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FLIGHT SIMULATION OF ORBITAL AND REENTRY VEHICLES

PART I — DEVELOPMENT OF EQUATIONS OF MOTION IN SIX DEGREES OF FREEDOM

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FOREWORD

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This report is Part I of a series of several parts under the general title, "Flight Simulation of Orbital and Reentry Vehicles." Research covered in it began in May 1959 and was completed in October, 1960.

ABSTRACT

Equations governing the motion of a lifting reentry vehicle in six degrees of freedom are developed for simulation purposes. Effects of earth rotation, earth oblateness and wind velocity are included. The rotational equations are conventional equations involving angular rates referred to body axes. Two different formulations of the translational equations are presented, using different coordinate systems. One involves spherical coordinates referred to the equatorial plane of the rotating earth; the other involves spherical coordinates referred to a nominal trajectory plane fixed with respect to inertial axes. The former has the advantage that earth rotation and oblateness are taken into account more simply, but an indeterminacy is encountered in the event of flight over a pole. The latter formulation eliminates this indeterminacy but is otherwise more complex.

It is concluded that the equations developed are suitable for digital or hybrid analog-digital computation, but that accuracy requirements would be excessive for analog computation exclusively. An alternative formulation, suitable for analog computation, is presented in Part II of the present series.

PUBLICATION REVIEW

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CONTENTS

LIST OF SYMBOLS

$\overline{a}$	Acceleration vector of vehicle relative to inertial axes
f	Parameter describing earth oblateness, defined in Eq. (31)
$\overline{F}$	Aerodynamic force vector
F	With appropriate subscripts, components of $\overline{F}$ referred to selected axes
$\overline{G}$	Gravity force vector
h	Altitude
$\overline{H}$	Control force vector
H	With appropriate subscripts, components of $\overline{H}$ referred to selected axes
i, j, k	Unit vectors along the X, Y, Z axes, respectively; with appropriate subscripts, unit vectors along axes in other sets
I_{xx}, I_{yy}, I_{zz}	Moments of inertia of vehicle about the X_B, Y_B, Z_B axes, respectively
I_{xz}	Product of inertia of vehicle about X_B and Z_B axes
J	With appropriate subscripts, components of control moment
K	Gravity constant defined in Eq. (22)
L, M, N	Components of the aerodynamic moment, referred to the X_B, Y_B, Z_B axes, respectively
m	Vehicle mass
$\overline{P}$	Propulsive force vector
P	With appropriate subscripts, components of $\overline{P}$ referred to selected axes
P, Q, R	Components of angular velocity of vehicle relative to inertial axes, referred to the X_B, Y_B, Z_B axes respectively

$\bar{r}$	Radius vector from earth center to vehicle centroid
r	Length of $\bar{r}$
δr	Difference between r and R_o
R_o	Earth radius in the equatorial plane
R_e	Earth radius at a local point on the earth's surface
R_1	Arithmetic mean of the earth's radius at pole and equator
t	Time
T	With appropriate subscripts, components of propulsive moment
$\bar{v}$	Velocity vector of vehicle relative to inertial axes
$\bar{v}_a$	Velocity vector of vehicle relative to the air
v_a	With appropriate subscripts, components of $\bar{v}_a$ referred to selected axes
$\bar{v}_E$	Velocity vector of vehicle relative to the X_E, Y_E, Z_E frame
$\bar{v}_w$	Wind velocity vector relative to the X_E, Y_E, Z_E frame
v_w	With appropriate subscripts, components of $\bar{v}_w$ referred to selected axes
V_a	Magnitude of $\bar{v}_a$
X, Y, Z	Inertial axes. With appropriate subscripts, other axis systems
α	Angle of attack (see Fig. 1c)
β	Angle of sideslip (see Fig. 1c)
θ	Euler angle establishing vehicle orientation (see Fib. 1b)
λ	Angle establishing initial direction of nominal trajectory (see Sect. 3.1)
λ_F	Angle establishing local direction of nominal trajectory (see Sect. 3.4)
μ	Gravity constant (see Eq. (22))

Φ	Geocentric latitude of vehicle
Φ_i	Geocentric latitude of launch point
Φ_F	Spherical angular coordinate of vehicle measured normal to nominal trajectory plane
ϕ	Euler angle establishing vehicle orientation (see Fig. 1b)
Ψ	Geocentric longitude of vehicle measured from launch point
Ψ_F	Spherical angular coordinate of vehicle measured in nominal trajectory plane
ψ	Euler angle establishing vehicle orientation (see Fig. 1b)
Ω_E	Earth rotational velocity
$\bar{\omega}_B$	Angular velocity vector of vehicle relative to inertial frame
$\bar{\omega}_E$	Angular velocity vector of X_E, Y_E, Z_E frame relative to inertial frame
$\bar{\omega}_G$	Angular velocity vector of X_G, Y_G, Z_G frame relative to inertial frame
$\bar{\omega}_V$	Angular velocity vector of X_V, Y_V, Z_V frame relative to inertial frame
$\bar{\omega}_F$	Angular velocity vector of X_F, Y_F, Z_F frame relative to inertial frame
$\bar{\omega}_{FE}$	Angular velocity vector of X_F, Y_F, Z_F frame relative to X_E, Y_E, Z_E frame

Subscripts

B	Body axes
E	Earth rotating axes
F	Earth-vehicle geocentric axes referred to nominal trajectory plane
G	Earth-vehicle geocentric axes referred to equatorial plane

V	Vehicle geocentric axes
W	Vehicle wind axes
x, y, z	Components along body axes, X_B, Y_B, Z_B, respectively
r, Φ, Ψ	Components along X_G, Y_G, Z_G axes respectively
r, Φ_F, Ψ_F	Components along X_F, Y_F, Z_F axes respectively

1. INTRODUCTION

The advent of the manned space vehicle is introducing new problems and new opportunities for development in the field of flight simulation. The need for crew training for such vehicles by means of flight simulators is much greater yet than in the case of aircraft, since opportunities for training in actual flight are essentially non-existent.

While the simulation problem is similar in some respects to that of aircraft, it is substantially more complex and can be expected to make greater demands on computing equipment. For this reason the present research program was undertaken, with the objective of providing information of value in the development of flight simulators for space vehicles. It has been concerned specifically with the development of an appropriate mathematical model for use in such simulation and with problems associated with its mechanization by means of computing equipment. Attention has been limited to orbital and reentry vehicles.

In the present report, which is a part of a comprehensive report to be issued on this research, equations governing the motion of a lifting reentry vehicle are developed in six degrees of freedom, including effects of earth rotation and oblateness. Two different formulations are presented, one in which the vehicle is located by means of spherical coordinates referred to the earth's equatorial plane and axes which rotate with the earth, the other involving spherical coordinates referred to a nominal trajectory plane which is fixed with respect to inertial axes with origin at the earth's center.

The former is advantageous in that the coordinates provide directly the geocentric latitude and longitude and permit the introduction of earth rotation and oblateness effects in a rather straightforward manner. A serious disadvantage in some cases is that the angle, Ψ, representing the geocentric latitude, becomes indeterminate when the vehicle passes over a pole, resulting in computational difficulties.

The latter formulation avoids this difficulty in that the vehicle will not depart very far from the nominal trajectory plane. It is, however, otherwise considerably more complex mathematically.

A somewhat different formulation, in which the translational equations are referred to axes aligned with the horizontal component of the vehicle's velocity vector, is presented in Part II of the present report. It is shown to have specific advantages in an analog computer mechanization.

2. FORMULATION INVOLVING COORDINATES REFERRED TO THE EQUATORIAL PLANE

2.1 Reference Axes and Coordinates

Various sets of reference axes are now defined. Some of these are for the purpose of establishing coordinates, others to facilitate coordinate transformations. Each is a set of right-handed orthogonal axes. They are illustrated in Fig. 1.

Inertial axes, X, Y, Z (unit vectors, i, j, k):

Origin at earth center. Z-axis coincident with earth polar axis, positive north. XZ-plane contains initial position of vehicle.

Earth rotating axes, X_E, Y_E, Z_E (unit vectors, i_E, j_E, k_E):

Origin at earth center. Z_E-axis coincident with earth polar-axis, positive north. Initially coincident with inertial axes.

Earth-vehicle geocentric axes, X_G, Y_G, Z_G (unit vectors, i_G, j_G, k_G):

Origin at earth center. X_G-axis passes through vehicle centroid.

X_GZ_G-plane contains earth polar axis. Z_G-axis positive north of the equatorial plane.

Vehicle-geocentric axes, X_V, Y_V, Z_V (unit vectors, i_V, j_V, k_V):

Origin at vehicle centroid. Z_V-axis passes through earth center. X_vZ_V-plane contains earth polar axis. X_V-axis positive north, Y_V-axis positive east.

Vehicle body axes, X_B, Y_B, Z_B (unit vectors i_B, j_B, k_B):

Origin at vehicle centroid. X_B Z_B - plane coincident with plane of symmetry of vehicle. X_B-axis positive forward, Y_B-axis positive to the right, Z_B-axis positive down.

Vehicle wind axes, X_W, Y_W, Z_W (unit vectors k_W, j_W, k_W):

Origin at vehicle centroid. X_W-axis points in direction of vehicle velocity relative to air. Z_W-axis lies in plane of symmetry of vehicle. These axes are reached from the X_B, Y_B, Z_B axes by rotating first about the Y_B-axis through the angle $-\alpha$, and then about the Z_W axis through the angle β.

