NONLINEAR STRUCTURAL ANALYSIS; TENSOR FORMULATION

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Using the approach of the stiffness method of matrix structural analysis, the mathematical model of a general nonlinear structure is derived in Cartesian tensor notation. Included in the formulation are temperature expansion,pre-stresses,pre-strains,and general anisotropies. The basis of the derivations is the fundamental strain expression of nonlinear theory of elasticity which allows rotations of arbitrary magnitude. Consequently, the method may be used for the analysis of highly flexible structures or even elastic linkages. The structural elements used are topological simplexes, i.e. bars, triangles, and tetrahedrons; their linearized models are compared with previously known results. Due to the discrete point and element representation of numerical methods, some non-tensor properties evolve which require a generalization of tensor algebra. The resulting index notation allows a direct translation into computer program languages and provides several computational shortcuts over matrix notation. The system equations are of the third order in the displacements. Stability criteria are derived and suitable iteration schemes for a static solution are discussed using a simple example problem for illustration.

INTRODUCTION

Nonlinear structural analysis in its modern form is an outgrowth of aircraft, missile, and space applications. Due to the complexity of most problems, numerical methods provide the only practical tool to arrive at meaningful solutions. Nonlinearities in the relation between the applied forces and the displacements may be attributed to two sources: material behavior and geometric changes. The first one, the material nonl nearity, is independent of the magnitude of the displacements but solely dependent on the magnitude of the strains in the structural members and their stress-strain relations. The second one, the geometric nonlinearity, has nothing to do with the magnitude of strains but is only dependent on the geometric change of the structure due to the displacements. In many cases, both phenomena may be observed simultaneously, however, it should be understood that they represent independent mechanisms. This point may be further elucidated by comparing the respective mathematical models.

$$\mathbf{P} = \mathbf{A}^T \mathbf{K} \mathbf{A} \mathbf{U}$$

(in matrix notation) gives the dependency of the forces **P** on the displacements **U** for a linear structure (References 5 and 10). The matrices **A** and **K** are constants. For material nonlinearities, **K** which is a function of the stress-strain relation of the structural members, becomes dependent on the strains (Reference 11) and possibly time (Reference 12). In the case of geometric nonlinearities, **A** as a function of the geometry and topology of the structure, becomes dependent on the displacements (Reference 7).

Physically, every structure possesses at least geometric nonlinearities because even the smallest load produces displacements which necessarily change the initial geometry. In many

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cases, the influence of these changes may be discarded as small of higher order. Generally, this simplification is made from the very beginning in the formulation of a method, however, it may also be made afterwards in a formulation which includes geometric nonlinearities. Therefore, the equations to be developed in this paper may always be simplified for linear applications without loss of accuracy. On the contrary, when linearizing at the end rather than at the beginning, the relative importance of nonlinear terms to be omitted may be evaluated to justify the step.

Material nonlinearities will not be dealt with explicitly. In the following developments it should be understood, though, that the stress-strain law of the structural elements can be, at all times, a function of the magnitude of the strains as well as time. The form of the equations remains completely unaffected and suitable iteration procedures to include this effect can be easily incorporated. Since these are nonlinear procedures, simple superpositioning may not be employed, i.e. geometric and material nonlinearities may not be evaluated separately and the results added.

NOTATION

The linear matrix algebra is at its best in the treatment of linear structural problems. For nonlinear applications, especially for structures with large deflections, matrix algebra becomes an unhandy tool because its symbolism has to be extended beyond its intended realm of application. Although it is always possible to state in an equation that a matrix, say M, is a function of some vector, say v, as M(v), it is impossible within the matrix notation itself to describe this dependency. Mathematically, such dependencies are given as summations over repeated indices (References 8 and 9) or by defining some three-dimensional matrix symbolism (Reference 13); in the computer, some pseudo-matrix operations are easily set up which, for instance, convert a vector into a diagonal matrix (Reference 15). No matter, what the adopted symbolism may be, the matrix notation forces the equations into a linear form which is then provided with some flags to signal the connection with some side equations, not of the matrix form, which account for the nonlinear influences.

For example, the mathematical model of a geometrically nonlinear structure may be written in matrix form as

$$\mathbf{P} = \mathbf{A}^{T}_{(X+U)}\,\mathbf{K}\,\mathbf{A}_{(X+\frac{1}{2}U)}\,\mathbf{U}$$

$\mathbf{A}_{(X+U)}$ indicates that the matrix $\mathbf{A}$ is itself a function of the vector $\mathbf{X}+\mathbf{U}$. Using indices, we may write this function as

$$A_{ij} = \sum_{k} B_{ijk}\,(X+U)_k$$

Obviously, the mathematical model may be given in one equation, if we use indices throughout, viz.

$$P_i = \sum_{j}\sum_{k}\sum_{l}\sum_{m}\sum_{n} B_{jik}\,(X+U)_k\,K_{jl}\,B_{lnm}\,(X+\tfrac{1}{2}U)_m\,U_n$$

This equation is truly nonlinear, i. e. it happens to be of the third order in the displacements $\mathbf{U}$. This notation is as easily adopted to the use on computers as the matrix notation. In fact, modern programming languages such as a revised Fortran IV or Algol will, or do already, provide for more than three subscripts. But even with the use of older "software", multiple indices provide absolutely no difficulties on the computers (References 17 and 19).

Indices are the tool of tensor notation to identify the way in which different tensors are combined. However, not every equation written in indices is necessarily a tensor equation. In order to be a true tensor, an ordered array must obey the transformation laws specified for

tensors (References 2 and 3). In this paper, both proper tensors and "generalized matrices", i.e. non-tensors, will be encountered. To separate these quantities conceptually, non-tensor indices will be written in parentheses (Reference 3). For instance, $a_{ij(e)}$ will be a tensor quantity a_{ij} for every value of the index (e); however, the total array $a_{ij(e)}$ is a "three-dimensional generalized matrix".

To avoid the frequent and repeated use of summation signs, tensor equations are generally written with the Einstein summation convention (Reference 3). Since this convention does not fit for all cases in which nontensor indices occur, a new summation convention is introduced here:

> An operation contains the summation over all possible values of an index, if this index does not appear on both sides of an equation.

Exceptions to this rule will be marked "no sum". This statement contains as a special case the Einstein summation convention.

