AFFDL-TR-66-215

# APPLICATION OF THE SUBDOMAIN (BIEZENO-KOCH) METHOD TO CIRCULAR PLATE BENDING

*O. E. ADAMS JR.*



This document is subject to special export controls and each transmittal to foreign governments or foreign nationals may be made only with prior approval of the Theoretical Mechanics Branch, Structures Division, Air Force Flight Dynamics Laboratory, Wright-Patterson AFB, Ohio 45433.

# FOREWORD

This report was prepared in-house under Project 1467, "Structural Analysis Methods," Task 146702, "Thermoelastic Structural Analysis Methods." The work was conducted by the Theoretical Mechanics Branch, Structures Division, Air Force Flight Dynamics Laboratory, with Mr. Robert M. Bader acting as Project Engineer.

Manuscript released by the author September 1966 for publication as an RTD Technical Report. This report covers work conducted from June 1966 to September 1966.

The author wishes to thank Mr. Gene E. Maddux of the Air Force Flight Dynamics Laboratory for his suggestions and comments concerning the various techniques of numerical solution and the interpretation of results. He is grateful to Mr. Robert M. Engle for his assistance in generating the computer programs.

This technical report has been reviewed and is approved.

FRANCIS J. JANIK, Jr.
Chief, Theoretical Mechanics Branch
Structures Division
AF Flight Dynamics Laboratory

ABSTRACT

The subdomain or Biezeno-Koch method is employed to obtain approximate solutions to the bending of a uniformly loaded, simply supported circular plate. The details of arriving at a trial function are discussed and the feasibility of automating this method by means of the digital computer is demonstrated. The effect of varying the limits of integration for the residual integral, including a weighting function, and employing a least squares solution technique are investigated.

This abstract is subject to special export controls and each transmittal to foreign governments or foreign nationals may be made only with prior approval of the Theoretical Mechanics Branch, Structures Division, Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio 45433.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS (Cont'd)

LIST OF ILLUSTRATIONS (Cont'd)

## LIST OF SYMBOLS

| | |
|---|---|
| $A_o$, $A_k$ | Undertermined parameters (k = 1, 2, . . . n) |
| a | Radius of circular plate |
| B, L | Differential operators |
| C, $\beta$ | Constants |
| V | Two-dimensional, continuum domain |
| $V_i$ | Subdomain (i = 1, . . . , n) |
| $\overline{D}$ | Flexural rigidity of a plate |
| dS | Subdomain differential |
| f, g | Functions |
| $M_r$ | Bending moment per unit length of a circumferential section of plate |
| Q | Shearing force per unit length of a circumferential section of plate |
| q | Uniform loading on plate surface |
| R | Residual function |
| $r_b$, $\theta_b$ | Coordinates of points on plate boundary |
| $r_1$, $r_2$ | Radial limits for the residual integral |
| w | Vertical deflection (or displacement) of the plate, positive in downward direction |
| $w_n$ | Trial plate deflection or displacement |
| x, y or r, $\theta$ | Coordinate axes |
| X | Exact value of a quantity |
| $X_M$ | Maximum value of quantity in exact solution |
| $X_n$ | Quantity whose error is being evaluated |
| $\nabla^4$ | Biharmonic or "del-fourth" operator |
| $\epsilon_w$, $\epsilon_M$ | Deflection error, Moment error |
| $\epsilon_X$ | Error of quantity X |
| $\gamma(r)$ | Weighting function |
| $\nu$ | Poisson's ratio, assumed = 0.30 |
| $\psi_o$, $\psi_k$ | Functions (k = 1, 2, . . . , n) |

# SECTION I

# APPLICATION OF THE SUBDOMAIN (BIEZENO-KOCH) METHOD TO CIRCULAR PLATE BENDING

## INTRODUCTION

Considerable interest has been generated in the applicability of the subdomain or Biezeno-Koch method to the solution of boundary value problems. Much of this interest is due to the satisfactory results obtained from this method as compared to other approximate methods such as: Ritz, Galerkin, collocation, point-matching, and Mikklin (Reference 1.). Although the subdomain method originated in 1923 (References 2, 3, 4) no widespread application of this technique has been noted in the technical literature during the intervening years up to the present time.

For the purpose of this study, the bending of the simply supported circular plate subjected to a uniformly distributed load was analyzed. This particular problem was selected since its solution is well known and is presented in Theory of Plates and Shells by Timoshenko and Woinowsky-Krieger (Reference 5). Therefore, there is a ready check for the accuracy of the solution obtained by the subdomain method. In addition the circular plate problem is somewhat easier to treat by approximate methods than is the rectangular or square plate since there is only one independent parameter, the radius, that is encountered in the expression for the deflection of a circular plate.

# SECTION II

## THE SUBDOMAIN (BIEZENO-KOCH) METHOD

The general method is described with reference to a finite, two-dimensional, elastic body in which the domain can be expressed as V (x, y) in cartesian coordinates or as V (r, $\theta$ ) for polar coordinates. The solution of all plate bending problems involves the satisfaction of a governing differential equation of equilibrium (or a set of differential equations) within the domain of the body. Thus:

$$Lw(r,\theta) = f(r,\theta) \text{ in } V \tag{1}$$

where L is a linear differential operator and w (r, $\theta$) or w (x, y) are the deflections of the plate as a function of the coordinates of any point on the domain. In addition, a set of boundary conditions must also be satisfied for all points falling on the boundary of the domain. These conditions may be expressed as:

$$Bw(r_b, \theta_b) = g(r_b, \theta_b) \text{ on the boundary} \tag{2}$$

where $(r_b, \theta_b)$ = coordinates of points on the boundary

B = a linear differential operator

According to the subdomain method a trial displacement function, $w_n$, is assumed that satisfies all boundary conditions as expressed by Equation 2.

This approximation to w frequently is cast in the form:

$$w_n = A_o \psi_o(r,\theta) + \sum_{k=1}^{n} A_k \psi_k(r,\theta) \tag{3}$$

In general the functions $\psi_k$ are linearly independent known functions within the domain, V, while $\psi_o$ is selected in some suitable manner depending upon the form of the differential equation of equilibrium, (Equation 1). $A_o$ and the $A_k$'s are the undetermined constants which are independent of the coordinates, (r, $\theta$ ), but nevertheless they are variable parameters of the solution.

A residual function is defined on the basis of Equation 1:

$$R(r,\theta) = Lw_n - f(r,\theta) \text{ in } V \tag{4}$$

The domain V is divided into subdomains $V_i$ according to some simple pattern and then the residual function is integrated over each subdomain and is set equal to zero. This process will yield n+1 linear equations which may be solved for the n+1 parameters $A_o, A_1, A_2, \ldots A_n$, thus:

$$\int_{V_i}\int R \, dS = 0 \tag{5}$$

$$i = 0, 1, 2 \text{ --- } n$$

where dS = subdomain differential, (dx dy) or (rd $\theta$ , dr)

An alternate approach may be applied to the method of subdomains by evaluating the integral of the residual function at more subdomains than there are undetermined constants. Then the constants are evaluated by the least squares approximation technique described in References 6 and 7.

# SECTION III

# APPLICATIONS OF THE METHOD

## 1. SIMPLY SUPPORTED CIRCULAR PLATE WITH A UNIFORM LOAD

Since the load acting on the circular plate is symmetrically distributed about the axis perpendicular to the plate through its center, the deflection function, w, of the plate will also be symmetrical. Therefore the vertical deflection, w, will be independent of $\theta$ (see below) and the subdomain differential becomes dr.

Equations governing this problem are (Reference 5):

a. The differential equation of equilibrium:

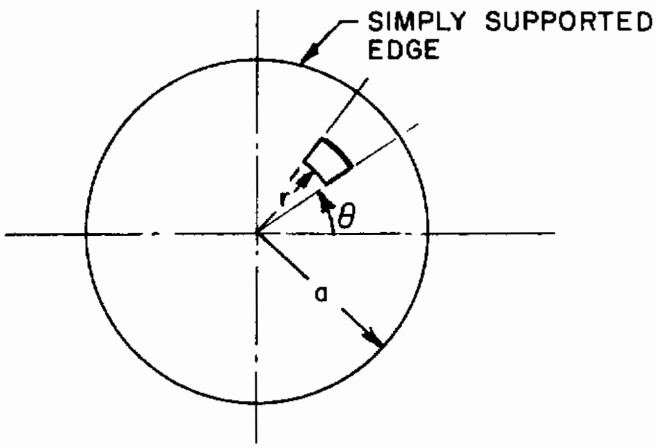
$$L\,w = \frac{d^3 w}{dr^3} + \frac{1}{r}\,\frac{d^2 w}{dr^2} - \frac{1}{r^2}\,\frac{dw}{dr} = \frac{Q}{\overline{D}} \tag{6a}$$

or

$$\frac{d}{dr}\left(\frac{1}{r}\,\frac{d}{dr}\left(r\,\frac{dw}{dr}\right)\right) = \frac{Q}{\overline{D}} \quad \text{within } V \tag{6b}$$

where Q = the shearing force per unit length of cylindrical section of radius, r. For a uniform loading, q, Q = qr/2

$\overline{D}$ = the flexural rigidity of a plate.




At this stage it is convenient to introduce the residual equation (Equation 4) for this problem in terms of the trial function, $w_n$, thus:

$$R = \frac{d^3 w_n}{dr^3} + \frac{1}{r}\,\frac{d^2 w_n}{dr^2} - \frac{1}{r^2}\,\frac{dw_n}{dr} - \frac{qr}{2\overline{D}} \tag{7}$$

b. The boundary conditions:

$$w = 0 \quad \text{at} \quad r = a \tag{8}$$

$$M_r = 0 = -\overline{D}\left(\frac{d^2 w}{dr^2} + \frac{\nu}{r}\,\frac{dw}{dr}\right) \quad \text{at} \quad r = a \tag{9}$$

$$\frac{dw}{dr} = 0 \quad \text{at} \quad r = 0 \qquad \text{(condition of symmetry)} \tag{10}$$

where $M_r$ = the bending moment per unit length along circumferential sections

$\nu$ = Poisson's ratio, is taken as 0.30

The most crucial step in the method is the selection of an admissible trial function, $w_n$, which is not overly restrictive. For example, no point within the domain should be limited to a constant value of deflection regardless of the number of terms assumed in the trial function unless, of course, the point is on a fixed boundary. Selection of a trial function is based largely upon intuition and experience so considerations of the type mentioned below will be referred to as intuitive conditions for lack of a better designation. The following intuitive conditions should be considered in connection with this problem.

$$w_n \neq 0 \quad \text{at} \quad r = 0 \tag{11}$$

$$\frac{dw_n}{dr} \neq 0 \quad \text{at} \quad r = a \quad \text{(Boundary condition resulting in a clamped edge)} \tag{12}$$

$$M_r \neq 0 \quad \text{at} \quad r = 0 \tag{13}$$

The residual, R should remains finite at r = 0

The above conditions represent some of the pitfalls the analyst may encounter and they should serve merely as a guide in selecting a suitable trial function.

The trial function employed in this analysis can initially be written in the form

$$w_n = A_0 \,(a^2 - r^2) + Cr^m \,\psi(r) + (a^2 - r^2)^3 \sum_{k=1}^{n} A_k \, r^k \tag{14}$$

where $\psi(r)$ is yet to be defined. An examination of Equation 14 shows that the first term and the series terms all satisfy Equation 8 and that all terms satisfy Equation 10 if the series terms begin at k = 2 and $m \geq 2$. The first term in the trial function was selected since it is a homogeneous solution of Equation 6 and there would be no residual arising from it. In addition it satisfies Equations 11, 12, and 13.

With regard to the moment boundary condition (Equation 9), the series terms satisfy the condition because the terms of the series are multiplied by $(a^2 - r^2)^3$ thus enabling the first and second derivatives of $w_n$ to equal zero at r = a. The first term of the trial function does not satisfy Equation 9 so that a moment exists at r = a. Therefore, the second term must be defined so that it not only satisfies Equation 8 but that the moment computed from it at r = a cancels the moment from the first term. Thus according to Equation 8:

$$\Big[\psi(r)\Big]_{r=a} = \psi(a) = 0 \tag{15}$$

and from Equation 9

$$\left[\frac{d^2w}{dr^2} + \frac{\nu}{r}\,\frac{dw}{dr}\right]_{r=a} = 0$$

$$-2A_0(1+\nu) + C\left(a^m \left.\frac{d^2\psi}{dr^2}\right|_{r=a} + (2m+\nu)\,a^{m-1}\left.\frac{d\psi}{dr}\right|_{r=a}\right) = 0$$

$$\left.\frac{d^2\psi}{dr^2}\right|_{r=a} + \frac{(2m+\nu)}{a}\left.\frac{d\psi}{dr}\right|_{r=a} = \frac{2A_0(1+\nu)}{a^m\,C} \tag{16}$$

A function that satisfies Equations 15 and 16 is

$$\psi = \frac{A_0 \beta (r-a)}{C a^{m-1}}$$

where

$$\beta = \frac{2(1+\nu)}{(2m+\nu)}$$

therefore the trial function becomes:

$$w_n = A_0 \left[ (a^2 - r^2) + \frac{\beta}{a^{m-1}} (r-a) r^m \right] + (a^2 - r^2)^3 \sum_{k=1}^{n} A_k r^k \tag{17}$$

The calculation of the residual from Equation 7 gives:

$$R = \frac{A_0 \beta}{a^{m-1}} \left[ (m+1)(m^2-1) r^{m-2} - a m^2 (m-2) r^{m-3} \right] + (a^2 - r^2)^3 \left[ \sum_{k=3}^{n} k^2 (k-2) A_k r^{k-3} - \frac{A_1}{r^2} \right] - (a^2 - r^2)^2 \left[ \sum_{k=2}^{n} 6k(3k+2) A_k r^{k-1} - 30 A_1 \right] + (a^2 - r^2) \sum_{k=1}^{n} 24(3k+4) A_k r^{k+1} - 48 \sum_{k=1}^{n} A_k r^{k+3} - \frac{q}{D} \left( \frac{r}{2} \right) \tag{18}$$

At all values of r (including r = 0), the residual R must be finite. This will be true only if $m \geq 3$ and $k \geq 2$. Including these restrictions, Equation 17 becomes:

$$w_n = A_0 \left[ (a^2 - r^2) + \frac{\beta}{a^2} (r-a) r^3 \right] + (a^2 - r^2)^3 \sum_{k=2}^{n} A_k r^k \tag{19}$$

where

$$\beta = \frac{2(1+\nu)}{6+\nu}$$

and the residual changes to:

$$R = \frac{A_0 \beta}{a^2} (32r - 9a) + (a^2 - r^2)^3 \sum_{k=3}^{n} k^2 (k-2) A_k r^{k-3} - (a^2 - r^2)^2 \sum_{k=2}^{n} 6k(3k+2) A_k r^{k-1} + (a^2 - r^2) \sum_{k=2}^{n} 24(3k+4) A_k r^{k+1} - 48 \sum_{k=2}^{n} A_k r^{k+3} - \frac{q}{D} \left( \frac{r}{2} \right) \tag{20}$$

Integrating Equation 20 over a radial subdomain between limits $r_1$ and $r_2$ and setting the resulting equation equal to zero (Equation 5) gives:

$$\int_{r_1}^{r_2} R \; dr = 0 \tag{21}$$

$$\left\{ \frac{A_0 \beta}{a^2} (16 r^2 - 9 a r) + \sum_{k=3}^{n} k^2 (k-2) A_k \left[ \frac{(a^2 - r^2)^3 r^{k-2}}{k-2} + \frac{6 (a^2 - r^2)^2 r^k}{(k-2)(k)} \right.\right.$$

$$\left. + \frac{24 (a^2 - r^2) r^{k+2}}{(k-2)(k)(k+2)} + \frac{(48) r^{k+4}}{(k-2)(k)(k+2)(k+4)} \right] - \sum_{k=2}^{n} 6k(3k+2) A_k \left[ \frac{(a^2 - r^2)^2 r^k}{k} \right.$$

$$\left. + \frac{4 (a^2 - r^2) r^{k+2}}{k(k+2)} + \frac{8 r^{k+4}}{(k)(k+2)(k+4)} \right] + \sum_{k=2}^{n} 24(3k+4) A_k \left[ \frac{(a^2 - r^2) r^{k+2}}{k+2} \right.$$

$$\left.\left. + \frac{2 r^{k+4}}{(k+2)(k+4)} \right] - 48 \sum_{k=2}^{n} A_k \frac{r^{k+4}}{k+4} - \frac{q r^2}{\bar{D} 4} \right\}_{r_1}^{r_2} = 0 \tag{22}$$

In order to obtain a solution there must be at least as many subdomains as there are "A" constants to evaluate. Hence Equation 22 was evaluated for each subdomain requiring that only the limits of integration be changed. These limits were chosen to correspond to either equal increments of radius or to equal annular areas. The digital computer was programmed to evaluate Equation 22 and to solve for the "A" constants. Both a simultaneous equation solver and a least squares equation solver are included in a subroutine. The results of this analysis are included the figures. The computer program may be obtained from the Air Force Flight Dynamics Laboratory upon request.

Other approaches to obtaining solutions to this problem are discussed in Appendix II.

The equation for integrating the residual over the subdomain, (Equation 20) can be written to include some weighting function $\gamma(r)$. Thus it becomes:

$$\int_{r_1}^{r_2} R \gamma (r) \; dr = 0 \tag{23}$$

For the purpose of this analysis $\gamma(r) = r$ and the same residual (Equation 20) was used. Equation 23 becomes:

$$\int_{r_1}^{r_2} R \, r \; dr = 0 \tag{24}$$

In addition different spacings of the subdomain were used; one set of spacings for equal radial subdivisions and another set based on equal areas. A computer program similar to the one described previously was used and the results are included in the figures.

# SECTION IV

# DISCUSSION

## 1. COMPARISON OF APPROXIMATE SOLUTION WITH THE TIMOSHENKO SOLUTION OF THIS PROBLEM

For purposes of comparison the errors in deflection and moment, and the residual defined by Equation 20 were plotted versus the radius of the plate. The errors were defined by the following equation:

$$\epsilon_x = \frac{X_n(r) - X(r)}{X_M(r)}$$

where

$X_n$ = the quantity whose accuracy is under evaluation

$X$ = the exact value at the same r (from Timoshenko solution)

$X_M$ = the maximum value of the exact solution

For the case where the residual is integrated over equal increments of radius (no weighting function, Equation 21), Figures 1 - 11 show similar curves for the deflection error. The deflection errors at the center of the plate improve progressively with an increase in the number of undetermined constants, $A_k$'s, up to an eight parameter trial function. Here the deflection error at the center seems to converge to an error of about 2.15 percent. A further increase in constants causes the error to diverge; however, a minimum central deflection error of 1.65 percent occurs for a trial function with 30 constants. In like manner the moment error at the center decreases to a value of approximately 1.57 percent for an eight constant solution and then diverges; however, the maximum error (in the outer portion of the plate) increases progressively from 24 percent for three constants to 119 percent for 30 constants. Although these maximum moment errors are high they occur in that portion of the plate where the moments are relatively small ($M/D < 0.074$). The behavior of the residuals shows a flattening effect in the central regions of the plate with an increase in the number of constants in the solution up to four. With a further increase in constants the residual tends to increase in magnitude at both ends. It is interesting to note that the best solution obtained for the deflection occurs when the residual curve is flattest throughout the largest portion of the plate. This occurs for the case when 30 constants are used.

Results where the limits of integration for the residual integral correspond to equal areas (all other conditions the same as above) are shown in Figures 12 - 18. A comparison of these results with those described in the preceding paragraph show that poorer solutions were obtained for integration limits based on equal areas rather than equal increments of radius. This is particularly true for the central region of the plate possibly because the integration for the innermost subdomain is over a relatively large increment of radius. In effect, this implies that there is a greater possibility for a wide fluctuation in the value of the residual in this subdomain.

Error curves resulting from the inclusion of a weighting function ($\gamma(r) = r$, in the integral expression Equation 24) are shown in Figures 19 - 27. The limits of integration for these runs correspond to equal increments of radius. The figures show an improvement in the deflection accuracy compared to the runs where the weighting function was not used. Deflection errors at the center of the plate decrease with an increase in the number of constants to about four before they begin to diverge. In the case of four constants the deflection error at the center is

-0.70 percent. A crude indicator of the most overall satisfactory solution appears to be the degree of flatness of the corresponding residual curve. For the same case of four constants the moment error at the center is zero percent with a maximum moment error of about 25 percent. These moment results are similar to the results obtained for a four or five constant trial function without the weighting function.

The results from the least squares solutions where the residual is integrated over equal increments of radius both with and without the weighting function, r, are shown in Figures 28 - 32 and Figures 33 - 37, respectively. All least squares solutions are for the case of a four constant trial function. Comparing the results for the least squares solutions with the previous solutions (for four constants) one may conclude that, in general, the least squares approach did not improve the accuracy of the deflections and moments, particularly at the center of the plate. However, the extremes of variation of the residual have been reduced by the least squares approach. The average values of the residual for the least squares approach appears to be greater in magnitude than that for the four constant case where this approach was not used.

There is one disadvantage that arises from using the residual equation as shown by Equation 7 where $q/\bar{D}$ represents the load intensity divided by the flexural rigidity. For an approximate $w_n$ the residual or error that occurs at a particular radius cannot be conveniently expressed in terms of a hypothetical or residual load intensity, q, since the pertinent relationship is

$$Q = \frac{\int_0^r qr \, dr}{r}$$

In order to evaluate this expression, q must be known as a function of r; however, it is generally unknown. It would be somewhat more satisfactory to employ the residual equation (for symmetric stress) in the form:

$$R = \nabla^4 w_n - \frac{q}{\bar{D}} \tag{25}$$

where

$$\nabla^4 w_n = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2 w_n}{dr^2} + \frac{1}{r}\frac{d w_n}{dr}\right)$$

In this case the resultant residual computed at a point is directly related to a hypothetical load intensity at that point. If the solution were exact at every point within the domain, R would equal zero everywhere within the domain ($\nabla^4 w_n = q/\bar{D}$).

A solution based on the application of Equation 25 appears to be more restrictive with regard to the selection of $w_n$. It has been observed that if the $w_n$ used in this analysis is differentiated further and substituted in Equation 25 there are two terms that are not finite at every point in the domain. These terms approach infinity at r = 0. This implies that, on the basis of Equation 25 the trial function used in this analysis is not suitable. In fact, this may account for the lack of convergence on the part of the approximate deflection and the approximate moment to their exact values.

## 2. CONCLUSIONS

The subdomain method can be adapted conveniently to digital computer programming when a series form of the trial function is employed. This results in a repetitive form of the residual or error integral.

