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FLIGHT SIMULATION OF ORBITAL AND REENTRY VEHICLES

PART IV — A STUDY OF EARTH OBLATENESS EFFECTS AND CHARACTERISTIC OSCILLATORY MOTIONS OF A LIFTING REENTRY VEHICLE

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FOREWORD

The research on which this report is based was performed by the Aeronautical and Astronautical Engineering Department of the The University of Michigan, Ann Arbor, Michigan, under Air Force Contract AF 33(616)-8385, Project No. 6114, "Training Equipment, Simulators, and Techniques for Air Force Systems," and Task No. 611407, "Mathematical Models." Professors G. Isakson and R. M. Howe directed the research for the University of Michigan. Mr. L. J. Kummeth of the Simulation Techniques Section, Training Research Branch, Behavioral Sciences Laboratory, Aerospace Medical Research Laboratories, was project engineer for the Aeronautical Systems Division.

This report is Part IV of a series of several parts under the general title, "Flight Simulation of Orbital and Reentry Vehicles." Research covered began in June 1960 and was completed in November 1961.

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ABSTRACT

The present report is concerned with two distinct aspects of the simulation problem for a lifting reentry vehicle.

The first relates to the effects of earth oblateness on the flight path of such a vehicle. The effects of oblateness of the gravity field on the orbital motion of a satellite are surveyed and are found to be small for a single orbit, probably sufficiently small to be neglected in a simulation for training purposes. Some of these effects are cumulative and would become substantial in a flight of long duration. The effects of oblateness of the earth's surface and atmosphere on the reentry trajectory are studied independently of the gravity oblateness effects. Typical trajectories are computed for the case of a nonrotating earth. Under the most extreme conditions, there is found to be a change in range of 690 nautical miles for a nominal once-around flight.

The second aspect relates to the characteristic oscillatory motion of the vehicle during reentry. The pertinent literature on this topic is surveyed. Approximate expressions for the period of the characteristic long-period and short-period oscillations are presented.

PUBLICATION REVIEW

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A	reference area in definition of aerodynamic coefficients
C_L	lift coefficient
C_D	drag coefficient
C_{m_α}	derivative of aerodynamic pitching moment coefficient, C_m, with respect to α
$\bar{d}$	non-dimensional range
D	aerodynamic drag on vehicle
f	oblateness geometric constant (Eq. (9))
g	acceleration due to gravity
g_o	value of g at $r = R_o$
h	altitude
I_x, I_y, I_z	principal moments of inertia of vehicle
ℓ	reference length in definition of aerodynamic moment coefficient
L	aerodynamic lift on vehicle
m	mass of vehicle
M	moment about pitch axis of vehicle due to gravity variation
M_Θ	derivative of M with respect to Θ
M_D	drag mass parameter, $\frac{C_D A \rho_o}{2m}$
q	dynamic pressure, $\frac{1}{2} \rho V^2$
r	distance of vehicle from earth's center
$\bar{r}$	mean value of r in a satellite orbit
δr	$r - R_o$
R	distance from center to surface of the earth
R_o	earth radius in the equatorial plane

s	distance along flight path
t	time
$\bar{t}$	variable defined in Eq. (4. 2)
T	period of oscillation
V	vehicle velocity
$\bar{V}$	non-dimensional velocity $\frac{V}{rg}$
α	angle of attack of vehicle
β	constant in Eq. (3. 1)
γ	angle of descent of vehicle (Fig. 1)
θ	polar coordinate of flight path measured from injection point (Fig. 1)
Θ	pitch angle of x-axis of vehicle above local horizontal
μ	gravity constant for spherical earth
ρ	earth density
ρ_o	reference value of ρ at surface of earth
ϕ	polar coordinate of flight path measured from ascending node (Fig. 2)
ψ	geocentric latitude of vehicle
ω	circular frequency of oscillation

I. Introduction

The present part of a comprehensive report on "Flight Simulation of Orbital and Reentry Vehicles" is concerned with two distinct aspects of the simulation problem for a lifting reentry vehicle.

The first relates to the effects of earth oblateness on the flight path of such a vehicle and provides information of value in assessing the importance of such effects and the need for their inclusion in a simulation for crew training purposes.

These effects are of two different types. Firstly, there is the departure of the earth's gravitational field from spherical symmetry. Secondly, there is the departure of the earth's atmosphere from spherical symmetry and its effect on air density variation along a particular flight path. While these two factors will be present simultaneously and exert a mutual influence, it is conjectured here that this mutual influence will not be large, and, consequently, that the importance of the two factors can be assessed independently. This is done in the following sections.

The second aspect of the flight simulation problem relates to the oscillatory nature of the reentry flight path when disturbances to a so-called "equilibrium" flight path are present. If a faithful simulation is to be achieved, the computing equipment used must be capable of generating such oscillations accurately. Toward this end, a knowledge of the frequencies of the characteristic oscillatory motions is of value. A survey of the available literature bearing on this topic is included in the present report.

II. Effect of Oblateness of the Earth's Gravitational Field on the Flight Path of an Orbiting Vehicle

In the absence of aerodynamic forces, the oblateness of the earth and its attendant distortion of the earth's gravity field has four main effects on the orbit of a near earth-satellite. These are:

(1) A rotation of the orbital plane about the earth's axis in a direction opposite to the motion of the satellite.

(2) A change in the period of the orbit with change in inclination of the orbital plane.

(3) A change in mean radial distance from the earth's center with change in inclination of the orbital plane, angular momentum of the satellite being maintained constant. In addition, there is a periodic variation of the radial distance apart from that associated with eccentricity of the orbit, so that the orbit departs from an elliptical shape.

(4) A rotation of the principal axes of the orbit in the orbital plane.

The rotation of the orbital plane about the earth's axis is given in Ref. 1 as being approximately equal to $10.0\,(R_o/\bar{r})^{3.5}\cos\alpha$ degrees/day, where R_o is the equatorial radius of the earth, $\bar{r}$ is the mean value of the satellite's distance from the earth's center, and α is the inclination of the orbital plane to the earth's equatorial plane. For a relatively low orbit, this expression can further be simplified by setting $R/\bar{r}$ equal to unity, and, introducing an approximate value of 90 minutes for the orbital period, it becomes $0.62 \cos\alpha$ deg./revolution. This represents a lateral displacement of the flight path of $37.3 \sin\alpha \cos\alpha$ nautical miles at successive equatorial crossings in a given direction. The lateral displacement achieves a maximum when $\alpha = 45^\circ$ and is then approximately 18.6 nautical miles.

