ON A THEORY OF COMPLEX DAMPING

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ABSTRACT

In non-proportionally damped structures, both energy dissipation and energy transformation exist. To characterize such aspects, in this paper, a new concept of complex damping ratio is introduced by means of generalizing the concept of Rayleigh quotient. The real part of this new quantity is the traditionally defined damping ratio, which reflects the modal energy dissipation per cycle; whereas, the imaginary part describes a ratio of energy transformation of a virtual mode per cycle. With this new concept, modal equations are set up and other relevant theoretical results are developed. Such a theory of complex damping is not only an alternativeway to describe the phenomena of complex modes, but also a useful tool, with strong physical meaning, for solving many theoretical and engineering problems of non-proportionally damped systems.

1 INTRODUCTION

In recent years considerable progress has been made in the field of mechanical vibrations and structural dynamics. However, many important questions remain to be answered in particularly concerning non-proportionally damped systems. For example, using proportional damping to describe real structures may result in severe errors (see Sigh 1986). Under what conditions do we have to change our models, Where do these errors come from and how can we minimize these errors? These questions are often asked in considering whether the structure responses should be calculated of estimated in the design of dampers; in the measurement of the damping matrix to evaluate the capability of energy dissipation of structures and in the construction of a valid damping matrix in finite element modeling. In the area of modal testing, we also face similar questions when we deal with damping ratios and with the measurement of damping matrix (see Liang and Lee 1991). Most of these unanswered questions are due to the lack of knowledge on energy relationship in non-proportionally damped structures. They may be systemically answered by using a theory of complex damping introduced in this paper. This theory unifies energy dissipation and energy transformation by means of one complex quantity. The real part of this complex quantity is the traditional damping ratio, describing the ratio of energy dissipation in a period; The imaginary part is the ratio of energy transmission in the same period.

One important advantage of using complex damping theory is as follows: When a structure is in vibration, the energy dissipation and transmission often bring the same results to a local region. Therefore, they are difficult to be distinguished. Traditionally, these energy terms are thought to be undecoupleable for a general damped system in N-dimensional space. We can now introduce a complex valued quantity, the complex damping ratio, to study each specific mode of the system. In so doing, we also may realize the physical meanings of the quantities.

2 CONCEPT OF COMPLEX DAMPING RATIOS, MATHEMATICAL TREATMENT

2.1 INTRODUCTION OF COMPLEX DAMPING COEFFICIENT

In this section we will introduce the quantity of complex damping coefficient. The equation of motion for a general MDOF system can be written as

$$M\,X''(t) + C\,X'(t) + K\,X(t) = F(t)$$

without loss of generality, consider the monic homogeneous form:

$$I\,X''(t) + \tilde{C}\,X'(t) + \tilde{K}\,X(t) = 0$$

with order n.

The above equation has eigenvalue matrix

$$\Lambda = \mathrm{diag}(\lambda_i) = \mathrm{diag}(-\xi_i\omega_i + j\sqrt{1-\xi_i^2}\,\omega_i)$$

and eigenvector matrix P_1. Then we have

$$P_1\,\Lambda^2 + \tilde{C}\,P_1\,\Lambda + \tilde{K}\,P_1 = 0 \qquad (1)$$

$$\tilde{K} = Q\,\Lambda_k\,Q^T \qquad (2)$$

$$\Lambda_k = \mathrm{diag}(\,\omega_{ni}^2\,) \qquad (3)$$

Pre-multiplying Q^T of equation (1) results in

$$R\,\Lambda^2 + Q^T\tilde{C}\,P_1\,\Lambda + \Lambda_k\,R = 0$$

where

$$R = Q^T P_1 = \begin{bmatrix} r_{11} & r_{12} \cdots & r_{1n} \\ r_{21} & r_{22} \cdots & r_{2n} \\ \vdots & \vdots & \vdots \\ r_{n1} & r_{n2} & r_{nn} \end{bmatrix}$$

So, using the notations (1) and (2), we can have the following expression of equations for the iith entries, that is,

$$\lambda_i^2\,r_{ii} + Q_i^T\,\tilde{C}\,P_{ii}\,\lambda_i + \omega_{ni}^2\,r_{ii} = 0 \qquad (4)$$

where Q_i and P_i are the ith column of matrices Q and P_1 respectively. Note that,

$$r_{ii} = Q_i^T\,P_{ii}$$

If the system is proportionally damped, r_{ii} must not be zero. We can also show that at least two of the terms r_{ij}, i = 1,...,n, are not equal to zero for at least one P_j in the case of non-proportionally damped system. With

$$\omega_{ni}^2\,Q_i = \tilde{K}\,Q_i \quad \text{or} \quad Q_i^T = \frac{1}{\omega_{ni}^2}\,Q_i^T\,\tilde{K}$$

we have

$$r_{ii} = \frac{1}{\omega_{ni}^{2}} Q_i^T \tilde{K} P_{ii} \tag{5}$$

Suppose $r_{ii} \neq 0$. Letting equation (4) be divided by r_{ii} yields

$$\lambda_i^2 + d_i\lambda + \omega_{ni}^2 = 0 \tag{6}$$

Define the term

$$(Q_i^T \tilde{C} P_j)/ Q_i^T P_j = ж_{ij} \tag{7}$$

to be a *generalized Rayleigh quotient* .

