results which reduce discontinuities and show convergence as the nodal network is refined. The equivalence between stress averaging and nodal force methods can be demonstrated theoretically (Appendix I). In conclusion, the shear stress results obtained with model 4 shows no improvement with respect to the results of reference 8.
The average values are nevertheless in good agreement with measurements as shown on figure (4.I0) for the field between rings I and 2.
Turning to the σ_x fields in the panels, the performance of model 4 is quite satisfactory. Figure (4.II) shows the root σ_x stresses obtained, compared to those of reference 8. If the latter are interpreted according to the stress matrix procedure [8] , the local agreement with our results is quite good.
Table I shows the normal traction results adjacent to the rings. The differences across the rings are responsible for the out of plane bending of the rings. They are small and for this reason the average values only have been represented on figure (4.I2). They are in excellent agreement with the stress matrix interpreted values of reference 8 (which neglect ring bending stiffness out of their plane) and the measurements of reference I0. Also to be observed as a measure of quality of the stress output are the very small discrepencies in σ_x when passing from a panel to an adjacent one.
The analysis of real fuselage structures justifies the use of models with the capability of representing distortions of the rings out of their plane. An actual fuselage idealization to which the model was applied is represented on figure (4.I3). It presents skin thickness variations in both the axial and circumferential directions together with a double system of stringer reinforcements.
The ring frames have important sections justified by the presence of a hinged after section for cargo loading facility. Account was taken of the presence of the floor by a similar idealization in beams and panels. The loading case considered here is a vertical force and a horizontal bending moment introduced as a the Saint Venant equivalent end force system. In the ring frame selected on figures (4.I3) and (4.I4) it induces in plane and out of plane bending moments of the same order

of magnitude. The discontinuities in the diagrams are due to jumps in gauge thicknesses and boundary conditions.

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Appendix.

A GENERAL FUSELAGE MEMBRANE ELEMENT.

The most general displacement field considered here and called model 5 is written

$$u = (1-\xi)(A_1 + A_2\,\eta + A_3\,\eta^2 + A_4\,\eta^3) + (1+\xi)(A_5 + A_6\,\eta + A_7\,\eta^2 + A_8\,\eta^3) + (1-\xi^2)\big(A_9(1+\eta) + A_{10}(1-\eta)\big)$$

$$v = (1-\eta)(B_1 + B_2\,\xi + B_3\,\xi^2 + B_4\,\xi^3) + (1+\eta)(B_5 + B_6\,\xi + B_7\,\xi^2 + B_8\,\xi^3) + (1-\eta^2)\big(B_9(1+\xi) + B_{10}(1-\xi)\big) \qquad \text{(A.I)}$$

It applies to the rectangular element sketched in figure A.I in the local non dimensional coordinates

$$\xi = \frac{x}{a} \qquad \eta = \frac{y}{b} \qquad \text{(A.2)}$$

By dropping some coefficients of this field the model can be reduced to the less refined ones mentionned above or even to spar displacement field. In the general case (model 5) the set of generalized displacements consists in

- three local values of the displacement component parallel to the edge, in the direction of positive axis,
- two slopes at each vertex

$$\psi = -\,b\,\frac{\partial u}{\partial y} = -\,\frac{\partial u}{\partial \eta} \qquad \phi = a\,\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi} \qquad \text{(A.3)}$$

These 20 displacements are shown figure A.I and define uniquely the twenty field parameters. The 20 corresponding linear equations are easily solved analytically for the parameters. The result is noted

$$\alpha = T^{-1}\,q \qquad \text{(A.4)}$$

where

$$\alpha' = (A_1 \;\dots\; A_{10} \,,\, B_1 \;\dots\; B_{10}) \qquad \text{(A.5)}$$

$$q' = (u_1 \;\dots\; u_6 \,,\, v_1 \;\dots\; v_6 \,,\, \psi_1 \;\dots\; \psi_4 \,,\, \phi_1 \;\dots\; \phi_4) \qquad \text{(A.6)}$$

represent the transposed column of the parameters and of the displacements, while T^{-1} is the inverse of the displacement connection matrix which is given in figure A.2.
By their definition the strain components can be written in terms of the field parameters

$$\varepsilon = \begin{vmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{vmatrix} = N \alpha \qquad (A.6)$$

and N is shown on figure A.3.
The strain energy is then written

$$U = \frac{1}{2} \int \varepsilon' M \varepsilon \, dxdy = \frac{1}{2} \alpha' \left(\int N' M N \, dxdy \right) \alpha \qquad (A.7)$$

The matrix M is Hooke's matrix resolved for surface tractions

$$\sigma = \begin{vmatrix} S_x \\ S_y \\ S_{xy} \end{vmatrix} = \frac{Et}{1-\nu^2} \begin{vmatrix} 1 & \nu & o \\ \nu & 1 & o \\ o & o & \frac{1-\nu}{2} \end{vmatrix} \begin{vmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{vmatrix} \qquad (A.8)$$

The tractions of the reinforcing set of stringers disposed in the x direction can be replaced by an equivalent thickness t_1

$$t_1 = \frac{\Omega_1}{a_1} \qquad (A.9)$$

where Ω_1 is the cross section area of a stringer and a_1 the pitch of the reinforcement. The stringers being in an axial state of stress only, their stress strain relations reduce to :

$$S_{x1} = E_1 \, t_1 \, \varepsilon_x \qquad (A.10)$$

Considering a similar set of stringers in the perpendicular y direction with a corresponding equivalent thickness t_2 it produces traction given by

$$S_{y2} = E_2\, t_2\, \varepsilon_y \qquad \text{(A.I1)}$$

Summing the three contributions of energy per unit area of the three members leads to an equivalent panel where the surface tractions are related to the deformations by the orthotropic Hooke's relations

$$\sigma = \begin{vmatrix} S_x \\ S_y \\ S_{xy} \end{vmatrix} = \frac{Et}{1-\nu^2} \begin{vmatrix} 1 + \dfrac{E_1 t_1 (1-\nu^2)}{Et} & \nu & o \\ \nu & 1 + \dfrac{E_2 t_2 (1-\nu^2)}{Et} & o \\ o & o & \dfrac{1-\nu}{2} \end{vmatrix} \begin{vmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{vmatrix} \qquad \text{(A.I2)}$$

or

$$\sigma = M\, \varepsilon \qquad \text{(A.I3)}$$

The integral matrix of the element

$$I = \int N'\, M\, N\, dxdy \qquad \text{(A.I4)}$$

can now be calculated and is given in figure A.4. The strain energy is still to be expressed in terms of generalized displacements. By relations (A.6) and (A.4) it turns out that

$$U = \frac{1}{2}\, q' (T^{-1})'\, I\, T^{-1}\, q = \frac{1}{2}\, q'\, K\, q \qquad \text{(A.I5)}$$

where

$$K = (T^{-1})'\, I\, T^{-1} \qquad \text{(A.I6)}$$

is the stiffness matrix of the element. The operations represented by (A.I6) are achieved numerically. The stresses are computed by

$$\sigma = M\, N\, T^{-1}\, q = S\, q \qquad \text{(A.I7)}$$

at convenient locations.

To apply the element to fuselage analysis, the local displacements have to be expressed in a global cylindrical coordinate system.

The set of displacements and the stiffness matrix are completed by the contributions of the beam along the edges $x = \pm a$, that leads to 6 generalized displacements at each vertex and four tangential displacements along the four edges, as shown on figure A.5.

It is now easy to express the displacements in cylindrical coordinates in terms of local displacements, each node possessing a particular coordinate transformation.

As mentionned above this general model can degenerate in more simple ones if some parameters are dropped or expressed in terms of others. This procedure can be applied not only at the analytical level, but also later numerically. In this case it is convenient to simplify the model by introducing relations between the original set of generalized displacements $\bar{q}$ and the new ones q corresponding to the simplified elements. Denoting such an operation by

$$\bar{q} = R\, q \qquad \text{(A.I8)}$$

where the matrix R is a reduction matrix, the new stiffness matrix K is simply obtained from the original $\bar{K}$ as

$$K = R' \, \bar{K} \, R \qquad \text{(A.I9)}$$

By this procedure the same general computer program can be use to produce the whole set of elements derived from the original by simplification of the displacement field.

Another important remark concerns the type of connection between elements.

If used as plane membrane elements, all the generalized displacements are connected to their corresponding ones in adjacent elements.

But when used in a polygonal cylinder, like in a fuselage or box beam analysis, it has been pointed out that the v displacements (figure A.I) are only transmitted at the vertices. The ϕ slopes are therefore to be left unconnected between two elements and are eliminated at the element level. The same situation was pointed out for spar elements and these membrane elements can indeed be considered as generalized spars. Different ways can be followed to eliminate the unconnected displacements. They are free displacements which can be subject to independant variations in the energy principle at any stage.

If, before building the element stiffness matrix, the energy principle is written in terms of the displacement field parameters, those corresponding to unconnected displacements can be subject to variation at this stage. The outcome are analytical relations expressing these parameters in terms of the others. However, to preserve the generality of the stiffness matrix formulation of the element, the variations can also be applied after building it. This turns out to be equivalent to the usual condensation procedure. If required, the variation can also be applied after assembling the elements into the structure, this is equivalent to the condensation procedure used in the substructure technique.

The second procedure was followed there. An optionnal condensation can be devised to program the stiffness matrix, so that the element can be used indifferenly for polygonal cylinders or plane membrane problems.

A final remark concerns the interpretation of the shearing stresses.

Calculating the shear force in terms of the parameters

$$T_v = \frac{1}{2\,b} \int_{-b}^{+b} \tau_{xy}\, dy \tag{A.20}$$

yields in this model the same expression as in the pure membrane rectangle using a quadratic field or as in the "Sparta" elements, that is

$$T_v(x) = (A_1 + x\, A_2 + x^2\, A_3)\alpha \tag{A.2I}$$

where A_1 , A_2 , A_3 are row matrices. In a previous report [9] it was shown that even in the case of a physically constant shear force (like in a cantilever beam under a tip load) a pronounced parabolic oscillation of shear force is produced by these elements and introduces difficulties in the interpretation of τ_{xy} stresses. The only solution was to consider the average value over the element

$$T_v = \frac{1}{2\,a} \int T_v(x)\, dx \tag{A.22}$$

which proves to be exact.

In the present new element, however, it can be shown that provided the loading be restricted to the end cross section, the shear force T_v is constant and exact

$$A_2\, \alpha = A_3\, \alpha = 0 \tag{A.23}$$

$$T_v(x) = A_1\, \alpha = \text{cste} \tag{A.24}$$

This means that the displacement field satisfies in the particular loading case, the transverse equilibrium equation in each section instead of globally. This conclusion is of particular importance : the local τ_{xy} values are now realistic and can be used directly for interpretation of the stress field.