Besides the fact that tensor notation allows a compact and straightforward representation of a geometrically nonlinear structure, it offers the advantage of much greater differentiation between the components of an array than matrix notation. For instance, a displacement may be doubly subscripted as $u_{i(q)}$ where the first index determines the direction, the second one the point at which it is measured; this property of the indicial notation is especially valuable when the index counting the structural elements is introduced. Thus, the resulting equations are much more compact and involve less computational steps than those built upon the common matrix algebra. For comparison, see the linear stiffness statement of a tetrahedron (Equation 33 for c = 3) and that given in Reference 12; they are exactly equal in content.

The greatest boon in using tensor notation lies in the fact that we can use the nonlinear theory of elasticity as the basis of the formulation rather than any strength-of-materials concepts. This has several consequences:

a. General anisotropic stress-strain and temperature-expansion laws;

b. The formulation may be adapted for plastic and visco-elastic phenomena using strain invariants;

c. Unrestricted magnitude of the translations and rotations.

All equations are written in orthogonal Cartesian coordinate systems. Consequently, only indices written as subscripts will be encountered. The field equations of Part I could have been written in general curvilinear coordinate systems. However, the introduction of finite structural elements in Part II demands that Euclidian element spaces be employed, because otherwise a perceivable curvature may exist in an element, which is not reflected in the equations. Since the elements may be one- two- or three-dimensional in an external space of up to three dimensions, only rectilinear, i.e. straight line, coordinates may be used. The whole formulation could have been given in oblique rectilinear (general Cartesian) coordinates, however, their advantage over orthogonal Cartesian coordinates is so small that for the sake of simplicity in the presentation, orthogonal Cartesian coordinates have been chosen.

PART I: FIELD EQUATIONS

COORDINATE SYSTEMS

The following orthogonal Cartesian coordinate systems will be distinguished:

x_i : (i=1,2. . . c) internal (or element) coordinates before deformation

X_I ; (I=1,2. . . C) external (or structure) coordinates before deformation

Y_I; (I=1,2. . . C) external coordinates after deformation.

To conserve space in writing equations involving tensor indices, we will adopt the convention that tensor indices designated by small letters assume the values 1, 2. . .c, whereas tensor indices represented by capital letters go 1, 2. . .C, and omit these statements from the equations.

The internal coordinates x provide the space in which the c-dimensional, (c=1,2,3), structural elements are described (Simultaneous elements may be of different dimension). The structure, i.e. the assembly of elements, is described in C-dimensional, ($c \leq C$=1,2,3), space X. The displacements U are given in the external coordinates X, such that

$$Y_I = (X + U)_I \tag{1}$$

(The index behind the parentheses indicates that both X and U carry this index.)

STRAINS

The square of a small element of length ds_0 before deformation expressed in internal coordinates is (References 2 and 3).

$$(ds_0)^2 = dx_i \; dx_i$$

The dx are the projections of ds_0 on the axes x. The square of the same arc length after deformation expressed in external coordinates is

$$(ds)^2 = dY_I \; dY_I$$

The coordinates Y are related to the coordinates x through

$$dY_I = Y_{I,i} \; dx_i \tag{2}$$

where $Y_{I,i}$ is the partial derivative of Y_I with respect to x_i. The difference of the squares of the lengths is now

$$(ds)^2 - (ds_0)^2 = (Y_{I,i} Y_{I,j} - \delta_{ij}) \, dx_i \; dx_j$$

The Kronecker delta $\delta_{ij} = 1$ for i = j, and $\delta_{ij} = 0$ for $i \neq j$. The term in parentheses provides a convenient way to express the deformation of the medium relative to the internal coordinates x. By definition of the Lagrangean form of the strain tensor (Reference 2)

$$\epsilon_{ij} = \frac{1}{2} (Y_{I,i} Y_{I,j} - \delta_{ij}) \tag{3}$$

If the external coordinates Y do not describe any deformation, the partial derivatives of Equation 2 become orthogonal transformations such that

$$\epsilon_{ij} = 0$$

With Equation 3 follows the symmetry of the strain tensor

$$\epsilon_{ij} = \epsilon_{ji} \tag{4}$$

STRESSES

A stress tensor σ is related to the strain tensor ϵ through a tensor k of rank four. If we include a temperature-expansion tensor α with a temperature interval θ and some possible prestresses σ_0, a relation of the following form exists

$$\sigma_{ij} = k_{ijkl} (\epsilon_{kl} - \alpha_{kl} \theta) + \sigma_{oij} \qquad (5)$$

All quantities are measured relative to internal coordinates x. Besides Equation 4, these tensors exhibit the following symmetries:

$$\begin{aligned} \sigma_{ij} &= \sigma_{ji} \quad ; \quad \sigma_{oij} = \sigma_{oji} \\ \alpha_{ij} &= \alpha_{ji} \\ k_{ijkl} &= k_{jikl} = k_{ijlk} = k_{klij} \end{aligned} \qquad (6)$$

The third symmetry in k follows from the reciprocal theorem for conservative systems (Reference 1), all others are consequence of this and Equation 4)

The material properties k and α are introduced here as constants; their possible dependency on the strains, time, and temperature may be considered in a stepwise adjustment which is tied to the iteration process of solution.

EQUILIBRIUM

The system is subjected to the external force field P, measured relative to the coordinates X. When these forces act through the virtual displacements δU, the increment of the external work is

$$\delta W = P_I \quad \delta U_I \qquad (7)$$

The energy increment transmitted into the system by a synchronous deformation can be written as (Reference 10)

$$\delta w = \int_V \sigma_{ij} \quad \delta \epsilon_{ij} \quad dV \qquad (8)$$

The integral extends over the total volume V of the system. The expression, (Equation 8) is usually considered to be an approximation; however, it is exact if it is used as the defining equation of the stresses (see "Physical Interpretation").

The strain increment $\delta\epsilon$ can be expressed as a function of the virtual displacements δU of Equation 7. Observing that δU is independent of location

$$\delta w = \delta U_I \int_V \sigma_{ij} \quad \frac{\partial \epsilon_{ij}}{\partial U_I} \quad dV \qquad (9)$$

If the force field P includes all external forces, reactions, and body forces and if the strains express all deformations including those of the constraints, the increments δW and δw must be equal, i.e. represent an external and internal energy which is invariant under transformation. With Equations 7 and 9 follows

$$P_I = \int_V \sigma_{ij} \quad \frac{\partial \epsilon_{ij}}{\partial U_I} \quad dV \qquad (10)$$

where the virtual displacements have been omitted from both sides of the equation as arbitrary.

The structural system is now fully described by the field Equations 1,3,5, and 10.