Since P_{ii} is generally a complex vector, so is the generalized Rayleigh quotient ω_{ni}^2 $(Q_i^T \tilde{C}P_{ii})/(Q_i^T \tilde{K}P_{ii})$. We therefore use a complex number $d_i = a_i + jb_i$ to describe this quantity,

$$d_i = a_i + jb_i = \omega_{ni}^2 (Q_i^T \tilde{C} P_{ii})/ (Q_i^T \tilde{K}P_{ii}) \tag{8}$$

For convenience, equation (7) and its corresponding differential equation

$$u_i'' + (a_i + j\,b_i)\,u_i' + u_i = 0 \tag{9}$$

is called the characteristic equation and differential equation of the i^{th} *virtual mode.*

Now consider the physical meaning of Pre-multiplying Q^T of equation (b). We may call the term $P_i\Lambda^2$, $\tilde{C} P_i \Lambda$ and $\tilde{K} P_i$ the inertial force, damping force and spring force respectively. (see Clough, 1985). Therefore $Q^TP_i\Lambda^2$, $Q^T\tilde{C}P_i\Lambda$ and $Q^T\tilde{K}P_i$ are the virtual work done along the virtual displacement Q. Under this consideration, the quantity d_i is a kind of ratio of damping virtual work (Q_i^T $\tilde{C}$ P_{ii}) and inertial virtual work (Q_i^T P_{ii}) or spring virtual work ($Q_i^T \tilde{K} P_{ii}$). In a later section, we will see that, the quantity d can play an important role the vibration analyses, we therefore name d_i to be the i^{th} *complex damping coefficient*.

2.2 SOME CHARACTERISTICS OF COMPLEX DAMPING

Substituting notations of λ_i and λ_i^2 into equation (7) and rearranging the results in two equations, for the real part, we have:

$$(2\xi_i^2 - 1)\,\omega_i^2 + \omega_{ni}^2 = \xi_i\omega_i a_i + \sqrt{1 - \xi_i^2}\;\omega_i\,b_i \tag{10a}$$

and for the imaginary part, we have:

$$-2\,\xi_i\sqrt{1 - \xi_i^2}\;\omega_i^2 = -\sqrt{1 - \xi_i^2}\;\omega_i\,a_i + \xi_i\,\omega_i\,b_i \tag{10b}$$

Combining the above two equations yields

$$a_i = \frac{\xi_i}{\omega_i} \left(\omega_{ni}^2 + \omega_i^2 \right) \tag{11a}$$

$$b_i = \frac{\sqrt{1 - \xi_i^2}}{\omega_i} \left(\omega_{ni}^2 - \omega_i^2 \right) \tag{11b}$$

From equation (11a), it is easy to see that, a_i is always greater than zero if ξ_i is non-zero. However, b_i appears to be undefined. We will show that, for a given M-C-K system, the sign of b is uniquely determined.

Taking the complex conjugate of equation (9), we have

$$d_i^* = a_i - jb_i = \omega_{ni}^2 \, (Q_i^T \tilde{C} P_{ii}^*) / (Q_i^T \tilde{K} P_{ii}^*)$$

Therefore, the sign of b_i is completely determined by the i^{th} eigenvector P_{ii}. In other words, only one of the i^{th} complex conjugate pair of the i^{th} eigenvectors can give the correct value of b_i. We thus define this eigenvector as the i^{th} *principal eigenvector* of the M-C-K system, and define the corresponding eigenvalue the i^{th} *principal eigenvalue* of the M-C-K system. We also define the eigen-matrix which consists of all n principal eigenvectors and eigenvalues the ***principal eigen-matrix***.

2.3 CRITERIA FOR PROPORTIONAL DAMPING, FIRST APPLICATION OF THE COMPLEX DAMPING THEORY

It is easy to show that b_i can be used as an index to calculate the difference between the undamped natural frequency of the system, ω_i, and that of corresponding non-proportionally damped system, ω_{ni}. In fact, we have the following theorem:

Theorem 1: The following facts are equivalent:

1) The system has proportional damping, that is,
   $\mathbf{C}\,\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\,\mathbf{M}^{-1}\mathbf{C}$ or $\tilde{\mathbf{C}}\,\tilde{\mathbf{K}} = \tilde{\mathbf{K}}\,\tilde{\mathbf{C}}$
2) The eigen-matrix has following properties:
   $\mathrm{Re}(\mathbf{A})\ \mathrm{Im}(\mathbf{A}) = \mathrm{Im}(\mathbf{A})\ \mathrm{Re}(\mathbf{A})$
   $\mathbf{A}\ \mathrm{Re}(\mathbf{A}) = \mathrm{Re}(\mathbf{A})\ \mathbf{A}$
3) The system has only normal mode, (all eigenvectors of system are weakly complex) that is, $\mathbf{P} = \mathbf{Q}\ \mathbf{A}$.