The angles Ψ and Φ which locate the X_G, Y_G, Z_G axes relative to the X_E, Y_E, Z_E axes, as shown in Fig. 1, and the distance r between the earth center and vehicle centroid, constitute a set of spherical coordinates locating the vehicle centroid relative to the earth.

A set of Euler angles θ, ϕ, and ψ, specifies the orientation of the vehicle body axes, X_B, Y_B, Z_B, relative to the vehicle geocentric axes. These are not the conventional Euler angles as used in aircraft analysis. Here body axes are reached from vehicle geocentric axes by placing the vehicle initially with the body axes coincident with vehicle geocentric axes and then rotating successively about the Z_B, X_B and Y_B axes through the angles ψ, ϕ and θ respectively. The Euler angles are defined in this manner in order to avoid the indeterminacy, or so-called "gimbal lock," which occurs with conventional Euler angles when the X_B-axis is vertical. Such an avoidance is necessary if the launch phase of the flight is to be simulated. A similar indeterminacy occurs with the presently defined Euler angles when the Y_B-axis is vertical, that is, when the vehicle is banked at 90°. This should not present difficulties here, since such an orientation would be highly abnormal for vehicles of the type under consideration, and it is not likely that it would need to be simulated.

2.2 Angular Transformation of Reference Axes

The angular relationship between the various axis systems may be expressed as transformation of the sets of unit vectors. These transformations, written in matrix form, are as follows (Ref. 1),

$$\begin{Bmatrix} i_E \\ j_E \\ k_E \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\Omega_E t & \sin\Omega_E t & 0 \\ -\sin\Omega_E t & \cos\Omega_E t & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[\,①\,]} \begin{Bmatrix} i \\ j \\ k \end{Bmatrix} \quad (1)$$

$$\begin{Bmatrix} i_G \\ j_G \\ k_G \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\Phi & 0 & \sin\Phi \\ 0 & 1 & 0 \\ -\sin\Phi & 0 & \cos\Phi \end{bmatrix}}_{[\textcircled{3}]} \underbrace{\begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[\textcircled{2}]} \begin{Bmatrix} i_E \\ j_E \\ k_E \end{Bmatrix} \tag{2}$$

$$\begin{Bmatrix} i_V \\ j_V \\ k_V \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{[\textcircled{4}]} \begin{Bmatrix} i_G \\ j_G \\ k_G \end{Bmatrix} \tag{3}$$

$$\begin{Bmatrix} i_B \\ j_B \\ k_B \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}}_{[\textcircled{7}]} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}}_{[\textcircled{6}]} \underbrace{\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[\textcircled{5}]} \begin{Bmatrix} i_V \\ j_V \\ k_V \end{Bmatrix} \tag{4}$$

$$\begin{Bmatrix} i_W \\ j_W \\ k_W \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[\textcircled{9}]} \underbrace{\begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}}_{[\textcircled{8}]} \begin{Bmatrix} i_B \\ j_B \\ k_B \end{Bmatrix} \tag{5}$$

2.3 Differential Equations for the Euler Angles

A set of differential equations governing the Euler angles, θ, ϕ, ψ, is now formulated.

The angular velocity, $\bar{\omega}_B$, of the vehicle relative to the inertial axes may be written as follows,

$$\bar{\omega}_B = Pi_B + Qj_B + Rk_B \tag{6}$$

where P, Q, and R are the components about the X_B, Y_B and Z_B axes respectively.

Equation (6) may be written in the matrix form,

$$\left\{\bar{\omega}_B\right\}_B = \begin{Bmatrix} P \\ Q \\ R \end{Bmatrix} \tag{7}$$

in which the elements are components of the vector and the subscript outside the bracket identifies the axis system with respect to which these components are taken.

We now proceed to relate these components to the Euler angles, θ, ϕ, ψ, the geocentric coordinates Φ and Ψ, and their derivatives. In doing this we set up an alternative representation for $\vec{\omega}_B$.

We first write, in matrix form, the angular velocity, $\vec{\omega}_E$, of the earth rotating axes, X_E, Y_E, Z_E, relative to the inertial axes, but resolved abcut the earth rotating axes. Thus,

$$\left\{\bar{\omega}_E\right\}_E = [①] \begin{Bmatrix} 0 \\ 0 \\ \Omega_E \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \Omega_E \end{Bmatrix} \tag{8}$$

Numbered matrices in this and subsequent equations are defined in Section 2. 2.

The angular velocity of the earth geocentric axes, X_G, Y_G, Z_G, relative to the inertial axes, but resolved about earth geocentric axes, is

$$\left\{\bar{\omega}_G\right\}_G = [③]\,[②]\left\{\bar{\omega}_E\right\}_E + [③]\,[②]\begin{Bmatrix} 0 \\ 0 \\ \dot{\Psi} \end{Bmatrix} + [③]\begin{Bmatrix} 0 \\ -\dot{\Phi} \\ 0 \end{Bmatrix}$$

$$= [③]\begin{Bmatrix} 0 \\ 0 \\ \Omega_E + \dot{\Psi} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\dot{\Phi} \\ 0 \end{Bmatrix} \tag{9}$$

The angular velocity of the vehicle geocentric axes, X_V, Y_V, Z_V relative to the inertial axes, resolved about the vehicle geocentric axes, is

$$\left\{\bar{\omega}_V\right\}_V = [④]\left\{\bar{\omega}_G\right\}_G = [④]\,[③]\begin{Bmatrix} 0 \\ 0 \\ \Omega_E + \dot{\Psi} \end{Bmatrix} + [④]\begin{Bmatrix} 0 \\ -\dot{\Phi} \\ 0 \end{Bmatrix} \tag{10}$$

Finally, the angular velocity of the vehicle body axes, X_B, Y_B, Z_B, relative to the inertial axes, resolved about the vehicle body axes, is

$$\left\{\bar{\omega}_B\right\}_B = [⑦][⑥][⑤]\left\{\bar{\omega}_v\right\}_v + [⑦][⑥][⑤]\begin{Bmatrix}0\\0\\\dot{\psi}\end{Bmatrix} + [⑦][⑥]\begin{Bmatrix}\dot{\phi}\\0\\0\end{Bmatrix} + [⑦]\begin{Bmatrix}0\\\dot{\theta}\\0\end{Bmatrix}$$

$$= [⑦][⑥][⑤][④][③]\begin{Bmatrix}0\\0\\\Omega_E+\dot{\Psi}\end{Bmatrix} + [⑦][⑥][⑤][④]\begin{Bmatrix}0\\-\dot{\Phi}\\0\end{Bmatrix}$$

$$+ [⑦][⑥]\begin{Bmatrix}0\\0\\\dot{\psi}\end{Bmatrix} + [⑦]\begin{Bmatrix}\dot{\phi}\\0\\0\end{Bmatrix} + \begin{Bmatrix}0\\\dot{\theta}\\0\end{Bmatrix} \tag{11}$$

Equating the right-hand sides of Eqs. (7) and (11) and reducing, we have

$$\begin{Bmatrix}P\\Q\\R\end{Bmatrix} = [⑦][⑥][⑤][④][③]\begin{Bmatrix}0\\0\\\Omega_E+\dot{\Psi}\end{Bmatrix} + [⑦][⑥][⑤][④]\begin{Bmatrix}0\\-\dot{\Phi}\\0\end{Bmatrix}$$

$$+ \begin{bmatrix}\cos\theta & 0 & -\sin\theta\cos\phi\\ 0 & 1 & \sin\phi\\ \sin\theta & 0 & \cos\theta\cos\phi\end{bmatrix}\begin{Bmatrix}\dot{\phi}\\\dot{\theta}\\\dot{\psi}\end{Bmatrix} \tag{12}$$

or,

$$\begin{Bmatrix}\dot{\phi}\\\dot{\theta}\\\dot{\psi}\end{Bmatrix} = \begin{bmatrix}\cos\theta & 0 & -\sin\theta\cos\phi\\ 0 & 1 & \sin\phi\\ \sin\theta & 0 & \cos\theta\cos\phi\end{bmatrix}^{-1}\left\{\begin{Bmatrix}P\\Q\\R\end{Bmatrix}\right.$$

$$\left. - [⑦][⑥][⑤][④][③]\begin{Bmatrix}0\\0\\\Omega_E+\dot{\Psi}\end{Bmatrix} - [⑦][⑥][⑤][④]\begin{Bmatrix}0\\-\dot{\Phi}\\0\end{Bmatrix}\right\} \tag{13}$$

Performing the indicated matrix inversion and matrix multiplications, and reducing, we obtain finally in scalar form the equations,

$$\dot{\theta} = P\sin\theta\tan\phi + Q - R\cos\theta\tan\phi + \dot{\Phi}\cos\psi\sec\phi + (\Omega_E+\dot{\Psi})\cos\Phi\sin\psi\sec\phi \tag{14}$$

$$\dot{\phi} = P \cos\theta + R \sin\theta + \dot{\Phi} \sin\psi - (\Omega_E + \dot{\Psi}) \cos\Phi \cos\psi \tag{15}$$

$$\dot{\psi} = -P \sin\theta \sec\phi + R \cos\theta \sec\phi - \dot{\Phi} \cos\psi \tan\phi - (\Omega_E + \dot{\Psi}) \cos\Phi \sin\psi \tan\phi + (\Omega_E + \dot{\Psi}) \sin\Phi \tag{16}$$