The difference in orbital period between an equatorial orbit and an inclined orbit, for the same mean radius to the earth's center, is shown in Ref. 1 to be approximately $14.5\sqrt{R/\bar{r}}\,\sin^2\alpha$ seconds. The maximum difference is thus seen to be between an equatorial and polar orbit, and is then about 14.5 seconds, or about 0.28 percent of the period.

For a given angular momentum of the satellite, the mean radial distance from the earth's center increases continuously as α varies from 0° to 90°. It is about 14 nautical miles, or 85,000 feet, greater for a polar orbit than for an equatorial orbit (Ref. 1).

The oscillation in radial distance has a period equal to one-half the orbital period and an amplitude of approximately $0.94\,(R/\bar{r})\sin^2\alpha$ nautical miles or, setting $R/\bar{r} \cong 1$, about $5700 \sin^2\alpha$ feet (Ref. 1). The points of maximum radial distance occur at the equatorial crossings and the points of minimum radial distance correspond to the points of maximum latitude. There is thus seen to be some tendency for the orbit to follow the curvature of the earth's surface. Since the cylic variation of the earth's radius in the plane of the orbit has an amplitude of 35, 230 $\sin^2 \alpha$ feet, it is seen that this tendency is only partially realized.

The rotation of the principal axes of the orbit in the orbital plane is given approximately by 5.0 $(R/\bar{r})^{3.5}$ $(5 \cos^2 \alpha - 1)$ degrees/day (Ref. 1). This represents a rotation of about 20 degrees/day, or 1.25 degree/revolution, in the direction of the satellite's motion for near-equatorial orbits, and 5 degrees/day, or 0.31 degree/revolution, in a direction opposite to the satellite's motion for a polar orbit.

These effects are small, or can be compensated for by an appropriate adjustment of conditions at injection in a simulation based on a spherical earth, in the case of a once-around or shorter flight. If, however, the flight involves many circuits of the earth, two of the effects, namely the rotation of the orbital plane about the earth's axis and the rotation of the principal axes of the orbit in the orbital plane, are cumulative and become substantial. Thus, it appears that oblateness gravity effects could be omitted in the simulation of a once-around or shorter flight with little sacrifice in the faithfulness of the simulation. On the other hand, it would likely be desirable to include such effects if an entire flight of much longer duration was to be simulated.

III. Effect of Oblateness of the Earth's Atmosphere on the Reentry Flight Path of a Lifting Vehicle

In this section we disregard the gravity effects of oblateness and concentrate attention on the geometric effects of oblateness of the earth's surface and the associated departure of the earth's atmosphere from spherical symmetry. The extent of the oblateness may be realized by noting that the radial distance from the center to the surface of the earth is estimated to be 70, 459 feet less at the poles than at the equator.

Atmospheric density is assumed here to be a function only of altitude above the surface of the earth, described as an oblate spheroid, and consequently independent of latitude and longitude. It has been found (Ref. 2) that in the altitude range of interest in reentry problems (0 - 400, 000 feet) the density variation can be approximated satisfactory by the following exponential form,

$$\rho = \rho_o \, e^{-\beta h} \tag{3.1}$$

where $\rho_o = 0.0027$ slug/ft.^{3} and is seen to differ somewhat from standard sea level density, and $\beta = 1/23,500$ ft.$^{-1}$.

Under these circumstances, the variation of air density along a given flight path referred to axes with origin at the earth's center will be different in the case of an oblate earth as compared with a spherical earth, since altitude variation will be different in the two cases. It follows that, with the conditions at injection the same, the reentry flight path in the case of an oblate earth will differ from that for a spherical earth.

We now proceed to evaluate this difference on the basis of a simplified analysis of the reentry flight path in the two cases. In this analysis it is assumed that the flight path lies in a plane passing through the earth's center. This assumption stems from the neglect of oblateness gravity effects, the assumed absence of rolling and yawing motion of the vehicle, and the neglect of earth rotation. It is furthermore assumed that the vehicle is at all times in moment equilibrium about its pitch axis and that the lift and drag coefficients are constant.

The coordinates and motion variables of the vehicle are shown in Fig. 1.

The following two kinetic equations for forces normal and parallel to the velocity vector may now be written,