4) The undamped natural frequencies of the system and the corresponding eigenvalues of the generalized stiffness $\tilde{K}$ are all equal. Namely, $\omega_i = \omega_{ni}$, $i = 1, \ldots, n$

5) All the imaginary parts of the generalized Rayleigh quotient $\Re_{ij}$ are zero. That is, $b_i = 0$, $i = 1, \ldots, n$

Since statements 1) and 2) 3) are well established, (see the paper "A Strong Criterion For Testing Proportionally Damped Systems", Theorem 6 and Corollary 6) we will only prove 4) and 5).

PROOF:

If a system has no complex mode, then b_i's must be zero. This is an obvious sufficient condition. From the condition, all the P_{ii}'s must be all non-complex valued, that is,

$$Q_i = P_{ii} \ , \quad i = 1,\ldots N$$

Then it is easy to see, from equation (8), d_i is a real scalar, or, $b_i = 0$; Also, from the argument $Q_i = P_i$ we know that other generalized Rayleigh Quotient $\Re_{ij}$'s , $j \neq i$, are zero.

Next, consider the the necessary condition. It is clear from equation (10b), that this condition is equivalent to the following:

$$\omega_i = \omega_{ni} \ , \ i = 1,\ldots,n \tag{12}$$

Or, in this case, we have

$$\Lambda \Lambda^* = Q^T \tilde{K} Q = \Lambda_k \tag{13}$$

We know that, (see the paper "A Strong Criterion of Proportionally Damped Systems" by the same authors) for an M-C-K system, if equation (12) holds, it has real-valued eigenvector matrix. Therefore, we know the necessary condition is also true here. ■

Theorem 1 is important. It provides two new criteria to judge whether a system is proportionally damped. Namely, if all the imaginary part of $\Re_{ij} = 0$ or if $\omega_{ni} = \omega_i$, $i = 1, \ldots n$. In addition it confirms the sufficient and necessary relationship between the complex damping coefficient and the damping property of the system. Also, from the equation (10b), we have a simple but import corollary.

Corollary 1: For a damped system, if its i^{th} complex damping coefficient is real, its i^{th} undamped natural frequency is equal to that of corresponding proportionally damped system.

2.4 INVARIANTS OF SYSTEMS

A general damping matrix can always be written as

$$C = C_p + C_n \tag{14}$$

where C_p contains all the exact damping ratios of the system, ξ_i's, by means of the relationship

$$\xi_i = \frac{d_{ii}}{2|\lambda_i|} = \frac{d_{ii}}{2\,\omega_i} \tag{15}$$

where d_{ii} is the ii^{th} entry of matrix $Q^T C_p Q$, which is a real number. Now, consider the i^{th} eigenvalue of the system with damping C, denoted by $\lambda_i(C)$, and that of the system with damping C_p, denoted by $\lambda_i(C_p)$, both systems have the same generalized stiffness matrices $\tilde{K}$. For convenience, the second system is called the *corresponding proportionally damped system* or simply the *corresponding system*, of the first system,; and denote the first system by H(C) and the corresponding system by $H(C_p)$. If $b_i = 0$, or $\omega_{ni} = \omega_i$, for all $i = 1,\ldots,N$, the system H(C) and the corresponding proportionally damped system $H(C_p)$ will have the same eigenvalue matrices.

Next let us consider some invariants of system H(C) by comparing with its corresponding system $H(C_p)$ where H(.) is the state matrix of the system, i.e.

$$H = \begin{bmatrix} -M^{-1}C & -M^{-1}K \\ I & 0 \end{bmatrix}$$

First, consider the proportionally damped system, we have the following corollary:

Corollary 2: For a M-C-K system with proportional damping, no matter how the damping matrix C changes, as long as the system is proportionally damped, all undamped natural frequencies remain unchanged. That is,

$$\omega_i = \text{constant} \quad i = 1,\ldots,N \tag{16}$$

This corollary is a direct deduction from theorem 1, condition (4). From this fact, we can state, for any proportionally damped system, if the mass and stiffness matrices remain unchanged but only the damping matrix varies, that system will have the invariant in undamped natural frequencies. Otherwise, we know that, equation (16) will be no longer valid if the system becomes non-proportionally damped. However, we have:

Lemma 1: The determinant of the state matrix and the corresponding generalized stiffness matrix are identical. That is,

$$\det(\, H \,) = \det(\, \tilde{K} \,) \tag{17}$$

From the second formula of **H** matrix given on the previous page, using simple manipulation of linear algebra, we can easily establish this lemma.