PHYSICAL INTERPRETATION

In engineering mechanics, the strains are defined as linear changes of lengths and angles. If e_i defines a direct engineering strain in x_i - direction and if γ_{ij}, $(i \neq j)$, stands for an engineering shear strain between the directions x_i and x_j, the following relations hold for strains which are small in comparison with unity

$$\epsilon_{ii} = e_i$$

$$\epsilon_{ij} = \frac{1}{2}\gamma_{ij} \quad ; \quad (i \neq j) \tag{11}$$

Allowing 1 percent strain error, (Equation 11) is valid (Reference 14) up to $e_i = 14$ percent and $\gamma_{ij} = 7$ percent. This means that for most applications (metals, etc.) the engineering definition of strains may be employed. Using Equation 11 does not limit the displacements themselves but only their derivatives. Therefore, even within the engineering assumptions concerning the deformation, the tensor representation allows for unlimited displacements and rotations. This means, that highly flexible structures or even elastic linkages may be analyzed.

The tensor k of Equation 5 describes Hooke's law for a general anisotropic material. Owing to the symmetries given in Equation 6, it possesses 21 independent coefficients for a three-dimensional body, (c=3), six for c = 2, and one for c = 1. For complete isotropy, the number of independent parameters is reduced to two (Reference 4), (c>1):

$$k_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \tag{12}$$

where λ and μ are the Lame constants (Reference 1). The non-zero coefficients of k in Equation 12 are, using E, Young's modulus, and ν, Poisson's ratio:

		c = 1	c = 2	c = 3
k_{iiii}	=	E	$\frac{E}{1-\nu^2}$	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$
k_{iijj}	=	–	$\frac{E}{1-\nu^2}$	$\frac{E}{(1+\nu)(1-2\nu)}$
$k_{ijij} = k_{ijji}$	=	–	$\frac{E}{2(1+\nu)}$	$\frac{E}{2(1+\nu)}$

$(i \neq j$; no sum$)$

The temperature-expansion tensor α of Equation 5 describes a general anisotropic material. It possesses six, three, one independent coefficients for c = 3,2,1 due to the symmetry given in Equation 6. For complete isotropy it has the form (Reference 4)

$$\alpha_{ij} = \alpha \delta_{ij} \tag{13}$$

with the single coefficient α.

Equation 8 may be used for the definition of the stresses, it is then valid for strains and stresses of arbitrary magnitude and the material tensor k will contain higher order terms in the displacement derivatives. However, within the regime of small strains assumed in

Equation 11, the stresses may be defined as surface tractions on undeformed surfaces in order to fulfill Equation 5. The connection with the engineering stresses is then

$$\begin{aligned} \sigma_{ii} &= \sigma_i \\ \sigma_{ij} &= \tau_{ij}\,, \quad (i \neq j) \end{aligned} \tag{14}$$

where σ_i are the longitudinal stresses parallel to x_i and τ_{ij} the shear stresses between x_i and x_j. Within the range of validity of Equations 11 and 14, we may also introduce V of Equation 10 as the undeformed volume of the structure.

PART II: DISCRETE POINT ELEMENT PRESENTATIONS

GRID

Except for very simple applications, it is impossible or impractical to employ the continuous field equations directly to a problem. It is possible, however, to avoid often unsurmountable mathematical difficulties by replacing the continuous structure by a finite gridwork of simple structures (elements) which all by themselves obey these field equations. This step replaces quality with quantity, i.e. the quality of higher mathematical functions (which may not even have been invented yet) with the sheer quantity of numerical data. Modern computing facilities offer the possibility to handle these data in a reasonably short time and at a moderate cost, although it happens every day that even the biggest present day computers are still too small and slow for the more advanced applications, especially in the field of dynamics. Economy becomes a major consideration in the development of a computational method and has been used as one of the principal guidelines in the following breakdown of the continuous representation into a discrete, finite system of simultaneous equations.

An arbitrary and possibly completely irregular grid of points is laid over the entire structure, its inside and its boundaries. It is impossible for the general case, to give any criterion concerning the fineness of the grid in relation to the desired numerical accuracy and to certain geometric parameters of the structure. Generally, the greater the changes of geometric and deformation characteristics become, the finer the grid has to be.

The number of grid (node) points is defined as N. The structure now possesses CN degrees of freedom instead of infinitely many for the continuous structure. Imbedded between the node points are n volume elements of finite magnitude, each one of which may be considered to be a simple structure for which the field equations of Part I apply. They represent the influence of the local material properties. All quantities measured on the (external) grid points are referred to the undeformed external coordinates X, whereas the quantities measured in the structural elements are referred to internal coordinates x_e , (e = 1.2. . .n), where each element possesses its own set.

SIMPLEX ELEMENTS

Assuming that we are given the deformed external coordinates Y and the undeformed internal coordinates x, we need the partial derivatives of Y with respect to x for use in Equation 3. Given a node point (0) and several others (q) = 1.2. . . in its vicinity, we can approximate the differentials of the coordinates by

$$\begin{aligned} dY_I &= Y_{I(q)} - Y_{I(o)} \\ dx_i &= x_{i(q)} - x_{i(o)} \end{aligned}$$

This goes into Equation 2

$$Y_{I(q)} = R_I + Y_{I,i}\, x_{i(q)} \tag{15}$$

where

$$R_I = Y_{I(o)} - Y_{I,i}\, x_{i(o)}$$

is an unknown vector. Equation 15 can be solved for $Y_{I,i}$ if (q) = 1,2. . .c+1 grid points are employed. With the substitution

$$\begin{aligned} d_{i(q)} &= x_{i(q)} \quad ; \quad (q = 1, 2, \cdots c+1) \\ d_{c+1(q)} &= 1 \end{aligned} \tag{16}$$

and the definition of a matrix "a" as the inverse transpose of the matrix "d" i.e. if "a" obeys the equation

$$d_{r(q)}\, a_{s(q)} = \delta_{rs} \quad ; \quad (r, s, q = 1, 2, .. c+1) \tag{17}$$

we get from Equation 15

$$Y_{I,i} = a_{i(q)}\, Y_{I(q)} \quad ; \quad (q = 1, 2, \ldots c+1) \tag{18}$$

If we postulate that a structural element possesses only one average strain tensor, the condition for the solution of Equation 15 leads to the conclusion that a c-dimensional structural element is an ordered c-dimensional topological simplex (Reference 23), i.e. a c-dimensional polyhedron with c+1 vertices (node points). Consequently, in an external space of C=3 dimensions, we may encounter structural elements of c=1,2,3 dimensions, i.e. bars, triangular plates, and tetrahedrons. These structural elements will be called simplex elements.

