ORTHOGONALIZATION OF INTERNAL FORCE AND STRAIN SYSTEMS

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A specialization of the Schmidt method of orthogonalization in structural analysis is developed matrically. In the force method of analysis the orthogonalization of the internal force systems leads to a diagonal flexibility matrix. In the displacement method the orthogonalization of the internal strain systems leads to a diagonal stiffness matrix. A comparative analysis concerning the aspects of computational procedure and computer memory requirements is made between the above method and the one of finding the eigenvalues and eigenvectors of the flexibility or stiffness matrices. A computer program flow chart for orthogonalization by the Schmidt Method and a simple example of the building of a set of orthogonal strain systems are presented.

INTRODUCTION

The theory of groups (systems) of redundant forces and displacements was formulated in (Reference 1). The final objective in the use of groups (systems) of redundant forces or displacements is to have a well conditioned flexibility or stiffness matrix which is the matrix to be inverted in the computational procedure.

The best conditioning of the flexibility or stiffness matrix that can be reached is the diagonalization. This is obtained when the so called orthogonal systems of internal forces or strains are used. The use of these systems — at least those of internal forces, in the force method — is very old and well-known in structural analyses (see for example References 3, 4 and 5. In Reference 4 a specialization of the Schmidt method of orthogonalization to structural analysis is developed for the generation of orthogonal systems of internal forces. In Reference 5 the method of generating orthogonal systems of internal forces by finding the eigenvalues and eigenvectors of the flexibility matrix is developed, strictly by matrix formulation.

In this paper the Schmidt method of orthogonalization in structural analysis is developed matrically following the theory presented in Reference 1. When the force method is used the orthogonalization of the internal force systems leads to a diagonal flexibility matrix. In the displacement method, on the other hand, one is lead to a diagonal stiffness matrix. In the force (displacement) method one starts with any transformation matrix of redundant forces (redundant displacements) into internal forces (strains), the columns of which constitute a set of systems of internal forces (strains). At the end of the process one obtains another transformation matrix, the columns of which constitute a set of orthogonal systems of internal forces (strains). From this set, a diagonal flexibility (stiffness) matrix is obtained.

A comparative analysis concerning the aspects of computational procedure and the requirements of the computer memory is made between the two methods of orthogonalization referred

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to above. This analysis shows that the specialization of the Schmidt method has great advantages over the method of finding the eigenvalues and eigenvectors.

A computer-program flow chart based on the above theory and a simple example of the building of a set of orthogonal strain systems are presented.

The increased use of the displacement method in recent years, specially in the analysis of discrete idealizations of continuous structural systems, such as plates and shells (Reference 6), and solids (Reference 7) seems to indicate the desirability to explore the use of orthogonal strain systems.

THE FLEXIBILITY AND THE STIFFNESS MATRIX

The flexibility matrix relative to the X unknowns (redundancies) is given by

$$D_X = b_{IX}^T f b_{IX} \tag{1}$$

where b_{IX} represents the matrix of element forces due to unit values of the redundancies X and f is flexibility matrix of the unassembled structural elements. The flexibility matrix relative to the Z unknowns (redundancies) is

$$D_Z = (b_{IX} Q_{ZX})^T f (b_{IX} Q_{ZX}) \tag{2}$$

where the redundancies X are related to Z by the matrix equation

$$X = Q_{XZ} Z \tag{2a}$$

Applying the rules of the transpose of a product of matrices and using Equation 1, Equation 2 is transformed into

$$D_Z = Q_{XZ}^T D_X Q_{XZ} \tag{3}$$

Alternatively, the matrix D_Z can be written as

$$D_Z = b_{IZ}^T f b_{IZ} \tag{4}$$

where

$$b_{IZ} = b_{IZ} Q_{XZ} \tag{4a}$$

By the rules of matrix multiplication the typical term of D_Z is

$$D_{Z ij} = D_{Z ji} = (b_{IZ})_i^T f (b_{IZ})_j \qquad (i, j = 1, 2 \cdots n) \tag{5}$$

where the indexes i and j indicate the order of the columns of b_{IZ} involved.

Similarly the stiffness matrices relative to the **X** and **Z** unknowns are calculated from

$$K_X = a_X^T \, r \, a_X \tag{6}$$

where a_X represents the matrix of element strains due to unit values of the redundant displacements **X**, and **r** is the stiffness matrix of the unassembled structural elements and

$$K_Z = Q_{ZX}^T K_X Q_{ZX} = a_Z^T \, r \, a_Z \tag{7}$$

where, as before, the unknowns **X** and **Z** are related by the equation $X = Q_{XZ} Z$.