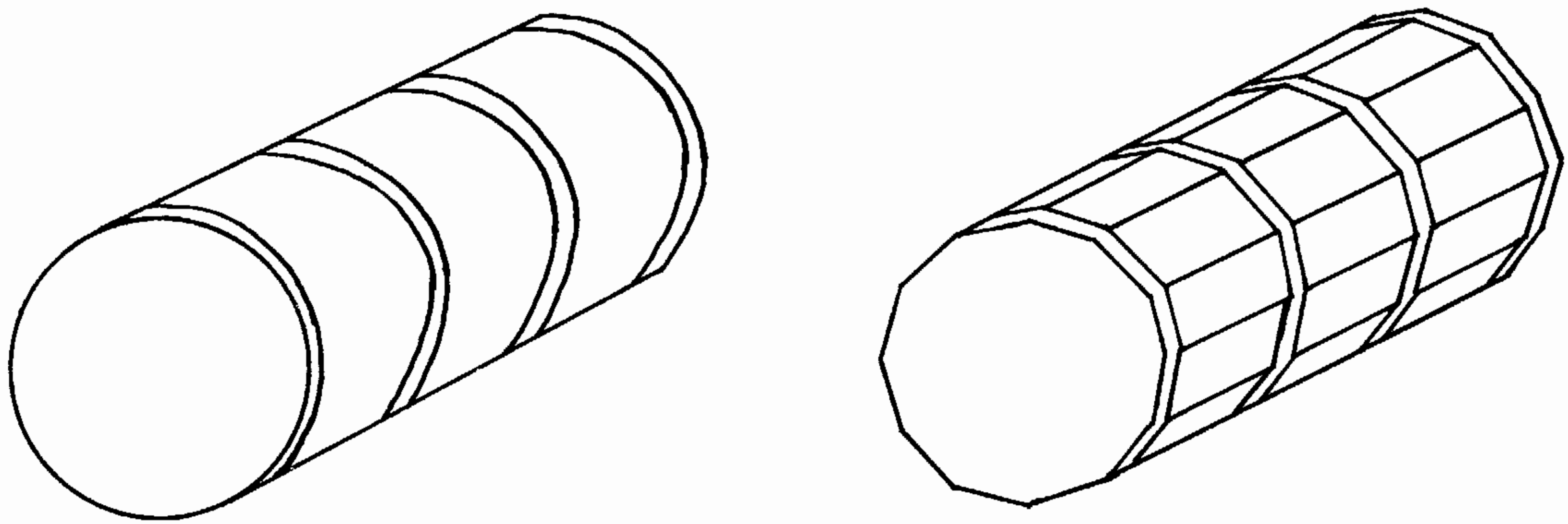


Figure 2.1 - Reinforced cylinder and idealization

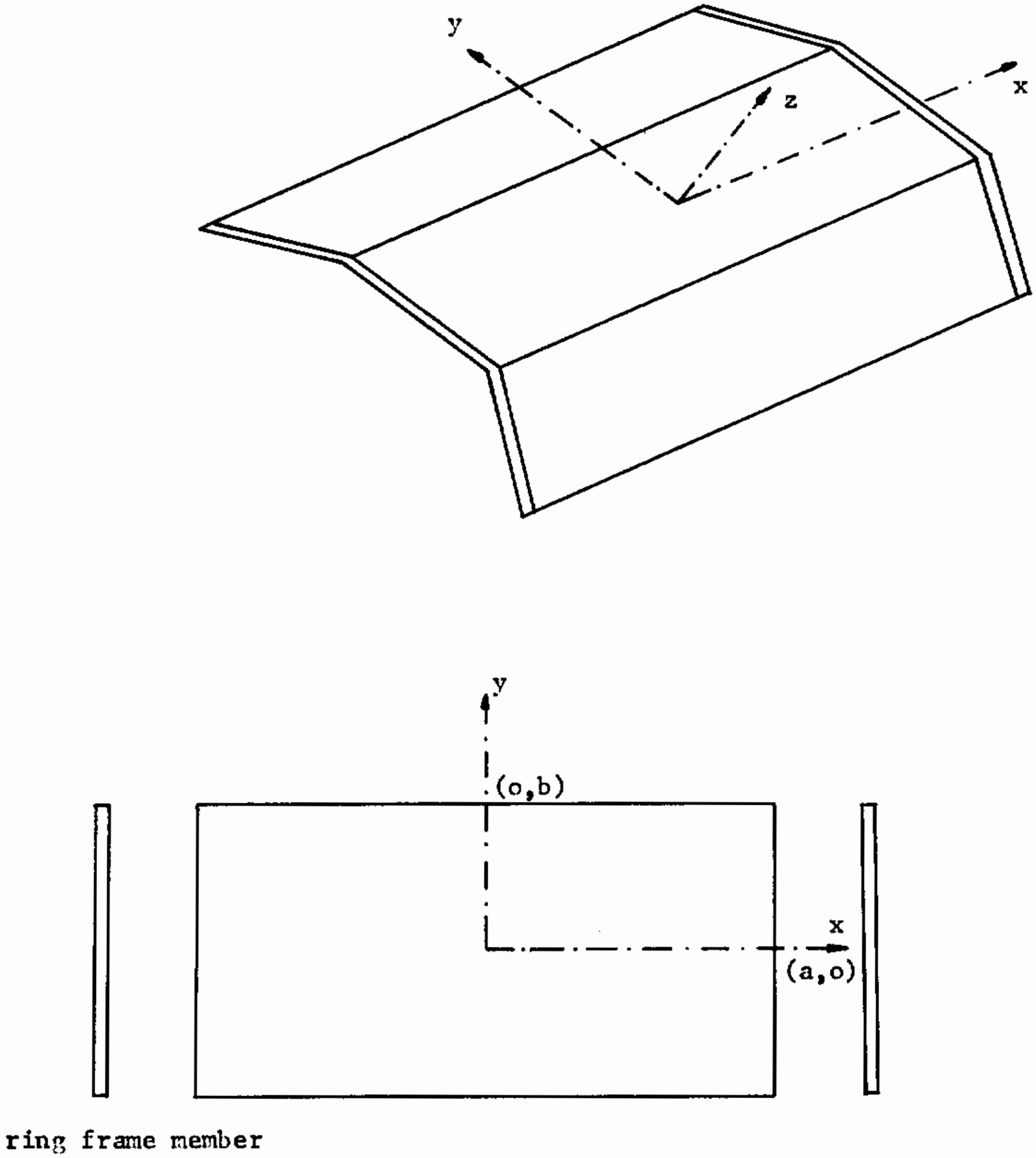


Figure 2.2 - Membrane element reference axes

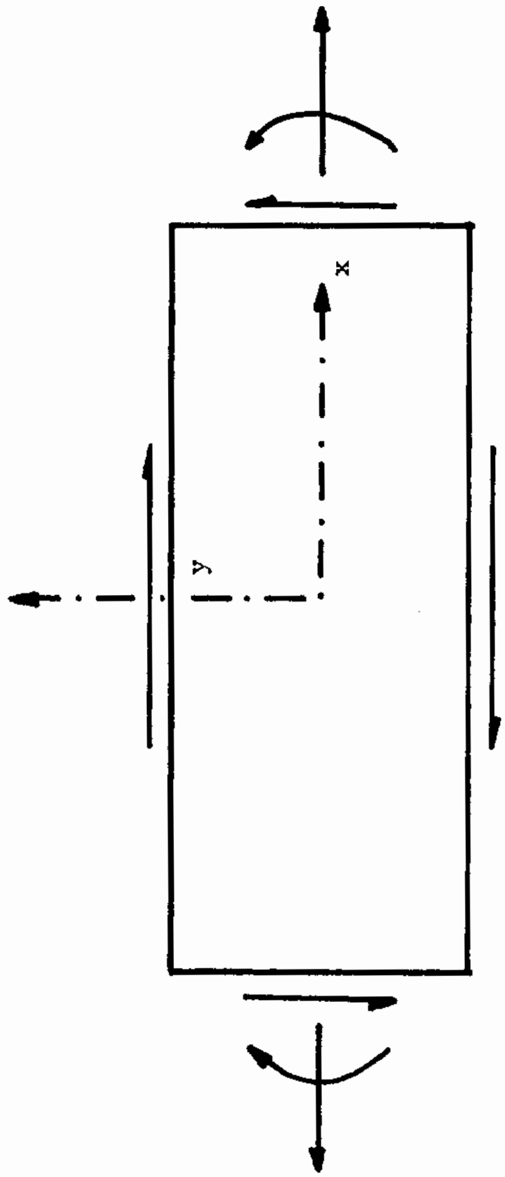
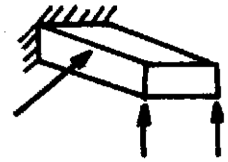