Equation 18 is not yet in the desired form for the application, because the coordinates Y are measured on the N points of the assembled structure rather than the c+1 points of an element. Introducing the index (e)=1,2. . .n to separate the structural elements, we can write

$$\begin{aligned} Y_{I(qe)} &= t_{(Qqe)}\, Y_{I(Q)} \\ &(q = 1, 2, \ldots c_{(e)}+1; \quad e = 1, 2, \ldots n; \quad Q = 1, 2, \ldots N) \end{aligned} \tag{19}$$

where "t" is a homomorphic mapping function (coincidence matrix) which maps the external point set into the individual element point sets. Since it establishes identities, its derivation is

$t_{(Qqe)}$ = 1 if node point Q of the assembled structure conincides with node point q of element e;

= 0 in all other cases.

The coincidence matrix t possesses very many zero elements. In large scale computer applications, it is advantageous to replace the multiplicative operation Equation 19 with a simple "bookkeeping" procedure.

The integral Equation 10 extends over the entire volume of the structure; it may be broken down into a sum of integrals, each one over the volume of a simplex element. Since stresses and strains are constants within a simplex element, the integration yields the element volumes themselves. According to the observation made under "Physical Interpretation" we may assume these to be the initial volumes. We find (Reference 3)

$$v_{(e)} = \left(\frac{b}{c!} |d| \right)_{(e)} \tag{20}$$

where

$|d|_{(e)}$ is the absolute value of the determinant of the matrix d of element e

$b_{(e)} = 1$ for $c_{(e)} = 3$

= element thickness for $c_{(e)} = 2$

= element cross section for $c_{(e)} = 1$

The volume $v_{(e)}$ will be zero if d = 0; i.e. if the matrix d is singular. This is true for degenerated elements, for instance, if the four node points of a three-dimensional element fall into a plane.

For the computation of the matrices "d" and "a" of Equations 16 and 17 the knowledge of the internal coordinates $x_{i(q)}$ is essential. Sometimes, another way of computation is more practical.

Assuming that the partial derivatives $X_{I,i}$ (direction cosines) of the external axes X_I with respect to the internal axes x_i are given, and also the external coordinates $X_{I(q)}$ which may be computed through Equation 19 a statement analog to Equation 15 exists:

$$x_{i(q)} = r_i + X_{I,i} X_{I(q)}$$

The vector r is unknown, however, when evaluating the matrix "a" and the determinant of "d", only the differences of $x_{I(q)}$ survive. Therefore, we may give r any desired value, say zero, and compute $x_{i(q)}$ from

$$x_{i(q)} = X_{I,i} X_{I(q)} \tag{21}$$

These values of $x_{i(q)}$ are used as before in Equation 16.

COLLECTED EQUATIONS

The field equations developed in part I are adapted to a structure composed of simplex elements: Constant date (Equations 16, 17 and 20):

$$\begin{aligned}
&d_{i(qe)} = x_{i(qe)} \quad ; \quad d_{c+1(qe)} = 1 \\
&d_{r(qe)} a_{s(qe)} = \delta_{rs(e)} \\
&v_{(e)} = \left(\frac{b}{c!} |d| \right)_{(e)} \\
&A_{i(Qe)} = a_{i(qe)} t_{(Qqe)} \\
&\qquad (i = 1, 2, .. c_{(e)} ; q, r, s = 1, 2, .. c_{(e)} + 1 ; \\
&\qquad e = 1, 2, .. n ; Q = 1, 2, .. N)
\end{aligned} \tag{22}$$

If the internal coordinates $x_{i(qe)}$ have to be computed from the external coordinates and the orientation of the element (Equation 21):

$$x_{i(qe)} = X_{I,i(e)}\ t_{(Qqe)}\ X_{I(Q)} \tag{23}$$

Variable data (Equations 1, 18, 19, 3, 5 and 10):

$$\begin{aligned}
Y_I &= (X + U)_I \\
Y_{I,i(e)} &= A_{i(Qe)}\ Y_{I(Q)} \\
\epsilon_{ij(e)} &= \tfrac{1}{2}(Y_{I,i}\ Y_{I,j} - \delta_{ij})_{(e)} \\
\sigma_{ij(e)} &= \left[k_{ijkl}\ (\epsilon_{kl} - \alpha_{kl}\ \theta) + \sigma_{oij}\right]_{(e)} \\
P_{I(Q)} &= (A_{i(Q)}\ Y_{I,j}\ \sigma_{ij}\ v)_{(e)} \\
&\qquad (i,j,k,l = 1,2,..c_{(e)}\ ;\ I = 1,2,..C;\ e = 1,2,..n;\ Q = 1,2,..N)
\end{aligned} \tag{24}$$

The internal forces $P_{I(Qe)}$ may be obtained from the last Equation(24) before the summation over the index (e) is performed. The last three equations of Equation 24 when combined into one yield:

$$\begin{aligned}
P_{I(Q)} = \Big\{ & v A_{i(Q)}\ Y_{I,j} \\
& \cdot \left[k_{ijkl}\ \left(\tfrac{1}{2} Y_{J,l}\ Y_{J,k} - \tfrac{1}{2}\delta_{kl} - \alpha_{kl}\ \theta\right)\ \sigma_{oij}\ \right]\Big\}_{(e)}
\end{aligned} \tag{25}$$

SEPARATION OF INITIAL AND FINAL STRAINS

So far, the undeformed coordinates X and the displacements U have been added in Equation 1 respectively in Equation 24. For the numerical accuracy, it is better to separate the possibly large coordinates X, from the displacements U which may be relatively small in the vicinity of the unloaded configuration. At the same time we will assume that the initial coordinates describe some intentional or unintentional deformation and the displacements are measured starting from the initially deformed state. Using an asterisk to signify these modified definitions, we get instead of the first three Equations of 24

$$\begin{aligned}
X^*_{I,i(e)} &= A_{i(Qe)}\ X^*_{I(Q)} \\
U^*_{I,i(e)} &= A_{i(Qe)}\ U^*_{I(Q)} \\
\epsilon_{oij(e)} &= \tfrac{1}{2}(X^*_{I,i}\ X^*_{I,j} - \delta_{ij})_{(e)} \\
\epsilon^*_{ij(e)} &= \tfrac{1}{2}(X^*_{I,i}\ U^*_{I,j} + U^*_{I,i}\ X^*_{I,j} + U^*_{I,i}\ U^*_{I,j})_{(e)} \\
\epsilon_{ij(e)} &= (\epsilon_o + \epsilon^*)_{ij(e)}
\end{aligned} \tag{26}$$