Results for the deflection of the uniformly loaded, simply supported plate attain engineering accuracy as shown by Figure 38. In addition the results for the bending moment at the center of the plate are satisfactory. However, no firm conclusions can be made regarding the overall effect of the residual on the accuracy of the solution. This is particularly evident when the lack of improvement in both deflection and bending moment accuracy is noted for the least squares solutions. In spite of a reduction in the extremes of residual variation by the least squares technique the accuracy does not improve.

REFERENCES


1. I. U. Ojalvo and K. F. Russell, "Approximate Structural Analysis Techniques Employing Undetermined Parameters", Tech. Report No. AFFDL-TR-66-59, May 1966.

2. C. B. Biezeno and R. Grammel, Engineering Dynamics, D. Van Nostrand Co., 1954, Vol I.

3. C. B. Biezeno, "Graphical and Numerical Methods for Solving Stress Problems", Proceedings of the First International Congress of Applied Mechanics, Delft (1924) p. 3-17.

4. P. Shuleshko, "Generalization of Previous Methods of Solving Boundary Value Problems of Mathematical Physics", Aust. J. of Applied Science, Vol. 10, No. 1, Mar. 1959, p. 9-16.

5. S. Timoshenko and S. Woinowsky-Krieger, Theory of Plates and Shells, McGraw-Hill Book Co., Second Edition, 1959, p. 51-57.

6. J. B. Scarborough, Numerical Mathematical Analysis, Johns Hopkins Press, 1962, p. 475-476, 478, 527-535.

7. R. G. Stanton, Numerical Methods for Science and Engineering, Prentice-Hall, Inc., 1961, p. 52-62

8. L. V. Kantorovich and U. I. Krylov, Approximate Methods of Higher Analysis, P. Noordhoff Ltd, 1958, p. 304-325.

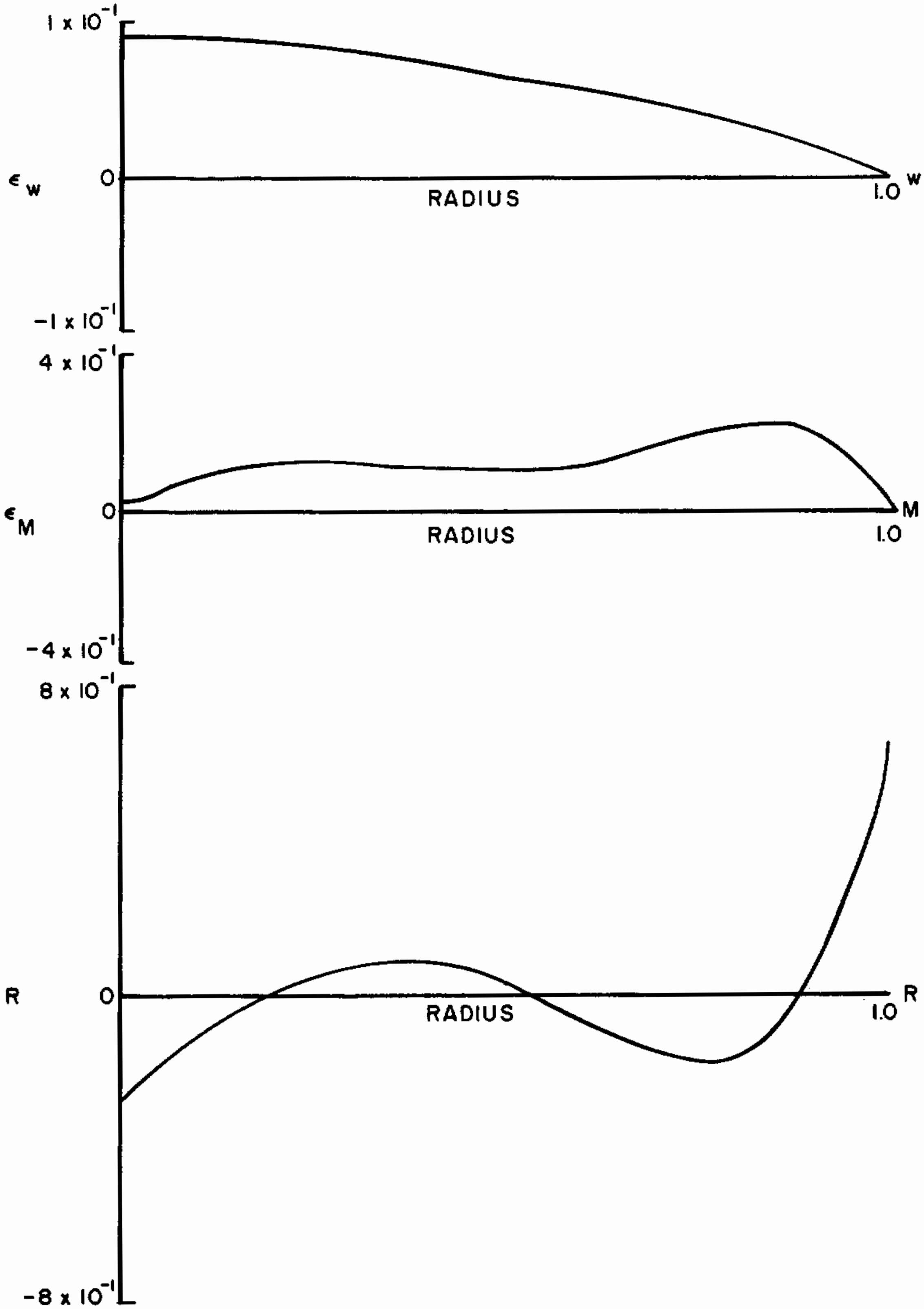



Figure 1. Deflection Error, Moment Error and Residual vs. Radius for Two Equal Radial Increments

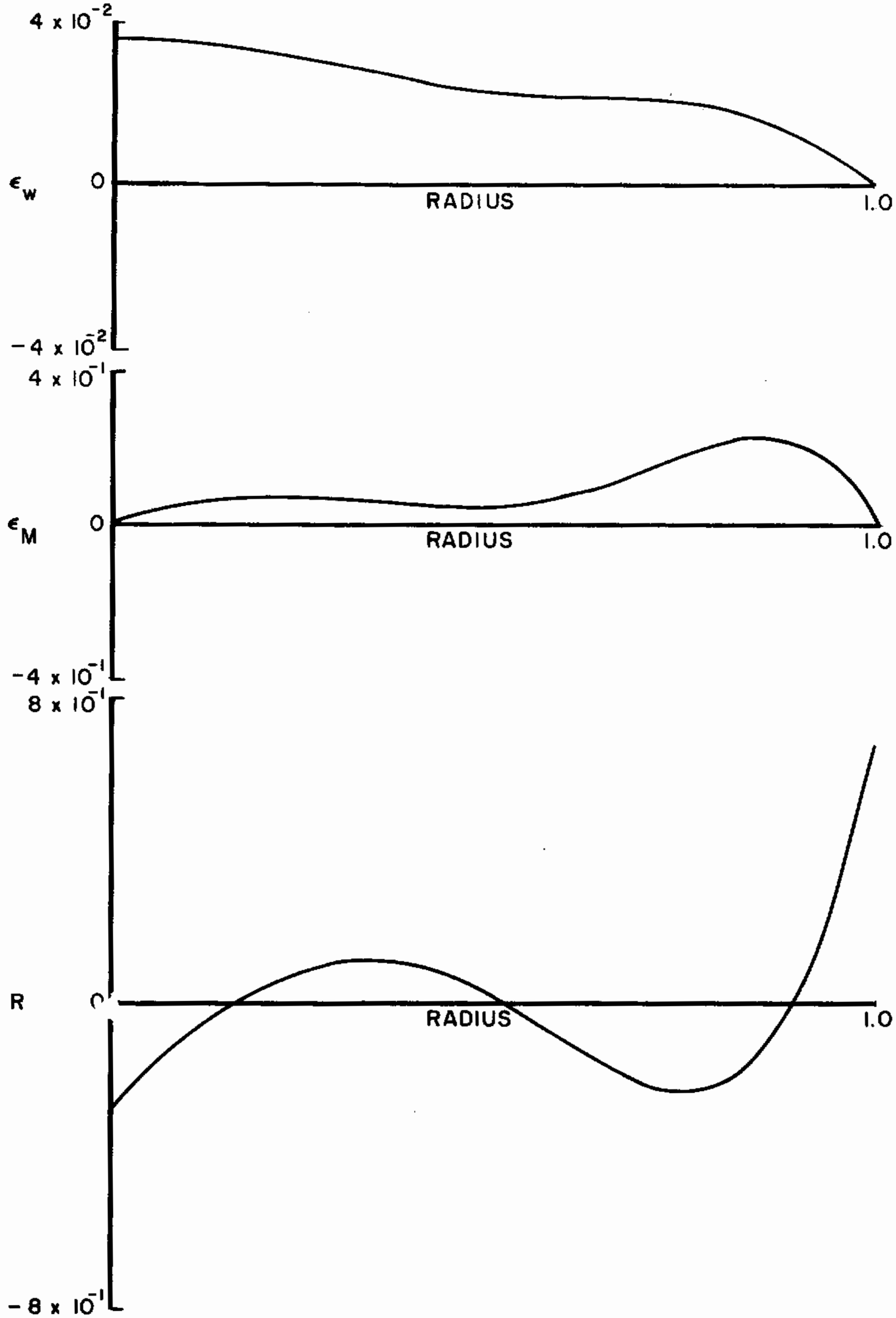



Figure 2. Deflection Error, Moment Error and Residual vs. Radius for Three Equal Radial Increments

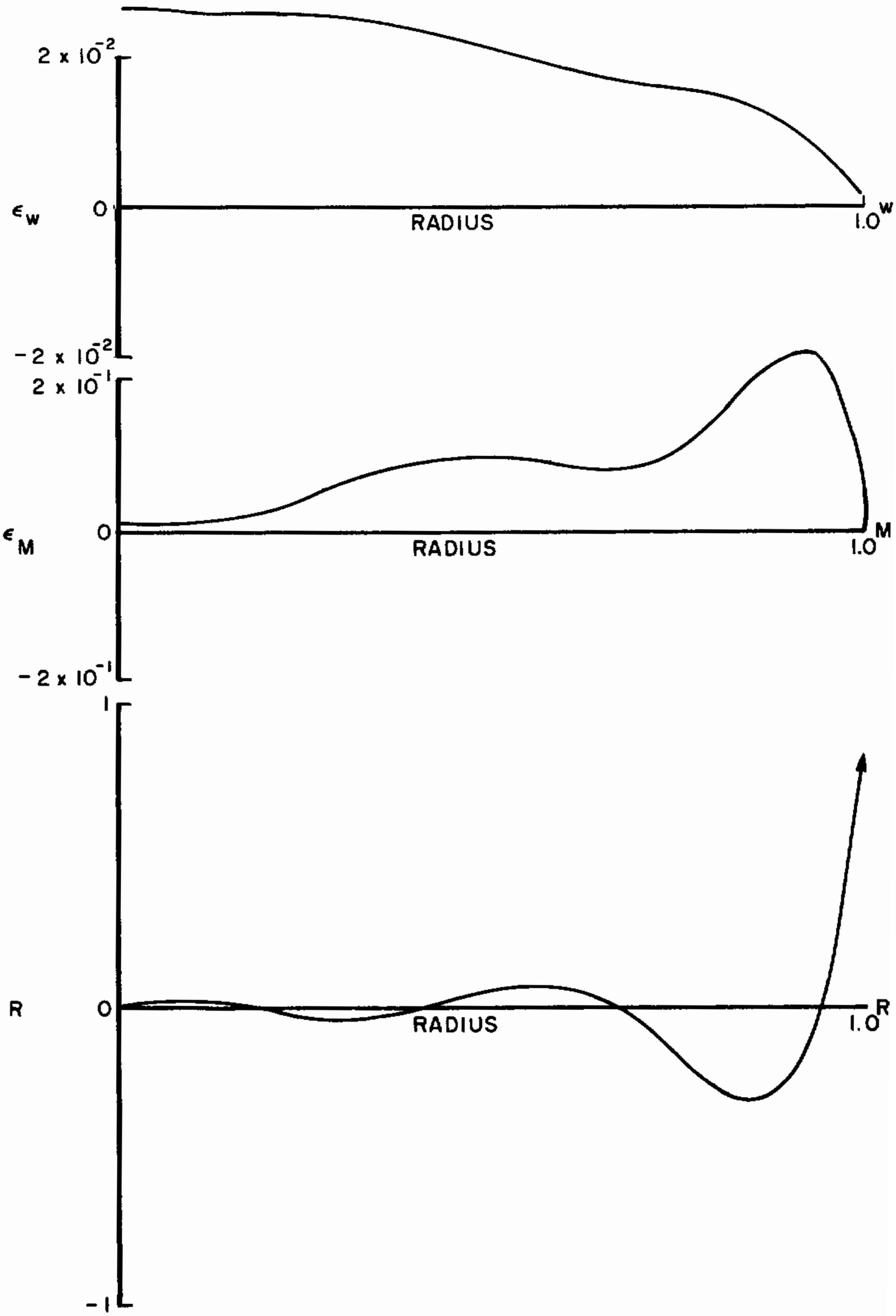



**Figure 3. Deflection Error, Moment Error and Residual vs. Radius for Four Equal Radial Increments**

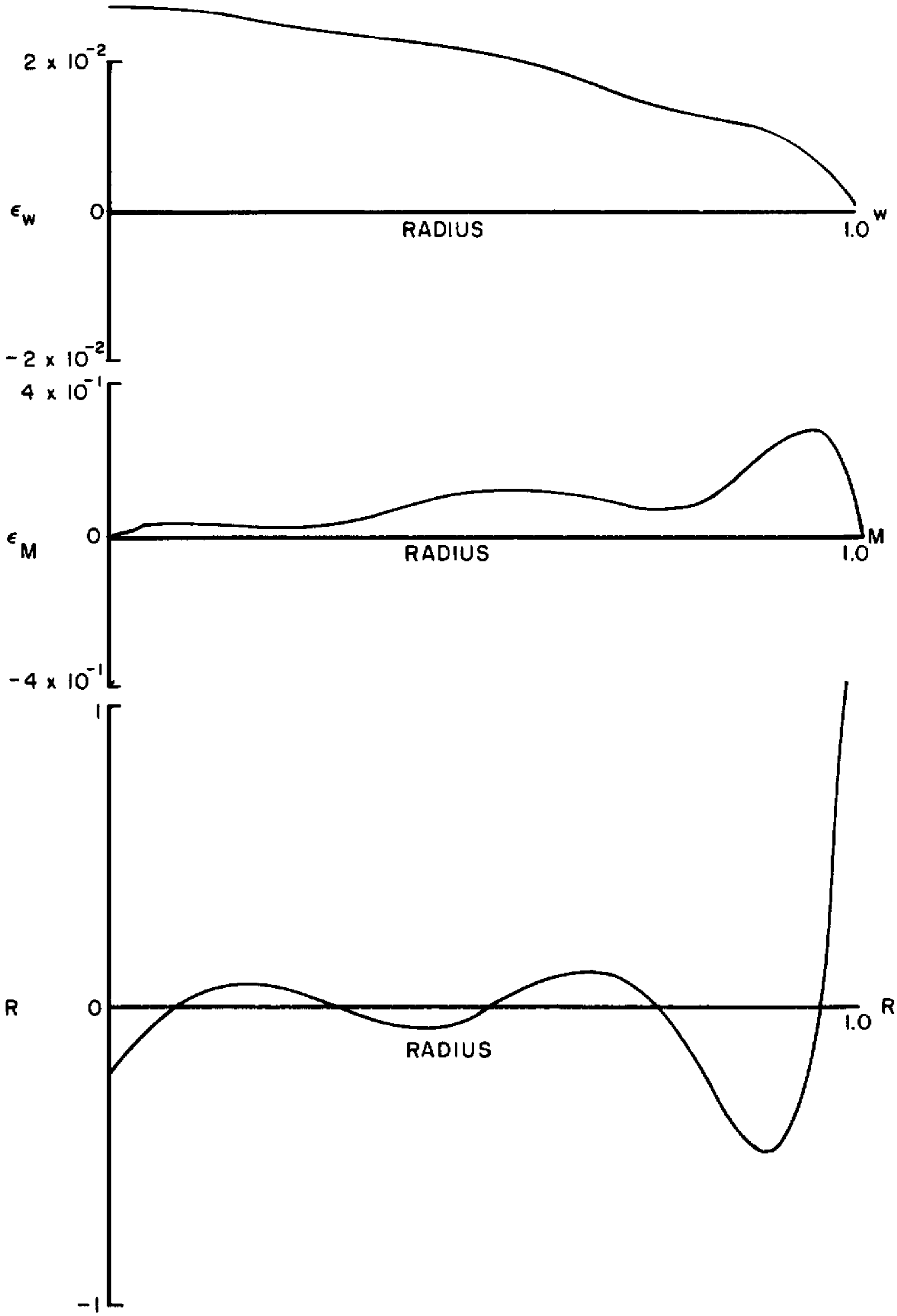



Figure 4. Deflection Error, Moment Error and Residual vs. Radius for Five Equal Radial Increments

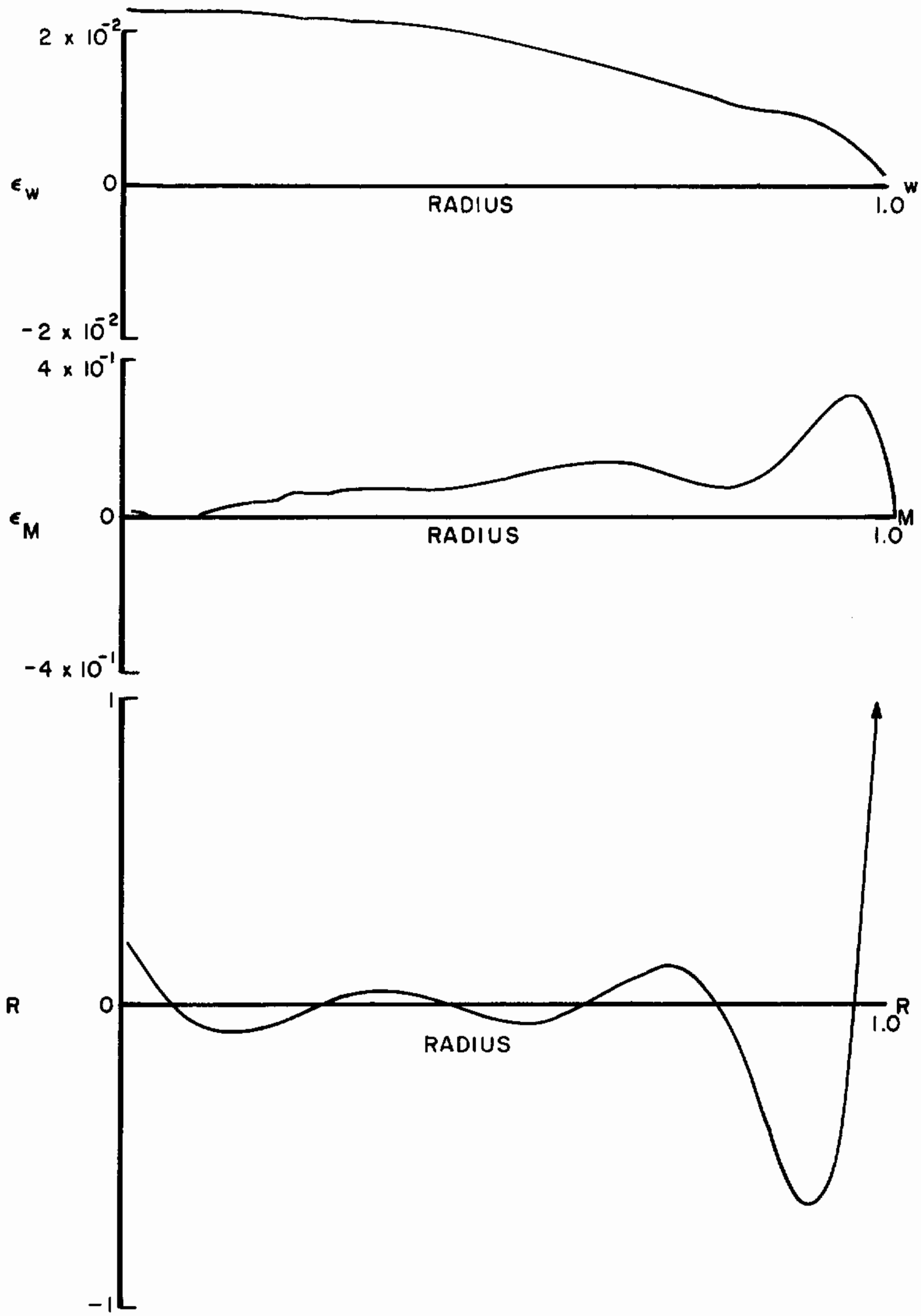



Figure 5. Deflection Error, Moment Error and Residual vs. Radius for Six Equal Radial Increments