THE METHOD OF ORTHOGONALIZATION

The method will be developed for the flexibility matrix but it is exactly the same when applied to the stiffness matrix.

Starting with any transformation matrix b_{1X} which leads to the flexibility D_X from Equation 1, one obtains D_Z, Equations 3 or 4, which is a diagonal matrix. Postmultiplying both sides of Equation 4a by Q_{ZX}^{-1} results in:

$$b_{1X} = b_{1Z} \, Q_{ZX}^{-1} \tag{8}$$

For Q_{ZX}^{-1} the following triangular matrix will be assumed:

$$Q_{XZ}^{-1} = \left[\begin{array}{cccccc}
1 & -Q_{12} & -Q_{13}\cdots & -Q_{1i}\cdots & -Q_{1j}\cdots & -Q_{1n} \\
0 & 1 & -Q_{23}\cdots & -Q_{2i}\cdots & -Q_{2j}\cdots & -Q_{2n} \\
0 & 0 & 1\cdots\cdots & -Q_{3i}\cdots & -Q_{3j}\cdots & -Q_{3n} \\
\hline
0 & 0 & 0\cdots\cdots & 1\cdots\cdots & -Q_{ij}\cdots & -Q_{in} \\
\hline
0 & 0 & 0\cdots\cdots & 0\cdots\cdots & 1\cdots\cdots & -Q_{jn} \\
\hline
0 & 0 & 0\cdots\cdots & 0\cdots\cdots & 0\cdots\cdots & 1
\end{array}\right] \tag{9}$$

The elements of Q_{XZ}^{-1} have to be calculated in order to produce a diagonal matrix D_Z. This condition is expressed by:

$$D_{Zij} = D_{Zji} = 0 \qquad (i = 1, 2 \cdots n, \text{ and } i \neq j) \tag{10}$$

Considering Equations 8 and 9 the j^{th} column of $\mathbf{b}_{1Z}$ is developed as:

$$(\mathbf{b}_{1Z})_j = Q_{1j}(\mathbf{b}_{1Z})_1 + Q_{2j}(\mathbf{b}_{1Z})_2 + Q_{3j}(\mathbf{b}_{1Z})_3 \cdots$$
$$+ Q_{ij}(\mathbf{b}_{1Z})_i \cdots + Q_{j-1,j}(\mathbf{b}_{1Z})_{j-1} + (\mathbf{b}_{1X})_j \qquad (11)$$

Substituting into Equation 5 the value of $(\mathbf{b}_{1Z})_j$ from Equation 11 it follows that

$$D_{Zij} = Q_{1j}(\mathbf{b}_{1Z})_i^T \mathbf{f}(\mathbf{b}_{1Z})_1 + Q_{2j}(\mathbf{b}_{1Z})_i^T \mathbf{f}(\mathbf{b}_{1Z})_2 + Q_{3j}(\mathbf{b}_{1Z})_i^T \mathbf{f}(\mathbf{b}_{1Z})_3 \cdots$$
$$+ Q_{ij}(\mathbf{b}_{1Z})_i^T \mathbf{f}(\mathbf{b}_{1Z})_i \cdots + Q_{j-1,j}(\mathbf{b}_{1Z})_i^T \mathbf{f}(\mathbf{b}_{1Z})_{j-1} + (\mathbf{b}_{1Z})_i^T \mathbf{f}(\mathbf{b}_{1X})_j \qquad (12)$$

Introducing Equation 5 into the right-hand side terms of Equation 12

$$D_{Zij} = Q_{1j} D_{Zi1} + Q_{2j} D_{Zi2} + Q_{3j} D_{Zi3} \cdots + Q_{ij} D_{Zii} \cdots$$
$$+ Q_{j-1,j} D_{Zi,j-1} + (\mathbf{b}_{1Z})_i^T \mathbf{f}(\mathbf{b}_{1X})_j \qquad (13)$$

Applying now the diagonalisation condition, (Equation 10), to the terms of Equation 13 it is seen that the left-hand side term and the right-hand side terms, but the one which has Q_{ij} and the last, are zero. This leads finally to:

$$Q_{ij} = -\frac{(\mathbf{b}_{1Z})_i^T \mathbf{f}(\mathbf{b}_{1X})_j}{D_{Zii}} \qquad (14)$$

With matrix $\mathbf{Q}_{ZX}^{-1}$, the elements of which are computed by Equation 14, a diagonal flexibility matrix $\mathbf{D}_Z$ is obtained, using Equation 3.

