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SECTION 4

STABILITY ANALYSIS OF A SIMPLIFIED PITCH-CONTROLLED FLEXIBLE AERODYNAMIC VEHICLE VIA LYAPUNOV'S DIRECT METHOD

by

P. K. C. Wang

IBM San Jose Research Laboratory
San Jose, California

ABSTRACT: Lyapunov's direct method is used to derive a sufficient condition for asymptotic stability of equilibrium of a simplified flexible vehicle with a pitch autopilot. The basic mathematical model corresponds to that of a non-uniform cantilever beam in plane bending along with a rigid body rotation. It is assumed that the autopilot is composed of a simple proportional plus rate controller. The analytical method used here has the advantage of dealing directly with the system's partial differential equations without resorting to approximations.

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4.1 INTRODUCTION

Perhaps one of the most common approaches to the stability analysis of aeroelastic systems is the modal approach in which the system's motion is assumed to be adequately described by a finite number of modes. No doubt, this approach has led to useful results for numerous problems of physical important; however, because of the approximations involved, it is conceivable that in certain cases a stability analysis based on a finite number of modes does not reflect any information on the behavior of all the remaining modes. Therefore, it is desirable to have methods for stability analysis which deal directly with the full system equations without introducing any approximations.

One of the most useful methods for the stability analysis of dynamical systems having a finite degree of freedom is the direct method of Lyapunov. Recently, the application of this method to stability problems associated with elastic [1,2] and aeroelastic[3] systems has been investigated. Although the successful application of this method to a particular problem usually requires a certain amount of effort and ingenuity on the part of the analyst, the resulting conditions will at least be sufficient to guarantee the stability of equilibrium of the mathematical model of the system under consideration.

Here, we shall apply Lyapunov's direct method to derive a sufficient condition for asymptotic stability of equilibrium of a simplified flexible vehicle with a pitch autopilot.

4.2 MATHEMATICAL MODEL

Consider a simplified pitch controlled flexible aerodynamic vehicle as shown in Figure 4-1. To derive a mathematical model for this vehicle, the following main assumptions are made:

4.2.1 The vehicle's motion is restricted to a plane, and its center of mass moves with a uniform horizontal velocity v_o.

4.2.2 The main lift coincides with the vertical axis passing through the center of mass, and the aerodynamic moment acting on the main lifting surface is negligible.*

4.2.3 The tail portion of the vehicle is flexible. Its motion corresponds approximately to that of a non-uniform cantilever beam in plane bending. The fixed end of the beam is considered to be coinciding with the center of mass as shown in Figure 4-1. Moreover, the deflection is small so that the shift in the center of mass is negligible.

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* The addition of aerodynamic moment, due to the main lifting surfaces, does not introduce any mathematical difficulties in the present analysis.

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4.2.4 The control surface is taken to be a flat plate attached to the tail, whose angle-of-attack can be varied. The aerodynamic moment acting on the control surface is negligible.

Further assumptions will be mentioned in the ensuing discussion.

Let the deflection of the tail section be defined with respect to a rotating Cartesian coordinate system with its origin at the center of mass and one of its axis coinciding with the undeformed elastic axis of the tail section. Assuming that the apparent forces due to coordinate rotation are small, the linearized dimensionless equation of motion for the flexible tail portion of the vehicle is given by:

$$m(x)v_o^2 \ell^2 \frac{\partial^2 w(t,x)}{\partial t^2} + v_o \ell^3 k_d(t,x) \frac{\partial w(t,x)}{\partial t} = - \frac{\partial^2}{\partial x^2} EI(x) \frac{\partial^2 w(t,x)}{\partial x^2} \tag{4.2-1}$$

where both the deflection w and the spatial coordinate x have been normalized with respect to the length of the tail section ℓ ; t is the dimensionless time normalized with respect to the quantity ℓ/v_o ; m, k_d , and EI are linear mass density, distributed damping coefficient, and bending rigidity, respectively.