Theorem 2: For a M-C-K system, if only the damping matrix C changes while both **M** and **K** matrices remain unchanged, the product of all undamped natural frequencies also remains unchanged. That is,

$$\prod_{i=1}^{2N} \omega_i = \text{constant} \tag{18}$$

With the help of lemma 1, this result is quite clear, since

$$\prod_{i=1}^{2N} \omega_i = \det(H) = \det(\tilde{K})$$

These invariants will play an important role in the energy analysis of the damped systems. The energy analysis will, in turn, help us to understand the physical meanings of these invariants, and also give the physical meaning of the complex damping coefficients.

3 ENERGY METHODS AND DYNAMICS MEANINGS

In the above section, we mathematically pointed out that the quantity d_i can describe the complex property of a system. We now try to interpret its physical meaning by the simplest case of SDOF system.

3.1 ENERGY DISSIPATION

Consider a SDOF system with free decay vibration:

$$m\, x'' + c\, x' + k\, x = 0$$

where m, c, and k are all real scalars. We can rewrite this equation in another form:

$$x'' + 2\xi\omega\, x' + \omega^2\, x = 0 \tag{19}$$

where ξ and ω have the same standard meaning in vibrational analysis. Solution of Eq (1a) requires certain initial conditions. For example, we can have

$$x = \exp(\lambda t) \tag{20}$$

Now, consider the work done by the inertial force, denoted by W_m, by the damping force, denoted by W_c, and by the spring force, denoted by W_k. Since the displacement (20) is in general complex valued, so will be the work done. It is important to know that, the work done can also be considered as virtual work. because the displacement x is in fact the virtual displacement. For convenience, we may call it *the complex virtual work* or simply the *complex work*. Consider the work done in one cycle, with period of $T = 2\pi/\omega$. We have

$$\left.\begin{aligned} W_m &= \int_0^{2\pi/\omega} x'' \, x \, dt = \frac{\lambda}{2}\,\eta \\ W_c &= \int_0^{2\pi/\omega} 2\xi\omega \, x' \, x \, dt = 2\,\xi\omega\,\eta \\ W_k &= \int_0^{2\pi/\omega} k \, x \, x \, dt = \frac{\omega^2}{2\lambda}\,\eta \end{aligned}\right\} \tag{21}$$

where η is a complex number

$$\eta = (\exp(2\lambda T) - 1) \tag{22}$$

Now denote η by equation (23)

$$\eta = \eta_o \, e^{j\phi} \tag{23}$$

in the normal complex plane, denoted by $C_p^{(n)}$. This complex work is shown in Fig. 1. If a number is mapped onto this normal complex plane, its module will be multiplied by η_o with an angle ϕ of rotation counterclockwise. To simplify the matter, we may use a modified complex plane $C_p^{(m)}$ with units measured by η_o and x-axis coincident with a line of angle ϕ in $C_p^{(n)}$. In this plane, we have

$$\left.\begin{aligned} W_m &= \frac{\lambda}{2} = \frac{1}{2}\left(-\xi\omega + j\sqrt{1-\xi^2}\,\omega\right) \\ W_c &= \xi\omega \\ W_k &= \frac{\omega^2}{2\lambda} = \frac{1}{2}\left(-\xi\omega - j\sqrt{1-\xi^2}\,\omega\right) \end{aligned}\right\} \tag{24}$$

The energy equation (24) satisfies the law of conservation of energy, or the law of virtual work. That is,

$$W_m + W_c + W_k = 0$$

Furthermore, the dissipated energy, W_c , is real-valued. In other words, W_c lies along the x-axis of the $C_p^{(m)}$ plane. However, both W_m and W_k are complex work done. The sum of their real parts, $\xi\omega$, are just the energy given from the system in the specific circle to the damper. The damper dissipates the exact amount of energy $\xi\omega$. Both W_m and W_k will have an angle γ to the x-axis, which is called the *loss angle*. If the damping ratio ξ is small enough, we can have the following relation:

$$\gamma = \xi = \tan\left(\frac{Im(\lambda)}{Re(\lambda)}\right) = \tan\left[\frac{Im(W_m)}{Re(W_m)}\right] = \tan\left[\frac{Im(W_k)}{Re(W_k)}\right] \qquad (25)$$

where the least three tangent forms are called *loss tangent*.

Theorem 3: For a SDOF system with real valued damping coefficient, denoted by (19), its damping ratio equals to the ratio of work done by damping force and the geometric sum of work done by inertia and spring forces during a cycle. That is,

$$\xi = \frac{W_c}{2\sqrt{W_m W_k}} \qquad (26)$$

Equations (25) and (26) can be also be obtained in figure 1.