$$L + m\,V\,(\dot{\theta} + \dot{\gamma}) - mg\cos\gamma = 0 \tag{3.2}$$

$$-D + mg\sin\gamma - m\,\dot{V} = 0 \tag{3.3}$$

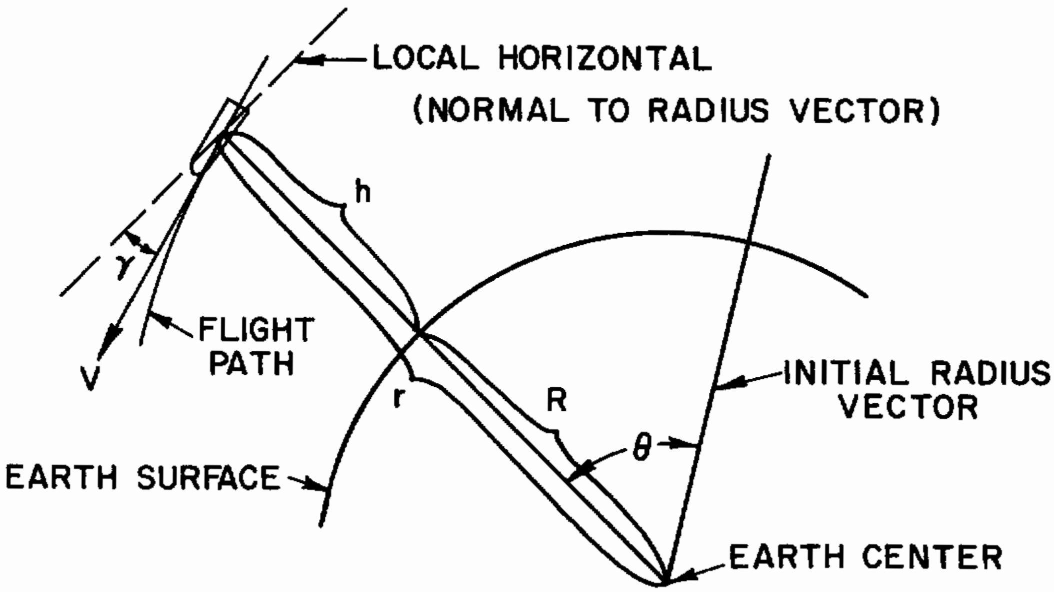

Fig. 1 - Coordinates and motion variables of vehicle

In addition, Eq. (3.1) and the following three relations are needed,

$$\dot{r} = -V \sin \gamma \tag{3.4}$$

$$r\dot{\theta} = V \cos \gamma \tag{3.5}$$

$$r = R + h \tag{3.6}$$

as well as the aerodynamic relations,

$$L = \frac{1}{2} \rho V^2 A C_L = \frac{1}{2} \rho V^2 A C_D \left(\frac{L}{D}\right) \tag{3.7}$$

$$D = \frac{1}{2} \rho V^2 A C_D \tag{3.8}$$

Assuming a spherical earth (R = constant) and making the following additional simplifying assumptions applicable to a shallow glide trajectory,

$$\gamma \quad \text{small} \qquad (\cos \gamma = 1,\ \sin \gamma = \gamma)$$

$$\dot{\gamma} = 0$$

$$g \sin \gamma \ll \dot{V}$$

several authors (e. g. , Ref. 3) have obtained a simple approximate analytical solution yielding a so-called "equilibrium trajectory." This trajectory is characterized by the fact that the gravity force is in equilibrium with the lift and the centrifugal force associated with motion at constant radius from the earth's center, and deceleration along the flight path is caused only by drag forces. Furthermore, no oscillatory component is present. For a vehicle with given L/D ratio and drag mass parameter $M_D = C_D A \rho_o/2m$, only one such trajectory exists, and conditions at injection (h, V, γ) must correspond to conditions at some point along this standard trajectory. If they do not, the trajectory will have an oscillatory characteristic and a more comprehensive analysis is necessary. An approximate solution for the oscillatory motion is discussed in Section 3.

It should be noted that an exact solution of the governing equations will also yield an equilibrium, or non-oscillatory, trajectory, provided, of course, that conditions at injection are appropriate. It is somewhat different from the approximate analytical solution discussed above because of the inclusion of the small terms neglected in the simpler analysis. These include the effect of curvature of the flight path associated with a varying angle of descent ($\dot{\gamma}$) and the component of the gravity force along the flight path.

The simplicity of the approximate analytical solution for the equilibrium trajectory raises the hope that a corresponding analysis may be performed in the case of the oblate earth and the effects of oblateness determined in that way. A closer examination of the nature of the approximate analytical result, however, reveals that such a hope is not well-founded. This result indicates that the velocity and angle of descent at a given altitude along the flight path depend almost entirely on the air density and are very insensitive to small changes in radial distance from the earth's center. Consequently, the small changes in radial distance from the earth's center associated with oblateness cannot be expected to cause significant changes in the trajectory. This is shown in Ref. 4, where the trajectories obtained in the two cases from such an analysis are essentially the same. The analysis is repeated in the Appendix of the present report.

It must be concluded that if oblateness introduces substantial effects, they will be associated with the influence of the small terms neglected in the simpler analysis, particularly the term in $\dot{\gamma}$. For this reason, an exact solution of the governing equations, Eqs. (3. 1) to (3. 6) inclusive, was undertaken.

The radial distance R from the center to the surface of an oblate earth can be related to the geocentric latitude as follows (Ref. 5),

$$R = R_o (1 - f \sin^2 \psi) \tag{3. 9}$$

where

$$f = 0.003367$$

$$R_o = 20,926,428 \text{ feet.}$$

From Fig. 2 it can be seen that ψ can be related to the angular coordinate in the orbital plane as follows,

$$\sin \psi = \sin \alpha \sin \emptyset \tag{3.10}$$

permitting the rewriting of Eq. (3.9) in the form,

$$R = R_o (1 - f \sin^2 \alpha \sin^2 \emptyset) \tag{3.11}$$

Furthermore,

$$\emptyset = \emptyset_o + \theta \tag{3.12}$$

Combining Eqs. (3.6), (3.9) and (3.10), we have

$$h = r - R_o \left\{ 1 - f \sin^2 \alpha \sin^2 (\emptyset_o + \theta) \right\} \tag{3.13}$$

Eqs. (3.1) to (3.5) inclusive and Eqs. (3.7), (3.8) and (3.13) now comprise the set to be solved. A reduction in the number of dependent variables can be effected by a transformation in which time is replaced by θ as the independent variable. Thus, from Eq. (3.5),

$$\dot{\theta} = \frac{V \cos \gamma}{r} \tag{3.14}$$

and, applying this relation,

$$\dot{V} = \frac{dV}{d\theta} \dot{\theta} = \frac{dV}{d\theta} \frac{V \cos \gamma}{r} = (V^2)' \frac{\cos \gamma}{2r} \tag{3.15}$$

$$\dot{\gamma} = \frac{d\gamma}{d\theta} \dot{\theta} = \gamma' \frac{V \cos \gamma}{r} \tag{3.16}$$

$$\dot{r} = \frac{dr}{d\theta} \dot{\theta} = r' \frac{V \cos \gamma}{r} \tag{3.17}$$

where primes denote differentiation with respect to θ.

Substituting Eqs. (3.1), (3.7), (3.8), (3.15), (3.16) and (3.17) into Eqs. (3.2) to (3.5) inclusive, and defining the drag mass parameter $M_D = C_D A \rho_o / 2m$, we have

$$\frac{L}{D} M_D e^{-\beta h} V^2 + \frac{V^2}{r} \cos \gamma (1 + \gamma') - g \cos \gamma = 0 \tag{3.18}$$

$$-M_D e^{-\beta h} V^2 + g \sin \gamma - (V^2)' \frac{\cos \gamma}{2r} = 0 \tag{3.19}$$

$$r' = - r \tan \gamma \tag{3.20}$$

which, together with Eq. (3.13), comprise the set to be solved.