Let the angle between the control surface and the natural axis of the tail end at $x = 1$ be denoted by δ_c. Using the simplified expression for the aerodynamic force acting on a thin flat plate in an incompressible flow,[4] and assuming the aerodynamic loading due to vertical translational motion of the vehicle is negligible, the linearized boundary conditions have the form:

$$w(t,0) = \left. \frac{\partial w(t,x)}{\partial x} \right|_{x=0} = 0 \tag{4.2-2}$$

and

$$\left. EI(x) \frac{\partial^2 w(t,x)}{\partial x^2} \right|_{x=1} = 0 \quad ,$$

$$\left. \frac{\partial}{\partial x} EI(x) \frac{\partial^2 w(t,x)}{\partial x^2} \right|_{x=1} =$$

$$- 2\pi \rho_a v_o^2 \ell^2 b a \left[\dot{\theta}(t) - \left. \frac{\partial w(t,x)}{\partial t} \right|_{x=1} + \theta(t) - \left. \frac{\partial w(t,x)}{\partial x} \right|_{x=1} + \delta_c \right] \tag{4.2-3}$$

where ρ_a is the mass density of the undisturbed air, a and $2b$ are the total length and width of the control surface, respectively, and $\dot{\theta}(t) = d\theta(t)/dt$.

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The pitching motion of the vehicle about its center of mass can be described by an ordinary differential equation of the form:

$$I_o \frac{d\dot{\theta}(t)}{dt} = 2\pi \rho_a ba \ell^3 \left[\dot{\theta}(t) - \frac{\partial w(t,x)}{\partial t}\bigg|_{x=1} + \theta(t) - \frac{\partial w(t,x)}{\partial x}\bigg|_{x=1} + \delta_c \right] \tag{4.2-4}$$

where the pitch angle θ is referenced with respect to the fixed horizontal axis, and I_o is the moment of inertia of the vehicle about its center of mass.

The simplified flexible vehicle considered here is basically a mixed distributed and lumped parameter system whose dynamical behavior is describable by a partial differential equation which is coupled with an ordinary differential equation. The state of the system at any time t can be specified by $\theta(t)$, $\dot{\theta}(t)$ and the functions $w(t,x)$ and $\partial w(t,x)/\partial t$ defined for all $x \in (0, 1)$. Hence, the state space Γ is the Cartesian product of a function space and a finite dimensional vector space.

4.3 STABILITY ANALYSIS

The problem of analytical design of an optimum pitch controller for the simplified flexible aerodynamic vehicle on the basis of the mathematical model given by Eq. (4.2-1) - (4.2-4) and the minimization of certain performance index is by no means a simple one. This problem is further complicated by the fact that the state variables describing the flexible motion of the vehicle cannot be accurately measured in practice. An ideal way to simplify the design problem is to reference the control surface angle with respect to the velocity vector v_o so that the effect of flexibility of the tail section on the control surface can be completely avoided. However, this approach may not be feasible from the physical standpoint.

Another possible approach is to design the pitch controller with the assumption that the vehicle is a rigid body. However, this approach does not guarantee that the equilibrium state of the controlled vehicle will be asymptotically stable under the influence of flexibility of the tail section. Here, we shall be concerned only with the problem of establishing conditions for asymptotic stability of equilibrium of the flexible vehicle with a specified control law.

