THEORETICAL INVESTIGATIONS OF BOUNDARY LAYER STABILITY

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FOREWORD

This report was prepared by Gibbs S. Raetz and W. Byron Brown of the Boundary Layer Research Section under the direction of Dr. Werner Pfenninger, Northrop Norair, a Division of Northrop Corporation, Hawthorne, California, and covers research investigations performed from July 1963 to August 1964. This work was performed under Air Force Contract AF33(657)-11618, Project Number 1366, Task Number 136612, "Application of Laminar Flow Control Technology to Optimum Supersonic Cruise."

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ABSTRACT

The mathematical analysis underlying a Fortran program for calculating the proper solutions of the Orr-Sommerfeld system with sufficient accuracy and economy for applying the resonance theory of transition is described. This program covers spacewise growths, rather than timewise growths as in previous computations, of mainly two-dimensional Fourier components of the motion. It employs various innovations providing as much accuracy from efficient single-precision arithmetic as would be obtained from awkward multiple-precision arithmetic in previous calculation schemes. The source programs and some sample calculations, for the principal mode of oscillation of the Blasius boundary layer, are included.

The Lees-Lin stability equations for compressible flow have been extended to include the terms involving the component of the mean boundary layer flow perpendicular to the flat plate. At Mach 5 this more than doubled the critical Reynolds number. Allowance was then made for the three-dimensional aspect of the disturbance velocity. The final result was to give good agreement with observed data in the lower branch of the neutral stability curve at Mach 2.2 and Mach 5, fair agreement with the upper branch at Mach 2.2 and large discrepancies with the data in the upper branch at Mach 5.

Comparison of experimentally determined neutral stability curves with those computed by simplified approximations have disagreed considerably at high Mach numbers on the upper branch, even when agreement was fairly good on the lower branch. To improve the calculations, the complete set of three-dimensional stability equations, including all three momentum equations and also the component of the mean flow in the boundary layer normal to the surface, are solved numerically. This set of equations can be reduced to a set of eight linear equations with complex coefficients. The theoretical solutions for Mach 2.2 and Mach 5 are compared with experimental data and show good agreement in both upper and lower branches.

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PART 1

CALCULATION OF PRECISE PROPER SOLUTIONS FOR THE RESONANCE THEORY OF TRANSITION

Gibbs S. Raetz

BASIC NOTATION

a	frequency ratio (streamwise/principal)
c	timewise frequency
f_m	amplitude coefficient (canonical)**
g_m	amplitude coefficient (logarithmic)**
h_m	amplitude coefficient (global)**
i	unit imaginary number
p	amplitude coefficient (pressure)
s	metrical coefficient denominator (local)
s_n	expansion coefficient of metrical coefficient denominator (local)***
t	metrical coefficient numerator (local)
t_n	expansion coefficient of metrical coefficient numerator (local)***
u	amplitude coefficient (principal velocity component)
u_ℓ	amplitude coefficient (original velocity component)*
w	amplitude coefficient (normal velocity component)
x	coordinate (local)
x_ℓ	coordinate (Cartesian)
y	coordinate (global)
y_o	origin (local)
z	coordinate (normal)

* $\ell = 1,2,3$ (terms summed over this range where same index appears twice)

** $m = 1,2,3,4$

*** $n = 0,1,\ldots,3M$ (M given)

A_ℓ	spacewise frequency*
F_m	amplitude coefficient (asymptotic)**
L	formal series degree
R	Reynolds number (local)
U	basic flow velocity component (principal)
U_ℓ	basic flow velocity component (original)*
α	principal frequency
β	frictional frequency
γ	phase velocity parameter (local)
δ	expansion radius (local)
η_m	secular determinant (local)**
$\varkappa$	metrical coefficient (global)
λ	metrical coefficient (local)
μ	viscosity (shearing)
ξ_m	fundamental solution (local)**
ρ	density
σ	Reynolds number parameter (local)
τ	phase velocity
φ_m	amplitude coefficient (local)**
Γ	frictional parameter
Θ	secular determinant (global)
Λ	reference length
Υ	reference velocity

* ℓ = 1,2,3 (terms summed over this range where same index appears twice)

** m = 1,2,3,4

$(\hat{\ })$	dimensionless value
$(\tilde{\ })$	complex conjugate value
$(\ )'$	total derivative
$(\ )^{(m)}$	fundamental solution**

** m = 1,2,3,4

I. INTRODUCTION

The resonance theory of transition is a non-linear theory of boundary layer oscillations intended to explain the principal motions during transition and to provide a practical technique for calculating such motions. Its present form is described in detail in Reference 1.

This theory was deduced from the continuity and Navier-Stokes equations using a decomposition of the whole motion into a perturbation series and then a decomposition of each perturbation into a Fourier spectrum with aperiodically varying coefficients. The first perturbation is produced by boundary irregularities such as wall waviness or suction uneveness and external turbulence or sound. The higher perturbations are generated by a coupling with lower perturbations through the non-linear terms in the Navier-Stokes equations. Each Fourier component of the higher perturbations is driven by a pair of Fourier components from lower perturbations in such a way that a partial resonance can occur in certain situations. During the stronger partial resonances, the driven Fourier component undergoes a rapid and large growth in amplitude and a gradual shift in phase. Otherwise, it somewhat follows a trend indicated by linear stability theory. Altogether, the recomposed motion is distinctly non-linear and, insofar as has been observed, should explain various transition phenomena beyond the scope of previous concepts.

In calculating the motion, the first perturbation spectrum is to be ascertained directly from the boundary irregularities, and the higher perturbation spectra are to be estimated recursively from lower perturbation spectra. For such calculations, precise proper solutions of the actual and adjoint Orr-Sommerfeld systems or their generalizations are essential. The actual and adjoint proper functions are required to evaluate the coupling between the driven and driving Fourier components, and the proper values are needed to determine the amplitude and phase modulations of the driven Fourier component. Once such data are readily obtainable, the estimation and analysis of various important phenomena should become relatively easy.

Thus, as an initial step in the implementation of the theory, a Fortran program for calculating the proper solutions of the actual Orr-Sommerfeld system with sufficient accuracy and efficiency was developed. The underlying mathematical analysis together with the program and some sample calculations are presented here.

In this particular program, to minimize distractions by secondary details, just a flat wall, a two-dimensional basic flow, and an incompressible fluid are considered. Also, only two-dimensional perturbations with waves traveling in the same direction as the basic flow are covered. However, the proper values for three-dimensional perturbations with waves traveling in all directions can be ascertained and, after minor extensions, the corresponding proper functions could be obtained as well. Contrary to custom, spacewise rather than timewise modulations of the Fourier components are allowed, since only the spacewise variations are important in most actual problems. Besides the principal mode of oscillation usually considered in stability theory, some higher modes also can be investigated.

In the sample calculations, a set of proper values which also could be used in linear stability theory and some typical proper functions are obtained. The Blasius basic flow, two-dimensional perturbations, and the principal mode are considered.

Previous methods of solving the Orr-Sommerfeld system such as those described in References 2 thru 6, which were developed for ordinary linear stability theory, were not entirely appropriate for the present application. As one example, the method of asymptotic expansions (References 2, 3, and 4) neglects all but the first term of an expansion which probably diverges, and it allows some excessive errors in that term. As another example, the method of numerical integration (Reference 5) can entail a large accumulative error, due to a spurious solution which enters the numerical solution through truncation errors and then tends to grow excessively. As a further example, the previous methods generally pertain to timewise rather than spacewise growths of Fourier components.

The present method avoids such inadequacies and also utilizes some innovations which further improve the precision. For example, instead of finding the secular determinant from extremely slight differences between fundamental solutions as in most previous methods, it employs a supplementary differential system for the secular determinant. As a result, it needs just single-precision arithmetic for calculations which in other schemes would require awkward multiple-precision arithmetic. Analogous innovations should be especially helpful in similar calculations for supersonic and hypersonic boundary layers, where the precision is even more critical. They should reduce the computing cost substantially in some cases and enable otherwise impractical or impossible analyses in other cases.

In developing and applying the program, the author was aided by Mr. Lester Pickett and Mrs. Dorothy McHugh, whose assistance is acknowledged with gratitude.

II. DIFFERENTIAL SYSTEMS

A. ORR-SOMMERFELD

The basic objective is to solve the approximate differential system for resonance amplitude coefficients deduced in Reference 1. Starting from Equations (114) and (115) of that reference and omitting superscript indices, this system may be expressed as

$$\begin{aligned} &iu_jA_j + u_3' = 0 \\ &i\rho u_j(A_kU_k + c) + \rho U_j' u_3 + ipA_j = \mu(u_j'' - u_jA_kA_k) \\ &i\rho u_3(A_kU_k + c) + p' = \mu(u_3'' - u_3A_kA_k) \end{aligned} \tag{1}$$

and

$$\begin{aligned} &u_j(0) = u_3(0) = 0 \\ &u_j(\infty) = u_3(\infty) = p(\infty) = 0 \end{aligned} \tag{2}$$

where $j,k = 1,2$ and the primes denote ordinary derivatives with respect to x_3. Here, Cartesian coordinates x_ℓ ($\ell = 1,2,3$) have been used, with the wall surface at $x_3 = 0$ and the adjoining flow at $x_3 = \infty$. The basic flow velocity components U_ℓ are to be given, while the resonance amplitude coefficients u_ℓ and p, of the perturbation velocity components and pressure, respectively, are to be sought. The density ρ and viscosity μ are known constants, whereas the spacewise frequencies A_ℓ and timewise frequency c are partly unknown parameters. Also, A_3 has been eliminated from the system by a transformation described in Part (G-3) of Reference 1, leaving just the value $A_3 = 0$ to be considered here. Consequently, if A_3 really is non-zero and the actual coefficients are needed, but not otherwise, the inverse of that transformation must be applied to the solution. Dimensionless quantities are chosen as

$$\begin{aligned} &\hat{\rho} = \rho/\rho = 1 && \hat{\mu} = \mu/\rho\Lambda T = 1/R \\ &\hat{A}_j = A_j\Lambda && \hat{c} = c\Lambda/T \\ &\hat{x}_j = x_j/\Lambda && \hat{x}_3 = x_3/\Lambda \\ &\hat{U}_j = U_j/T && \hat{p} = p/\rho T T \\ &\hat{u}_j = u_j/T && \hat{u}_3 = u_3/T \end{aligned} \tag{3}$$

with the result that, disregarding the overscript, Equations (1) and (2) apply directly to dimensionless as well as dimensional quantities.

For definiteness, the basic flow is taken in the x_1-direction, so that $\hat{U}_2$ vanishes, and it necessarily is regarded as unseparated. The reference length Λ may be any characteristic thickness of the boundary layer, but for the Blasius basic flow considered in the sample calculations it is chosen as x_1/R. The reference velocity Υ is chosen as the adjoining flow velocity, so that $\hat{U}_1$ varies from 0 to 1 monotonically as $\hat{x}_3$ varies from 0 to ∞. The streamwise frequency $\hat{A}_1$ is to be complex, whereas the crosswise frequency $\hat{A}_2$ and the timewise frequency $\hat{c}$ are to be real. Otherwise, only dimensionless quantities with the overscript omitted are considered further.

For present purposes, an alternate form of the differential system is preferred, which involves the additional parameters

$$\begin{aligned} \alpha &= (A_j A_j)^{1/2} \\ a &= A_1/\alpha \\ b &= A_2/\alpha \\ \tau &= -c/A_1 \end{aligned} \tag{4}$$

with $Re(\alpha) > 0$ and the additional variables

$$\begin{aligned} z &= x_3 \\ U &= U_1 \\ u &= A_j u_j/\alpha \\ w &= u_3 \\ \omega &= iA_2u_1 - iA_1u_2 \end{aligned} \tag{5}$$

Substituting these in Equations (1) and (2), the alternate system is obtained as

$$\begin{aligned} &i\alpha u + w' = 0 \\ &i\rho a\alpha(U - \tau)u + \rho aU'w + i\alpha p = \mu(u'' - \alpha^2 u) \\ &i\rho a\alpha(U - \tau)w + p' = \mu(w'' - \alpha^2 w) \\ &i\rho a\alpha(U - \tau)\omega + i\rho\alpha bU'w = \mu(\omega'' - \alpha^2\omega) \end{aligned} \tag{6}$$

and

$$\begin{aligned} &u(0) = w(0) = \omega(0) = 0 \\ &u(\infty) = w(\infty) = \omega(\infty) = p(\infty) = 0 \end{aligned} \tag{7}$$

Clearly, the first three of Equations (6) may be solved for u, w, p independently of the fourth. As a result, the first three suffice to determine the proper values and functions for two-dimensional perturbations (with $A_2 = 0$) and also the proper values for three-dimensional perturbations (with $A_2 \neq 0$), which are the quantities sought here. Hence, the fourth is not considered further.

To reduce the order of the remaining system, u is eliminated, yielding

$$
\begin{aligned}
w''' - \alpha^2 w' &= i\alpha R[a(U - \tau)w' - aU'w - i\alpha p] \\
p' &= -i\alpha a(U - \tau)w + \mu(w'' - \alpha^2 w)
\end{aligned}
\tag{8}
$$

and

$$
\begin{aligned}
w(0) &= w'(0) = 0 \\
w(\infty) &= p(\infty) = 0
\end{aligned}
\tag{9}
$$

Elimination of p from this system yields the customary form of the Orr-Sommerfeld system

$$
\begin{aligned}
& w^{IV} - 2\alpha^2 w'' + \alpha^4 w \\
& \qquad = i\rho a\alpha R[(U - \tau)(w'' - \alpha^2 w) - U''w]
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
w(0) &= w'(0) = 0 \\
w(\infty) &= w'(\infty) = 0
\end{aligned}
\tag{11}
$$

(when $\rho = a = 1$) which however is not needed here.

To obtain a first-order set of equations, the new variables

$$
\begin{aligned}
f_1 &= w \\
f_2 &= w' \\
f_3 &= w'' - \alpha^2 w \\
f_4 &= \alpha^2 R p
\end{aligned}
\tag{12}
$$

are introduced, whereupon Equations (8) and (9) become

$$
\begin{aligned}
f_1' &= f_2 \\
f_2' &= f_3 + \alpha^2 f_1 \\
f_3' &= f_4 + i\rho a\alpha R[(U - \tau)f_2 - U'f_1] \\
f_4' &= \alpha^2[f_3 - i\rho a\alpha R(U - \tau)f_1]
\end{aligned}
\tag{13}
$$

and

$$f_1(0) = f_2(0) = 0$$
$$f_1(\infty) = f_4(\infty) = 0 \tag{14}$$

which are the basis for the ensuing analysis.

Outside the basic flow, as z approaches ∞ and U′ becomes negligible, Equations (13) approach the asymptotic system

$$\begin{aligned} f_1' &= f_2 \\ f_2' &= f_3 + \alpha^2 f_1 \\ f_3' &= f_4 + i\rho a\alpha R(1 - \tau)f_2 \\ f_4' &= \alpha^2[f_3 - i\rho a\alpha R(1 - \tau)f_1] \end{aligned} \tag{15}$$

with constant coefficients. This has the four fundamental solutions

$$\begin{aligned} F_1^{(1)} &= \exp(-\beta z) \\ F_1^{(2)} &= \exp(-\alpha z) \\ F_1^{(3)} &= \exp(+\alpha z) \\ F_1^{(4)} &= \exp(+\beta z) \end{aligned} \tag{16}$$

where

$$\beta = [\alpha^2 + i\rho a\alpha R(1 - \tau)]^{1/2} \tag{17}$$

with $Re(\beta) > 0$. The corresponding general solution is

$$F_1^* = \Sigma_1^4 B_m F_1^{(m)} \tag{18}$$

where each B_m is an arbitrary constant. However, to conform with Equations (14), the restriction $B_3 = B_4 = 0$ is necessary, so that the corresponding complete solution is

$$F_1 = B_1 F_1^{(1)} + B_2 F_1^{(2)} \tag{19}$$

Inside the basic flow, where U′ is significant, Equations (13) have four fundamental solutions $f_1^{(m)}$ which approach $F_1^{(m)}$ as z approaches ∞ (m = 1,2,3,4). Thus, in that region, the general solution is

$$f_1^* = \Sigma_1^4 B_m f_1^{(m)} \tag{20}$$

and the applicable complete solution is

$$f_1 = B_1 f_1^{(1)} + B_2 f_1^{(2)} \tag{21}$$

Across much of the basic flow, at the more important conditions, both $f_1^{(1)}$ and $f_1^{(2)}$ tend to vary rapidly and greatly. Meanwhile, toward the wall, at least in the Blasius basic flow, f_1 tends to become an extremely slight difference between components of those functions. These and other properties greatly hinder actual calculations unless, as done here, special procedures are employed.

At the wall, where z = 0, to comply with Equations (14), the two quantities

$$\begin{aligned} f_1 &= B_1 f_1^{(1)} + B_2 f_1^{(2)} \\ f_2 &= B_1 f_2^{(1)} + B_2 f_2^{(2)} \end{aligned} \tag{22}$$

must vanish simultaneously, where $f_2^{(1)}$ and $f_2^{(2)}$ are the first derivatives of $f_1^{(1)}$ and $f_1^{(2)}$, respectively, with respect to z. But in a non-trivial solution this can happen only if the determinant of the system

$$\theta = f_1^{(1)} f_2^{(2)} - f_1^{(2)} f_2^{(1)} \tag{23}$$

called the secular determinant vanishes there. Quite obviously, the secular determinant is a function of the parameters of Equations (13). Thus, at z = 0, the secular equation

$$\theta(A_1, A_2, c, R) = 0 \tag{24}$$

must be satisfied, which determines the proper value of A_1 as an implicit function of A_2,c,R. This value is not unique but instead ranges over a sequence of discrete values called the proper value spectrum. The corresponding complete solutions are called the proper function spectrum, and both spectra together are called the proper solution spectrum. Usually only the proper value with the algebraically smallest imaginary part and the associated proper function, called the principal proper solution, are needed. In linear stability theory, this solution would represent the most unstable and therefore the principal mode of oscillation, and the corresponding variation of Re(A_1) with R at Im (A_1) = A_2 = 0 would be the customary neutral curve.

For the sample calculations, wherein A_2 = 0, the ranges of c and R are chosen as

$$\begin{aligned} -.005 \geq c \geq -.100 \\ 125 \leq R \leq 2500. \end{aligned} \tag{25}$$

so as to cover the main regions of most actual transitions. Consequently, the most difficult condition concerned here is at $c = -.100$ and $R = 2500$, where a proper value like $A_1 = 0.25 + i0.03$ and therefore values like

$$\alpha = 0.25 + i0.03$$

$$\beta = 19 + i2$$

are encountered.

At this condition, to illustrate an obstacle which often has been overlooked, suppose that the method of numerical integration is tried. For simplicity, assume that A_1 and $f_m(0)$ $(m = 1,2,3,4)$ have been ascertained in some way and that an integration for f_1 across the basic flow, from $z = 0$ to approximately $z = 5$, is sought. According to Equation (21), f_1 correctly contains components of only $f_1^{(1)}$ and $f_1^{(2)}$, which vanish as z becomes infinite. But as the integration proceeds, truncation errors unavoidably will occur, introducing spurious components of $f_1^{(3)}$ and $f_1^{(4)}$ into the numerical solution. Initially these unwanted components may be very small, but eventually they can become excessively large, since both $f_1^{(3)}$ and $f_1^{(4)}$ grow unboundedly as z becomes infinite. In particular, the components of $f_1^{(4)}$ finally will grow like $F_1^{(4)}$ does, so that the ratios of the final to the initial error magnitudes for $0 \le z \le 5$ will tend to resemble $|F_1^{(4)}(5)| \simeq (10)^{42}$ in magnitude. Consequently, unless extraordinarily precise arithmetic is used, the numerical solution will become meaningless before the integration is completed, although it deceptively may remain smooth enough to appear accurate. Furthermore, even if the necessary precision were provided, the integration then would be too cumbersome and costly to be really practical. For such reasons, a more sophisticated form of the differential equations, allowing a more dependable and efficient method of solution, is deduced and applied here.

B. FUNDAMENTAL SOLUTION

Thus, to overcome various obstacles, f_m (m = 1,2,3,4) are replaced by the new variables

$$\begin{aligned} g_1 &= \log (f_1) \\ g_2 &= f_2/f_1 \\ g_3 &= f_3/f_1 \\ g_4 &= f_4/f_1 \end{aligned} \tag{26}$$

whereupon Equations (13) transform to

$$\begin{aligned} g_1' &= g_2 \\ g_2' &= -g_2 g_2 + g_3 + \alpha^2 \\ g_3' &= -g_2 g_3 + g_4 + i\rho a\alpha R[(U - \tau)g_2 - U'] \\ g_4' &= -g_2 g_4 + \alpha\,[g_3 - i\rho a\alpha R(U - \tau)] \end{aligned} \tag{27}$$

Here the last three equations can be solved for g_2, g_3, g_4 independently of the first equation, which for simplicity is not considered further.

The remaining equations are to be integrated from $z = \infty$ to $z = 0$ for two fundamental solutions $g_2^{(1)}$ and $g_2^{(2)}$ corresponding to $f_1^{(1)}$ and $f_1^{(2)}$, respectively, which approach $F_1^{(1)}$ and $F_1^{(2)}$ as z approaches ∞. Hence, complying with Equations (15) and (16), the initial values at $z = \infty$ for the two solutions are taken as

$$\begin{aligned} g_2^{(1)} &= -\beta & \qquad g_2^{(2)} &= -\alpha \\ g_3^{(1)} &= \Gamma & \qquad g_3^{(2)} &= 0 \\ g_4^{(1)} &= 0 & \qquad g_4^{(2)} &= \alpha\Gamma \end{aligned} \tag{28}$$

where

$$\Gamma = i\rho a\alpha R(1 - \tau) \tag{29}$$

Also, θ is replaced by the re-normalized secular determinant

$$\Theta = \theta / f_1^{(1)} f_1^{(2)} = g_2^{(2)} - g_2^{(1)} \tag{30}$$

so that the secular equation becomes

$$\Theta(A_1, A_2, c, R) = 0 \tag{31}$$

at $z = 0$.

Next, to obtain a finite interval of integration, the new coordinate

$$y = U(z) \tag{32}$$

and the metrical coefficient

$$\varkappa(y) = U'(z) \tag{33}$$

together with the new unknowns

$$h_m(y) = g_m(z) \tag{34}$$

(m = 2,3,4) are introduced. Thereby, the last three of Equations (27) become

$$\begin{aligned} \varkappa h_2' &= -h_2h_2 + h_3 + \alpha^2 \\ \varkappa h_3' &= -h_2h_3 + h_4 + i\rho a\alpha R[(y - \tau)h_2 - \varkappa] \\ \varkappa h_4' &= -h_2h_4 + \alpha^2[h_3 - i\rho a\alpha R(y - \tau)] \end{aligned} \tag{35}$$

which are to be integrated from y = 1 to y = 0. The initial values at y = 1 for the two fundamental solutions are

$$\begin{aligned} h_2^{(1)} &= -\beta & h_2^{(2)} &= -\alpha \\ h_3^{(1)} &= \Gamma & h_3^{(2)} &= 0 \\ h_4^{(1)} &= 0 & h_4^{(2)} &= \alpha\Gamma \end{aligned} \tag{36}$$

and the final condition at y = 0 for a proper value is

$$\boldsymbol{\Theta} = h_2^{(2)} - h_2^{(1)} = 0 \tag{37}$$

Now, in place of the original fourth-order linear system over an infinite interval, a third-order non-linear system over a unit interval is involved. Also, whereas the original unknowns $f_m(z)$ (m = 1,2,3,4) are analytic and thus have only zeros over much of the complex z-plane, the new unknowns $h_m(y)$ (m = 2,3,4) have poles at points in the complex y-plane corresponding to those zeros, as evident from Equations (26) and (34). However, while $f_m(z)$ tend to vary rapidly and strongly in an oscillatory manner, $h_m(y)$ tend to vary more slowly and weakly in a more monotonic manner except near the poles, where they still vary rather simply. As a net result, in actual calculations, the advantages of the new system substantially outweigh the disadvantages.

In the present method, the integration is performed by expanding the unknowns in power series and then finding the coefficients of the series from the differential system*. However, at the conditions of interest, generally at least one singularity of $h_m(y)$ is close enough to the real interval $0 \le y \le 1$ to prevent a suitable representation over that interval by a single expansion in y. Therefore, a sequence of local expansions, adjoined to provide an analytic continuation across the interval, necessarily is employed.

In each local expansion, the local coordinate

$$x = (y - y_o)/\delta \tag{38}$$

is used, where y_o is a local origin on the real y-axis** and δ is a local expansion radius chosen so that $0 \le x \le 1$. Also, the local parameters

$$\begin{aligned} \gamma &= (\tau - y_o)/\delta \\ \sigma &= i\rho a\alpha R\delta \end{aligned} \tag{39}$$

along with the local metrical coefficient

$$\lambda(x) = \varkappa(y)/\delta \tag{40}$$

and the local unknowns

$$\varphi_m(x) = h_m(y) \tag{41}$$

(m = 2,3,4) are employed. Substituting these quantities, Equations (35) become

$$\begin{aligned} \lambda\varphi_2' &= -\varphi_2\varphi_2 + \varphi_3 + \alpha^2 \\ \lambda\varphi_3' &= -\varphi_2\varphi_3 + \varphi_4 + \sigma[(x - \gamma)\varphi_2 - \lambda] \\ \lambda\varphi_4' &= -\varphi_2\varphi_4 + \alpha^2[\varphi_3 - \sigma(x - \gamma)] \end{aligned} \tag{42}$$

For the first expansion (with $y_o = 1$), the initial values (at x = 0) are

$$\begin{aligned} \varphi_2^{(1)} &= -\beta & \qquad \varphi_2^{(2)} &= -\alpha \\ \varphi_3^{(1)} &= \Gamma & \qquad \varphi_3^{(2)} &= 0 \\ \varphi_4^{(1)} &= 0 & \qquad \varphi_4^{(2)} &= \alpha\Gamma \end{aligned} \tag{43}$$

*For integrating from y = 1 (or $z = \infty$), this process seems to be preferable to the simpler method of numerical integration, which would entail a troublesome numerical instability at the start of the integration unless excessively many steps were taken there.

**Recent experience suggests that use of a suitable sequence of complex rather than real values of y_o could substantially reduce the required number of local expansions.

whereas, for each subsequent expansion, the initial values are the final values of the preceding expansion. For a proper solution, the final values of the last expansion must satisfy

$$\Theta = \varphi_2^{(2)} - \varphi_2^{(1)} = 0 \qquad (44)$$

At the more important conditions, at least for the Blasius basic flow, the two fundamental solutions tend to become almost identical toward the wall, even in quite improper solutions. Indeed, near the wall, the secular determinant often amounts to such a slight difference in those solutions that it cannot be evaluated at all by just the single-precision arithmetic used in most Fortran programs. In previous techniques, such as the method of numerical integration, this obstacle or its equivalent sometimes is partially overcome by utilizing double-precision arithmetic, which however is cumbersome and costly and still is not adequate in many important situations. Here the obstacle is overcome simply by deducing and applying a supplementary differential system for the secular determinant, which generally can be solved with adequate accuracy by just single-precision arithmetic.

Therefore, only the more gradually varying fundamental solution, found to be $\varphi_2^{(1)}$, is obtained directly from Equations (42) and (43). Then, using that solution in the secular determinant system to be derived, Θ rather than $\varphi_2^{(2)}$ is determined.

C. SECULAR DETERMINANT

To deduce the secular determinant system, the supplementary local unknowns

$$\eta_m = \varphi_m^{(2)} - \xi_m \tag{45}$$

(m = 2,3,4) are introduced, where $\xi_m = \varphi_m^{(1)}$ and $\eta_2 = \Theta$. Substituting these in Equations (42) and noting that ξ_m satisfy those relations, the supplementary local equations

$$\begin{aligned} \lambda\eta_2' &= -\xi_2\eta_2 - \xi_2\eta_2 - \eta_2\eta_2 + \eta_3 \\ \lambda\eta_3' &= -\xi_2\eta_3 - \xi_3\eta_2 - \eta_2\eta_3 + \eta_4 + \sigma(x - \gamma)\eta_2 \\ \lambda\eta_4' &= -\xi_2\eta_4 - \xi_4\eta_2 - \eta_2\eta_4 + \alpha^2\eta_3 \end{aligned} \tag{46}$$

are obtained, in which ξ_m are to be regarded as known. For the first expansion (with y_o = 1), the initial values are

$$\begin{aligned} \eta_2 &= \beta - \alpha \\ \eta_3 &= -\Gamma \\ \eta_4 &= \alpha\Gamma \end{aligned} \tag{47}$$

whereas, for each subsequent expansion, the initial values are the final values of the preceding expansion. For a proper solution, the final value of the last expansion must satisfy

$$\Theta = \eta_2 = 0 \tag{48}$$

As η_m and thus Θ become small, Equations (46) approach linear equations with variable coefficients, which vary rather gradually in the critical situations concerned here. Consequently, with due care for a rather abrupt transition from large initial values at y = 1 to very small final values near y = 0, the supplementary system can be solved quite satisfactorily by just single-precision arithmetic*.

*To cover the wide ranges of η_m more easily, a further transformation to unknowns of a more logarithmic nature perhaps would be desirable.

III. EXPANSION SYSTEMS

A. BASIC FLOW

For the integration, the basic flow must be ascertained in a suitable form, which as an illustration now is done for the Blasius basic flow considered in the sample calculations. As explained in Reference 1, this flow is governed by the Blasius differential system

$$2X''' + XX'' = 0 \qquad (49)$$

and

$$X(0) = X'(0) = 0$$
$$X'(\infty) = 1 \qquad (50)$$

where X is a dimensionless stream function of z such that $U = X'$. Here z and X are replaced as variables by y and $\varkappa$, yielding the more pertinent system

$$2\varkappa\varkappa'' + y = 0 \qquad (51)$$

and

$$\varkappa'(0) = \varkappa(1) = 0 \qquad (52)$$

The latter system may be solved by expanding $\varkappa$ in the formal series

$$\varkappa = \Sigma_0^N \varkappa_{3n} y^{3n} \qquad (53)$$

where N is a sufficiently large integer and, by a rather elementary process, evaluating the constants $\varkappa_{3n}$ $(n = 0,1,\ldots,N)$ from the system. This series is quite simple, and as N increases it converges over the whole interval $0 \le y \le 1$. Thus, at least for present purposes, it is preferable to previous representations of the Blasius basic flow, which involve either a more complicated series as in the method of steepest descent (Reference 4), two adjoined series as in the method of Blasius (Reference 2), or tabular data as in the method of numerical integration. However, its rate of convergence is quite slow, due to the presence of a weak singularity at $y = 1$.

Therefore, a more efficient representation in the form of the Pade approximant

$$\varkappa = \Sigma_0^M t_{3n}^o y^{3n} \Big/ \Sigma_0^M s_{3n}^o y^{3n} \qquad (54)$$

actually is used, where $s_0^o = 1$ and M is a sufficiently large integer. The constants t_{3n}^o $(n = 0,1,\ldots,M)$ and s_{3n}^o $(n = 1,2,\ldots,M)$ could be evaluated from $\varkappa_{3n}$ $(n = 0,1,\ldots,2M + 1)$ by the method of Pade (Reference 7). However, to satisfy the boundary conditions more suitably, they actually are ascertained more directly from the differential system, so that the numerator series of the Pade approximant rather than the formal series vanishes exactly at $y = 1$.

The resulting coefficients for $M = 3,5,\ldots,13$ are listed in Table 1, and the corresponding approximants are plotted against y^3 (which presently is more convenient than y) in Figure 1. Also, the numerator zeros in the y^3-plane, which will produce singularities in the fundamental solution and secular determinant, and the denominator zeros in that plane are listed in Table 2. These zeros all occur in the real interval $1 \le y^3 < \infty$ and become more dense as M increases, the approximants in the y^3-plane evidently converging to a solution with a branch cut along that interval. Meanwhile, as M increases, the smallest denominator zero approaches 1, and the slope $\varkappa'(y)$ at $y = 1$ approaches $-\infty$, which is the correct value. As a net result, even with relatively few terms, the Pade approximant is quite accurate over the interval $0 \le y \le 1$, although its slope $\varkappa'(y)$ is finite at $y = 1$ and thus is not entirely typical near that point. Ordinarily, this slight deficiency will not significantly affect the proper solutions, which depend mainly on the nature of the basic flow closer to $y = 0$. In fact, the deficiency prevails over only a tiny range of y, which may be regarded as merely the external flow for the boundary layer in the y-coordinate in analogy to the boundary layer for the external flow in the z-coordinate.