The variable V^2 can be put in a non-dimensional form by dividing it by the square of the circular orbital velocity corresponding to radial distance r from the

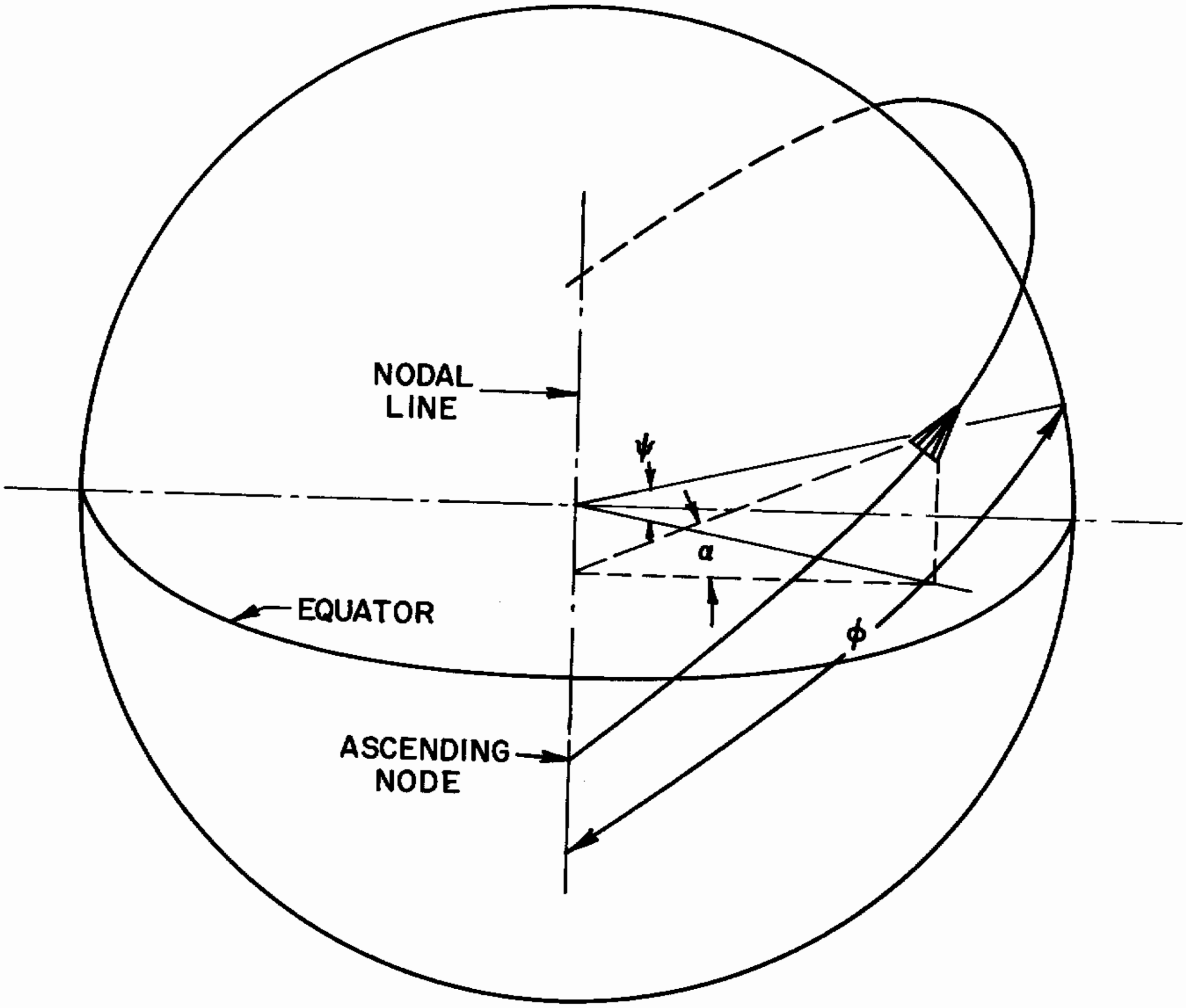

Fig. 2 - Geometric Parameters for an Inclined Orbit

center of the earth. Thus, we define

$$\overline{V}^2 = \frac{V^2}{rg} = \frac{V^2 r}{\mu} \tag{3.21}$$

where $g = \mu/r^2$, with μ the gravitational constant neglecting oblateness effects.

We can now write

$$\begin{aligned}(V^2)' &= (\overline{V}^2)' \frac{\mu}{r} - \overline{V}^2 \mu \frac{r'}{r^2} \\ &= (\overline{V}^2)' g r - \overline{V}^2 g r'\end{aligned} \tag{3.22}$$

Substituting Eqs. (3.21) and (3.22) into Eqs. (3.18) and (3.19), and rearranging, we have

$$\gamma' = \frac{1}{\overline{V}^2} - 1 - \frac{\frac{L}{D} M_D e^{-\beta h} r}{\cos \gamma} \tag{3.23}$$

$$(\overline{V}^2)' = (2 - \overline{V}^2) \tan \gamma - \frac{2 M_D e^{-\beta h} r \overline{V}^2}{\cos \gamma} \tag{3.24}$$

which, together with Eqs. (3.20) and (3.13), comprise the set of equations to be solved.

Round-off error in a digital solution can be greatly reduced by replacing Eq. (3.20) by

$$(\delta r)' = -(\delta r + R_o) \tan \gamma \tag{3.25}$$

where

$$\delta r = r - R_o \tag{3.26}$$

and is small in magnitude as compared with r. Eq. (3.13) can now be replaced by

$$h = \delta r + R_o f \sin^2 \alpha \sin^2 (\emptyset_o + \theta) \tag{3.27}$$

Rewriting Eq. (3.26) in the form

$$r = \delta r + R_o \tag{3.28}$$

we now have finally the set of equations (3.23), (3.24), (3.25), (3.27) and (3.28).