If all elements fit without deformation into the grid provided by the coordinates X*, the initial strains ϵ_0 will be zero; this fact may be used as a check for the compatibility of the input data. The strains ϵ^* are those measured above the initial strains; the sum of ϵ_0 and ϵ^* gives the total strains ϵ. Stresses and forces will be computed in the last two Equations of 24, where

$$Y_{I,i(e)} = (X^* + U^*)_{I,i(e)} \tag{27}$$

The combined equations analog to Equation 25 are

$$P_{I(Q)} = \left\{ vA_{i(Q)} Y_{I,j(e)} \cdot \left[k_{ijkl} \left((X^* + \tfrac{1}{2} U^*)_{j,l} U^*_{J,k} + \epsilon_{okl} - \alpha_{kl} \theta \right) + \sigma_{oij} \right] \right\}_{(e)} \tag{28}$$

LINEARIZATION

The Equations 25 or 26 are developed into a Taylor series with respect to the independent variables and truncated after the linear terms. The increments Δ of the variables at any given load configuration are related as:

$$\Delta P_{I(Q)} = S_{I(Q)J(R)} \Delta U_{J(R)} + \Delta P(\theta, \sigma_o)_{I(Q)} \qquad (I, J = 1, 2, \ldots C;\ Q, R = 1, 2, \ldots N) \tag{29}$$

where

$$S_{I(Q)J(R)} = \delta_{ij} \left[vA_{i(Q)} A_{j(R)} \sigma_{ij} \right]_{(e)} + \left[vA_{i(Q)} Y_{I,j} k'_{ijkl} Y_{J,k} A_{l(R)} \right]_{(e)}$$

$$\Delta P(\theta, \sigma_o)_{I(Q)} = \left[vA_{i(Q)} Y_{I,j} \left(\Delta\sigma_{oij} - k'_{ijkl} \alpha'_{kl} \Delta\theta \right) \right]_{(e)} \tag{30}$$

In these equations the modified material laws k' and α' have been introduced. They are defined similarly to a "tangent" modulus in the deformed state, whereas k and α assume the role of "secant" moduli between the origin and the deformed configuration. For linear material behavior, k' = k and $\alpha' = \alpha$.

When linearizing the equations for the undeformed configuration, the first term of the stiffness matrix **S** vanishes because $\sigma = 0$; for the remainder follows with U = 0 that $Y_{I,i(e)} = X_{I,i(e)}$.

For the nonlinear force-displacement relation, a Maxwell reciprocity does not exist: take two arbitrary directions X_I and X_J and two arbitrary points Q and R, then Equation 25 or 28 shows that the symmetry condition

$$P_{I(Q)}(\delta U_{J(R)}) = P_{J(R)}(\delta U_{I(Q)})$$

for non-small displacement increments $\delta U_{J(R)} = \delta U_{I(Q)}$ is not fulfilled. It holds, however, for small variations; therefore, the stiffness matrix S of Equation 30 is symmetric:

$$S_{I(Q)J(R)} = S_{J(R)I(Q)} \tag{31}$$

STIFFNESS MATRIX OF A SIMPLEX ELEMENT

For comparison with known results, the stiffness matrix of an undeformed simplex element is derived. For this purpose, let the entire structure consist of one element only and let the numbering of internal and external node points coincide. With the first equation of 30 follows for the undeformed state:

$$s_{I(q)J(r)} = {}^{v}a_{i(q)}\ X_{I,j}\ k_{ijkl}\ X_{J,k}\ a_{l(r)} \qquad (32)$$

$$(i,j,k,l = 1,2,..c;\ I,J = 1,2,..C;\ q,r = 1,2,..c+1)$$

This is the stiffness matrix of a simplex element in arbitrary orientation. Rotation of the external frame such that X_I coincides with X_i for $I \leq c$ yields

$$s_{j(q)k(r)} = {}^{v}a_{i(q)}\ k_{ijkl}\ a_{l(r)} \qquad (33)$$

This element stiffness matrix s is identical with previously obtained formulations: for $c = 1$ it describes a bar undergoing longitudinal deflections; for $c = 2$ it agrees with the formulation given in Reference 6 for a triangular plate element in general orientation; for $c = 3$ it is identical with the stiffness matrix of a tetrahedron of Reference 12.

PART III: SOLUTION

BOUNDARY CONDITIONS

The equations may be solved numerically for any system of unknowns provided that the necessary and sufficient input is available. To limit the discussion of the "load" conditions, we will assume that the entire initial geometry, the material behavior, the temperatures, and the pre-stresses are known. The external coordinates which are used to measure the forces and displacements are partitioned into the following classes:

(1) known forces

(2) known displacements.

Each one of these classes may be partitioned again for zero and non-zero components to reduce the amount of computations; however, this is avoided here so as not to confuse the issue. This second partitioning may be introduced any time into the final equations, if desired.

The classification of coordinates is of importance in Equation 29 because, with sufficient coordinates in class (2) to provide the constraints, it may be solved directly for the increments of the unknown displacements:

$$\Delta U^{(1)}_{K(S)} = F^{(11)}_{K(S)I(Q)} \cdot \left[\Delta P^{(1)}_{I(Q)} - s^{(12)}_{I(Q)J(R)}\ \Delta U^{(2)}_{J(R)} - \Delta P(\theta, \sigma_o)^{(1)}_{I(Q)}\right] \qquad (34)$$

The superscripts indicate the boundary classes; e.g. $\Delta U^{(1)}_{K(S)}$ describes the displacement increments in those directions X_K at those points S for which the forces are given. Obviously, within each partitioned array, the indices do not run through all those values which they were

allowed to assume before partitioning. The array $F^{(II)}$ is the flexibility matrix in the coordinates of the known forces; it is obtained by inversion of the corresponding stiffness matrix:

$$F^{(II)}_{K(S)I(Q)} \; S^{(II)}_{I(Q)J(R)} = \delta^{(II)}_{K(S)J(R)} \qquad (35)$$

STABILITY

A nonlinear structure may possess more than one equilibrium configuration for a given load system. Two consecutive equilibrium positions must be separated by an unstable region in which no additional forces have to be applied to produce finite deformations. By definition, this applies only to coordinates in which the forces, not the displacements, are controlled. An increment of work done on the entire structure by the action of an increment of the controlled forces is

$$\delta W = \left(P + \delta P\right)^{(I)}_{I(Q)} \; \delta U^{(I)}_{I(Q)}$$

As long as the structure is stable under the action of the forces P, the external work must represent a minimum, therefore, the stability criterion may be written as

$$\delta P^{(I)}_{I(Q)} \; \delta U^{(I)}_{I(Q)} > 0$$

valid for the immediate neighborhood of the equilibrium configuration, i.e. for small increments. For this purpose we may linearize the force-displacement relation at a given point and use Equation 34 to express $\delta U^{(I)}$ as a function of $\delta P^{(I)}$.

$$F^{(II)}_{I(Q)J(R)} \; \delta P^{(I)}_{I(Q)} \; \delta P^{(I)}_{J(R)} > 0 \qquad (36)$$

This must be valid for any arbitrary force increments; therefore, Equation 36 is a condition on $\mathbf{F}^{(II)}$: the structure is stable if the flexibility matrix $\mathbf{F}^{(II)}$, which corresponds with a given deformed state, is positive definite (Reference 2).