It should be remarked that, if desired, Euler angles as conventionally defined may be used in place of the Euler angles used in the present analysis. That is, the vehicle's orientation would be reached by successive rotations about the Z_B, Y_B and X_B-axes through the redefined angles ψ, θ and ϕ respectively. It can easily be seen that this change could be effected simply by interchanging the matrices [⑥] and [⑦] where they occur as a product and replacing the last two terms in Eq. (11) by

$$[⑥] \begin{Bmatrix} 0 \\ \dot{\theta} \\ 0 \end{Bmatrix} + \begin{Bmatrix} \dot{\phi} \\ 0 \\ 0 \end{Bmatrix} .$$

2.4 Translational Equations of Motion

Let the radius vector from the earth center to the vehicle centroid be $\bar{r}$. Then the velocity vector relative to the inertial axes is given by

$$\bar{v} = \frac{d\bar{r}}{dt} = \left(\frac{\delta \bar{r}}{\delta t}\right)_G + \bar{\omega}_G \times \bar{r} \tag{17}$$

where $\left(\frac{\delta}{\delta t}\right)_G$ denotes a partial differentiation in which i_G, j_G, k_G are held fixed.

With
$$\bar{r} = r i_G \tag{18}$$

where $r = |\bar{r}|$,

and from Eq. (9),

$$\bar{\omega}_G = (\Omega_E + \dot{\Psi}) \sin\Phi \, i_G - \dot{\Phi} \, j_G + (\Omega_E + \dot{\Psi}) \cos\Phi \, k_G \tag{19}$$

Eq. (17) becomes

$$\bar{v} = \dot{r} \, i_G + r(\Omega_E + \dot{\Psi}) \cos\Phi \, j_G + r\dot{\Phi} \, k_G \tag{20}$$

The acceleration vector relative to the inertial axes may now be written as follows,

$$\bar{a} = \frac{d\bar{v}}{dt}$$

$$= \left(\frac{\delta v}{\delta t}\right)_G + \bar{\omega}_G \times \bar{v}$$

$$= \left[\ddot{r} - r\left\{\dot{\Phi}^2 + (\Omega_E + \dot{\Psi})^2 \cos\Phi\right\}\right] i_G$$

$$+ \left[\left\{2\dot{r}(\Omega_E + \dot{\Psi}) + r\ddot{\Psi}\right\}\cos\Phi - 2r\dot{\Phi}(\Omega_E + \dot{\Psi})\sin\Phi\right] j_G$$

$$+ \left[r\ddot{\Phi} + 2\dot{r}\dot{\Phi} + r(\Omega_E + \dot{\Psi})^2 \sin\Phi\cos\Phi\right] k_G \tag{21}$$

The gravity acceleration vector, including the effect of earth oblateness, is written as follows (Ref. 2),

$$\frac{\bar{G}}{m} = \left[-\frac{K}{r^2} + \frac{6\mu K R_o^2}{r^4}(2 - 3\cos^2\Phi)\right] i_G$$

$$+ \left[-\frac{12\mu K R_o^2}{r^4}\sin\Phi\cos\Phi\right] k_G \tag{22}$$

where G is the gravity force vector, m is the vehicle mass, R_o is the radius of the earth at the equator (R_o = 20,926,428 feet), and K and μ are gravity constants with the following values,

$K = 0.14077500 \times 10^{17}$ ft.^{3} sec.$^{-2}$

$6\mu = 1.638 \times 10^{-3}$

The aerodynamic, propulsive and control forces are first computed with reference to vehicle body axes, and a transformation to earth-vehicle geocentric axes is then effected.

The aerodynamic force vector is written as follows,

$$\bar{F} = F_x i_B + F_y j_B + F_z k_B$$

$$= F_r i_G + F_\Psi j_G + F_\Phi k_G \tag{23}$$

The propulsive force vector is,

$$\bar{P} = P_x i_B + P_y j_B + P_z k_B$$
$$= P_r i_G + P_\Psi j_G + P_\Phi k_G \quad (24)$$

and the control force vector is

$$\bar{H} = H_x i_B + H_y j_B + H_z k_B$$
$$= H_r i_G + H_\Psi j_G + H_\Phi k_G \quad (25)$$

Summing corresponding components from Eqs. (23), (24), and (25), the following matrix equation represents the transformation from body axes to earth geocentric axes,

$$\begin{Bmatrix} F_r + P_r + H_r \\ F_\Psi + P_\Psi + H_\Psi \\ F_\Phi + P_\Phi + H_\Phi \end{Bmatrix} = [④]'[⑤]'[⑥]'[⑦]' \begin{Bmatrix} F_x + P_x + H_x \\ F_y + P_y + H_y \\ F_z + P_z + H_z \end{Bmatrix} \quad (26)$$

where the primes denote matrix transposition. Upon multiplication, the transformation matrix becomes,

$$[④]'[⑤]'[⑥]'[⑦]' =$$

$$\begin{bmatrix} \sin\theta \cos\phi & -\sin\phi & -\cos\theta\cos\phi \\ \sin\psi\cos\theta + \cos\psi\sin\theta\sin\phi & \cos\psi\cos\phi & \sin\psi\sin\theta - \cos\psi\cos\theta\sin\phi \\ \cos\psi\cos\theta - \sin\psi\sin\theta\sin\phi & -\sin\psi\cos\phi & \cos\psi\sin\theta + \sin\psi\cos\theta\sin\phi \end{bmatrix} \quad (27)$$

The corresponding transformation matrix in the case of conventional Euler angles is obtained by interchanging matrices [⑥]′ and [⑦]′ in Eq. (27).

Summing applied, gravity and inertia force components in the direction of vehicle geocentric axes, we have the following three translational equations of motion in terms of the coordinates r, Ψ and Φ,

$$\ddot{r} - r\left\{\dot{\Phi}^2 + (\Omega_E + \dot{\Psi})^2 \cos^2\Phi\right\}$$

$$= -\frac{K}{r^2} + \frac{6\mu K R_o^2}{r^4}(2 - 3\cos^2\Phi) + \frac{1}{m}(F_r + P_r + H_r) \tag{28}$$

$$\left\{r\ddot{\Psi} + 2\dot{r}(\Omega_E + \dot{\Psi})\right\}\cos\Phi - 2r\dot{\Phi}(\Omega_E + \dot{\Psi})\sin\Phi$$

$$= \frac{1}{m}(F_\Psi + P_\Psi + H_\Psi) \tag{29}$$

$$r\ddot{\Phi} + 2\dot{r}\dot{\Phi} + r(\Omega_E + \dot{\Psi})^2 \sin\Phi\cos\Phi$$

$$= -\frac{12\mu K R_o^2}{r^4}\sin\Phi\cos\Phi + \frac{1}{m}(F_\Phi + P_\Phi + H_\Phi) \tag{30}$$

Given the shape of the oblate earth, approximated as follows (Ref. 3, App. I),

$$R_e = R_o(1 - f\sin^2\Phi) \tag{31}$$

where R_e is the distance from the earth's center to a local point on the earth's surface, and (Ref. 2),

$$f = 0.0033670034$$

the altitude h can be determined from the relation,

$$h = r - R_o(1 - f\sin^2\Phi) \tag{32}$$

It can be seen from Eq. (29) that Ψ and its derivatives are indeterminate at $\Phi = 90^o$, thus precluding the use of the present equations for simulation of flight over a pole.

2.5 Rotational Equations of Motion

The rotational equations of motion developed on the basis of moment equilibrium about the body axes are the same as those familiar in aircraft analysis. For a vehicle with the $X_B Z_B$-plane a plane of symmetry, they are

$$-[\dot{P}I_{xx} - (I_{yy} - I_{zz})QR - I_{xz}(\dot{R} + PQ)] + L + T_x + J_x = 0 \tag{33}$$

$$-[\dot{Q}I_{yy} - (I_{zz} - I_{xx})RP - I_{xz}(R^2 - P^2)] + M + T_y + J_y = 0 \tag{34}$$

$$-[\dot{R}I_{zz} - (I_{xx} - I_{yy})PQ - I_{xz}(\dot{P} - QR)] + N + T_z + J_z = 0 \tag{35}$$

where I_{xx}, I_{yy}, I_{zz}, I_{xy} are moments and products of inertia referred to the body axes, L, M and N are components of the aerodynamic moment, T_x, T_y and T_z are components of the propulsive moment, and J_x, J_y and J_z are components of the control moments, all referred to the X_B, Y_B and Z_B axes respectively.