Concluding, it is important to mention that in the actual computation it is not necessary to invert matrix $\mathbf{Q}_{XZ}^{-1}$ to calculate $\mathbf{D}_Z$ from Equation 3. In fact, in computing row by row, the elements of $\mathbf{Q}_{XZ}^{-1}$ in Equation 14, the columns of b_{1Z} are found successively from Equation 11. These constitute a set of n orthogonal systems of internal stresses. Moreover, the elements of $\mathbf{D}_Z$ which is a diagonal matrix are also found during the computational procedure. To check the accuracy of the diagonalization, $\mathbf{D}_Z$ can be computed by Equation 4.

COMPARISON BETWEEN THE TWO METHODS OF ORTHOGONALIZATION

The problem is: given the matrices $\mathbf{b}_{1X}$ and $\mathbf{f}$, find $\mathbf{b}_{1Z}$, whose columns constitute a set of orthogonal systems of internal forces or strains, and also find $\mathbf{D}_Z$ which is a diagonal matrix.

In both methods matrices $\mathbf{b}_{1X}$ and $\mathbf{f}$ are initially stored in the computer and the same storage is provided for $\mathbf{D}_X$ (in the method of finding the eigenvalues and eigenvectors), for the diagonal elements of matrix $\mathbf{D}_Z$ and for the elements of $\mathbf{Q}_{XZ}^{-1}$ (in the present method).

In the method of finding the eigenvalues and eigenvectors, matrix $\mathbf{D}_X$ (Equation 1) is set up using Equation 5. A program of finding eigenvalues and eigenvectors is then applied to matrix $\mathbf{D}_X$. The diagonal matrix of the eigenvalues is matrix $\mathbf{D}_Z$ and the matrix of the eigenvectors is matrix $\mathbf{Q}_{XZ}$. If matrix $\mathbf{b}_{1Z}$ is desired it has to be computed from Equation 4a.

In the present method the diagonal elements of $\mathbf{D}_Z$ and the numerators of the elements of $\mathbf{Q}^{-1}_{XZ}$ (Equation 14) are calculated by the same kind of computation which is used in the other method to compute $\mathbf{D}_X$ (Equation 5). The only additional work are the divisions needed to compute the elements of $\mathbf{Q}^{-1}_{XZ}$ (Equation 14). It is to be noticed that the total number of elements of $\mathbf{D}_Z$ and $\mathbf{Q}^{-1}_{XZ}$ that have to be computed is the same as in $\mathbf{D}_X$. The columns of $\mathbf{b}_{1Z}$ are successively computed by Equation 11 and stored in the same locations of the columns of $\mathbf{b}_{1X}$ which are no longer needed. The drawback of this method is that if matrix $\mathbf{Q}_{XZ}$ is needed it has to be computed by the inversion of $\mathbf{Q}^{-1}_{XZ}$.

Summing up, with slightly more computational work (the divisions in Equation 14 and the multiplications and sums in Equation 11) than the work of building up matrix $\mathbf{D}_X$, the present method gives matrices $\mathbf{b}_{1Z}$ and $\mathbf{D}_Z$.

COMPUTER PROGRAM AND EXAMPLE

A computer program based on the method of orthogonalization was developed using Basic Fortran language for the IBM 1620 computer at the Instituto Tecnológico de Aeronáutica. The flow chart of this program is presented in Figure 1.

The orthogonalization of the strain systems of the structure of Figure 2 is presented as example. Matrix $\mathbf{a}_X$ which represents the strains associated to the degrees of freedom indicated in Figure 2 is:

$$\mathbf{a}_X = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \qquad (15)$$

Matrix $\mathbf{r}$ is:

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r}_5 & \mathbf{r}_6 \end{bmatrix} \tag{16}$$

with

$$\mathbf{r}_1 = \mathbf{r}_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{17}$$

$$\mathbf{r}_3 = \mathbf{r}_4 = \mathbf{r}_5 = \mathbf{r}_6 = \begin{bmatrix} 2 & 1 & -0.3 \\ 1 & 2 & -0.3 \\ -0.3 & -0.3 & 0.06 \end{bmatrix} \tag{18}$$