Initial conditions were established on the basis of the approximate solution for the equilibrium trajectory (Appendix). A range was selected, and the appropriate initial altitude determined from the approximate analysis. Corresponding initial values of velocity and descent angle were then determined from the following approximate expressions developed in the Appendix,

$$\overline{V}^2 = \frac{1}{1 + \frac{L}{D} M_D e^{-\beta h} r} \tag{3.29}$$

$$\gamma = \frac{1}{\dfrac{1}{1 + \frac{L}{D} M_D e^{-\beta h} r} + \beta r} \left\{ \frac{2D}{L} \left(1 + \frac{L}{D} M_D e^{-\beta h} r\right) + \beta R_o f \sin^2 \alpha \sin 2(\phi_o + \theta) \right\} \quad (3.30)$$

Digital computer solutions were obtained using a Runge-Kutta method of integration. The results in terms of range are shown in Table 1 and some typical trajectories are shown in Fig. 3. It is seen that for a given inclination of the orbital plane with respect to the equatorial plane, the range for a given initial altitude and corresponding values of velocity and descent angle, as determined from Eqs. (3. 29) and (3. 30), is moderately sensitive to latitude of the injection point. The longest and shortest range occur in the case of injection at the equator and pole respectively for a polar orbit, the difference in range for these two cases being 1260 n. miles in a nominal once-around flight. The corresponding range for a spherical earth is intermediate in value, being 690 n. miles less than the range for the oblate earth with injection at the equator and 570 n. miles greater than the range for the oblate earth with injection at a pole.

Oblateness effects decrease markedly with decrease in range, and for a nominal range of 180^{o} the effect on range is only about one-tenth as large as in the case of a nominal range of 360^{o}.

Fig. 3 shows that the application of initial conditions based on the approximate equilibrium trajectory results in a non-oscillatory trajectory in the case of a spherical earth, but introduces an oscillatory component into the oblate earth trajectories. The program of computations was not carried far enough to indicate whether an equilibrium trajectory, in the sense of being non-oscillatory, does in fact exist in the case of the oblate-earth for constant lift and drag coefficients as assumed here.

TABLE I

EFFECT OF OBLATENESS ON RANGE

(Gravity Oblateness Effect Neglected)

Earth Shape	Orbit Incl., α	Initial Conditions $\overline{V}^2$	h - ft	γ - rad	Injection Lat., ϕ_o	Range deg.	Range Differential* n. mi.
Nominal Range = 360°							
Spherical	-	.98483	329,764	.000698	-	369.0	-
Oblate	90°	.98483	"	.000698	0°	380.5	+690
"	90°	.98488	"	.000700	90°	359.5	-570
"	90°	.98486	"	.003796	45°	363.0	-360
"	90°	.98486	"	-.002398	-45°	376.0	+420
"	45°	.98483	"	.000698	0°	374.5	+330
Nominal Range = 180°							
Spherical	-	.87685	277,759	.000836	-	183.0	-
Oblate	90°	.87685	"	.000836	0°	184.2	+72
"	90°	.87721	"	.000838	90°	182.0	-60

*Departure from corresponding spherical earth range

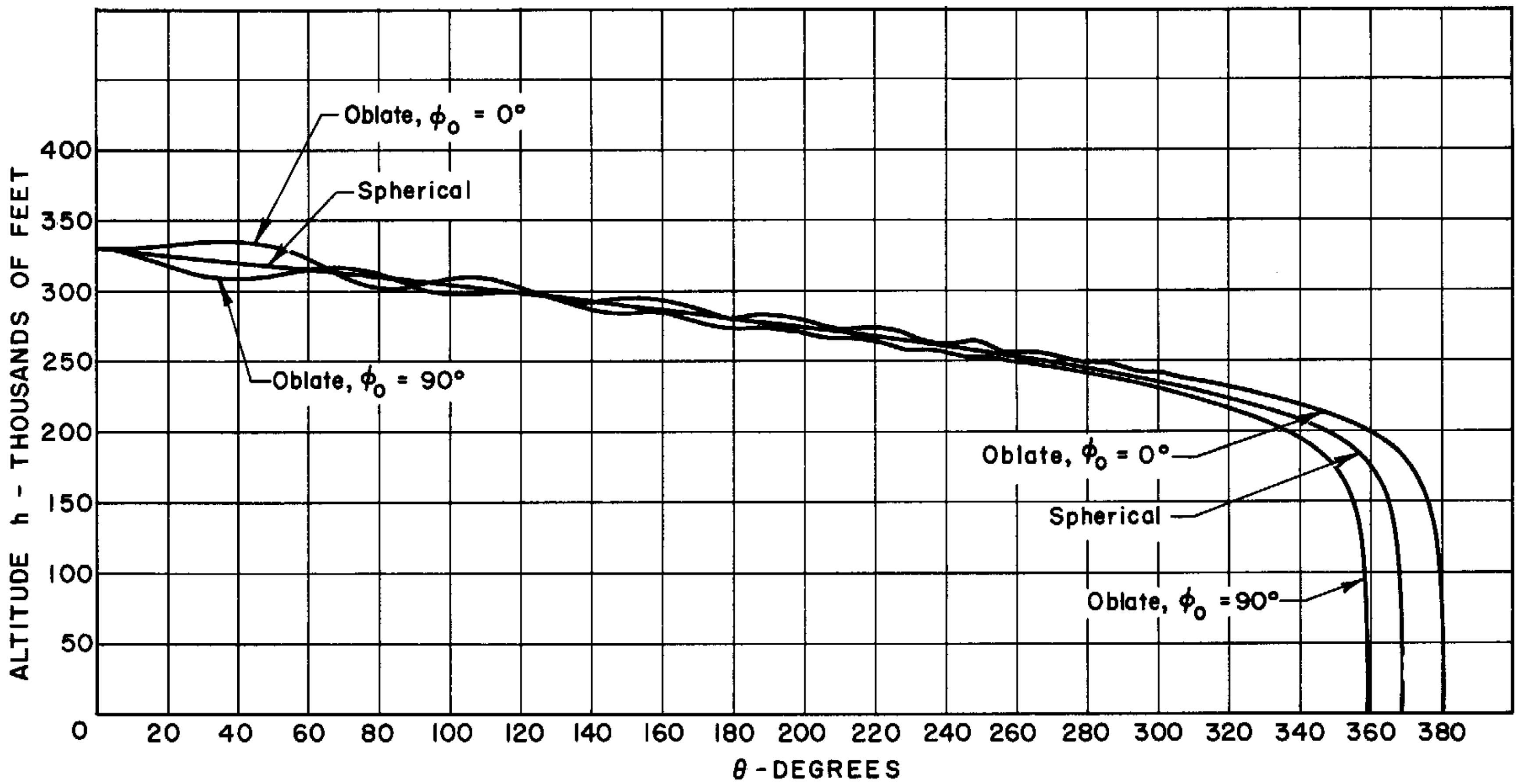

Fig. 3 - Comparison of Spherical and Oblate Earth Trajectories for a Once-Around Polar Orbit