In the sample calculations, just the data for $M = 5$ are used. In other calculations for other boundary layers, the basic flow generally should be representable by similar Pade approximants. In some cases, such as for an asymptotic suction boundary layer, the basic flow can be expressed exactly by a finite Pade approximant.

Therefore, in each local expansion, λ is represented as

$$\lambda = t/s \qquad (55)$$

where

$$t = \Sigma_o^{3M} t_m x^m$$
$$s = \Sigma_o^{3M} s_m x^m \qquad (56)$$

and $s_o = 1$. Substituting from Equations (38) and (40) into Equation (54) and applying the binomial theorem, the local constants are evaluated as

$$t_m = t_m^* / s_o^* \delta$$
$$s_m = s_m^* / s_o^* \qquad (57)$$

where

$$t_m^* = \left(\Sigma_m^{3M} t_n^o y_o^n g_m^n\right) \delta^m / y_o^m$$
$$s_m^* = \left(\Sigma_m^{3M} s_n^o y_o^n g_m^n\right) \delta^m / y_o^m \qquad (58)$$

Here g_m^n are the binomial coefficients, which are obtained recursively as

$$g_n^{n-1} = 0$$
$$g_o^n = 1 \qquad (59)$$
$$g_m^n = g_m^{n-1} + g_{m-1}^{n-1}$$

for m = 0,1,...,3M and n = m,...,3M. For the first expansion (with $y_o = 1$), $t_o = 0$ from Equations (52); whereas, for the subsequent expansions, $t_o \neq 0$.

B. FUNDAMENTAL SOLUTION

With the basic flow represented as a rational function, Equations (42) now are expressed in the more convenient form

$$t\varphi_2' = s\Phi_2$$

$$t\varphi_3' = s\Phi_3 - \sigma t \qquad (60)$$

$$t\varphi_4' = s\Phi_4$$

where

$$\Phi_2 = -\varphi_2\varphi_2 + \varphi_3 + \alpha^2$$

$$\Phi_3 = -\varphi_2\varphi_3 + \varphi_4 + \sigma(x - \gamma)\varphi_2 \qquad (61)$$

$$\Phi_4 = -\varphi_2\varphi_4 + \alpha^2 [\varphi_3 - \sigma(x - \gamma)]$$

which will entail only quadratic products of series.

Next, the unknowns are expressed as the formal series

$$\varphi_2 = \Sigma_o^L p_n x^n \qquad \Phi_2 = \Sigma_o^L P_n x^n$$

$$\varphi_3 = \Sigma_o^L q_n x^n \qquad \Phi_3 = \Sigma_o^L Q_n x^n \qquad (62)$$

$$\varphi_4 = \Sigma_o^L r_n x^n \qquad \Phi_4 = \Sigma_o^L R_n x^n$$

where L is to be a sufficiently large integer. Substituting these relations and Equations (56) into Equations (60) and (61) and then equating coefficients with the same power of x, the expansion system

$$\Sigma_o^{n+1} t_{n-k+1} k p_k - \Sigma_o^n s_{n-k} P_k = 0$$

$$\Sigma_o^{n+1} t_{n-k+1} k q_k - \Sigma_o^n s_{n-k} Q_k + \sigma t_n = 0 \qquad (63)$$

$$\Sigma_o^{n+1} t_{n-k+1} k r_k - \Sigma_o^n s_{n-k} R_k = 0$$

where $n \geq 0$ and

$$P_n + \Sigma_o^n p_{n-k} p_k - q_n$$

$$= \alpha^2 \qquad (n = 0) \qquad (64)\ \text{(cont.)}$$

$$= 0 \qquad (n \geq 1)$$

$$Q_n + \Sigma_o^n p_{n-k} q_k - r_n + \sigma\gamma p_n$$

$$= 0 \qquad (n = 0) \qquad \text{(cont.)}$$

$$= \sigma p_{n-1} \qquad (n \geq 1) \qquad (64)$$

$$R_n + \Sigma_o^n p_{n-k} r_k - \alpha^2 q_n$$

$$= \alpha^2 \sigma\gamma \qquad (n = 0)$$

$$= -\alpha^2 \sigma \qquad (n = 1)$$

$$= 0 \qquad (n \geq 2)$$

is obtained. Both the case $t_o = 0$ for the first expansion and the case $t_o \neq 0$ for each subsequent expansion necessarily are considered.

Later, in solving the expansion system, the summations

$$a_n = \Sigma_o^n (t_{n-k+1} k p_k - s_{n-k} P_k)$$

$$b_n = \Sigma_o^n (t_{n-k+1} k q_k - s_{n-k} Q_k) \qquad (65)$$

$$c_n = \Sigma_o^n (t_{n-k+1} k r_k - s_{n-k} R_k)$$

where $n \geq 0$ and

$$A_n = 0 \qquad (n = 1)$$

$$= \Sigma_1^{n-1} p_{n-k} p_k \qquad (n \geq 2)$$

$$B_n = -\sigma p_{n-1} \qquad (n = 1)$$

$$= \Sigma_1^{n-1} p_{n-k} q_k \qquad (n \geq 2) \qquad (66)$$

$$C_n = \alpha^2 \sigma \qquad (n = 1)$$

$$= \Sigma_1^{n-1} p_{n-k} r_k \qquad (n \geq 2)$$

along with

$$D_n = \Sigma_o^n p_{n-k} p_k - q_n$$

$$E_n = \Sigma_o^n p_{n-k} q_k - r_n + \sigma\gamma p_n \qquad (67)$$

$$F_n = \Sigma_o^n p_{n-k} r_k - \alpha^2 q_n$$

where $n \geq 0$ are used.

For $t_o = 0$ and $n = 0$, Equations (63) and (64) degenerate to

$$P_o = 0$$
$$Q_o = 0 \qquad (68)$$
$$R_o = 0$$

and

$$p_o p_o = q_o + \alpha^2$$
$$p_o q_o = r_o - \sigma\gamma p_o \qquad (69)$$
$$p_o r_o = \alpha^2 (q_o + \sigma\gamma)$$

which have four solutions corresponding to $F_1^{(m)}$ $(m = 1,2,3,4)$. Here only the solution corresponding to $F_1^{(1)}$ and thus $\varphi_2^{(1)}$

$$p_o = -\beta$$
$$q_o = \Gamma \qquad (70)$$
$$r_o = 0$$

is needed, which also could be obtained by applying Equations (43) to Equations (62).

For $t_o = 0$ and $n \geq 1$, Equations (63) and (64) become

$$nt_1 p_n - s_o P_n = -a_{n-1}$$
$$nt_1 q_n - s_o Q_n = -b_{n-1} - \sigma t_n \qquad (71)$$
$$nt_1 r_n - s_o R_n = -c_{n-1}$$

and

$$P_n + p_o p_n + p_n p_o - q_n = -A_n$$
$$Q_n + p_o q_n + p_n q_o - r_n + \sigma\gamma p_n = -B_n \qquad (72)$$
$$R_n + p_o r_n + p_n r_o - \alpha^2 q_n = -C_n$$

which necessarily are solved recursively for $n = 1,2,\ldots,L$. For each value of n, the right-hand terms are ascertained from preceding data, so that the six equations always constitute an inhomogeneous linear system for the six coefficients p_n, q_n, r_n, P_n, Q_n, R_n.

The determinant of this system is found to be

$$\Delta_n = (T_n + 2p_o) \left[(T_n + p_o)^2 - \alpha^2 + p_o^2 - \beta^2\right] \qquad (73)$$

where

$$T_n = nt_1/s_o \tag{74}$$

For the particular solution concerned here, in which $p_o = -\beta$, Δ_n reduces to

$$\Delta_n = (T_n - 2\beta)\,(T_n - \beta + \alpha)\,(T_n - \beta - \alpha) \tag{75}$$

and thus has zeros at $n = 2\beta s_o/t_1$ and $n = (\beta \pm \alpha)s_o/t_1$. However, in a normal solution, which is the only kind considered here, none of these zeros coincide with $n = 1,2,\ldots,\infty$ and the six coefficients always can be determined, the formal series then being valid over their circle of convergence.

In an abnormal solution, to allow a zero at one of the values taken by n, the parameters α and β and therefore A_1, A_2, c, R must have special values depending on t_1, which itself must be finite because α and β are finite in the pertinent situations. At these special values, the coefficients for that and the higher values of n cannot be determined, the formal series then being invalid unless generalized appropriately. For some basic flows, like that in the asymptotic suction boundary layer, t_1 actually is finite and the abnormal solutions perhaps have a physical significance. However, for the Blasius basic flow, t_1 really is infinite and the abnormal solutions apparently do not exist. Nevertheless, if t_1 were too small in the approximate representation of that flow, a zero of Δ_n perhaps could occur at one of the values taken by n, causing a misleading result.

Thus, solving Equations (71) and (72) with $\Delta_n \neq 0$, the six coefficients are calculated as

$$\begin{aligned} p_n &= \Gamma_n/\Delta_n \\ q_n &= -d_n + (T_n + 2p_o)p_n \\ r_n &= -e_n + (T_n + p_o)q_n + (p_o^2 - \beta^2)p_n \end{aligned} \tag{76}$$

and

$$\begin{aligned} P_n &= T_n p_n + s_o^{-1}a_{n-1} \\ Q_n &= T_n q_n + s_o^{-1}(b_{n-1} + \sigma t_n) \\ R_n &= T_n r_n + s_o^{-1}c_{n-1} \end{aligned} \tag{77}$$

where

$$\begin{aligned} d_n &= -A_n - s_o^{-1}a_{n-1} \\ e_n &= -B_n - s_o^{-1}(b_{n-1} + \sigma t_n) \\ f_n &= -C_n - s_o^{-1}c_{n-1} \end{aligned} \tag{78}$$

and

$$\Gamma_n = [(T_n + p_o)^2 - \alpha^2]d_n + (T_n + p_o)e_n + f_n \qquad (79)$$

Together with Equations (68) and (70), the resulting data provide the initial expansion for the fundamental solution.

For each subsequent expansion, in which $t_o \neq 0$, the conditions

$$\begin{aligned} p_o &= p_o^* \\ q_o &= q_o^* \\ r_o &= r_o^* \end{aligned} \qquad (80)$$

necessarily are observed, where p_o^*, q_o^*, r_o^* are the final values at $x = 1$ of the preceding expansion. Then, complying with Equations (63) and (64), the remaining coefficients are calculated recursively from

$$\begin{aligned} P_n &= -D_n + \alpha^2 && (n = 0) \\ &= -D_n && (n \geq 1) \\ Q_n &= -E_n && (n = 0) \\ &= -E_n + \sigma p_{n-1} && (n \geq 1) \\ R_n &= -F_n + \alpha^2\sigma\gamma && (n = 0) \\ &= -F_n - \alpha^2\sigma && (n = 1) \\ &= -F_n && (n \geq 2) \end{aligned} \qquad (81)$$

and

$$\begin{aligned} p_{n+1} &= -a_n/(n + 1)t_o \\ q_{n+1} &= -(b_n + \sigma t_n)/(n + 1)t_o \\ r_{n+1} &= -c_n/(n + 1)t_o \end{aligned} \qquad (82)$$

wherein $n \geq 0$.

C. SECULAR DETERMINANT

Using the expansion coefficients of both the basic flow and the fundamental solution, the secular determinant is ascertained in a manner similar to that just described. In fact, in the Fortran program, the same subroutines are used for both the fundamental solution and the secular determinant. Hence, just the counterparts of the main equations of the preceding section, identified by the same numbers with an asterisk, are listed here.

Thus, Equations (46) are expressed as

$$t\eta_2' = sH_2$$
$$t\eta_3' = sH_3 \qquad (60^*)$$
$$t\eta_4' = sH_4$$

where

$$H_2 = -\xi_2\eta_2 - \xi_2\eta_2 - \eta_2\eta_2 + \eta_3$$
$$H_3 = -\xi_2\eta_3 - \xi_3\eta_2 - \eta_2\eta_3 + \eta_4 + \sigma(x - \gamma)\eta_2 \qquad (61^*)$$
$$H_4 = -\xi_2\eta_4 - \xi_4\eta_2 - \eta_2\eta_4 + \alpha^2\eta_3$$

while the unknowns are represented as

$$\eta_2 = \Sigma_o^L u_n x^n \qquad H_2 = \Sigma_o^L U_n x^n$$
$$\eta_3 = \Sigma_o^L v_n x^n \qquad H_3 = \Sigma_o^L V_n x^n \qquad (62^*)$$
$$\eta_4 = \Sigma_o^L w_n x^n \qquad H_4 = \Sigma_o^L W_n x^n$$

Then, along with Equations (56) and (62), these relations are substituted into Equations (60*) and (61*), yielding the expansion system

$$\Sigma_o^{n+1} t_{n-k+1} k u_k - \Sigma_o^n s_{n-k} U_k = 0$$
$$\Sigma_o^{n+1} t_{n-k+1} k v_k - \Sigma_o^n s_{n-k} V_k = 0 \qquad (63^*)$$
$$\Sigma_o^{n+1} t_{n-k+1} k w_k - \Sigma_o^n s_{n-k} W_k = 0$$

where $n \geq 0$ and

$$U_n + \Sigma_o^n (p_{n-k}u_k + p_k u_{n-k} + u_{n-k}u_k) - v_n$$
$$= 0 \qquad (n \geq 0)$$

$$V_n + \Sigma_o^n (p_{n-k}v_k + q_k u_{n-k} + u_{n-k}v_k) - w_n + \sigma\gamma u_n \qquad (64^*) \text{ (cont.)}$$
$$= 0 \qquad (n = 0)$$
$$= \sigma u_{n-1} \qquad (n \geq 1)$$

$$W_n + \Sigma_o^n(p_{n-k}w_k + r_k u_{n-k} + u_{n-k}w_k) - \alpha^2 v_n$$ (cont.) (64*)

$$= 0 \qquad (n \geq 0)$$

As before, the cases $t_o = 0$ and $t_o \neq 0$ for the first and subsequent expansions, respectively, necessarily are considered.

Later, the summations

$$a_n^* = \Sigma_o^n(t_{n-k+1}ku_k - s_{n-k}U_k)$$
$$b_n^* = \Sigma_o^n(t_{n-k+1}kv_k - s_{n-k}V_k) \qquad (65^*)$$
$$c_n^* = \Sigma_o^n(t_{n-k+1}kw_k - s_{n-k}W_k)$$

where $n \geq 0$ and

$$A_n^* = 0 \qquad (n = 1)$$
$$= \Sigma_1^{n-1}(p_{n-k}u_k + p_k u_{n-k} + u_{n-k}u_k) \qquad (n \geq 2)$$
$$B_n^* = -\sigma u_{n-1} \qquad (n = 1)$$
$$= \Sigma_1^{n-1}(p_{n-k}v_k + q_k u_{n-k} + u_{n-k}v_k) - \sigma u_{n-1} \qquad (n \geq 2) \qquad (66^*)$$
$$C_n^* = 0 \qquad (n = 1)$$
$$= \Sigma_1^{n-1}(p_{n-k}w_k + r_k u_{n-k} + u_{n-k}w_k) \qquad (n \geq 2)$$

along with

$$D_n^* = \Sigma_o^n(p_{n-k}u_k + p_k u_{n-k} + u_{n-k}u_k) - v_n$$
$$E_n^* = \Sigma_o^n(p_{n-k}v_k + q_k u_{n-k} + u_{n-k}v_k) - w_n + \sigma\gamma u_n \qquad (67^*)$$
$$F_n^* = \Sigma_o^n(p_{n-k}w_k + r_k u_{n-k} + u_{n-k}w_k) - \alpha^2 v_n$$

where $n \geq 0$ are used.

For $t_o = 0$, the solution at $n = 0$ is

$$U_o = 0$$
$$V_o = 0 \qquad (68^*)$$
$$W_o = 0$$

and

$$u_o = \beta - \alpha$$
$$v_o = -\Gamma \qquad (70^*)$$
$$w_o = \alpha\Gamma$$

whereas the recursive system at n = 1,2,...,L is

$$nt_1u_n - s_oU_n = -a^*_{n-1}$$
$$nt_1v_n - s_oV_n = -b^*_{n-1} \qquad (71^*)$$
$$nt_1w_n - s_oW_n = -c^*_{n-1}$$

and

$$U_n + p_ou_n + p_ou_n + u_ou_n$$
$$+ p_nu_o + p_nu_o + u_nu_o - v_n = -A^*_n$$
$$V_n + p_ov_n + q_ou_n + u_ov_n \qquad (72^*)$$
$$+ p_nv_o + q_nu_o + u_nv_o - w_n + \sigma\gamma u_n = -B^*_n$$
$$W_n + p_ow_n + r_ou_n + u_ow_n$$
$$+ p_nw_o + r_nu_o + u_nw_o - \alpha^2v_n = -C^*_n$$

which is solved for u_n, v_n, w_n, U_n, V_n, W_n.

Here the determinant of the system is

$$\Delta^*_n = (T_n + 2p_o + 2u_o)\ [(T_n + p_o + u_o)^2 - \alpha^2 + (p_o + u_o)^2 - \beta^2] \qquad (73^*)$$

which for the solution concerned reduces to

$$\Delta^*_n = (T_n - 2\alpha)(T_n - \alpha + \beta)(T_n - \alpha - \beta) \qquad (75^*)$$

and has zeros at $n = 2\alpha s_o/t_1$ and $n = (\alpha \pm \beta)s_o/t_1$. Again, only a normal solution avoiding such zeros is considered, the formal series then being valid over their circle of convergence. However, in singular circumstances like those discussed before, an abnormal solution possibly could occur.

Accordingly, for the first expansion, the remaining coefficients (at $n \geq 1$) are calculated as

$$u_n = \Gamma^*_n/\Delta^*_n$$
$$v_n = -d^*_n + (T_n + 2p_o + 2u_o)u_n \qquad (76^*)$$
$$w_n = -e^*_n + (T_n + p_o + u_o)v_n + [(p_o + u_o)^2 - \beta^2]u_n$$

and

$$U_n = T_nu_n + s_o^{-1}a^*_{n-1}$$
$$V_n = T_nv_n + s_o^{-1}b^*_{n-1} \qquad (77^*)$$
$$W_n = T_nw_n + s_o^{-1}c^*_{n-1}$$

where

$$d_n^* = -A_n^* - s_o^{-1} a_{n-1}^* - u_o p_n - u_o p_n$$

$$e_n^* = -B_n^* - s_o^{-1} b_{n-1}^* - v_o p_n - u_o q_n \qquad (78^*)$$

$$f_n^* = -C_n^* - s_o^{-1} c_{n-1}^* - w_o p_n - u_o r_n$$

and

$$\Gamma_n^* = [(T_n + p_o + u_o)^2 - \alpha^2] d_n^* + (T_n + p_o + u_o) e_n^* + f_n^* \qquad (79^*)$$

For each subsequent expansion, wherein $t_o \neq 0$, the conditions

$$u_o = u_o^*$$

$$v_o = v_o^* \qquad (80^*)$$

$$w_o = w_o^*$$

are observed, where u_o^*, v_o^*, w_o^* are the final values at $x = 1$ of the preceding expansion, and the remaining coefficients are calculated from

$$U_n = -D_n^* \qquad (n \geq 0)$$

$$V_n = -E_n^* \qquad (n = 0)$$

$$= -E_n^* + \sigma u_{n-1} \qquad (n \geq 1) \qquad (81^*)$$

$$W_n = -F_n^* \qquad (n \geq 0)$$

and

$$u_{n+1} = -a_n^*/(n + 1) t_o$$

$$v_{n+1} = -b_n^*/(n + 1) t_o \qquad (82^*)$$

$$w_{n+1} = -c_n^*/(n + 1) t_o$$

wherein $n \geq 0$. For a proper solution, the final value u_o^* at $x = 1$ of the last expansion, which is the value of θ at $y = z = 0$, must vanish.

IV. AUXILIARY RELATIONS

A. RATIONAL EXPANSIONS

In each expansion, whereas the origin y_o is predetermined, the radius δ is somewhat flexible and can be chosen best only after the expansion coefficients for a trial value have been found. Therefore, in the Fortran program, the expansion coefficients first are computed from the preceding relations for a tentative value of δ, which is selected so as to avoid troublesome truncation and overflow or underflow errors and to keep the expansion interval within the applicable range of y. Then, a convergence test is applied to these coefficients to find a preferred value of δ, which would provide a moderate rate of convergence at $x = 1$. Finally, the expansion coefficients either are left unchanged or are scaled down by an elementary process to the preferred value, according to whether the tentative value is smaller or larger than the preferred value. Thereby, the final formal series always converge at least moderately over $0 \le x \le 1$.

However, to accelerate the convergence and thus minimize the number of terms required for a given precision, the formal series subsequently are converted into Pade approximants. Thus, representing each final formal series as

$$f = \Sigma_o^L f_n x^n \tag{83}$$

where f_n $(n = 0,1,\ldots,L)$ are known constants, the rational function

$$f = (\Sigma_o^\mu p_n x^n + \epsilon_o x^{\mu+\nu+1}) / \Sigma_o^\nu q_n x^n \tag{84}$$

always is constructed. Here $\mu = L - 1 - \nu$ and $q_o = 1$ whereas ν is an arbitrary positive interger and p_n $(n = 0,1,\ldots,\mu)$, q_n $(n = 1,2,\ldots,\nu)$, and ϵ_o are constants determined from f_n $(n = 0,1,\ldots,L)$. The last numerator term is the leading term of the truncation error and is omitted in the actual evaluation of the approximant, the coefficient ϵ_o serving only as an error index.

To determine the rational coefficients, Equations (83) and (84) are multiplied by the denominator of Equation (84), and the resulting coefficients with the same power of x are equated. This process yields

$$p_n = \Sigma_o^{n_o} f_{n-k} q_k \tag{85}$$

for $0 \le n \le \mu$, where $n_o = \min(n, \nu)$, and

$$\Sigma_1^\nu f_{n-k} q_k = -f_n q_o \tag{86}$$

for $\mu + 1 \le n \le \mu + \nu$ together with

$$\epsilon_o = \Sigma_o^\nu f_{n-k} q_k \tag{87}$$

for $n = \mu + \nu + 1 = L$. Equations (86) constitute an inhomogeneous linear system for the unknowns q_n ($n = 1,2,\ldots,\nu$), which is solved by the elimination method (Reference 8) assuming that the determinant of the system does not vanish. Then, using that solution, p_n ($n = 0,1,\ldots,\mu$) and ϵ_o are computed directly from Equations (85) and (87), respectively.

In the complex x-plane, as L increases, the formal series converges only within the circle centered at the origin and extended to the nearest singularity. Meanwhile, as μ and ν increase, the rational function (even though constructed from the formal series) converges over a much larger region, which apparently includes the whole finite plane except for infinitesimal regions around the poles and around the rays emanating from the other singularities to infinity. Furthermore, the superior convergence of the rational function persists into the circle of convergence of the formal series. As a net result, in the present computations, a given precision can be attained more economically by obtaining and using the rational function, despite the extra relations entailed. In fact, in place of a sequence of local expansions, just a single rational expansion at $y = 1$ would suffice theoretically, since it would converge at $y = 0$. However, in most cases of interest, the rounding and other errors then would be too troublesome, the procedure followed here being a practical compromise.

In the present Fortran program, to limit the calculations and errors, the restrictions $L = \min(L_o, L_*)$ and $\nu = \min(6, \mu + 1)$ are imposed. Here, L_o is the maximum degree of the formal series, which is chosen from $3 \le L_o \le 29$, and L_* is the minimum degree of that series providing moderate convergence, which is calculated along with the expansion coefficients. In the sample calculations, the value $L_o = 24$ generally was used, and the condition $L = L_o$ commonly occurred. As a result, the number of local expansions varied from about 11 to over 30, depending on the values of A_1, A_2, c, and R.

B. ROOT EXTRACTION

As indicated earlier, when A_2, c, R are specified, the proper values of A_1 are just the zeros of Θ at $z = 0$, to be denoted as $\Theta^*(A_1)$. However, these zeros cannot be expressed explicitly in a tractable way, owing to the intricacy of $\Theta^*(A_1)$. Therefore, to ascertain a proper value, first a local polynomial approximation of $\Theta^*(A_1)$ is established. Then, the pertinent root of that polynomial is located and used as an approximation of the proper value.

Thus, for each proper value, a small set of adjacent values of A_1, denoted as $A_1^{(j)}$ $(j = 0,1,\ldots,n-1)$ where n is the number of such values, is selected. To minimize the computations for a given precision, these values are distributed equiangularly around a circle in the complex A_1-plane chosen so that the center $\bar{A}_1$ is as close to the proper value as possible and the radius $\bar{R}_1$ is as small as tolerable. Thus, the adjacent values are

$$A_1^{(j)} = \bar{A}_1 + \bar{R}_1 E_j \tag{88}$$

where each E_j is a unit circle value

$$E_j = \exp(i2\pi j/n) \tag{89}$$

of another complex variable E. Next, the quantities $\Theta_j = \Theta^*(A_1^{(j)})$ are calculated by the preceding method and used to construct the interpolation polynomial

$$\Theta_* = \Sigma_0^{n-1} c_k E^k \tag{90}$$

where c_k $(k = 0,1,\ldots,n-1)$ are constants such that $\Theta_* = \Theta_j$ at $E = E_j$. In this construction, the identity

$$\begin{aligned} \Sigma_{j=0}^{n-1} E_j^{\ell} &= n \qquad (\ell = 0) \\ &= 0 \qquad (\ell \neq 0) \end{aligned} \tag{91}$$

for $\ell = 0, \pm 1, \ldots, \pm(n-1)$ is employed, yielding

$$\begin{aligned} \Sigma_{j=0}^{n-1} \Theta_j E_j^{-k} &= \Sigma_{j=0}^{n-1} (\Sigma_{m=0}^{n-1} c_m E_j^m) E_j^{-k} \\ &= \Sigma_{m=0}^{n-1} c_m (\Sigma_{j=0}^{n-1} E_j^{m-k}) \\ &= n c_k \end{aligned}$$

whereupon

$$c_k = n^{-1} \Sigma_{j=0}^{n-1} \Theta_j E_j^{-k} \tag{92}$$

Except as limited by the precision of the arithmetic, the resulting polynomial approximates the first n terms of a complex Taylor series about $E = 0$ and thus, as n increases, converges in the circle centered at that point and extended to the nearest singularity (generally a pole). This particular interpolation scheme, perhaps somewhat new, is a complex-variable counterpart of the highly-efficient real-variable Chebychev interpolation scheme. At the larger values of n, to further improve the convergence and thereby the accuracy of the resulting root (particularly near a pole), the interpolation polynomial is converted into the Pade approximant

$$\Theta_{**} = \Sigma_o^{n-1-\nu} d_k E^k \Big/ \Sigma_o^{\nu} e_k E^k \tag{93}$$

Here, $e_o = 1$ and ν is an arbitrary integer, while $d_k (k = 0,1,\ldots,n-1-\nu)$ and $e_k (k = 1,2,\ldots,\nu)$ are constants determined from $c_k (k = 0,1,\ldots,n-1)$ by the method described in Section (F-1). Finally, the pertinent root E_* of

$$\Sigma_o^{n-1} c_k E^k = 0 \tag{94}$$

or

$$\Sigma_o^{n-1-\nu} d_k E^k = 0 \tag{95}$$

as applicable is located by Newton's method (for simple roots) and, if within the range of the adjacent values, is used in the approximation

$$A_1^* = \bar{A}_1 + \bar{R}_1 E_* \tag{96}$$

for the desired proper value. Subsequently, as necessary, the process is repeated until sufficient accuracy is attained. With due care for truncation errors, particularly those inherent in each Θ_j, all calculations generally can be performed satisfactorily with just single-precision arithmetic.

In the present Fortran program, the root is extracted from the polynomial if $2 \le n \le 4$ and from the rational function if $5 \le n \le 16$ (a maximum of 16 adjacent values being allowed). In the latter case, to further improve the accuracy, a second extraction from a polynomial centered at the root from the rational function is included (with $2 \le n \le 8$). In both cases, as a check, the value of $\Theta^*(A_1)$ at the final root, which in an exact calculation would vanish, also is obtained. When this check value significantly exceeds the inherent computational errors, the program is rerun, using the final root of the preceding run as the initial center. In the sample calculations, for many runs, the value $n = 6$ was chosen (with $n = 3$ in the second extraction). For most runs, the interpolation radii were taken as rather small fractions of the proper value magnitude.

Ordinarily, just the principal proper value, which has the algebraically smallest imaginary part, is sought. However, at least in principle, several higher proper values also could be ascertained by the foregoing procedure. In general, the significance of the higher proper values has not yet been adequately assessed, especially in regard to the resonance theory of transition. Some instructive explorations of such values for relatively simple basic flows are described in References 9 thru 11.

V. FORTRAN PROGRAM

As in most other methods of solving the Orr-Sommerfeld system, the calculations entailed here are too extensive to perform manually in a practical way, even for a single proper solution. Therefore, a Fortran program (References 12 and 13) for conducting them on an automatic computer (the IBM 7090 data processing system at the Norair Division) necessarily was developed.

At present, the whole program includes one main program (MPA) using eleven subroutines (SRB thru SRL) and another main program (MPM) using the same subroutines plus two additional subroutines (SRN and SRØ). The titles, common notation, and source statements of these routines are listed in Appendix I. This information together with the preceding analysis and the following remarks indicates the general nature of the program. Complete details would be too lengthy to describe here*.

In its present form, the program includes a few vestiges from earlier programs that were tried without adequate success. Also, it omits labels from output data, which therefore must be identified from the listed notation and output statements. Moreover, various generalizations, such as to include the useful bypassed equations and to cover the adjoint as well as the actual Orr-Sommerfeld system, are underway or contemplated. Consequently, the program is somewhat tentative and later may be refined and revised.

A. ADJOINTED POWER SERIES SOLUTION

The main purpose of program MPA (adjoined power series solution) is to obtain the expansion coefficients of the fundamental solution and secular determinant for specified values of A_1, A_2, c, R. When A_1 has a proper value, these coefficients readily yield the corresponding proper function. Another purpose is to provide the values of Θ at $z = 0$ for sets of values of A_1, A_2, c, R from which the proper values can be estimated in a preliminary manner.

The principal subroutine is SRG, which is performed once for each local expansion. It calculatesthe formal expansion coefficients of both the fundamental solution and the secular determinant by the relations of Sections (E-2) and (E-3), using SRH thru SRJ for the summations involved. Then, from those coefficients, it calculates the corresponding rational expansion coefficients by the relations of Section (F-1), using SRK for this purpose. In turn, the last routine employes SRL to solve the complex linear system thus encountered.

The local expansions are adjoined by SRE, which also provides the local basic flow coefficients and the local parameters for SRG by use of SRF. The latter routine uses basic flow coefficients and binomial coefficients supplied by SRB along with parameters computed in SRD. The secular determinant at the wall is calculated by SRC.

*Further details can be supplied upon request.

B. PROPER VALUE LOCATION

The main purpose of program MPM (proper value location) is to find the proper values of A_1 when A_2,c,R are specified, which are needed to evaluate the resonance growth functions of Reference 1. Another purpose is to find the proper values of R when A_1,A_2,c are specified, which are needed to evaluate the resonance coefficients of Reference 1.

Here, the primary subroutine is SRC, which employs SRD thru SRL to provide the value of Θ at z = 0 for each adjacent value of A_1 (or R). From these data, the proper value is extracted by the relations of Section (F-2), using SRN for the interpolation polynomial coefficients and SRK for the rational function coefficients together with SRØ for the zero location.