Figure 2.3 - Load system on membrane element

$$\mu = \frac{\tau_{xy}\ \text{local}}{\tau_{xy}\ \text{average}}$$



Level surfaces of τ_{xy} stress upper half of web displacement model 5

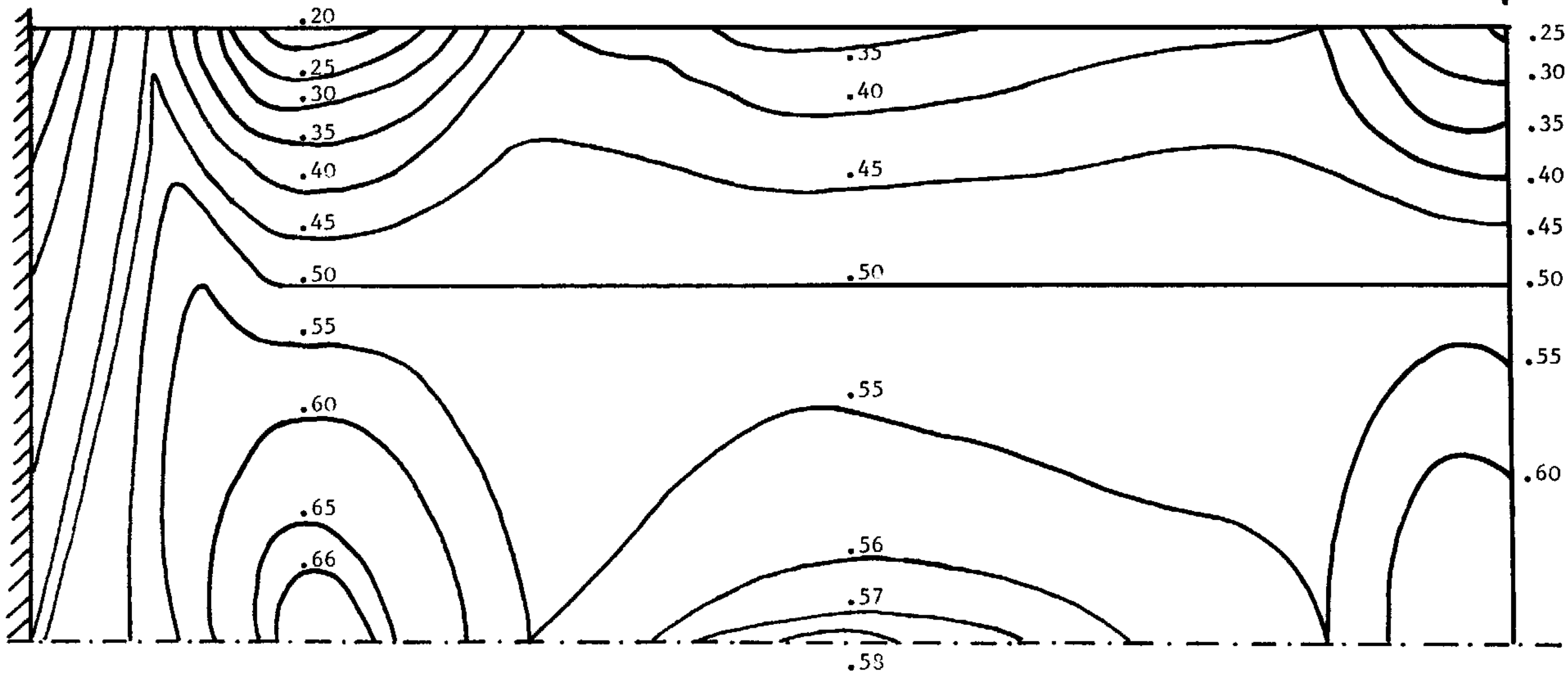


Figure 3.1 - Shear stress level surfaces for model 5 in a spar bending mode

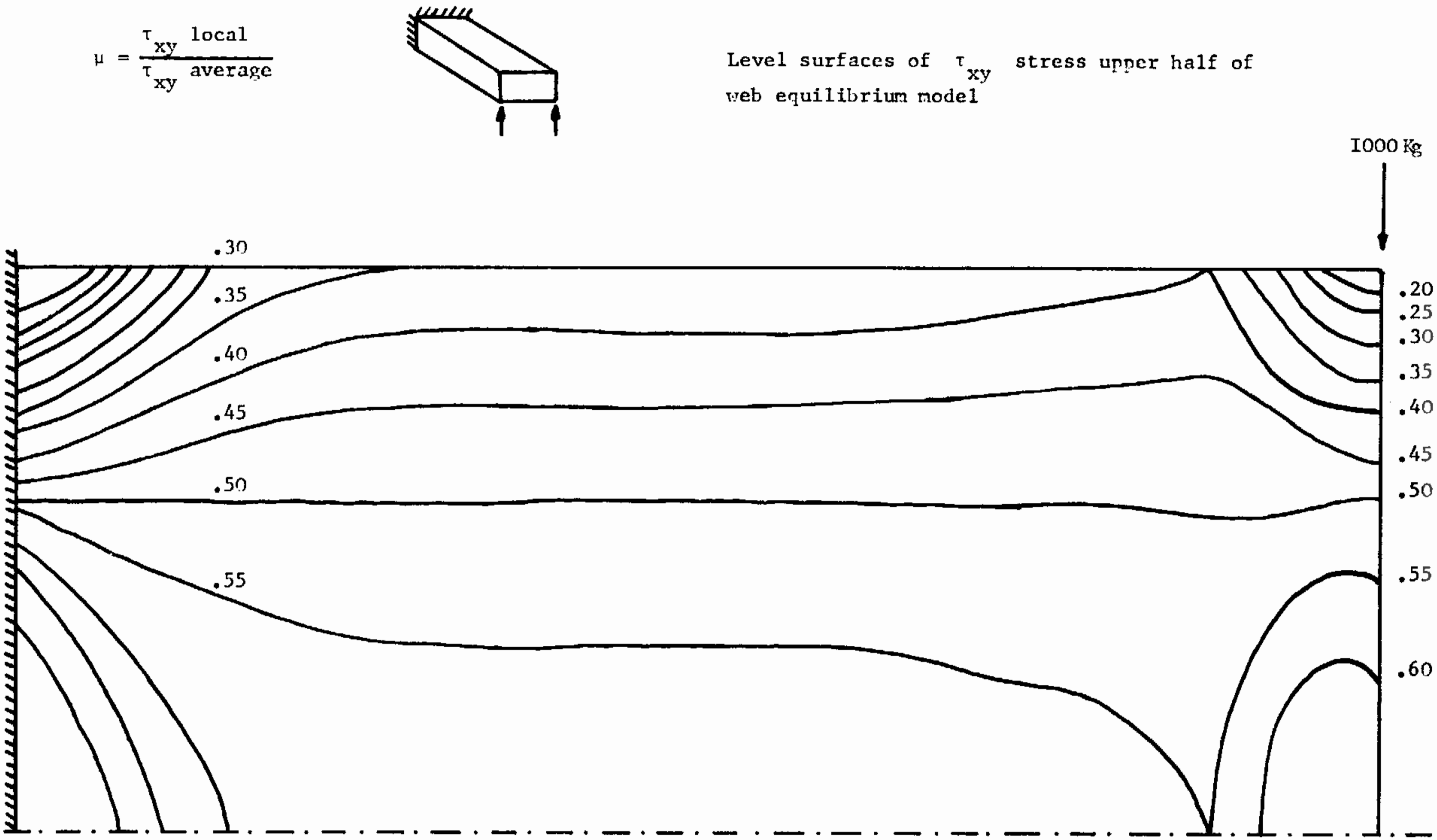
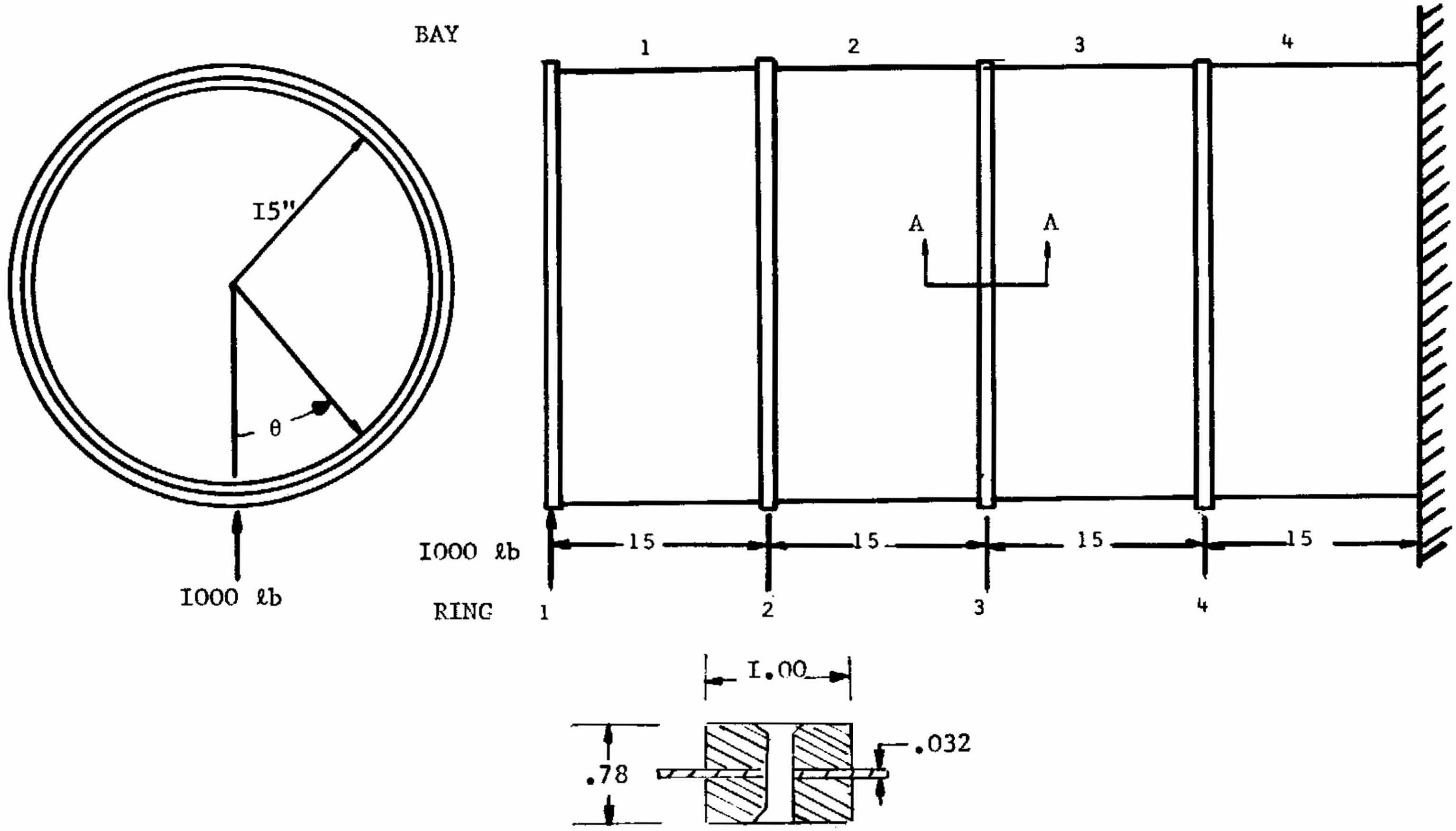


Figure 3.2 - Shear stress level surfaces for equilibrium model in a spar bending mode