2.6 Angle of Attack and Angle of Sideslip

With appropriate modification of Eq. (20), the velocity of the vehicle relative to the earth rotating axes, X_E, Y_E, Z_E, becomes

$$\bar{v}_E = \dot{r}i_G + r\dot{\Psi}\cos\Phi\, j_G + r\dot{\Phi}k_G \tag{36}$$

If we now allow a wind velocity given by

$$\bar{v}_w = v_{w_r} i_G + v_{w_\Psi} j_G + v_{w_\Phi} k_G \tag{37}$$

the velocity of the vehicle relative to the air becomes

$$\bar{v}_a = v_{a_r} i_G + v_{a_\Psi} j_G + v_{a_\Phi} k_G \tag{38}$$

where

$$v_{a_r} = \dot{r} - v_{w_r}$$

$$v_{a_\Psi} = r\dot{\Psi}\cos\Phi - v_{w_\Psi}$$

$$v_{a_\Phi} = r\dot{\Phi} - v_{w_\Phi}$$

and the magnitude of this velocity is

$$V_a = \sqrt{v_{a_r}^2 + v_{a_\Psi}^2 + v_{a_\Phi}^2} \tag{39}$$

A transformation to body axes may be effected as follows,

$$\begin{Bmatrix} v_{a_x} \\ v_{a_y} \\ v_{a_z} \end{Bmatrix} = [\,⑦\,]\,[\,⑥\,]\,[\,⑤\,]\,[\,④\,] \begin{Bmatrix} v_{a_r} \\ v_{a_\Psi} \\ v_{a_\Phi} \end{Bmatrix} \tag{40}$$

Noting that

$$\bar{v}_a = V_a i_w \tag{41}$$

a transformation from wind axes to body axes is given by

$$\begin{Bmatrix} v_{a_x} \\ v_{a_y} \\ v_{a_z} \end{Bmatrix} = [\,⑧\,]'\;[\,⑨\,]' \begin{Bmatrix} V_a \\ 0 \\ 0 \end{Bmatrix}$$

$$= V_a \begin{Bmatrix} \cos\alpha\cos\beta \\ \sin\beta \\ \sin\alpha\cos\beta \end{Bmatrix} \tag{42}$$

Equating the right-hand sides of Eqs. (40) and (42), we have,

$$\begin{Bmatrix} \cos\alpha\cos\beta \\ \sin\beta \\ \sin\alpha\cos\beta \end{Bmatrix} = [\,⑦\,]\,[\,⑥\,]\,[\,⑤\,]\,[\,④\,] \begin{Bmatrix} v_{a_r}/V_a \\ v_{a_\Psi}/V_a \\ v_{a_\Phi}/V_a \end{Bmatrix} \tag{43}$$

from which α and β may be determined. The transformation matrix in this equation is seen to be the transpose of the matrix given in Eq. (27). The corresponding transformation matrix in the case of conventional Euler angles is obtained by interchanging matrices [⑥] and [⑦] in Eq. (43).

2.7 Block Diagram of the Equations of Motion

The equations developed in the preceding sections are summarized and presented in block diagram form in Fig. 2. This diagram illustrates the flow of information between segments of the computation, and facilitates a comparison with other formulations of the problem.

2.8 Modification of Translational Equations with Altitude as a Basic Variable

It is seen from Eq. (32) that altitude is computed as a small difference between large numbers for near-orbital and reentry flight. Accuracy in its determination would require computations to be carried out to at least seven or eight significant figures.

This difficulty can be avoided by using the altitude, h, directly as a coordinate in the equations of motion, replacing r. It should be pointed out, however, that under certain flight conditions, especially near-circular orbital flight, small differences between large numbers still occur in Eq. (28) and considerable accuracy is required in the computations.

From Eq. (32) we can write

$$r = h + R_o(1 - f \sin^2\Phi) \tag{44}$$

Differentiating with respect to time,

$$\dot{r} = \dot{h} - 2R_o f \dot{\Phi} \sin\Phi \cos\Phi \tag{45}$$

$$\ddot{r} = \ddot{h} - 2R_o f \ddot{\Phi} \sin\Phi \cos\Phi - 2R_o f \dot{\Phi}^2 (2\cos^2\Phi - 1) \tag{46}$$

With $h - R_o f \sin^2\Phi << R_o$, we can introduce the followi ng approximations,

$$\frac{1}{r^2} = \frac{1}{R_o^2}\left\{1 + 2f \sin^2\Phi - 2\frac{h}{R_o} + 3(f \sin^2\Phi - \frac{h}{R_o})^2\right\} \tag{47}$$

$$\frac{1}{r^4} = \frac{1}{R_o^4}(1 + 4f \sin^2\Phi - 4\frac{h}{R_o}) \tag{48}$$

More terms in the binomial expansion are retained in Eq. (47) than in Eq. (48) because the latt er is multiplied by the small quantity μ.

Introducing Eqs. (44) to (48) inclusive into Eq. (28), (29) and (30), we obtain the modified translational equations,

$$\ddot{h} - 2R_o f\ddot{\Phi} \sin\Phi \cos\Phi - R_o(1 + 2f)\dot{\Phi}^2 + 5R_o f \dot{\Phi}^2 \sin^2\Phi$$
$$- R_o(\Omega_E + \dot{\Psi})^2(1 - f \sin^2\Phi) \cos^2\Phi - h \left\{ \dot{\Phi}^2 + (\Omega_E + \dot{\Psi})^2 \cos\Phi \right\}$$
$$= - \frac{K}{R_o^2} \left\{ 1 + 2f \sin^2\Phi - 2\frac{h}{R_o} + 3(f \sin^2\Phi - \frac{h}{R_o})^2 \right\}$$
$$+ \frac{6\mu K}{R_o^2} \left\{ (1 + 4f \sin^2\Phi)(2 - 3\cos^2\Phi) - 4\frac{h}{R_o}(2 - 3\cos^2\Phi) \right\}$$
$$+ \frac{1}{m}(F_r + P_r + H_r) \tag{49}$$

$$\ddot{\Psi} R_o \cos\Phi (1 - f\sin^2\Phi + \frac{h}{R_o})$$
$$-2R_o\dot{\Phi}(\Omega_E + \dot{\Psi}) \sin\Phi \left\{ 1 + \frac{h}{R_o} + f(2 - 3\sin^2\Phi) \right\}$$
$$+ 2\dot{h}(\Omega_E + \dot{\Psi}) \cos\Phi = \frac{1}{m}(F_\Psi + P_\Psi + H_\Psi) \tag{50}$$

$$\ddot{\Phi} \left\{ R_o(1 - f\sin^2\Phi) + h \right\} - 4R_o f\dot{\Phi}^2 \sin\Phi \cos\Phi + 2\dot{h}\dot{\Phi}$$
$$+ R_o(\Omega_E + \dot{\Psi})^2(1 - f\sin^2\Phi) \sin\Phi \cos\Phi + h(\Omega_E + \dot{\Psi})^2 \sin\Phi \cos\Phi$$
$$= - \frac{12\mu K}{R_o^2}(1 + 4f\sin^2\Phi - 4\frac{h}{R_o}) \sin\Phi \cos\Phi + \frac{1}{m}(F_\Phi + P_\Phi + H_\Phi) \tag{51}$$

In addition, the components in Eq. (38) are modified as follows,

$$v_{a_r} = -2R_o f\dot{\Phi} \sin\Phi \cos\Phi + \dot{h} - v_{w_r}$$
$$v_{a_\Psi} = R_o \dot{\Psi}(1 - f\sin^2\Phi) \cos\Phi + h\dot{\Psi}\cos\Phi - v_{w_\Psi} \tag{52}$$
$$v_{a_\Phi} = R_o\dot{\Phi}(1 - f\sin^2\Phi) + h\dot{\Phi} - v_{w_\Phi}$$

It is seen that the gain in accuracy associated with the use of h rather than r as a basic variable would be achieved at the expense of considerable complication in the equations.

An alternative modification, based on the substitution,

$$r = R_o + \delta r \tag{53}$$

in the translational equations of motion, would permit a gain in accuracy in the computation of altitude with less added complication than that indicated above when h is used as a basic variable. h would then be computed from the relation,

$$h = \delta r + R_o f \sin^2 \Phi \tag{54}$$

2.9 Simplification of the Modified Translational Equations

A number of terms in Eqs. (49), (50) and (51) are small in relation to the principal terms in these equations. This suggests that considerable simplification might be achieved by identifying these terms and eliminating them.

Proceeding in this direction, we assume the following maximum values for the dependent variables and their derivatives,

$h = 10^6$ ft.

$\dot{h} = 10,000$ ft/sec.

$\ddot{h} = 5\ g = 160$ ft/sec^2

$\dot{\Phi} = (\Omega_E + \dot{\Psi}) \cos \Phi = \sqrt{\frac{g}{R_o}} = 0.0012$ rad/sec.

$\Omega_E + \dot{\Psi}$ (where it does not appear in combination with $\cos \Phi$)

$= 0.005$ rad/sec.