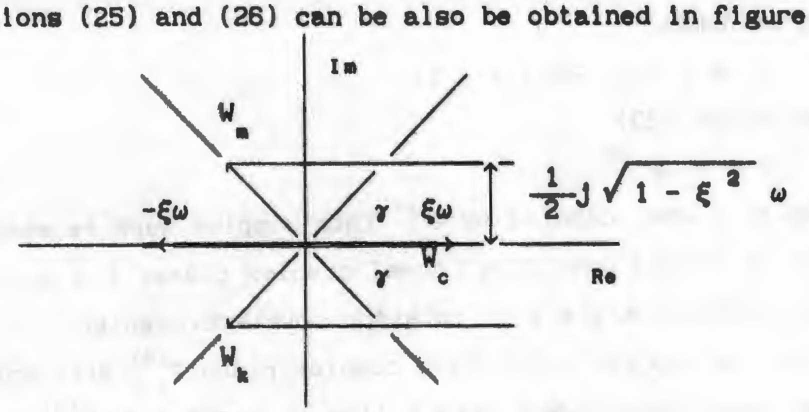


Figure 1 complex $C_p^{(m)}$ plane

The module of the complex work W_m (or W_k) is equal to ω. If we return to the $C_p^{(n)}$ plane, and suppose at the beginning of the circle, the amplitude of the displacement is one, then, the quantity ω represents the amount of the kinetic energy at this moment. So, for convenience, we also call the undamped natural frequency *virtual energy* , denoted by $\mathcal{E}_v$ and have the following corollary.

Corollary 3: For a SDOF system with real valued damping coefficient, denoted by (19), its virtual energy equals the square root of generalized stiffness,

$$\mathcal{E}_v = \sqrt{m^{-1}k} = \sqrt{\omega^2} = \omega \tag{27}$$

Compare corollary 1 with corollary 3, we see that, the virtual energy of system (19) is an invariant. In other words, for a system (19), no matter how the damping coefficient c changes, the virtual energy remains unchanged. Also, an MDOF proportionally damped system can be decoupled into n-real modes of n-individual equations, like equation (19), as stated by the following corollary.

Corollary 4: If an MDOF system with proportional damping is decoupled into n-real modes, then each mode has invariant virtual energy, regardless whether or not the damping matrix changes.

Now, consider the imaginary parts of the complex work W_m and W_k. The work done by the inertial and the spring force contains the real part, the energy to have been dissipated, contains the imaginary part. Only when this later part is included, the virtual energy is equal to ω. And this amount of energy is the conservative portion of the energy during this circle, (from kinetic energy to potential energy) . If the damping is equal to zero, the conservative portion of the energy is equal to ω. However, as the damping ratio ξ becomes larger, this amount of energy will become smaller by the factor $\sqrt{1 - \xi^2}$, because a certain amount energy is dissipated. The interesting thing is, the portion of energy or work done, under the notation of complex work or the $C_p^{(m)}$ plane, is perpendicular to W_c . This important conclusion also holds for MDOF system.

3.2 ENERGY TRANSMISSION

Now, let us consider the imaginary coefficient of the velocity term in the equation of motion

$$m\,x'' + j\,c\,x' + k = 0$$

Or, in a more familiar form,

$$x'' + 2j\,\zeta\omega\,x' + \omega^2 x = 0 \tag{28}$$

where m, c and k are real scalars. In practice, equation (28) has no real meaning, if it describes a SDOF system. However, it will have clear physical meaning if it is used to express a virtual mode of a MDOF system. Suppose the solution to equation (28) is

$$x = \exp(\nu t),$$

then

$$x' = \nu \exp(\nu t) \quad \text{and} \quad x'' = \nu^2 \exp(\nu t)$$

Therefore we have the characteristic equation as follows

$$\nu^2 + 2 j \zeta\omega \nu + \omega^2 = 0 \tag{29}$$

The solution of equation (28) is given by

$$\nu = j \omega (- \zeta \pm \sqrt{1 + \zeta^2})$$

Without loss of generality, we may write

$$\nu = j \omega (- \zeta + \sqrt{1 + \zeta^2})$$

The solution does not have real part.