Sect. A - A through Ring

Figure 4.1 - Data for reinforced circular cylinder

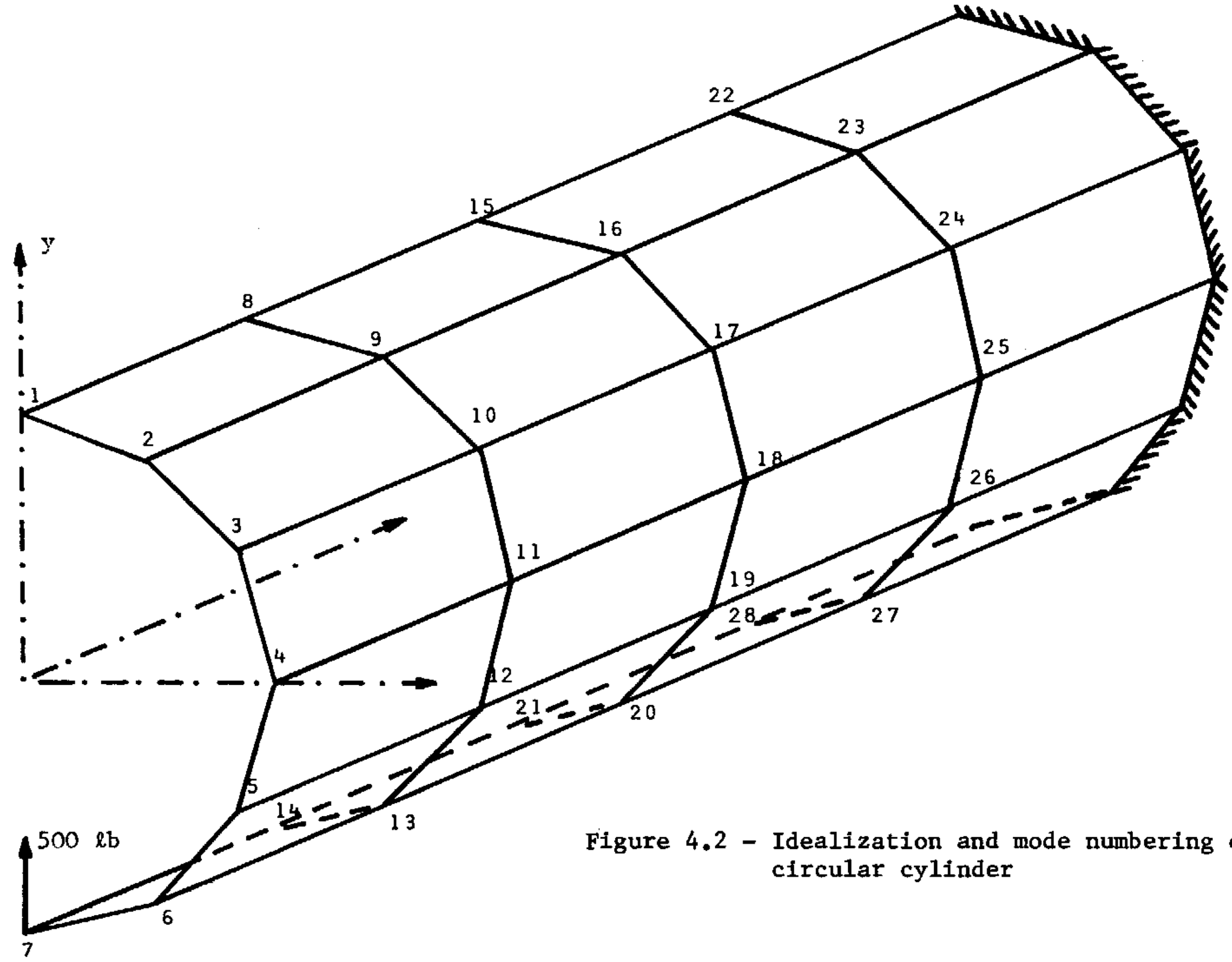


Figure 4.2 - Idealization and mode numbering of reinforced circular cylinder

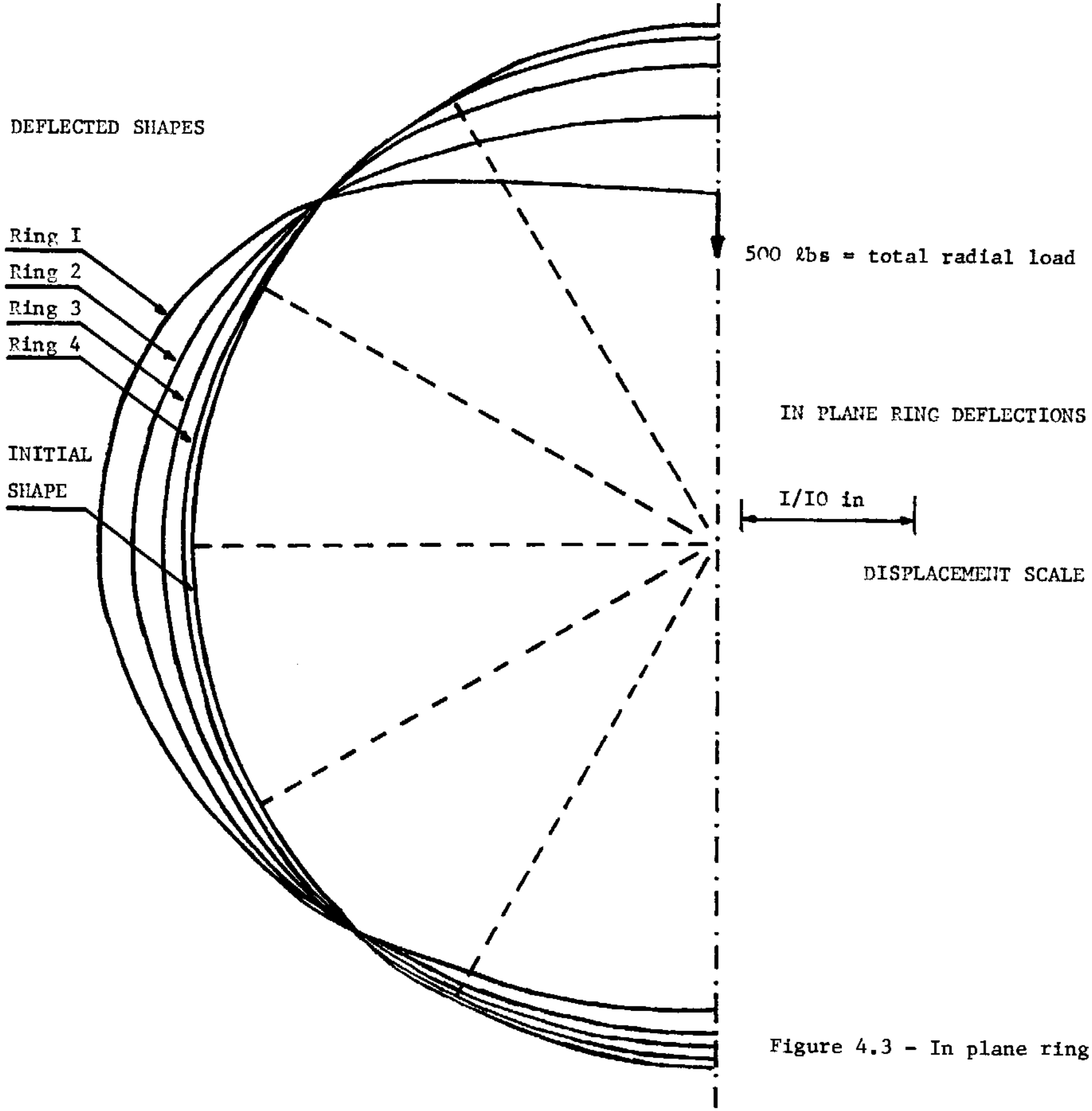


Figure 4.3 - In plane ring deflections

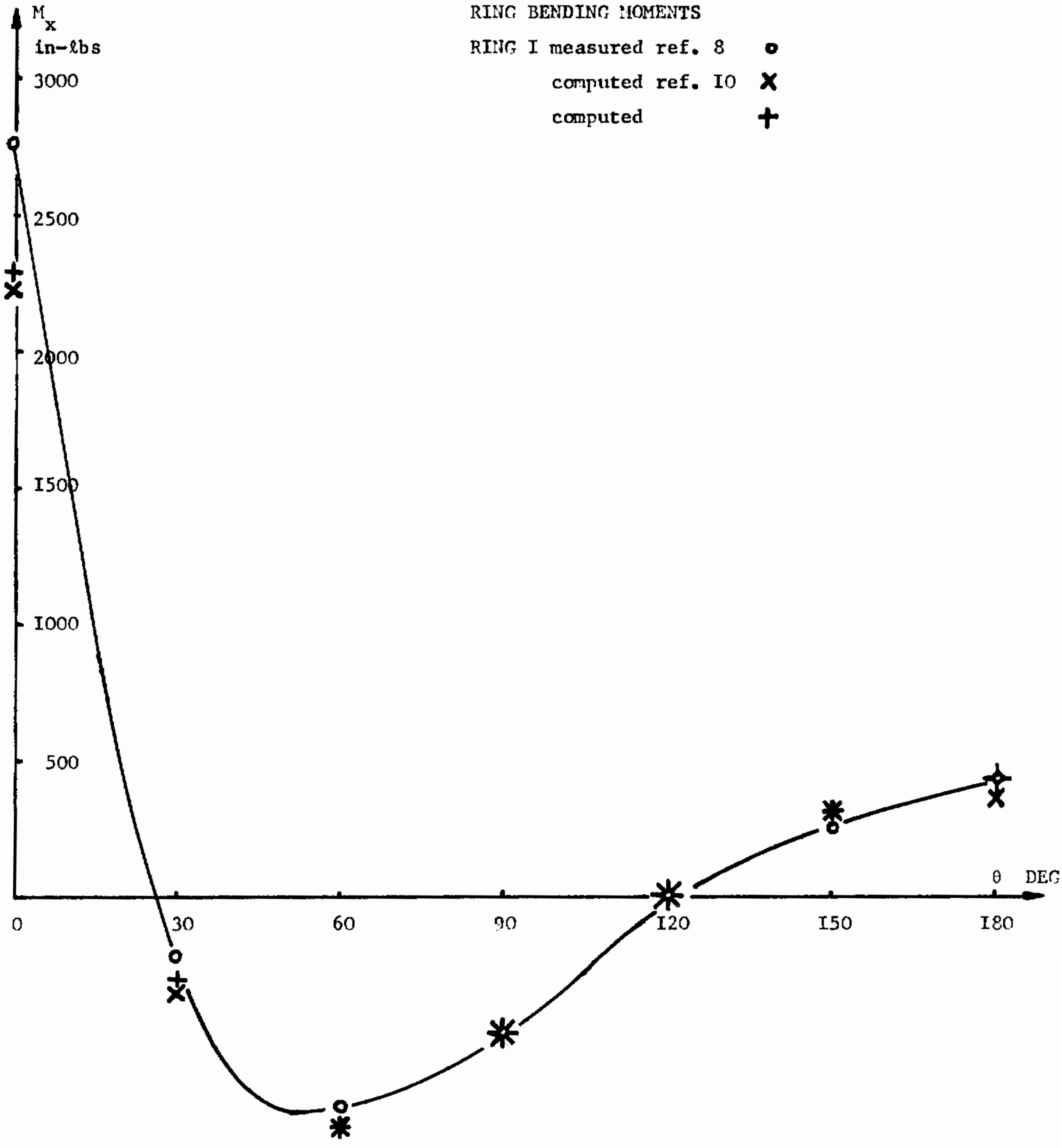


Figure 4.4 - Ring bending moments, Ring I

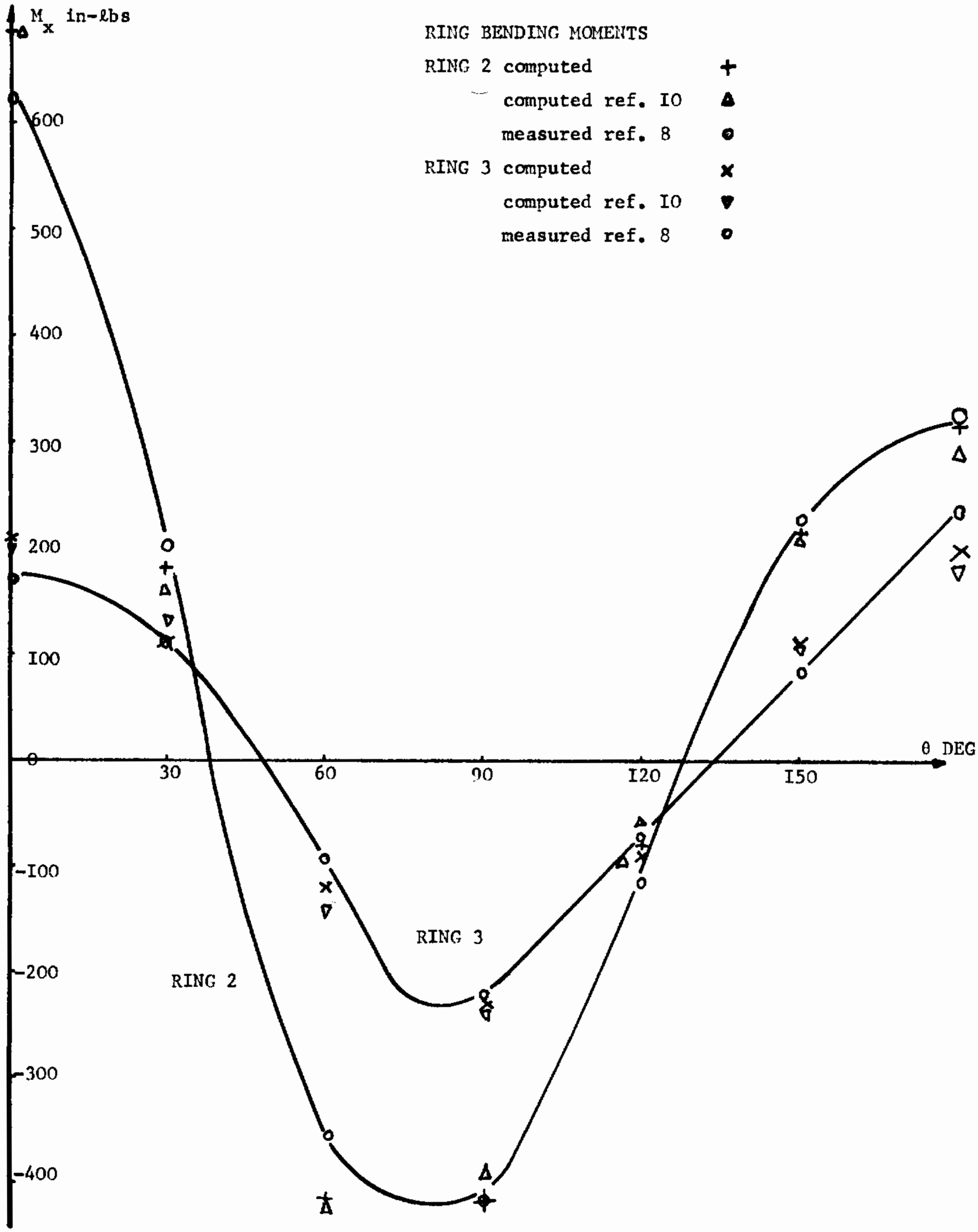


Figure 4.5 - Ring bending moments, Ring 2 and 3

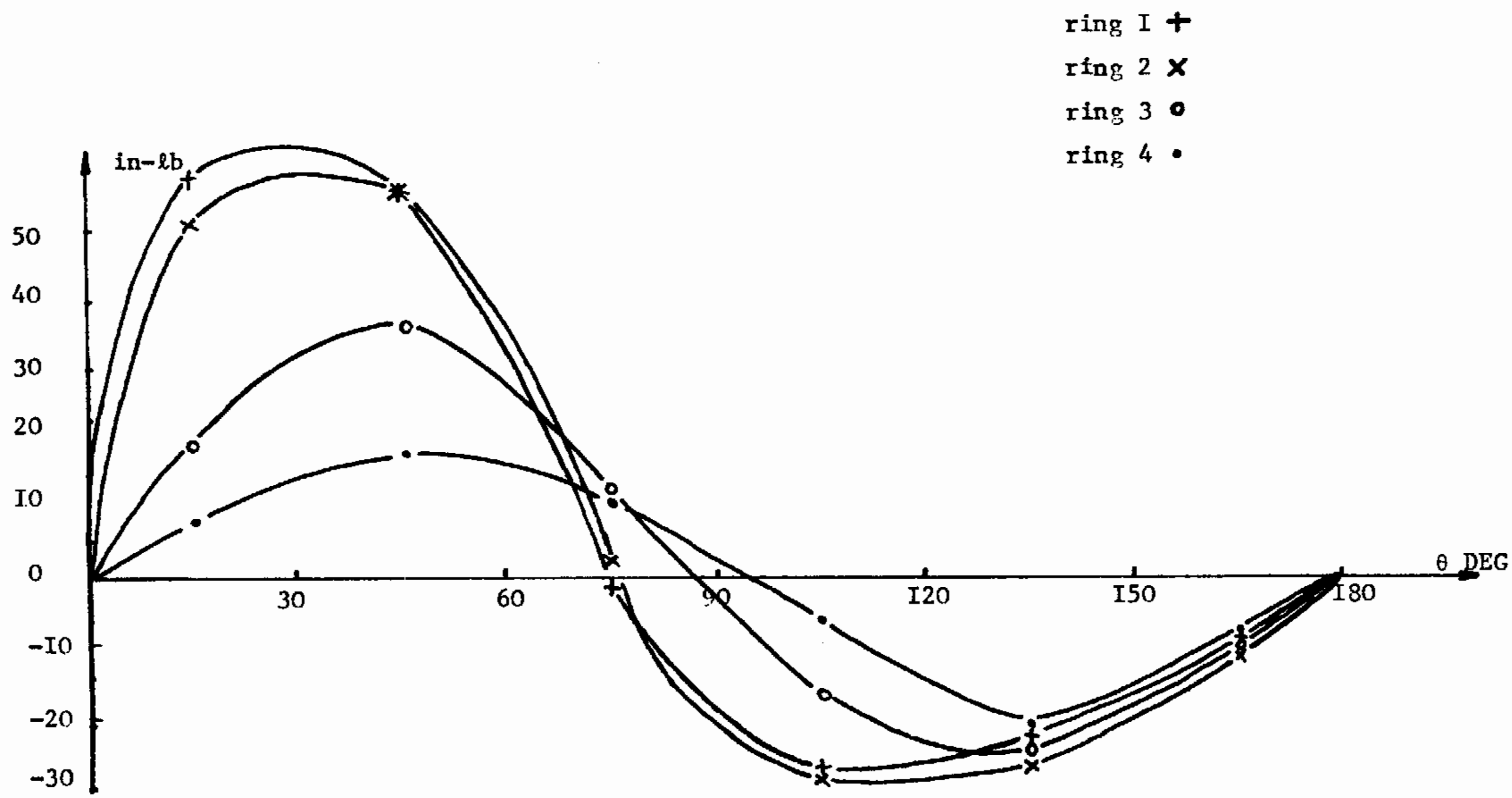


Figure 4.6 - Ring torsional moments

RADIAL BENDING MOMENT

RING I

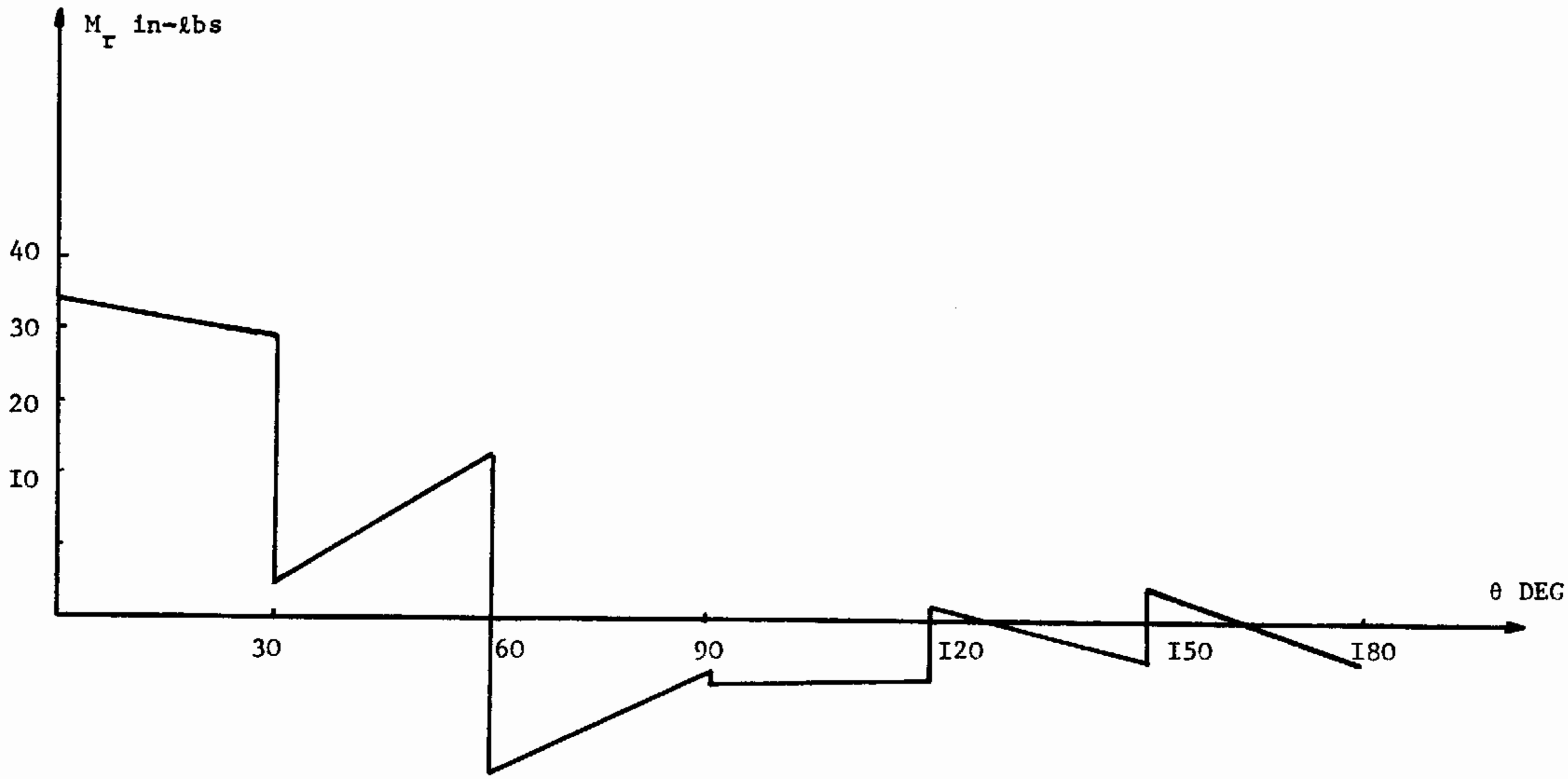