$\ddot{\Phi} = \ddot{\Psi} \cos \Phi = \frac{5g}{R_o} = 8 \times 10^{-6}$ rad/sec.^{2}

On this basis, the largest terms in each equation are of order 160 ft/sec.^{2}, and all terms which are of order 0.10 ft/sec.^{2} or less are listed as follows,

$$3\frac{K}{R_o^2} f^2 = 0.0011 \qquad \frac{48\mu K}{R_o^2} f = 0.0014$$

$$6\frac{K}{R_o^2} f \frac{h}{R_o} = 0.033 \qquad \frac{48\mu K}{R_o^2}\frac{h}{R_o} = 0.021$$

If all of these terms are eliminated, Eqs. (49), (50) and (51) become

$$\ddot{h} - 2R_o f \ddot{\Phi} \sin\Phi \cos\Phi - R_o(1 + 2f)\dot{\Phi}^2 + 5R_o f \dot{\Phi}^2 \sin^2\Phi$$

$$- R_o(\Omega_E + \dot{\Psi})^2(1 - f\sin^2\Phi)\cos^2\Phi - h\left\{\dot{\Phi}^2 + (\Omega_E + \dot{\Psi})^2\cos^2\Phi\right\}$$

$$= -\frac{K}{R_o^2}\left\{1 + 2f\sin^2\Phi - 2\frac{h}{R_o} + 3\left(\frac{h}{R_o}\right)^2\right\}$$

$$+ \frac{6\mu K}{R_o^2}(2 - 3\cos^2\Phi) + \frac{1}{m}(F_r + P_r + H_r) \tag{55}$$

$$\ddot{\Psi} R_o \cos\Phi\left(1 - f\sin^2\Phi + \frac{h}{R_o}\right)$$

$$- 2R_o\dot{\Phi}(\Omega_E + \dot{\Psi})\sin\Phi\left\{1 + \frac{h}{R_o} + f(2 - 3\sin^2\Phi)\right\} + 2\dot{h}(\Omega_E + \dot{\Psi})\cos\Phi$$

$$= \frac{1}{m}(F_\Psi + P_\Psi + H_\Psi) \tag{56}$$

$$\ddot{\Phi} R_o\left(1 - f\sin^2\Phi + \frac{h}{R_o}\right) - 4R_o f\dot{\Phi}^2 \sin\Phi\cos\Phi + 2\dot{h}\dot{\Phi}$$

$$+ R_o(\Omega_E + \dot{\Psi})^2 \sin\Phi\cos\Phi\left(1 - f\sin^2\Phi + \frac{h}{R_o}\right)$$

$$= -\frac{12\mu K}{R_o^2}\sin\Phi\cos\Phi + \frac{1}{m}(F_\Phi + P_\Phi + H_\Phi) \tag{57}$$

The oblateness gravity terms, that is, the terms involving the constant μ, still remaining in the equations are not much larger than terms eliminated, and these too could be eliminated as a second step in the process of simplification.

A still more drastic simplification could be effected by eliminating all oblateness effects by setting the constant f equal to zero and using a mean value for the earth's radius in place of R_o. This introduces appreciable error in range in the case of a complete orbit, as found in another study to be reported in a subsequent part of this report, but such error could be compensated for by appropriate adjustment of conditions at injection. It is likely that the system representation obtained in this manner would be adequate for simulation purposes.

The equations would then reduce to the simpler form,

$$\ddot{h} - R_1\dot{\Phi}^2 - R_1(\Omega_E + \dot{\Psi})^2\cos^2\Phi - h\left\{\dot{\Phi}^2 + (\Omega_E + \dot{\Psi})^2\cos^2\Phi\right\}$$

$$= -\frac{K}{R_1^2}\left\{1 - 2\frac{h}{R_1} + 3\left(\frac{h}{R_1}\right)^2\right\} + \frac{1}{m}(F_r + P_r + H_r) \tag{58}$$

$$\ddot{\Psi} R_1\cos\Phi\left(1 + \frac{h}{R_1}\right) - 2R_i\dot{\Phi}(\Omega_E + \dot{\Psi})\sin\Phi\left(1 + \frac{h}{R_1}\right) + 2\dot{h}(\Omega_E + \dot{\Psi})\cos\Phi$$

$$= \frac{1}{m}(F_\Psi + P_\Psi + H_\Psi) \tag{59}$$

$$\ddot{\Phi} R_1\left(1 + \frac{h}{R_1}\right) + 2\dot{h}\dot{\Phi} + R_1(\Omega_E + \dot{\Psi})^2\sin\Phi\cos\Phi\left(1 + \frac{h}{R_1}\right)$$

$$= \frac{1}{m}(F_\Phi + P_\Phi + H_\Phi) \tag{60}$$

where R_1 is the arithmetic mean of the earth's radius at the equator and at the poles and may be taken to be 20,891,198 feet (Ref. 2).

3. FORMULATION INVOLVING COORDINATES REFERRED TO A NOMINAL TRAJECTORY PLANE

3.1 Reference Axes and Coordinates

The present formulation differs from the preceding one primarily in that the angular coordinates establishing the orientation of the earth-vehicle geocentric axes relative to the inertial axes are referred to a nominal trajectory plane, fixed with respect to the inertial axes, rather than to the earth's equatorial plane.

The nominal trajectory plane contains the position of the vehicle at launch, identified by geocentric latitude Φ_i, and is parallel to the plane in which the trajectory is designed to lie just after being bent over from the vertical, the direction of flight at that point being given by the angle λ north of east.

The new earth-vehicle geocentric axes are denoted by X_F, Y_F, Z_F, and the corresponding unit vectors are i_F, j_F, k_F. The origin is again at the earth center, X_F passes through the vehicle centroid, the $X_F\,Y_F$-plane is the nominal trajectory plane and Y_F points essentially in the direction of flight. The Z_F-axis points to the left when looking in the direction of flight.

The orientation of the X_F, Y_F, Z_F axes may thus be reached from the inertial axes, X, Y, Z, by successive rotations (a) about the y-axis through the angle $-\Phi_i$, (b) about the x-axis through the angle λ, (c) about the z-axis through the angle Ψ_F, and (d) about the y-axis through the angle $-\Phi_F$, small letters denoting intermediate positions of the axes. Thus,

$$\begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix} = [⑬]\,[⑫]\,[⑪]\,[⑩] \begin{Bmatrix} i \\ j \\ k \end{Bmatrix} \tag{61}$$

where

$$[⑩] = \begin{bmatrix} \cos\Phi_i & 0 & \sin\Phi_i \\ 0 & 1 & 0 \\ -\sin\Phi_i & 0 & \cos\Phi_i \end{bmatrix}$$

$$[⑪] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda & \sin\lambda \\ 0 & -\sin\lambda & \cos\lambda \end{bmatrix}$$

$$[⑫] = \begin{bmatrix} \cos\Psi_F & \sin\Psi_F & 0 \\ -\sin\Psi_F & \cos\Psi_F & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[⑬] = \begin{bmatrix} \cos\Phi_F & 0 & \sin\Phi_F \\ 0 & 1 & 0 \\ -\sin\Phi_F & 0 & \cos\Phi_F \end{bmatrix}$$

The orientation of the X_F, Y_F, Z_F axes relative to the earth rotating axes, X_E, Y_E, Z_E, may be obtained by means of the transformation,

$$\begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix} = [⑬]\,[⑫]\,[⑪]\,[⑩]\,[①]' \begin{Bmatrix} i_E \\ j_E \\ k_E \end{Bmatrix} \tag{62}$$

The orientation of the vehicle body axes, X_B, Y_B, Z_B, relative to the new geocentric axes is determined by means of Euler angles ψ, ϕ, θ similar to those used previously, except that they are now referred to the new geocentric axes. Thus,

$$\begin{Bmatrix} i_B \\ j_B \\ k_B \end{Bmatrix} = [⑦]\,[⑥]\,[⑤]\,[④] \begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix} \tag{63}$$

The orientation of the wind axes, X_W, Y_W, Z_W, relative to the body axes is, as before, given by the relation,

$$\begin{Bmatrix} i_w \\ j_w \\ k_w \end{Bmatrix} = [⑨]\,[⑧] \begin{Bmatrix} i_B \\ j_B \\ k_B \end{Bmatrix} \tag{64}$$

In the course of the analysis it will be desirable to determine the geocentric latitude Φ and longitude Ψ. A relationship between Φ and Ψ on the one hand and Φ_F and Ψ_F on the other is thus required. We proceed to establish such a relationship by writing the transformation,

$$\begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix} = [⑬]\,[⑫]\,[⑪]\,[⑩]\,[①]'\,[②]'\,[③]' \begin{Bmatrix} i_G \\ j_G \\ k_G \end{Bmatrix} \tag{65}$$

Since $i_F = i_G$, we may write

$$\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = [⑬]\,[⑫]\,[⑪]\,[⑩]\,[①]'\,[②]'\,[③]' \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \tag{66}$$

or

$$[①]'\,[②]'\,[③]' \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = [⑩]'\,[⑪]'\,[⑫]'\,[⑬]' \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \tag{67}$$

Eq. (67) yields the relations,

$$\cos(\Psi + \Omega_E t)\cos\Phi = \cos\Phi_i \cos\Psi_F \cos\Phi_F$$
$$-\sin\Phi_i \sin\lambda \sin\Psi_F \cos\Phi_F - \sin\Phi_i \cos\lambda \sin\Phi_F \tag{68}$$

$$\sin(\Psi + \Omega_E t)\cos\Phi = \cos\lambda \sin\Psi_F \cos\Phi_F - \sin\lambda \sin\Phi_F \tag{69}$$

$$\sin\Phi = \sin\Phi_i \cos\Psi_F \cos\Phi_F + \cos\Phi_i \sin\lambda \sin\Psi_F \cos\Phi_F$$
$$+ \cos\Phi_i \cos\lambda \sin\Phi_F \tag{70}$$

from which the coordinates Ψ and Φ may be determined.