A necessary, but not a sufficient condition for Equation 36 to be true is the test of the diagonal elements of $\mathbf{F}^{(II)}$:

$$F^{(II)}_{I(Q)I(Q)} > 0 \; ; \; (\text{no sum}) \qquad (37)$$

which will suffice for most practical applications. A complete test is provided by the Sylvester determinant (References 24 and 25), i.e. the corner minors of the matrix must all be positive. The orthogonal instability modes are the eigenvectors associated with the negative eigenvalues of $\mathbf{F}^{(II)}$.

Instead of subjecting the flexibility matrix $\mathbf{F}^{(II)}$ to the stability test, the stiffness matrix $\mathbf{S}^{(II)}$ may be chosen. However, the incomplete test analogous to Equation 37 is of little value for global instabilities.

ITERATION PROCESSES FOR STATIC SOLUTIONS

The nonlinear equations are of the third order in the displacements as can be verified in Equations 25 and 28. A practical way of a direct analytical solution does not exist for simultaneous systems of any significant order. Therefore, trial and error methods have to be

employed which, when combined with some rational error reduction scheme, are called iteration processes. There are at least two different possible kinds of iteration processes:

direct iteration

progressive iteration

A direct iteration process aims immediately at the solution for the full values of the independent variables, whereas a progressive process reaches the final solution via approximate intermediate solutions which are more or less evenly distributed between the initial and final configurations. Besides giving valuable information about the behavior at intermediate load levels, the progressive processes may provide the only means of solution for cases in which instabilities occur. An unstable structure processes more than one equilibrium configuration for a given load level, and to decide which one represents the correct solution of the problem requires the knowledge and the use of the time history of the independent variables. This is most conveniently simulated in a progressive iteration process.

As a simple example for the explanation of the various iteration processes, a plane two-bar structure (Figure 1) will be employed. The structure is loaded by a single force P; temperature and pre-stresses are omitted. Equation 28 describes the behavior of this structure as

$$P = \frac{2bE}{\ell^3}(U-h)(\frac{1}{2}U-h)U \tag{38}$$

where b and E are the cross section and Young's modulus of the two bars. The load deflection curve has the two extremes

$$P_{max} = \frac{2bEh^3}{3\ell^3\sqrt{3}} \quad \text{for} \quad U = h(1-\frac{1}{\sqrt{3}})$$

$$P_{min} = -\frac{2bEh^3}{3\ell^3\sqrt{3}} \quad \text{for} \quad U = h(1+\frac{1}{\sqrt{3}})$$

Between these values of the displacement U, the structure is instable. The stiffness matrix (in this case a single coefficient) follows from Equation 2.16 as

$$S = \frac{2bE}{\ell^3}\left[3U(\frac{1}{2}U-h)+h^2\right] \tag{39}$$

For the computations, the numerical values

$$\frac{2bE}{\ell^3} = 1\,; \quad h = 1$$

will be used.

The simplest progressive iteration process (Reference 18) is obtained by using the linearized Equation 34 with the stiffness matrix computed from Equation 30. In this process, the tangent (Equation 30) to the nonlinear load-deflection curve is used in Equations 35 and 34 to find a displacement increment for an assumed force increment. The displacement increment is added to the previous displacement and a new tangent (Equation 30) is computed.

In Figure 2, this iteration is carried out in six steps ($\tau = 2$ to 7). As long as the nonlinearity is not too pronounced and step size is sufficiently small, the results stay reasonably close to the theoretical values. In regions of stronger nonlinearity, the cumulative error builds up to unacceptable levels and in regions of instability, the process fails completely (step $\tau = 7$ in Figure 2).

An improved progressive iteration process (Reference 18) is obtained when after each forward step on the tangent to the load-deflection curve, a back-substitution into the nonlinear equations is added which restores the accuracy. This implies the use of Equations 30, 35, and 34 as before, however, when the new displacements have been found, they are put into the set of Equations 24 respectively Equations 26 (or Equations 25 respectively Equation 23) to compute the true forces which are needed to sustain the deflected shape of the structure. The force increments for the next iteration step are then computed on the basis of the actual sustained forces.

In Figure 3, this iteration is carried out in six steps. In the first four steps the path follows the theoretical curve very closely. In the fifth step it overshoots the point of the beginning instability so that the sixth step moves in the wrong direction with a positive load increment. Although all points of back-substitution represent true solutions, the process is inherently incapable of passing beyond the zone of instability in a rational manner.

In a process that is laid out to pass through an unstable region (Reference 18), several additional provisions have to be made. The first and most important one is that before the load increments are applied to the tangent equation, a check is made (for most applications Equation 37 should be sufficient) if the structure is stable. If it is not, arbitrary load increments are selected with the signs reversed from those prior to the instability and applied to the unstable coordinates (those coordinates for which test (Equation 37) is not fulfilled). This avoids the type of failure encountered in the two previous iteration processes. In addition to this, there has to be some device to limit the magnitude of the increments of the variables. This is necessary because with nearly horizontal tangents, i.e. nearly singular matrices, exceedingly large displacement increments may be encountered for moderate force increments. Also, large force increments may occur when the force level has been temporarily decreased in or after an instability.

In Figure 4, this iteration process is carried through the entire unstable region of the example problem. The bounds on the displacement increments were selected as 0.3. If this value was exceeded (steps after $\tau = 4, 5, 8, 9$) the magnitude of the forward step was reduced to this value and the next step would not carry a higher intended load. Also bounded were the differences between intended load level and the load level found in the back-substitution. If this value exceeded 0.02 (steps after $\tau = 4, 9, 10, 11,$), the intended load level for the next step was not increased.

This relatively complicated iteration scheme, of which only the most important aspects are given here, will follow the behavior of a structure through an unstable region in completely automatic fashion. However, the selection of the force increments and bounds on the force deviation and displacement increments requires some insight into the behavior of the structure prior to the computer iteration.