Again using the concept of a modified $C_p^{(m)}$ plane, with the complex number, $\eta = (\exp(2\nu T) - 1)$, we may obtain the work done for one cycle $T = 2\pi/\omega$:

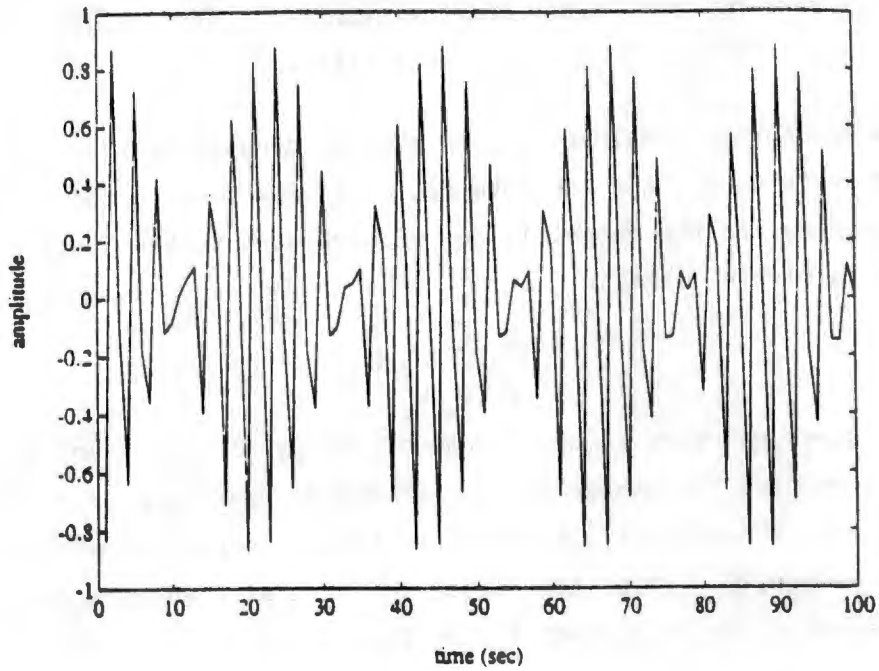
$$\left.\begin{aligned} W_m &= \frac{\nu}{2} = \frac{1}{2} j \omega (-\zeta + \sqrt{1 + \zeta^2}) \\ W_c &= j \zeta\omega \\ W_k &= \frac{\omega^2}{2\nu} = \frac{1}{2} j \omega (-\zeta - \sqrt{1 + \zeta^2}) \end{aligned}\right\} \tag{30}$$

The above quantities expressed in equations (30) satisfy the law of conservation of energy or the law of virtual work. That is,

$$W_m + W_c + W_k = 0$$

In addition, these work done quantities are all imaginary. Thus, during a cycle, no energy is dissipated. Figure 2 gives a typical response time history of system (28) subjected to an impulse. It is seen that, without real part of the damping coefficient the amplitude of vibration will not decrease. In other words, no energy is dissipated. The above described impulse response does not behave like a SDOF system. Rather, it behaves like an MDOF system with some more natural frequencies, (see figures 2). In a certain cycle the amplitude seems to decrease whereas in a different cycle it increases. It is easy to understand that, during a certain cycle, the energy, which is proportional to the square of the amplitude, is

Fig. 2 Time History of Imaginarily Damped System



different from that in another cycle. Energy changes or transfers from time to time. The interesting thing is, energy is also transferred within a system with real-valued damping. But the energy-transfer is essentially different from the case where the damping is only imaginary-valued.

Energy dissipation in real-valued and imaginary-valued systems are different. The portion of energy transferring between kinetic energy and potential energy is included in both the first case (24) and second case (30), In the first case it is the part of

$\pm\frac{1}{2}\, j\, \omega\, \sqrt{1-\xi^2}$; and in the second case, it is the part

$\pm\, \frac{1}{2} j\, \omega \sqrt{1+\zeta^2}$.

In the first case, the energy transfers or dissipates to the the damper is represented by the part $\xi\omega$. The energy transferred to an "imaginary" device in the second case is represented by the part $j\zeta\omega$. The major difference here is that the energy quantity $\xi\omega$ is changed from mechanical work to another type of energy, in most cases thermal energy, while the quantity $j\zeta\omega$ remains in the form of mechanical work. We may think of this energy is transferred to somewhere and stored there for a period of time and then it may be transferred back to the mass-spring system at a later time. Based on this concept we may call the quantity $j2\zeta\omega$ the *imaginary damping coefficient* and the quantity ζ the *imaginary damping ratio*. This is stated in the following theorem.

Theorem 4: For an imaginary damped system, denoted by (28), the imaginary damping ratio equals the absolute ratio of work done by the imaginary damping force and the geometric sum of work done by the inertia and spring force in a circle. That is,

$$\zeta = \left| \frac{W_c}{2\sqrt{W_m W_k}} \right| \tag{31}$$

In this paper we refer to the change of energy *of* $\xi\omega$ the *energy dissipation*, and the change of $j\zeta\omega$ the *energy transformation*. It is interesting to note that, with a given amount energy transferred, the virtual energy, $\mathcal{E}_v$, i.e. the "undamped" natural frequency $|\nu|$, is no longer equal to ω. It is modified by the factor

$$(- \zeta \pm \sqrt{1 + \zeta^2})$$

This tells us that, with the energy transformation, the total energy during cycles changes. It also suggests, that for a SDOF system, there appears to have two different values of virtual energy. But for a SDOF system, equation (28) does not have real meaning, neither does the virtual energy of SDOF system have real meaning. In fact, we refer a "SDOF system" to be a virtual mode of an MDOF system. For any given MDOF system, its N-undamped natural frequencies are uniquely determined. Therefore, we have the following corollary.