VI. SAMPLE CALCULATIONS

To demonstrate the method and provide basic data for research on transition, some proper values and functions for the Blasius basic flow were calculated. The principal mode, for which Im (A_1) has the algebraically smallest value, and two-dimensional perturbations, for which $A_2 = A_3 = 0$, were considered. To otherwise cover the main regions of most actual transitions in an efficient way, the values of c and R were selected as

$$c_j = -.005 \exp[\log(20) \sin^2(j\pi/2n)]$$
$$R_k = 125 \exp[\log(20) \sin^2(k\pi/2n)] \qquad (97)$$

where $j,k = 0,1,\ldots,n$ and $n = 6$. These 49 points encompass the ranges of Equations (25) and are distributed so as to allow Chebychev interpolations in the logarithms of c and R.

Each point required an automatic computer time of about 0.1 hour or more and thus was rather expensive, which emphasizes the advisability of economizing throughout the program and calculation. In previous computational schemes, some of the points would have required double- or triple-precision arithmetic, which perhaps would have increased the time for those points about four or nine times, respectively, or more.

The accuracy of the resulting data varies somewhat, being least where c and R are greatest. Near $c = -.100$ and $R = 2500$, which is the most critical region, the error in the proper values apparently is of the order of 0.001 percent of the absolute value. The accuracy of the proper functions should equal that of the proper values, since their greatest error tends to occur near the wall where the proper values are determined. However, in previous computational schemes, the proper functions evidently can be less accurate than the proper values.

A. PROPER VALUES

The proper value of A_1 as a function of c and R is listed in Table 3 and plotted in Figure 2. Here A_1 is complex while c and R are real, unlike conventional linear stability data in which c is complex while A_1 and R are real. Thus, these data pertain to spacewise modulations of Fourier components of the motion, which are needed in most applications, whereas the conventional data represent timewise modulations of those components. In previous analyses, the spacewise variations usually have been merely estimated from the timewise variations, which sometimes can be done without excessive error (Reference 14). However, at least as ordinarily performed, such estimates become poorer as the phase velocity of the Fourier component decreases and in fact are invalid where that velocity vanishes, which happens when stationary waves occur.

For the particular basic flow and conditions considered here, the proper values near and inside the neutral curve, on which $\mathrm{Im}(A_1) = 0$, agree reasonably well with previous double-precision calculations (Reference 15).

However, the proper functions themselves may differ more substantially, especially at the larger values of c and R, the present data presumably being the more accurate. The proper values further outside the neutral curve cannot be compared, because they have been omitted from the previous calculations. Nevertheless, such values are somewhat important, since some resonance growth can occur outside the neutral curve, contrary to linear stability theory which predicts only damping in that region.

B. PROPER FUNCTIONS

The fundamental solution ξ_2 and secular determinant η_2 composing each proper function, for the values of Equation (97) with j,k = 0,n/2,n only, are represented as functions of y in Figure 3. For convenience, in place of η_2, the more tractable parameter y log(η_2) is plotted.

Clearly, as anticipated, ξ_2 generally is remarkably smooth, whereas the original variable

$$f_1^{(1)} = \exp\left[\int_o^y \xi_2 dy/\varkappa\right] \tag{98}$$

varies greatly in an oscillatory manner, particularly at the higher values of c and R. Contrarily, η_2 itself varies rather strongly, although the parameter representing this quantity in Figure 3 is almost as smooth as ξ_2. These relatively simple and mild trends facilitate the calculations and thus help to justify the elaboration of the method. They also suggest that useful asymptotic approximations, differing from those of the familiar method of asymptotic expansions (References 2 thru 4), perhaps could be established by further investigation.

VII. CONCLUDING REMARKS

Insofar as observed, the Fortran program described here should be satisfactory as a basis for implementing the resonance theory of transition, and it incidently should be valuable for extending the linear theory of instability. To fully exploit the possibilities, though, various generalizations are desirable. The most urgent is the incorporation of the pertinent bypassed equations and the adjoint Orr-Sommerfeld system, so as to evaluate the resonance coefficient of Reference 1. Others include extensions to three-dimensional curvilinear coordinates, for handling curvilinear phenomena like Goertler and crossflow vortices, and to compressible and real fluids, for investigating supersonic and hypersonic transitions. In general, the present technique should be somewhat more economical and dependable than previous techniques, and it perhaps could be improved significantly by further development.

Meanwhile, continuation of the present calculations to broader conditions and additional basic flows would be appropriate. In particular, some higher modes and the three-dimensional perturbations should be covered. Indeed, as mentioned in Reference 1, the proper solutions for $Re(A_1) = A_3 = 0$, which represent streamwise (not crossflow) vortices, may provide insight into the nature of turbulent wedges. Also, systematic data for special basic flows such as the Hartree flows with suction and viscoelastic walls would be valuable for general reference. In this connection, the proper values for the asymptotic suction profile, as obtained by a predecessor of the present method, are included in Figure 4. For these calculations, just single-precision arithmetic (8 decimal places) was used, but the accuracy exceeds that attainable from quadruple-precision arithmetic (32 decimal places) in previous schemes. Altogether, considering the growing importance of transition in technology and the wide range of conditions encountered, such calculations could be continued in a worthwhile way rather indefinitely.

Also, separate programs are needed for actually evaluating the resonance coefficients and growth functions of Reference 1 from the proper solutions. To indicate the significance of such growth functions, the downstream modulations of a typical Fourier component of the motion according to the resonance and linear theories, as estimated from Figure 2 in an approximate way, are compared in Figure 5. In this particular case, the two theories differ greatly near the lower branch of the neutral curve. In other cases, they would differ substantially in other ways, yielding greatly different whole motions.

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TABLE 1

PADE APPROXIMANTS OF BLASIUS BASIC FLOW

n	M = 3	M = 5	M = 7	M = 9	M = 11	M = 13
			NUMERATOR COEFFICIENTS, t^{o}_{3n}			
0	0.32757200E-00	0.33096903E-00	0.33160199E-00	0.33181465E-00	0.33190898E-00	0.33195852E-00
1	-0.45868548E-00	-0.77873190E+00	-0.11068182E+01	-0.14369169E+01	-0.17678156E+01	-0.21006171E+01
2	0.13111347E-00	0.61371642E+00	0.14238728E+01	0.25651931E+01	0.40382836E+01	0.58521502E+01
3		-0.18050170E-00	-0.88166723E+00	-0.24333216E+01	-0.51672749E+01	-0.94381107E+01
4		0.14548149E-01	0.26641611E-00	0.13169408E+01	0.40579316E+01	0.97460579E+01
5			-0.34713268E-01	-0.40526959E-00	-0.20147107E+01	-0.67295320E+01
6			0.13079017E-02	0.66356314E-01	0.62793871E+00	0.31502946E+01
7				-0.49035216E-02	-0.11788957E-00	-0.99248555E+00
8				0.10676728E-03	0.12201311E-01	0.20460183E-00
9					-0.58171535E-03	-0.26130266E-01
10					0.82415373E-05	0.18744009E-02
11						-0.62649747E-04
12						0.62544522E-06
			DENOMINATOR COEFFICIENTS, s^{o}_{3n}			
0	0.09999999E+01	0.09999999E+01	0.09999999E+01	0.09999999E+01	0.09999999E+01	0.09999999E+01
1	-0.62364453E+00	-0.15921308E+01	-0.25799394E+01	-0.35735995E+01	-0.45697550E+01	-0.55717252E+01
2	0.36553286E-01	0.75883153E+00	0.24535765E+01	0.51405852E+01	0.88244886E+01	0.13530057E+02
3		-0.10834911E-00	-0.10521892E+01	-0.38086461E+01	-0.93727504E+01	-0.18793847E+02
4		0.21018477E-02	0.19804998E-00	0.15426160E+01	0.59704392E+01	0.16475727E+02
5			-0.13054046E-01	-0.33369502E-00	-0.23329208E+01	-0.94880256E+01
6			0.12752540E-03	0.34680687E-01	0.54903654E+00	0.36221441E+01
7				-0.13302821E-02	-0.73238564E-01	-0.90362266E+00
8				0.78543261E-05	0.48765983E-02	0.14166714E-00
9					-0.12289093E-03	-0.12965245E-01
10					0.48674239E-06	0.60466661E-03
11						-0.10831913E-04
12						0.30921377E-07

TABLE 2

SINGULAR POINTS OF PADE APPROXIMANTS

OF BLASIUS BASIC FLOW (IN y^3-PLANE)

M = 3	M = 5	M = 7	M = 9	M = 11	M = 13
NUMERATOR ZEROS					
0.09999999E+01	0.09999999E+01	0.99999999E+00	0.99999999E+00	0.99999999E+00	0.09999999E+01
0.24983854E+01	0.12718301E+01	0.11168176E+01	0.10653087E+01	0.10417578E+01	0.10289373E+01
	0.22759326E+01	0.14317130E+01	0.12245183E+01	0.11391436E+01	0.10948876E+01
	0.78594307E+01	0.21901134E+01	0.15348202E+01	0.13126343E+01	0.12069684E+01
		0.44190729E+01	0.21448667E+01	0.16064972E+01	0.13832153E+01
		0.16383472E+02	0.34954806E+01	0.21169628E+01	0.16573593E+01
			0.73699684E+01	0.30767792E+01	0.20948822E+01
			0.28092227E+02	0.51653680E+01	0.28338622E+01
				0.11123975E+02	0.42051671E+01
				0.43000236E+02	0.71723385E+01
					0.15623021E+02
					0.60867601E+02
DENOMINATOR ZEROS					
0.17916168E+01	0.11968607E+01	0.10907993E+01	0.10524815E+01	0.10342404E+01	0.10240617E+01
0.15269627E+02	0.19081376E+01	0.13497735E+01	0.11899979E+01	0.11204170E+01	0.10832789E+01
	0.47699933E+01	0.19426033E+01	0.14553662E+01	0.12744232E+01	0.11847983E+01
	0.43674470E+02	0.34752448E+01	0.19585484E+01	0.15319507E+01	0.13438457E+01
		0.92538712E+01	0.29972110E+01	0.19676333E+01	0.15883288E+01
		0.85251991E+02	0.55889170E+01	0.27512462E+01	0.19707873E+01
			0.15217055E+02	0.43322890E+01	0.25972073E+01
			0.13990977E+03	0.82363035E+01	0.37056612E+01
				0.22648154E+02	0.59234278E+01
				0.20757968E+03	0.11373988E+02
					0.31421522E+02
					0.28708810E+03

TABLE 3

PROPER VALUES FOR BLASIUS BASIC FLOW

$A_2 = A_3 = 0$

C	R	0.12500000E+03	0.15277859E+03	0.26434282E+03	0.55901699E+03	0.11821770E+04	0.20454437E+04	0.25000000E+04
-0.10000000E-00	Re (A_1)	0.21686852E-00	0.22160747E-00	0.23596253E-00	0.25439022E-00	0.24517559E-00	0.24603989E-00	0.24652200E-00
	Im (A_1)	0.11129852E-01	0.84501098E-02	0.50787261E-02	0.15705512E-01	0.26995083E-01	0.24979940E-01	0.25314984E-01
-0.81817747E-01	Re (A_1)	0.18361566E-00	0.18729601E-00	0.19871257E-00	0.21673702E-00	0.22211543E-00	0.21648271E-00	0.21689171E-00
	Im (A_1)	0.10992372E-01	0.78180569E-02	0.16329508E-02	0.25031101E-02	0.19713859E-01	0.18647963E-01	0.18122607E-01
-0.47287080E-01	Re (A_1)	0.11846527E-00	0.12049082E-00	0.12626956E-00	0.13619406E-00	0.14889636E-00	0.15764015E-00	0.15926751E-00
	Im (A_1)	0.14893263E-01	0.11777792E-01	0.42134376E-02	-0.35561898E-02	-0.58508192E-02	-0.10888631E-02	0.25105111E-02
-0.22360680E-01	Re (A_1)	0.64664940E-01	0.66380134E-01	0.70265529E-01	0.74822985E-01	0.80069748E-01	0.85042638E-01	0.87128898E-01
	Im (A_1)	0.20116798E-01	0.17328181E-01	0.11097494E-01	0.42045870E-02	-0.18314128E-02	-0.53143138E-02	-0.62350804E-02
-0.10573713E-01	Re (A_1)	0.30393575E-01	0.32576964E-01	0.37110192E-01	0.41496253E-01	0.44853160E-01	0.47117659E-01	0.48004983E-01
	Im (A_1)	0.24200945E-01	0.20928008E-01	0.14646326E-01	0.91249647E-02	0.48702142E-02	0.19371174E-02	0.91682477E-03
-0.61111436E-02	Re (A_1)	0.11756438E-01	0.14280590E-01	0.20023553E-01	0.25088498E-01	0.28716786E-01	0.30818158E-01	0.31506772E-01
	Im (A_1)	0.28318892E-01	0.23456597E-01	0.15732370E-01	0.10490891E-01	0.70913665E-02	0.49885029E-02	0.42400190E-02
-0.50000000E-02	Re (A_1)	0.63572866E-02	0.85893959E-02	0.14754717E-01	0.20155030E-01	0.23939757E-01	0.26114386E-01	0.26820645E-01
	Im (A_1)	0.30348890E-01	0.24691364E-01	0.15827350E-01	0.10613675E-01	0.74804458E-02	0.55967435E-02	0.49487246E-02

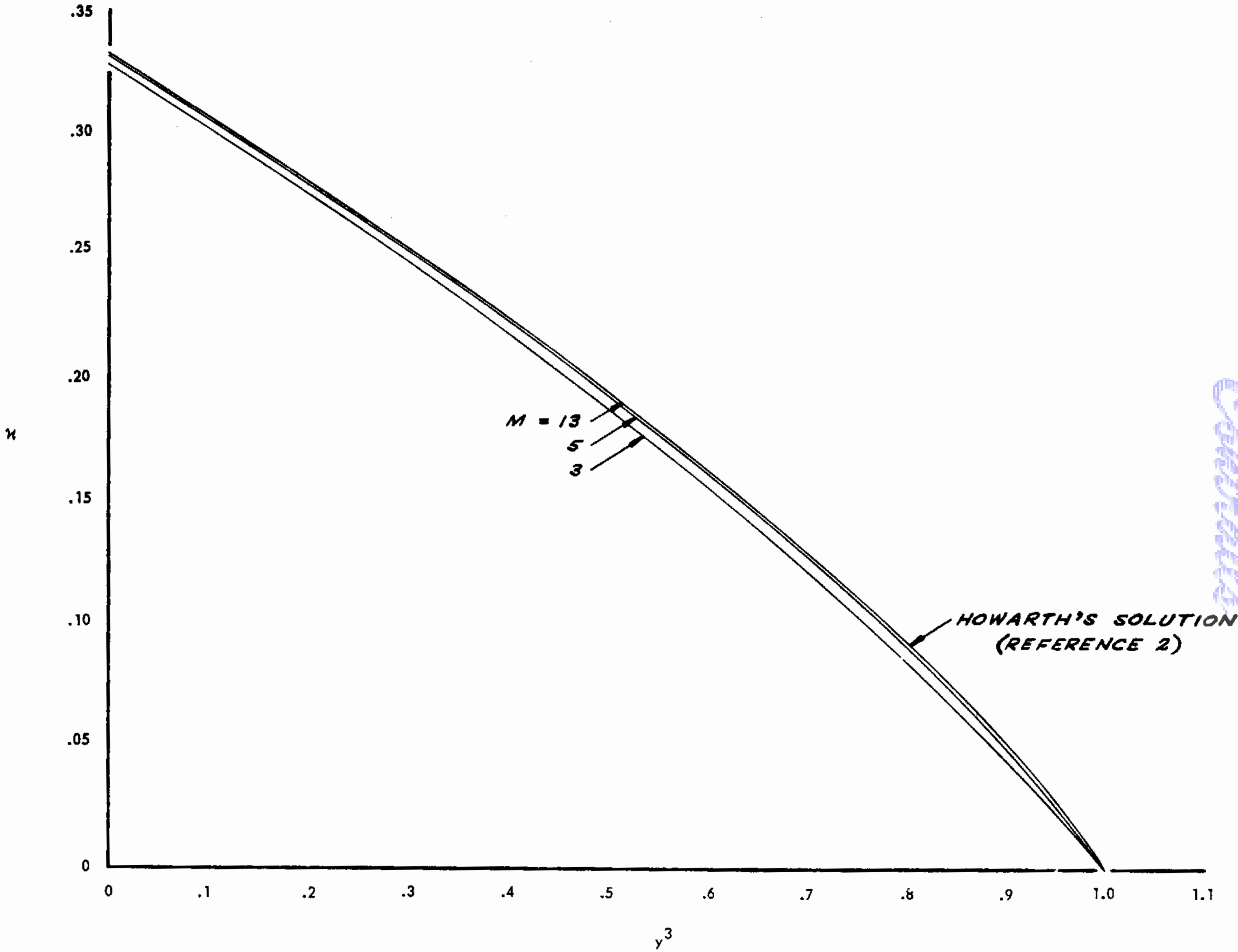
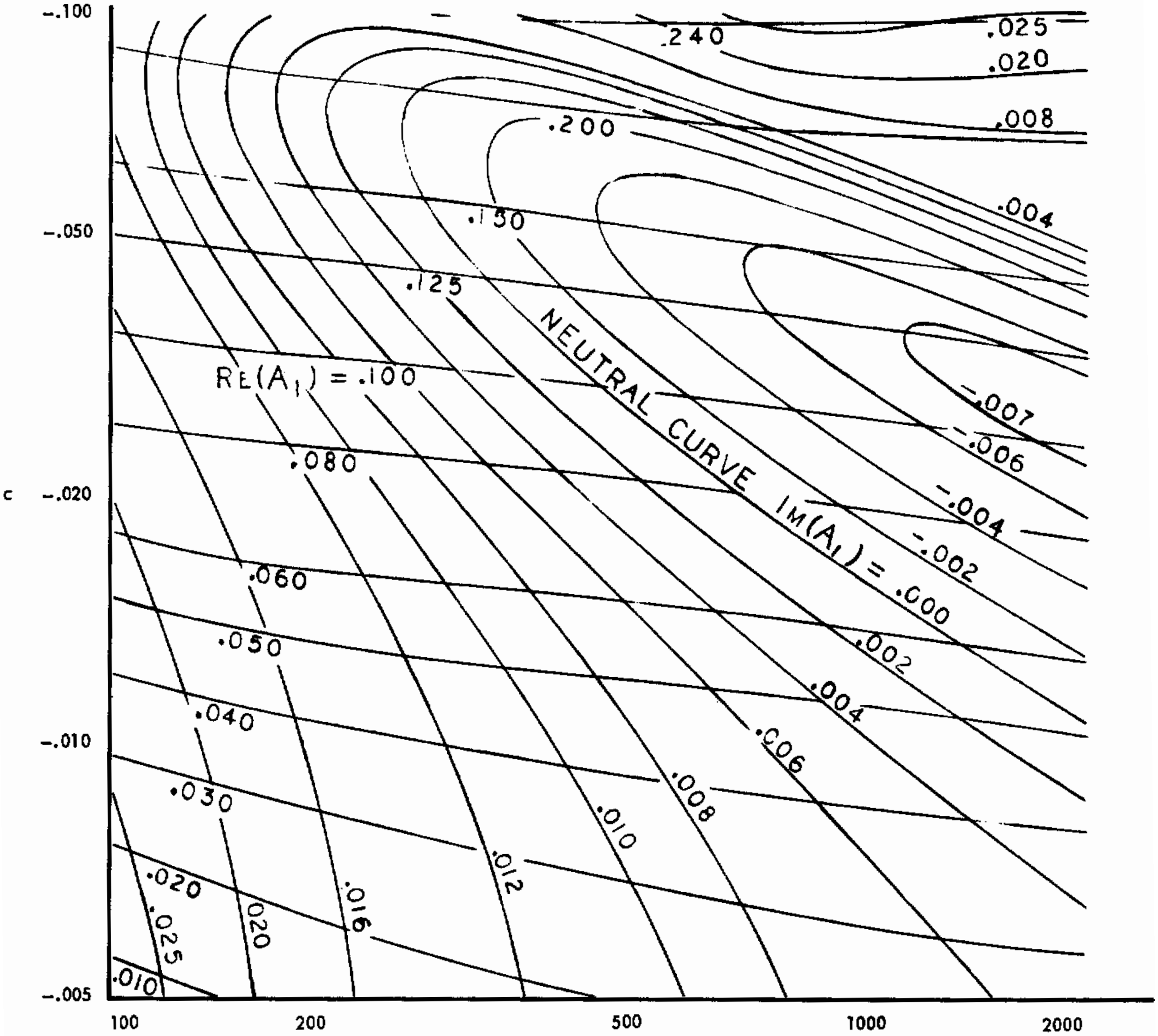


FIGURE 1 PADE APPROXIMANTS OF BLASIUS BASIC FLOW



$A_2 = A_3 = 0$

FIGURE 2 PROPER VALUES FOR BLASIUS BOUNDARY LAYER

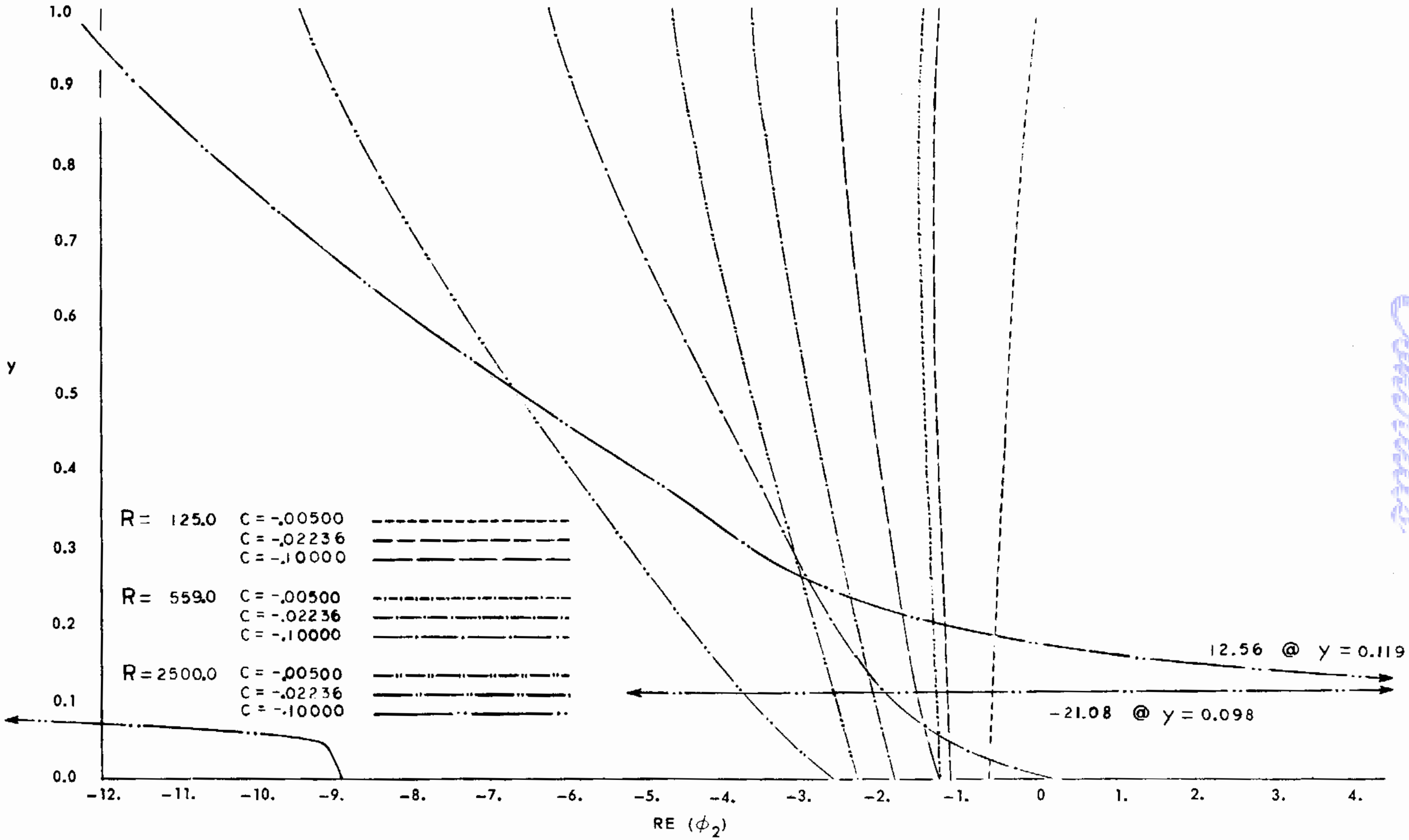


FIGURE 3 PROPER FUNCTIONS FOR BLASIUS BASIC FLOW
(A) FUNDAMENTAL SOLUTION (REAL PART)

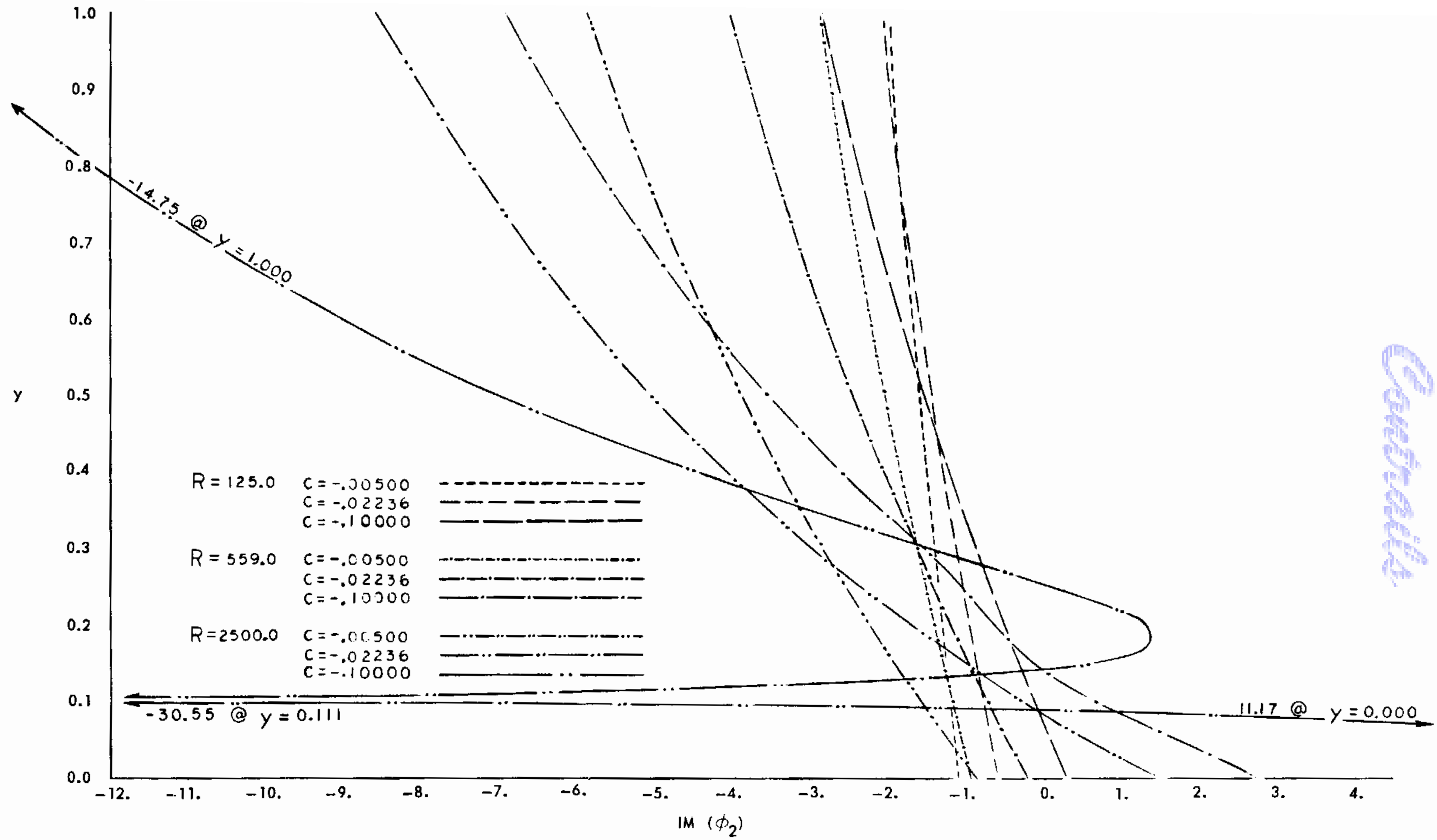


FIGURE 3 PROPER FUNCTIONS FOR BLASIUS BASIC FLOW
(A) FUNDAMENTAL SOLUTION (IMAGINARY PART)

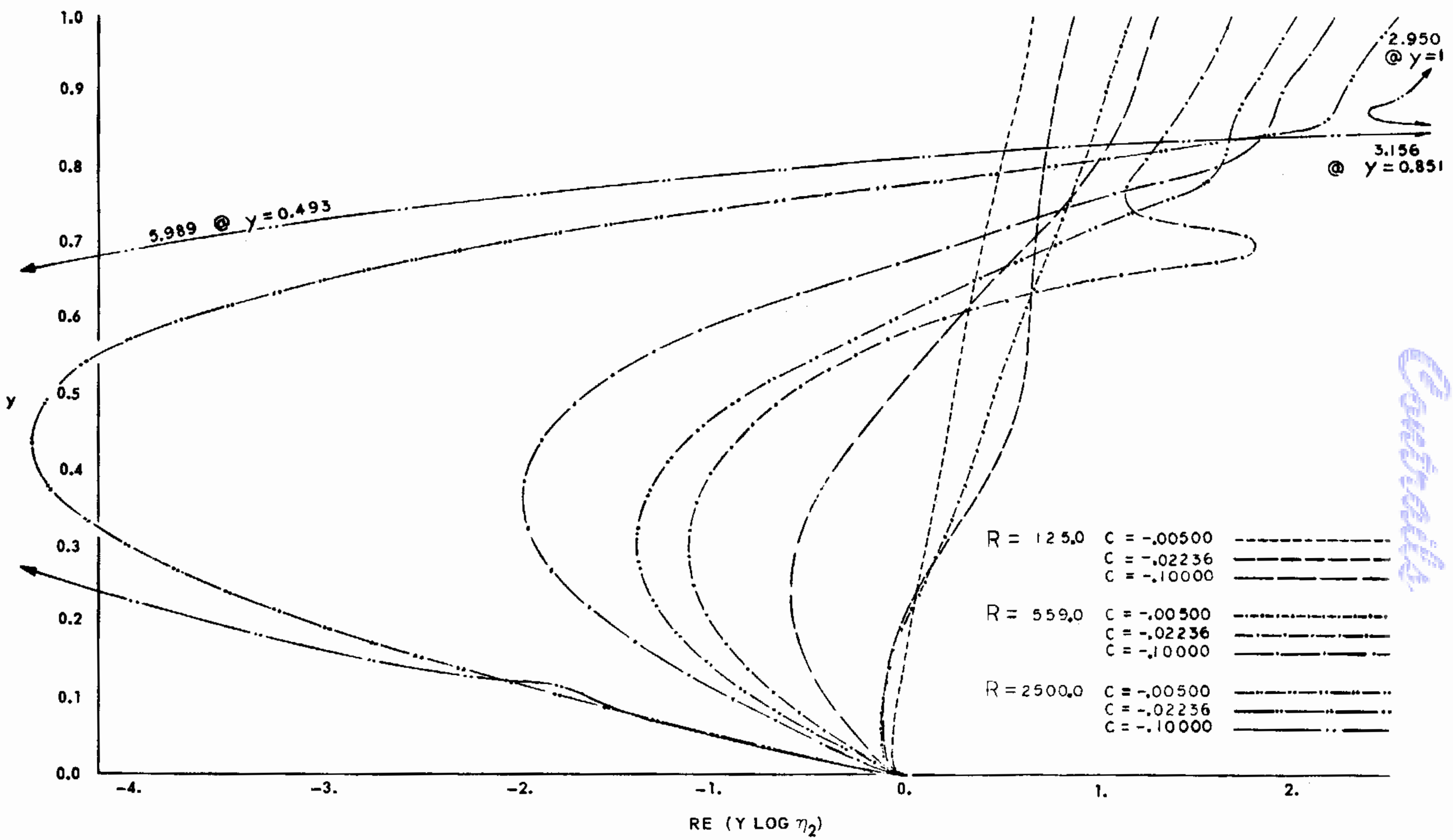


FIGURE 3 PROPER FUNCTIONS FOR BLASIUS BASIC FLOW
(B) SECULAR DETERMINANT (REAL PART)