First, consider the case where the vehicle is rigid. The equation for the pitching motion of the rigid vehicle is simply:

$$I_o \frac{d\dot{\theta}(t)}{dt} = -2\pi \rho_a ba \ell^3 \left[\theta(t) + \dot{\theta}(t) + \delta_c \right] \tag{4.3-1}$$

Clearly, the equilibrium state $(\theta = \dot{\theta} = 0)$ of the uncontrolled vehicle $(\delta_c = 0)$ is asymptotically stable. To alter the dynamic behavior of the uncontrolled vehicle, one might introduce a simple-minded proportional plus rate pitch controller, i.e., the control law has the form:

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$$\delta_c = k_1\theta + k_2 v_o \ell^{-1}\dot{\theta} \tag{4.3-2}$$

Obviously, in order to ensure asymptotic stability, the parameters k_1 and k_2 must be selected so that the following inequalities are satisfied:

$$(k_1 + 1) > 0 \quad ; \quad (k_2 v_o \ell^{-1} + 1) > 0 \tag{4.3-3}$$

Now, suppose one uses the same simple-minded pitch controller for a flexible vehicle, what conditions must k_1 and k_2 satisfy in order to ensure asymptotic stability? Note that in spite of the linearity of the system equations, the above question cannot be readily answered by conventional analysis. In the sequel, we shall derive a sufficient condition for asymptotic stability of equilibrium of the controlled flexible vehicle with control law (4.3-2) via Lyapunov's direct method.

Let us define asymptotic stability in the sense of Lyapunov with respect to a metric ρ given by:

$$\rho(S_t, 0) = \left\{ \theta^2(t) + \dot{\theta}^2(t) + \int_0^1 \left[\left(\frac{\partial w(t,x)}{\partial t} \right)^2 + \sum_{n=0}^{2} \left(\frac{\partial^n w(t,x)}{\partial x^n} \right)^2 \right] dx \right\}^{1/2} \tag{4.3-4}$$

In essence, $\rho(S_t, 0)$ establishes a measure for the closeness of the system state S_t from the equilibrium null state in terms of the pitch angle and velocity, the tail section's velocity distribution, the tail section's displacement distribution and its first two spatial derivatives.

Fundamental to Lyapunov's direct method is selecting a functional V which will give some estimate of $\rho(S_t, 0)$ at any time t. If it is possible to show that V evaluated along any perturbed system trajectory is small whenever $\rho(S_0, 0)$ is small and $V \to 0$ as $t \to +\infty$, then the equilibrium null state is asymptotically stable.

For the present system, we shall consider the following functional:

$$V = 1/2 \left\{ v_o^2 I_o \ell^{-1} \left(c_o \theta^2 + 2c_1 \theta \dot{\theta} + \dot{\theta}^2 \right) + \int_0^1 \left[m(x) v_o^2 \ell^2 \left(\frac{\partial w}{\partial t} \right)^2 + EI(x) \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2 c_2 m(x) v_o^2 \ell_o^2 \left(\frac{\partial w}{\partial t} \right) \left(\frac{\partial w}{\partial x} \right) \right] dx \right\} \tag{4.3-5}$$

where c_o, c_1, and c_2 are positive constants. It should be pointed out that due to the presence of boundary conditions, the selection of a useful form for V is not a straightforward task.

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From Lyapunov's stability theory,[3] a sufficient condition for asymptotic stability is that V is positive definite with respect to metric ρ and $dV/dt < 0$ along any perturbed motion.

To establish the positive-definiteness of V, we make use of the following inequalities:

$$\int_0^1 G(x)\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \geq \left[\underset{x \in (0,1)}{\text{Min}} G(x)\right] \int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 dx$$

$$\geq \left[\underset{x \in (0,1)}{\text{Min}} G(x)\right] \int_0^1 w^2 \, dx \qquad (4.3\text{-}6)$$

where w is a twice differentiable function of x defined on $(0,1)$, which satisfies boundary condition (4.2-2), and G is a positive function of x. These inequalities can be proved readily by integrating by parts and applying Schwarz inequality.