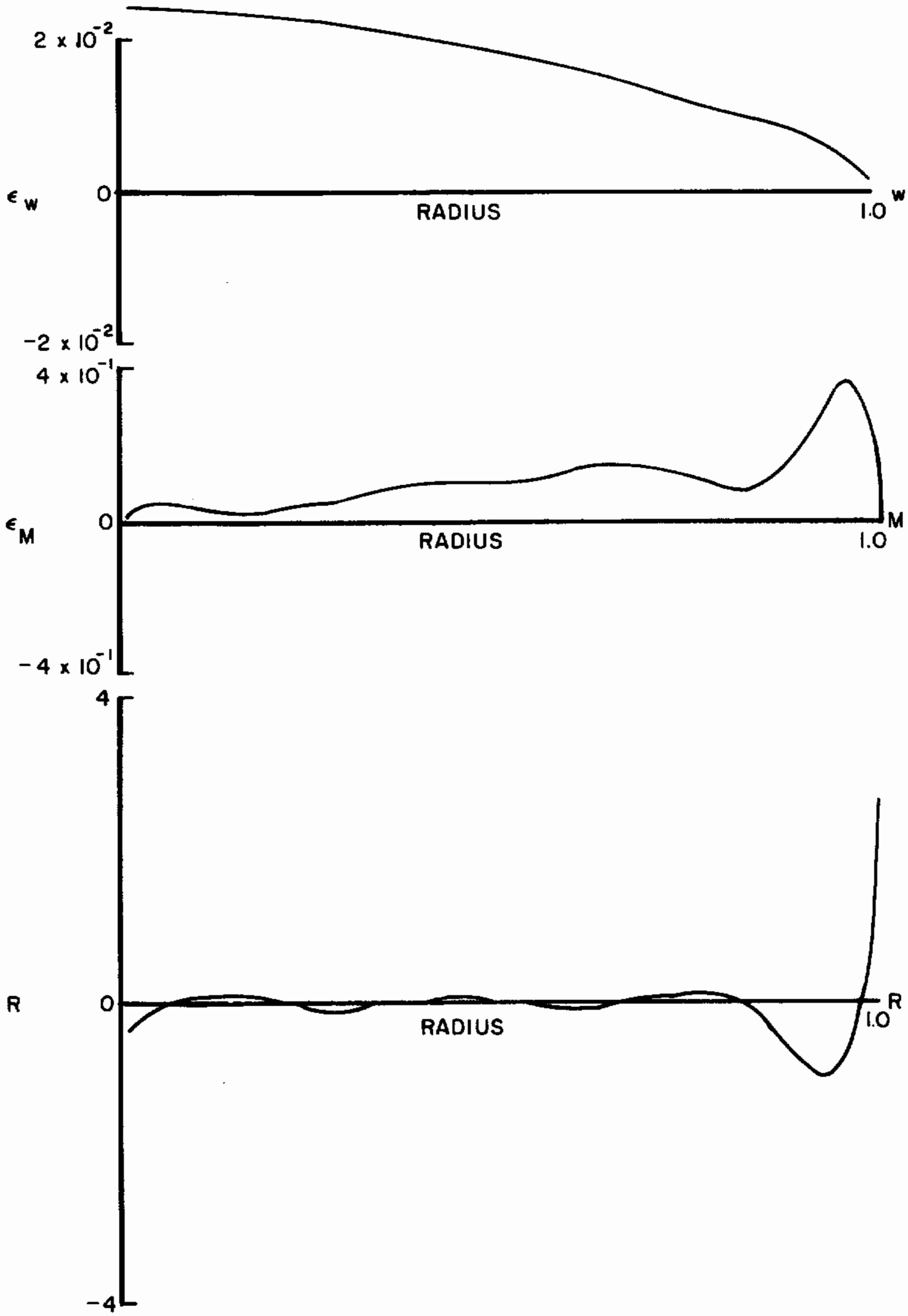



Figure 6. Deflection Error, Moment Error and Residual vs. Radius for Seven Equal Radial Increments

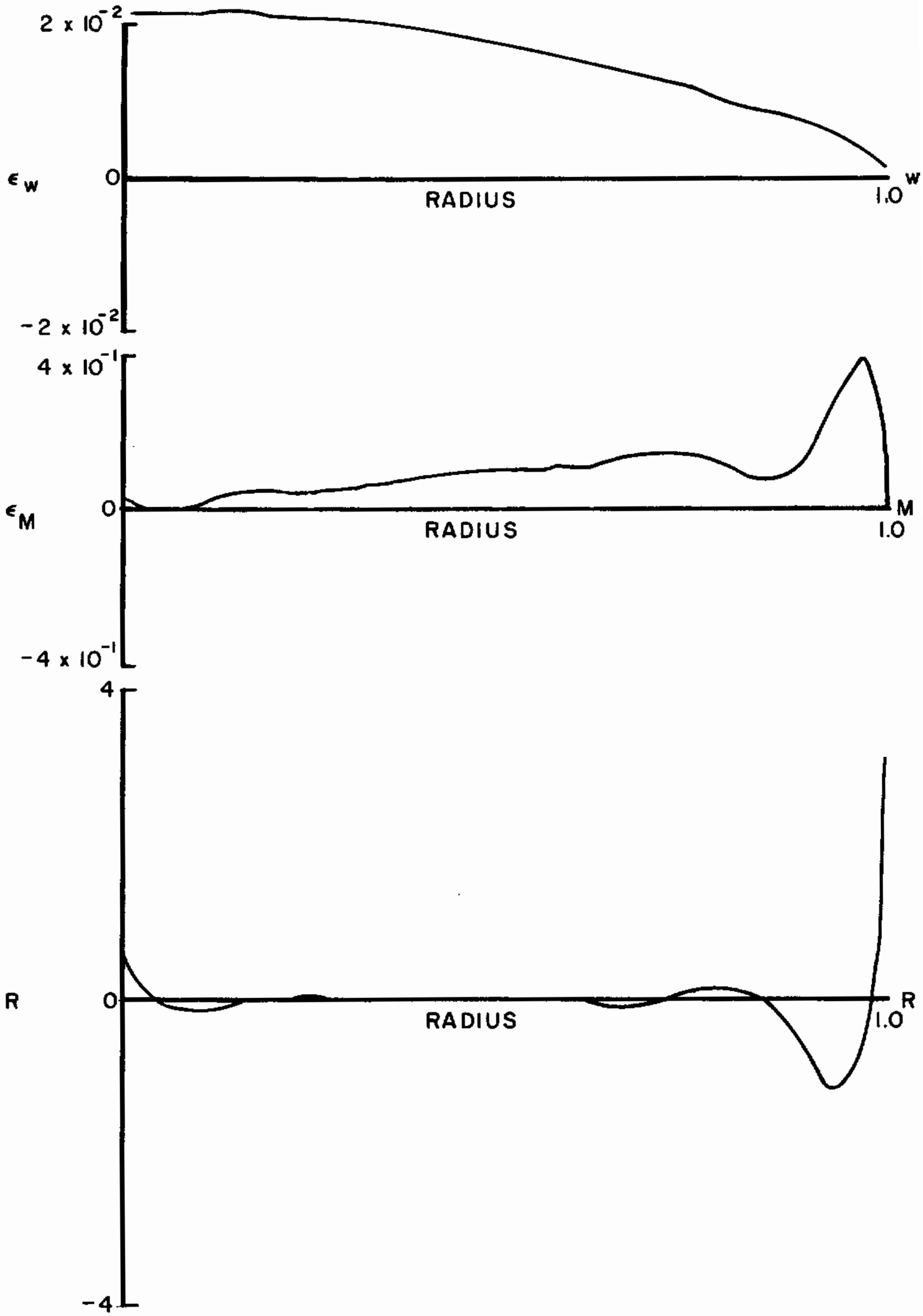



Figure 7. Deflection Error, Moment Error and Residual vs. Radius for Eight Equal Radial Increments

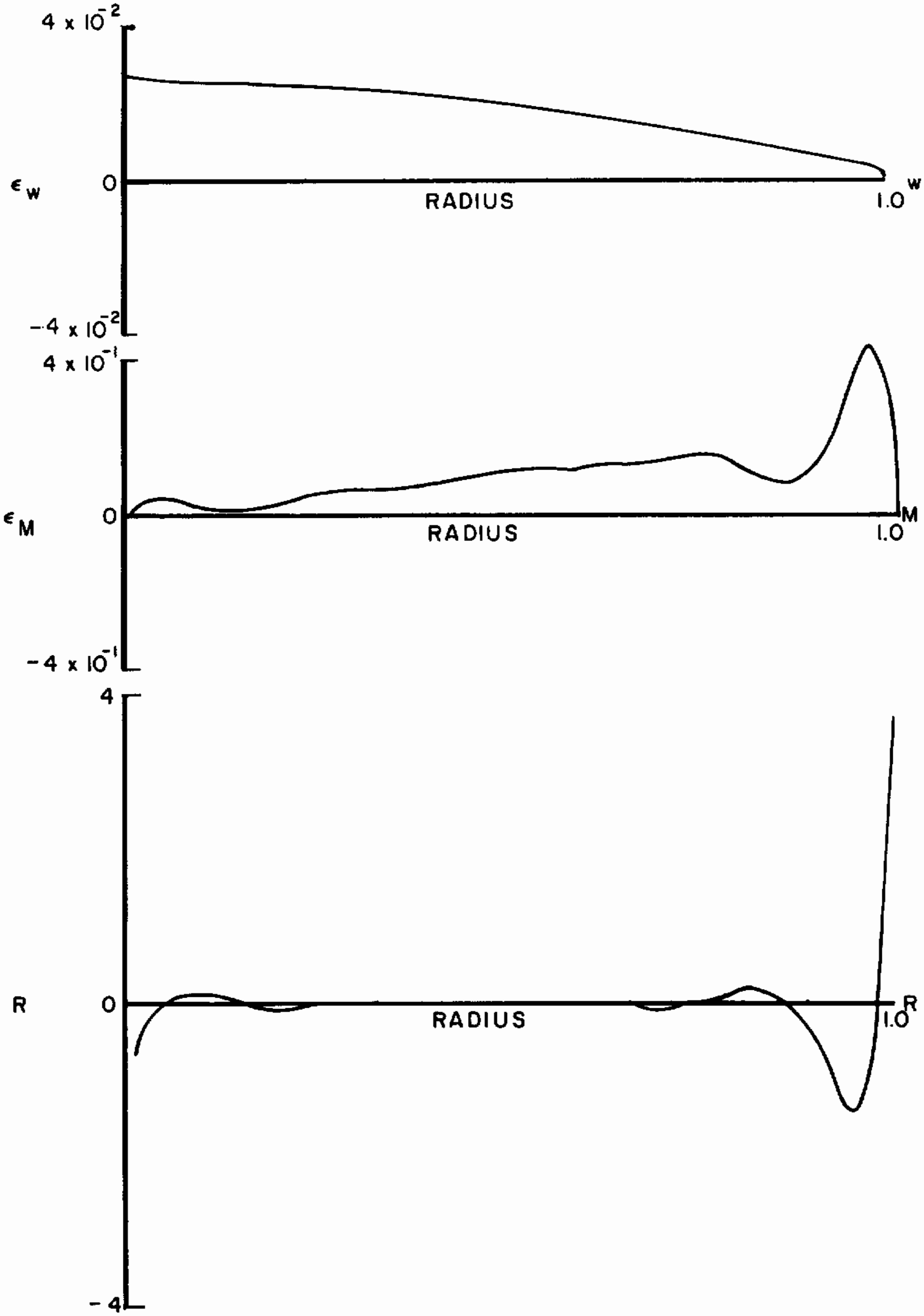



Figure 8. Deflection Error, Moment Error and Residual vs. Radius for Nine Equal Radial Increments

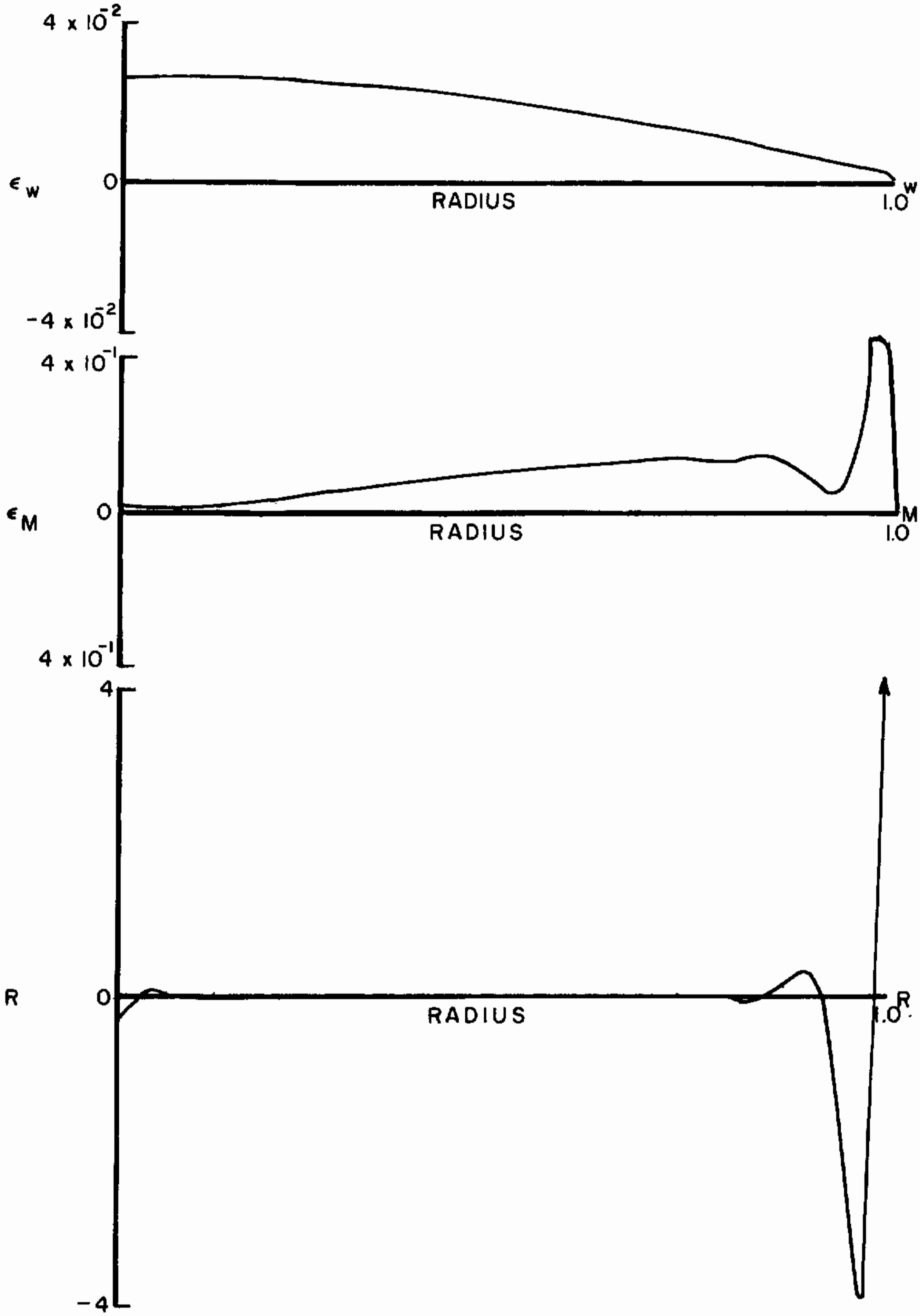



Figure 9. Deflection Error, Moment Error and Residual vs. Radius for Fifteen Equal Radial Increments

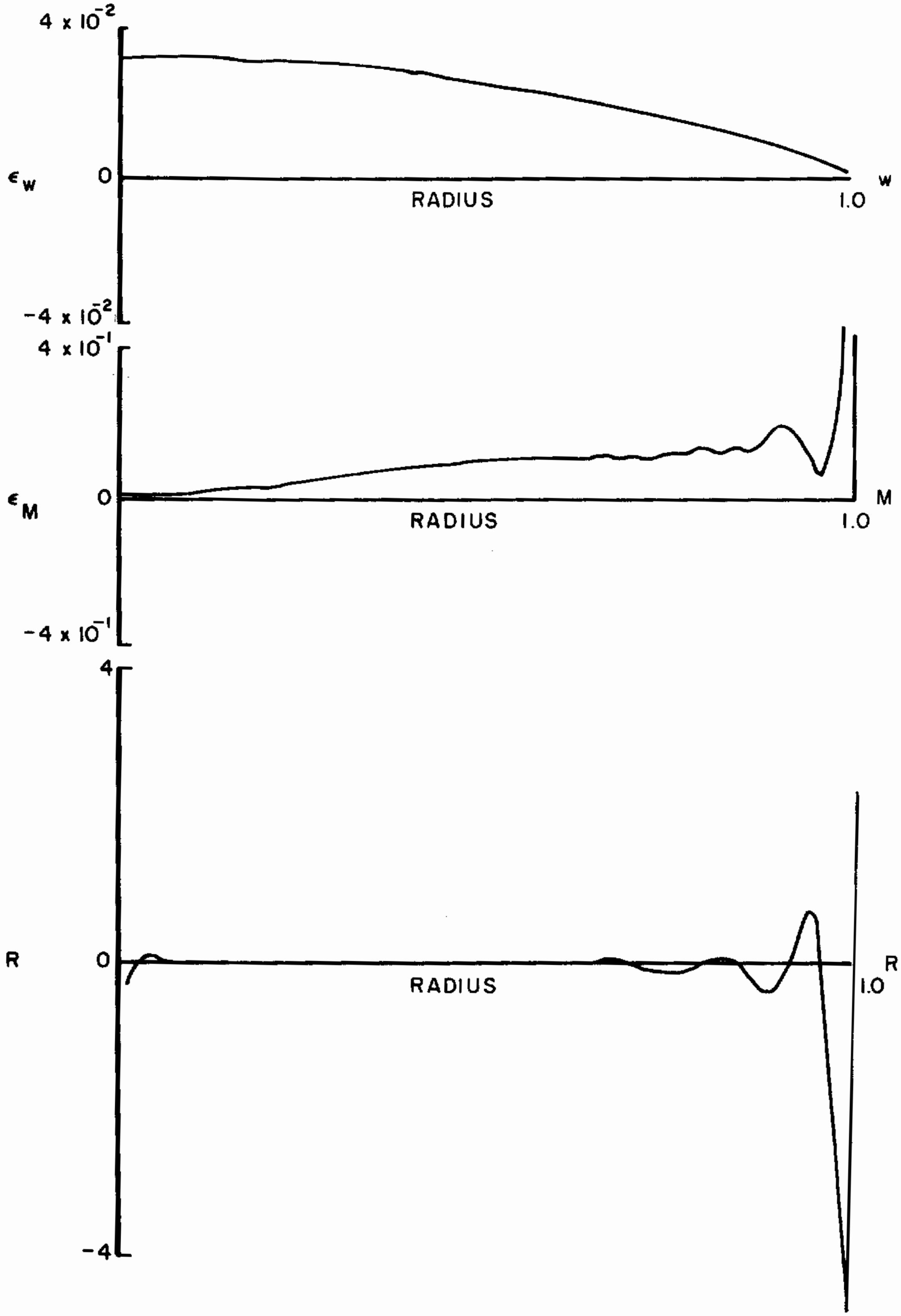



**Figure 10. Deflection Error, Moment Error and Residual vs. Radius for Nineteen Equal Radial Increments**

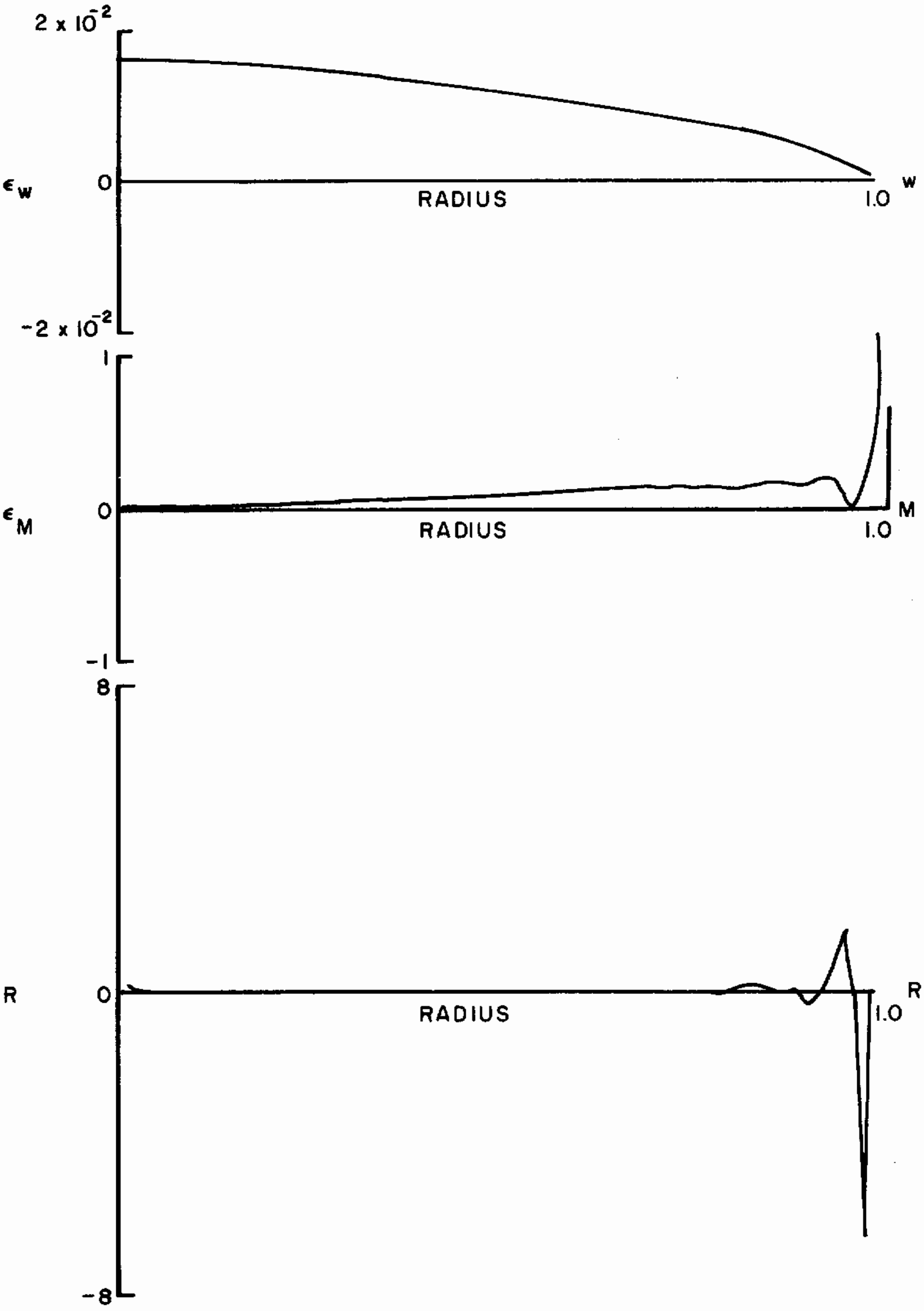



Figure 11. Deflection Error, Moment Error and Residual vs. Radius for Thirty Equal Radial Increments