Figure 4.7 - Out of plane bending moment, Ring I

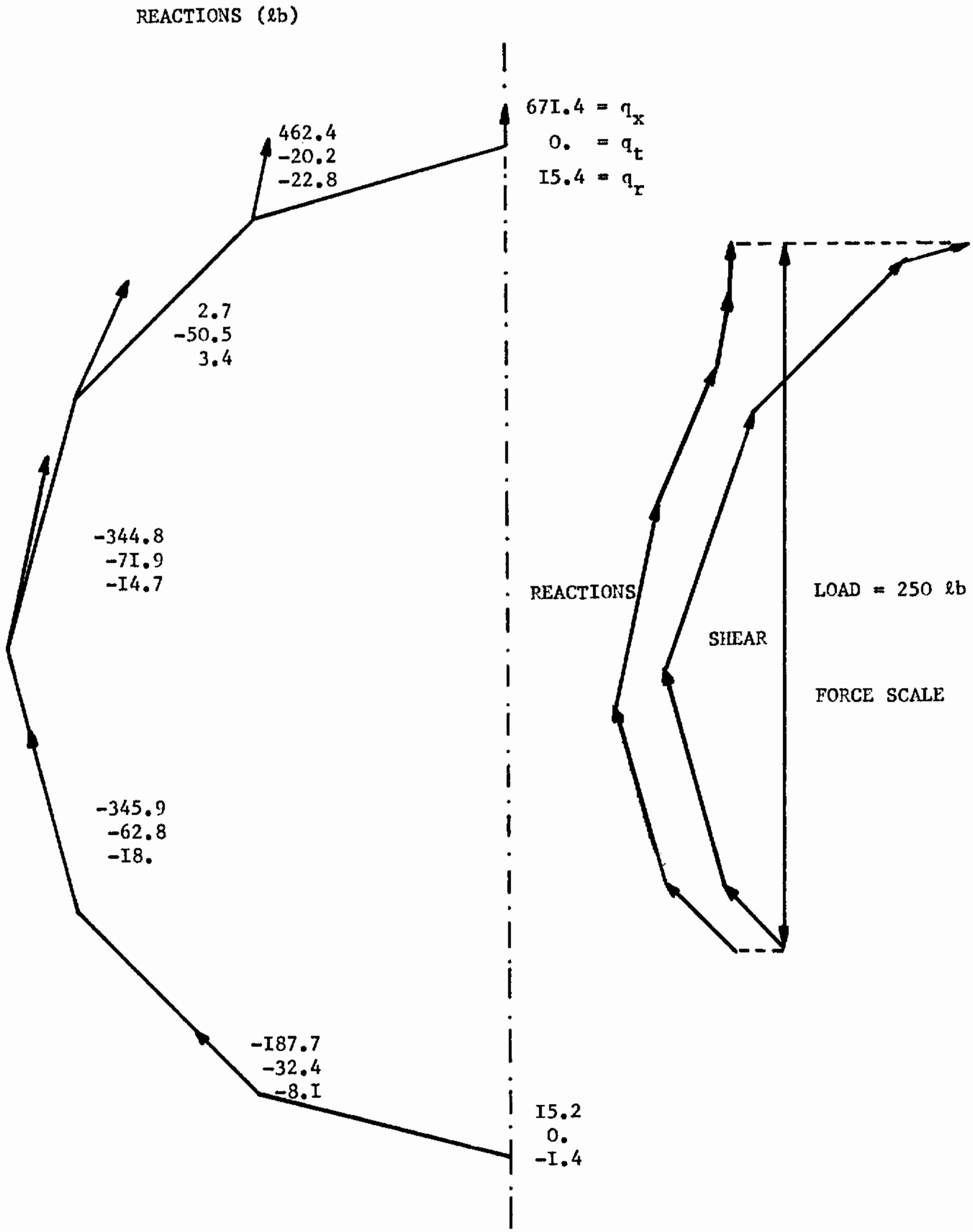


Figure 4.8 – Reaction loads in built-in section

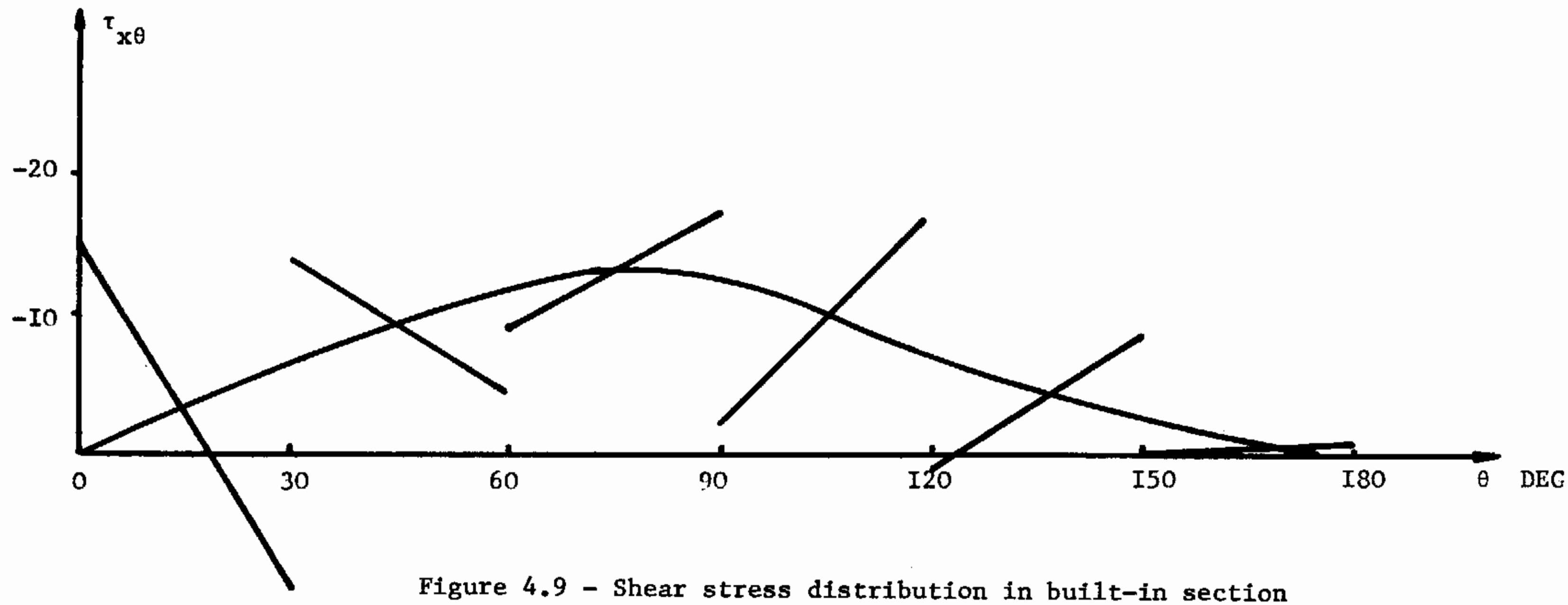


Figure 4.9 - Shear stress distribution in built-in section

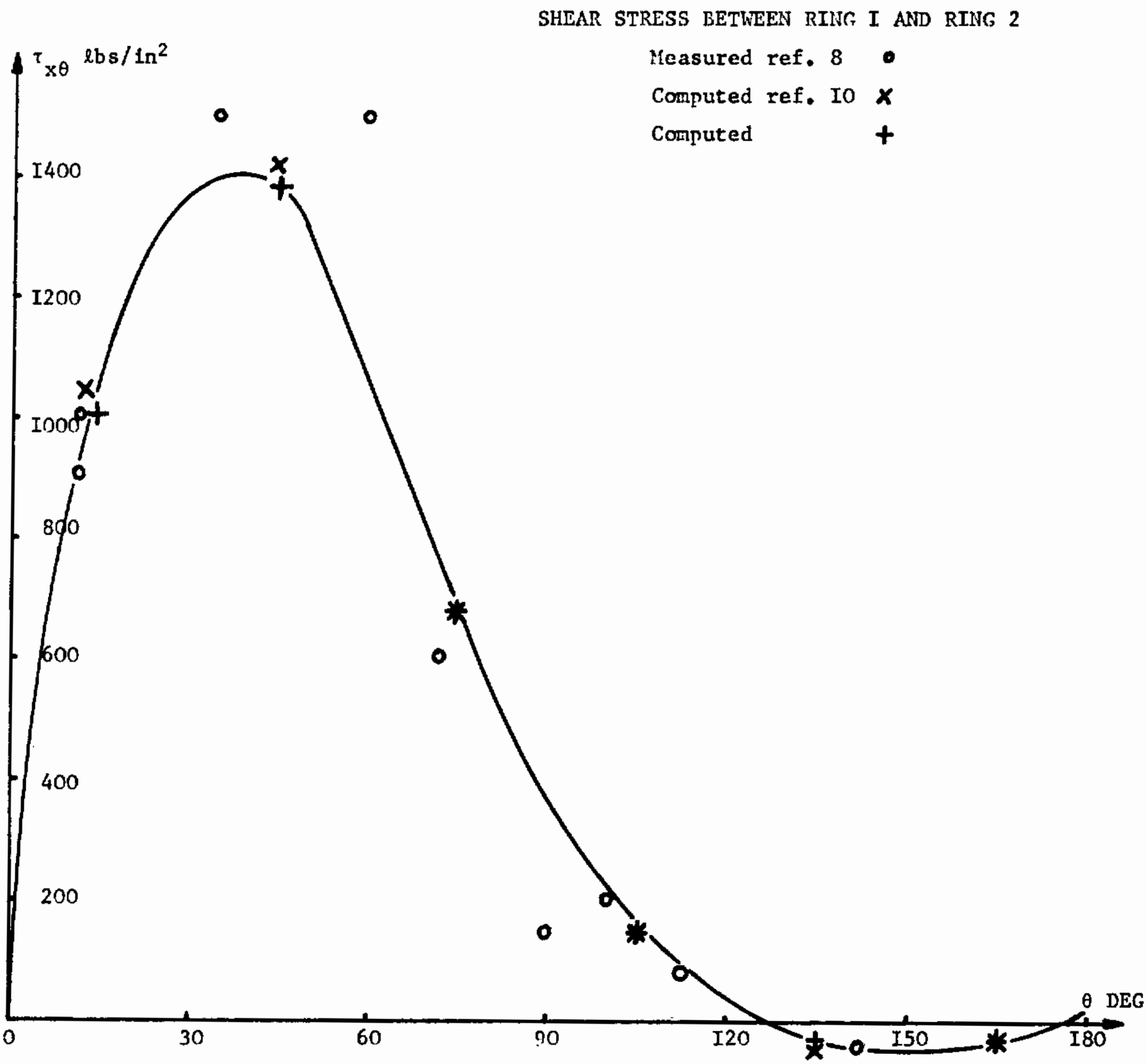


Figure 4.10 - Shear stress between rings I and 2

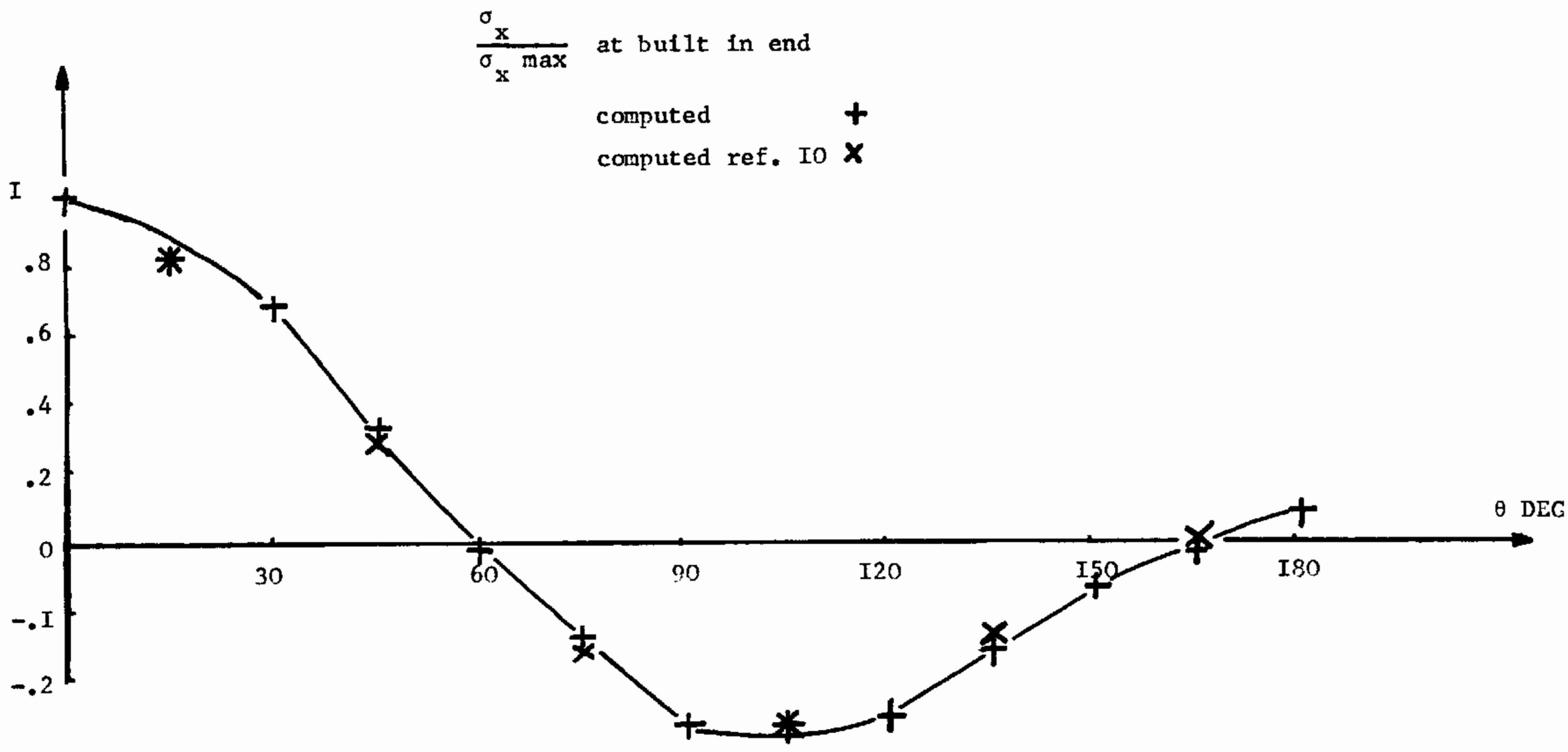


Figure 4.11 - Axial stresses in built-in section

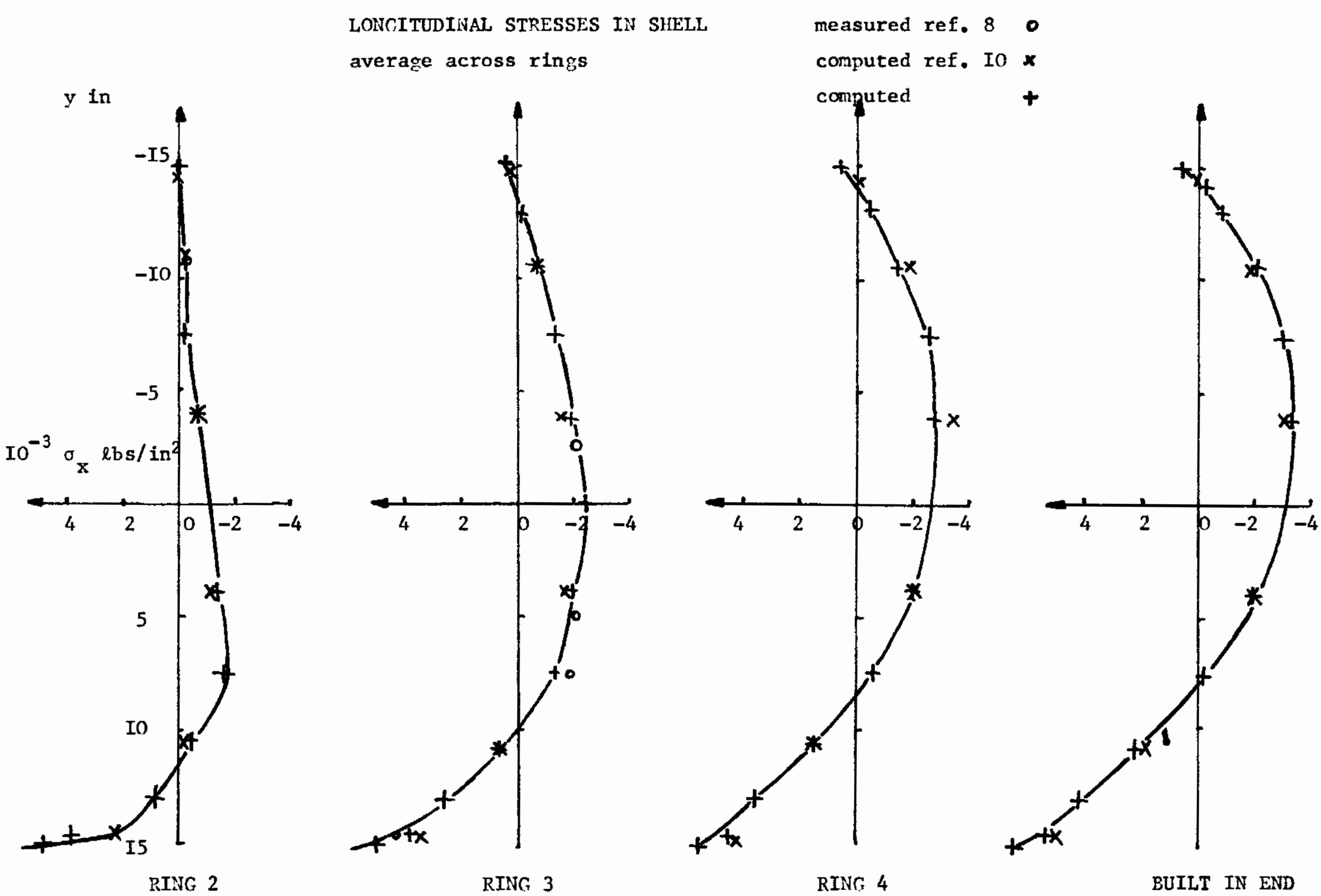


Figure 4.12 - Average axial stresses across rings

Table I Axial stress distribution

DEG												
I80												
	-.3	.08	I.64	2.50	4.I	6.75	6.36	7.4	9.I9	8.I6	8.24	8.28