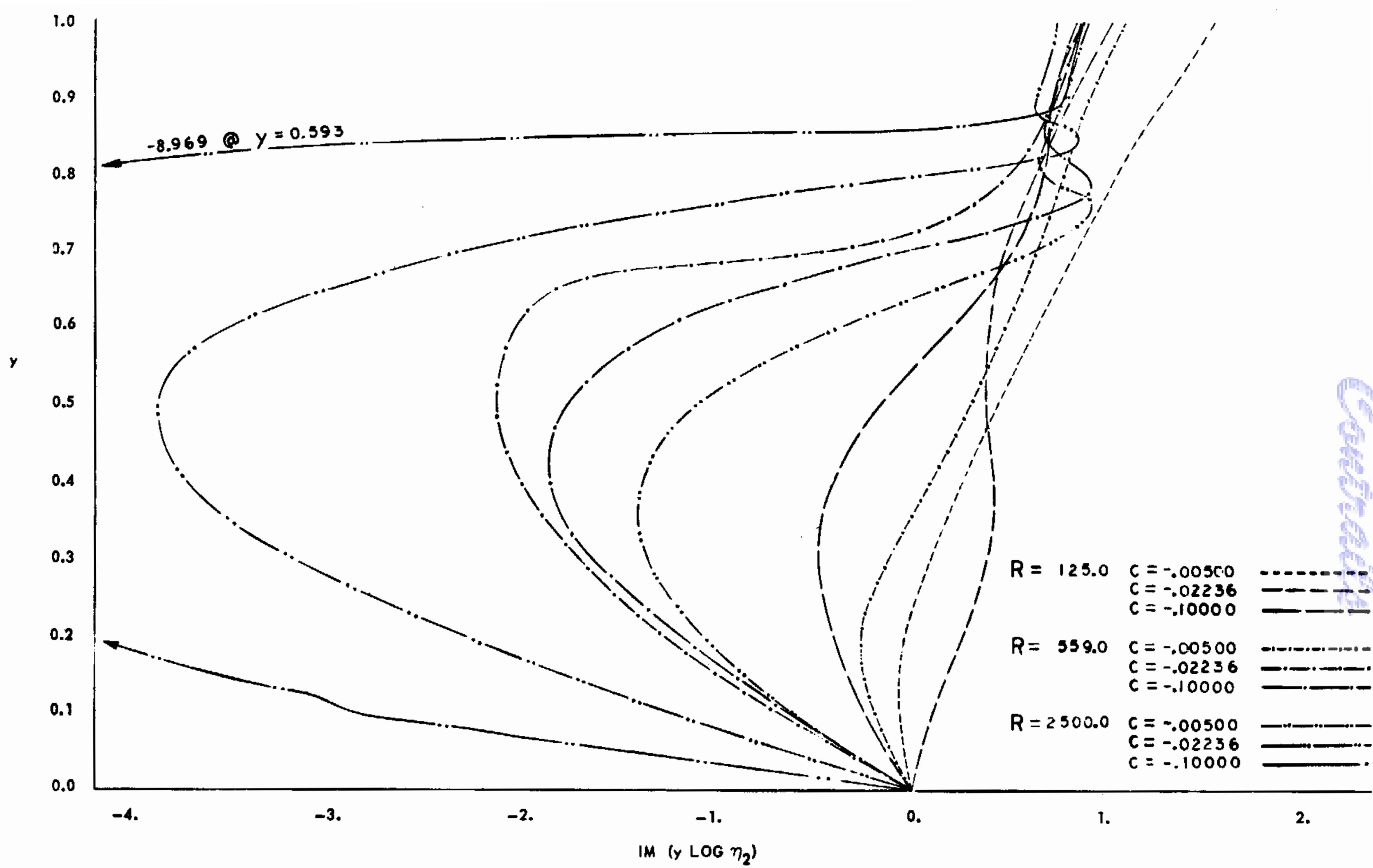


FIGURE 3 PROPER FUNCTIONS FOR BLASIUS BASIC FLOW
(B) SECULAR DETERMINANT (IMAGINARY PART)

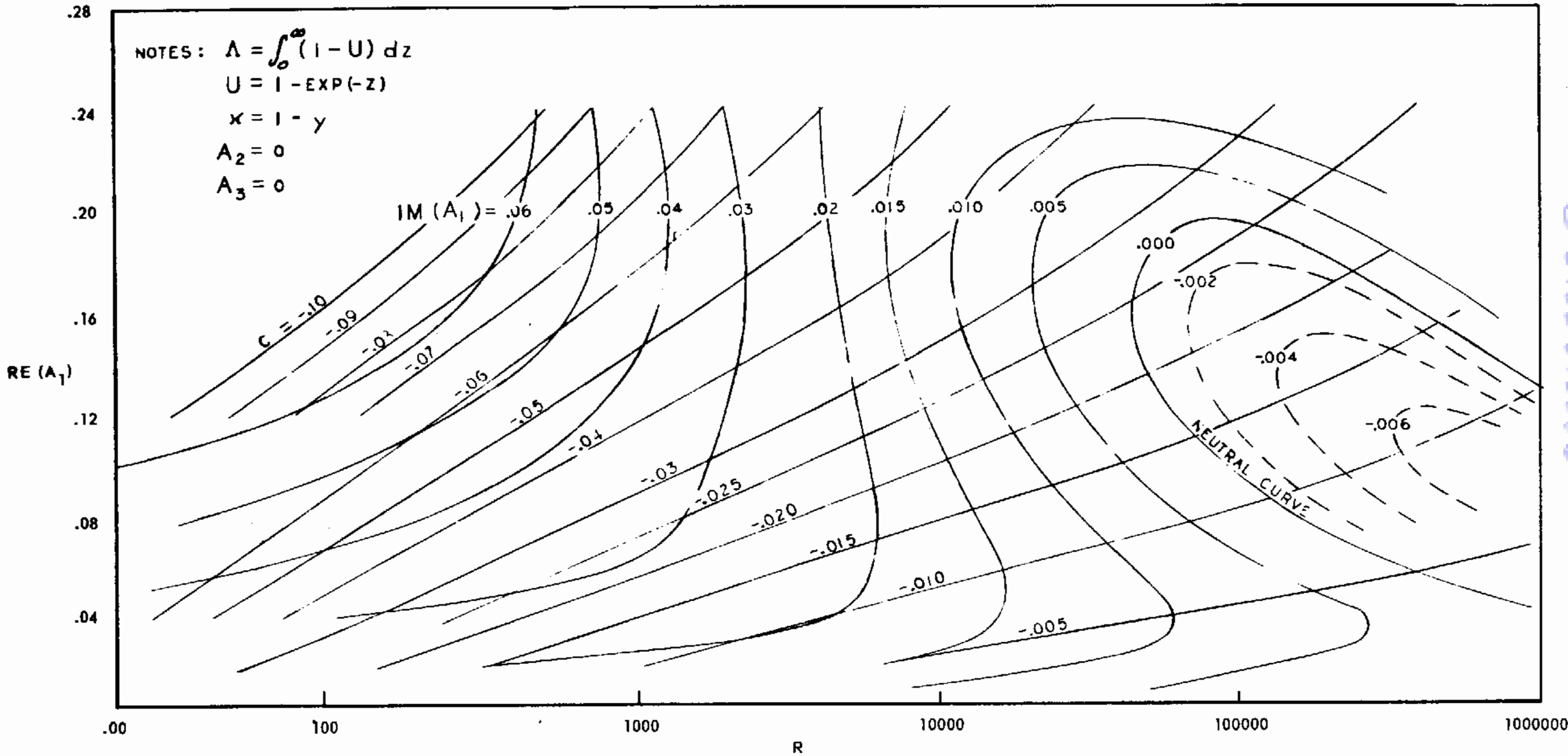


FIGURE 4 PROPER VALUES FOR ASYMPTOTIC SUCTION BOUNDARY LAYER

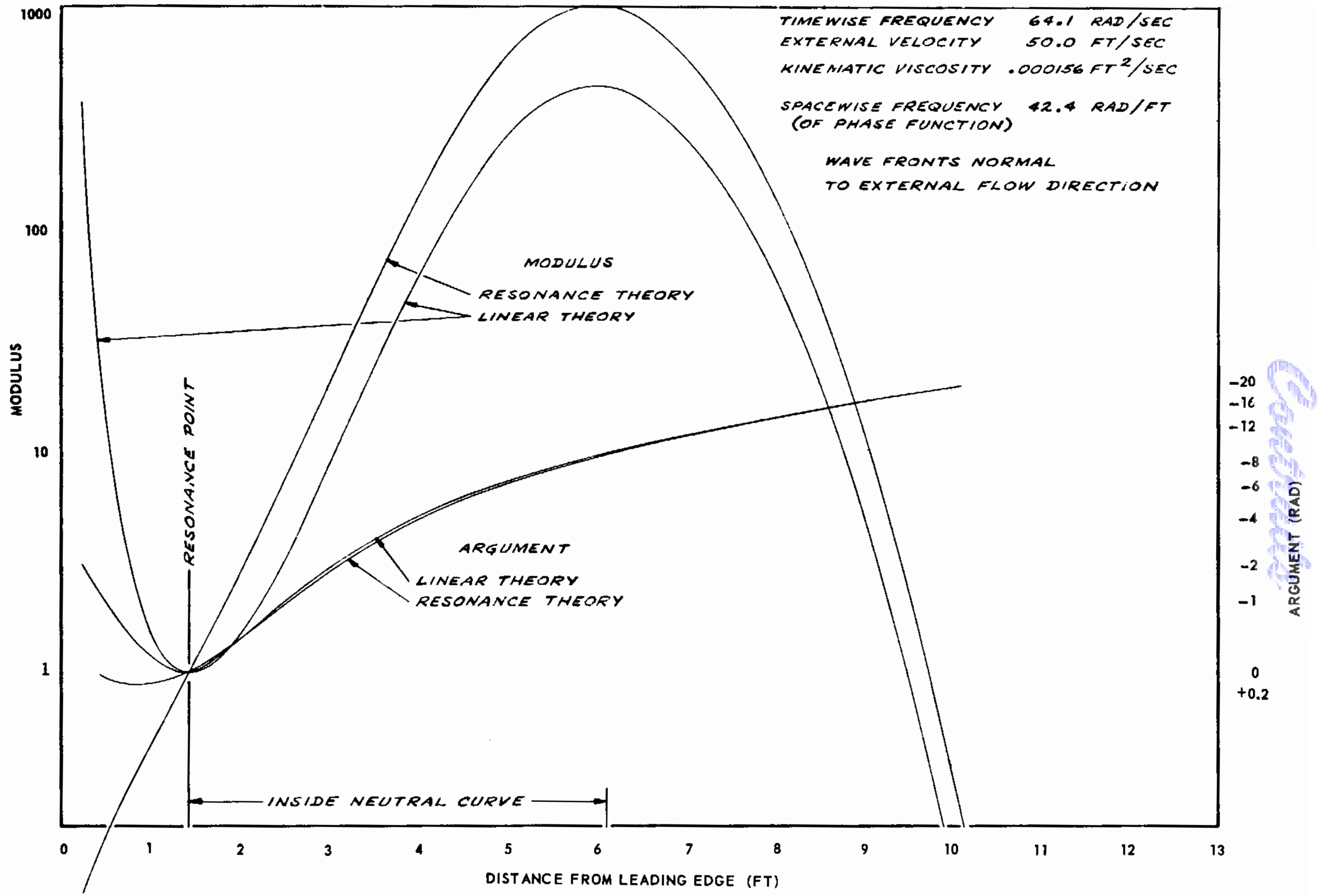


FIGURE 5 TYPICAL GROWTH FUNCTIONS IN BLASIUS BOUNDARY LAYER

APPENDIX

SOURCE PROGRAMS

A. <u>TITLES</u>

MPA	adjoined power series solution	930TA
SRB	basic flow and constants	930TB
SRC	secular determinant	930TC
SRD	parameters	930TD
SRE	adjoined expansions	930TE
SRF	basic flow and parameters (local)	930TF
SRG	formal and rational expansions	930TG
SRH	first summation	930TH
SRI	second summation	930TI
SRJ	third summation	930TJ
SRK	rational function approximation	930TK
SRL	linear system solution	930TL
MPM	proper value location	930TM
SRN	polynomial coefficients	930TN
SRØ	simple zeros near origin	930TØ

B. COMMON NOTATION

Subscripts used below: I = expansion number

J = j + 1

K = k + 1

L = root number

B (1) = A_1

B (2) = A_2

B (3) = c

B (4) = R

B (5) = α^2

B (6) = α

B (7) = α^{-1}

B (8) = a

B (9) = b

B (10) = c

B (11) = iR

B (12) = $ia\alpha R$

B (13) = $ia\alpha Rc$

B (14) = Γ

B (15) = β^2

B (16) = β

B (17) = $\alpha\Gamma$

B (18) = vacant

B (19) = vacant

B (20) = vacant

C (I,1) = γ

C (I,2) = σ

C (I,3) = $\gamma\sigma$

C (I,4) = $\alpha^2\sigma$

C (I,5) = $\alpha^2\gamma\sigma$

C (I,6) = vacant

C (I,7) = vacant

C (I,8) = vacant

D (1) = $h_2^{(1)}(0)$

D (2) = $h_2^{(2)}(0)$

D (3) = Θ*

D (4) = vacant

D (5) = vacant

D (6) = vacant

D (7) = vacant

D (8) = vacant

D (9) = (S2,HM)*

G (J,K) = g_j^k

H (I + 1,1) = $\varphi_2(1)$

H (I + 1,2) = $\varphi_3(1)$

H (I + 1,3) = $\varphi_4(1)$

H (I + 1,4) = $\eta_2(1)$

---

*S2 = $y_o + \delta$ of last expansion
HM = maximum error index of all expansions

H (I + 1,5) = $\eta_3(1)$

H (I + 1,6) = $\eta_4(1)$

H (I + 1,7) = error index for φ_2

H (I + 1,8) = error index for φ_3

H (I + 1,9) = error index for φ_4

H (I + 1,10) = error index for η_2

H (I + 1,11) = error index for η_3

H (I + 1,12) = error index for η_4

K (1) = adjacent values (initial interpolation)

K (2) = adjacent values (final interpolation)

K (3) = unknown (streamwise frequency or Reynolds number)

K (4) = roots sought (each run)

K (5) = iterations (root extraction)

K (6) = printing (maximum or minimum)

K (7) = interpolation (initial or final)

K (8) = terms (in polynomial or Pade numerator)

K (9) = roots calculated (each interpolation)

L (1) = expansion number

L (2) = expansion type (initial or subsequent)

L (3) = n (term degree)

L (4) = n + 1 (term number)

L (5) = n + 2

L (6) = rational functions (1 to 6)

M (1) = numerator terms (basic flow)

M (2) = denominator terms (basic flow)

M (3) = expansions (per solution)

M (4) = maximum terms (per expansion)

M (5) = printing (maximum, medium, or minimum)

M (6) = vacant

M (7) = vacant

M (8) = vacant

M (9) = vacant

M (10) = maximum of M (1) and M (2)

N (I,1) = expansion type (initial or subsequent)

N (I,2) = total terms

N (I,3) = numerator terms

N (I,4) = denominator terms (assigned)

Ø (1,K) = t_k^o

Ø (2,K) = s_k^o

P (I,1,K) = p_k

P (I,2,K) = q_k

P (I,3,K) = r_k

P (I,4,K) = u_k

P (I,5,K) = v_k

P (I,6,K) = w_k

Q (1,K) = P_k

Q (2,K) = Q_k

Q (3,K) = R_k

Q (4,K) = U_k

Q (5,K) = V_k

Q (6,K) = W_k

R (I,1,K) = t_k

R (I,2,K) = s_k

S (I,1) = δ (tentative value)

S (I,2) = y_o

S (I,3) = λ_o

S (I,4) = δ (final value)

U (1) = a_{n-1} or a_n

U (2) = $b_{n-1} + \sigma t_n$ or $b_n + \sigma t_n$

U (3) = c_{n-1} or c_n

U (4) = a^*_{n-1} or a^*_n

U (5) = b^*_{n-1} or b^*_n

U (6) = c^*_{n-1} or c^*_n

V (1) = d_n or D_n

V (2) = e_n or E_n

V (3) = f_n or F_n

V (4) = d_n^* or D_n^*

V (5) = e_n^* or E_n^*

V (6) = f_n^* or F_n^*

X (1,J) = $\widetilde{E}_j/n$ (for initial interpolation)

X (2,J) = $\widetilde{E}_j/n$ (for final interpolation)

Y (1,J) = θ_j

Y (2,J) = c_j

Y (3,J) = polynomial coefficients or Pade coefficients (numerator and denominator)

Z (1,L) = initial root (unit-circle or actual)

Z (2,L) = final root (unit-circle or actual)

Z (3,L) = error in final root

C. SOURCE STATEMENTS

```
C     MAIN PROGRAM MPA                                                   930TA010
C     ADJOINED POWER SERIES SOLUTION                                     930TA020
C                                                                        930TA030
C     USES SRB THRU SRL                                                  930TA040
C                                                                        930TA050
C     USES 10 DATA CARDS FOR BASIC FLOW.....                             930TA060
C     2...M(1)...29 NUMERATOR TERMS   (BASIC FLOW)                       930TA070
C     2...M(2)...29 DENOMINATOR TERMS (BASIC FLOW)                       930TA080
C                                                                        930TA090
C     PLUS 2 DATA CARDS FOR CONTROL.....                                 930TA130
C     4...M(4)...30 MAXIMUM TERMS (PER EXPANSION)                        930TA140
C     1...M(5)...3  PRINTING      (MAX, MED, MIN)                        930TA150
C                                                                        930TA250
C     PLUS 6 DATA CARDS FOR PARAMETERS.....                              930TA170
C     A(1,J) STREAMWISE FREQUENCY  (REAL PART)                           930TA180
C     A(2,J) STREAMWISE FREQUENCY  (IMAG PART)                           930TA190
C     A(3,J) CROSSWISE FREQUENCY   (REAL)                                930TA200
C     A(4,J) TIMEWISE FREQUENCY    (REAL)                                930TA210
C     A(5,J) REYNOLDS NUMBER       (REAL PART)                           930TA220
C     A(6,J) REYNOLDS NUMBER       (IMAG PART)                           930TA230
C                                                                        930TA240
      DIMENSION L(6),M(10),N(30,4),                                      930TA260
     1          R(30,2,29),S(30,4),G(29,29),O(2,29),                     930TA270
     2          A(6,5)                                                   930TA280
I     DIMENSION H(31,12),P(30,6,30),Q(6,48),                             930TA290
     1          B(20)                                                    930TA300
      COMMON L,M,N,H,P,Q,R,S,G,O,B                                       930TA310
 10   CALL SRB                                                           930TA320
 20   READ INPUT TAPE 5,30, (L(I),(A(I,J),J=1,5),I=1,6)                  930TA330
 30   FORMAT (I12,5E12.8)                                                930TA340
      L1=L(1)                                                            930TA350
      L2=L(2)                                                            930TA360
      L3=L(3)                                                            930TA370
      L4=L(4)                                                            930TA380
      L5=L(5)                                                            930TA390
      L6=L(6)                                                            930TA400
      L(6)=6                                                             930TA405
      B(22)=0.                                                           930TA410
```

```
      B(23)=0.                                                          930TA420
      DO 150 I1=1,L1                                                    930TA430
      B(1)=A(1,I1)                                                      930TA440
      DO 140 I2=1,L2                                                    930TA450
      B(21)=A(2,I2)                                                     930TA460
      DO 130 I3=1,L3                                                    930TA470
      B(2)=A(3,I3)                                                      930TA480
      DO 120 I4=1,L4                                                    930TA490
      B(3)=A(4,I4)                                                      930TA500
      DO 110 I5=1,L5                                                    930TA510
      B(4)=A(5,I5)                                                      930TA520
      DO 100 I6=1,L6                                                    930TA530
      B(24)=A(6,I6)                                                     930TA540
      CALL SRC                                                          930TA560
  100 CONTINUE                                                          930TA570
  110 CONTINUE                                                          930TA580
  120 CONTINUE                                                          930TA590
  130 CONTINUE                                                          930TA600
  140 CONTINUE                                                          930TA610
  150 CONTINUE                                                          930TA620
      GO TO 10                                                          930TA630
      END (0) 930TA000                                                  930TA640
```

```
      SUBROUTINE SRB                                                    930TB010
C     BASIC FLOW AND CONSTANTS                                          930TB020
C     O(2,1) MUST EQUAL 1                                               930TB030
C                                                                       930TB050
      DIMENSION L(6),M(10),N(30,4),                                     930TB060
     1           R(30,2,29),S(30,4),G(29,29),O(2,29)                    930TB070
I     DIMENSION H(31,12),P(30,6,30),Q(6,48)                             930TB090
      COMMON L,M,N,H,P,Q,R,S,G,O                                        930TB110
      READ INPUT TAPE 5,10,M(1),(O(1,K),K=1,29)                         930TB120
      READ INPUT TAPE 5,10,M(2),(O(2,K),K=1,29)                         930TB130
      READ INPUT TAPE 5,10,M(4)                                         930TB150
      READ INPUT TAPE 5,10,M(5)                                         930TB160
   10 FORMAT (I12,5E12.8/(6E12.8))                                      930TB170
      WRITE OUTPUT TAPE 6,20,                                           930TB180
   20 FORMAT (1H1)                                                      930TB190
   28 WRITE OUTPUT TAPE 6,30,M(1),(O(1,K),K=1,29)                       930TB200
      WRITE OUTPUT TAPE 6,30,M(2),(O(2,K),K=1,29)                       930TB210
   30 FORMAT (1H0,I19,5E20.8/(1H ,E19.8,5E20.8))                        930TB230
      M(1)=XMAXOF (M(1),2)                                              930TB235
      M(2)=XMAXOF(M(2),2)                                               930TB240
      M(4)=XMAXOF(M(4),4)                                               930TB250
      N(1,1)=1                                                          930TB253
      DO 40 I=2,30                                                      930TB260
   40 N(I,1)=2                                                          930TB270
      DO 60 J=7,12                                                      930TB330
I60   H(1,J)=(0.,0.)                                                    930TB340
      M12=XMAXOF(M(1),M(2))                                             930TB350
      M(10)=M12                                                         930TB355
      G(1,1)=1.                                                         930TB360
   70 DO 100 J=2,M12                                                    930TB370
      G(1,J)=1.                                                         930TB380
      J1=J-1                                                            930TB390
      G(J,J1)=0.                                                        930TB400
   80 DO 90 I=2,J                                                       930TB410
   90 G(I,J)=G(I,J1)+G(I-1,J1)                                          930TB420
  100 CONTINUE                                                          930TB430
      RETURN                                                            930TB440
```

```
FREQUENCY 70(20),80(10)                                                  930TB720
END (0) 930TB000                                                         930TB750
```

```
      SUBROUTINE SRC                                              930TC010
C     SECULAR DETERMINANT                                         930TC020
C     USES SRD THRU SRL                                           930TC030
C                                                                 930TC040
      DIMENSION L(6),M(10),N(30,4),                               930TC050
     1          R(30,2,29),S(30,4),G(29,29),O(2,29)               930TC060
I     DIMENSION H(31,12),P(30,6,30),Q(6,48),                      930TC070
     1          B(20),C(30,8),D(9)                                930TC080
      COMMON L,M,N,H,P,Q,R,S,G,O,B,C,D                            930TC090
      CALL SRD                                                    930TC100
I     H(1,1)=-B(16)                                               930TC110
I     H(1,2)=B(14)                                                930TC120
I     H(1,3)=(0.,0.)                                              930TC130
I     H(1,4)=B(16)-B(6)                                           930TC170
I     H(1,5)=-B(14)                                               930TC180
I     H(1,6)=B(17)                                                930TC190
      CALL SRE                                                    930TC200
      M5=M(5)                                                     930TC240
 10   GO TO (20,30,50),M5                                         930TC250
 20   CONTINUE                                                    930TC260
 30   WRITE OUTPUT TAPE 6,40, (D(I),D(I+9),I=1,3)                 930TC270
 40   FORMAT (1H0,E19.8,5E20.8)                                   930TC280
 50   RETURN                                                      930TC290
      FREQUENCY 10(1,10,100)                                      930TC300
      END (0) 930TC000                                            930TC310
```

```
      SUBROUTINE SRD                                                   930TD010
C     PARAMETERS                                                       930TC020
C                                                                      930TC030
      DIMENSION L(6),M(10),N(30,4),                                    930TD040
     1          R(30,2,29),S(30,4),G(29,29),O(2,29)                    930TD050
I     DIMENSION H(31,12),P(30,6,30),Q(6,48),                           930TD060
     1          B(20)                                                  930TC070
      COMMON L,M,N,H,P,Q,R,S,G,O,B                                     930TD080
I     B(5)=B(1)**2+B(2)**2                                             930TC090
I     B(6)=SQRTF(B(5))                                                 930TC100
I     B(7)=(1.,0.)/B(6)                                                930TC110
I     B(8)=B(1)*B(7)                                                   930TC120
I     B(9)=B(2)*B(7)                                                   930TC130
I     B(10)=-B(3)/B(1)                                                 930TC140
I     B(11)=(0.,1.)*B(4)                                               930TC150
I     B(12)=B(1)*B(11)                                                 930TC160
I     B(13)=-B(3)*B(11)                                                930TC170
I     B(14)=B(12)-B(13)                                                930TC180
I     B(15)=B(5)+B(14)                                                 930TC190
I     B(16)=SQRTF(B(15))                                               930TC200
I     B(17)=B(6)*B(14)                                                 930TC210
      M5=M(5)                                                          930TC330
   20 GO TO (30,40,50),M5                                              930TC340
   30 CONTINUE                                                         930TC350
   40 WRITE OUTPUT TAPE 6,60,                                          930TC370
     1   B(1),B(21),B(2),B(3),B(4),B(24),                              930TC380
     2   B(6),B(26),B(16),B(36),B(10),B(30)                            930TC390
   50 RETURN                                                           930TC400
   60 FORMAT (1H0,E19.8,5E20.8/1H ,E19.8,5E20.8)                       930TC410
      FREQUENCY 20(1,10,100)                                           930TC420
      END (0) 930TD000                                                 930TD430
```

```
      SUBROUTINE SRE                                                    930TE010
C     ADJOINED EXPANSIONS                                               930TE020
C     SIX POLES ALLOWED (MAXIMUM)                                       930TE025
C     USES SRF THRU SRL                                                 930TE030
C     STORAGE REUSED AFTER 30 EXPANSIONS                                930TE035
C     TO ALLOW ADDITIONAL 29 EXPANSIONS                                 930TE036
C                                                                       930TE040
      DIMENSION L(6),M(10),N(30,4),                                     930TE050
     1          R(30,2,29),S(30,4),G(29,29),O(2,29)                     930TE060
I     DIMENSION H(31,12),P(30,6,30),Q(6,48),                            930TE070
     1          B(20),C(30,8),D(9)                                      930TE075
      COMMON L,M,N,H,P,Q,R,S,G,O,B,C,D                                  930TE080
      L(1)=1                                                            930TE090
      L(2)=1                                                            930TE100
      N(1,4)=7                                                          930TE101
      S(1,1)=-0.1                                                       930TE103
      S(1,2)= 1.0                                                       930TE105
      CALL SRF                                                          930TE107
      CALL SRG                                                          930TE110
      L(2)=2                                                            930TE120
      DO 160 K=1,2                                                      930TE125
      DO 30 I=2,30                                                      930TE130
      N(I,4)=7                                                          930TE135
      S2=S(I-1,2)+S(I-1,4)                                              930TE140
      S1=1.0-S2                                                         930TE150
   10 IF (S2) 40,40,20                                                  930TE160
   20 S(I,1)=-MIN1F(0.1,S1,S2)                                          930TE170
      S(I,2)=S2                                                         930TE180
      L(1)=I                                                            930TE190
      CALL SRF                                                          930TE200
      CALL SRG                                                          930TE210
   30 CONTINUE                                                          930TE220
   40 M3=L(1)                                                           930TE222
      M(3)=M3                                                           930TE223
      M11=M3+1                                                          930TE224
I     D(1)=H(M11,1)                                                     930TE225
I     D(3)=H(M11,4)                                                     930TE226
I     D(2)=D(1)+D(3)                                                    930TE227
```

```
      HM=0.                                                             930TE228
      DO 47 I=2,M11                                                     930TE230
      DO 43 J=7,12                                                      930TE231
I     HA=ABSF(H(I,J))                                                   930TE232
 43   HM=MAX1F(HM,HA)                                                   930TE233
 47   CONTINUE                                                          930TE235
      D(9) =S(M3,2)+S(M3,4)                                             930TE237
      D(18)=HM                                                          930TE238
      M5=M(5)                                                           930TE240
 50   GO TO (60,90,130),M5                                              930TE250
 60   WRITE OUTPUT TAPE 6,70,((S(I,J),J=1,4),I=K,M3)                    930TE280
      DO 65 J1=1,10,3                                                   930TE285
      J2=J1+2                                                           930TE290
      WRITE OUTPUT TAPE 6,80,                                           930TE295
     1((H(I,J),H(I,J+12),J=J1,J2),I=K,M11)                              930TE296
 65   CONTINUE                                                          930TE300
 70   FORMAT (1H0,E19.8,3E20.8/(1H ,E19.8,3E20.8))                      930TE310
 80   FORMAT (1H0,E19.8,5E20.8/(1H ,E19.8,5E20.8))                      930TE320
 90   WRITE OUTPUT TAPE 6,120, M3,D(9),D(18)                            930TE420
 120  FORMAT (1H0,I19,2E40.8)                                           930TE430
 130  IF(D(9))170,170,140                                               930TE432
 140  DO 150 J=1,12                                                     930TE434
I150  H(2,J)=H(31,J)                                                    930TE436
      S(1,2)=S(30,2)                                                    930TE438
 160  S(1,4)=S(30,4)                                                    930TE440
 170  RETURN                                                            930TE445
      FREQUENCY 10(0,1,7),50(1,10,100)                                  930TE450
      END (0) 930TE000                                                  930TE460
```

```
      SUBROUTINE SRF                                              930TF010
C     BASIC FLOW AND PARAMETERS (LOCAL)                           930TM020
C     S(I,1),S(I,2) MUST NOT VANISH                               930TM030
C                                                                 930TM040
      DIMENSION L(6),M(10),N(30,4),                               930TF050
     1          R(30,2,29),S(30,4),G(29,29),O(2,29),              930TF060
     2          F(2,29),E(29),D(29),A(2)                          930TM070
I     DIMENSION H(31,12),P(30,6,30),Q(6,48),                      930TF080
     1          B(20),C(30,8)                                     930TF090
      COMMON L,M,N,H,P,Q,R,S,G,O,B,C                              930TF100
      I=L(1)                                                      930TM110
      M12=M(10)                                                   930TM120
      S1=S(I,1)                                                   930TM130
      S2=S(I,2)                                                   930TM140
      S4=S1/S2                                                    930TM150
      E(1)=1.                                                     930TM490
      D(1)=1.                                                     930TM500
  110 DO 120 J=2,M12                                              930TM510
      J1=J-1                                                      930TM520
      E(J)=S2*E(J1)                                               930TM530
  120 D(J)=S4*D(J1)                                               930TM540
      DO 160 J=1,2                                                930TM550
      MO=M(J)                                                     930TM555
  140 DO 150 K=1,MO                                               930TM560
  150 F(J,K)=O(J,K)*E(K)                                          930TM570
  160 CONTINUE                                                    930TM580
      DO 210 J=1,2                                                930TM590
      MO=M(J)                                                     930TM600
  170 DO 200 K1=1,MO                                              930TM610
      RO=0.                                                       930TM620
  180 DO 190 K2=K1,MO                                             930TM630
  190 RO=RO+F(J,K2)*G(K1,K2)                                      930TM640
  200 R(I,J,K1)=RO*D(K1)                                          930TM650
  210 CONTINUE                                                    930TM660
      A(2)=1.0/R(I,2,1)                                           930TM670
      A(1)=A(2)/S1                                                930TM675
      DO 240 J=1,2                                                930TM680
```

```
      M0=M(J)                                                   930TM690
      R2=A(J)                                                   930TM695
  220 DO 230 K=1,M0                                             930TM700
  230 R(I,J,K)=R2*R(I,J,K)                                      930TM710
  240 CONTINUE                                                  930TM720
      S3=R(I,1,1)                                               930TM740
      S(I,3)=S3                                                 930TM760
      C(I,1)=(B(10)-S(I,2))/S1                                  930TM880
      C(I,9)=B(30)/S1                                           930TM890
      C(I,2)=S1*B(12)                                           930TM900
      C(I,10)=S1*B(32)                                          930TM910
I     C(I,3)=C(I,1)*C(I,2)                                      930TM920
I     C(I,4)=B(5)*C(I,2)                                        930TM930
I     C(I,5)=B(5)*C(I,3)                                        930TM940
      RETURN                                                    930TM950
      FREQUENCY 110(20),140(20),                                930TM960
     1          170(20),180(10),220(20)                         930TM970
      END (0) 930TF000                                          930TF980
```