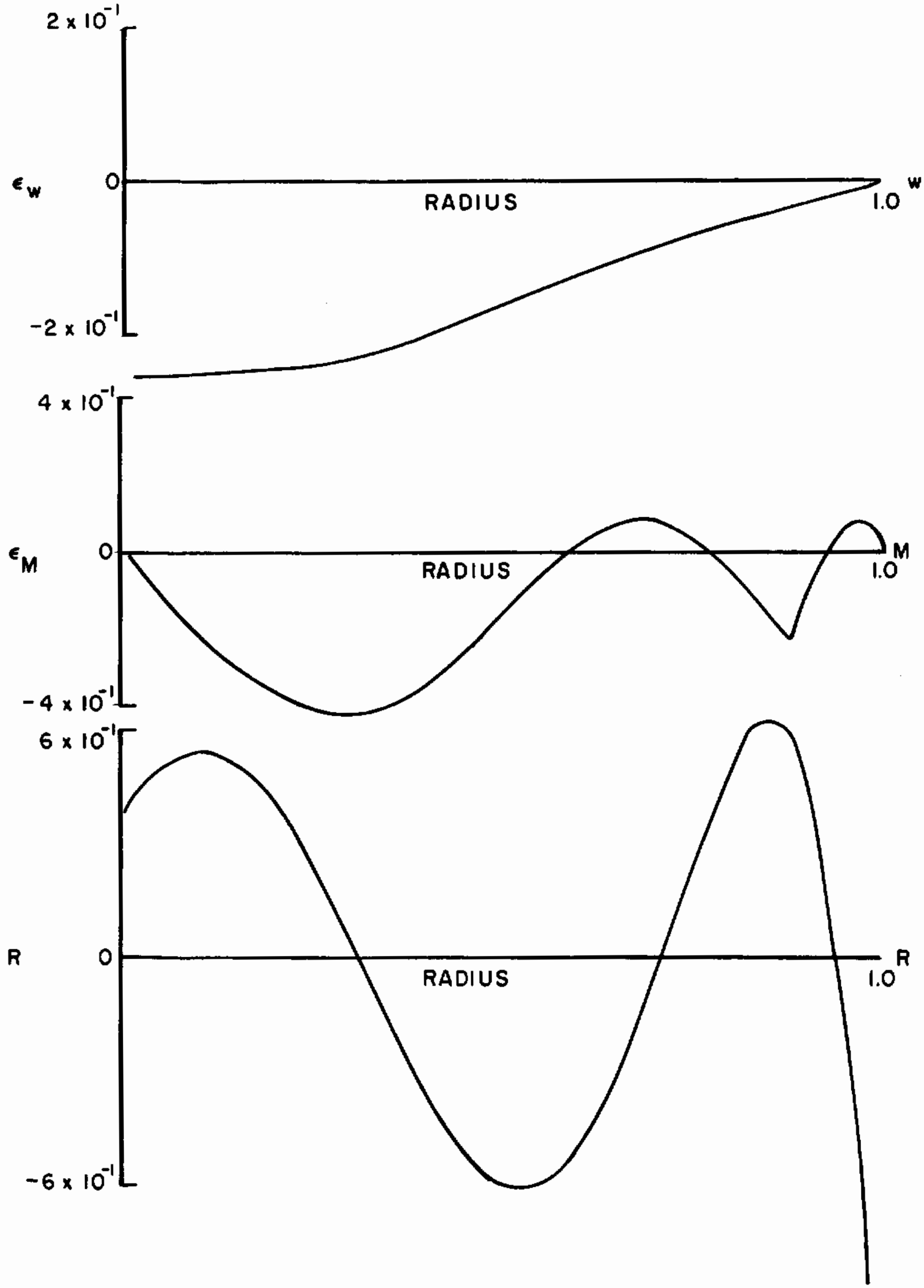



Figure 12. Deflection Error, Moment Error and Residual vs. Radius for Three Equal Area Increments

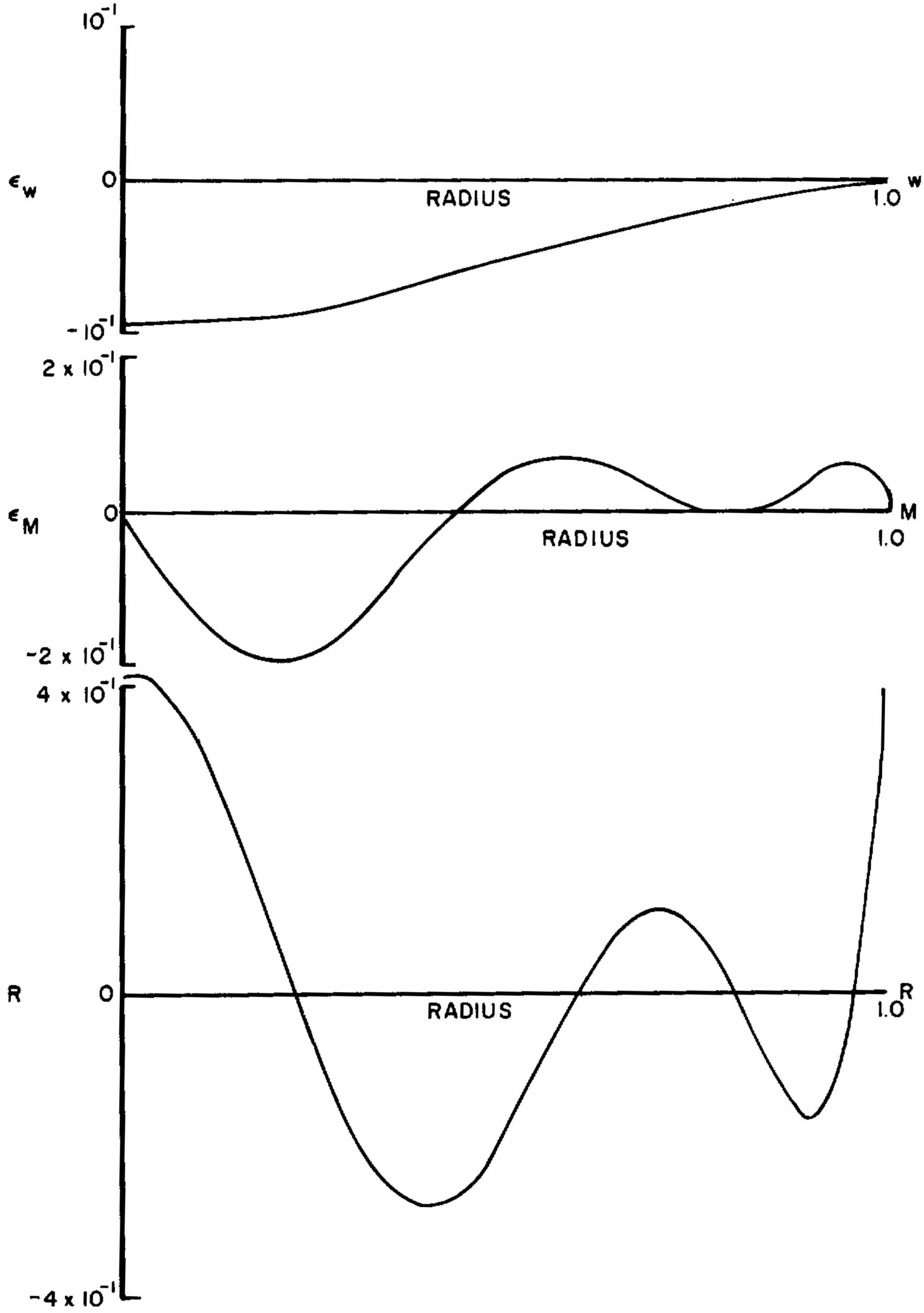



Figure 13. Deflection Error, Moment Error and Residual vs. Radius for Four Equal Area Increments

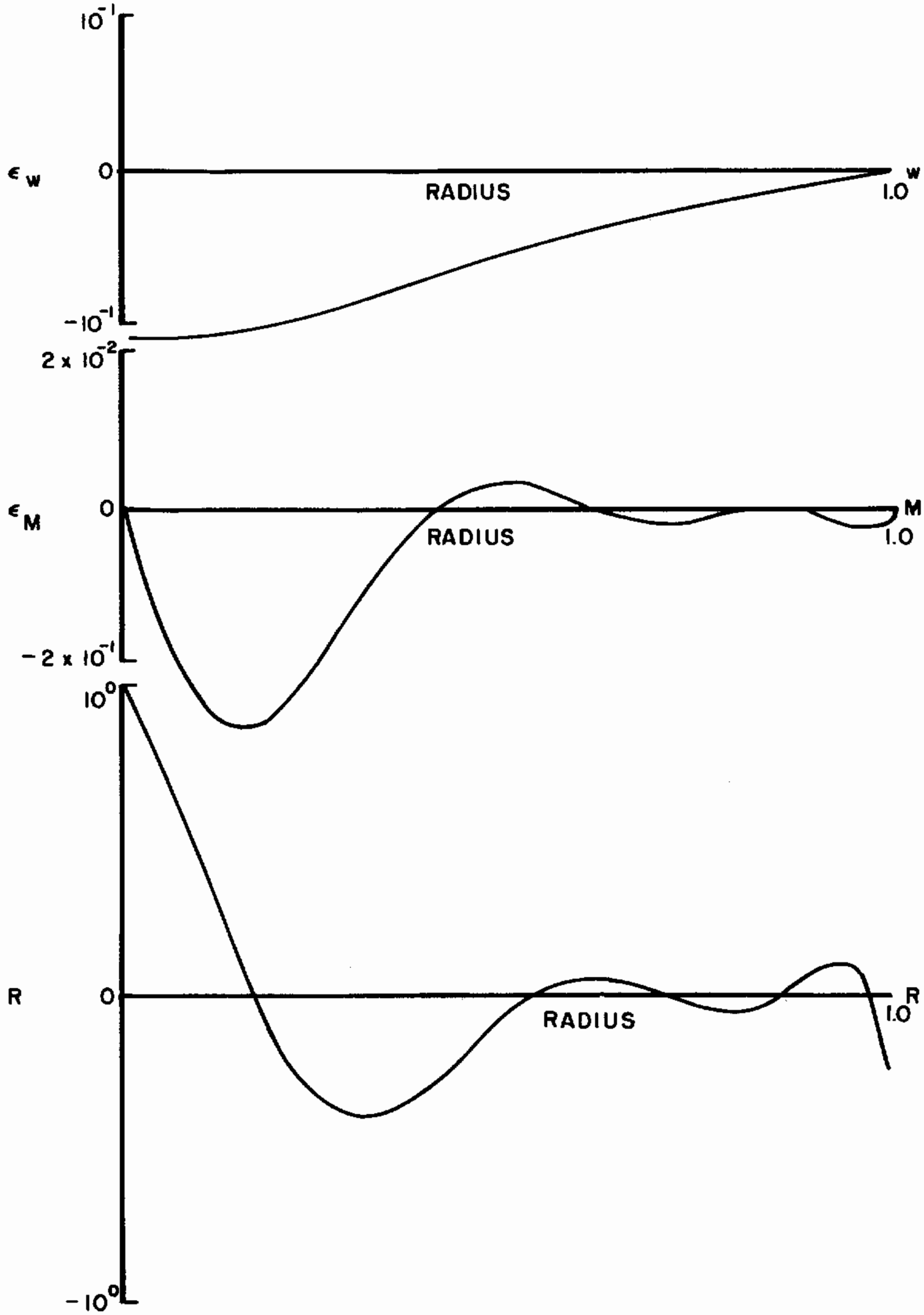



Figure 14. Deflection Error, Moment Error and Residual vs. Radius for Five Equal Area Increments

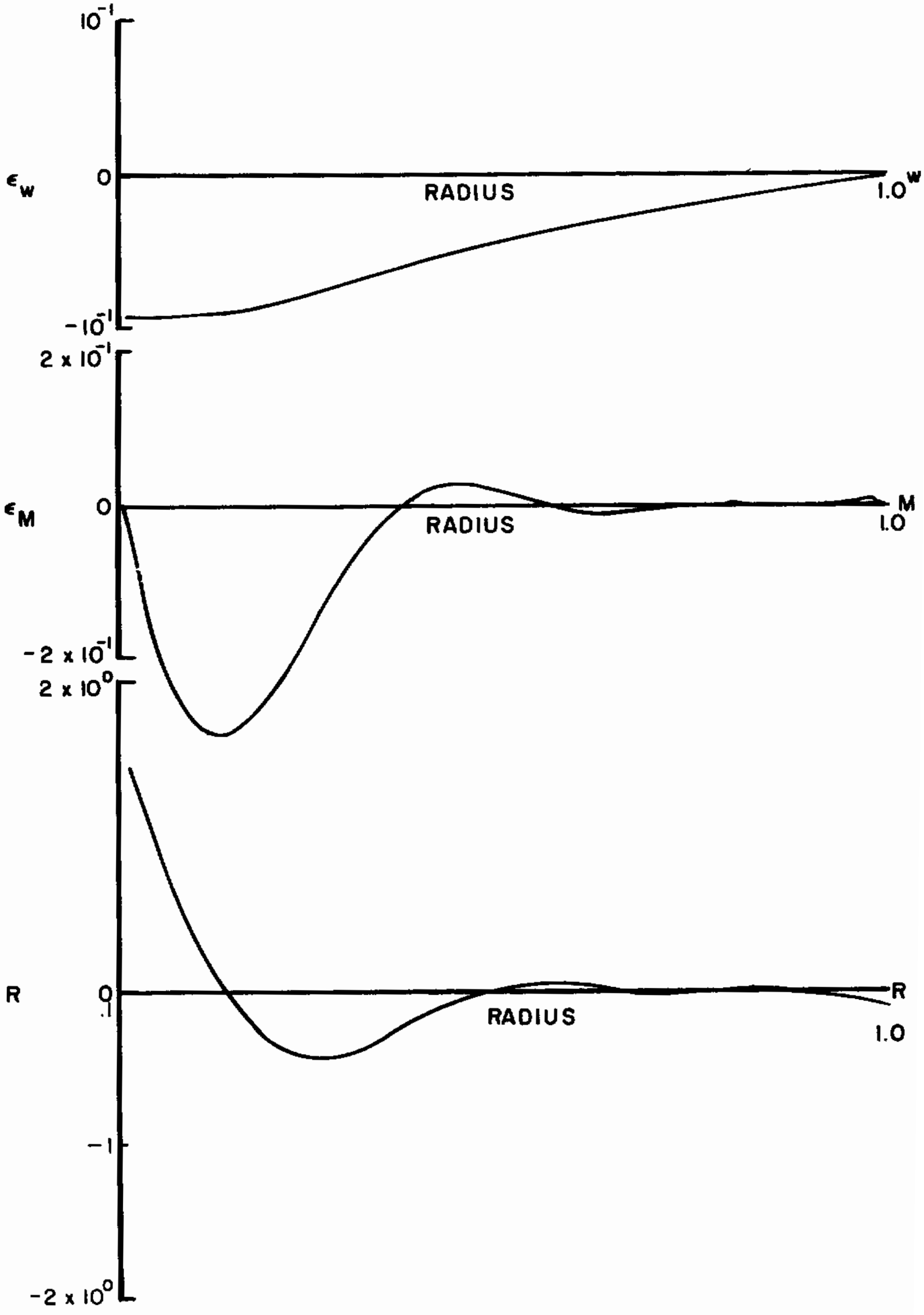



Figure 15. Deflection Error, Moment Error and Residual vs. Radius for Six Equal Area Increments

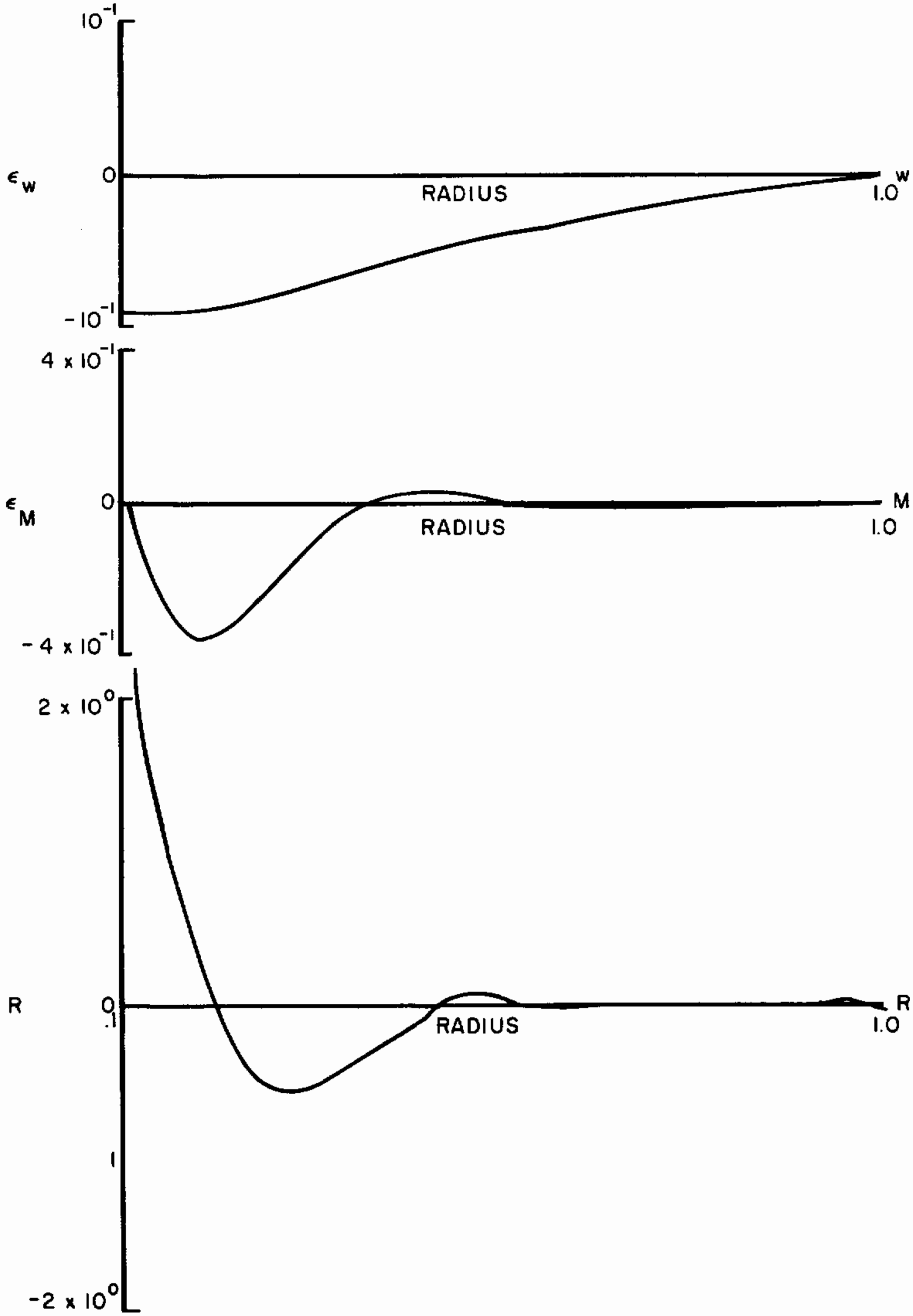



Figure 16. Deflection Error, Moment Error and Residual vs. Radius for Seven Equal Area Increments

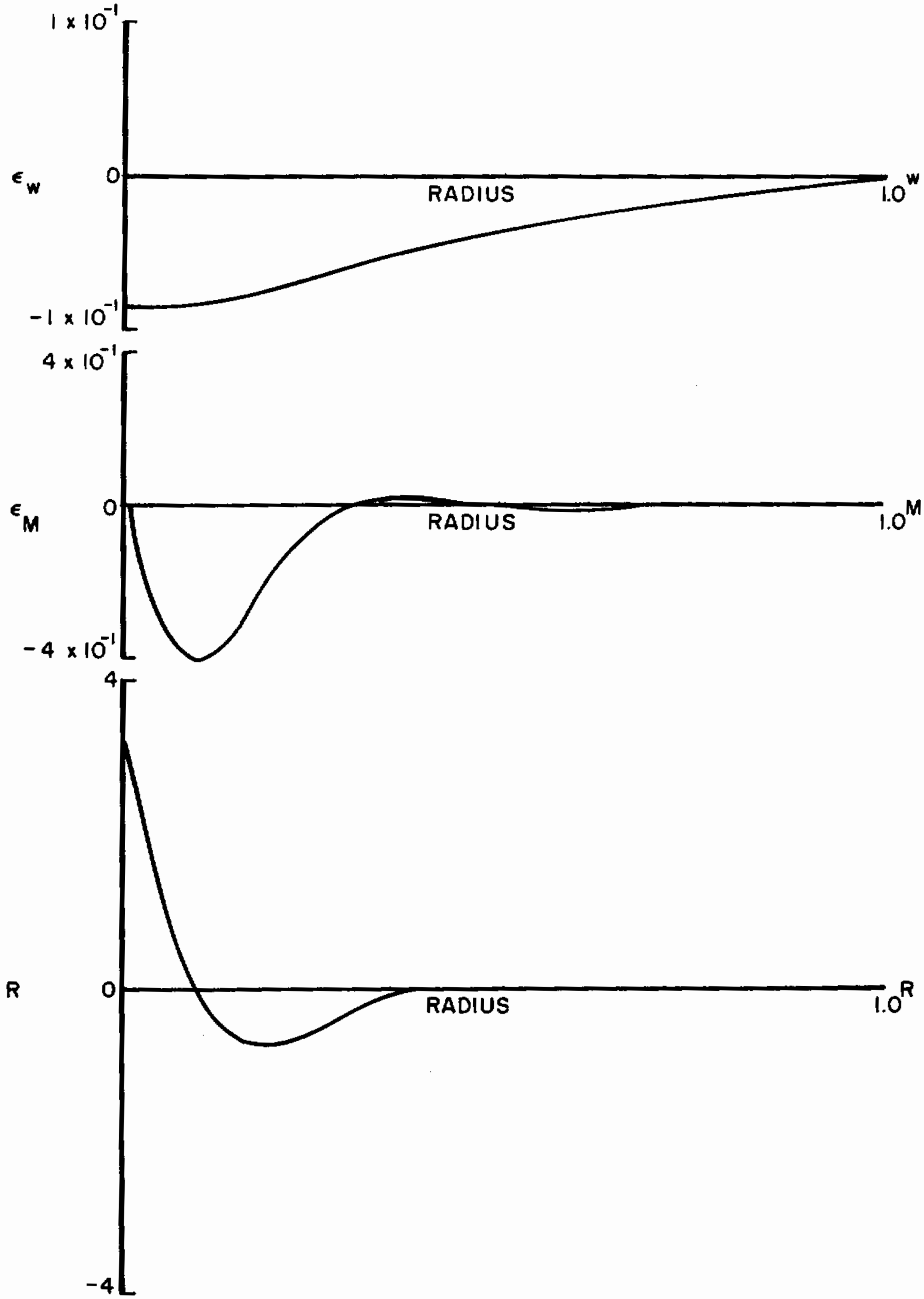



Figure 17. Deflection Error, Moment Error and Residual vs. Radius for Eight Equal Area Increments