3.2 Differential Equations for the Euler Angles

Following the notation of Section 2.3, the angular velocity of the X_F, Y_F, Z_F axes relative to the inertial axes, but resolved about the X_F, Y_F, Z_F axes, is given by

$$\left\{\bar{\omega}_F\right\}_F = [⑬] \begin{Bmatrix} 0 \\ 0 \\ \dot{\Psi}_F \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\dot{\Phi}_F \\ 0 \end{Bmatrix} \tag{71}$$

Similarly, the angular velocity of the body axes relative to the inertial axes, but resolved about the body axes, is given by

$$\left\{\bar{\omega}_B\right\}_B = [⑦]\,[⑥]\,[⑤]\,[④]\,[⑬] \begin{Bmatrix} 0 \\ 0 \\ \dot{\Psi}_F \end{Bmatrix} + [⑦]\,[⑥]\,[⑤]\,[④] \begin{Bmatrix} 0 \\ -\dot{\Phi}_F \\ 0 \end{Bmatrix}$$

$$+ [⑦]\,[⑥] \begin{Bmatrix} 0 \\ 0 \\ \dot{\psi} \end{Bmatrix} + [⑦] \begin{Bmatrix} \dot{\phi} \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \dot{\theta} \\ 0 \end{Bmatrix} \qquad (72)$$

But,

$$\left\{\bar{\omega}_B\right\}_B = \begin{Bmatrix} P \\ Q \\ R \end{Bmatrix} \qquad (73)$$

Therefore, equating the right-hand sides of Eqs. (72) and (73), and proceeding as in Section 2.3, we obtain the relations,

$$\dot{\theta} = P \sin\theta \tan\phi + Q - R \cos\theta \tan\phi + \dot{\Phi}_F \cos\psi \sec\phi + \dot{\Psi}_F \cos\Phi_F \sin\psi \sec\phi \qquad (74)$$

$$\dot{\phi} = P \cos\theta + R \sin\theta + \dot{\Phi}_F \sin\psi - \dot{\Psi}_F \cos\Phi_F \cos\psi \qquad (75)$$

$$\dot{\psi} = P \sin\theta \sec\phi + R \cos\theta \sec\phi - \dot{\Phi}_F \cos\psi \tan\phi - \dot{\Psi}_F \cos\Phi_F \sin\psi \tan\phi + \dot{\Psi}_F \sin\Phi_F \qquad (76)$$

3.3 Translational Equations of Motion

Following the analysis of Section 2.4, the radius vector, velocity and acceleration may be written as follows,

$$\bar{r} = r i_F \qquad (77)$$

$$\bar{v} = \dot{r} i_F + r \dot{\Psi}_F \cos\Phi_F j_F + r\dot{\Phi}_F k_F \qquad (78)$$

$$\bar{a} = [\ddot{r} - r(\dot{\Phi}_F^2 + \dot{\Psi}_F^2 \cos^2 \Phi_F)] i_F$$

$$+[(2\dot{r}\dot{\Psi}_F + r\ddot{\Psi}_F) \cos \Phi_F - 2r\dot{\Phi}_F\dot{\Psi}_F \sin \Phi_F] j_F$$

$$+[r\ddot{\Phi}_F + 2\dot{r}\dot{\Phi}_F + r\dot{\Psi}_F^2 \sin \Phi_F \cos \Phi_F] k_F \tag{79}$$

The gravity acceleration vector must now be resolved into components along the X_F, Y_F, Z_F axes. This necessitates a transformation from the original to the new earth-vehicle geocentric axes. Since the X_F-axis is coincident with the X_G-axis, this transformation involves simply a rotation about the X_G-axis through an angle which we will denote by λ_F. Thus,

$$\begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \lambda_F & \sin \lambda_F \\ 0 & -\sin \lambda_F & \cos \lambda_F \end{bmatrix} \begin{Bmatrix} i_G \\ j_G \\ k_G \end{Bmatrix} \tag{80}$$

and

$$\frac{\overline{G}}{m} = [-\frac{K}{r^2} + \frac{6\mu K R_o^2}{r^4} (2 - 3\cos^2\Phi)] i_F$$

$$+[-\frac{12K\mu R_o^2}{r^4} \sin \lambda_F \sin \Phi \cos \Phi] j_F$$

$$+[-\frac{12K\mu R_o^2}{r^4} \cos \lambda_F \sin \Phi \cos \Phi] k_F \tag{81}$$

From Eqs. (65) and (80) we can write,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \lambda_F & \sin \lambda_F \\ 0 & -\sin \lambda_F & \cos \lambda_F \end{bmatrix} = [⑬] [⑫] [⑪] [⑩] [①]' [②]' [③]' \tag{82}$$

or, inverting,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda_F & -\sin\lambda_F \\ 0 & \sin\lambda_F & \cos\lambda_F \end{bmatrix} = [\textcircled{3}]\,[\textcircled{2}]\,[\textcircled{1}]\,[\textcircled{10}]'\,[\textcircled{11}]'\,[\textcircled{12}]'\,[\textcircled{13}]' \tag{83}$$

This may be rearranged in the form,

$$[\textcircled{2}]'\,[\textcircled{3}]' \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda_F & -\sin\lambda_F \\ 0 & \sin\lambda_F & \cos\lambda_F \end{bmatrix} = [\textcircled{1}]\,[\textcircled{10}]'\,[\textcircled{11}]'\,[\textcircled{12}]'\,[\textcircled{13}]' \tag{84}$$

Equating the elements in the last row and second and third columns of the product matrices on both sides of Eq. (84), we have finally,

$$\cos\Phi \sin\lambda_F = -\sin\Phi_i \sin\Psi_F + \cos\Phi_i \sin\lambda \cos\Psi_F \tag{85}$$

$$\cos\Phi \cos\lambda_F = -\sin\Phi_i \sin\Phi_F \cos\Psi_F - \cos\Phi_i \sin\lambda \sin\Phi_F \sin\Psi_F + \cos\Phi_i \cos\lambda \cos\Phi_F \tag{86}$$

We can now substitute Eqs. (85) and (86) into Eq. (81) to obtain the gravity acceleration in the form,

$$\begin{aligned} \frac{\overline{G}}{m} = & \left[-\frac{K}{r^2} + \frac{6\mu K R_o^2}{r^4} (2 - 3\cos^2\Phi) \right] i_F \\ & + \left[\frac{12K\mu R_o^2}{r^4} \sin\Phi (\sin\Phi_i \sin\Psi_F - \cos\Phi_i \sin\lambda \cos\Psi_F) \right] j_F \\ & + \left[\frac{12K\mu R_o^2}{r^4} \sin\Phi (\sin\Phi_i \sin\Phi_F \cos\Psi_F + \cos\Phi_i \sin\lambda \sin\Phi_F \sin\Psi_F \right. \\ & \left. - \cos\Phi_i \cos\lambda \cos\Phi_F) \right] k_F \end{aligned} \tag{87}$$

The aerodynamic, propulsive and control forces are again determined with reference to body axes and then transformed to vehicle geocentric axes,

X_F, Y_F, Z_F, using the transformation matrix of Eq. (26), but recognizing that the Euler angles here are referred to the X_F, Y_F, Z_F axes.

The translational equations of motion may now be written as follows,

$$\ddot{r} - r(\dot{\Phi}_F^2 + \dot{\Psi}_F^2 \cos^2\Phi_F)$$

$$= -\frac{K}{r^2} + \frac{6\mu K R_o^2}{r^4}(2 - 3\cos^2\Phi) + \frac{1}{m}(F_r + P_r + H_r) \tag{88}$$

$$r\ddot{\Psi}_F \cos\Phi_F + 2\dot{r}\dot{\Psi}_F \cos\Phi_F - 2r\dot{\Phi}_F\dot{\Psi}_F \sin\Phi_F$$

$$= \frac{12K\mu R_o^2}{r^4} \sin\Phi(\sin\Phi_i \sin\Psi_F - \cos\Phi_i \sin\lambda \cos\Psi_F)$$

$$+ \frac{1}{m}(F_{\Psi_F} + P_{\Psi_F} + H_{\Psi_F}) \tag{89}$$

$$r\ddot{\Phi}_F + 2\dot{r}\dot{\Phi}_F + r\dot{\Psi}_F^2 \sin\Phi_F \cos\Phi_F$$

$$= \frac{12K\mu R_o^2}{r^4} \sin\Phi(\sin\Phi_i \sin\Phi_F \cos\Psi_F$$

$$+ \cos\Phi_i \sin\lambda \sin\Phi_F \sin\Psi_F - \cos\Phi_i \cos\lambda \cos\Phi_F)$$

$$+ \frac{1}{m}(F_{\Phi_F} + P_{\Phi_F} + H_{\Phi_F}) \tag{90}$$

Upon solution of these equations, the altitude may again be determined from Eq. (32). Because of the increased complexity of the equations, even with r as a basic variable, a reformulation to introduce h as a basic variable is not carried out. However, the substitution indicated in Eq. (53) may be made.

3.4 Rotational Equations of Motion

The change in coordinate system does not affect the rotational equations of motion, and Eqs. (33), (34) and (35) remain applicable.