Corollary 5: A given N-dimensional MDOF system has and only has N-virtual energy. That is,

$$\mathcal{E}_{vi} = \omega_i \qquad \text{for } i = 1,\ldots,N$$

Typically, "virtual energy" is not used in energy considerations of a SDOF system. However, for convenience, we state the following corollary using the virtual energy of SDOF system to actually express the energy relationship between different modes of an MDOF system.

Corollary 6: For a SDOF system (28) with an imaginary damping coefficient jc, if the value of jc changes, the virtual energy of the system will also be changed.

3.3 COMPLEX WORK DONE

Consider now the case when a SDOF system has both real and imaginary damping coefficient,

$$x'' + 2(\xi + j\zeta)\,\omega\, x' + \omega^2 x = 0 \tag{32}$$

That is, the system has complex damping ratio. From theorems 4 and 5, it is not difficult to qualitatively understand the results of complex work done. Complex work done consists both energy dissipation and energy transformation. And, its virtual energy or undamped natural frequency will not be an invariant. Also, we know that, an MDOF system can only be "decoupled" with the form (32). Therefore, each equation (28) will no longer have invariant virtual energy. On the other hand, from theorem 3, we know that, the product of total virtual energy is still an invariant. Fig.3 shows the typical time history of complex damped systems, Fig. 4(a) is a real structure, one of its time histories is given in Fig 4(b) which shows the complex damping effects.

To quantitatively describe the complex work done and all its implications is rather complicated. However, it is possible for special cases. One example is the complex damping of lightly damped structures (see the paper "Lightly Damped Systems" by the same authors). Figure 3 (a) gives a response of the complex damped system. Figure 3(b) shows a time history of a real structure.

CONCLUSIONS

1) If a vibrational structure is non-proportionally damped, there exist both energy dissipation and energy transformation resulted by damping effects. The energy transformation is essentially the conservative energy. If the non-proportionality is heavy, this amount energy cannot be simply neglected nor be mistaken as dissipative energy. Otherwise severe errors may be caused.

2) To describe both the energy dissipation and the energy transformation quantitatively, a complex-valued generalized Rayleigh quotient can be used, which is obtained from natural parameters of the structure. This quantity is called complex damping, whose real part is traditional damping