In view of inequality (4.3-6), V can be bounded below by:

$$V \geq 1/2 \left\{ v_o^2 I_o \ell^{-1} \left(c_o \theta^2 + 2 c_1 \theta \dot{\theta} + \dot{\theta}^2\right) + \int_0^1 \left[m(x) v_o^2 \ell^2 \left(\frac{\partial w}{\partial t}\right)^2 + 2c_2 m(x) v_o^2 \ell^2 \left(\frac{\partial w}{\partial t}\right)\left(\frac{\partial w}{\partial x}\right) + \left[\underset{x \in (1,0)}{\text{Min}} EI(x)\right]\left[\kappa_1 \left(\frac{\partial w}{\partial x}\right)^2 + \kappa_2 w^2\right] + (1 - \kappa_1 - \kappa_2) \, EI(x) \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \right] \right\} dx \qquad (4.3\text{-}7)$$

where κ_1 and κ_2 are positive constants satisfying $\kappa_1 + \kappa_2 < 1$. Clearly, V will be positive definite with respect to ρ, if the constants c_i, $i = 0,1,2$, are selected so that the following inequalities are satisfied:

$$\left.\begin{aligned} & c_o > c_1 , \\ & \kappa_1 \underset{x \in (0,1)}{\text{Min}} EI(x) > c_2 m(x) v_o^2 \ell^2 \end{aligned}\right\} \qquad (4.3\text{-}8)$$

The derivative of V with respect to t, after integrating by parts, can be expressed in the form:

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$$\frac{dV}{dt} = \int_0^1 \left[m(x)\, v_o^2 \ell^2 \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2} EI(x) \frac{\partial^2 w}{\partial x^2} \right] \frac{\partial w}{\partial t} \, dx$$

$$+ \int_0^1 \left[c_2 m(x)\, v_o^2 \ell^2 \frac{\partial^2 w}{\partial t^2} \frac{\partial w}{\partial x} - \frac{1}{2} c_2 v_o^2 \ell^2 \frac{dm(x)}{dx} \left(\frac{\partial w}{\partial t}\right)^2 \right] dx$$

$$+ v_o^2 I_o \ell^{-1} \left[c_o \theta \dot{\theta} + (\dot{\theta} + c_1 \theta) \frac{d\dot{\theta}}{dt} + c_1 \dot{\theta}^2 \right]$$

$$+ \left[EI(x) \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial x \partial t} - \frac{\partial}{\partial x} EI(x) \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial t} + \frac{1}{2} c_2 m(x) v_o^2 \ell^2 \left(\frac{\partial w}{\partial t}\right)^2 \right]\Bigg|_0^1$$

(4.3-9)

The above equation, in view of system equations (4.2-1), (4.2-4), (4.3-2), and boundary conditions (4.2-2) and (4.2-3), can be rewritten as:

$$\frac{dV}{dt} = - \int_0^1 \left[k_d(t,x) v_o \ell^3 + 1/2\, c_2 v_o^2 \ell^2 \frac{dm(x)}{dx} \right] \left(\frac{\partial w}{\partial t}\right)^2 dx + \int_0^1 c_2 m(x) v_o^2 \ell^2 \frac{\partial^2 w}{\partial t^2} \frac{\partial w}{\partial x} \, dx$$

$$+ \mu \Bigg\{ \left(\frac{\partial w}{\partial t}\Big|_{x=1} \right) \left[\left(\frac{1}{4} m(1) \alpha c_2 - 1 \right) \left(\frac{\partial w}{\partial t}\Big|_{x=1} \right) + (\gamma + c_1)\theta + (\beta + 1)\dot{\theta} - \frac{\partial w}{\partial x}\Big|_{x=1} \right]$$

$$+ (\dot{\theta} + c_1 \theta) \frac{\partial w}{\partial x}\Big|_{x=1} - c_1 \gamma \theta^2 \Bigg\} + (c_1 v_o^2 I_o \ell^{-1} - \mu\beta)\dot{\theta}^2 + (c_o v_o^2 I_o \ell^{-1} - \mu(\gamma + \beta c_1))\theta\dot{\theta}$$