3.5 Angle of Attack and Angle of Sideslip

The velocity of the vehicle relative to the earth rotating axes, X_E, Y_E, Z_E, may be written,

$$\bar{v}_E = \left(\frac{\delta \bar{r}}{\delta t}\right)_F + \bar{\omega}_{FE} \times \bar{r} \tag{91}$$

in which $\bar{r}$ is given by Eq. (77), $\left(\frac{\delta}{\delta t}\right)_F$ denotes a partial differentiation in which i_F, j_F, k_F are held fixed, and $\bar{\omega}_{FE}$ is the rotational velocity of the X_F, Y_F, Z_F frame relative to the X_E, Y_E, Z_E frame. In terms of components about the X_F, Y_F, Z_F axes, $\bar{\omega}_{FE}$ is given by,

$$\left\{\bar{\omega}_{FE}\right\}_F = [⑬] \begin{Bmatrix} 0 \\ 0 \\ \dot{\Psi}_F \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\dot{\Phi}_F \\ 0 \end{Bmatrix} - [⑬]\,[⑫]\,[⑪]\,[⑩] \begin{Bmatrix} 0 \\ 0 \\ \Omega_E \end{Bmatrix} \tag{92}$$

or, after performing the indicated matrix multiplications,

$$\begin{aligned} \bar{\omega}_{FE} = &\left\{\dot{\Psi}_F \sin \Phi_F - \Omega_E \sin \Phi_i \cos \Psi_F \cos \Phi_F \right. \\ &\left. -\Omega_E \cos \Phi_i \sin \lambda \sin \Psi_F \cos \Phi_F - \Omega_E \cos \Phi_i \cos \lambda \sin \Phi_F \right\} i_F \\ &+ \left\{ -\dot{\Phi}_F + \Omega_E \sin \Phi_i \sin \Psi_F - \Omega_E \cos \Phi_i \sin \lambda \cos \Psi_F \right\} j_F \\ &+ \left\{ \dot{\Psi}_F \cos \Phi_F + \Omega_E \sin \Phi_i \cos \Psi_F \sin \Phi_F \right. \\ &\left. + \Omega_E \cos \Phi_i \sin \lambda \sin \Psi_F \sin \Phi_F - \Omega_E \cos \Phi_i \cos \lambda \cos \Phi_F \right\} k_F \end{aligned} \tag{93}$$

If we now allow a wind velocity given by,

$$\bar{v}_w = v_{w_r} i_F + v_{w_{\Psi_F}} j_F + v_{w_{\Phi_F}} k_F \tag{94}$$

the velocity of the vehicle relative to the air becomes,

$$\bar{v}_a = \left(\frac{\delta \bar{r}}{\delta t}\right)_F + \bar{\omega}_{FE} \times \bar{r} - \bar{v}_w \tag{95}$$

Introducing Eqs. (77), (93) and (94) into Eq. (95), we have finally,

$$\bar{v}_a = v_{a_r} i_F + v_{a_{\Psi_F}} j_F + v_{a_{\Phi_F}} k_F \qquad (96)$$

where

$$v_{a_r} = \dot{r} - v_{w_r}$$

$$v_{a_{\Psi_F}} = r(\dot{\Psi}_F \cos \Phi_F + \Omega_E \sin \Phi_i \cos \Psi_F \sin \Phi_F + \Omega_E \cos \Phi_i \sin \lambda \sin \Psi_F \sin \Phi_F - \Omega_E \cos \Phi_i \cos \lambda \cos \Phi_F) - v_{w_{\Psi_F}}$$

$$v_{a_{\Phi_F}} = r(\dot{\Phi}_F - \Omega_E \sin \Phi_i \sin \Psi_F + \Omega_E \cos \Phi_i \sin \lambda \cos \Psi_F) - v_{w_{\Phi_F}}$$

and

$$V_a = \sqrt{v_{a_r}^2 + v_{a_{\Psi_F}}^2 + v_{a_{\Phi_F}}^2} \qquad (97)$$

α and β may now be determined as in Section 2.6, Eq. (43) being replaced by the relation

$$\begin{Bmatrix} \cos \alpha \cos \beta \\ \sin \beta \\ \sin \alpha \cos \beta \end{Bmatrix} = [⑦] [⑥] [⑤] [④] \begin{Bmatrix} v_{a_r}/V_a \\ v_{a_{\Psi_F}}/V_a \\ v_{a_{\Phi_F}}/V_a \end{Bmatrix} \qquad (98)$$

where again the transformation matrix is the transpose of the matrix in Eq. (27).

4. CONCLUDING REMARKS

The present part of the comprehensive report has presented two alternative formulations of the translational equations of motion of an orbital and reentry vehicle. Both involve the use of spherical coordinates referred to the earth's center as basic variables.

Both have the desirable feature that the coordinates used in the translational equations do not depend upon the vehicle's angular orientation.

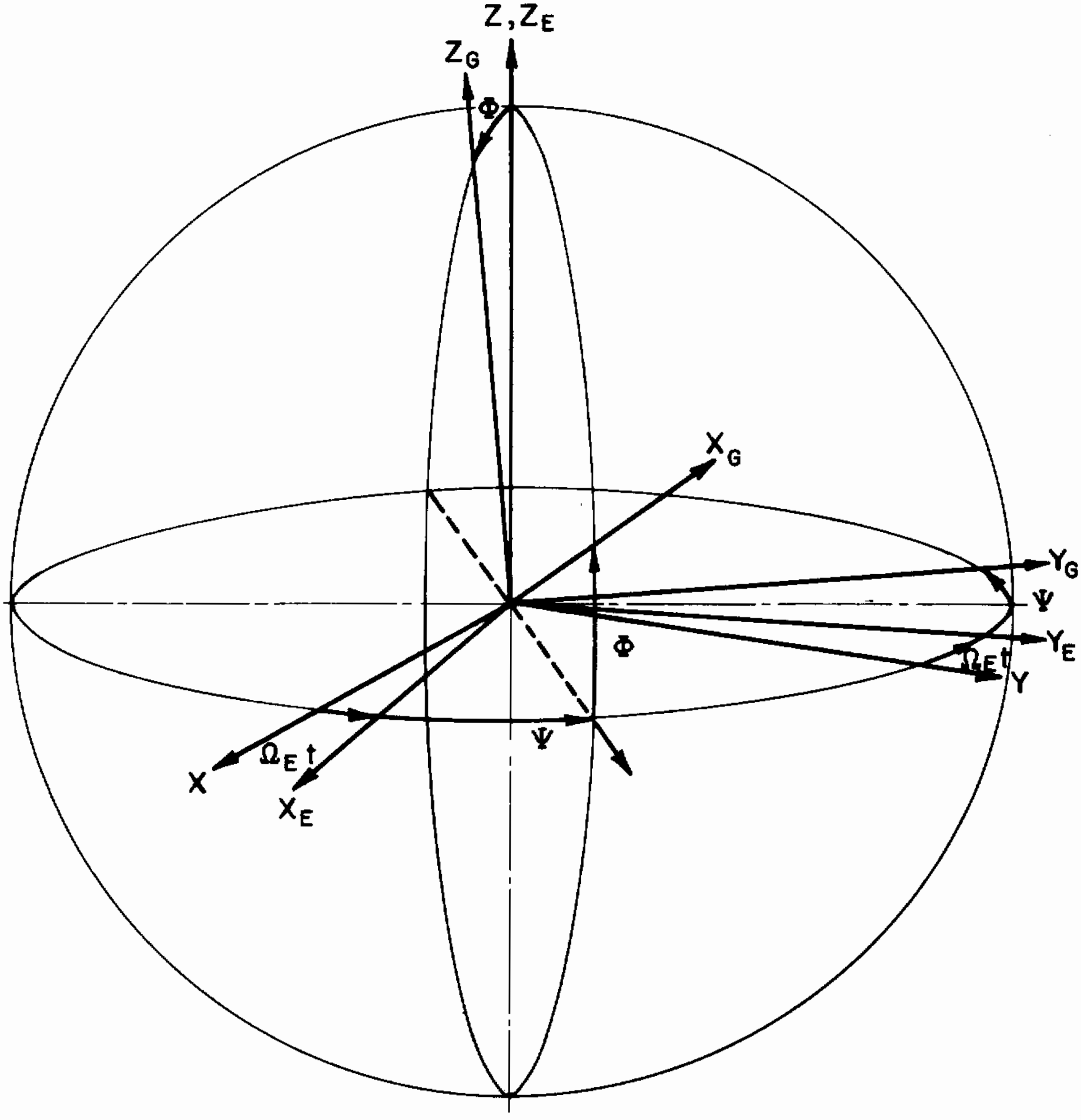
As a result, the components of the vehicle's velocity in this coordinate system cannot change value rapidly unless the velocity vector of the vehicle changes magnitude or direction rapidly. This situation is distinctly different from that existing when translational velocity components are referred to body axes, for in that case rapid changes in the angular orientation of the vehicle introduce correspondingly rapid changes in translational velocity components referred to such axes, irrespective of whether the velocity vector is itself changing.

Another desirable feature of the formulation presented in Section 2 is that the solution of the translational equations yields directly the velocity of the vehicle relative to the rotating atmosphere. This improves the accuracy with which aerodynamic forces can be computed at relatively low speeds as compared with the use of an inertial reference frame as in the formulation of Section 3 and that of Part II of the comprehensive report. In the latter cases the velocity associated with the earth's rotation must be subtracted out before the velocity relative to the atmosphere can be determined, leading to small differences between large numbers in the case of low-speed flight. It should be remarked that this advantage is actually of limited significance in view of the rather severe accuracy requirements on velocity in the orbital and suborbital phases of flight.