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      SUBROUTINE SRG                                                   930TG005
C     FORMAL AND RATIONAL EXPANSIONS                                   930TG010
C     M(4) MUST EXCEED 3                                               930TG015
C     R(I,2,1) MUST EQUAL 1                                            930TG020
C     POLE ON REAL AXIS BETWEEN TERMINAL POINTS NOT COVERED            930TG025
C     USES SRH THRU SRL                                                930TG030
C                                                                      930TG035
      DIMENSION L(6),M(10),N(30,4),                                    930TG040
     1          R(30,2,29),S(30,4),G(29,29),O(2,29)                    930TG045
I     DIMENSION H(31,12),P(30,6,30),Q(6,48),                           930TG050
     1          B(20),C(30,8),D(9),                                    930TG055
     2          U(6),V(6),                                             930TG060
     3          A1(1),A2(1),D1(1),D2(1),PA(1),PB(1)                    930TG065
      COMMON L,M,N,H,P,Q,R,S,G,O,B,C,D,U,V                             930TG070
      I=L(1)                                                           930TG075
      L2=L(2)                                                          930TG080
I     C2=C(I,2)                                                        930TG082
 10   GO TO (20,110),L2                                                930TG085
 20   T1=R(1,1,2)                                                      930TG090
I     P0=H(1,1)                                                        930TG092
I     U0=H(1,4)                                                        930TG094
I     V0=H(1,5)                                                        930TG095
I     W0=H(1,6)                                                        930TG096
I     PU=P0+U0                                                         930TG097
I     UU=U0+U0                                                         930TG098
I     A0=P0**2-B(15)                                                   930TG100
I     D0=PU**2-B(15)                                                   930TG102
I     A1=P0                                                            930TG105
I     D1=PU                                                            930TG107
I     A2=P0+P0                                                         930TG110
I     D2=PU+PU                                                         930TG112
I     B5=B(5)                                                          930TG115
      DO 30 J=1,6                                                      930TG120
I     P(1,J,1)=H(1,J)                                                  930TG125
I30   Q(J,1)=(0.,0.)                                                   930TG130
      L(4)=2                                                           930TG135
      CALL SRH                                                         930TG140
```

```
I       V(1)=-U(1)                                     930TG145
I       V(2)=-U(2)+P0*C2                               930TG150
I       V(3)=-U(3)-C(I,4)                              930TG155
        A1(1)=A1(1)+T1                                 930TG160
        A2(1)=A2(1)+T1                                 930TG165
I       A3=A1**2-B5                                    930TG170
I       A4=A3*V(1)+A1*V(2)+V(3)                        930TG175
I       A5=A2*(A0+A3)                                  930TG180
I       P1=A4/A5                                       930TG185
I       P2=-V(1)+A2*P1                                 930TG190
I       P3=-V(2)+A1*P2+A0*P1                           930TG192
I       P(1,1,2)=P1                                    930TG195
I       P(1,2,2)=P2                                    930TG200
I       P(1,3,2)=P3                                    930TG201
I       V(4)=-U(4)-P1*UU                               930TG203
I       V(5)=-U(5)-P1*V0-U0*(P2-C2)                    930TG204
I       V(6)=-U(6)-P1*W0-U0*P3                         930TG205
        D1(1)=D1(1)+T1                                 930TG207
        D2(1)=D2(1)+T1                                 930TG208
I       D3=D1**2-B5                                    930TG209
I       D4=D3*V(4)+D1*V(5)+V(6)                        930TG210
I       D5=D2*(D0+D3)                                  930TG211
I       P4=D4/D5                                       930TG212
I       P5=-V(4)+D2*P4                                 930TG213
I       P(1,4,2)=P4                                    930TG214
I       P(1,5,2)=P5                                    930TG215
I       P(1,6,2)=-V(5)+D1*P5+D0*P4                     930TG216
        DO 40 J1=1,6                                   930TG218
        J2=J1+6                                        930TG219
        Q(J1,2) =U(J1)+T1*P(1,J1,2)                    930TG220
 40     Q(J1,50)=U(J2)+T1*P(1,J1,32)                   930TG225
I       PA=ABSF(P4)                                    930TG230
        PM=PA(1)                                       930TG235
        TN=T1                                          930TG240
        M4=M(4)                                        930TG245
 50     DO 90 K1=3,M4                                  930TG250
        K2=K1+30                                       930TG255
        K3=K1+48                                       930TG257
        L(3)=K1-1                                      930TG260
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```
          L(4)=K1                                                       930TG265
          L(5)=K1+1                                                     930TG270
          TN=TN+T1                                                      930TG275
          CALL SRH                                                      930TG280
          CALL SRI                                                      930TG285
          A1(1)=A1(1)+T1                                                930TG290
          A2(1)=A2(1)+T1                                                930TG295
I         A3=A1**2-B5                                                   930TG300
I         A4=A3*V(1)+A1*V(2)+V(3)                                       930TG305
I         A5=A2*(A0+A3)                                                 930TG310
I         P1=A4/A5                                                      930TG315
I         P2=-V(1)+A2*P1                                                930TG320
I         P3=-V(2)+A1*P2+A0*P1                                          930TG322
I         P(1,1,K1)=P1                                                  930TG325
I         P(1,2,K1)=P2                                                  930TG330
I         P(1,3,K1)=P3                                                  930TG331
I         V(4)=V(4)-P1*UU                                               930TG333
I         V(5)=V(5)-P1*V0-U0*P2                                         930TG334
I         V(6)=V(6)-P1*W0-U0*P3                                         930TG335
          D1(1)=D1(1)+T1                                                930TG337
          D2(1)=D2(1)+T1                                                930TG338
I         D3=D1**2-B5                                                   930TG339
I         D4=D3*V(4)+D1*V(5)+V(6)                                       930TG340
I         D5=D2*(D0+D3)                                                 930TG341
I         P4=D4/D5                                                      930TG342
I         P5=-V(4)+D2*P4                                                930TG343
I         P(1,4,K1)=P4                                                  930TG344
I         P(1,5,K1)=P5                                                  930TG345
I         P(1,6,K1)=-V(5)+D1*P5+D0*P4                                   930TG346
          DO 60 J1=1,6                                                  930TG348
          J2=J1+6                                                       930TG349
          Q(J1,K1)=U(J1)+TN*P(1,J1,K1)                                  930TG350
 60       Q(J1,K3)=U(J2)+TN*P(1,J1,K2)                                  930TG355
I         PA=ABSF(P4)                                                   930TG365
          PM=MAX1F(PM,PA(1))                                            930TG370
 70       IF (PA(1)-(1.0E+15))     80,80,100                            930TG375
 80       IF (PM-(1.0E+05)*PA(1)) 90,90,100                             930TG380
 90       CONTINUE                                                      930TG385
```

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 100  C0=1.0E-05                                                930TG390
      GO TO 200                                                 930TG392
 110  R0=1.0/R(I,1,1)                                           930TG395
      DO 120 J=1,6                                              930TG405
I120  P(I,J,1)=H(I,J)                                           930TG410
      L(4)=1                                                    930TG415
      L(5)=2                                                    930TG420
      CALL SRJ                                                  930TG425
I     Q(1,1)=-V(1)+B(5)                                         930TG430
I     Q(2,1)=-V(2)                                              930TG435
I     Q(3,1)=-V(3)+C(I,5)                                       930TG440
      DO 125 J=4,6                                              930TG442
I125  Q(J,1)=-V(J)                                              930TG443
      CALL SRH                                                  930TG445
      DO 130 J1=1,6                                             930TG450
      J2=J1+6                                                   930TG455
      P(I,J1,2) =-R0*U(J1)                                      930TG460
 130  P(I,J1,32)=-R0*U(J2)                                      930TG465
      L(4)=2                                                    930TG470
      L(5)=3                                                    930TG475
      CALL SRJ                                                  930TG480
I     Q(1,2)=-V(1)                                              930TG485
I     Q(2,2)=-V(2)+C2*P(I,1,1)                                  930TG490
I     Q(3,2)=-V(3)-C(I,4)                                       930TG495
I     Q(4,2)=-V(4)                                              930TG496
I     Q(5,2)=-V(5)+C2*P(I,4,1)                                  930TG497
I     Q(6,2)=-V(6)                                              930TG498
      CALL SRH                                                  930TG500
      R1=0.5*R0                                                 930TG505
      DO 140 J1=1,6                                             930TG510
      J2=J1+6                                                   930TG515
      P(I,J1,3) =-R1*U(J1)                                      930TG520
 140  P(I,J1,33)=-R1*U(J2)                                      930TG525
I     PA=ABSF(P(I,4,3))                                         930TG530
      PM=PA(1)                                                  930TG535
      M0=M(4)-1                                                 930TG540
 150  DO 190 K=3,M0                                             930TG545
      K1=K+1                                                    930TG550
      K2=K1+30                                                  930TG555
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      L(4)=K                                                            930TG560
      L(5)=K1                                                           930TG565
      CALL SRJ                                                          930TG570
     I      Q(1,K)=-V(1)                                                930TG575
     I      Q(2,K)=-V(2)+C2*P(I,1,K-1)                                  930TG580
     I      Q(3,K)=-V(3)                                                930TG585
     I      Q(4,K)=-V(4)                                                930TG586
     I      Q(5,K)=-V(5)+C2*P(I,4,K-1)                                  930TG587
     I      Q(6,K)=-V(6)                                                930TG588
      CALL SRH                                                          930TG590
      RN=RO/FLOATF(K)                                                   930TG595
      DO 160 J1=1,6                                                     930TG600
      J2=J1+6                                                           930TG605
      P(I,J1,K1)=-RN*U(J1)                                              930TG610
  160 P(I,J1,K2)=-RN*U(J2)                                              930TG615
     I      PA=ABSF(P(I,4,K1))                                          930TG625
      PM=MAX1F(PM,PA(1))                                                930TG630
  170 IF (PA(1)-(1.0E+15))     180,180,195                              930TG635
  180 IF (PM-(1.0E+07)*PA(1)) 190,190,195                               930TG640
  190 CONTINUE                                                          930TG645
  195 CO=1.0E-03                                                        930TG647
  200 L4=L(4)+L(2)-1                                                    930TG650
      N(I,2)=L4                                                         930TG655
     I      PB=ABSF(P(I,4,1))                                           930TG658
      E0=1.0/FLOATF(L4-1)                                               930TG660
      S0=(CO*(PB(1)/PA(1)))**E0                                         930TG662
  210 IF (S0-1.0) 220,260,260                                           930TG665
  220 S(I,4)=S0*S(I,1)                                                  930TG670
      F0=1.0                                                            930TG675
  230 DO 250 K1=2,L4                                                    930TG680
      K2=K1+30                                                          930TG685
      F0=F0*S0                                                          930TG690
      DO 240 J=1,6                                                      930TG695
      P(I,J,K1)=F0*P(I,J,K1)                                            930TG700
  240 P(I,J,K2)=F0*P(I,J,K2)                                            930TG705
  250 CONTINUE                                                          930TG710
      GO TO 270                                                         930TG715
  260 S(I,4)=S(I,1)                                                     930TG720
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270  M5=M(5)                                                    930TG725
280  GO TO (290,310,320),M5                                     930TG730
290  WRITE OUTPUT TAPE 6,300, ((P(I,J,K),P(I,J,K+30),J=1,3),K=1,L4)   930TG735
     WRITE OUTPUT TAPE 6,300, ((P(I,J,K),P(I,J,K+30),J=4,6),K=1,L4)   930TG737
300  FORMAT (1H0,E19.8,5E20.8/(1H ,E19.8,5E20.8))               930TG740
310  CONTINUE                                                   930TG745
320  CALL SRK                                                   930TG750
     RETURN                                                     930TG755
     FREQUENCY 10(1,6),                                         930TG760
    1          50(27),70(100,0,1),80(100,0,1),                  930TG765
    2          150(26),170(100,0,1),180(100,0,1),               930TG770
    3          210(3,1,3),230(29),280(1,10,100)                 930TG775
     END (0) 930TG000                                           930TG780
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```
      SUBROUTINE SRH                                                    930TH010
C     FIRST SUMMATION                                                   930TH020
C     1...L(2)...2                                                      930TH030
C                                                                       930TH040
      DIMENSION L(6),M(10),N(30,4),                                     930TH050
     1          R(30,2,29),S(30,4),G(29,29),O(2,29)                     930TH060
I     DIMENSION H(31,12),P(30,6,30),Q(6,48),                            930TH070
     1          B(20),C(30,8),D(9),                                     930TH080
     2          U(6)                                                    930TH090
      COMMON L,M,N,H,P,Q,R,S,G,O,B,C,D,U                                930TH100
      I=L(1)                                                            930TH110
      N1=L(4)+L(2)-2                                                    930TH120
      N2=N1+1                                                           930TH130
      N3=N2+1                                                           930TH140
I     U(1)=(0.,0.)                                                      930TH150
   10 IF (L(4)-M(1)) 20,20,30                                           930TH160
   20 L4=L(4)                                                           930TH170
      R0=R(I,1,L4)                                                      930TH180
      U(2)=R0*C(I,2)                                                    930TH190
      U(8)=R0*C(I,10)                                                   930TH200
      GO TO 40                                                          930TH210
I  30 U(2)=(0.,0.)                                                      930TH220
   40 DO 45 J=3,6                                                       930TH230
I  45 U(J)=(0.,0.)                                                      930TH235
      M1=XMINOF(N2,M(1))                                                930TH240
   50 DO 70 K=2,M1                                                      930TH250
      K1=N3-K                                                           930TH260
      K2=K1+30                                                          930TH270
      R1=R(I,1,K)*FLOATF(K1-1)                                          930TH280
      DO 60 J1=1,6                                                      930TH290
      J2=J1+6                                                           930TH300
      U(J1)=U(J1)+R1*P(I,J1,K1)                                         930TH310
   60 U(J2)=U(J2)+R1*P(I,J1,K2)                                         930TH320
   70 CONTINUE                                                          930TH330
      M2=XMINOF(N1,M(2))                                                930TH340
   80 DO 100 K=1,M2                                                     930TH350
      K1=N2-K                                                           930TH360
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      K2=K1+48                                                          930TH370
      R2=R(I,2,K)                                                       930TH380
      DO 90 J1=1,6                                                      930TH390
      J2=J1+6                                                           930TH400
      U(J1)=U(J1)-R2*Q(J1,K1)                                           930TH410
   90 U(J2)=U(J2)-R2*Q(J1,K2)                                           930TH420
  100 CONTINUE                                                          930TH430
      RETURN                                                            930TH440
      FREQUENCY 10(10,1,20),50(10),80(10)                               930TH450
      END (0) 930TH000                                                  930TH460
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```
      SUBROUTINE SRI                                                     930TI010
C     SECOND SUMMATION                                                   930TI020
C     L(4) MUST EXCEED 2                                                 930TI030
C                                                                        930TI040
      DIMENSION L(6),M(10),N(30,4),                                      930TI050
     1          R(30,2,29),S(30,4),G(29,29),O(2,29)                      930TI060
I     DIMENSION H(31,12),P(30,6,30),Q(6,48),                             930TI070
     1          B(20),C(30,8),D(9),                                      930TI080
     2          U(6),V(6)                                                930TI090
      COMMON L,M,N,H,P,Q,R,S,G,O,B,C,D,U,V                               930TI100
      I=L(1)                                                             930TI110
      L3=L(3)                                                            930TI120
      L5=L(5)                                                            930TI130
I     C2=C(I,2)                                                          930TI135
I     V(1)=-U(1)                                                         930TI140
I     V(2)=-U(2)+C2*P(I,1,L3)                                            930TI150
I     V(3)=-U(3)                                                         930TI160
I     V(4)=-U(4)                                                         930TI163
I     V(5)=-U(5)+C2*P(I,4,L3)                                            930TI165
I     V(6)=-U(6)                                                         930TI167
 10   DO 30 K1=2,L3                                                      930TI170
      K2=L5-K1                                                           930TI180
I     P1=P(I,1,K1)                                                       930TI190
I     P4=P(I,4,K1)                                                       930TI193
I     P7=P1+P4                                                           930TI197
      DO 20 J1=1,3                                                       930TI200
      J2=J1+3                                                            930TI203
I     P0=P(I,J1,K2)                                                      930TI207
I     V(J1)=V(J1)-P1*P0                                                  930TI210
I20   V(J2)=V(J2)-P4*P0-P7*P(I,J2,K2)                                    930TI215
 30   CONTINUE                                                           930TI220
      RETURN                                                             930TI230
      FREQUENCY 10(13)                                                   930TI240
      END (0) 930TI000                                                   930TI250
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```
      SUBROUTINE SRJ                                                    930TJ010
C     THIRD SUMMATION                                                   930TJ020
C                                                                       930TJ030
      DIMENSION L(6),M(10),N(30,4),                                     930TJ040
     1          R(30,2,29),S(30,4),G(29,29),O(2,29)                     930TJ050
I     DIMENSION H(31,12),P(30,6,30),Q(6,48),                            930TJ060
     1          B(20),C(30,8),D(9),                                     930TJ070
     2          U(6),V(6)                                               930TJ080
      COMMON L,M,N,H,P,Q,R,S,G,O,B,C,D,U,V                              930TJ090
      I=L(1)                                                            930TJ100
      L4=L(4)                                                           930TJ110
      L5=L(5)                                                           930TJ120
I     C3=C(I,3)                                                         930TJ125
I     P2=P(I,2,L4)                                                      930TJ130
I     P5=P(I,5,L4)                                                      930TJ135
I     V(1)=-P2                                                          930TJ140
I     V(2)=-P(I,3,L4)+C3*P(I,1,L4)                                      930TJ150
I     V(3)=-P2*B(5)                                                     930TJ160
I     V(4)=-P5                                                          930TJ163
I     V(5)=-P(I,6,L4)+C3*P(I,4,L4)                                      930TJ165
I     V(6)=-P5*B(5)                                                     930TJ167
 10   DO 30 K1=1,L4                                                     930TJ170
      K2=L5-K1                                                          930TJ180
I     P1=P(I,1,K1)                                                      930TJ190
I     P4=P(I,4,K1)                                                      930TJ193
I     P7=P1+P4                                                          930TJ197
      DO 20 J1=1,3                                                      930TJ200
      J2=J1+3                                                           930TJ203
I     P0=P(I,J1,K2)                                                     930TJ207
I     V(J1)=V(J1)+P1*P0                                                 930TJ210
I20   V(J2)=V(J2)+P4*P0+P7*P(I,J2,K2)                                   930TJ215
 30   CONTINUE                                                          930TJ220
      RETURN                                                            930TJ230
      FREQUENCY 10(15)                                                  930TJ240
      END (0) 930TJ000                                                  930TJ250
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      SUBROUTINE SRK                                            930TK010
C     RATIONAL FUNCTION APPROXIMATION                           930TK020
C     1...L(6)...6                                              930TK023
C     N(I,2) MUST EXCEED 4                                      930TK025
C     N(I,3) MUST EXCEED N(I,4)-2                               930TK030
C     N(I,4) MUST EXCEED 2                                      930TK040
C     USES SRL (ONE ITERATION)                                  930TK050
C                                                               930TK060
      DIMENSION L(6),M(10),N(30,4)                              930TK070
I     DIMENSION H(31,12),P(30,6,30),                            930TK080
     1            A(16,16),B(16),X(16),                         930TK090
     2            W(30),Y(16),Z(3,6)                            930TK100
      COMMON L,M,N,H,P,A,B,X                                    930TK110
      I=L(1)                                                    930TK120
      I1=I+1                                                    930TK125
      L6=L(6)                                                   930TK127
      N2=N(I,2)                                                 930TK130
      N4=XMINOF(N(I,4),(N2+1)/2)                                930TK140
      N3=N2-N4                                                  930TK150
      N(I,3)=N3                                                 930TK160
      N(I,4)=N4                                                 930TK165
      N5=N4-1                                                   930TK170
      N6=N3+1                                                   930TK180
      DO 130 J0=1,L6                                            930TK190
 10   DO 20 K=1,N2                                              930TK210
I20   W(K)=P(I,J0,K)                                            930TK220
      DO 40 J=1,N5                                              930TK230
      K1=J+N3                                                   930TK240
      DO 30 K=1,N5                                              930TK250
      K0=K1-K                                                   930TK260
I30   A(J,K)=W(K0)                                              930TK270
I40   B(J)=-W(K1)                                               930TK280
      CALL SRL                                                  930TK290
      DO 50 J=1,N5                                              930TK300
I50   Y(J)=X(J)                                                 930TK310
      DO 70 J=1,N5                                              930TK320
I     B0=B(J)                                                   930TK330
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      DO 60 K=1,N5                                                      930TK340
I60   B0=B0-A(J,K)*Y(K)                                                 930TK350
I70   B(J)=B0                                                           930TK360
      CALL SRL                                                          930TK370
      DO 80 J=1,N5                                                      930TK380
I80   Y(J)=Y(J)+X(J)                                                    930TK390
I     P1=W(1)                                                           930TK400
 90   DO 110 J=2,N3                                                     930TK410
I     P0=W(J)                                                           930TK420
      N0=XMINOF(J,N4)-1                                                 930TK430
      DO 100 K=1,N0                                                     930TK440
      K0=J-K                                                            930TK450
I100  P0=P0+W(K0)*Y(K)                                                  930TK460
I     P1=P1+P0                                                          930TK470
I110  P(I,J0,J)=P0                                                      930TK480
I     P(I,J0,N6)=(1.,0.)                                                930TK490
I     Q1=(1.,0.)                                                        930TK500
I     E0=W(N2)                                                          930TK510
      DO 120 K=1,N5                                                     930TK520
      K0=N2-K                                                           930TK530
      K1=K+N6                                                           930TK540
I     Y0=Y(K)                                                           930TK550
I     Q1=Q1+Y0                                                          930TK560
I     E0=E0+W(K0)*Y0                                                    930TK570
I120  P(I,J0,K1)=Y0                                                     930TK580
I     F1=P1/Q1                                                          930TK583
I     F0=ABSF(W(1))+ABSF(F1)                                            930TK587
I     Z(1,J0)=P1                                                        930TK590
I     Z(2,J0)=Q1                                                        930TK600
I     Z(3,J0)=E0                                                        930TK610
I     H(I1,J0)=F1                                                       930TK620
I130  H(I1,J0+6)=E0/(Q1*F0)                                             930TK630
      M5=M(5)                                                           930TK640
 140  GO TO (150,160,170),M5                                            930TK650
 150  DO 155 J1=1,4,3                                                   930TK653
      J2=J1+2                                                           930TK657
      WRITE OUTPUT TAPE 6,180, ((P(I,J,K),P(I,J,K+30),J=J1,J2),K=1,N3)  930TK660
      WRITE OUTPUT TAPE 6,180, ((P(I,J,K),P(I,J,K+30),J=J1,J2),K=N6,N2) 930TK670
      WRITE OUTPUT TAPE 6,180, ((Z(I,J),Z(I,J+6),J=J1,J2),I=1,3)        930TK680
```

```
155   CONTINUE                                                    930TK685
160   CONTINUE                                                    930TK690
170   RETURN                                                      930TK700
180   FORMAT (1H0,E19.8,5E20.8/(1H ,E19.8,5E20.8))                930TK710
      FREQUENCY 10(30),90(30),140(1,10,100)                       930TK720
      END (0) 930TK000                                            930TK730
```

```
      SUBROUTINE SRL                                              930TL010
C     LINEAR SYSTEM SOLUTION                                      930TL020
C     SOLVES A*X=B                                                930TL025
C     3...N(I,4)...17 ONLY                                        930TL030
C                                                                 930TL040
C     DET(A) MUST NOT VANISH                                      930TL045
      DIMENSION L(6),M(10),N(30,4)                                930TL050
I     DIMENSION H(31,12),P(30,6,30),                              930TL060
     1          A(16,16),B(16),X(16),                             930TL070
     2          C(16,17)                                          930TL080
      COMMON L,M,N,H,P,A,B,X                                      930TL090
      L0=L(1)                                                     930TL100
      K0=N(L0,4)                                                  930TL110
      J0=K0-1                                                     930TL120
      I0=J0-1                                                     930TL130
      DO 20 J=1,J0                                                930TL140
      DO 10 K=1,J0                                                930TL150
I10   C(J,K)=A(J,K)                                               930TL160
I20   C(J,K0)=B(J)                                                930TL170
      DO 50 I=1,I0                                                930TL180
      I1=I+1                                                      930TL190
I     C0=(1.,0.)/C(I,I)                                           930TL200
      DO 40 K=I1,K0                                               930TL210
I     C1=C0*C(I,K)                                                930TL220
I     C(I,K)=C1                                                   930TL225
      DO 30 J=I1,J0                                               930TL230
I30   C(J,K)=C(J,K)-C(J,I)*C1                                     930TL240
 40   CONTINUE                                                    930TL250
 50   CONTINUE                                                    930TL260
      DO 55 J=1,J0                                                930TL270
I55   X(J)=C(J,K0)                                                930TL275
I     X(J0)=X(J0)/C(J0,J0)                                        930TL277
      DO 70 I=1,I0                                                930TL280
      K=K0-I                                                      930TL290
      K1=K-1                                                      930TL300
I     X1=X(K)                                                     930TL305
      DO 60 J=1,K1                                                930TL310
I60   X(J)=X(J)-C(J,K)*X1                                         930TL320
```

```
70    CONTINUE                                                  930TL330
      RETURN                                                    930TL360
      END (0) 930TL000                                          930TL370
```

```
C     MAIN PROGRAM MPM                                                  930TM010
C     PROPER VALUE LOCATION                                             930TM015
C                                                                       930TM020
C     USES SRB THRU SRL                                                 930TM025
C     PLUS SRN THRU SRO                                                 930TM030
C                                                                       930TM035
C     USES 10 DATA CARDS FOR BASIC FLOW.....                            930TM040
C     2...M(1)...29 NUMERATOR TERMS   (BASIC FLOW)                      930TM045
C     2...M(2)...29 DENOMINATOR TERMS (BASIC FLOW)                      930TM050
C                                                                       930TM055
C     PLUS 2 DATA CARDS FOR SOLUTION CONTROL.....                       930TM060
C     4...M(4)...30 MAXIMUM TERMS (PER EXPANSION)                       930TM065
C     3...M(5)...3  PRINTING      (MIN ONLY)                            930TM070
C                                                                       930TM075
C     PLUS 6 DATA CARDS FOR INTERPOLATION CONTROL.....                  930TM080
C     2...K(1)...16 POINTS (INITIAL INTERPOLATION)                      930TM085
C     2...K(2)...8  POINTS     (FINAL INTERPOLATION)                    930TM090
C     1...K(3)...2  UNKNOWN    (STREAM FREQ,REY NO)                     930TM095
C     1...K(4)...7  ROOTS      (PER POINT)                              930TM100
C     1...K(5)....  ITERATIONS (ROOT EXTRACTION)                        930TM105
C     1...K(6)...2  PRINTING   (MAX,MIN)                                930TM110
C                                                                       930TM115
C     PLUS 2 DATA CARDS FOR INTERPOLATION RANGE.....                    930TM120
C     T(1)          RADIUS (INITIAL INTERPOLATION)                      930TM125
C     T(2)          RADIUS (FINAL INTERPOLATION)                        930TM130
C                                                                       930TM135
C     PLUS 6 DATA CARDS FOR PARAMETERS (PER POINT).....                 930TM140
C     A(1)          STREAMWISE FREQUENCY (REAL PART)                    930TM145
C     A(2)          STREAMWISE FREQUENCY (IMAG PART)                    930TM150
C     A(3)          CROSSWISE FREQUENCY  (REAL)                         930TM155
C     A(4)          TIMEWISE FREQUENCY   (REAL)                         930TM160
C     A(5)          REYNOLDS NUMBER      (REAL PART)                    930TM165
C     A(6)          REYNOLDS NUMBER      (IMAG PART)                    930TM170
C                                                                       930TM175
C     2...K(1)...4 YIELDS JUST INITIAL INTERPOLATION                    930TM180
C                                                                       930TM185
      DIMENSION L(6),M(10),N(30,4),                                     930TM215
     1          R(30,2,29),S(30,4),G(29,29),O(2,29)                     930TM220
```

```
I     DIMENSION H(31,12),P(30,6,30),Q(6,48),                            930TM225
     1          B(20),C(30,8),D(9),                                     930TM230
     2          U(6),V(6)                                               930TM235
      DIMENSION K(9),                                                   930TM240
     1          T(2),A(6),                                              930TM245
     2          E(6)                                                    930TM247
I     DIMENSION X(2,16),Y(3,16),Z(3,7),                                 930TM250
     1          W(2,16),B1(1),B2(1),B3(1)                               930TM255
      COMMON L,M,N,H,P,Q,R,S,G,O,B,C,D,U,V,                             930TM260
     1       K,X,Y,Z                                                    930TM265
      CALL SRB                                                          930TM270
      M(5)=3                                                            930TM275
      READ INPUT TAPE 5,10, (K(I),I=1,6)                                930TM280
   10 FORMAT (I12)                                                      930TM285
      K1=K(1)                                                           930TM290
      K2=K(2)                                                           930TM295
      K3=K(3)                                                           930TM300
      K0=3*K3-2                                                         930TM302
      IF (K1-5) 13,15,15                                                930TM304
   13 L0=1                                                              930TM305
      GO TO 17                                                          930TM306
   15 L0=2                                                              930TM307
   17 L1=K1-1                                                           930TM308
      READ INPUT TAPE 5,20, (T(I),I=1,2)                                930TM310
   20 FORMAT (E12.8)                                                    930TM315
      WRITE OUTPUT TAPE 6,25, M(4),(K(I),I=1,5),(T(I),I=1,2)            930TM317
   25 FORMAT (1H0,I19/1H0,I19,4I20/1H0,E19.8,E20.8)                     930TM318
      DO 40 I=1,L0                                                      930TM320
      J0=K(I)                                                           930TM325
      T0=T(I)                                                           930TM330
      C0=1.0/FLOATF(J0)                                                 930TM335
      C1=6.28318531*C0                                                  930TM340
      DO 30 J1=1,J0                                                     930TM345
      J2=J1+16                                                          930TM350
      C2=C1*FLOATF(J1-1)                                                930TM355
      C3=COSF(C2)                                                       930TM360
      C4=SINF(C2)                                                       930TM365
      W(I,J1)=T0*C3                                                     930TM370
```

```
      W(I,J2)=T0*C4                                              930TM375
      X(I,J1)=+C0*C3                                             930TM380
   30 X(I,J2)=-C0*C4                                             930TM385
   40 CONTINUE                                                   930TM390
      B(22)=0.                                                   930TM395
      B(23)=0.                                                   930TM400
   50 READ INPUT TAPE 5,60, (A(I),I=1,6)                         930TM405
   60 FORMAT (E12.8)                                             930TM410
      WRITE OUTPUT TAPE 6,70, (A(I),I=1,6)                       930TM415
   70 FORMAT (1H0,E19.8,5E20.8)                                  930TM420
      B(1) =A(1)                                                 930TM425
      B(21)=A(2)                                                 930TM430
      B(2) =A(3)                                                 930TM435
      B(3) =A(4)                                                 930TM440
      B(4) =A(5)                                                 930TM445
      B(24)=A(6)                                                 930TM450
      DO 75 I=1,6                                                930TM452
   75 E(I)=0.                                                    930TM453
      K(7)=1                                                     930TM455
      L(6)=6                                                     930TM460
  I   B1=B(K0)                                                   930TM465
      DO 80 J=1,K1                                               930TM470
  I   B(K0)=B1+W(1,J)                                            930TM475
      CALL SRC                                                   930TM480
  I   Y(1,J)=D(3)                                                930TM483
      E(1)=MAX1F(E(1),D(9))                                      930TM485
   80 E(2)=MAX1F(E(2),D(18))                                     930TM487
      CALL SRN                                                   930TM489
      DO 85 J=1,K1                                               930TM490
 I85  Y(3,J)=Y(2,J)                                              930TM491
      K(8)=K1                                                    930TM492
      K(9)=L1                                                    930TM493
      CALL SRO                                                   930TM494
      GO TO (103,88),L0                                          930TM496
   88 DO 90 J=1,K1                                               930TM498
 I90  P(1,1,J)=Y(2,J)                                            930TM500
      L(1)=1                                                     930TM505
      L(6)=1                                                     930TM510
      N(1,2)=K1                                                  930TM515
```

```
      N(1,4)=3                                                          930TM520
      CALL SRK                                                          930TM525
      K(8)=N(1,3)                                                       930TM530
      K4=XMINOF(K(4),K1-3)                                              930TM535
      DO 100 J=1,K1                                                     930TM540
I100  Y(3,J)=P(1,1,J)                                                   930TM545
      K(9)=K4                                                           930TM550
      CALL SRO                                                          930TM555
      K(7)=2                                                            930TM560
      L(6)=6                                                            930TM565
 103  DO 130 I1=1,K4                                                    930TM570
      I2=I1+7                                                           930TM575
      B2(1)=B1(1)+T(1)*Z(1,I1)                                          930TM580
      B2(2)=B1(2)+T(1)*Z(1,I2)                                          930TM585
I     Z(1,I1)=B2                                                        930TM590
      GO TO (123,107),LO                                                930TM593
 107  DO 110 J=1,K2                                                     930TM595
I     B(K0)=B2+W(2,J)                                                   930TM600
      CALL SRC                                                          930TM605
I     Y(1,J)=D(3)                                                       930TM608
      E(3)=MAX1F(E(3),D(9))                                             930TM610
 110  E(4)=MAX1F(E(4),D(18))                                            930TM612
      CALL SRN                                                          930TM615
      K(8)=K2                                                           930TM620
      DO 120 J=1,K2                                                     930TM625
I120  Y(3,J)=Y(2,J)                                                     930TM630
      K(9)=1                                                            930TM635
      CALL SRO                                                          930TM640
      B3(1)=B2(1)+T(2)*Z(2,I1)                                          930TM645
      B3(2)=B2(2)+T(2)*Z(2,I2)                                          930TM650
      GO TO 127                                                         930TM653
I123  B3=B2                                                             930TM655
I127  Z(2,I1)=B3                                                        930TM657
I     B(K0)=B3                                                          930TM660
      CALL SRC                                                          930TM665
I     Z(3,I1)=D(3)                                                      930TM668
      E(5)=MAX1F(E(5),D(9))                                             930TM670
 130  E(6)=MAX1F(E(6),D(18))                                            930TM672
```

```
      WRITE OUTPUT TAPE 6,140, ((Z(I,J),Z(I,J+7),I=1,3),J=1,K4)          930TM675
      WRITE OUTPUT TAPE 6,140, (E(I),I=1,6)                              930TM677
  140 FORMAT (1H0,E19.8,5E20.8/(1H ,E19.8,5E20.8))                       930TM680
      GO TO 50                                                           930TM685
      END (0) 930TM000                                                   930TM690
```

```
      SUBROUTINE SRN                                                    930TN010
C     POLYNOMIAL COEFFICIENTS                                           930TN020
C     UNIT CIRCLE DATA                                                  930TN030
C                                                                       930TN040
      DIMENSION L(6),M(10),N(30,4),                                     930TN050
     1          R(30,2,29),S(30,4),G(29,29),O(2,29)                     930TN060
I     DIMENSION H(31,12),P(30,6,30),Q(6,48),                            930TN070
     1          B(20),C(30,8),D(9),                                     930TN080
     2          U(6),V(6)                                               930TN090
      DIMENSION K(9)                                                    930TN100
I     DIMENSION X(2,16),Y(3,16)                                         930TN110
      COMMON L,M,N,H,P,Q,R,S,G,O,B,C,D,U,V,                             930TN120
     1       K,X,Y                                                      930TN130
      IO=K(7)                                                           930TN140
      JO=K(IO)                                                          930TN150
      DO 20 I=1,JC                                                      930TN160
I     CO=(0.,0.)                                                        930TN170
      DO 10 J=1,JO                                                      930TN180
      KO=XMODF((I-1)*(J-1),JO)+1                                        930TN190
I10   CO=CO+X(IO,KO)*Y(1,J)                                             930TN200
I20   Y(2,I)=CO                                                         930TN210
      RETURN                                                            930TN220
      END (0) 930TN000                                                  930TN230
```

```
      SUBROUTINE SRO                                                    930T0010
C     SIMPLE ZEROS NEAR ORIGIN                                          930T0020
C                                                                       930T0030
      DIMENSION L(6),M(10),N(30,4),                                     930T0040
     1          R(30,2,29),S(30,4),G(29,29),O(2,29)                     930T0050
I     DIMENSION H(31,12),P(30,6,30),Q(6,48),                            930T0060
     1          B(20),C(30,8),D(9),                                     930T0070
     2          U(6),V(6)                                               930T0080
      DIMENSION K(9)                                                    930T0090
I     DIMENSION X(2,16),Y(3,16),Z(3,7),                                 930T0100
     1          E(3,16)                                                 930T0110
      COMMON L,M,N,H,P,Q,R,S,G,O,B,C,D,U,V,                             930T0120
     1       K,X,Y,Z                                                    930T0130
      K5=K(5)                                                           930T0140
      K6=K(6)                                                           930T0150
      K7=K(7)                                                           930T0160
      K8=K(8)                                                           930T0170
      K9=K(9)                                                           930T0180
      K10=K8+1                                                          930T0190
      K11=K(K7)                                                         930T0195
      DO 10 J=1,K8                                                      930T0200
I10   E(1,J)=Y(3,J)                                                     930T0210
      DO 110 I1=1,K9                                                    930T0230
      N0=K10-I1                                                         930T0240
      N1=N0-1                                                           930T0250
      N2=N1-1                                                           930T0260
 20   IF (N0-3) 30,40,60                                                930T0270
I30   Z(K7,I1)=-E(1,1)/E(1,2)                                           930T0280
      GO TO 120                                                         930T0290
I40   E(2,2)=E(1,3)                                                     930T0300
I     E(3,1)=E(2,2)                                                     930T0310
I     Z0=(0.,0.)                                                        930T0315
      DO 50 I=1,K5                                                      930T0320
I     E(2,1)=E(1,2)+Z0*E(2,2)                                           930T0330
I     E2=E(1,1)+Z0*E(2,1)                                               930T0340
I     E3=E(2,1)+Z0*E(3,1)                                               930T0350
I50   Z0=Z0-E2/E3                                                       930T0360
      GO TO 90                                                          930T0370
```

```
I60    E(2,N1)=E(1,N0)                                                  930T0380
I      E(3,N2)=E(2,N1)                                                  930T0390
I      Z0=(0.,0.)                                                       930T0395
       DO 80 I=1,K5                                                     930T0400
I      E(2,N2)=E(1,N1)+Z0*E(2,N1)                                       930T0410
       DO 70 J=2,N2                                                     930T0420
       J0=N1-J                                                          930T0430
       J1=J0+1                                                          930T0440
I      E(2,J0)=E(1,J1)+Z0*E(2,J1)                                       930T0450
I70    E(3,J0)=E(2,J1)+Z0*E(3,J1)                                       930T0460
I      E2=E(1,1)+Z0*E(2,1)                                              930T0470
I      E3=E(2,1)+Z0*E(3,1)                                              930T0480
I80    Z0=Z0-E2/E3                                                      930T0490
I90    Z(K7,I1)=Z0                                                      930T0500
       DO 100 J=1,N1                                                    930T0510
I100   E(1,J)=E(2,J)                                                    930T0520
 110   CONTINUE                                                         930T0530
 120   GO TO (130,160),K6                                               930T0540
 130   WRITE OUTPUT TAPE 6,140, ((Y(I,J),Y(I,J+16),I=1,3),J=1,K11)      930T0550
 140   FORMAT (1H0,E19.8,5E20.8/(1H ,E19.8,5E20.8))                     930T0560
       WRITE OUTPUT TAPE 6,150, (Z(K7,J),Z(K7,J+7),J=1,K9)              930T0570
 150   FORMAT (1H0,E19.8,E20.8/(1H ,E19.8,E20.8))                       930T0580
 160   RETURN                                                           930T0590
       FREQUENCY 20(1,1,100),120(1,100)                                 930T0600
       END (0) 930T0000                                                 930T0610
```