	.7	.7	I.93	I.3	I.5	2.76	2.43	I.4	I.I9	.96	-.68	-2.35
I50	-.2	-.5	.34	.69	-.5	-.66	-I.36	-4.4	-6.67	-7.79	-II.2	-I4.56
	-.25	-.2	.28	.63	-.65	-.65	-I.36	-2.	-6.44	-7.55	-7.7	-I4.56
	-.0I	-I.2	-2.05	-2.94	-.6	-II.62	-II.63	-I5.3	-22.7I	-22.57	-24.7	-33.46
I20	-.3	-2.8	-4.92	-6.08	-I2.9	-2I.I4	-2I.79	-28.4	-38.86	-39.58	-43.6	-54.35
	-.6	-2.02	-5.22	-6.38	-II.I3	-2I.I7	-2I.82	-25.9	-38.63	-39.35	-40.92	-54.35
	-I.3	-5.9	-I2.2I	-I2.69	-I8.9	-30.39	-30.5I	-33.	-44.09	-43.7I	-43.35	-54.86
90	-.3	-8.	-I7.46	-I9.7I	-27.II	-40.3I	-39.05	-40.	-49.39	-49.I6	-46.87	-56.46
	-I.3	-7.I5	-I7.86	-20.I2	-26.7	-40.37	-39.I	-40.76	-49.55	-49.3I	-49.3	-56.46
	-.9	-I0.33	-24.64	-22.47	-23.	-30.60	-30.I	-26.54	-30.II	-29.79	-26.II	-29.59
60	3.4	-9.6	-27.49	-26.77	-2I.3	-22.79	-20.24	-II.46	-9.8	-9.27	-I.9	-I.7I
	I.57	-I0.62	-27.34	-26.62	-24.85	-23.04	-20.49	-I7.3I	-I0.5	-9.96	-9.2	-I.7I
	I.74	-2.28	-I0.83	-6.07	I.8	9.67	9.8	I4.38	2I.6	22.67	24.8	33.65
30	4.98	9.I4	8.77	I3.79	27.7	4I.69	40.85	46.83	56.44	57.3	60.78	7I.
	3.76	5.85	9.42	I4.44	2I.49	4I.2	40.37	40.42	55.7	56.56	54.28	7I.
	-5.26	I9.75	46.24	43.00	45.07	59.8	59.99	58.93	73.I	72.43	70.02	86.6I
0	-I7.	30.94	80.35	73.34	70.42	80.I7	77.69	75.5I	88.56	89.94	87.4I	I03.9