REFERENCES

1. Love, A.E.H., A Treatise of the Mathematical Theory of Elasticity, Dover, New York, 1944.
2. Sokolnikoff, I.S., Tensor Analysis, John Wiley and Sons, New York, 1951.
3. McConnell, A.J., Application of Tensor Analysis, Dover, New York, 1957.
4. Jeffreys, H., Cartesian Tensors, Cambridge University Press, London, 1957.
5. Pestel, E.C., Leckie, F.A., Matrix Methods in Elastomechanics, McGraw-Hill Book Co., New York, 1963.

REFERENCES (Cont'd)

6. Turner, M.J., Clough, R.W., Martin, H.C., Topp, L.J., Journal of Aeronautical Sciences. Vol. 23, p. 805, 1956.

7. Turner, M.J., Dill, E.H., Martin, H.C., Melosh, R.C., Journal of Aeronautical Sciences, Vol. 27, p. 97, 1960.

8. Argyris, J.H., AIAA Journal, Vol. 3, No. 1, 1965.

9. Argyris, J.H., "Recent Advances in Matrix Methods of Structural Analysis," Progress in Aeronautical Sciences, Pergamon Press, London, Vol. 4, 1963.

10. Argyris, J. H.,Aircraft Engineering, Vol. 27, p. 42, 1955.

11. Denke, P. H., Journal of Aeronautical Sciences, Vol. 23, p. 231, 1956.

12. Gallagher, R. H., Padlog, J., Bijlaard, P. P., ARS Journal, Vol. 32, p. 700, 1962.

13. Lansing, W., Jones, I. W., Ratner, P., AIAA Journal, Vol. 1, p. 1619.

14. Wissmann, Numerische Berechnung nichtlinearer eleastischer Koerper, Dissertation to Technische Hochschule, Hannover, 1963.

15. --, Nonlinear Analysis of Elastic Structures, GD/Convair ERR-SD-069, 1960.

16. Calculation of the Membrane Forces of an Initially Curved Wing, GD/Convair ERR-SD-124, 1961.

17. --, Mains, W.R., Basic Operations in General Matrix Algebra GD/Astronautics ERR-AN-125, 1962.

18. --, Numerical Analysis of Nonlinear Elastic Bodies, GD/Astronautics ERR-AN-157, 1963.

19. --, The Primitive and Auxiliary Operations of General Matrix Algebra, GD/Astronautics ERR-AN-289, 1963.

20. --, Notes on the Numerical Analysis of Nonlinear Elastic Bodies, GD/Astronautics ERR-AN-290, 1963.

21. --, Initial Strains in Structural Elements, GD/Astronautice ERR-AN-571, 1964.

22. --, Simplex Decompositions for Uniform Beams, GD/Astronautics ERR-AN-715, 1965.