PART 2

STABILITY OF COMPRESSIBLE FLOW OVER A FLAT PLATE

W. Byron Brown

NOTATION

Dimensionless Quantities		Characteristic Measure
ρ	gas density	$\bar{\rho}_o^*$
T	gas temperature	$\bar{T}_o^*$
μ_1	first viscosity coefficient	$\bar{\mu}_{10}^*$
μ_2	second viscosity coefficient	$\bar{\mu}_{20}^*$
α	disturbance wave number	ℓ^{-1}
c	phase velocity of the disturbance	$\bar{U}_o^*$
γ	specific heat ratio	c_p/c_v
R	Reynolds number	$\frac{\bar{\rho}_o^* U_o^* \ell}{\bar{\mu}_{10}^*}$
M	Mach number (R* gas constant per gram)	$\frac{\bar{U}_o^*}{\sqrt{\gamma \cdot \bar{T}_o^* R^*}}$
$\bar{M}$	Mach number ($\bar{M} = M \cos \Psi$)	
σ	Prandtl number	$\frac{c_p \bar{\mu}_{10}^*}{\bar{k}_1^*}$
Ψ	angle between main velocity and the disturbance velocity	
k	thermal conductivity	$c_p \bar{\mu}_{10}^*$
ℓ	length unit	$\frac{x^*}{R}$
x^*	distance from the stagnation point	

Dimensionless Quantities		Characteristic Measure
x	non-dimensional distance	ℓ
y	distance from the flat plate	ℓ
w	undisturbed velocity parallel to the plate in component boundary layer	$\bar{U}_o^*$
v	undisturbed velocity component in boundary layer perpendicular to the wall	$\bar{U}_o^*$
f	velocity disturbance amplitude in x direction	
φ	velocity disturbance amplitude in y direction	
π	amplitude of the pressure disturbance	
r	amplitude of the density disturbance	
θ	amplitude of the temperature disturbance	

A bar over a quantity denotes average value. A prime denotes differentiation with respect to y. Subscript o denotes free-stream value; subscripts r and i denote real and imaginary parts.

I. INTRODUCTION

The exact numerical solution of the Lees-Lin equations (Reference 1) at Mach 5.8 showed better agreement with experimental data (Reference 2) than previous approximate solutions (References 3 and 4), but it was still 25% low at the critical Reynolds number and differed much more with the upper branch data. An improved calculation therefore was attempted by dropping the usual assumption that the flow was parallel to the flat plate and that the velocity component of the mean flow perpendicular to the plate could be safely neglected.

These calculations restored to the system of stability equations the terms involving the velocity component of the mean flow perpendicular to the flat plate. Calculations made with the more complete equations (References 5 and 6) showed that the expected increase in critical Reynolds number was much too large and that, in order to produce an agreement with experimental data, the three-dimensional aspect of the disturbance velocities would have to be taken into account. This was done by the method suggested by Dunn (Reference 3). This report gives the new equations and the new results. Dunn considers the Lees-Lin equations as obtained from the complete three-dimensional set (3 momentum equations instead of 2) by a transformation in direction so that the flow makes an angle Ψ with the x-axis. Then

$$\begin{aligned} \bar{M} &= M \cos \Psi \\ \bar{\alpha}\,\bar{M} &= \alpha M \\ \bar{\alpha}\,\bar{R} &= \alpha R \\ \bar{c} &= c \end{aligned} \tag{1}$$

The solution of the equations two dimensional in form yields eigenvalues for $\bar{\alpha}$ and $\bar{R}$, once a value of $\bar{M}$ has been chosen. The values of α and R (corresponding to the real flow) are found from the transformation equations (1).

II. ANALYSIS

The disturbance forms assumed are those of Dunn, Reference 6, namely

$$u_1 = w(y) + f(y) \exp[i(\alpha_1 x + \alpha_3 z - \alpha_1 ct)]$$

$$u_2 = v(y) + \alpha_1 \varphi(y) \exp[i(\alpha_1 x + \alpha_3 z - \alpha_1 ct)]$$

$$u_3 = h(y) \exp[i(\alpha_1 x + \alpha_3 z - \alpha_1 ct)]$$

$$\rho = \rho(y) + r(y) \exp[i(\alpha_1 x + \alpha_3 z - \alpha_1 ct)]$$

$$p = p(y) + \pi(y) \exp[i(\alpha_1 x + \alpha_3 z - \alpha_1 ct)]$$

$$T = T(y) + \theta(y) \exp[i(\alpha_1 x + \alpha_3 z - \alpha_1 ct)]$$

$$\mu_1 = \mu_1(y) + \overline{\frac{d\mu}{dT}}\, \theta(y) \exp[i(\alpha_1 x_1 + \alpha_3 z - \alpha_1 ct)]$$

In Reference 6, these are substituted into the equations of motion and reduced to the two-dimensional form given by the transformation

$$\bar{\alpha}\,\bar{f} = \alpha_1 f + \alpha_3 h$$

$$\bar{\alpha}\,\bar{\theta} = \alpha\,\varphi$$

$$\bar{\alpha}^2 = \alpha_1^2 + \alpha_3^2$$

A. EQUATIONS

By Reference 7 (page 37) $\mu_2 = -\frac{2}{3}\mu_1$. This substitution has been made in all the equations. Then the first momentum equation (x-direction) becomes

$$f\left[\frac{\alpha}{T} i(w - c) + \frac{8}{9}\frac{\mu}{R}\alpha^2\right] + f'\left(\rho v - \frac{T'}{R}\frac{d\mu}{dT}\right)$$
$$+ \varphi\left(w'\frac{\alpha}{T} - i\frac{T'}{R}\alpha^2\frac{d\mu}{dT}\right) + \frac{\pi}{M_1^2}\left(\frac{vw'M^2}{T} + \frac{i\alpha}{\gamma}\right)$$
$$+ \theta\left(-\frac{vw'}{T^2} + i\frac{10}{9}\frac{v'\alpha}{R}\frac{d\mu}{dT} - \frac{w''}{R}\frac{d\mu}{dT} - \frac{w'}{R}T'\frac{d^2\mu}{dT^2}\right)$$
$$+ \theta'\left(-\frac{w'}{R}\frac{d\mu}{dT}\right) = f''\frac{\mu}{R} + \varphi'\left(-\frac{\mu\alpha^2}{9R}i\right) \qquad (2)$$

The second momentum equation (y-direction) is

$$
\begin{aligned}
& f\left(i\,\frac{10}{9}\,\frac{T'\alpha}{R}\,\frac{d\mu}{dT}\right) + f'\left(-\frac{i\alpha\mu}{R} + i\,\frac{10}{9}\,\frac{\alpha\mu}{R}\right) \\
& + \varphi\left[\frac{i\alpha^2}{T}(w - c) + \frac{\alpha v'}{T} + \frac{\mu\alpha^3}{R}\right] + \frac{\pi}{M^2}\left(\frac{v\;v'M^2}{T}\right) \\
& + \theta\left(-\frac{v\;v'}{T^2} - \frac{i\alpha w'}{R}\,\frac{d\mu}{dT} - \frac{8}{9}\,\frac{v''}{R}\,\frac{d\mu}{dT}\right) + \theta'\left(-\frac{8}{9}\,\frac{v'}{R}\,\frac{d\mu}{dT}\right) \\
& = \varphi'\left(-\frac{\alpha v}{T} + \frac{8}{9}\,\frac{T'\alpha}{R}\,\frac{d\mu}{dT}\right) + \varphi''\left(\frac{8}{9}\,\frac{\alpha\mu}{R}\right) - \frac{\pi'}{\gamma M^2}
\end{aligned}
\tag{3}
$$

The energy equation is

$$
\begin{aligned}
& f\left[i\alpha(\gamma - 1) + i\,\frac{20}{9}\,\frac{\mu}{R}\,\gamma(\gamma - 1)\,M^2\alpha v'\right] + f'\left[-\frac{2w'\,\mu(\gamma - 1)\,M^2}{R}\right] \\
& + \varphi\left[\alpha(\ell nT)' - \frac{2iw'\alpha^2M\,\gamma(\gamma - 1)M^2}{R}\right] + \frac{\pi}{M^2}\left[vM^2(\ell nT)' + (\gamma - 1)\,M\overset{2}{v}'\right] \\
& + \theta\left[\frac{i\alpha}{T}(w - c) - \frac{v}{T}(\ell nT)' - \frac{\gamma(\gamma - 1)M^2}{R}\left(\frac{8}{9}v'^2 + w'^2\right)\frac{d\mu}{dT}\right. \\
& \left. + \frac{\gamma\mu}{\sigma R}\alpha^2 - \frac{\gamma}{\sigma R}T''\frac{d\mu}{dT} - \frac{\gamma}{\sigma R}T'^2\frac{d^2\mu}{dT^2}\right] + \theta'\left(\rho v - \frac{2T'\gamma}{\sigma R}\frac{d\mu}{dT}\right) \\
& = \varphi'\left[-\alpha(\gamma - 1) + \frac{16}{9}\,\frac{\alpha v'\mu\gamma(\gamma - 1)}{}\;M^2\right] + \theta''\frac{\gamma\mu}{\sigma R}
\end{aligned}
\tag{4}
$$

The continuity equation is

$$
\begin{aligned}
& fi - \varphi(\ell nT)' + \frac{\pi}{M^2}\left[\frac{M^2}{\alpha}\left(-v(\ell nT)' + i\alpha(w - c) + v'\right)\right] \\
& + \theta\left[\frac{1}{\alpha T}\left(2v(\ell nT)' - i\alpha(w - c) - v'\right)\right] - \theta'\frac{v}{\alpha T} = -\varphi' - \pi'\frac{v}{\alpha}
\end{aligned}
\tag{5}
$$

The equation of state has been used to replace r and r′ in the continuity equation

$$
\begin{aligned}
r &= \frac{\pi}{T} - \frac{\theta}{T^2} \\
r' &= \frac{M^2\left(\frac{\pi'}{M^2}T - \frac{\pi}{M^2}T'\right)}{T^2} - \frac{\theta' T^2 - 2\theta T\,T'}{T^4}
\end{aligned}
\tag{6}
$$

When φ'' in the second momentum equation is replaced by φ'' as obtained by differentiating the continuity equation, the term $\frac{\pi'' v}{\alpha}$ occurs. This term is dropped as negligibly small (π changes but very little through the boundary layer). Thus a system of 6 first order differential equations may be obtained.

B. SOLUTION OF THE EQUATIONS

These equations are solved in the same manner as the abbreviated equations of Reference 8, except that the characteristic equation is more complicated so that it has to be solved numerically rather than by formulae.

In order to write these in the standard form, six linear first order equations, the following substitutions are made

$$Z_1 = f$$

$$Z_2 = f' = Z_1'$$

$$Z_3 = \varphi$$

$$Z_4 = \frac{\Pi}{M^2}$$

$$Z_5 = \theta$$

$$Z_6 = \theta' = Z_5'$$

Boundary conditions are

$$Z_1 = Z_3 = Z_5 = 0$$

when

$y = 0$ and Z_1, Z_3, Z_5 bounded as $y \to \infty$

These may be written

$$\sum = a_{ij} Z_j' = \sum b_{ij} Z_j \quad (i = 1, 2, \ldots 6)$$

where the row index is assigned to the six equations as follows

i	Equation	
1	$Z_1' = Z_2$	
2	First momentum	(2)
3	Continuity	(5)
4	Second momentum	(3)
5	$Z_5' = Z_6$	
6	Energy	(4)

The a_{ij} and b_{ij} are found from the preceding equations. They are

$$a_{11} = 1$$

$$a_{22} = \frac{\mu}{R}$$

$$a_{23} = -\frac{i\mu\alpha^2}{9R}$$

$$a_{33} = -1$$

$$a_{34} = -\frac{vM^2}{\alpha}$$

$$a_{41} = -i\,\frac{8}{9}\,\alpha\,\frac{\mu}{R}$$

$$a_{43} = -\frac{\alpha v}{T} + \frac{8}{9}\,\frac{T'\alpha}{R}\,\frac{d\mu}{dT} - \frac{8}{9}\,\frac{\alpha\mu}{R}\,[-(\ell nT)']$$

$$a_{44} = -\frac{8}{9}\,\frac{\mu}{R}\left[M^2\left|-v(\ell nT)' + i\alpha\,(w-c) + v'\right| + v'M^2\right] - \frac{1}{\gamma}$$

$$a_{45} = -\frac{8}{9}\,\frac{\mu}{R}\left[\frac{1}{T}\left\{2v(\ell nT)' - i\alpha\,(w-c) - v'\right\}\right]$$

$$a_{46} = \frac{8}{9}\,\frac{v}{T}\,\frac{\mu}{R}$$

$$a_{55} = 1$$

$$a_{63} = -\alpha(\gamma-1) + \frac{16}{9}\,\frac{\alpha v'\mu}{R}\,\gamma\,(\gamma-1)\,M^2$$

$$a_{66} = \frac{\gamma\mu}{\sigma R}$$

$$b_{12} = 1$$

$$b_{21} = i\,\frac{\alpha}{T}\,(w-c) + \frac{8}{9}\,\frac{\mu}{R}\,\alpha^2$$

$$b_{22} = \frac{v}{T} - \frac{T'}{R}\,\frac{d\mu}{dT}$$

$$b_{23} = \frac{w'\alpha}{T} - \frac{iT'\alpha^2}{R}\,\frac{d\mu}{dT}$$

$$b_{24} = \frac{vw'M^2}{T} + \frac{i\alpha}{\gamma}$$

$$b_{25} = -\frac{vw'}{T^2} + \frac{i10}{9}\,\frac{v'\alpha}{R}\,\frac{d\mu}{dT} - \frac{w''}{R}\,\frac{d\mu}{dT} - \frac{w'}{R}\,\frac{d^2\mu}{dT^2}\,T'$$

$$b_{26} = \frac{-w'}{R}\,\frac{d\mu}{dT}$$

$$b_{31} = i$$

$$b_{33} = -(\ell nT)'$$

$$b_{34} = \frac{M^2}{\alpha}\,[-v(\ell nT)' + i\alpha\,(w-c) + v']$$

$$b_{35} = \frac{1}{\alpha T}\,[2v(\ell nT)' - i\alpha\,(w-c) - v']$$

$$b_{36} = \frac{-v}{\alpha T}$$

$$b_{41} = i\,\frac{10}{9}\,\frac{T'\alpha}{R}\,\frac{d\mu}{dT}$$

$$b_{42} = \frac{i\alpha\mu}{9R}$$

$$b_{43} = \frac{i\alpha^2}{T}(w-c) + \frac{\alpha v'}{T} + \frac{\mu\alpha^3}{R} + \frac{8}{9}\,\alpha\,\frac{\mu}{R}\left[-(\ell nT)''\right]$$

$$b_{44} = \frac{vv'M^2}{T} + \frac{8}{9}\,\frac{\mu}{R}\left[M^2\left|-v'(\ell nT)' - v\,(\ell nT)'' + i\alpha w' + v''\right|\right]$$

$$b_{45} = \frac{-vv'}{T^2} - \frac{i\alpha w'}{R}\,\frac{d\mu}{dT} - \frac{8}{9}\,\frac{v''}{R}\,\frac{d\mu}{dT} + \frac{8}{9}\,\frac{\mu}{R}\left[\frac{-T'}{T^2}\left|2v(\ell nT)' - i\alpha\,(w-c) - v'\right|\right.$$

$$\left. + \frac{1}{T}\left|2v'\,(\ell nT)' + 2v\,(\ell nT)'' - i\alpha w' - v''\right|\right]$$

$$b_{46} = -\,\frac{8}{9}\,\frac{v'}{R}\,\frac{d\mu}{dT} + \frac{8}{9}\,\frac{\mu}{TR}\left[-\,v' + v\,(\ell nT)'\right]$$

$$b_{56} = 1$$

$$b_{61} = i\alpha\,(\gamma-1) + i\,\frac{20}{9}\,\frac{\mu}{R}\,\gamma\,(\gamma-1)\,M^2\,\alpha v'$$

$$b_{62} = \frac{-2w'\mu}{R}\,\gamma\,(\gamma-1)\,M^2$$

$$b_{63} = \alpha(\ell nT)' - i\,\frac{2w'\alpha^2}{R}\,\mu\gamma\,(\gamma-1)\,M^2$$

$$b_{64} = vM^2\,(\ell nT)' + (\gamma-1)\,M^2\,v'$$

$$b_{65} = \frac{i\alpha}{T}(w-c) - \frac{v}{T}(\ell nT)' - \gamma\,\frac{(\gamma-1)\,M^2}{R}\left(\frac{8}{9}\,v'^2 + w'^2\right)\frac{d\mu}{dT}$$

$$+ \frac{\gamma\mu}{\sigma R}\,\alpha^2 - \frac{\gamma}{\sigma R}\,T''\,\frac{d\mu}{dT} + \frac{\gamma}{\sigma R}\,T'^2\,\frac{d^2\mu}{dT^2}$$

$$b_{66} = \frac{v}{T} - \frac{2T'}{\sigma R}\,\gamma\,\frac{d\mu}{dT}$$

In order to obtain the Z_j', the system

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} & 0 & 0 \\ a_{41} & 0 & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & a_{63} & 0 & 0 & a_{66} \end{bmatrix} \begin{bmatrix} Z_1' \\ Z_2' \\ Z_3' \\ Z_4' \\ Z_5' \\ Z_6' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & 0 & b_{33} & b_{34} & b_{35} & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \end{bmatrix}$$

must be transformed into a system

$$Z_i' = \sum_{j=1}^{6} C_{ij}\, Z_j \qquad (i = 1, 2, \dots 6)$$

which may be achieved by simple elimination procedures. We have

$$Z_1' = Z_2$$

$$Z_2' = \frac{1}{a_{22}}\,(b_{2j}\, Z_j - a_{23}\, Z_3')$$

$$Z_3' = \frac{1}{a_{33}}\,(b_{3j}\, Z_j - a_{34}\, Z_4')$$

$$Z_4' = \frac{1}{a_{44}}\,(b_{4j}\, Z_j - a_{41}\, Z_1' - a_{43}\, Z_3' - a_{45}\, Z_5' - a_{46}\, Z_6')$$

$$Z_5' = Z_6$$

$$Z_6' = \frac{1}{a_{66}}\,(b_{6j}\, Z_j - a_{63}\, Z_3')$$

For solution outside the boundary layer, the determinant of the characteristic equation is now

$$\begin{vmatrix} -\lambda & 1 & 0 & 0 & 0 & 0 \\ C_{21} & C_{22}-\lambda & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33}-\lambda & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44}-\lambda & C_{45} & C_{46} \\ 0 & 0 & 0 & 0 & -\lambda & 1 \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}-\lambda \end{vmatrix} = 0$$