IV. Frequencies of the Characteristic Oscillatory Motions of a Lifting Reentry Vehicle

Lifting reentry vehicles, when disturbed from an equilibrium flight path, have been shown analytically to exhibit characteristic oscillatory motions which may be identified with those of a conventional airplane. That is, there is a characteristic long-period, or phugoid, motion in which the velocity, pitch angle and altitude vary periodically, while the angle of attack remains essentially constant, and there is a characteristic short-period motion in which the angle of attack and pitch angle vary periodically, while the velocity remains essentially constant.

A significant difference in the case of the reentry vehicle is the fact that the period of the so-called "short-period" motion increases greatly at altitudes approaching orbital conditions and may exceed that of the long-period motion, under certain circumstances even degenerating into a simple divergence at sufficiently high altitude (Ref. 6).

Different approaches have been used in the analysis of these motions. In Ref. 7 the long-period motion and in Ref. 8 the short-period motion are analyzed on the basis of small disturbances from the equilibrium, or non-oscillatory, reentry flight path. This leads to linear differential equations with time-varying coefficients, since conditions along the equilibrium flight path vary.

In Ref. 6, Etkin considers small disturbances from the steady orbital flight of a lifting self-propelled vehicle in a circular orbit. In such a flight the velocity of the vehicle will be less than true orbital velocity because of the presence of lift, and sufficient thrust must be provided to overcome drag. While this assumed flight is substantially different from the reentry flight of an unpowered glide vehicle, it is reasonable to expect that conditions at corresponding altitudes in the two cases will be sufficiently similar that at least the frequencies, if not the damping characteristics, of the oscillatory motions will be similar in the two cases. The principal advantage of this type of approach is the fact that it represents a simple extension of the classical methods used in airplane stability analysis.

Returning to the analysis of Ref. 7, where small disturbances in velocity, glide angle and altitude from an approximate analytically-determined equilibrium flight path, as discussed in Section 3, are considered, it is found that the frequency of the characteristic oscillatory motion, which in this case is the long-period motion, is given approximately by

$$\omega = \sqrt{\beta g} \quad \text{sech } \bar{t} \tag{4.1}$$

where

$$\bar{t} = \frac{D}{L}\sqrt{\frac{g}{r}}\,(t_c - t) \tag{4.2}$$

t_c is a constant of integration which may be determined by substituting Eq. (4.2) into the following relation applicable to the equilibrium trajectory,

$$\beta h = \ln \left(\frac{L}{D} M_D \, r \sinh^2 \bar{t}\right) \tag{4.3}$$

and introducing initial values of the variable quantities. This yields,

$$t_c = \frac{L}{D} \sqrt{\frac{r}{g}} \quad \sinh^{-1} \left(\frac{L}{D} \, M_D \, e^{-\beta h} \, r\right)^{-\frac{1}{2}} \tag{4.4}$$

in which the values of h, r and g are initial values.

Alternatively, by substitution of Eq. (4.3) into Eq. (4.1), and as a further approximation setting $r = R_o$, the period T can be expressed in terms of h as follows,

$$T = 2\pi \sqrt{\frac{1 + \frac{L}{D} M_D \, e^{-\beta h} \, R_o}{\frac{L}{D} M_D \, e^{-\beta h} \, R_o \, \beta \, g_o}} \tag{4.5}$$

For the parametric values used in the example computations of Section 3, that is, $L/D = 3$, $M_D = .0003$, we obtain the curve of period versus altitude shown in Fig. 4. From Eq. (4.5) it can be seen that the period decreases continuously as altitude decreases. Furthermore, it increases indefinitely as the air density approaches zero. This limiting value is not realistic and is a consequence of approximations introduced into the analysis. In a more precise analysis it would be found that the period approaches that of a single orbit, since the effect of a disturbance is to change the eccentricity of the orbit. In fact, when oblateness gravity effects are included, and the orbit is not equatorial, there is also an oscillation with period equal to half the orbital period, as discussed in Section 2.

Turning now to the analysis of Ref. 8, where velocity perturbations are suppressed and pitching perturbations relative to the flight path considered, it is found that the angle of attack variation may be approximated by an expression which contains the oscillatory factors $\cos \Phi(s)$ and $\sin \Phi(s)$, where

$$\Phi(s) = \int \sqrt{-\frac{\rho A \ell}{2 I_y} \, C_{m_\alpha}} \, ds \tag{4.6}$$

and s is the distance traversed along the flight path.

In the course of traversing one cycle of oscillation, the function $\Phi(s)$ will change by the amount 2π. Thus, changing to time variable t by substituting $ds = V\,dt$, we can write

$$\int_t^{t+T} \sqrt{-\frac{\rho A \ell}{2 I y} \, C_{m_\alpha}} \; V \, dt = 2\pi \tag{4.7}$$

where T is the period of the oscillation. Assuming that the integrand does not vary significantly during one cycle of oscillation, Eq. (4.7) yields finally

$$T = 2\pi \sqrt{-\frac{I_y}{q A \ell \, C_{m_\alpha}}} \tag{4.8}$$

where q is the dynamic pressure.