Fig. 3(a) Time History of Complex Damped System

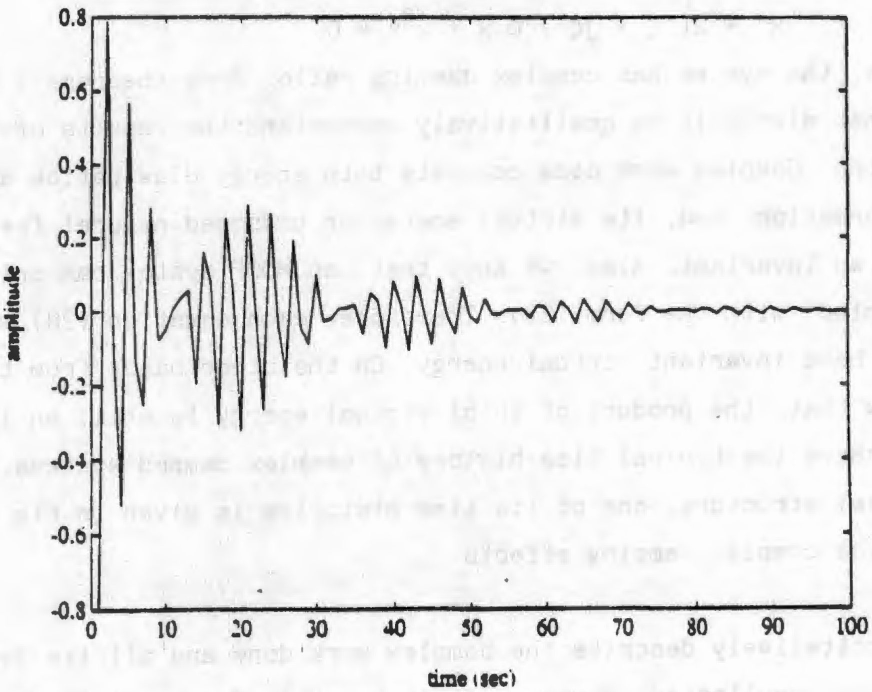


Fig. 3(b) Free response of Real Structures

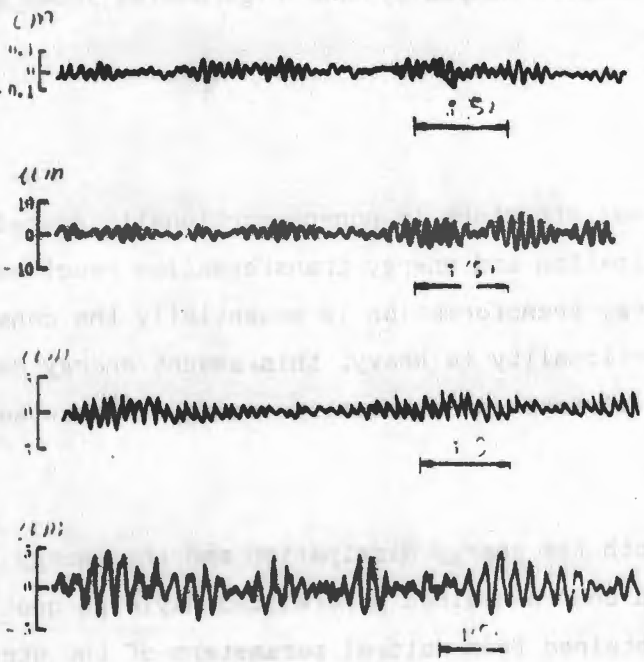


Fig. 4(a) 5-Floor Structure

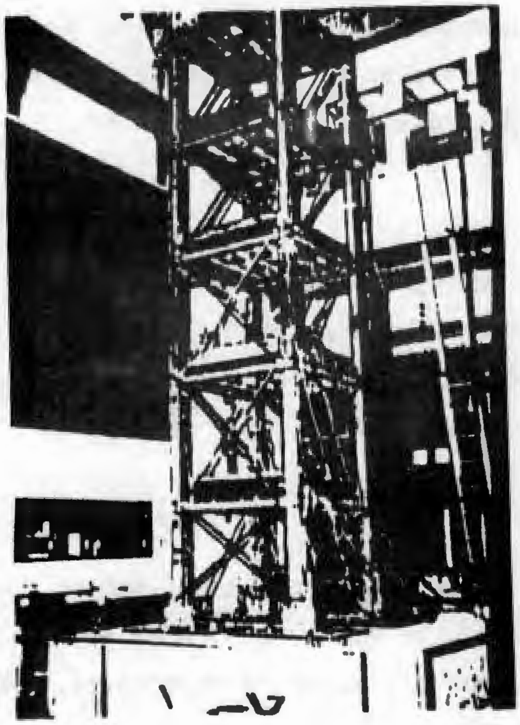
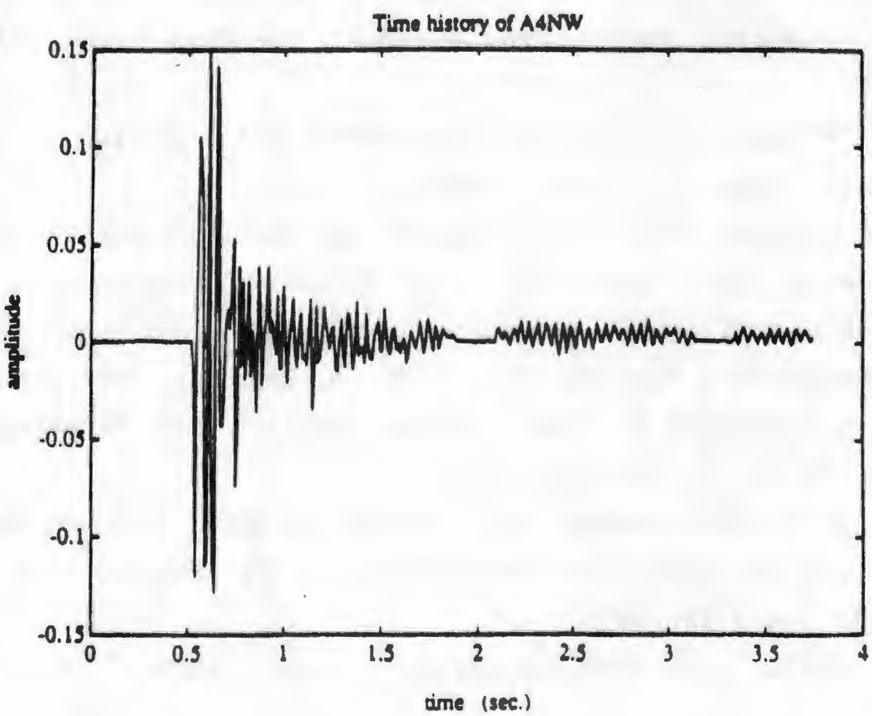


Fig. 4(b) Acceleration Time History (4th floor, north-west)



ratio and imaginary part stands for the ratio of energy transformation.

3) With the help of complex damping, the energy relationship of modal equation, the energy based invariants of systems, and therefore the physical meanings of complex damping itself are explained by the theory of complex damping described by this paper.

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