Expansion of this determinant yields the characteristic equation:

$$\lambda^6 - \lambda^5\,(C^*_{33} + C^*_{44} + C^*_{66} + C^*_{22})$$

$$+ \lambda^4 \Big[-C^*_{21} + C^*_{22}\,(C^*_{33} + C^*_{44} + C^*_{66}) + C^*_{33}\, C^*_{44} + C^*_{33}\, C^*_{66} + C^*_{44}\, C^*_{66}$$

$$- C^*_{65} - C^*_{36}\, C^*_{63} - C^*_{34}\, C^*_{43} - C^*_{46}\, C^*_{64} - C^*_{23}\, C^*_{32} - C^*_{24}\, C^*_{42} - C^*_{26}\, C^*_{62}\Big]$$

$$+ \lambda^3 \Big[C^*_{21}\,(C^*_{33} + C^*_{44} + C^*_{66}) - C^*_{22}\,(C^*_{33}\, C^*_{44} + C^*_{33}\, C^*_{66} + C^*_{44}\, C^*_{66}$$

$$- C^*_{65} - C^*_{36}\, C^*_{63} - C^*_{34}\, C^*_{43} - C^*_{46}\, C^*_{64}) - C^*_{33}\, C^*_{44}\, C^*_{66} + C^*_{33}\, C^*_{65}$$

$$+ C^*_{44}\, C^*_{65} - C^*_{36}\, C^*_{43}\, C^*_{64} - C^*_{34}\, C^*_{46}\, C^*_{63} + C^*_{36}\, C^*_{44}\, C^*_{63} - C^*_{35}\, C^*_{63}$$

$$+ C^*_{34} C^*_{43} C^*_{66} + C^*_{33} C^*_{46} C^*_{64} - C^*_{45} C^*_{64} - C^*_{23} C^*_{31}$$

$$+ C^*_{32} (- C^*_{26} C^*_{63} - C^*_{24} C^*_{43} + C^*_{23} C^*_{66} + C^*_{23} C^*_{44}) - C^*_{24} C^*_{41}$$

$$+ C^*_{42} (- C^*_{23} C^*_{34} - C^*_{26} C^*_{64} + C^*_{24} C^*_{33} - C^*_{24} C^*_{66}) - C^*_{26} C^*_{61}$$

$$+ C^*_{62} (- C^*_{24} C^*_{46} - C^*_{23} C^*_{36} + C^*_{26} C^*_{33} + C^*_{26} C^*_{44} - C^*_{28}) \Big]$$

$$+ \lambda^2 \Big[C^*_{21} (C^*_{36} C^*_{63} + C^*_{34} C^*_{43} + C^*_{46} C^*_{64} - C^*_{33} C^*_{44} - C^*_{33} C^*_{66}$$

$$- C^*_{44} C^*_{66} + C^*_{65}) + C^*_{22} (C^*_{33} C^*_{44} C^*_{66} - C^*_{33} C^*_{65} - C^*_{44} C^*_{65}$$

$$+ C^*_{36} C^*_{43} C^*_{64} + C^*_{34} C^*_{46} C^*_{63} - C^*_{36} C^*_{44} C^*_{63} + C^*_{35} C^*_{63} - C^*_{34} C^*_{43} C^*_{66}$$

$$- C^*_{33} C^*_{46} C^*_{64} + C^*_{45} C^*_{64}) - (C^*_{33} C^*_{44} C^*_{65} + C^*_{35} C^*_{43} C^*_{64} + C^*_{34} C^*_{45} C^*_{63}$$

$$- C^*_{35} C^*_{44} C^*_{63} - C^*_{34} C^*_{43} C^*_{65} - C^*_{33} C^*_{45} C^*_{64}) + C^*_{31} (- C^*_{26} C^*_{63}$$

$$- C^*_{24} C^*_{43} + C^*_{23} C^*_{66} + C^*_{23} C^*_{44}) + C^*_{32} (C^*_{26} C^*_{44} C^*_{63} - C^*_{25} C^*_{63}$$

$$+ C^*_{24} C^*_{43} C^*_{66} + C^*_{23} C^*_{46} C^*_{64} - C^*_{23} C^*_{66} C^*_{44} + C^*_{23} C^*_{65} - C^*_{26} C^*_{43} C^*_{64}$$

$$- C^*_{24} C^*_{46} C^*_{63}) + C^*_{41} (- C^*_{23} C^*_{34} - C^*_{26} C^*_{64} + C^*_{24} C^*_{33} + C^*_{24} C^*_{66})$$

$$+ C^*_{42} (C^*_{23} C^*_{34} C^*_{66} + C^*_{26} C^*_{33} C^*_{64} - C^*_{25} C^*_{64} + C^*_{24} C^*_{36} C^*_{63}$$

$$- C^*_{26} C^*_{34} C^*_{63} - C^*_{24} C^*_{33} C^*_{66} + C^*_{24} C^*_{65} - C^*_{23} C^*_{36} C^*_{64})$$

$$+ C^*_{61} (- C^*_{24} C^*_{46} - C^*_{23} C^*_{36} + C^*_{26} C^*_{33} + C^*_{26} C^*_{44} - C^*_{25})$$

$$+ C^*_{62} (C^*_{26} C^*_{43} C^*_{34} + C^*_{24} C^*_{33} C^*_{46} - C^*_{24} C^*_{45} + C^*_{23} C^*_{36} C^*_{44}$$

$$- C^*_{23} C^*_{35} - C^*_{23} C^*_{34} C^*_{46} + C^*_{25} C^*_{33} + C^*_{25} C^*_{44} - C^*_{26} C^*_{33} C^*_{44}$$

$$- C^*_{24} C^*_{36} C^*_{43}) \Big]$$

$$+ \lambda \Big[C^*_{21} (C^*_{33} C^*_{44} C^*_{66} - C^*_{33} C^*_{65} - C^*_{44} C^*_{65} + C^*_{36} C^*_{43} C^*_{64} + C^*_{34} C^*_{46} C^*_{63}$$

$$- C^*_{36} C^*_{44} C^*_{63} + C^*_{35} C^*_{63} - C^*_{34} C^*_{43} C^*_{66} - C^*_{33} C^*_{46} C^*_{64} + C^*_{45} C^*_{64})$$

$$+ C^*_{22} (C^*_{33} C^*_{44} C^*_{65} + C^*_{35} C^*_{43} C^*_{64} + C^*_{34} C^*_{45} C^*_{63} - C^*_{35} C^*_{44} C^*_{63}$$

$$- C^*_{34} C^*_{43} C^*_{65} - C^*_{33} C^*_{45} C^*_{64}) + C^*_{31} (C^*_{26} C^*_{44} C^*_{63} - C^*_{25} C^*_{63}$$

$$+ C^*_{24} C^*_{43} C^*_{66} + C^*_{23} C^*_{46} C^*_{64} - C^*_{23} C^*_{66} C^*_{44} + C^*_{23} C^*_{65} - C^*_{26} C^*_{43} C^*_{64}$$

$$- C^*_{24} C^*_{46} C^*_{63}) + C^*_{32} (C^*_{25} C^*_{44} C^*_{63} + C^*_{24} C^*_{43} C^*_{65} + C^*_{23} C^*_{45} C^*_{64}$$

$$- C^*_{23} C^*_{44} C^*_{65} - C^*_{25} C^*_{43} C^*_{64} - C^*_{24} C^*_{45} C^*_{63}) + C^*_{41} (C^*_{23} C^*_{34} C^*_{66}$$

$$+ C^*_{26} C^*_{33} C^*_{64} - C^*_{25} C^*_{64} + C^*_{24} C^*_{36} C^*_{63} - C^*_{26} C^*_{34} C^*_{63} - C^*_{24} C^*_{33} C^*_{66}$$

$$+ C^*_{24} C^*_{65} - C^*_{23} C^*_{36} C^*_{64}) + C^*_{42} (C^*_{23} C^*_{34} C^*_{65} + C^*_{25} C^*_{33} C^*_{64}$$

$$+ C^*_{24} C^*_{35} C^*_{63} - C^*_{25} C^*_{34} C^*_{63} - C^*_{24} C^*_{33} C^*_{65} - C^*_{23} C^*_{35} C^*_{64})$$

$$+ C^*_{61} (C^*_{26} C^*_{43} C^*_{34} + C^*_{24} C^*_{33} C^*_{46} - C^*_{24} C^*_{45} + C^*_{23} C^*_{36} C^*_{44}$$

$$- C^*_{23} C^*_{35} - C^*_{23} C^*_{34} C^*_{46} + C^*_{25} C^*_{33} + C^*_{25} C^*_{44} - C^*_{26} C^*_{33} C^*_{44}$$

$$- C^*_{24} C^*_{36} C^*_{43}) + C^*_{62} (C^*_{25} C^*_{34} C^*_{43} + C^*_{24} C^*_{33} C^*_{45} + C^*_{23} C^*_{35} C^*_{44}$$

$$- C^*_{23} C^*_{34} C^*_{45} - C^*_{25} C^*_{33} C^*_{44} - C^*_{24} C^*_{35} C^*_{43}) + C^*_{21} (C^*_{33} C^*_{44} C^*_{65}$$

$$+ C^*_{35} C^*_{43} C^*_{64} + C^*_{34} C^*_{45} C^*_{63} - C^*_{35} C^*_{44} C^*_{63} - C^*_{34} C^*_{43} C^*_{65}$$

$$- C^*_{33} C^*_{45} C^*_{64}) + C^*_{31} (C^*_{25} C^*_{44} C^*_{63} + C^*_{24} C^*_{43} C^*_{65} + C^*_{23} C^*_{45} C^*_{64}$$

$$- C^*_{23} C^*_{44} C^*_{65} - C^*_{25} C^*_{43} C^*_{64} - C^*_{24} C^*_{45} C^*_{63})$$

$$+ C^*_{41} (C^*_{23} C^*_{34} C^*_{65} + C^*_{25} C^*_{33} C^*_{64} + C^*_{24} C^*_{35} C^*_{63} - C^*_{25} C^*_{34} C^*_{63}$$

$$- C^*_{24} C^*_{33} C^*_{65} - C^*_{23} C^*_{35} C^*_{64}) + C^*_{61} (C^*_{25} C^*_{34} C^*_{43} + C^*_{23} C^*_{35} C^*_{44}$$

$$+ C^*_{24} C^*_{33} C^*_{45} - C^*_{23} C^*_{34} C^*_{45} - C^*_{24} C^*_{35} C^*_{43} - C^*_{25} C^*_{33} C^*_{44})\Big|$$

The six complex roots λ_s (s = 1, 2, . . . 6) are found by a numerical method.

Then outside the boundary layer where the coefficients are constant, the solution of the system is (Reference 14)

$$Z_i = \sum_{s=1}^{6} \bar{k}_{is} (K_s e^{\lambda sy})$$

where the $\bar{k}_{is}$ are the cofactors of the elements of the fourth row of the characteristic determinant outside the boundary layer $y = \delta$ to $y = \infty$. These values are

$$\bar{k}_{1s} = \left[C_{65} + \lambda_s (C_{66} - \lambda_s)\right] \left[C_{24} \lambda_s - C_{24} C_{33} + C_{23} C_{34}\right] + (C_{35} + C_{36} \lambda_s)$$
$$(C_{24} C_{63} - C_{23} C_{64}) + (C_{25} + C_{26} \lambda_s) (C_{64} C_{33} - C_{63} C_{34} - C_{64} \lambda_s)$$

$$\bar{k}_{2s} = \lambda_s \bar{k}_{1s}$$

$$\bar{k}_{3s} = \begin{vmatrix} \lambda_s C_{32} + C_{31} & C_{34} & C_{35} + \lambda_s C_{36} \\ \lambda_s C_{42} + C_{41} & C_{44} - \lambda_s & C_{45} + \lambda_s C_{46} \\ \lambda_s C_{62} + C_{61} & C_{64} & C_{65} + \lambda C_{66} - \lambda_s^2 \end{vmatrix}$$

$$\bar{k}_{4s} = \begin{vmatrix} C_{33} - \lambda_s & \lambda_s C_{32} + C_{31} & C_{35} + \lambda_s C_{36} \\ C_{44} & \lambda_s C_{42} + C_{41} & C_{45} + \lambda_s C_{46} \\ C_{63} & \lambda_s C_{62} + C_{61} & C_{65} + \lambda_s C_{66} - \lambda_s^2 \end{vmatrix}$$

$$\bar{k}_{5s} = \begin{vmatrix} C_{33} - \lambda_s & C_{34} & \lambda_s C_{32} + C_{31} \\ C_{43} & C_{44} - \lambda_s & \lambda_s C_{42} + C_{41} \\ C_{63} & C_{64} & \lambda_s C_{62} + C_{61} \end{vmatrix}$$

$$\bar{k}_{6s} = \lambda_s \bar{k}_{5s}$$

The rest of the solution is exactly like that of Reference 1.

III. CALCULATIONS

In the solution of equations 2 to 6, the velocity profiles parallel to the plate were taken from Reference 9 except for Mach 8, which was computed by the method of Reference 10. The perpendicular profiles were computed by the equation of Reference 9.

$$\rho^* v^* = \frac{1}{2}\frac{1}{R}\left(\rho^* u^* \eta - \int_0^{\eta} \rho^* u^* \, d\eta\right) \tag{7}$$

In our non-dimensional notation, this becomes

$$vR = \frac{1}{2}\left(w\eta - T\int_0^{\eta} \frac{w}{T}\, d\eta\right) \tag{8}$$

The viscosity variation was computed by the use of Sutherland's equation as given in Reference 9.

$$10^5 \mu^* = \frac{1.458 \, T^{*\frac{3}{2}}}{T^* + 110.4} \tag{9}$$

The non-dimensional form is therefore

$$\mu = T^{1/2} \frac{1 + \frac{110.4}{T_\infty}}{1 + \frac{1}{T}\frac{110.4}{T_\infty}} \tag{10}$$

The constant in the formula, $\frac{110.4}{T_\infty}$, depends on the freestream temperature and must be altered when this changes. Here T_∞ must be in degrees Kelvin. The Prandtl number was assumed constant through the layer. If the pressure change through the boundary layer is neglected, then

$$\rho T = 1 \tag{11}$$

Stability calculations were made for M = 2.2 and $\overline{M}$ = 2.2, 1.474, 1.232, 1.6852, corresponding to angles with the main flow of 0°, 48°, 50° and 40° respectively.

For M = 5, $\overline{M}$ was 5, 4.33, 3.5, 2.5, 3.0, 1.294, corresponding to angles of 0°, 30°, 45.5°, 60°, 53° and 75° respectively.

For M = 8, $\overline{M}$ was 8, 5.6, 4, corresponding to angles of 0°, 45.5° and 60°.

IV. RESULTS AND DISCUSSION

A comparison at Mach 5 of neutral stability curves computed by the Dunn-Lin equations, the Lees-Lin equations and the Dunn-Cheng equations is shown in Figure 1. The critical Reynolds number based on momentum thickness turned out to be 185 for Dunn-Lin, 325 for Lees-Lin. When the terms involving the velocity component perpendicular to the flat plate were included, the critical R_θ jumped to 1250. The experimental value of Reference 2 was about 550. When these last equations were used at a wave angle of 60°, R_θ critical became 625. None of the computed curves agreed very well with the data points on the upper branch, though the 60° curve was much closer than the others.

The directional effect is shown in Figure 2, where the critical Reynolds number is plotted against the wave angle.

When similar calculations were made for Mach 2.2, the results are shown in Figure 3. The Lees-Lin equations gave lower branch values of R about 6% below the data points of Reference 12 and upper branch values about 18% below the data points. When the Dunn and Cheng equations were used with a wave angle of 50°, the computed lower branch came almost exactly on the data points, while the upper branch fell about 10% below the data points.

A few results at Mach 8 are shown in Figure 4. The abscissa here is $R = \sqrt{\dfrac{u_0^* \times \bar{\rho}_0^*}{\bar{\mu}_{10}^*}}$. The Lees-Lin curve shows values of R around 600, the Dunn and Cheng around 5080, and the Dunn and Cheng with a wave angle of 45.5° about 1900. Also shown are some experimental transition measurements from Reference 13. These latter are somewhat (14%) above the directional computations.

V. CONCLUSIONS

The inclusion in the stability equations of the boundary layer velocity component and the allowance for a three-dimensional disturbance velocity improve the agreement between the observed and calculated neutral stability curves.

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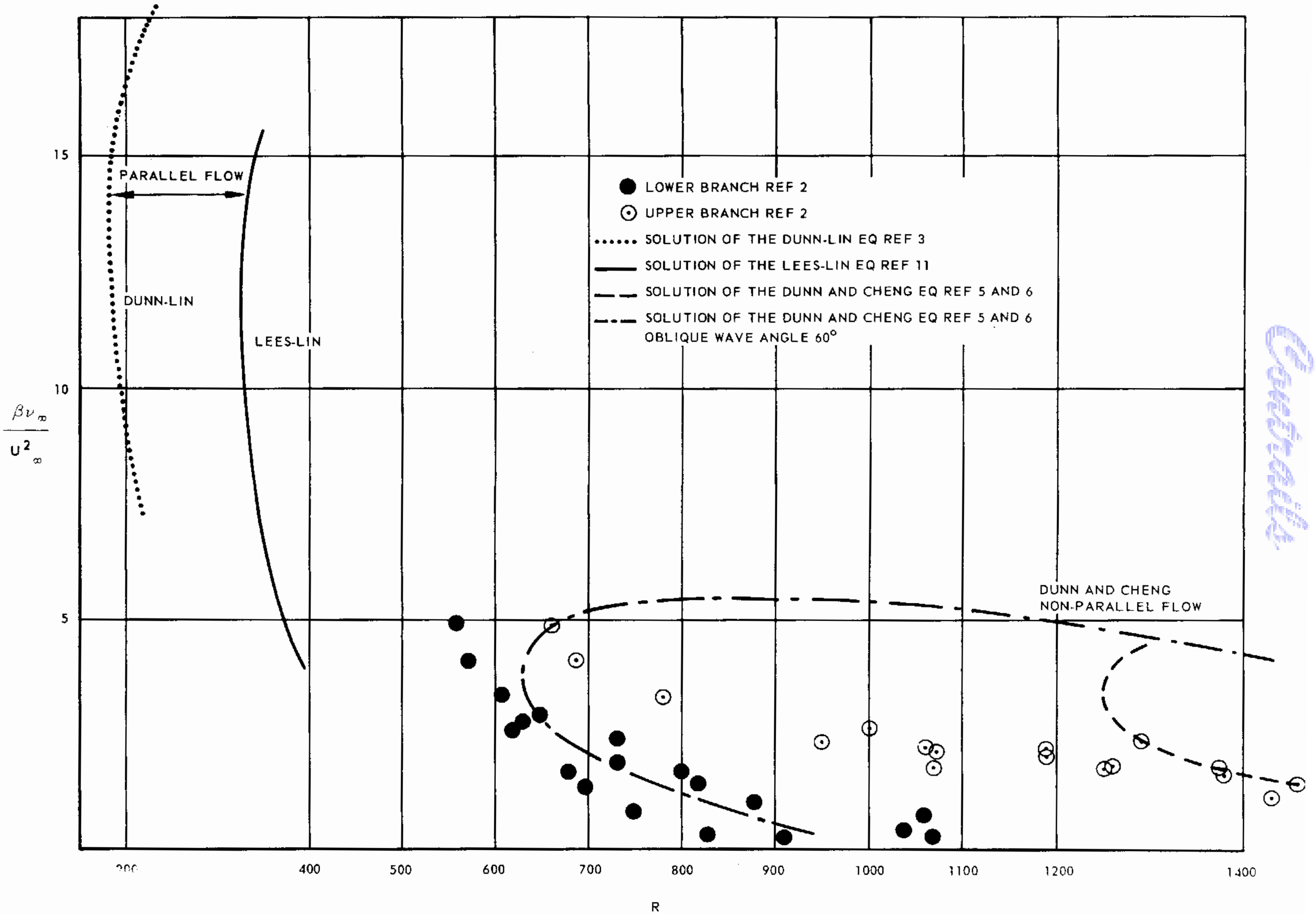

FIGURE 1 OBSERVED AND COMPUTED NEUTRAL STABILITY CURVES.

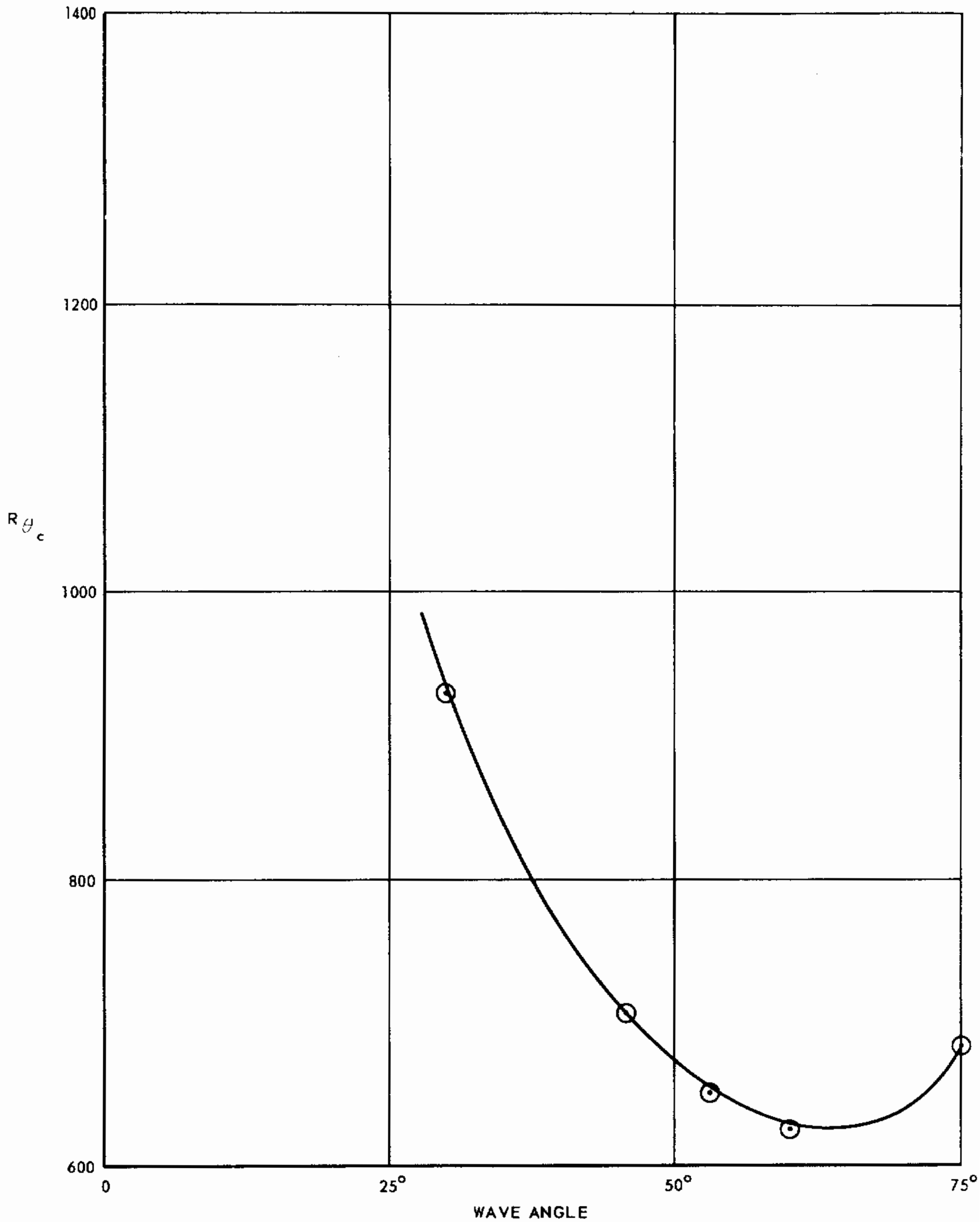

FIGURE 2 CRITICAL REYNOLDS NO. VS. DIRECTION.
M = 5

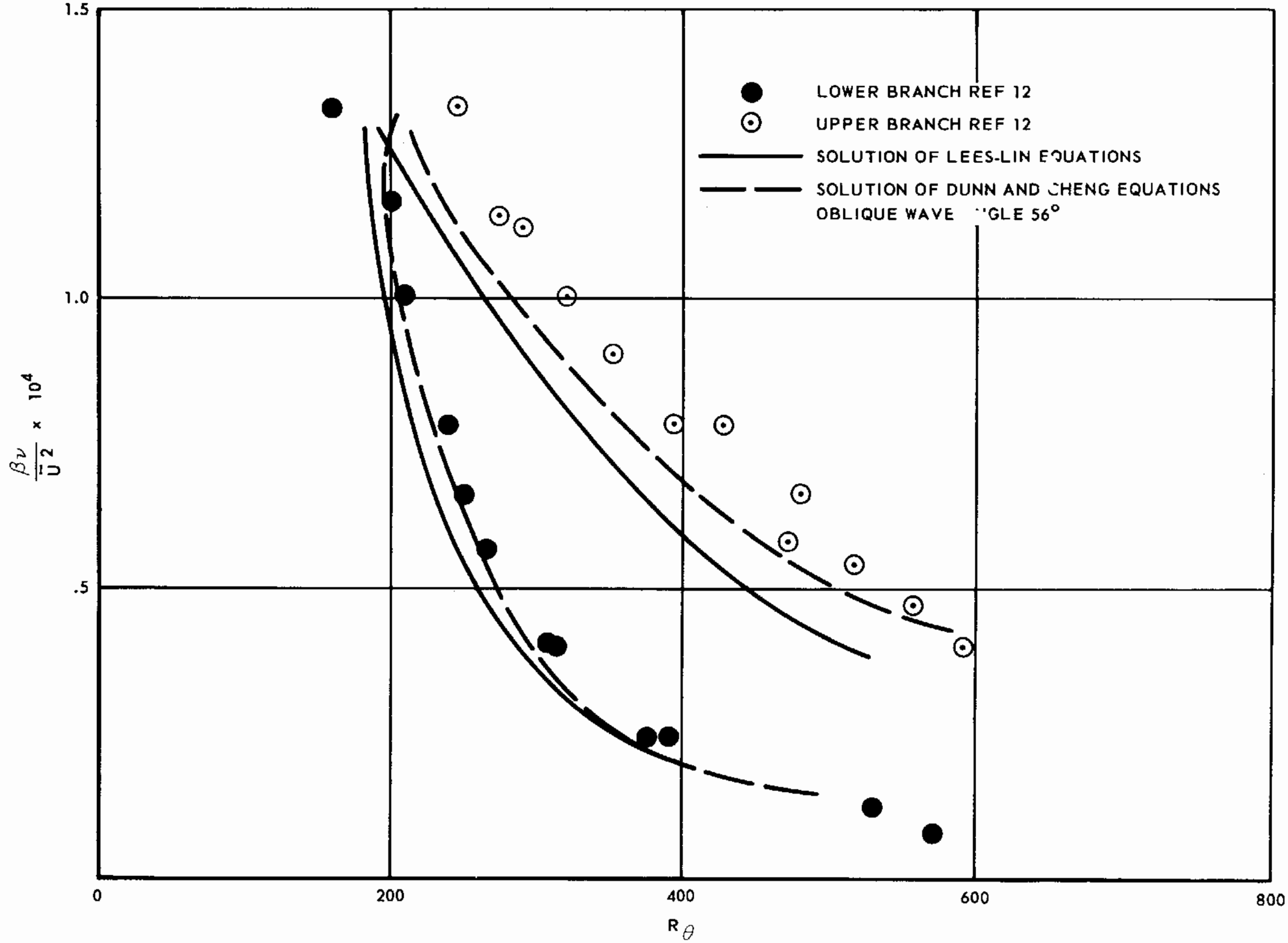

FIGURE 3 OBSERVED AND COMPUTED NEUTRAL STABILITY CURVES
MACH NO. 2.2

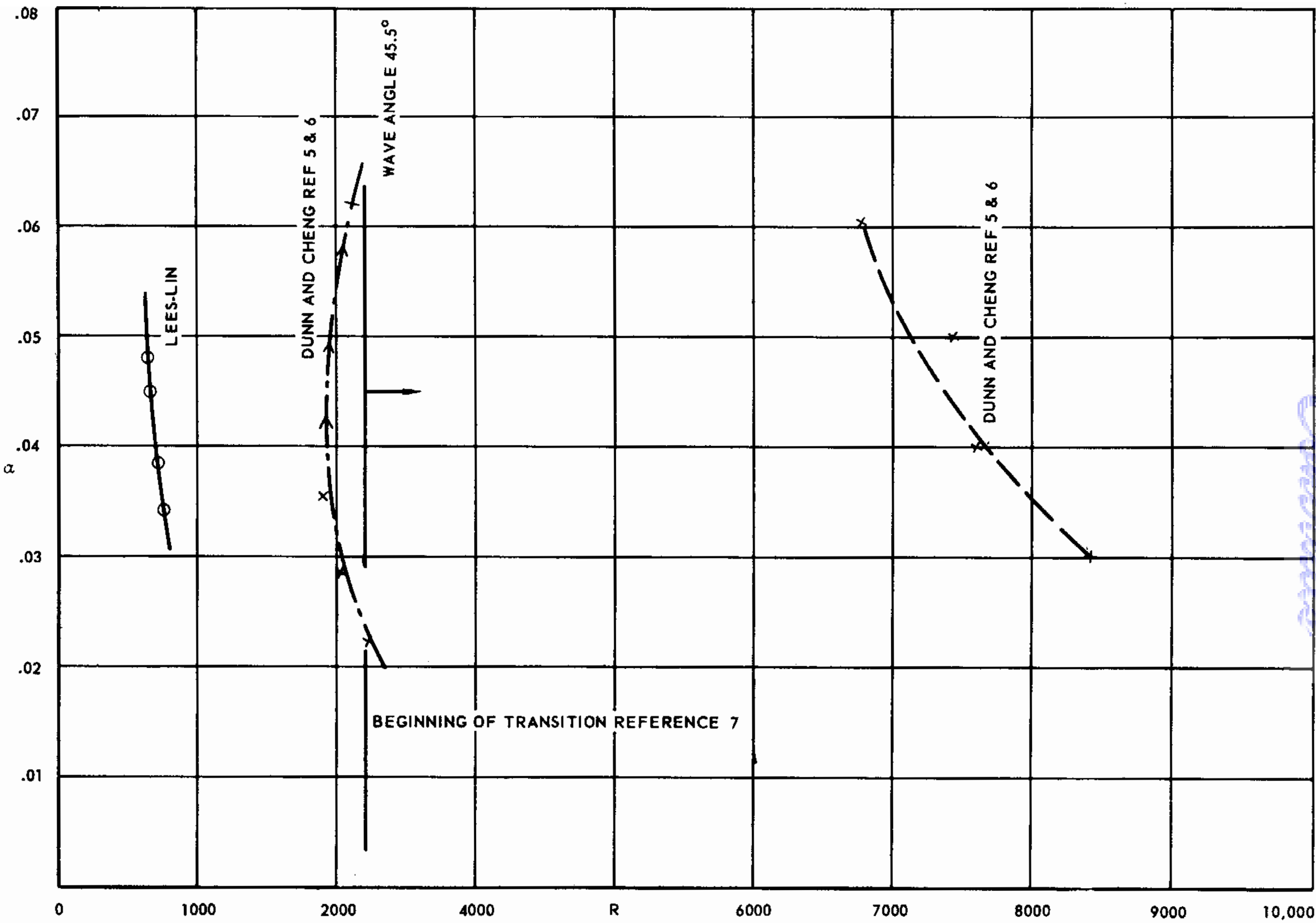

FIGURE 4 COMPUTED NEUTRAL STABILITY CURVES
MACH NO. 8

PART 3

NUMERICAL SOLUTION OF THE COMPLETE THREE DIMENSIONAL STABILITY EQUATIONS OF THE COMPRESSIBLE BOUNDARY LAYER ON A FLAT PLATE

W. Byron Brown

NOTATION

Dimensionless Quantities		Characteristic Measure
ρ	gas density	$\bar{\rho}^*_o$
T	gas temperature	$\bar{T}^*_o$
μ_1	first viscosity coefficient	$\bar{\mu}^*_{10}$
μ_2	second viscosity coefficient	$\bar{\mu}^*_{20}$
c	phase velocity of the disturbance	$\bar{U}^*_o$
γ	specific heat ratio	c_p/c_v
R	Reynolds number	$\dfrac{\bar{\rho}^*_o U^*_o \ell}{\bar{\mu}^*_{10}}$
M	Mach number (R* gas constant per gram)	$\dfrac{U^*_o}{\sqrt{\gamma \bar{T}^*_o R^*}}$
α_1	disturbance wave number	ℓ^{-1}
α_3	disturbance wave number	ℓ^{-1}
σ	Prandtl number	$\dfrac{c_p \bar{\mu}^*_{10}}{\bar{k}^*_1}$
ψ	angle between main velocity and the disturbance velocity	
k	thermal conductivity	$c_p \bar{\mu}^*_{10}$
ℓ	length unit	$\dfrac{x^*}{R}$

Dimensionless Quantities		Characteristic Measure
x^*	distance from the stagnation point	
x	non-dimensional distance	ℓ
y	distance from the flat plate	ℓ
w	undisturbed velocity parallel to the plate in component boundary layer	$\bar{U}^*_o$
v	undisturbed velocity component in boundary layer perpendicular to the wall	$\bar{U}^*_o$
f	velocity disturbance amplitude in x direction	
φ	velocity disturbance amplitude in y direction	
h	velocity disturbance amplitude in z direction	
π	amplitude of the pressure disturbance	
r	amplitude of the density disturbance	
θ	amplitude of the temperature disturbance	

A bar over a quantity denotes average value. A prime denotes differentiation with respect to y. Subscript o denotes free-stream value; subscripts r and i denote real and imaginary parts.

I. INTRODUCTION

It has been shown (Reference 1) that the approximate method (Reference 2) of acoustics for the three-dimensional aspect of the disturbance velocities is of only limited value and does not agree well with experimental data on the upper branches of the neutral stability curves, especially at Mach 5.

To remedy this defect and to obtain a calculation method that is more reliable at high Mach numbers, a direct solution of the linearized stability equations that contain all three momentum equations for the disturbance velocity has been obtained. The new calculations and the new results for Mach 2.2 and Mach 5 are given in this report.