(4.3-10)

where

$$\left.\begin{aligned} \alpha &= (\pi \rho_a b\, a)^{-1}, & \beta &= (1 + k_2 v_o \ell^{-1}), \\ \gamma &= (1 + k_1), & \mu &= 2\pi \rho_a v_o^2 \ell^2 ab\,. \end{aligned}\right\} \quad (4.3\text{-}11)$$

Using (4.2-1) and (4.2-2) and integrating by parts, the second integral in (4.3-10) can be expressed in the form:

$$\int_0^1 c_2 m(x) v_o^2 \ell^2 \frac{\partial^2 w}{\partial t^2} \frac{\partial w}{\partial x} \, dx =$$

$$- c_2 \int_0^1 \left[k_d(t,x) v_o \ell^3 \frac{\partial w}{\partial t} \frac{\partial w}{\partial x} - 1/2 \frac{d\, EI(x)}{dx} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] dx$$

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$$+ \mu c_2 \left(\frac{\partial w}{\partial x} \bigg|_{x=1} \right) \left[\beta \, \dot{\theta} + \gamma \theta - \left(\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} \right)_{x=1} \right]$$

$$- \frac{1}{2} c_2 EI(0) \left(\frac{\partial^2 w}{\partial x^2} \bigg|_{x=0} \right)^2 \qquad (4.3\text{-}12)$$

Substituting (4.3-12) into (4.3-10) and assuming that

$$\frac{dEI(x)}{dx} \overset{\bullet}{<} 0 \quad \text{for all } x \in (0,1) \quad , \qquad (4.3\text{-}13)$$

the following upper bound for dV/dt can be obtained by making use of inequality (4.3-6):

$$\frac{dV}{dt} \leq - \int_0^1 Y'(t,x)\, A(t,x)\, Y(t,x) dx - Z'(t)\, BZ(t) - \frac{1}{2} c_2 EI(0) \left(\frac{\partial^2 w}{\partial x^2} \bigg|_{x=0} \right)^2$$

(4.3-14)

where ( )' denotes transpose and

$$Y = \text{Col} \left[\frac{\partial w}{\partial t} \, , \; \frac{\partial w}{\partial x} \right]$$

$$Z = \text{Col} \left[\theta \, , \; \dot{\theta} \, , \; \frac{\partial w}{\partial t} \bigg|_{x=1} , \; \frac{\partial w}{\partial x} \bigg|_{x=1} \right]$$

$$A(t,x) = \begin{bmatrix} k_d(t,x) v_o \ell^3 + \frac{1}{2} c_2 v_o^2 \ell^2 \frac{dm(x)}{dx} & \frac{1}{2} k_d(t,x)\, v_o \ell^3 c_2 \\ \frac{1}{2} k_d(t,x)\, v_o \ell^3 c_2 & \frac{1}{2} c_2 \, \underset{x}{\text{Min}} \left| \frac{dEI(x)}{dx} \right| \end{bmatrix}$$

$$B = 1/2\mu \begin{bmatrix} 2\gamma c_1 & -\frac{1}{2} c_o I_o \ell^{-3} \alpha + (\gamma + \beta c_1) & -(c_1 + \gamma) & -(\gamma c_2 + c_1) \\ -\frac{1}{2} c_o I_o \ell^{-3} \alpha + (\gamma + \beta c_1) & 2\beta - c_1 I_o \ell^{-3} \alpha & -(1+\beta) & -(1 + \beta c_2) \\ -(c_1 + \gamma) & -(1+\beta) & 2 - \frac{1}{2} c_2 m(1) \alpha & (1 + c_2) \\ -(\gamma c_2 + c_1) & -(1 + \beta c_2) & (1 + c_2) & 2 c_2 \end{bmatrix}$$