With the input data of Equations 15, 16, 17, and 18 the following output is obtained:

$$a_Z = \left[\begin{array}{cccccc}
1 & -0.166 & 0 & -0.171 & 0.061 & 0.039 \\
0 & 1 & 0 & 0.028 & -0.176 & 0.039 \\
\hline
0 & 0 & 0 & 1 & -0.189 & 0.026 \\
0 & 0 & 0 & 0 & 1 & 0.026 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
1 & -0.166 & 0 & -0.171 & 0.061 & 0.039 \\
0 & 0 & 1 & -1.250 & -1.013 & 0.434 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0.028 & -0.176 & 0.039 \\
0 & 0 & 1 & -1.250 & -1.013 & 0.434 \\
\hline
1 & -0.166 & 0 & -0.171 & 0.061 & 0.039 \\
0 & 0 & 0 & 1 & -0.189 & 0.026 \\
0 & 0 & -1 & 1.250 & 1.013 & 0.565 \\
\hline
0 & 1 & 0 & 0.028 & -0.176 & 0.039 \\
0 & 0 & 0 & 0 & 1 & 0.026 \\
0 & 0 & -1 & 1.250 & 1.013 & 0.565
\end{array}\right] \tag{19}$$

and

$$K_Z = \begin{bmatrix} 6.000 & 5.833 & 0.240 & 3.453 & 3.329 & 0.028 \end{bmatrix} \tag{20}$$

REFERENCES

1. Venâncio Filho, F., "Groups of Unknowns in Structural Analysis," Journal of the Royal Aeronautical Society, Vol. 66, No. 617, p. 322, May 1962.

2. Mueller-Breslau, H., "Zentralblatt der Bauverwaltung," p. 513, 1897.

3. Kincaid, W. M., and Morkovin, W., "An Application of Orthogonal Moments to Problems in Statically Indeterminate Structures," Quart. Appl. Maths., Vol. 1, No. 4, pp. 334-340, January 1944.

4. Gillis, P., and Gerstle, K. H., "Analysis of Structures by Combining Redundants," ASCE Structural Division Journal, ASCE, Vol. 87, No. ST1, pp. 41-56, January 1961.

5. Argyris, J. H., and Kelsey, S., Modern Fuselage Analysis and the Elastic Aircraft, Butterworths, London, p. 101, 1963.

6. Turner, M. J., Martin, H. C., and Weikel, R. C., Further Developments and Applications of the Stiffness Methods, Boeing Documents No. D2-22061, The Boeing Company, 1962.

7. Argyris, J. H., "Matrix Analysis of Three-Dimensional Elastic Media Small and Large Displacements," AIAA Journal, Vol. 3, No. 1, pp. 45-51, January 1965.