23. Alexandroff, P., Elementary Concepts of Topology, Dover, New York, 1961.

24. Zurmuehl, R., Matrizen, Springer, Berlin, 1958.

25. Gantmakher, F. R., Applications of the Theory of Matrices, Interscience Publ, New York, 1959.

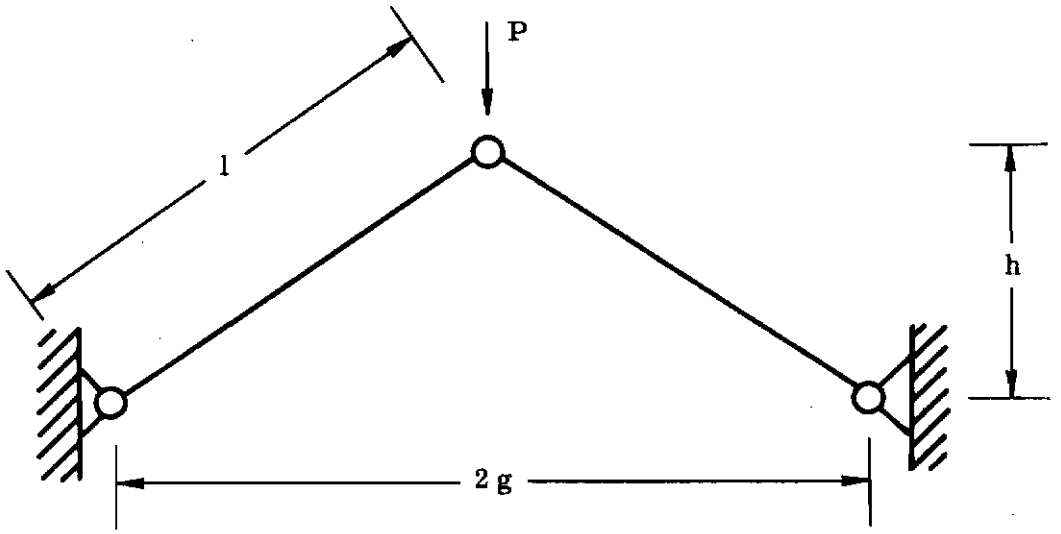

Figure 1. Example Problem for the Demonstration of the Iteration Processes

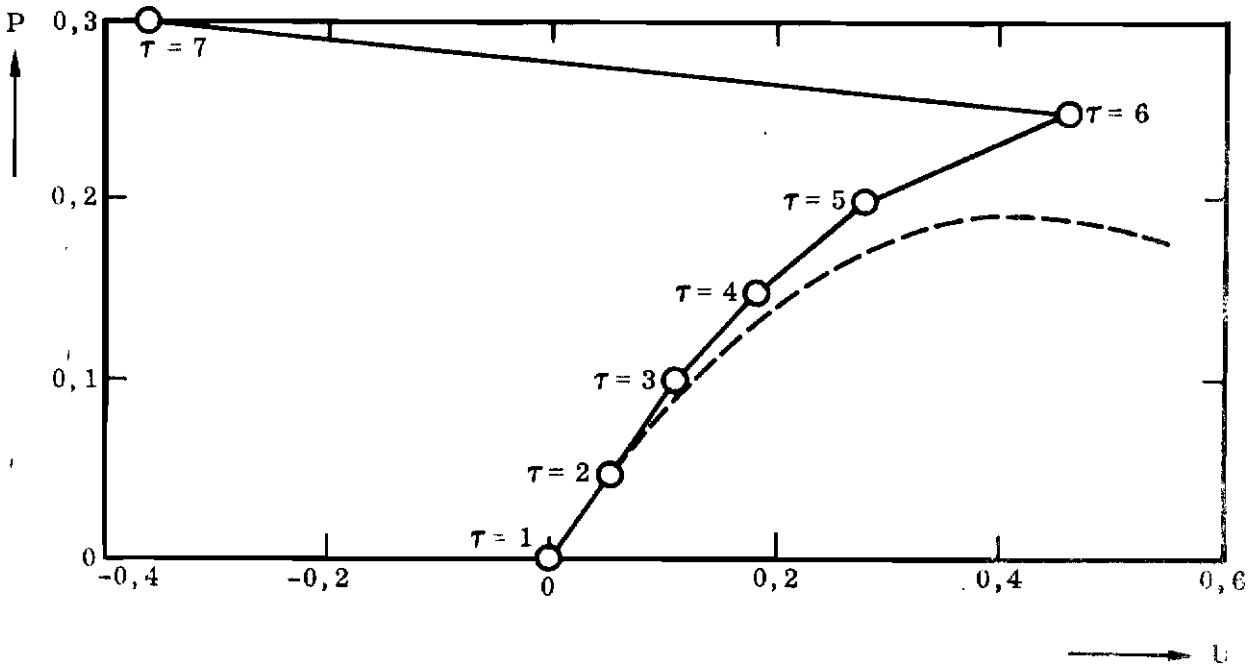

Figure 2. Progressive Iteration Without Back-Substitution

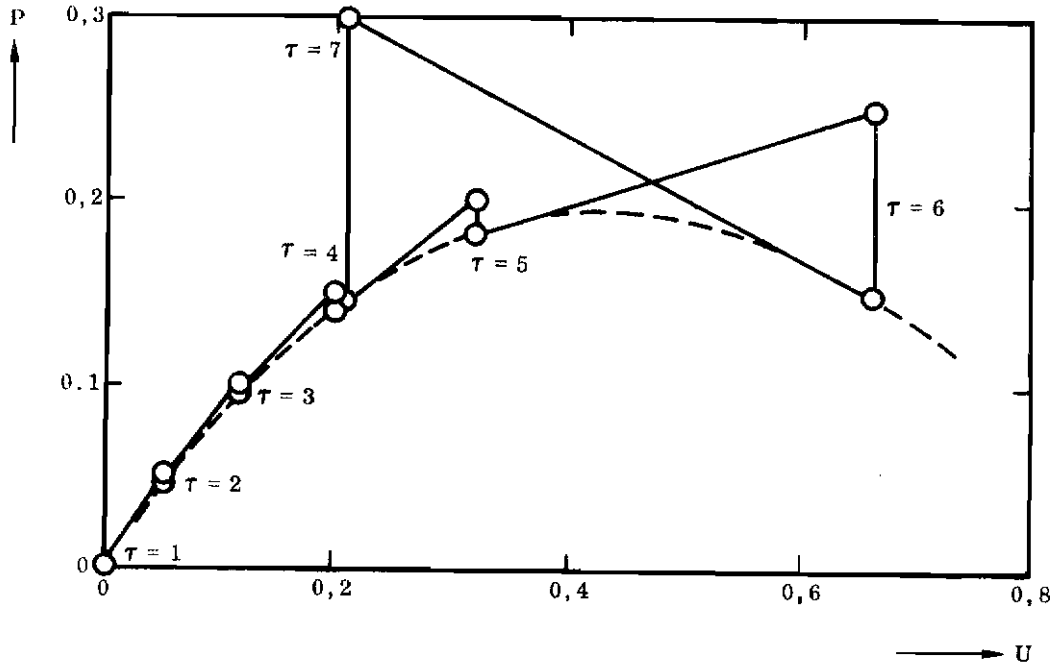

Figure 3. Progressive Iteration With Back-Substitution

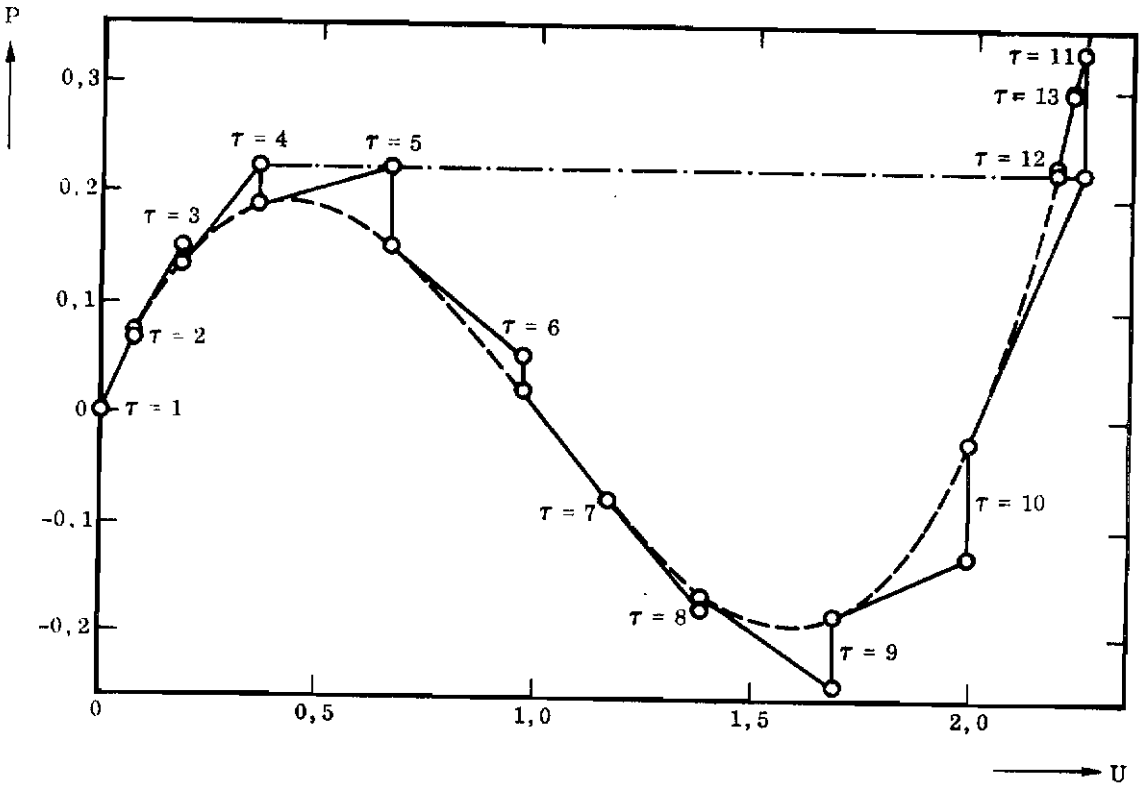

Figure 4. Progressive Iteration With Back-Substitution and Provisions for Instabilities