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Clearly, dV/dt will be < 0 if both A and B are positive definite or the following inequalities are satisfied:

$$\gamma > 0 \tag{4.3-15}$$

$$2 c_1 \gamma \zeta > \eta^2 \tag{4.3-16}$$

$$2\eta (1+\beta)(c_1+\gamma) > \zeta [m(1)\alpha\gamma c_1 c_2 + (c_1-\gamma)^2] + 2\gamma c_1 (1+\beta)^2 + \eta^2 (2 - \tfrac{1}{2} c_2 m(1)\alpha) \tag{4.3-17}$$

$$\begin{aligned} &\zeta [\tfrac{1}{2} m(1)\alpha c_2 (c_1 - \gamma c_2)^2 - c_1 c_2 (c_1 + 2\gamma)] \\ &+ 4\gamma c_1 [\beta - (\beta - \tfrac{1}{4} m(1)\alpha) c_2 + \beta(1+\beta) c_2^2] \\ &+ \eta^2 [c_2^2 m(1)\alpha + (1-c_2)^2] \\ &- 2\eta [(c_1-1)^2 (c_1\beta + \gamma) + \tfrac{1}{2} m(1)\alpha c_2 (1+\beta c_2)(c_2\gamma + c_1)] \\ &> (c_2-1)^2 (\beta c_1 - \gamma)^2 \end{aligned} \tag{4.3-18}$$

$$2 \left(\operatorname*{Min}_x \left| \frac{dEI(x)}{dx} \right| \right) \left(k_d(t,x) \ell + \frac{1}{2} c_2 v_o \frac{dm(x)}{dx} \right) > c_2 v_o \ell^4 k_d^2 (t,x) > 0$$

$$\text{for all } t > t_o \text{ and all } x \in (0,1) \tag{4.3-19}$$

where

$$\left.\begin{aligned} \eta &= (\gamma + \beta c_1) - \tfrac{1}{2} c_o I_o \ell^{-3} \alpha \\ \zeta &= 2\beta - c_1 I_o \ell^{-3} \alpha \end{aligned}\right\} \tag{4.3-20}$$

The above inequalities are reduced from those of Sylvester. By selecting a set of constants c_o, c_1, and c_2 which satisfy (4.3-8), the inequalities (4.3-13), (4.3-15) - (4.3-20) constitute a sufficient condition for asymptotic stability. Note that (4.3-15) is identical to the first inequality in (4.3-3), and (4.3-16) implies that $\zeta > 0$ or

$$(1 + k_2 v_o \ell^{-1}) > \frac{1}{2} c_1 I_o \ell^{-3} \alpha > 0 \tag{4.3-21}$$

Hence, the region represented by inequalities (4.3-15) - (4.3-18) in the $k_1 - k_2$ parameter plane is smaller than that represented by (4.3-3) as expected.

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Now, consider the case where the control law δ_c has a form different from (4.3-2). Using the same functional V given by (4.3-5), it can be readily deduced that dV/dt can be bounded by:

$$\frac{dV}{dt} \leq - \int_0^1 Y'(t,x)\ A(t,x)\ Y(t,x)\ dx - Z'(t)\ \hat{B}\ Z(t)$$
$$- \frac{1}{2}\ c_2 EI(0) \left(\frac{\partial^2 w}{\partial x^2} \bigg|_{x=0} \right)^2$$
$$- \mu \left(\dot{\theta} + c_1 \theta - \frac{\partial w}{\partial t} \bigg|_{x=1} - c_2 \frac{\partial w}{\partial x} \bigg|_{x=1} \right) \delta_c \qquad (4.3\text{-}22)$$

where $\hat{B}$ corresponds to B with $\beta = \gamma = 1$. For the case where δ_c is a linear function of θ, $\dot{\theta}$, $\frac{\partial w}{\partial t}\big|_{x=1}$ and $\frac{\partial w}{\partial x}\big|_{x=1}$, a sufficient condition for asymptotic stability can be derived in a straightforward manner.