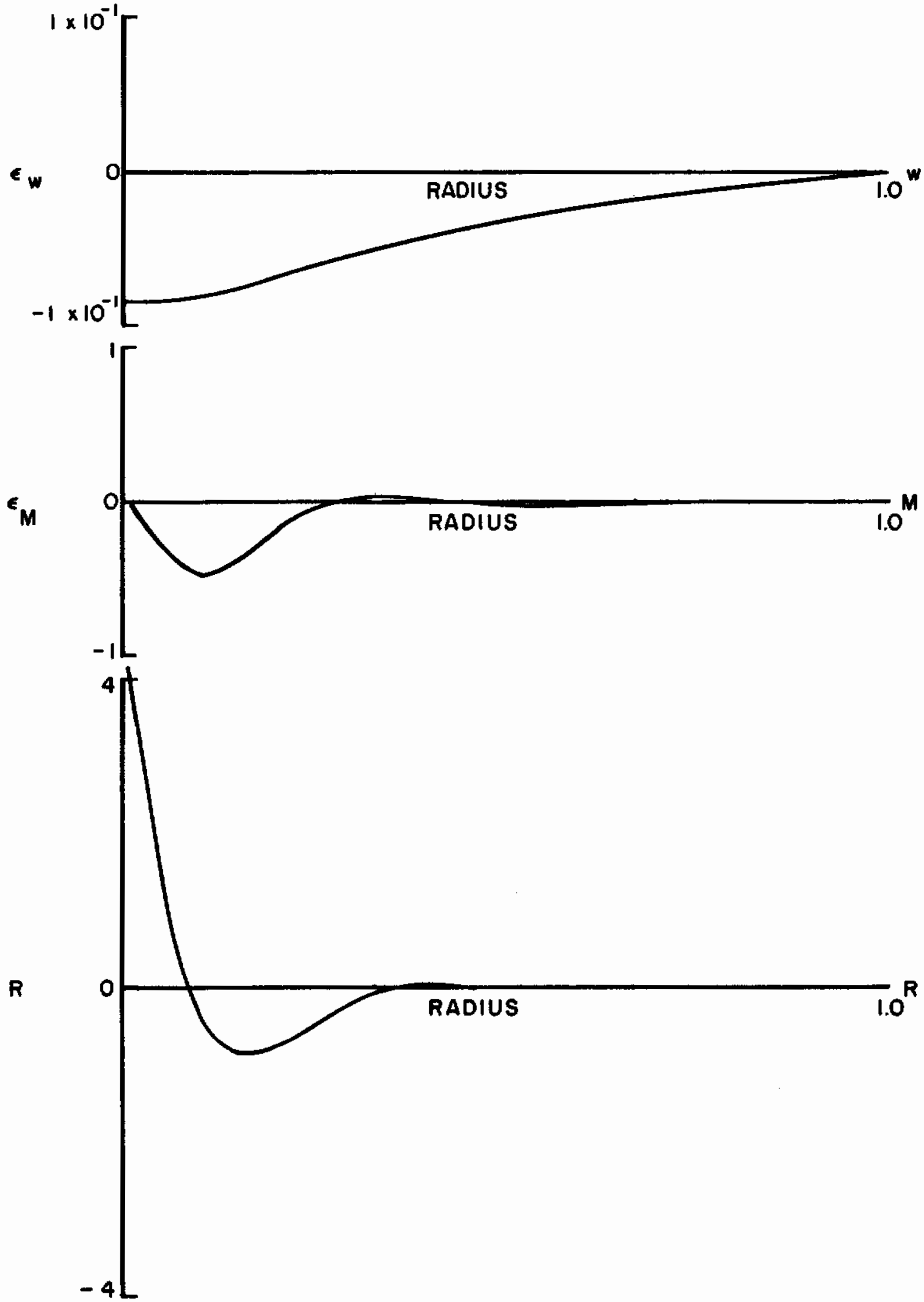



Figure 18. Deflection Error, Moment Error and Residual vs. Radius for Nine Equal Area Increments

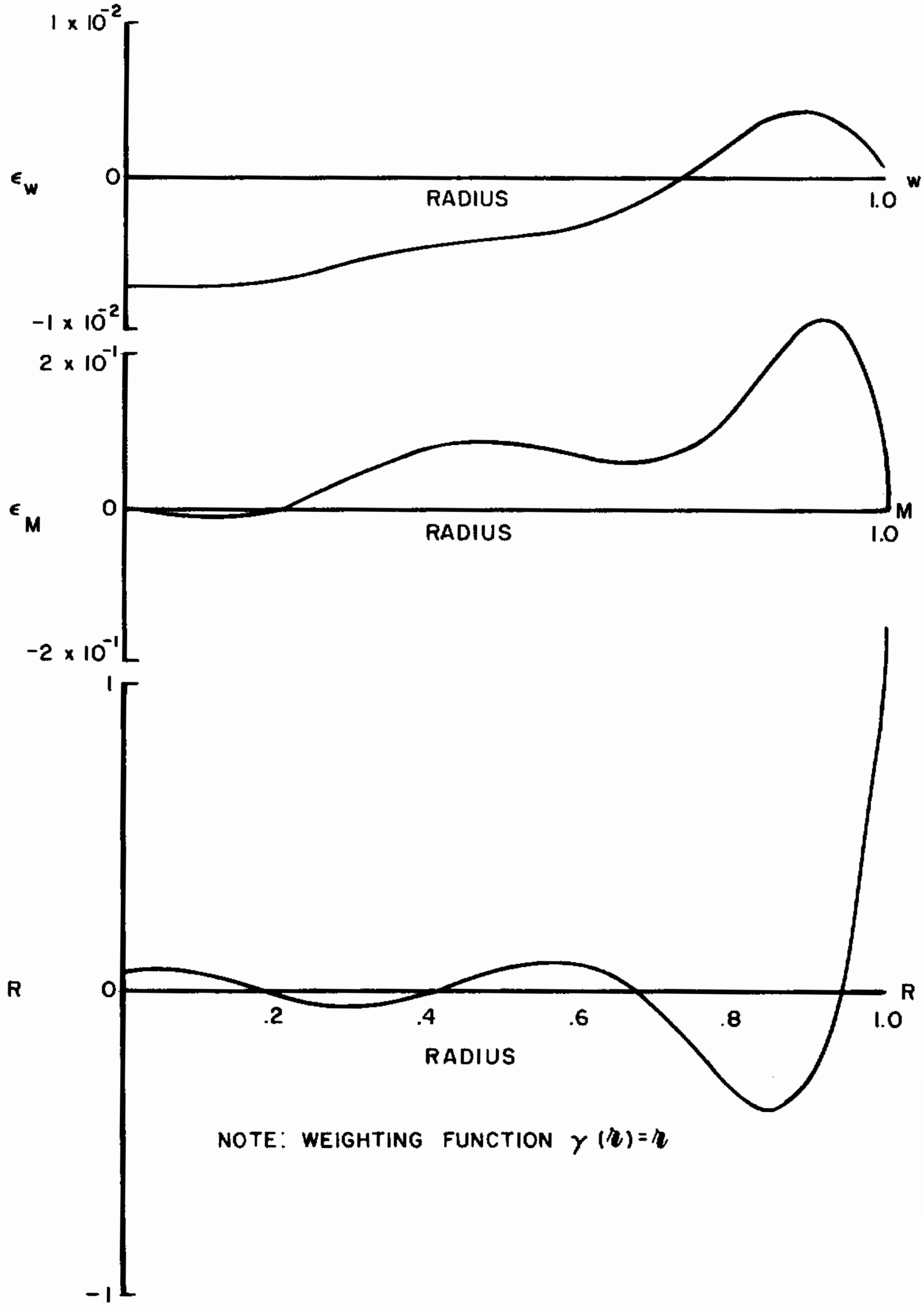



Figure 19. Deflection Error, Moment Error and Residual vs. Radius for Four Equal Radial Increments

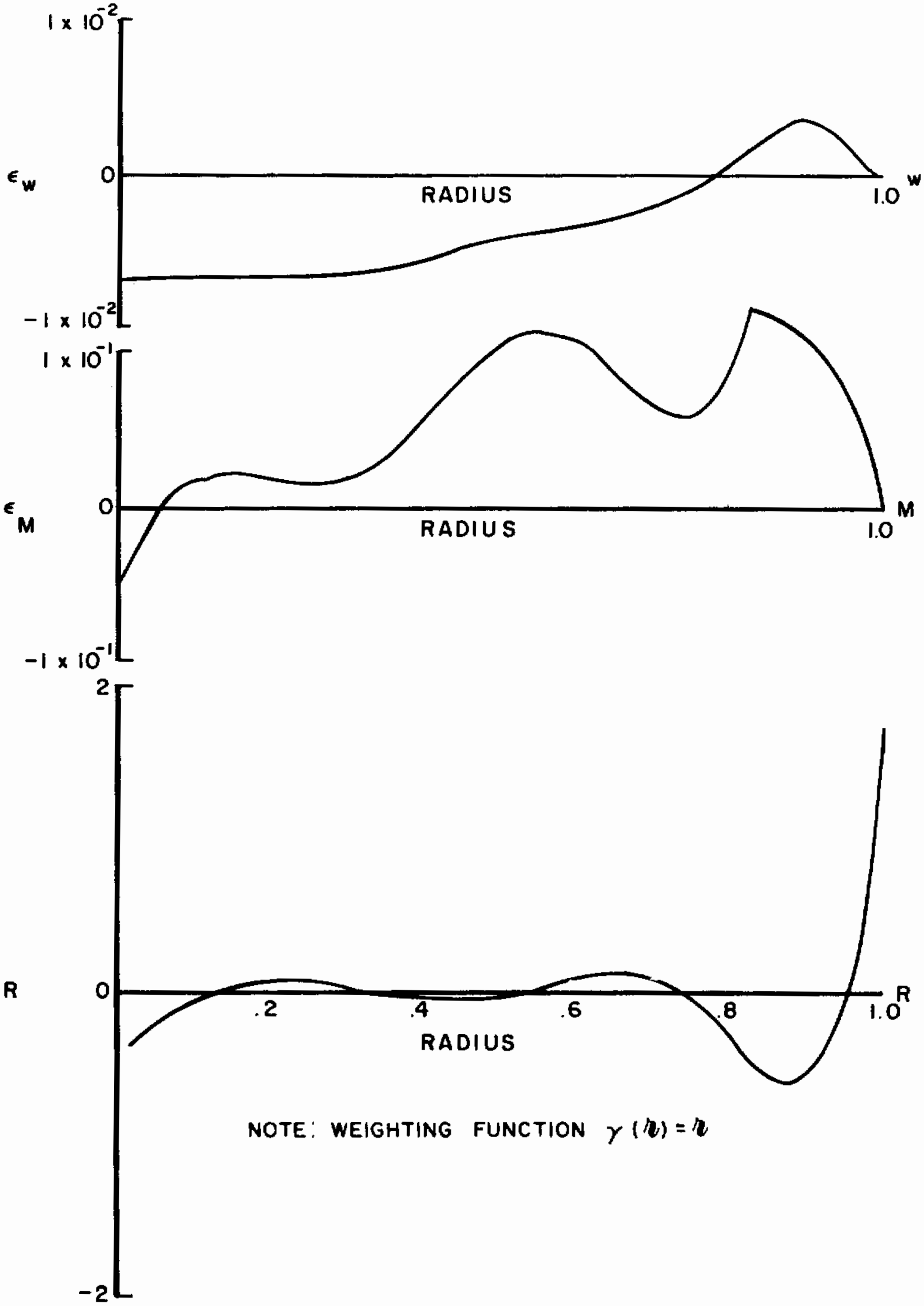



Figure 20. Deflection Error, Moment Error and Residual vs. Radius for Five Equal Radial Increments

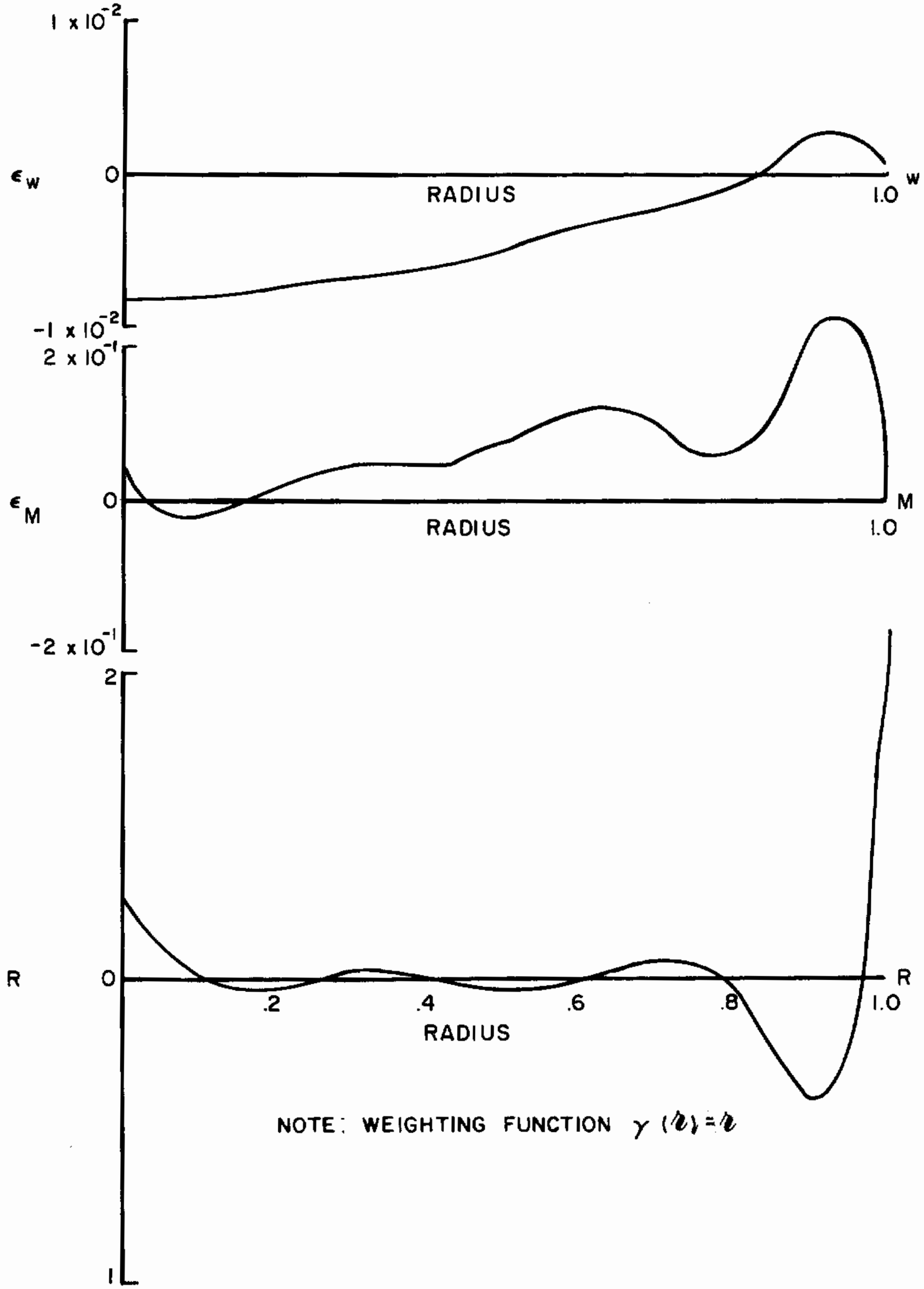



Figure 21. Deflection Error, Moment Error and Residual vs. Radius for Six Equal Radial Increments

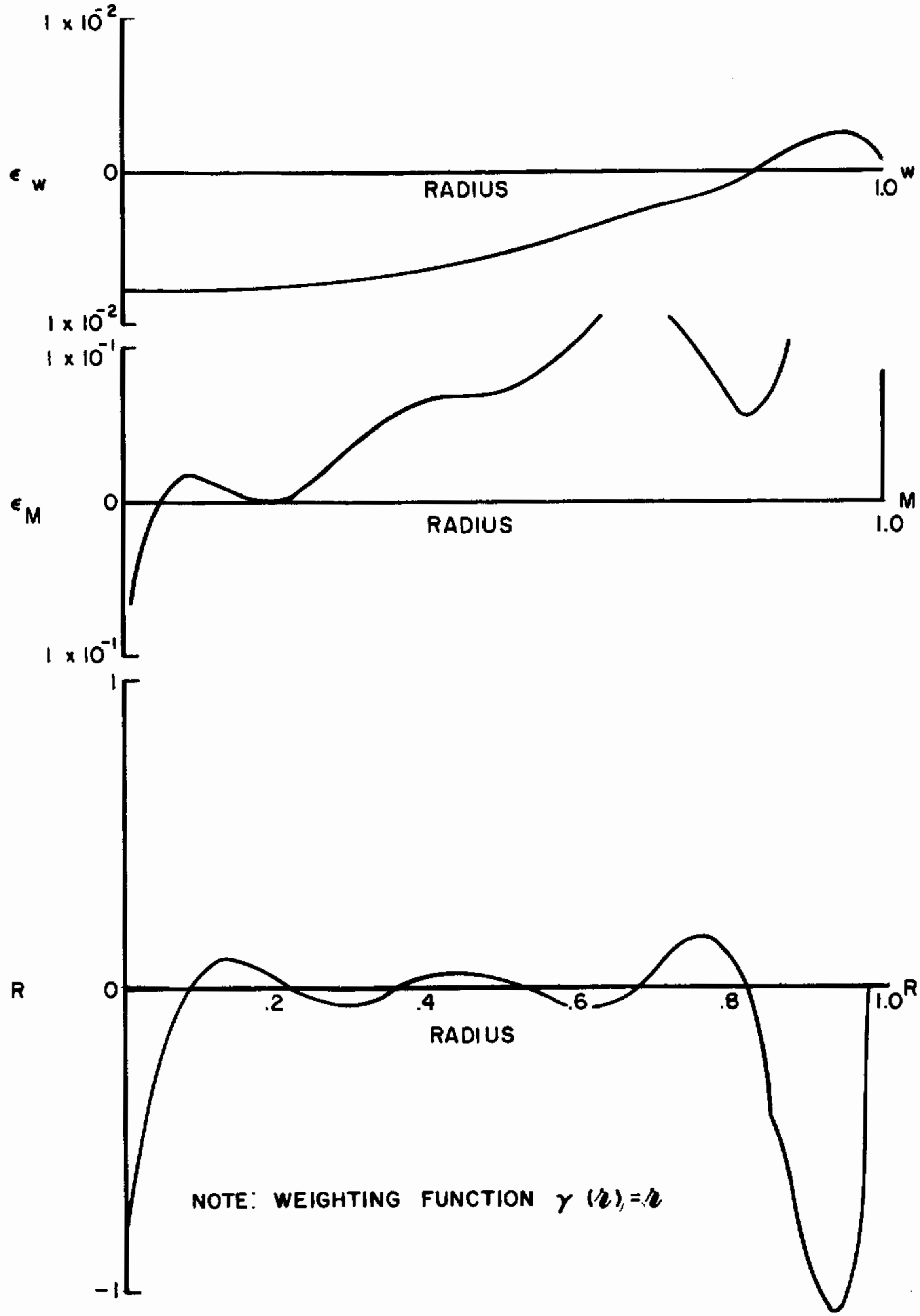



Figure 22. Deflection Error, Moment Error and Residual vs. Radius for Seven Equal Radial Increments

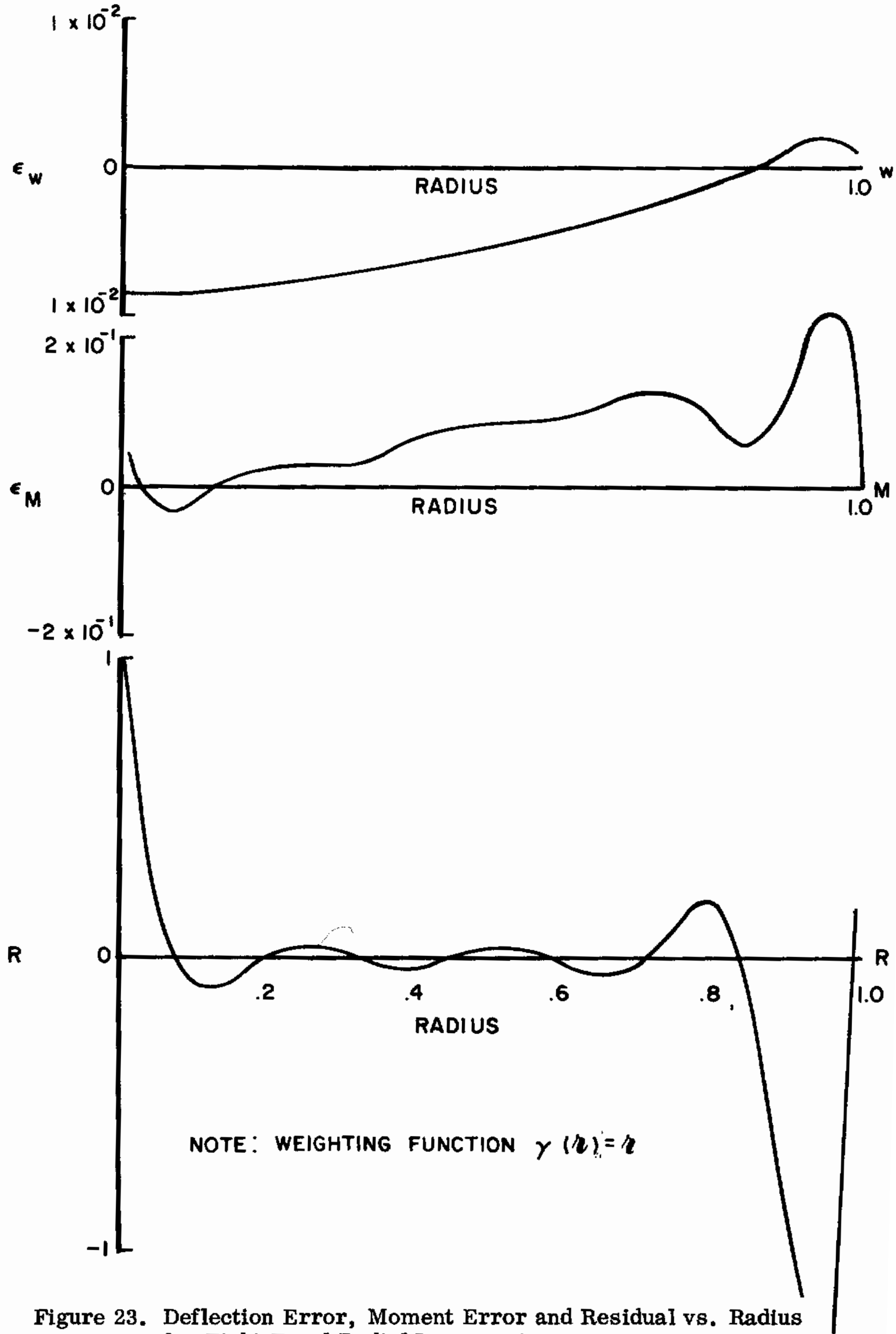



Figure 23. Deflection Error, Moment Error and Residual vs. Radius for Eight Equal Radial Increments