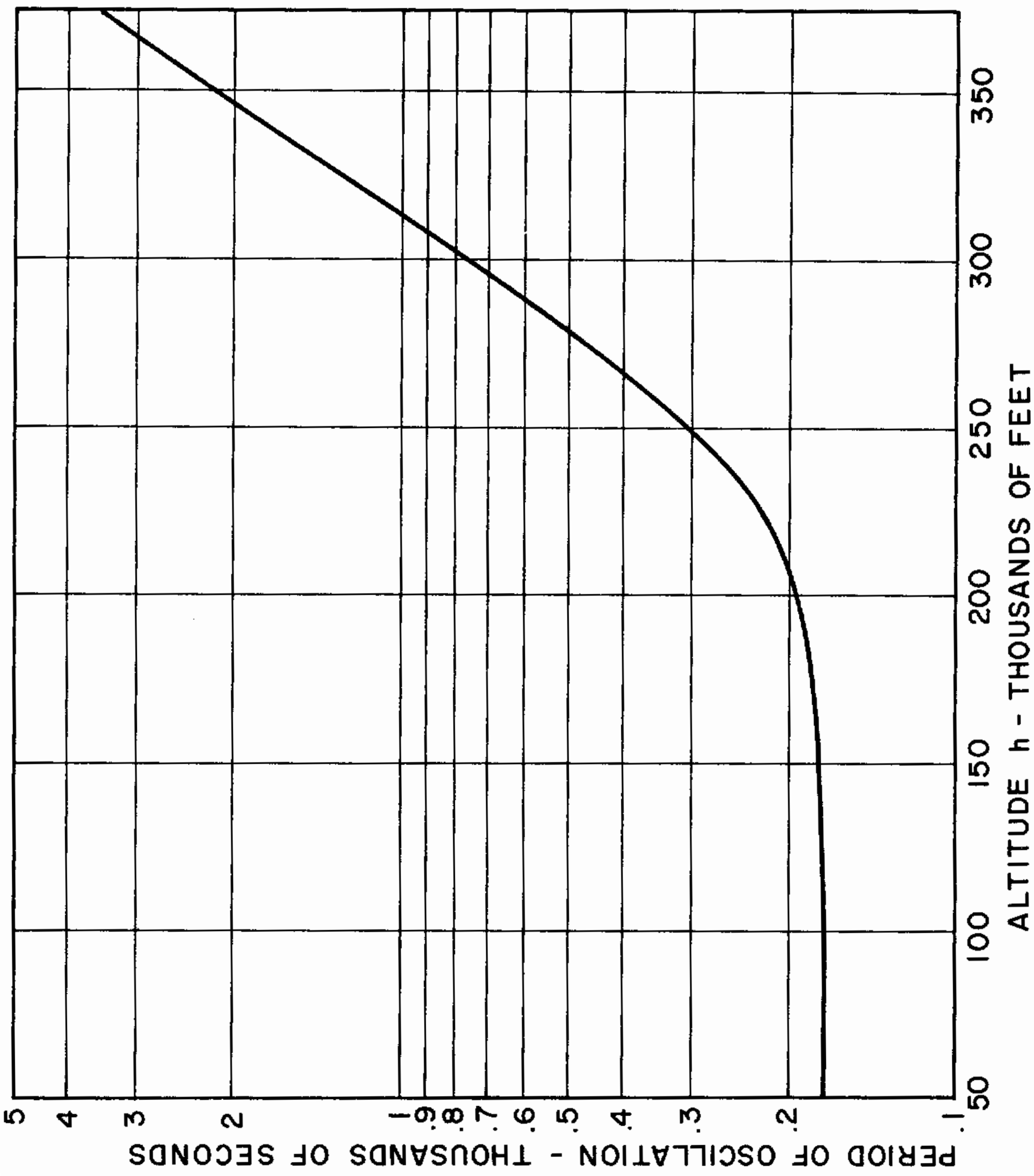

Fig. 4 - Variation of the Period of the Long-Period Motion with Altitude for $L/D = 3$, $M_D = 0.003$

It is seen that if the variation of C_{m_α} is disregarded, the period achieves a minimum value when the dynamic pressure is a maximum and increases indefinitely as the density approaches zero. This solution is identical with the classical approximate solution for an airplane when only the pitching degree of freedom is included. Significant differences occur only in the damping characteristics.

Etkin's results (Ref. 6) indicate trends which are in general consistent with those discussed above. The asymptotic behavior of the long period mode as air density approaches zero is more realistic in his analysis than that of Ref. 7, as already discussed. In the case of the short-period mode he obtains an approximate expression for the period which is similar to Eq. (4. 8) but includes an additional term in the denominator, as follows,

$$T = 2\pi \sqrt{- \frac{I_y}{q A \ell C_{m_\alpha} + M_\Theta}} \qquad (4.9)$$

The presence of the M_Θ derivative is associated with an effect of variation of gravity forces with distance from the center of the earth. Mass elements of the vehicle which are farther from the earth than others experience a smaller gravity force, and this can be shown to give rise to a pitching moment given by

$$M = - \frac{3}{2} \frac{g}{r} (I_x - I_z) \sin 2\Theta \qquad (4.10)$$

where g is the value of the gravitational acceleration at the centroid of the vehicle, Θ is the angle between the z-axis and the radius vector from the center of the earth. The x and z axes are presumed to be principal axes of the vehicle. It is seen that this moment tends to align the major principal axis with the radius vector from the center of the earth. Differentiating Eq. (4. 10) with respect to Θ,

$$M_\Theta = - 3 \frac{g}{r} (I_x - I_y) \cos 2\Theta \qquad (4.11)$$

M_Θ is small in comparison with $q A \ell C_{m_\alpha}$ except at very high altitudes. In the limiting case of zero air density, Eq. (4. 9) becomes

$$T = 2\pi \sqrt{- \frac{I_y}{M_\Theta}} \qquad (4.12)$$

When, as one would expect, $I_x < I_z$, M_Θ is negative for $\Theta < 45^o$, and the disturbed motion is a simple divergence. On the other hand, for large angles of incidence ($\Theta > 45^o$) M_Θ is positive and the disturbed motion is oscillatory with a period of the same order of magnitude as the orbital period.