II. ANALYSIS

The disturbance forms assumed are those of Dunn, Reference 2, namely

$$u_1 = w\,(y) + f\,(y)\,\exp\,[i\,(\alpha_1 x + \alpha_3 z - \alpha_1 ct)]$$

$$u_2 = v\,(y) + \alpha_1 \varphi\,(y)\,\exp\,[i\,(\alpha_1 x + \alpha_3 z - \alpha_1 ct)]$$

$$u_3 \qquad h\,(y)\,\exp\,[i\,(\alpha_1 x + \alpha_3 z - \alpha_1 ct)]$$

$$\rho = \rho\,(y) + r\,(y)\,\exp\,[i\,(\alpha_1 x + \alpha_3 z - \alpha_1 ct)]$$

$$p = p\,(y) + \pi\,(y)\,\exp\,[i\,(\alpha_1 x + \alpha_3 z - \alpha_1 ct)]$$

$$T = T\,(y) + \theta\,(y)\,\exp\,[i\,(\alpha_1 x + \alpha_3 z - \alpha_1 ct)]$$

$$\mu_1 = \mu_1\,(y) + \overline{\frac{d\mu}{dT}}\,\theta\,(y)\,\exp\,[i\,(\alpha_1 x_1 + \alpha_3 z - \alpha_1 ct)]$$

A. EQUATIONS

These are the same as in Reference 1, except that a third momentum equation is included; also another disturbance wave number, as in Reference 3. Thus the equations to be solved are now:

First momentum (x direction):

$$\begin{aligned} & f\left[\frac{\alpha_1}{T}\,i\,(w - c) + \frac{8}{9}\frac{\mu}{R}\left(\alpha_1^2 + \alpha_3^2\right)\right] + f'\left(\frac{v}{T} - \frac{T'}{R}\frac{d\mu}{dT}\right) \\ & + \varphi\left(\frac{w'\alpha_1}{T} - \frac{iT'}{R}\alpha_1^2\frac{d\mu}{dT}\right) + \frac{\pi}{M^2}\left(\frac{vw'M^2}{T} + \frac{i\alpha_1}{\gamma}\right) \\ & + \theta\left(-\frac{vw'}{T^2} + i\,\frac{10}{9}\,\frac{v'\alpha_1}{R}\frac{d\mu}{dT} - \frac{w''}{R}\frac{d\mu}{dT} - \frac{w'}{R}T'\frac{d^2\mu}{dT^2}\right) + \theta'\left(-\frac{w'}{R}\frac{d\mu}{dT}\right) \\ & = f''\frac{\mu}{R} + \varphi'\left(-\frac{\mu\alpha_1^2}{9R}\,i\right) \end{aligned} \qquad (1)$$

Second momentum (y direction):

$$f\left(i\,\frac{10}{9}\,\frac{T'\alpha_1}{R}\frac{d\mu}{dT}\right) + f'\left(-\frac{i\alpha_1\mu}{R} + i\,\frac{10}{9}\,\frac{\alpha_1\mu}{R}\right) \qquad (2)$$

$$+ \varphi \left(\frac{i\alpha_1^2}{T} (w - c) + \frac{\alpha_1 v'}{T} + \frac{\mu\alpha_1^3}{R} \right) + \frac{\pi}{M^2} \left(\frac{vv'M^2}{T} \right)$$

$$+ \theta \left(- \frac{vv'}{T^2} - \frac{i\alpha_1 w'}{R} \frac{d\mu}{dT} - \frac{8}{9} \frac{v''}{R} \frac{d\mu}{dT} \right) + \theta' \left(- \frac{8}{9} \frac{v'}{R} \frac{d\mu}{dT} \right) \qquad (2) \text{ (cont.)}$$

$$= \varphi' \left(- \frac{\alpha_1 v}{T} + \frac{8}{9} \frac{T'\alpha_1}{R} \frac{d\mu}{dT} \right) + \varphi'' \left(\frac{8}{9} \frac{\alpha_1 \mu}{R} \right) - \frac{\pi'}{\gamma M^2}$$

Third momentum (z direction):

$$\frac{i\alpha_1}{T} (w - c) h = \frac{-i\alpha_3 \pi}{\gamma M^2} + \frac{\mu}{R} h'' + \frac{h'}{R} \left(\frac{d\mu}{dT} T' - \frac{Rv}{T} \right) - \frac{\mu h}{R} \left(\alpha_1^2 + \alpha_3^2 \right) \qquad (3)$$

Energy equation is:

$$f \left[i\alpha_1 (\gamma - 1) + i \frac{20}{9} \frac{\mu\gamma}{R} (\gamma - 1) M^2 \alpha_1 v' \right] + f' \left[\frac{-2w'\mu\gamma (\gamma - 1) M^2}{R} \right]$$

$$+ \varphi \left[\alpha_1 (\ell nT)' - \frac{2iw'\alpha_1^2 \mu\gamma (\gamma - 1) M^2}{R} \right] + \frac{\pi}{M^2} \left[vM^2 (\ell nT)' + (\gamma - 1) M^2 v' \right]$$

$$+ \theta \left[\frac{i\alpha_1}{T} (w - c) - \frac{v}{T} (\ell nT)' - \frac{\gamma (\gamma - 1) M^2}{R} \left(\frac{8}{9} v'^2 + w'^2 \right) \frac{d\mu}{dT} \right.$$

$$\left. + \frac{\gamma\mu}{\sigma R} \left(\alpha_1^2 + \alpha_3^2 \right) - \frac{\gamma}{\sigma R} T'' \frac{d\mu}{dT} - \frac{\gamma}{\sigma R} T'^2 \frac{d^2\mu}{dT^2} \right] \qquad (4)$$

$$+ \theta' \left(\frac{v}{T} - \frac{2T'\gamma}{\sigma R} \frac{d\mu}{dT} \right) + h \, i \, \alpha_3 (\gamma - 1)$$

$$= \varphi' \left[- \alpha_1 (\gamma - 1) + \frac{16}{9} \frac{\alpha_1 v' \mu\gamma (\gamma - 1)}{R} M^2 \right] + \theta'' \frac{\gamma\mu}{\sigma R}$$

Continuity equation is:

$$fi - \varphi (\ell nT)' + \frac{\pi}{M^2} \left[\frac{M^2}{\alpha_1} \left\{ - v (\ell nT)' + i\alpha_1 (w - c) + v' \right\} \right] \qquad (5)$$

$$+ \frac{\theta}{\alpha_1 T} \left[2v (\ell nT)' - i\alpha_1 (w - c) - v' \right] - \theta' \frac{v}{\alpha_1 T} + h \frac{i\alpha_3}{\alpha_1} = - \varphi' - \frac{\pi' v}{\alpha_1}$$

The equation of state has been used to replace r and r′ in the continuity equation:

$$r = \frac{\pi}{T} - \frac{\theta}{T^2}$$
$$r' = \frac{M^2\left(\frac{\pi'}{M^2}T - \frac{\pi}{M^2}T'\right)}{T^2} - \frac{\theta' T^2 - 2\theta T T'}{T^4} \qquad (6)$$

When φ'' in the second momentum equation is replaced by φ'' as obtained by differentiating the continuity equation, the term $\pi'' v/\alpha$ occurs. Since both π'' and v are very small, this term is negligible and is dropped. Thus a system of 8 first order differential equations may be obtained.

B. SOLUTION OF EQUATIONS

This is done by substituting

$$Z_1 = f$$
$$Z_2 = f' = Z_1'$$
$$Z_3 = \varphi$$
$$Z_4 = \frac{\pi}{M^2}$$
$$Z_5 = \theta \qquad (7)$$
$$Z_6 = \theta' = Z_5'$$
$$Z_7 = h$$
$$Z_8 = h' = Z_7'$$

This system is set up as follows:

$[a]\,[Z'] = [b]\,[Z]$

Row Index	Equation
1	$Z_1' = Z_2$
2	First momentum
3	Continuity
4	Second momentum (modified by φ'' replacement)
5	$Z_5' = Z_6$
6	Energy
7	$Z_7' = Z_8$
8	Third momentum

The a_{ij} and b_{ij} matrices are then as follows:

$$a_{11} = 1$$

$$a_{22} = \frac{\mu}{R}$$

$$a_{23} = \frac{-i\mu\alpha_1^2}{9R}$$

$$a_{33} = -1$$

$$a_{34} = -\frac{vM^2}{\alpha_1}$$

$$a_{41} = -i\,\frac{8}{9}\,\frac{\alpha_1\mu}{R}$$

$$a_{43} = \frac{8}{9}\,\frac{\alpha_1\mu}{R}\,(\ell nT)' - \frac{\alpha_1 v}{T} + \frac{8}{9}\,\frac{T'\alpha_1}{R}\,\frac{d\mu}{dT}$$

$$a_{44} = -\frac{1}{\gamma} - \frac{8}{9}\,\frac{\mu M^2}{R}\left\{-v\,(\ell nT)' + i\alpha_1\,(w - c) + 2v'\right\}$$

$$a_{45} = -\frac{8}{9}\,\frac{\mu}{RT}\left\{2v\,(\ell nT)' - i\alpha_1\,(w - c) - v'\right\}$$

$$a_{46} = \frac{8}{9}\,\frac{v\mu}{TR}$$

$$a_{47} = -\frac{8}{9}\,\frac{i\alpha_3\mu}{R}$$

$$a_{55} = 1$$

$$a_{63} = -\alpha_1\,(\gamma - 1) + \frac{16}{9}\,\frac{\alpha_1 v'\mu\gamma\,(\gamma - 1)\,M^2}{R}$$

$$a_{66} = \frac{\gamma\mu}{\sigma R}$$

$$a_{77} = 1$$

$$a_{88} = \frac{\mu}{R}$$

$$b_{12} = 1$$

$$b_{21} = \frac{i\alpha_1}{T}\,(w - c) + \left(\alpha_1^2 + \alpha_3^2\right)\frac{\mu}{R}$$

$$b_{22} = \frac{v}{T} - \frac{T'}{R}\frac{d\mu}{dT}$$

$$b_{23} = \frac{w'\alpha_1}{T} - \frac{iT'\alpha_1^2}{R}\frac{d\mu}{dT}$$

$$b_{24} = \frac{i\alpha_1}{\gamma} + \frac{vw'M^2}{T}$$

$$b_{25} = -\frac{w'T'}{R}\frac{d^2\mu}{dT^2} - \frac{w''}{R}\frac{d\mu}{dT} - \frac{vw'}{T^2} + i\,\frac{10}{9}\frac{v'\alpha_1}{R}\frac{d\mu}{dT}$$

$$b_{26} = -\frac{w'}{R}\frac{d\mu}{dT}$$

$$b_{31} = i$$

$$b_{33} = -(\ell nT)'$$

$$b_{34} = \frac{M^2}{\alpha_1}\left\{-v(\ell nT)' + i\alpha_1(w - c) + v'\right\}$$

$$b_{35} = \frac{1}{\alpha_1 T}\left\{2v(\ell nT)' - i\alpha_1(w - c) - v'\right\}$$

$$b_{36} = -\frac{v}{T\alpha_1}$$

$$b_{37} = \frac{i\alpha_3}{\alpha_1}$$

$$b_{41} = i\,\frac{10}{9}\frac{T'\alpha_1}{R}\frac{d\mu}{dT}$$

$$b_{42} = \frac{i\alpha_1\mu}{9R}$$

$$b_{43} = \frac{i\alpha_1^2}{T}(w - c) - \frac{8}{9}\frac{\alpha_1\mu}{R}(\ell nT)'' + \frac{\alpha_1 v'}{T} + \frac{\mu\alpha_1^3}{R}$$

$$b_{44} = \frac{8}{9}\frac{\mu M^2}{R}\left\{-v'(\ell nT)'' + i\alpha_1 w' + v''\right\} + \frac{vv'M^2}{T}$$

$$b_{45} = \frac{8}{9}\frac{\mu}{R}\left[-\frac{T'}{T^2}\left\{2v(\ell nT)' - \alpha_1 i(w - c) - v'\right\} - \frac{vv'}{T^2}\right.$$

$$\left. + \frac{1}{T}\left\{2v'(\ell nT)' + 2v(\ell nT)'' - i\alpha_1 w' - v''\right\}\right]$$

$$- \frac{i\alpha_1 w'}{R}\frac{d\mu}{dT} - \frac{8}{9}\frac{v''}{R}\frac{d\mu}{dT}$$

$$b_{46} = \frac{8}{9}\,\frac{\mu}{TR}\left(\frac{vT'}{T} - v'\right) - \frac{8}{9}\,\frac{v'}{R}\,\frac{d\mu}{dT}$$

$$b_{56} = 1$$

$$b_{61} = i\alpha_1\,(\gamma - 1) + i\,\frac{20}{9}\,\frac{\mu}{R}\,\gamma\,(\gamma - 1)\,M^2\,\alpha_1 v'$$

$$b_{62} = -\,\frac{2w'\mu\gamma\,(\gamma - 1)\,M^2}{R}$$

$$b_{63} = \alpha_1\,(\ell nT)' - \frac{2iw'\alpha_1^2\mu\gamma\,(\gamma - 1)\,M^2}{R}$$

$$b_{64} = vM^2\,(\ell nT)' + (\gamma - 1)\,M^2\,v'$$

$$b_{65} = \frac{i\alpha_1(w-c)}{T} - \frac{\gamma T''}{\sigma R}\,\frac{d\mu}{dT} - \frac{\gamma T'^2}{\sigma R}\,\frac{d^2\mu}{dT^2} + \frac{\gamma\mu}{\sigma R}\left(\alpha_1^2 + \alpha_3^2\right) - \frac{v}{T}\,(\ell nT)' - \frac{\gamma\,(\gamma - 1)\,M^2}{R}\left(w'^2 + \frac{8}{9}\,v'^2\right)\frac{d\mu}{dT}$$

$$b_{66} = \frac{v}{T} - \frac{2T'\gamma}{\sigma R}\,\frac{d\mu}{dT}$$

$$b_{67} = i\,(\gamma - 1)\,\alpha_3$$

$$b_{78} = 1$$

$$b_{84} = \frac{i\alpha_3}{\gamma}$$

$$b_{87} = \frac{i\alpha_1\,(w - c)}{T} + \left(\alpha_1^2 + \alpha_3^2\right)\frac{\mu}{R}$$

$$b_{88} = -\,\frac{T'}{R}\,\frac{d\mu}{dT} + \frac{v}{T}$$

The system

$[a]\,[Z'] = [b]\,[Z]$

is reduced to a system

$[I]\,[Z'] = [C]\,[Z]$

by multiplying the right hand side by the reciprocal $[a]^{-1}$. Thus $[C] = [a]^{-1}\,[b]$.

Once the [C] matrix is computed, the characteristic equation is found and solved numerically. The rest of the solution is essentially the same as that used in Reference 4. The solution is carried out in two parts. If the boundary layer depth is δ (point where the boundary layer velocity is 99.9% of the free stream value), then within this distance the C_{ij} coefficients are variables. Hence the integration from $y = 0$ to $y = \delta$ is carried out numerically. When $y > \delta$, the C_{ij}'s are constant. Hence the solution is the sum of eight exponential terms, one for each root of the characteristic equation. Since Z_1, Z_3, Z_5 and Z_7 are bounded as $y \rightarrow \infty$, the coefficients of four of these terms, those in which the real part of the root is positive, must vanish. At the point $y = \delta$, the numerical solutions must of course match the exponential solutions. Thus four conditions must be satisfied here for the four coefficients to vanish. These suffice to determine the eigen values required. Specifically in this case, the numerical solutions are

$$Z_i = \sum_{j=1}^{8} C_j \, Z_i^{(j)} \qquad (i = 1, 2, \ldots 8) \tag{8}$$

The $Z_i^{(j)}$ are fundamental solutions defined by their initial conditions

$$Z_i^{(j)} (0) = \delta_{ij} \tag{9}$$

$$\delta_{ij} = 0 \text{ if } i \neq j \qquad \delta_{ij} = 1 \text{ if } i = j$$

The initial conditions ($y = 0$) are

$$\begin{aligned} Z_1 &= 0 \\ Z_2 &= C_2 = 1 \\ Z_3 &= 0 \\ Z_4 &= C_4 \\ Z_5 &= 0 \\ Z_6 &= C_6 \\ Z_7 &= 0 \\ Z_8 &= C_8 \end{aligned}$$

Thus

$$\begin{aligned} Z_1 (0) &= C_1 Z_1^{(1)} (0) = C_1 = 0 \\ Z_3 (0) &= C_3 Z_3^{(3)} (0) = C_3 = 0 \\ Z_5 (0) &= C_5 Z_5^{(5)} (0) = C_5 = 0 \\ Z_7 (0) &= C_7 Z_7^{(7)} (0) = C_7 = 0 \end{aligned} \tag{10}$$

Hence the general solution is

$$Z_i = Z_i^{(2)} + C_4 Z_i^{(4)} + C_6 Z_i^{(6)} + C_8 Z_i^{(8)} \tag{11}$$

and the problem is to determine C_4, C_6, C_8 and any two of the real parameters α , α, R, c_r, ci from the conditions at $y = \delta$, $K_5 = K_6 = K_7 = K_8 = 0$. The general exponential solution can be written

$$Z_i = \sum_{s=1}^{8} \bar{k}_{is} (K_s e^{\lambda s y}) \qquad (i = 1, 2, \dots 8) \tag{12}$$

where the $\bar{k}_{is}$ are the cofactors of the elements of the fourth row of the characteristic determinant

$$\det (C^*_{ij} - \lambda s \, \delta_{ij}) \qquad C^*_{ij} \text{ (values of } C_{ij} \text{ when } y > \delta)$$

Each of these is computed numerically by the machine.

Since Equations (12) applied at $y = \delta$ form a system of simultaneous linear equations for evaluating the 8 K_s's, upon solving and setting $K_s = 0$ $s = 5, 6, 7, 8$ four homogeneous linear functionals in the Z_i's result which must be satisfied when $y = \delta$.

Thus the boundary conditions can be written

$$\sum_{j=1}^{8} K_{ij} Z_j (\delta) = 0 \qquad i = 5, 6, 7, 8 \tag{13}$$

where the matrix K_{ij} is the inverse of the matrix $\bar{k}_{ij}$.

Thus, at $y = \delta$

$$\sum_{j=1}^{8} K_{ij} Z_j^{(2)} + C_4 \sum_{j=1}^{8} K_{ij} Z_j^{(4)} + C_6 \sum_{j=1}^{8} K_{ij} Z_j^{(6)} + C_8 \sum_{j=1}^{8} K_{ij} Z_j^{(8)} = 0 \tag{14}$$

Let K^*_i denote the linear functional

$$\sum_{j=1}^{8} K_{ij} Z_j (\delta)$$

and

$$K_i^{*(p)} = \sum_{j=1}^{8} K_{ij} Z_j^{(p)} (\delta)$$

Then Equation (14) can be written

$$K_i^* = K_i^{*(2)} + C_4 K_i^{*(4)} + C_6 K_i^{*(6)} + C_8 K_i^{*(8)} = 0$$

$$i = 5, 6, 7, 8$$

The first three of these can be used to compute C_4, C_6 and C_8. Substitution in the fourth yields a complex number for K_8^*. To make this number vanish, the parameters α_1, α_3, R, c_r and c_i must be adjusted. The result is a set of eigen-values. For a neutral curve, $c_i = 0$, α_3/α_1 is given a value (usually to make R a minimum) and R and c_r are computed for a series of values of α_1. The plot of α_1 against R is the usual neutral curve.

III. CALCULATIONS

The calculations were carried out in the same manner as in Reference 1 except that, of course, equations 1 to 6 were used. These contain another parameter α_3 so the complete parameter list consists of α_1 (wave number), R (Reynolds number), c_r (wave velocity), ci (amplification or damping factor), α_3 (wave number). After testing a few cases where α_1 was fixed and α_3 varied, as in Figure 1, it was decided to adopt a fixed value of the ratio $\frac{\alpha_3}{\alpha_1} = 1.428$. This corresponds to a flow angle of 55°; i.e., tan 55° = 1.428. This seems to be near the angle between the three-dimensional flow and the main flow where the critical Reynolds number is a minimum. (Reference 2 gives about 51° in a similar case for a lower Mach number.)

At this ratio, $\frac{\alpha_3}{\alpha_1} = 1.428$, neutral stability curves were computed for Mach numbers 2.2 and 5.

IV. RESULTS AND DISCUSSION

The results for Mach number 2.2 are shown in Figure 2, where the data of Reference 5 are plotted also. Agreement between theory and observed data is quite good on both upper and lower branches of the neutral curve.

The Mach 5 results are shown in Figure 3, where the data of Reference 6 are plotted. Here also, agreement is good on both branches of the neutral curve.

V. CONCLUSIONS

The addition of the third momentum equation to the usual set of stability equations for supersonic laminar boundary layers gives good agreement with observed data for both the upper and lower branches of the neutral stability curve.

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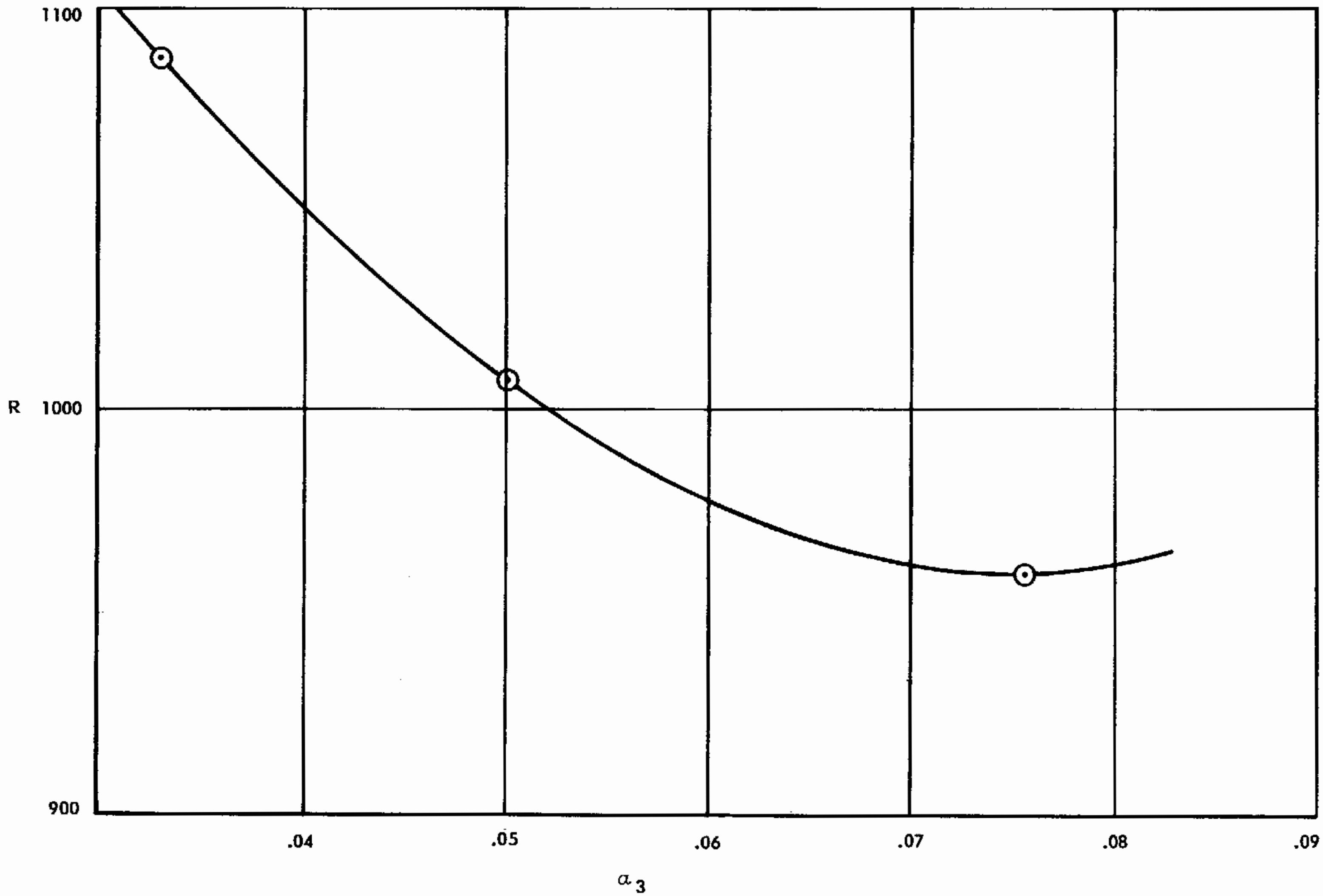

FIGURE 1 EFFECT ON THE REYNOLDS NO. OF VARYING THE WAVE NO. RATIO

$\alpha = .06 \quad M = 5$

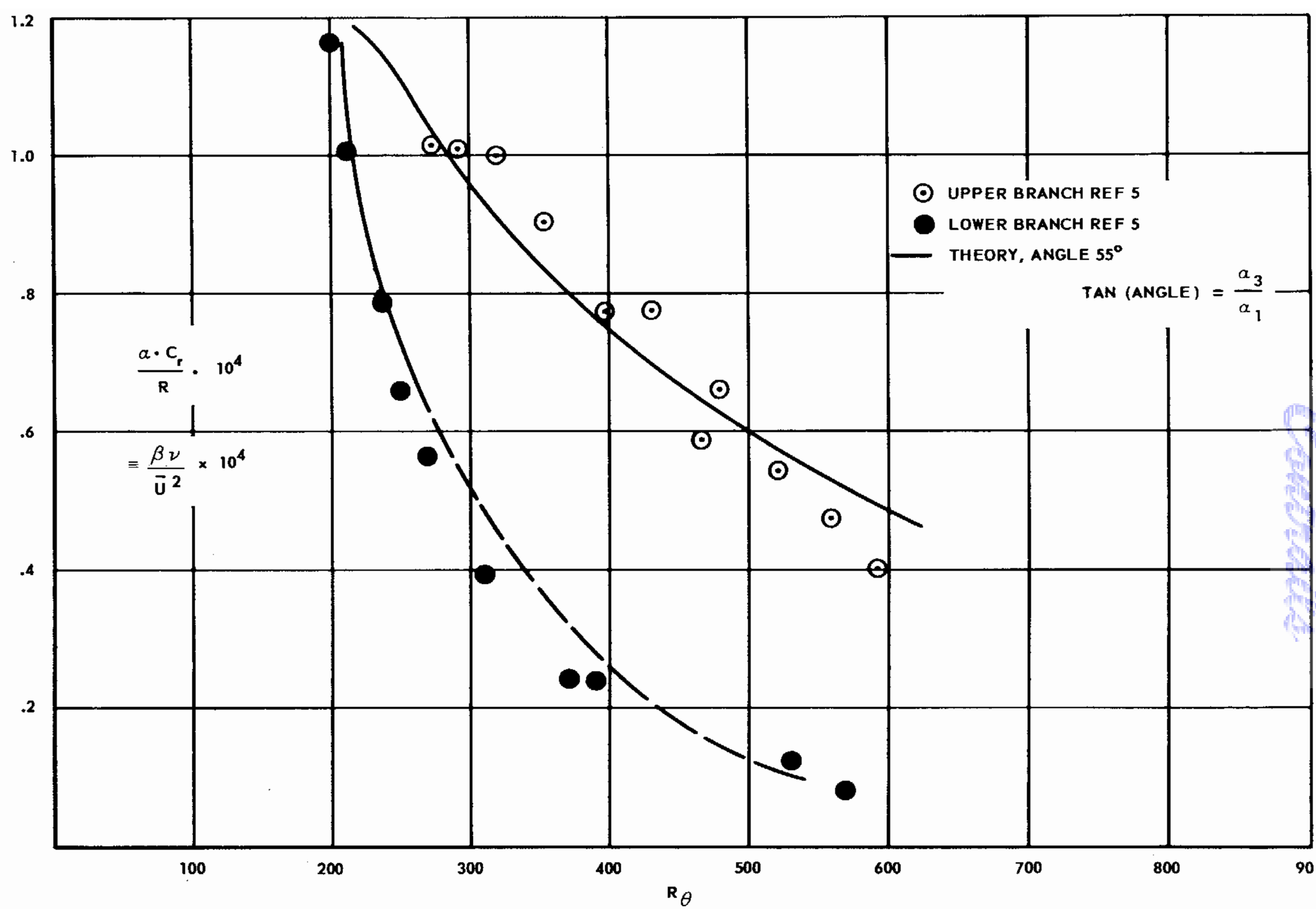

FIGURE 2 THEORETICAL AND EXPERIMENTAL NEUTRAL STABILITY CURVES
M = 2.2

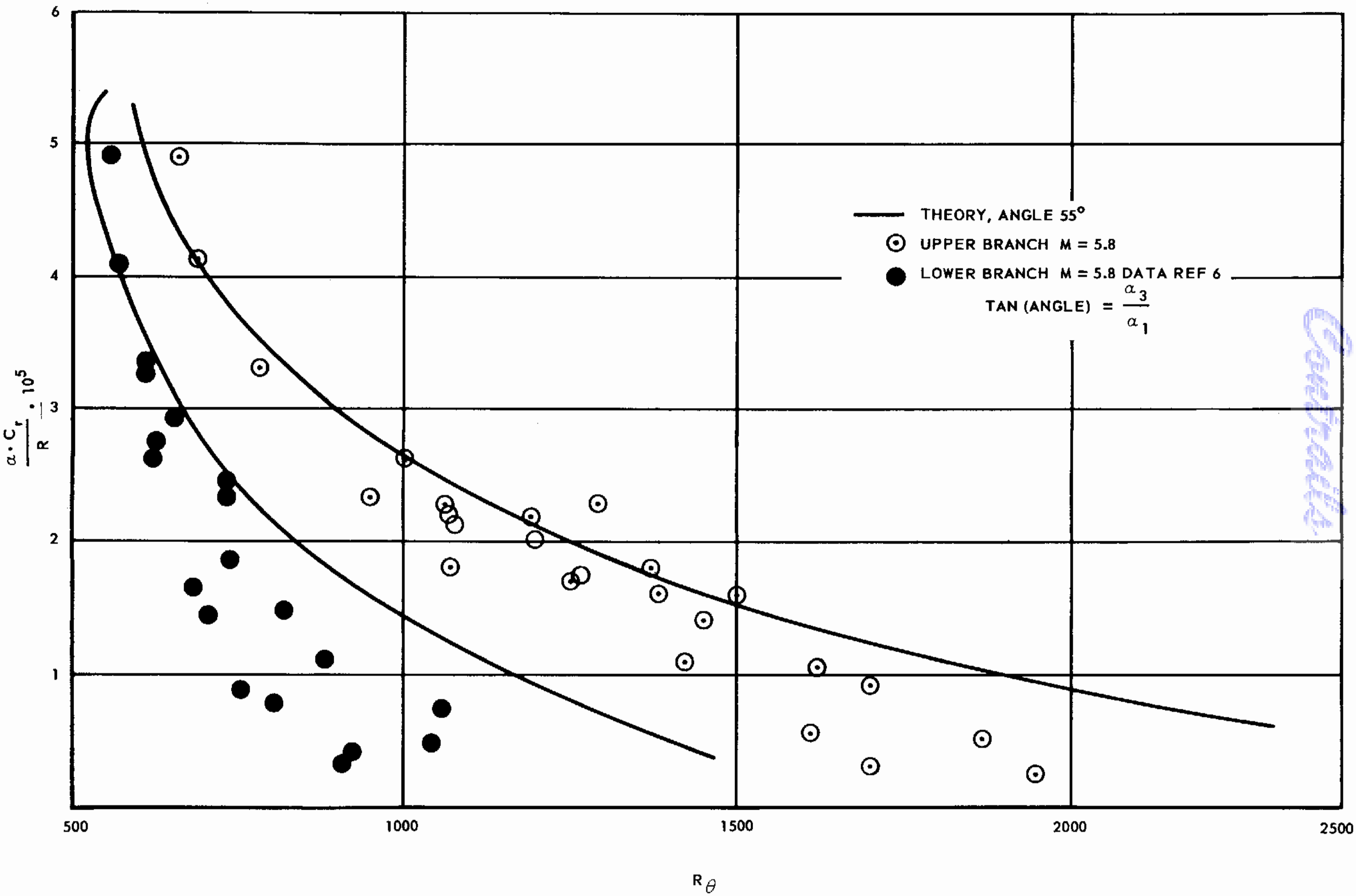

FIGURE 3 THEORETICAL AND EXPERIMENTAL NEUTRAL STABILITY CURVES
M = 5

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13. ABSTRACT

The mathematical analysis underlying a Fortran program for calculating the proper solutions of the Orr-Sommerfeld system with sufficient accuracy and economy for applying the resonance theory of transition is described. This program covers spacewise growths, rather than timewise growths as in previous computations, of mainly two-dimensional Fourier components of the motion. It employs various innovations providing as much accuracy from efficient single-precision arithmetic as would be obtained from awkward multiple-precision arithmetic in previous calculation schemes. The source programs and some sample calculations, for the principal mode of oscillation of the Blasius boundary layer, are included.

The Lees-Lin stability equations for compressible flow have been extended to include the terms involving the component of the mean boundary layer flow perpendicular to the flat plate. At Mach 5 this more than doubled the critical Reynolds number. Allowance was then made for the three-dimensional aspect of the disturbance velocity. The final result was to give good agreement with observed data in the lower branch of the neutral stability curve at Mach 2.2 and Mach 5, fair agreement with the upper branch at Mach 2.2 and large discrepancies with the data in the upper branch at Mach 5.

Comparison of experimentally determined neutral stability curves with those computed by simplified approximations have disagreed considerably at high Mach numbers on the upper branch, even when agreement was fairly good on the lower branch. To improve the calculations, the complete set of three-dimensional

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ABSTRACT

stability equations, including all three momentum equations and also the component of the mean flow in the boundary layer normal to the surface, are solved numerically. This set of equations can be reduced to a set of eight linear equations with complex coefficients. The theoretical solutions for Mach 2.2 and Mach 5 are compared with experimental data and show good agreement in both upper and lower branches.