RING I RING 2 RING 3 RING 4 BUILT IN END

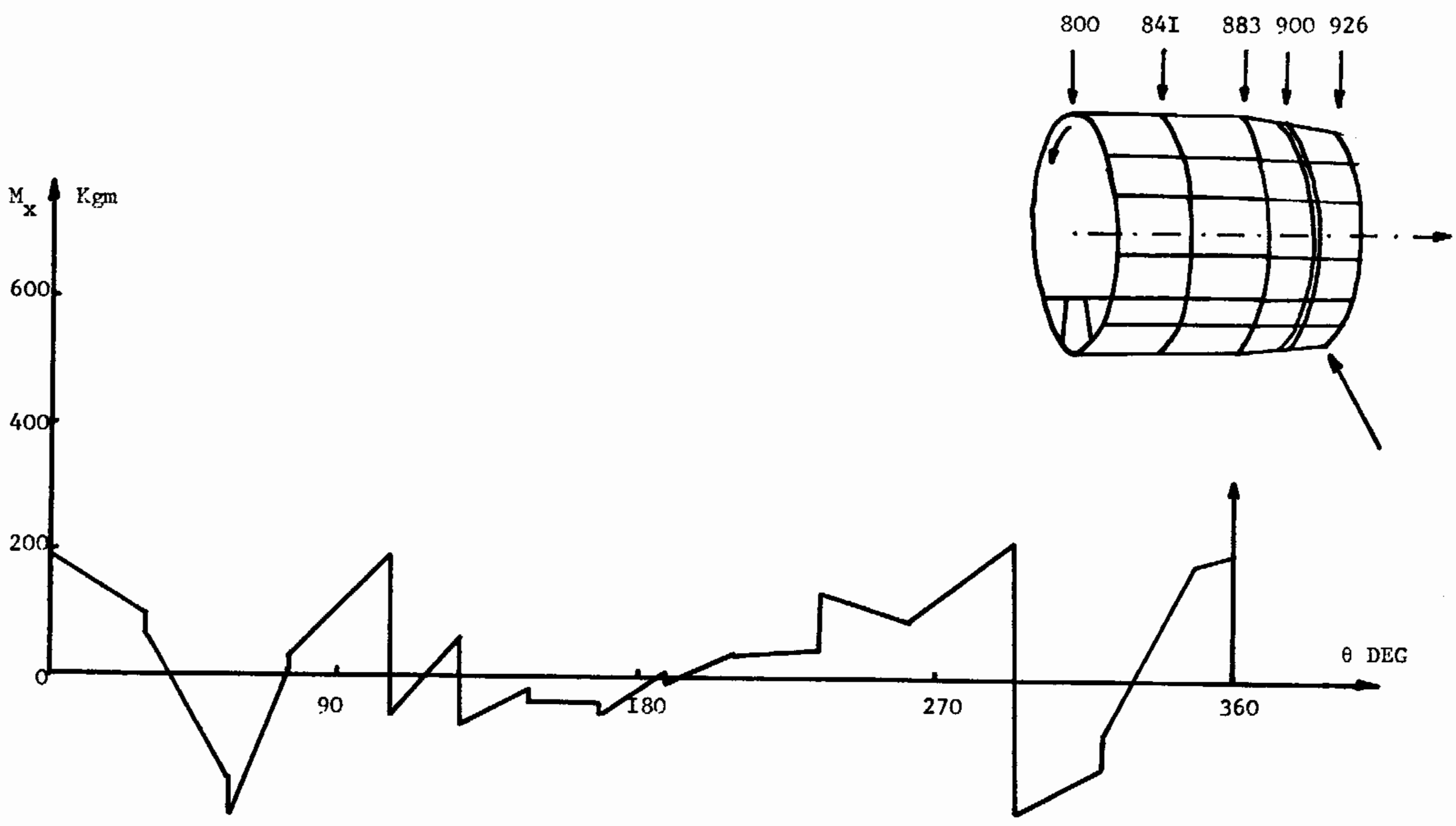


Figure 4.13 - In plane ring bending moment in real fuselage structure

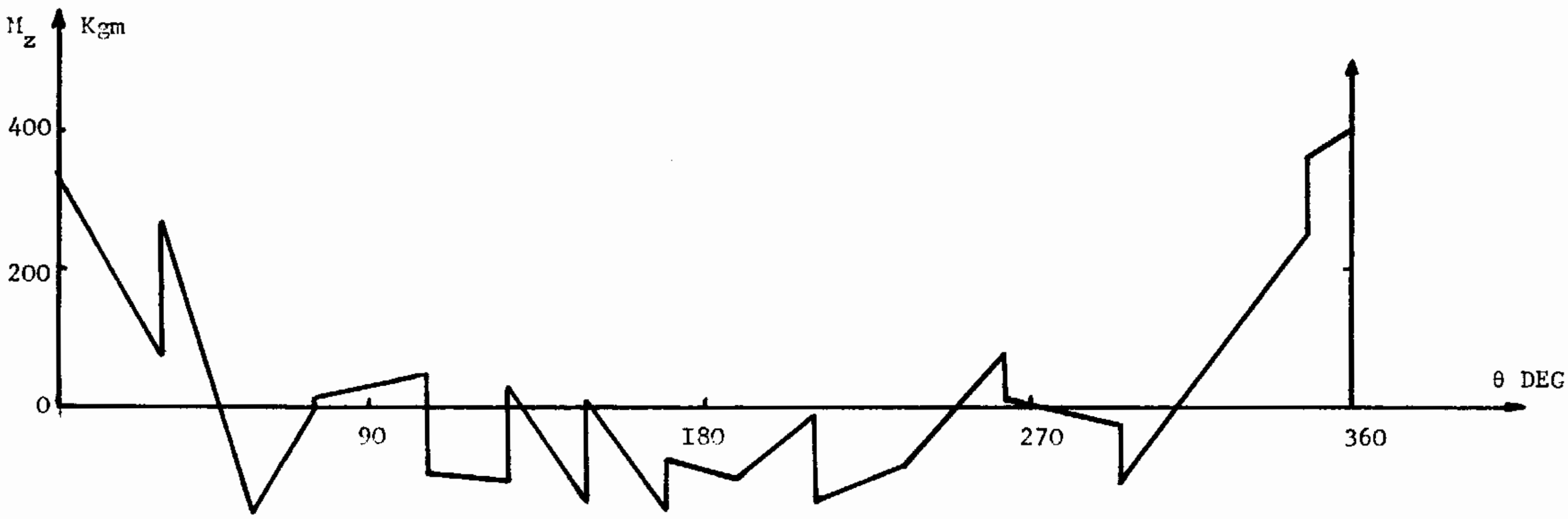


Figure 4.14 - Out of plane ring bending moment in real fuselage structure

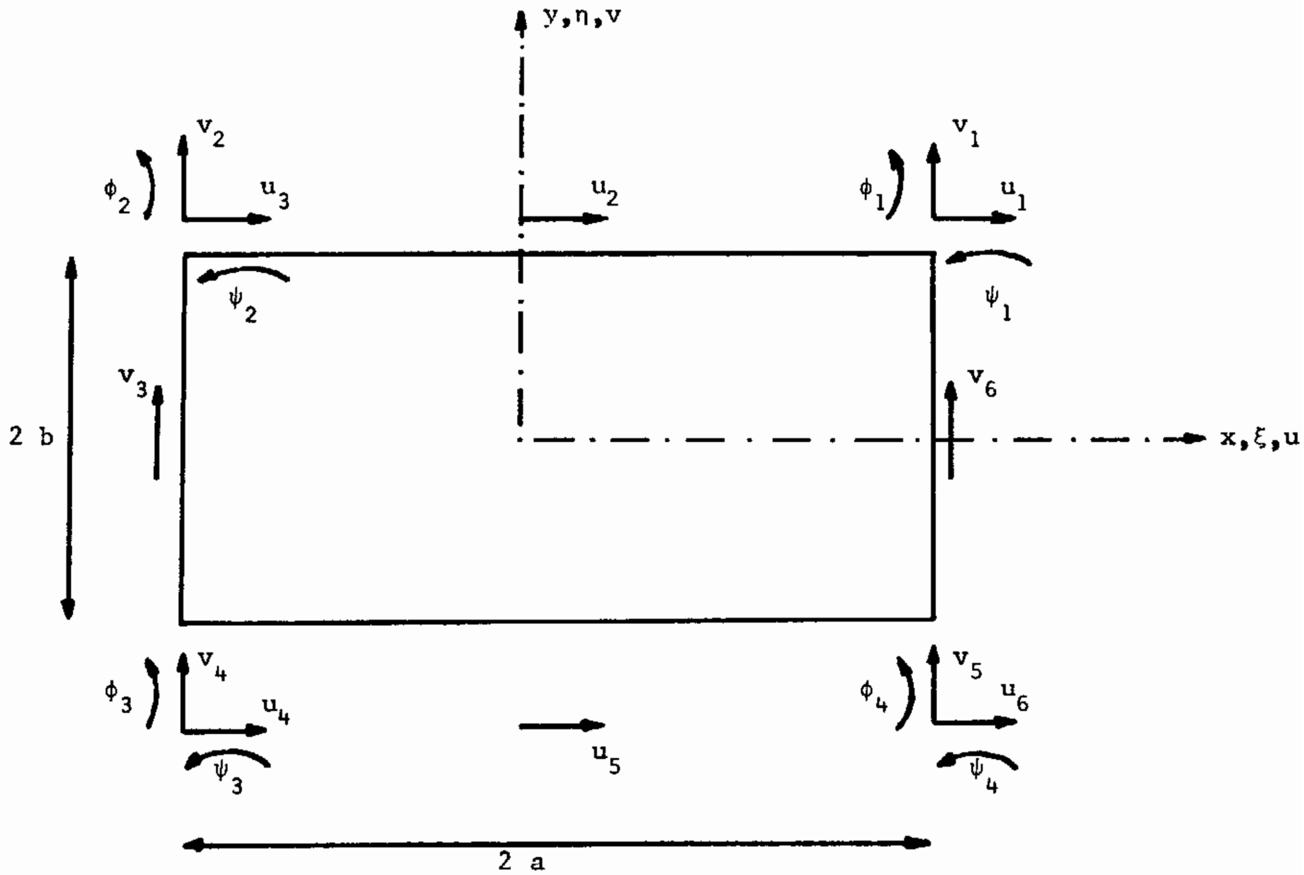


Figure A.1 - Panel geometry and local displacements for model 5

	u_1	u_2	u_3	u_4	u_5	u_6	v_1	v_2	v_3	v_4	v_5	v_6	ψ_1	ψ_2	ψ_3	ψ_4	ϕ_1	ϕ_2	ϕ_3	ϕ_4
A_1			2	2										1	-1					
A_2			3	-3										1	1					
A_3														-1	1					
A_4			-1	1										-1	-1					
A_5	2					2							1			-1				
A_6	3					-3							1			1				
A_7													-1			1				
A_8	-1					1							-1			-1				
A_9	-2	4	-2																	
A_{10}				-2	4	-2														
B_1										2	2								1	-1
B_2										-3	3								-1	-1
B_3																			-1	1
B_4										1	-1								1	1
B_5							2	2									-1	1		
B_6							3	-3									-1	-1		
B_7																	1	-1		
B_8							-1	1									1	1		
B_9							-2				-2	4								
B_{10}								-2	4	-2										

Figure A.2 - Transformation matrix for model 5

$$N = \begin{vmatrix} N_1 & N_2 \end{vmatrix}$$

$$N_1 = \begin{vmatrix} -\frac{1}{a} & -\frac{\eta}{a} & -\frac{\eta^2}{a} & -\frac{\eta^3}{a} & \frac{1}{a} & \frac{\eta}{a} & \frac{\eta^2}{a} & \frac{\eta^3}{a} & -\frac{2\xi(1+\eta)}{a} & -\frac{2\xi(1-\eta)}{a} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1-\xi}{b} & \frac{2\eta(1-\xi)}{b} & \frac{3\eta^2(1-\xi)}{b} & 0 & \frac{1+\xi}{b} & \frac{2\eta(1+\xi)}{b} & \frac{3\eta^2(1+\xi)}{b} & \frac{1-\xi^2}{b} & -\frac{1-\xi^2}{b} \end{vmatrix}$$

$$N_2 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{b} & -\frac{\xi}{b} & -\frac{\xi^2}{b} & -\frac{\xi^3}{b} & \frac{1}{b} & \frac{\xi}{b} & \frac{\xi^2}{b} & \frac{\xi^3}{b} & -\frac{2\eta(1+\xi)}{b} & -\frac{2\eta(1-\xi)}{b} \\ 0 & \frac{1-\eta}{a} & \frac{2\xi(1-\eta)}{a} & \frac{3\xi^2(1-\eta)}{a} & 0 & \frac{1+\eta}{a} & \frac{2\xi(1+\eta)}{a} & \frac{3\xi^2(1+\eta)}{a} & 1-\eta^2 & -(1-\eta^2) \end{vmatrix}$$

Figure A.3 – The N matrix of model 5

$$I = \begin{vmatrix} I_1 & 0 \\ I_3 & I_2 \end{vmatrix}$$

I_1 , I_2 and I_3 are symmetrical

I_2 is derived from I_1 by replacing $A = 4\ ab\left(\frac{Et}{1-\nu^2} + E_1 t_1\right)$ by $C = 4\ ab\left(\frac{Et}{1-\nu^2} + E_2 t_2\right)$

and interchanging a and b $\qquad B = 2\ ab \frac{Et}{1+\nu}$

$$I_1 = \begin{vmatrix}
\frac{A}{a^2} \\
0 & \frac{A}{3a^2}+\frac{4B}{3b^2} \\
\frac{A}{3a^2} & 0 & \frac{A}{5a^2}+\frac{16B}{9b^2} \\
0 & \frac{A}{5a^2}+\frac{4B}{3b^2} & 0 & \frac{A}{7a^2}+\frac{12B}{5b^2} \\
\frac{-A}{a^2} & 0 & \frac{-A}{3a^2} & 0 & \frac{A}{a^2} \\
0 & \frac{-A}{3a^2}+\frac{2B}{3b^2} & 0 & \frac{-A}{5a^2}+\frac{2B}{3b^2} & 0 & \frac{A}{3a^2}+\frac{4B}{3b^2} \\
\frac{-A}{3a^2} & 0 & \frac{-A}{5a^2}+\frac{8B}{9b^2} & 0 & \frac{A}{3a^2} & 0 & \frac{A}{5a^2}+\frac{16B}{9b^2} \\
0 & \frac{-A}{5a^2}+\frac{2B}{3b^2} & 0 & \frac{-A}{7a^2}+\frac{6B}{5b^2} & 0 & \frac{A}{5a^2}+\frac{4B}{3b^2} & 0 & \frac{A}{7a^2}+\frac{12B}{5b^2} \\
0 & \frac{2B}{3b^2} & 0 & \frac{2B}{3b^2} & 0 & \frac{2B}{3b^2} & 0 & \frac{2B}{3b^2} & \frac{16A}{9a^2}+\frac{8B}{15b^2} \\
0 & \frac{-2B}{3b^2} & 0 & \frac{-2B}{3b^2} & 0 & \frac{-2B}{3b^2} & 0 & \frac{-2B}{3b^2} & \frac{8A}{9a^2}-\frac{8B}{15b^2} & \frac{16A}{9a^2}+\frac{8B}{15b^2}
\end{vmatrix}$$

Figure A.4 - The integral matrix of model 5

$$
I_3 = \left|\begin{array}{cccccccccc}
4\nu & & & & & & & & & \\
0 & 2(1-\nu) & & & & & & & & \\
\frac{4}{3}\nu & -\frac{4}{3}(1-\nu) & \frac{4}{9}(2-\nu) & & & & & & & \\
0 & 2(1-\nu) & -\frac{4}{3}(1-\nu) & 2(1-\nu) & & & & & & \\
-4\nu & 0 & -\frac{4}{3}\nu & 0 & 4\nu & & & & & \\
0 & 2(1-\nu) & \frac{4}{3}(1-\nu) & 2(1-\nu) & 0 & 2(1-\nu) & & & & \\
-\frac{4}{3}\nu & -\frac{4}{3}(1-\nu) & -\frac{4}{9}(2-\nu) & -\frac{4}{3}(1-\nu) & \frac{4}{3}\nu & \frac{4}{3}(1-\nu) & \frac{4}{9}(2-\nu) & & & \\
0 & 2(1-\nu) & \frac{4}{3}(1-\nu) & 2(1-\nu) & 0 & 2(1-\nu) & \frac{4}{3}(1-\nu) & 2(1-\nu) & & \\
0 & \frac{4}{3}(1+\nu) & 0 & \frac{4}{5}(1+\nu) & 0 & \frac{4}{3}(1-2\nu) & 0 & \frac{4}{3}(1-2\nu) & \frac{8}{9}(1+\nu) & \\
0 & \frac{4}{3}(3\nu-1) & 0 & \frac{4}{5}(3\nu-1) & 0 & -\frac{4}{3}(1+\nu) & 0 & -\frac{4}{5}(1+\nu) & -\frac{8}{9}(1+\nu) & \frac{8}{9}(1+\nu)
\end{array}\right|
$$

Figure A.4. continued.

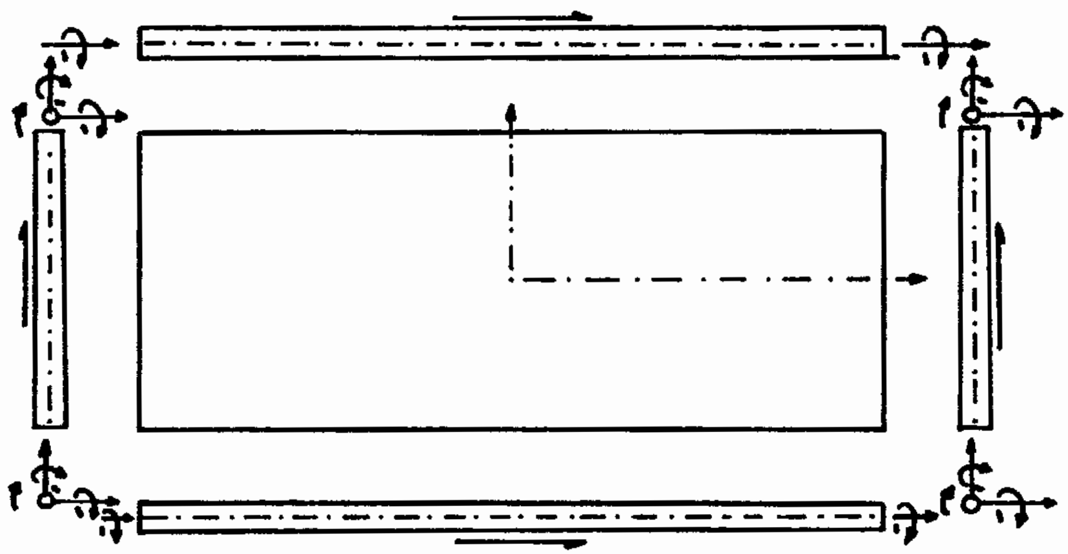


Figure A.5 - Ring-membrane connections

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	Air Force Flight Dynamics Laboratory (FDTR), Wright-Patterson AFB, Ohio 45433

13. ABSTRACT

The report is concerned with the development of membrane elements for cylindrical, fuselage-type structures. The proposed models include the out of plane bending modes of the reinforcing rings, an important feature in some applications to real situations. The performance of model 4 is tested on a circular cantilevered cylinder, loaded by a radial concentrated force on the end ring. Deflections and stresses are compared with previous analyses and test results. The conclusions are that, while good deflections are already obtained with simple models, more elaborate models are required to avoid stress output interpretations. Model 5 is satisfactory from this point of view.

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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