4.4 CONCLUSIONS

The direct method of Lyapunov has been used to derive a sufficient condition for asymptotic stability of equilibrium of a simplified pitch-controlled flexible aerodynamic vehicle. Due to the fact that the analysis makes use of a priori estimates rather than detailed knowledge of the solutions, the resulting condition may not give sharp estimates of the asymptotically stable region in the parameter space of interest. However, the analysis does illustrate how Lyapunov's direct method may be applied to an aeroelastic stability problem whose solution is not readily obtainable by conventional analytical methods without resorting to approximations.

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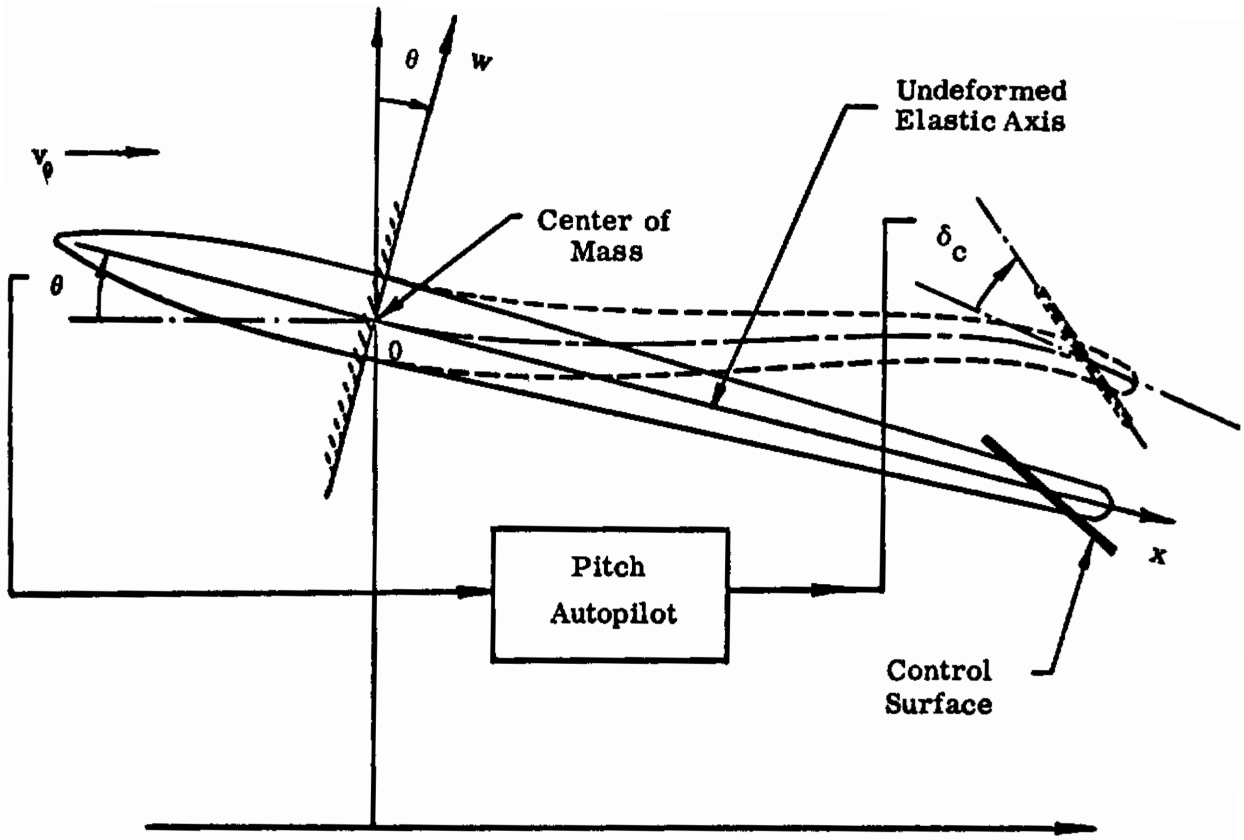

Figure 4.1

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