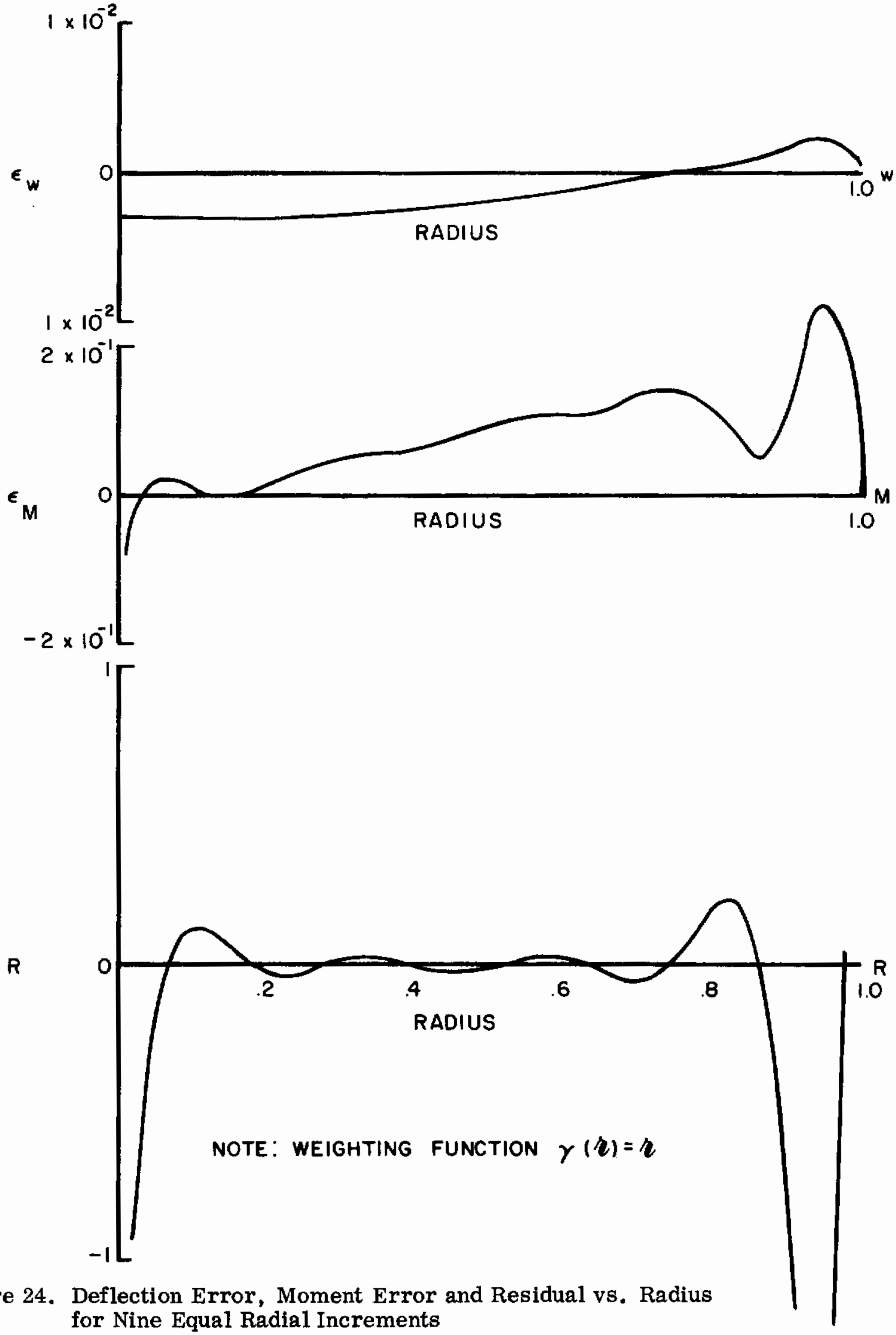



Figure 24. Deflection Error, Moment Error and Residual vs. Radius for Nine Equal Radial Increments

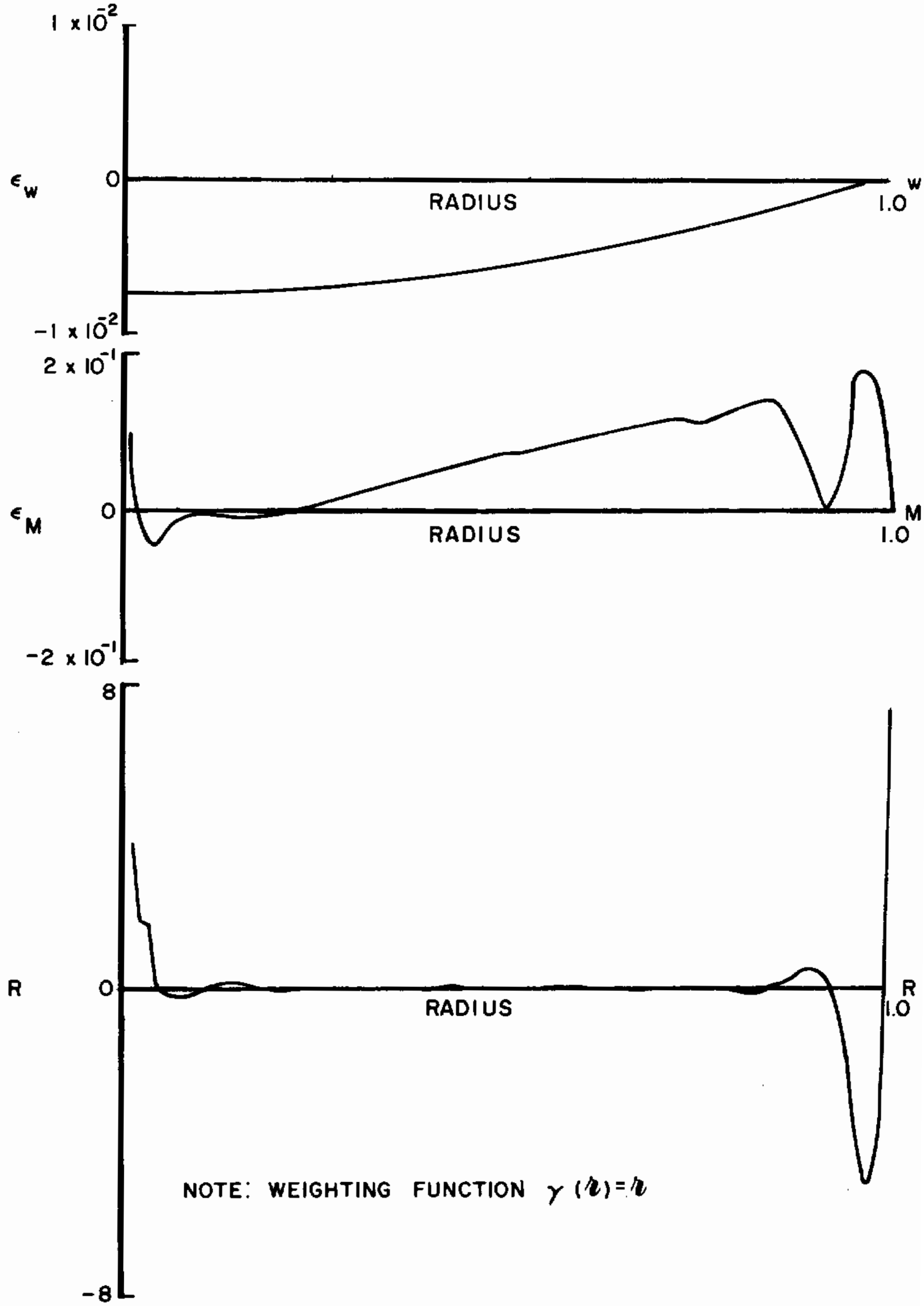



Figure 25. Deflection Error, Moment Error and Residual vs. Radius for Fifteen Equal Radial Increments

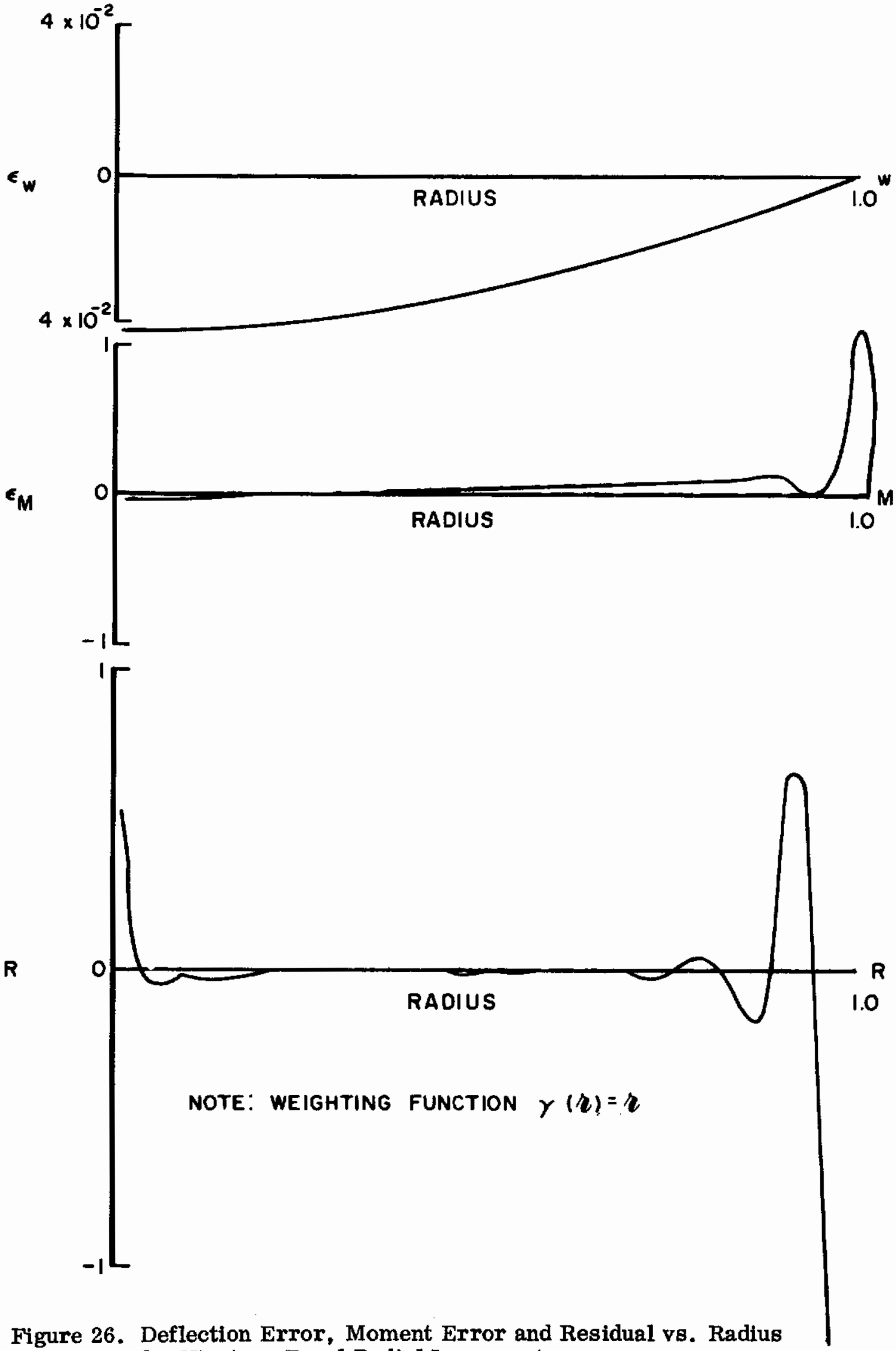



Figure 26. Deflection Error, Moment Error and Residual vs. Radius for Nineteen Equal Radial Increments

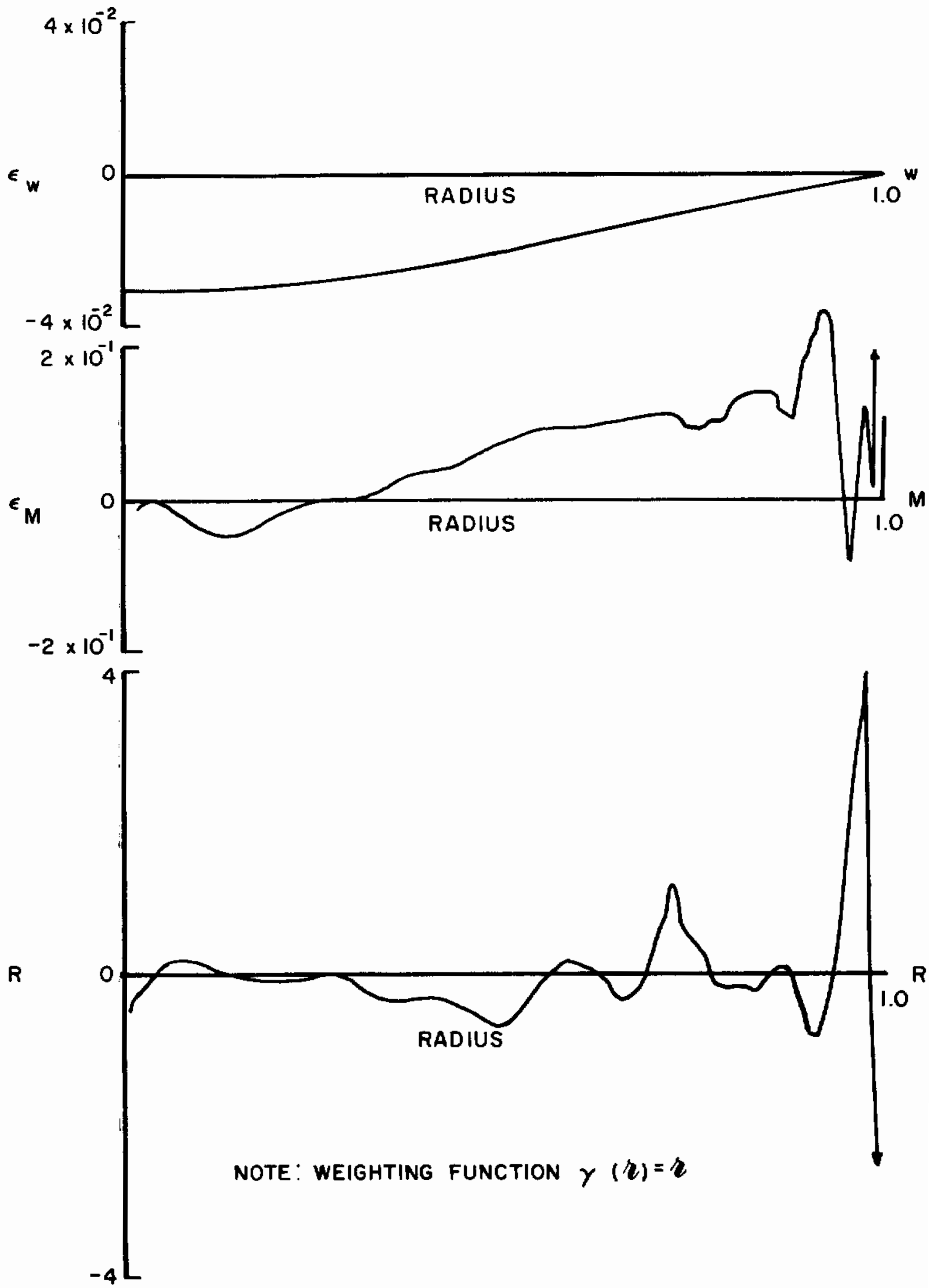



Figure 27. Deflection Error, Moment Error and Residual vs. Radius for Thirty Equal Radial Increments

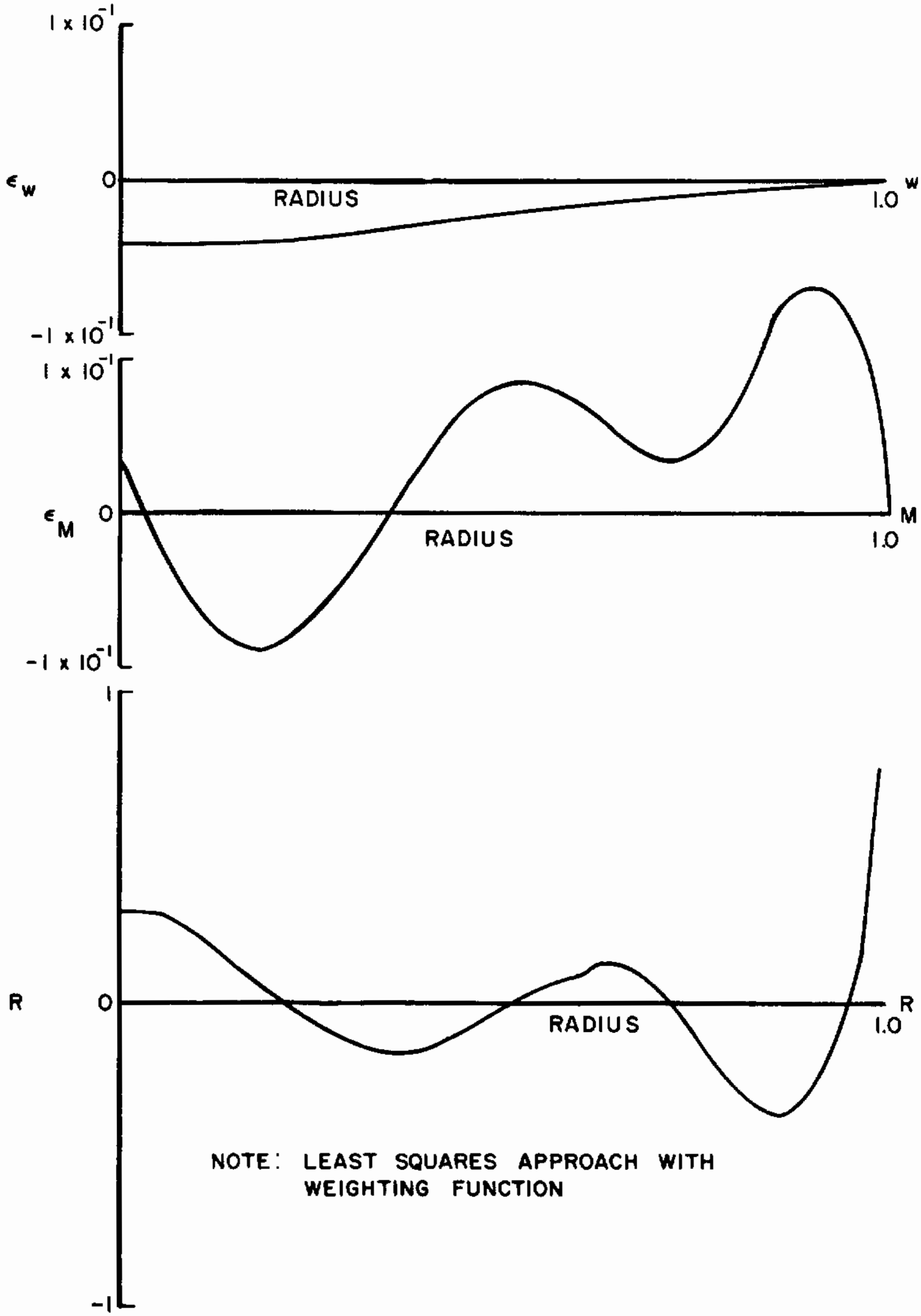



Figure 28. Deflection Error, Moment Error and Residual vs. Radius for Five Equal Radial Increments

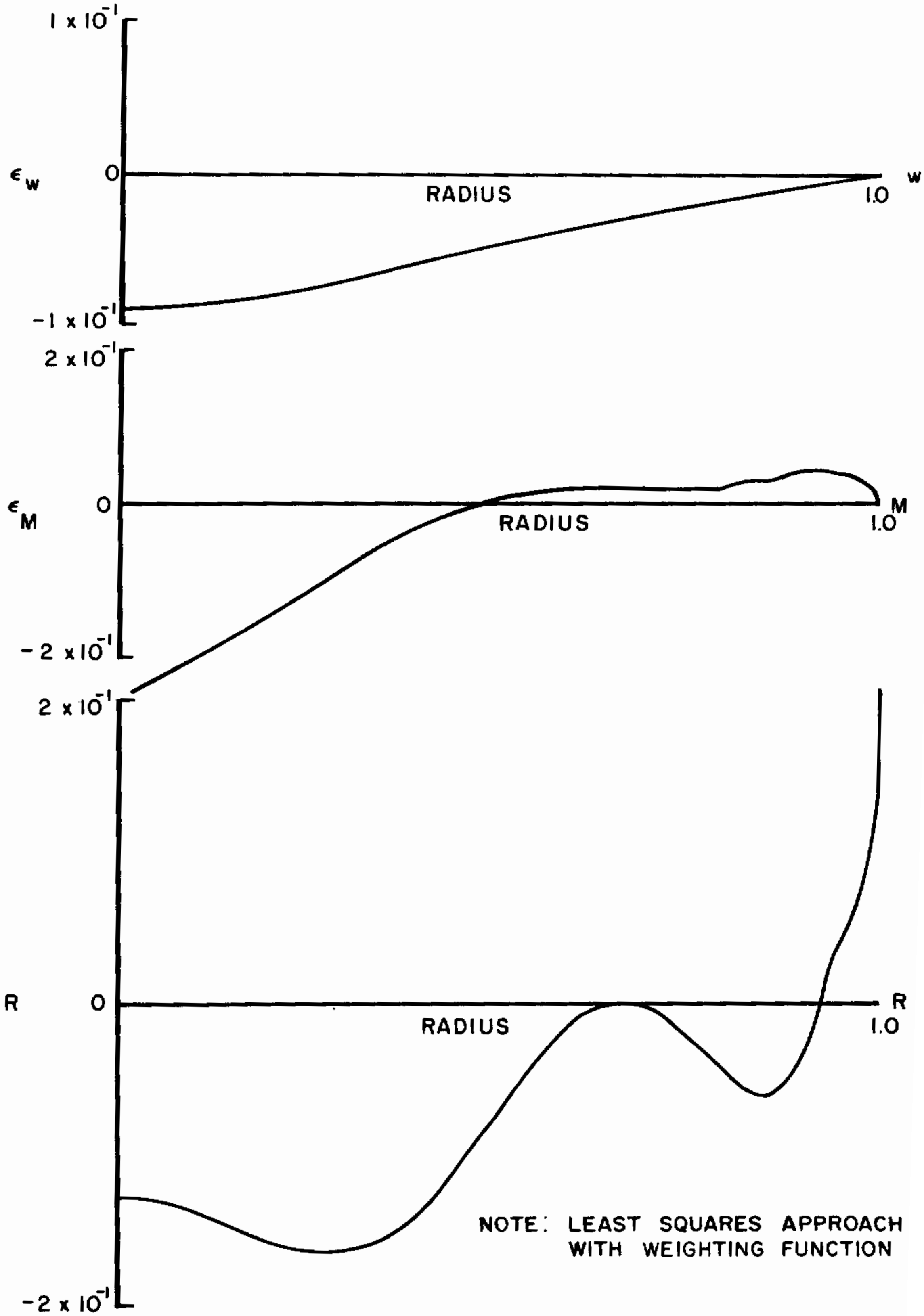



Figure 29. Deflection Error, Moment Error and Residual vs. Radius for Six Equal Radial Increments