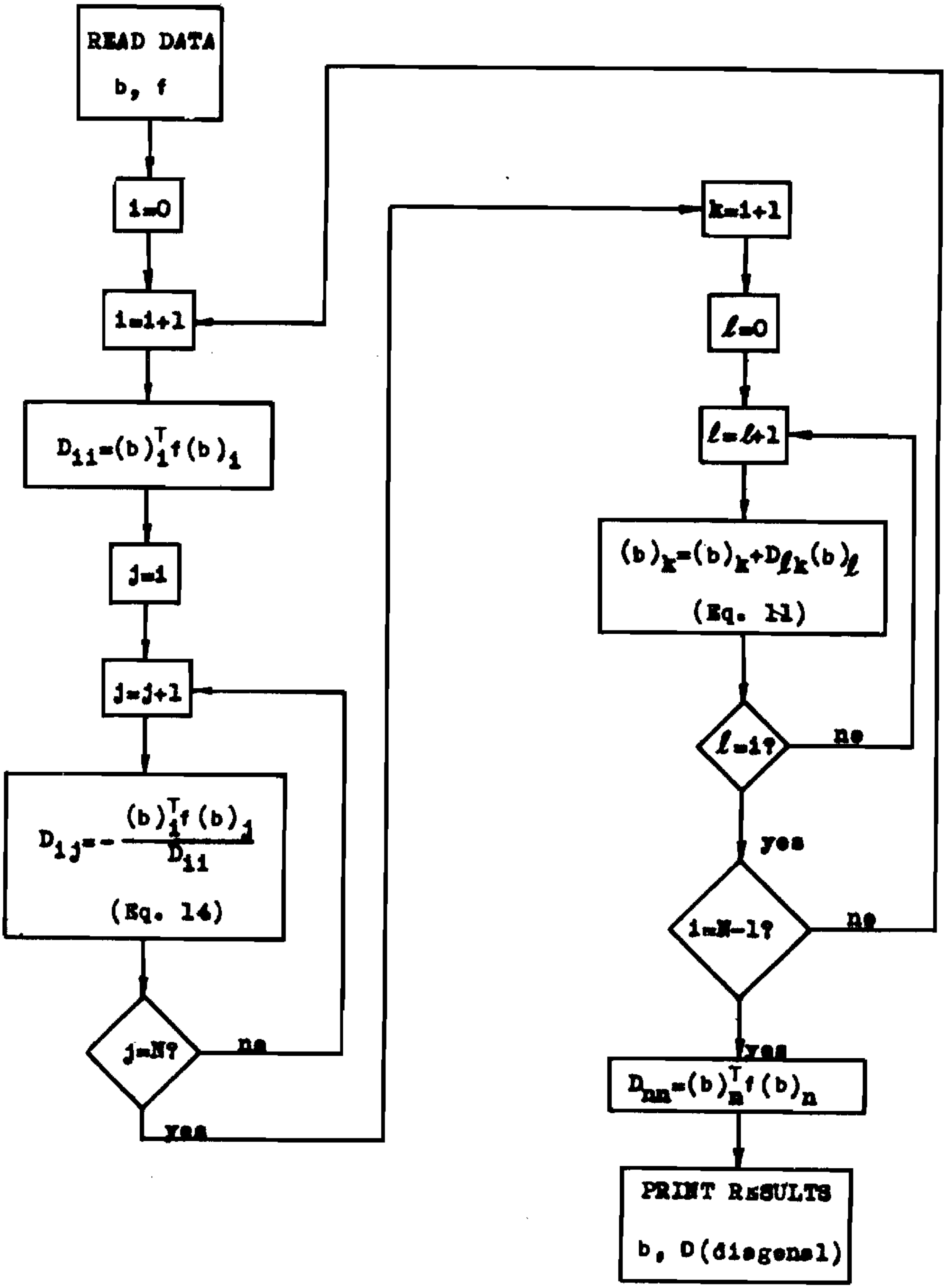

Figure 1. Flow Chart of the Program for Orthogonalization

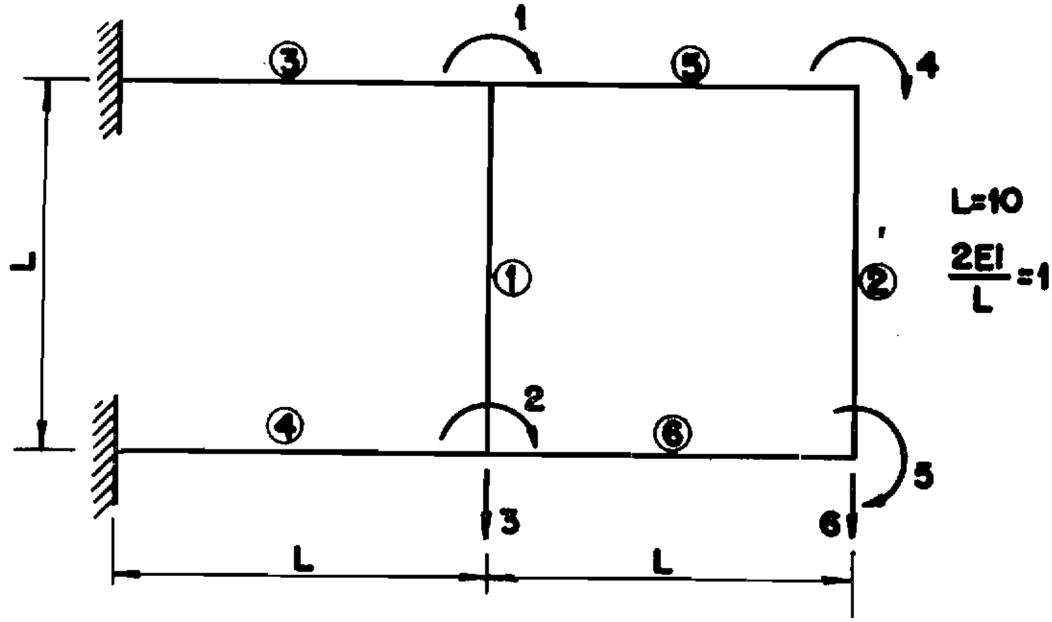

Figure 2. Structure of the Example