V. Concluding Remarks

The information provided in the present report relative to oblateness effects, both gravity and atmospheric, on the trajectory of a glide reentry vehicle indicates that these effects, while not trivial, could be adequately compensated for by appropriate adjustment of initial conditions in a simulation based on a spherical earth. This is especially true in the case of a once-around or shorter flight. In the case of flights of much longer duration, certain of the effects of oblateness of the gravity field are cumulative and become substantial. It is, however, doubtful whether there would be a need to simulate such flights in their entirety.

The only consideration which would dictate the inclusion of oblateness effects would be the use, in the simulation, of actual flight components, such as components of the guidance system, which are designed to account for earth oblateness. On the other hand, it should be noted that the inclusion of earth oblateness effects does not complicate the flight equations greatly as seen in Parts I and II of the present report, so that it may be desirable to include them for the sake of completeness.

The information on frequency of the characteristic oscillatory motions of the vehicle during reentry indicates that the period of the "long period" motion decreases continuously as the vehicle descends and asymptotically approaches the value $2\pi/\sqrt{\beta g_0}$ or 172 seconds. This result is based on small disturbances from an equilibrium trajectory when lift and drag coefficients remain constant and would not be applicable when the vehicle approaches a landing, particularly if the vehicle is powered in that phase of its flight.

The period of the "short period" motion achieves a minimum approximately when the dynamic pressure reaches a maximum. It may then be approximated by a simple expression based on the static longitudinal stability derivative of the vehicle.

The reader will have noted that the effect of earth rotation on vehicle trajectory has not been considered in the present report. The fact that this effect can be expected to be substantial in the case of a lifting vehicle and its inclusion in a simulation introduces little complication makes it advisable that it be included. For this reason no detailed study of it was undertaken.

APPENDIX

Approximate Equilibrium Trajectory For Oblate Earth

Introducing the following simplifying assumptions,

$$\gamma \quad \text{small} \quad (\cos\gamma = 1, \ \sin\gamma = \gamma)$$

$$\dot{\gamma} = 0$$

$$g \sin\gamma \ll \dot{V}$$

Eqs. (3. 2) to (3. 5) inclusive become

$$\frac{L}{m} + \frac{V^2}{r} = g \tag{A1}$$

$$\dot{V} = -\frac{D}{m} = -\frac{D}{L}\left(\frac{L}{D} M_D e^{-\beta h} V^2\right) \tag{A2}$$

$$\dot{r} = -V\gamma \tag{A3}$$

$$\dot{\theta} = \frac{V}{r} \tag{A4}$$

Substituting

$$\frac{L}{m} = \frac{L}{D} M_D e^{-\beta h} V^2$$

into Eq. (A1), we obtain

$$\frac{L}{D} M_D e^{-\beta h} V^2 + \frac{V^2}{r} = g \tag{A5}$$

or,

$$\overline{V}^2 = \frac{1}{1 + \frac{L}{D} M_D e^{-\beta h} r} \tag{A6}$$

This expression is seen not to depend explicitly on oblateness. However, there is a small implicit dependence on oblateness through the effect of oblateness on r for a given value of h.

Turning now to the descent angle, we differentiate Eq. (A1) with respect to t, neglecting the small variation in g with r,

$$\frac{L}{D} M_D\left(-\beta \dot{h} e^{-\beta h} V^2 + 2 e^{-\beta h} V \dot{V}\right) + \frac{2 V \dot{V} r - V^2 \dot{r}}{r^2} = 0 \tag{A7}$$

From Eq. (3. 13),

$$\dot{h} = \dot{r} + R_o f \sin^2\alpha \sin 2(\phi_o + \theta) \cdot \dot{\theta} \tag{A8}$$

or, substituting Eq. (A4),

$$\dot{h} = \dot{r} + R_o f \sin^2 \alpha \sin 2 (\phi_o + \theta) \cdot \frac{V}{r} \tag{A9}$$

Substituting Eqs. (A2) and (A9) into Eq. (A7) and solving for $\dot{r}$, we obtain

$$\dot{r} = \frac{V}{1 + \frac{L}{D} M_D e^{-\beta h} r (\beta r)} \left\{ - 2 \frac{D}{L} (\frac{L}{D} M_D e^{-\beta h} r) (\frac{L}{D} M_D e^{-\beta h} r + 1) - \frac{L}{D} M_D e^{-\beta h} r \beta R_o f \sin^2 \alpha \sin 2 (\phi_o + \theta) \right\} \tag{A10}$$

The angle of descent is now obtained by substituting Eq. (A10) into Eq. (A3),

$$\gamma = \frac{\dot{r}}{V}$$

$$= \frac{1}{\frac{1}{\frac{L}{D} M_D e^{-\beta h} r} + \beta r} \left\{ 2 \frac{D}{L} (\frac{L}{D} M_D e^{-\beta h} r + 1) + \beta R_o f \sin^2 \alpha \sin 2 (\phi_o + \theta) \right\} \tag{A11}$$

Except at very high altitudes, $\beta r >> (\frac{L}{D} M_D e^{-\beta h} r)^{-1}$, so that Eq. (A11) may be simplified to the form,

$$\gamma = 2 \frac{D}{L} (\frac{L}{D} M_D e^{-\beta h} r + 1)/\beta r + \frac{R_o}{r} f \sin^2 \alpha \sin 2 (\phi_o + \theta) \tag{A12}$$

in which the first term corresponds to the solution for a spherical earth, while the second term is very nearly equal to the angle between the local tangent to the earth's surface in the plane of the trajectory and the local horizontal, defined as the plane normal to the radius vector from the earth's center.

According to this result, the descent angle of the vehicle relative to a tangent to the earth's surface in the plane of the trajectory is virtually the same in the case of the oblate earth as it is in the case of the spherical earth. This indicates that the altitude variation along the flight path will be essentially the same in the two cases, since the velocity variation is also essentially the same in the two cases. Thus, on the basis of the present analysis the range corresponding to given initial conditions will not be appreciably influenced by oblateness. This range can be shown (Ref. 3) to be given by the expression,

$$\overline{d} = \frac{1}{2} \frac{L}{D} \ln \frac{1 - \overline{V}_F^2}{1 - \overline{V}_I^2} \tag{A13}$$

where $\overline{d}$ is in terms of the angular coordinate θ, and $\overline{V}_I$ and $\overline{V}_F$ are respectively the initial and final values of $\overline{V}$.

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