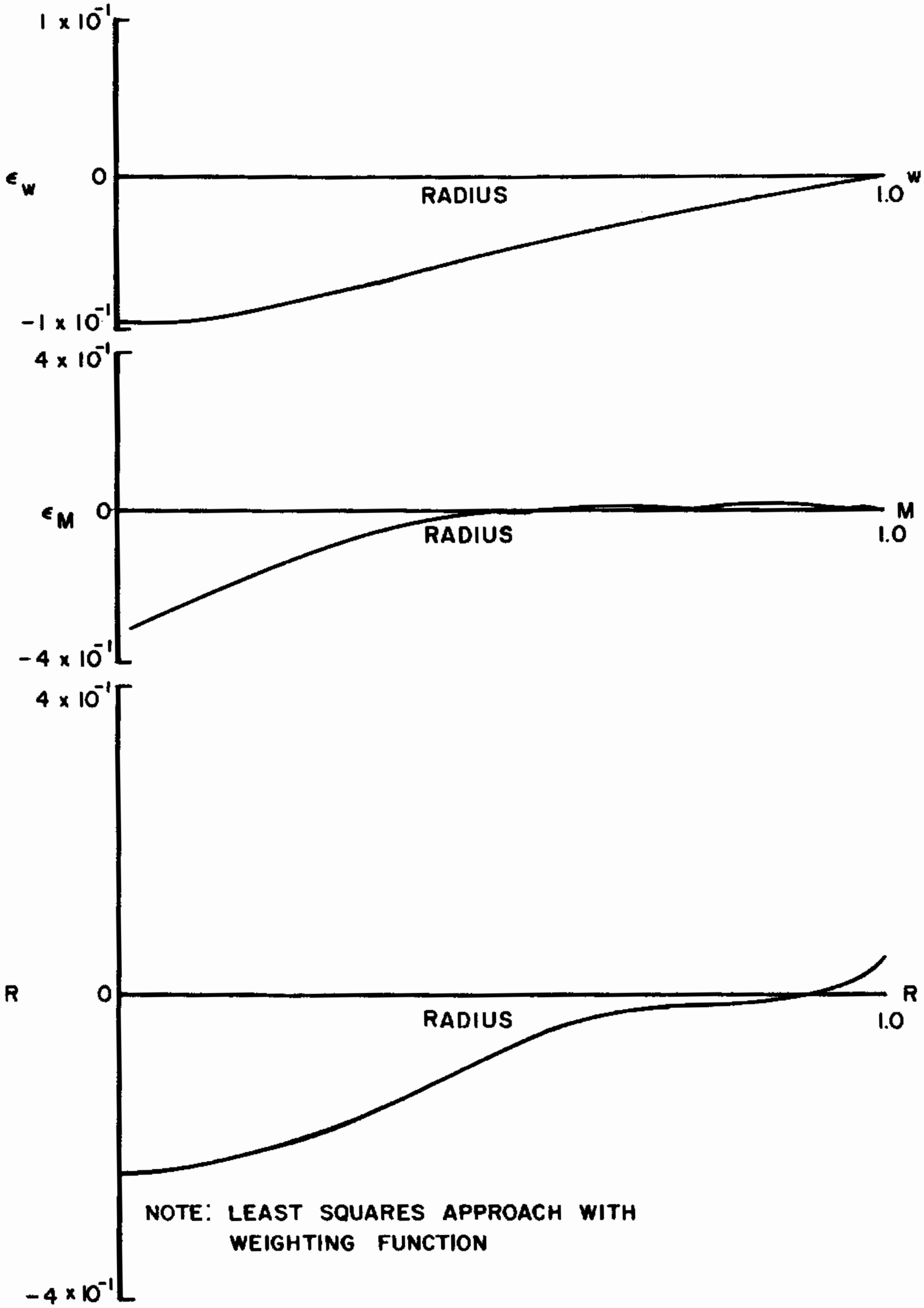



Figure 30. Deflection Error, Moment Error and Residual vs. Radius for Seven Equal Radial Increments

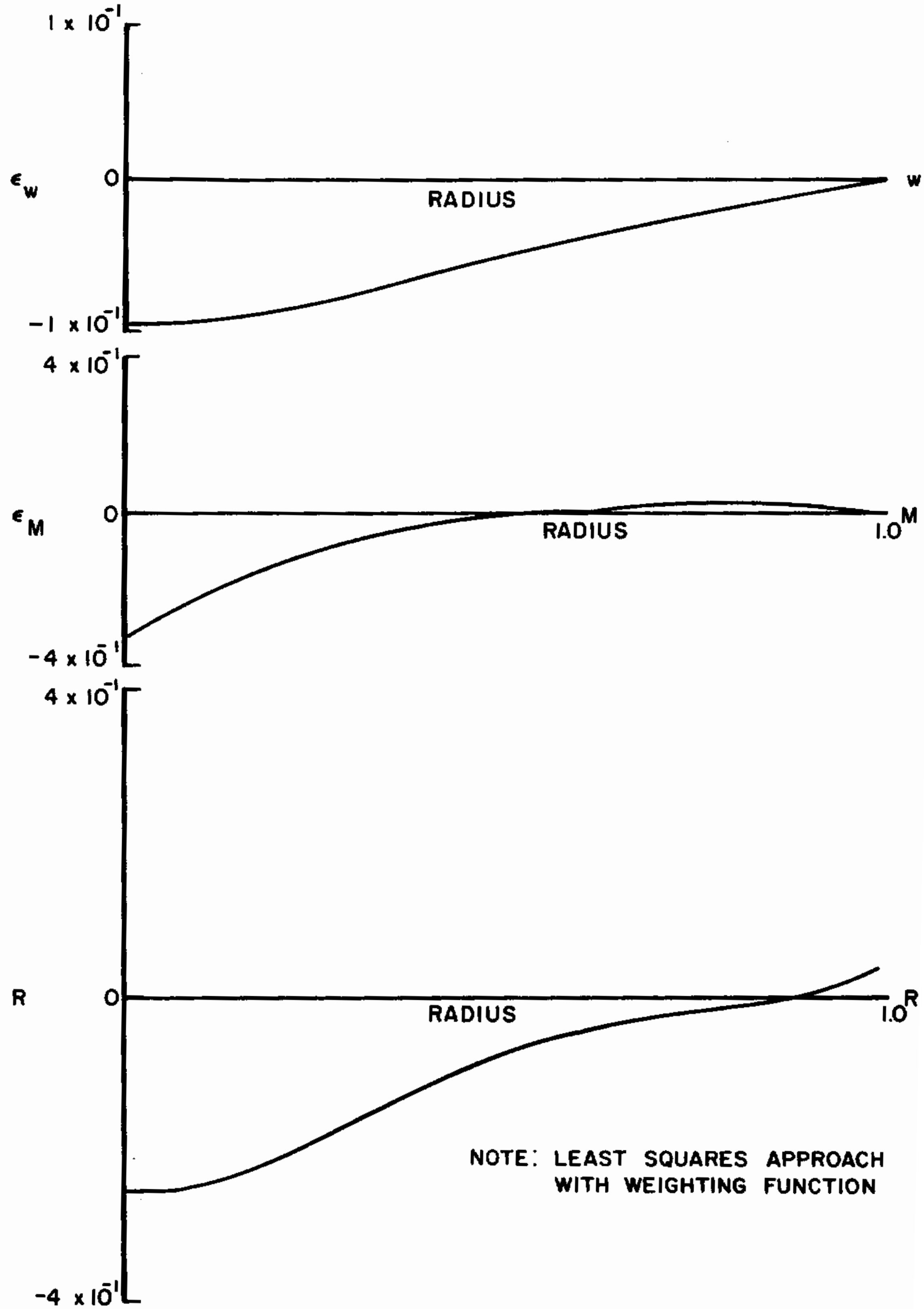



Figure 31. Deflection Error, Moment Error and Residual vs. Radius for Eight Equal Radial Increments

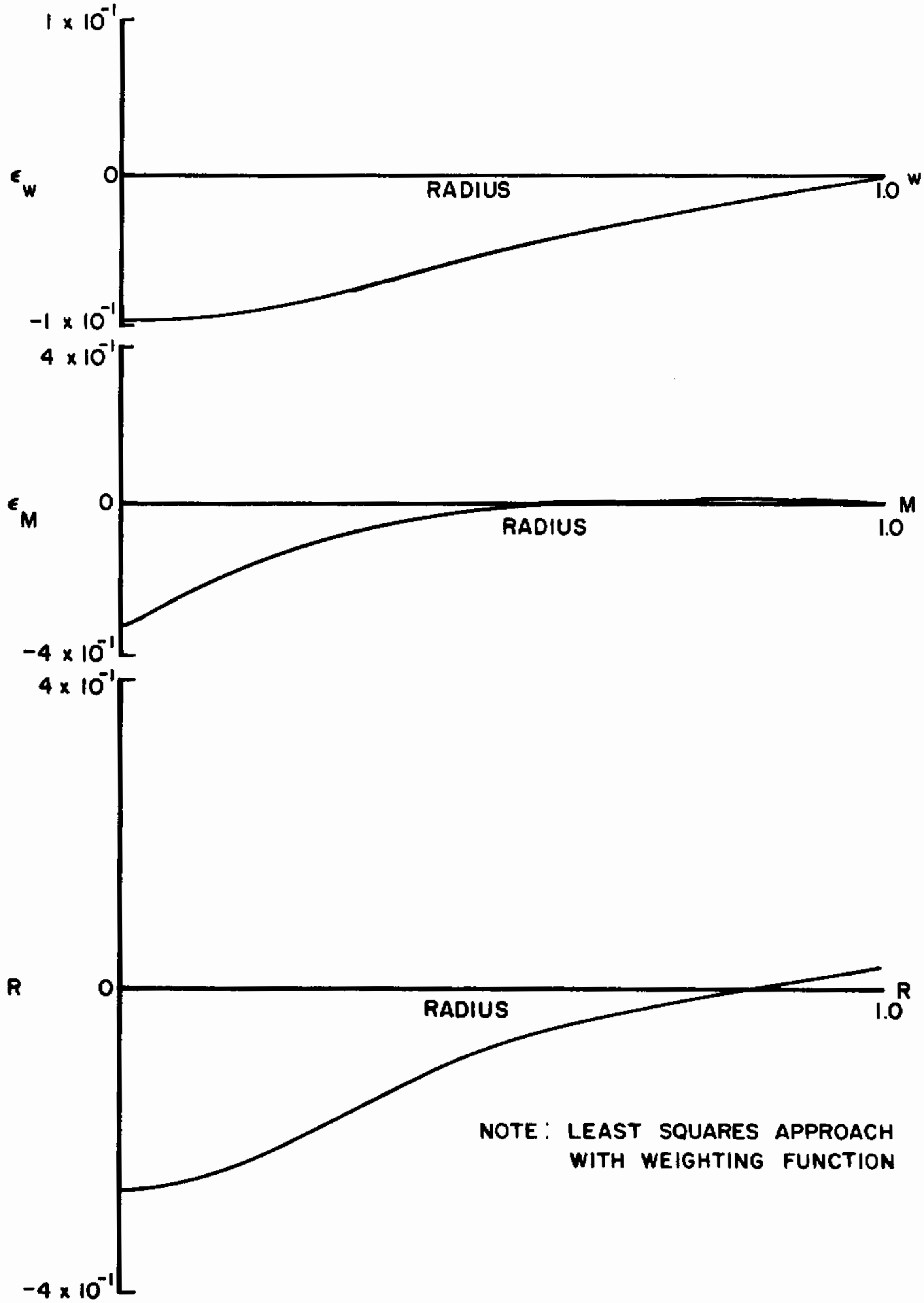



Figure 32. Deflection Error, Moment Error and Residual vs. Radius for Nine Equal Radial Increments

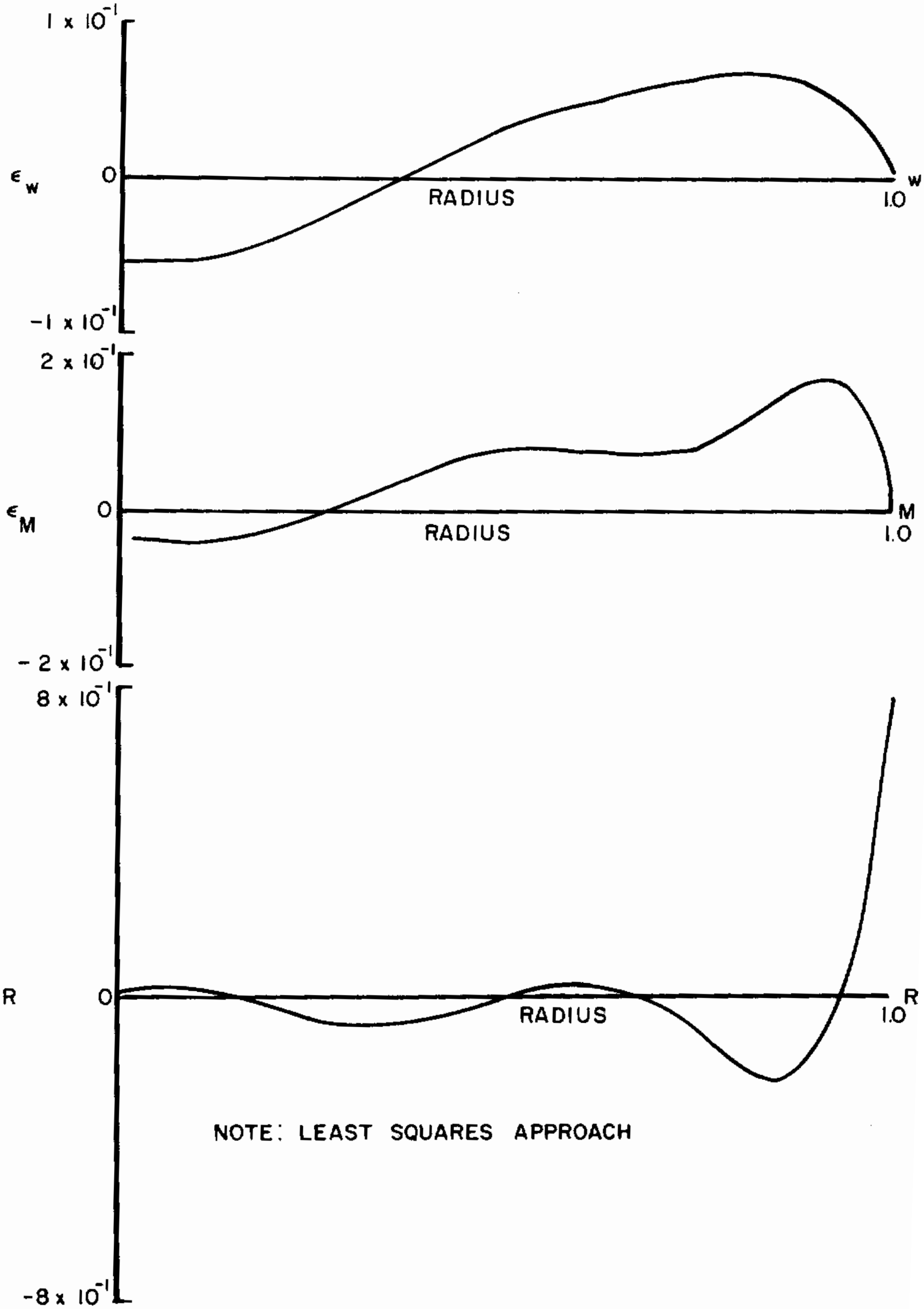



Figure 33. Deflection Error, Moment Error and Residual vs. Radius for Five Equal Radial Increments

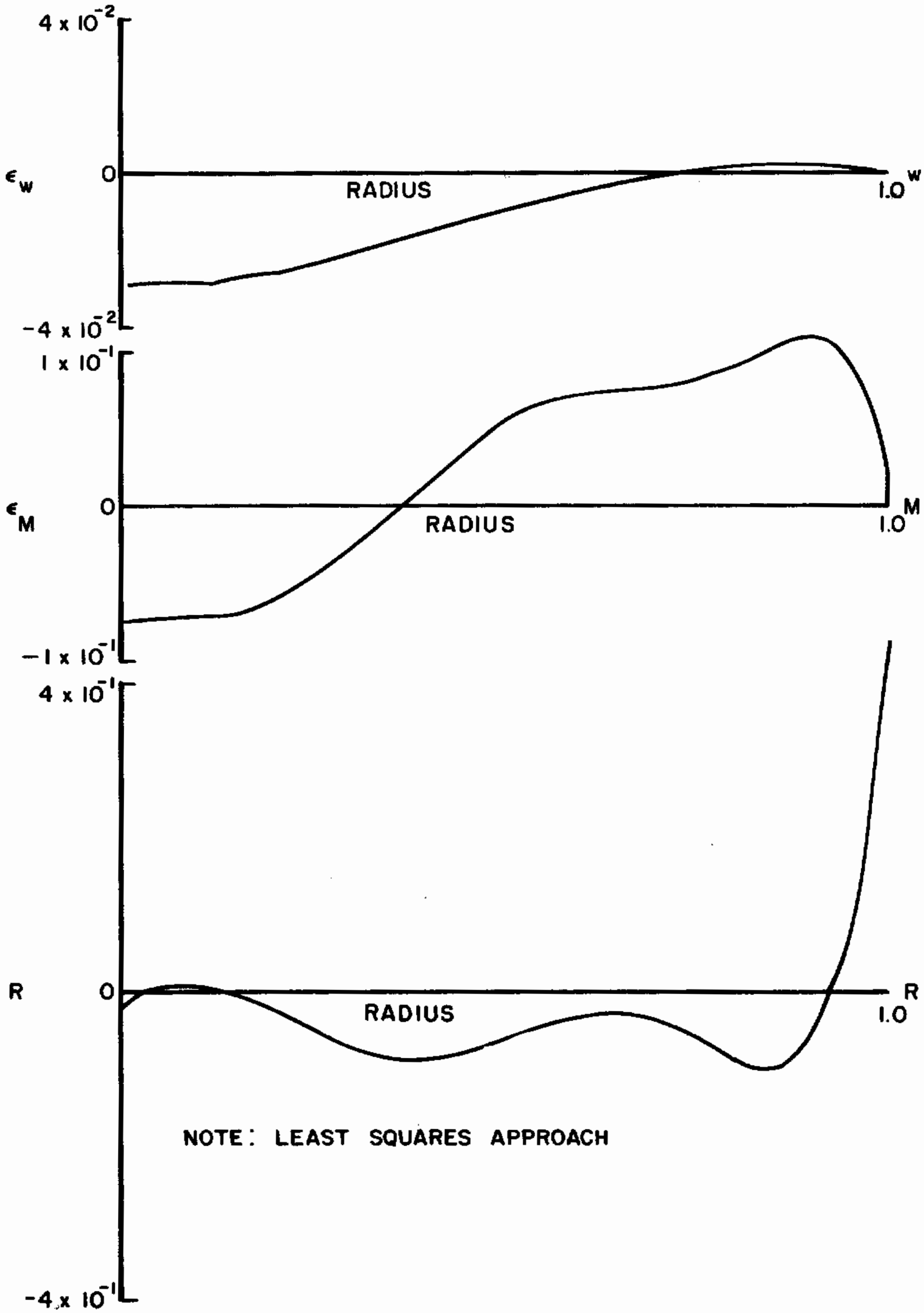



Figure 34. Deflection Error, Moment Error and Residual vs. Radius for Six Equal Radial Increments

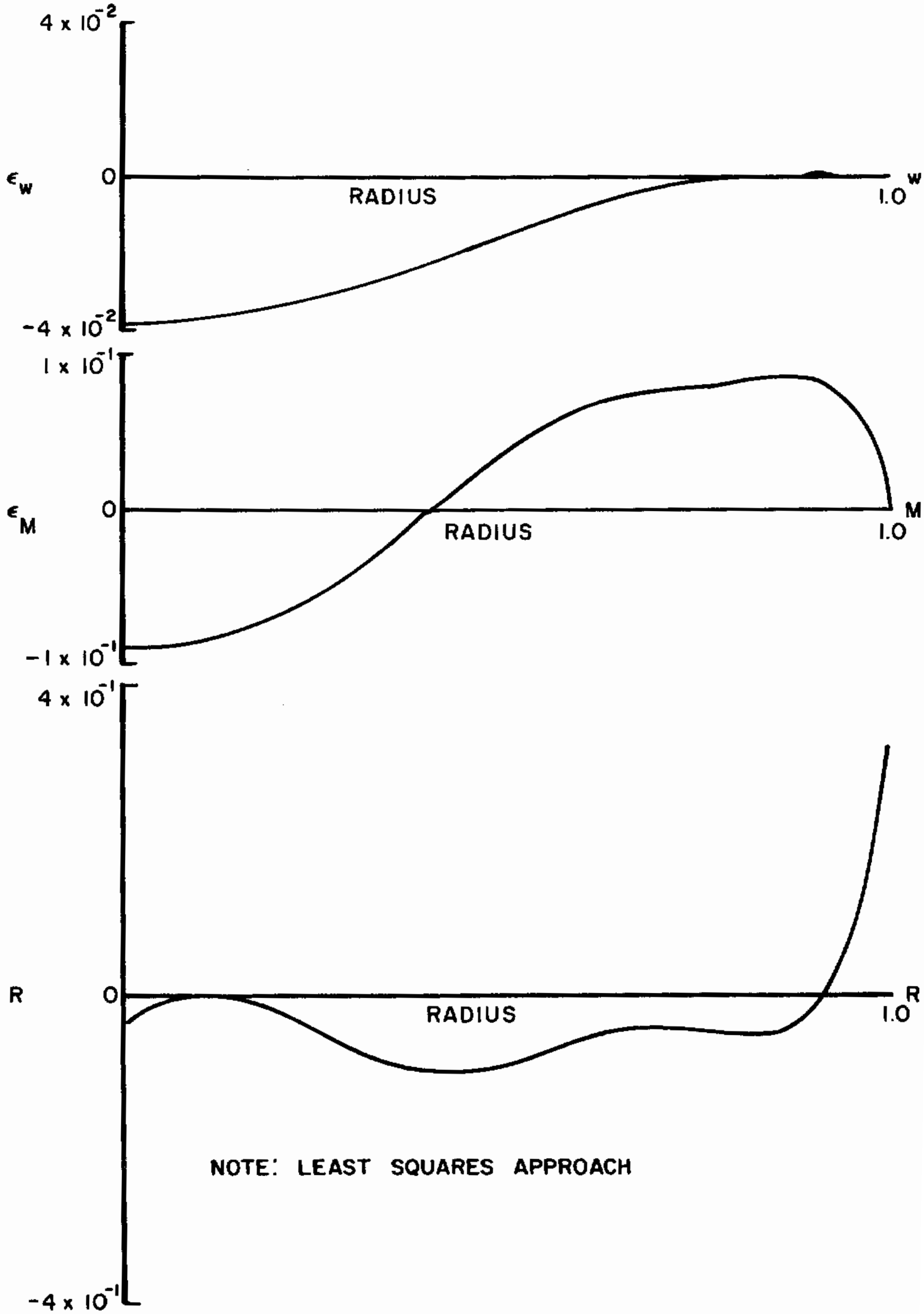



Figure 35. Deflection Error, Moment Error and Residual vs. Radius for Seven Equal Radial Increments