A further advantage of the formulation of Section 2, as compared with that of Section 3, is the fact that the geocentric latitude is a basic variable in the translational equations and does not have to be computed in a supplementary calculation. This facilitates the introduction of earth oblateness effects.

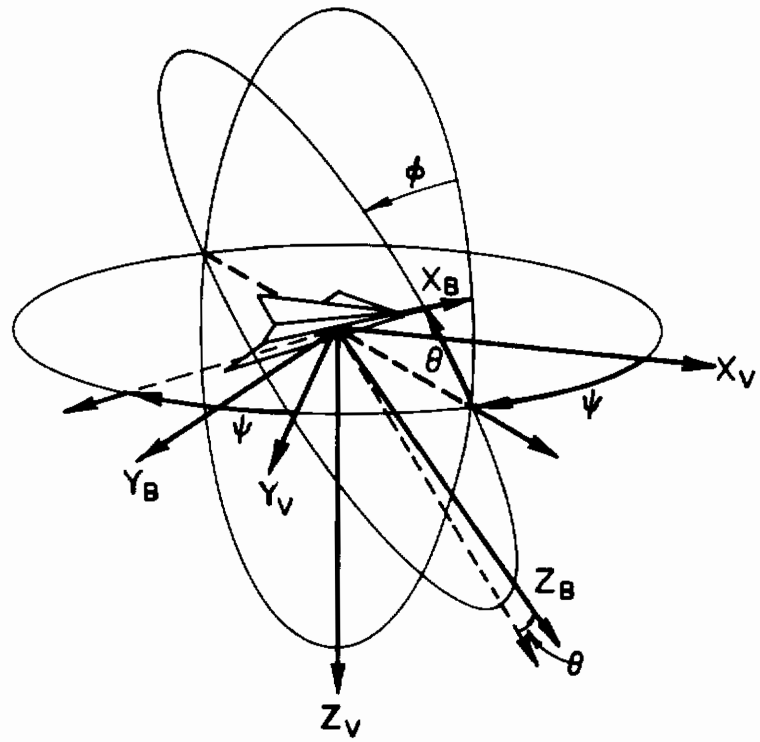
The formulations presented here are suitable for digital and for hybrid analog-digital computation, that of Section 2 being clearly preferable unless flight over, or close to, a pole is to be simulated, in which case it would be necessary to use the formulation of Section 3. In the case of hybrid analog-digital computation, the translational equations would be solved digitally, since accuracy requirements are greater while the variables involved vary more slowly than those in the rotational equations. On the other hand, the rotational equations, Euler angle equations and coordinate transformations involving body axes could be mechanized most efficiently by analog means because of the rapidity of the variations involved and the closed-loop nature of the portion of the system involved. The generation of aerodynamic forces and moments requires further study. It may be suggested, however, that the dynamic pressure could be computed efficiently in the digital phase, since it would not be subject to rapid variation. Furthermore, the aerodynamic coefficients could be expressed as combinations of functions of angle of attack and sideslip on the one hand and Mach number and Reynolds number on the other hand, the former being computed by analog means and the latter digitally.

It is expected that accuracy requirements for an exclusively analog mechanization of the equations developed here would be excessive. For this reason, alternative formulations which might be better suited to analog computation have been studied, and it has been found that a modified axis system aligned with the vertical and horizontal components of the vehicle's velocity vector has significant advantages in that direction. A formulation of the problem based on its use is presented in Part II of the present sequence. While this formulation of the problem is of advantage principally in an analog mechanization, it represents also an alternative to the formulations presented here in the case of digital or hybrid analog-digital mechanization. Further discussion of its merits relative to those of the formulations presented here may be found in Part II.

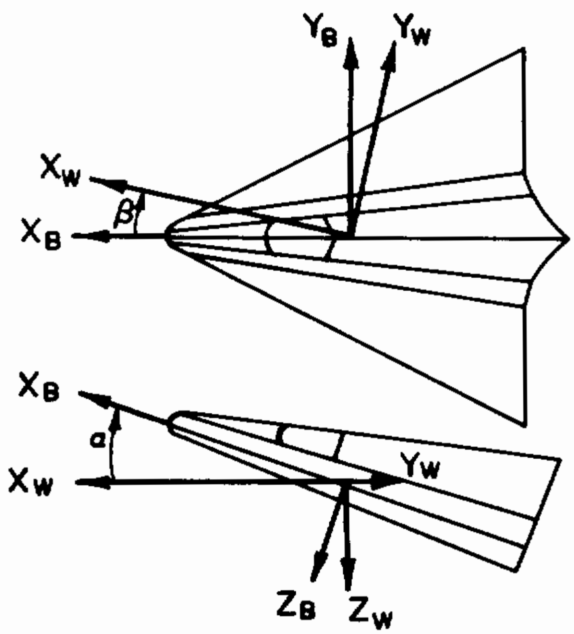


(a) Inertial, Earth Rotating and Earth-Vehicle Geocentric Axes

Figure 1 - Axis Systems



(b) Vehicle Geocentric and Body Axes



(c) Angle of Attack and Angle of Sideslip

Figure 1 - (continued)

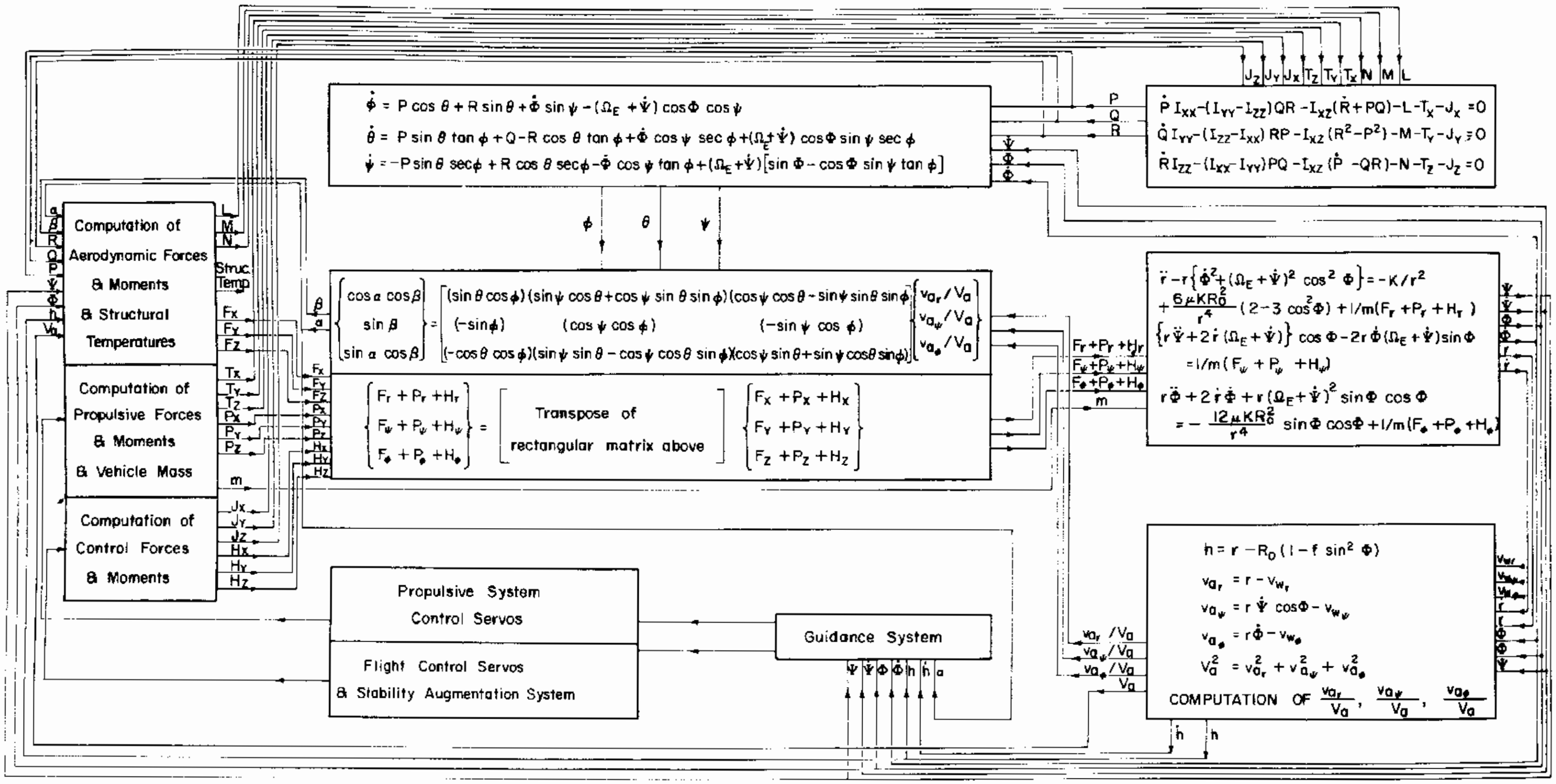


Figure 2 - Block Diagram of the Equations of Motion from the Formulation of Section 2

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