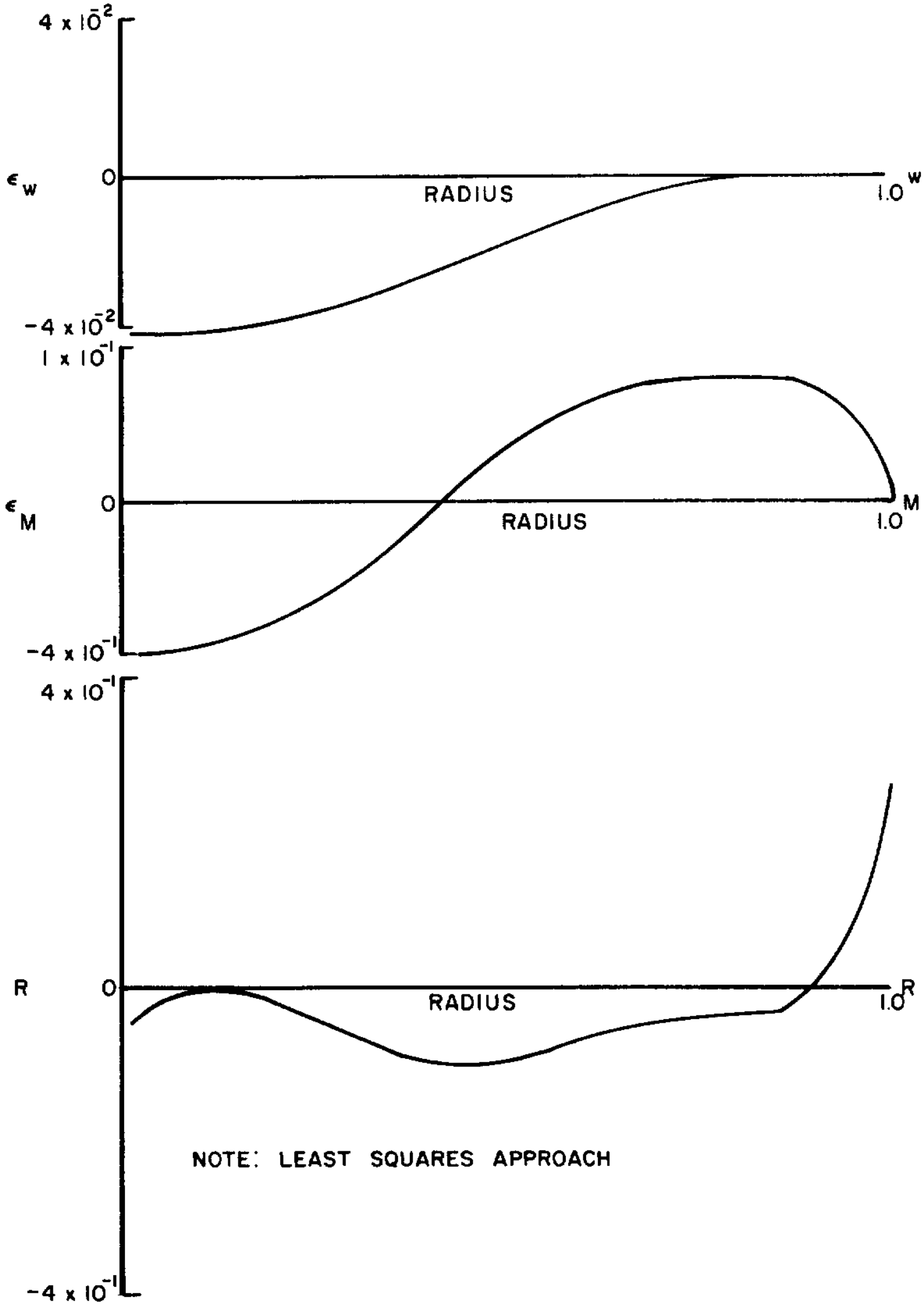



Figure 36. Deflection Error, Moment Error and Residual vs. Radius for Eight Equal Radial Increments

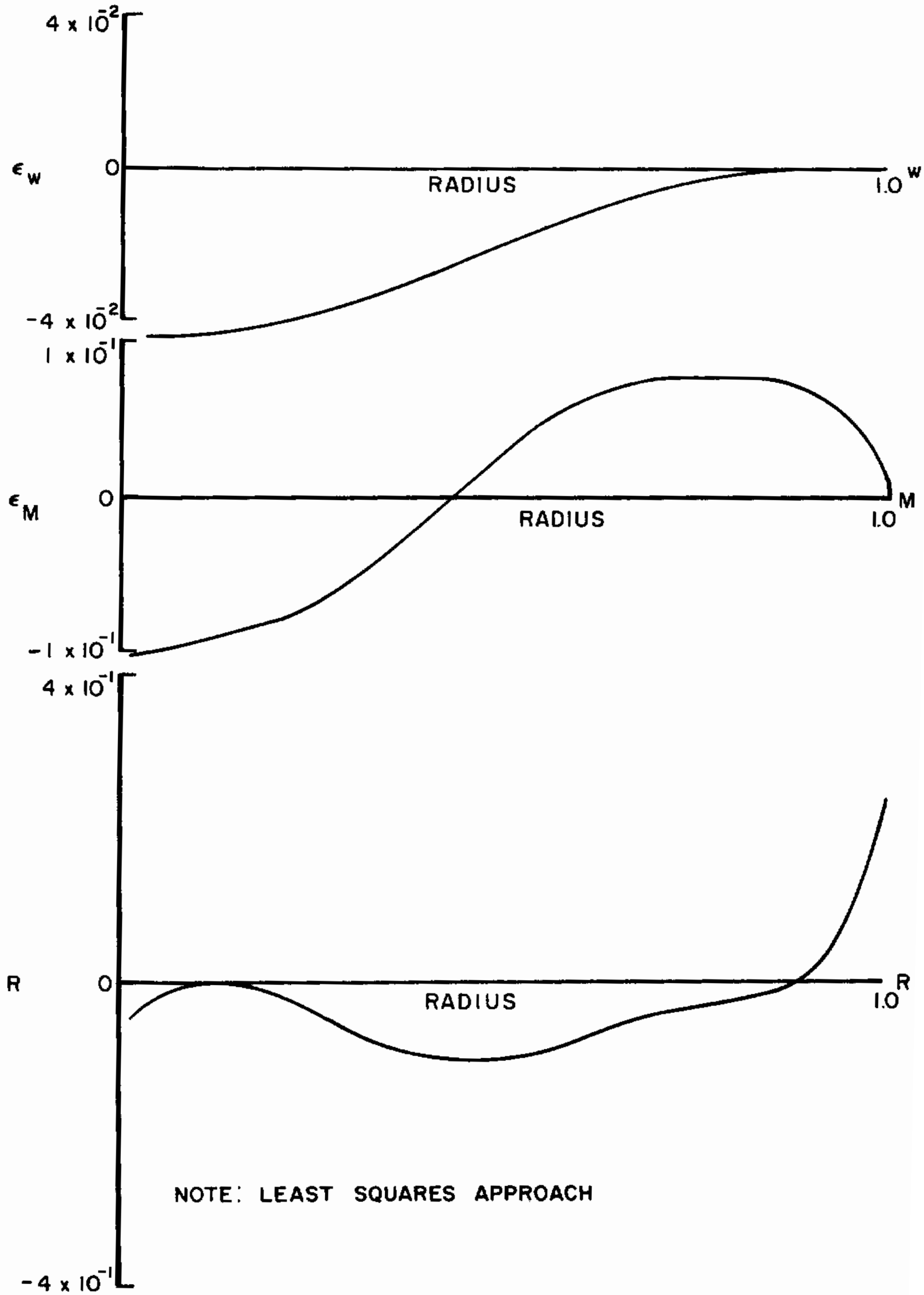



Figure 37. Deflection Error, Moment Error and Residual vs. Radius for Nine Equal Radial Increments

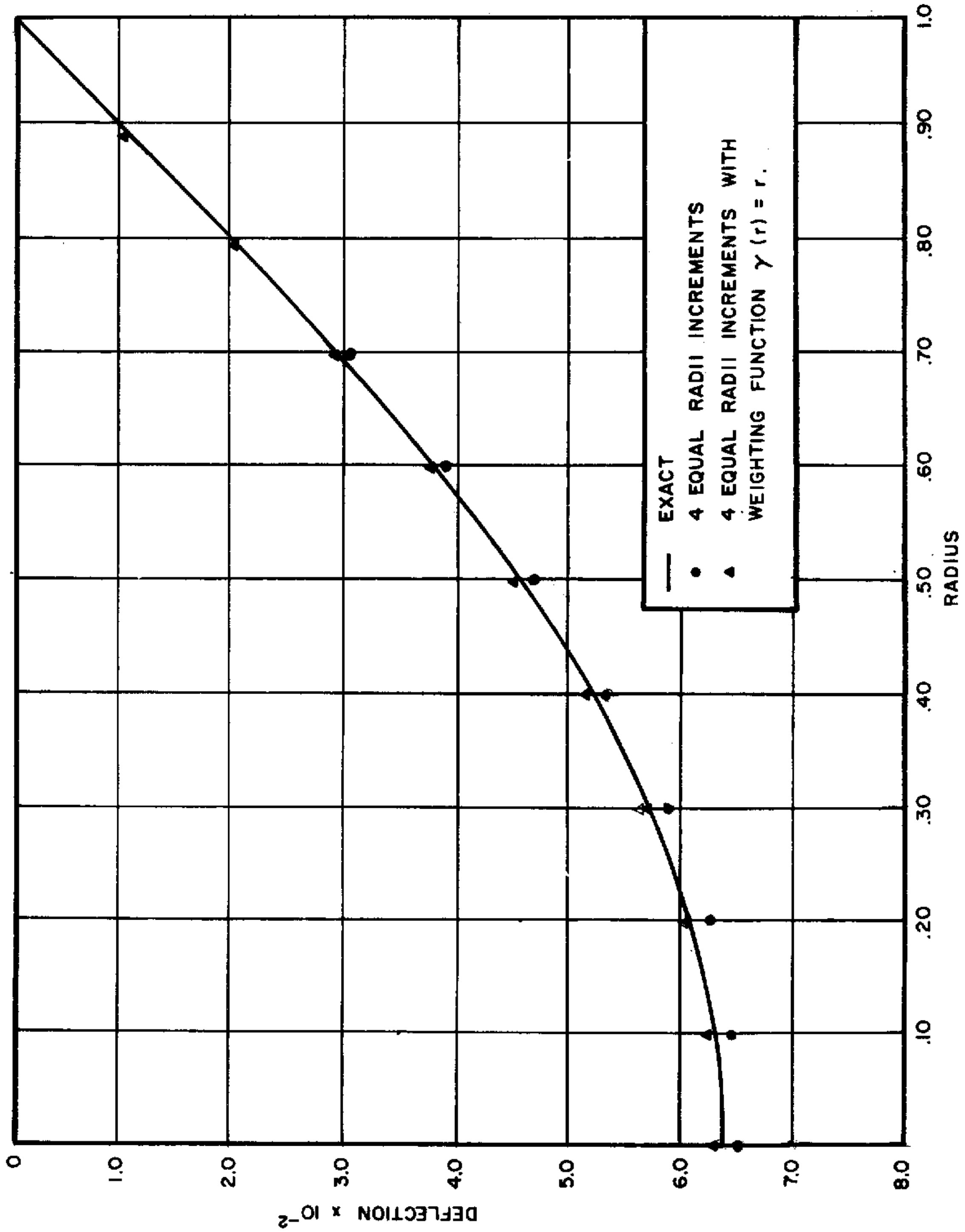



Figure 38. Deflection vs Radius

# APPENDIX II

# OTHER APPROACHES

## 1. TRANSFORMATION OF COORDINATE

By a suitable transformation of the independent coordinate, it is sometimes possible to obtain a simplified differential equation. Thus let:

$$x = \left(\frac{r}{a}\right)^{1/n} \tag{26}$$

Determine the first, second, and third derivatives and substitute in the residual equation:

$$R = \frac{d^3 w}{dr^3} + \frac{1}{r}\frac{d^2 w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr} - \frac{qr}{2\bar{D}} \tag{27}$$

The transformed residual equation becomes:

$$R = \frac{x^{3(1-n)}}{(an)^3}\frac{d^3 w}{dx^3} + \frac{3-2n}{(an)^3}\, x^{2-3n}\frac{d^2 w}{dx^3} + \frac{1-2n}{(an)^3}\, x^{1-3n}\frac{dw}{dx} - \left(\frac{ax^n}{2}\right)\frac{q}{\bar{D}} \tag{28}$$

An examination of Equation 28 shows that if $0 < n \leq 1/3$ none of the terms will have an x coefficient with a negative exponent. Therefore an infinite residual will not exist at $x = 0$ (for all powers of x in $w_n(x) = Cx^k$ where $k \geq 0$).

Assuming $n = 1/3$, Equation 26 becomes:

$$x = \left(\frac{r}{a}\right)^3$$

and the residual equation becomes:

$$R = \frac{27x^2}{a^3}\frac{d^3 w}{dx^3} + \frac{63x}{a^3}\frac{d^2 w}{dx^2} + \frac{9}{a^3}\frac{dw}{dx} - \frac{ax^{1/3}}{2} \tag{29}$$

By means of the transform Equation 29 the boundary conditions are changed to:

$$w = 0 \quad \text{at} \quad x = 1 \quad \text{or} \quad r = a \tag{30}$$

and

$$M_r = 0 = -\frac{\bar{D}}{a^2}\left[9x^{4/3}\frac{d^2 w}{dx^2} + 3(2+\nu)\, x^{1/3}\frac{dw}{dx}\right] \tag{31}$$

$$\text{at} \quad x = 1 \quad \text{or} \quad r = a$$

$$\frac{dw}{dr} = 0 = \frac{3}{a}\, x^{2/3}\frac{dw}{dx} \quad \text{at} \quad x = 0 \quad \text{or} \quad r = 0 \tag{32}$$

if $w_n = Cx^k$, Equation 32 is satisfied if $k > \frac{1}{3}$ or $= 0$.

In addition the intuitive conditions become:

$$w \neq 0 \quad \text{at} \quad x = 0 \quad \text{or} \quad r = 0 \tag{33}$$

$$\frac{dw}{dr} \neq 0 = \frac{3}{a} x^{2/3} \frac{dw}{dx} \quad \text{at} \quad x = 1 \quad \text{or} \quad r = a$$

therefore:

$$\frac{dw}{dx} \neq 0 \quad \text{at} \quad x = 1 \tag{34}$$

$$M_r \neq 0 \quad \text{at} \quad x = 0 \tag{35}$$

The integral of the residual must be transformed to the following form:

$$\int_{ax_1^{1/3}}^{ax_2^{1/3}} R x^{-2/3} \, dx = 0$$

Some of the relatively compact and simple expressions for $w_n$ that satisfy all conditions are as follows:

$$W_n = A_0 x^{2/3} + \sum_{k=1}^{n} A_k (x-1)^k$$

where

$$2(1+\nu) A_0 + 3(2+\nu) A_1 + 18 A_2 = 0$$

in order for Equation 31 to be satisfied, for $n \geq 3$.

$$W_n = A_0 x^{2/3} + (x-1) \sum_{k=0}^{n} C_k x^k$$

where

$$2(1+\nu) A_0 + 18 \sum_{k=1}^{n} k C_k + (6+3\nu) \sum_{k=0}^{n} C_k = 0$$

in order for Equation 31 to be satisfied.

2. USE OF TRIGONOMETRIC SERIES

An investigation was made of the following trial function

$$W_n = \sum_{k=0}^{n} A_{2k+1} \cos\left(\frac{2k+1}{2a}\right) \pi r \tag{36}$$

This trial function will satisfy all conditions including the moment boundary condition, Equation 9 if

$$\sum_{k=0}^{n} (-1)^k \ (2k+1) \ A_{(2k+1)} = 0$$

The difficulty arises when trying to evaluate the integral of the residual over a subdomain, for example:

$$\int_{r_1}^{r_2} R \ dr = 0$$

$$\int_{r_1}^{r_2} \left\{ \sum_{k=0}^{n} \frac{(2k+1)^3 \pi^3}{(2a)^3} A_{2k+1} \sin\left(\frac{2k+1}{2a}\right)\pi r - \sum_{k=0}^{n} \frac{(2k+1)^2 \pi^2}{(2a)^2} A_{2k+1} \frac{\cos\left(\frac{2k+1}{2a}\right)\pi r}{r} \right.$$

$$\left. + \sum_{k=0}^{n} \frac{(2k+1)\pi}{2a} A_{2k+1} \frac{\sin\left(\frac{2k+1}{2a}\right)\pi r}{r^2} \right\} dr - \int_{r_1}^{r_2} \frac{qr}{2D} \ dr = 0$$

The integration of the second and third terms in the above equation cannot be integrated directly but must be either expressed as a series and integrated term by term (which converges slowly) or integrated by a numerical method such as Simpson's Rule.

## 3. HYBRID METHODS

### a. Combination of the Subdomain and Collocation Methods

According to this concept n - i linear equations are obtained by the subdomain technique and i linear equations are obtained by setting the residual equal to zero at i locations (collocation). This system of n linear equations in terms of n constants is solved for the constants. The residual probably should be collocated at those points where the residual is expected to be largest (i.e. at $r = a$ and $r = 0$ for the circular plate). However, the choice of collocation points and the effect on results should be investigated.

### b. Combination of Subdomain and Point Matching Methods

The approach might involve using the subdomain method to satisfy all conditions except the moment boundary condition ($M_r = 0$ at $r = a$). This would yield n - 1 linear equations in n undetermined constants. An additional linear equation would be obtained by satisfying the moment condition using the point matching technique. Depending upon the trial function used this approach for the circular plate may work out to be the same as the basic subdomain method. This approach should prove useful to problems involving irregular boundaries.

### c. A Kantorovich Approach to the Subdomain Method

The Kantorovich method is applied to the potential energy to obtain a differential equation in terms of an unknown function (Reference 8). This equation is then solved for the function.

The method as now proposed for the circular plate would involve assuming a trial function $w_n = (a - r)^3 \psi(r)$ where $\psi(r)$ is undetermined as yet. After the integral of the residual is minimized over the entire domain a differential equation is obtained in terms of $\psi(r)$. Solution of this differential equation should yield admissible $\psi(r)$ functions. With these functions known the subdomain method can be applied in the usual manner. Hopefully, a more accurate solution may be obtained since the functions should be closer to the solution of the equilibrium differential equation.

Unclassified
Security Classification

**DOCUMENT CONTROL DATA - R&D**

*(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)*

| | |
|---|---|
| 1. ORIGINATING ACTIVITY (Corporate author)<br>Air Force Flight Dynamics Laboratory<br>Research and Technology Division<br>Wright-Patterson Air Force Base, Ohio | 2a. REPORT SECURITY CLASSIFICATION<br>Unclassified<br>2b. GROUP |

3. REPORT TITLE
APPLICATION OF THE SUBDOMAIN (BIEZENO-KOCH) METHOD TO CIRCULAR PLATE BENDING

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (Last name, first name, initial)
Adams, Jr., O. E.

| 6. REPORT DATE | 7a. TOTAL NO. OF PAGES | 7b. NO. OF REFS |
|---|---|---|
| June 1967 | 62 | 8 |

| | |
|---|---|
| 8a. CONTRACT OR GRANT NO.<br>b. PROJECT NO. 1467<br>c. Task No. 146702<br>d. | 9a. ORIGINATOR'S REPORT NUMBER(S)<br>AFFDL-TR-66-215<br>9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) |

10. AVAILABILITY/LIMITATION NOTICES This document is subject to special export controls and each transmittal to foreign governments or foreign nationals may be made only with prior approval of the Theoretical Mechanics Branch, Structures Division, Air Force Flight Dynamics Laboratory, Wright-Patterson AFB, Ohio 45433.

| | |
|---|---|
| 11. SUPPLEMENTARY NOTES | 12. SPONSORING MILITARY ACTIVITY<br>Air Force Flight Dynamics Laboratory (FDTR)<br>Research and Technology Division<br>Wright-Patterson Air Force Base, Ohio 45433 |

13. ABSTRACT

The subdomain or Biezeno-Koch method is employed to obtain approximate solutions to the bending of a uniformly loaded, simply supported circular plate. The details of arriving at a trial function are discussed and the feasibility of automating this method by means of the digital computer is demonstrated. The effect of varying the limits of integration for the residual integral, including a weighting function, and employing a least squares solution technique are investigated.

This abstract is subject to special export controls and each transmittal to foreign governments or foreign nationals may be made only with prior approval of the Theoretical Mechanics Branch, Structures Division, Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio 45433.

DD FORM 1 JAN 64 1473

Unclassified
Security Classification

Unclassified
Security Classification

| 14. KEY WORDS | LINK A | | LINK B | | LINK C | |
|---|---|---|---|---|---|---|
| | ROLE | WT | ROLE | WT | ROLE | WT |
| Circular Plate Bending | | | | | | |
| Least Squares | | | | | | |
| Numerical Analysis | | | | | | |
| Biezeno-Koch Method | | | | | | |
| Subdomain Method | | | | | | |

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through
________________________________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through
________________________________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through
________________________________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.

Unclassified
Security Classification