SESSION 7. NON–LINEAR EFFECTS

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A PROCEDURE FOR FINITE ELEMENT PLATE AND SHELL PRE- AND POST-BUCKLING ANALYSIS

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A procedure for finite element analysis of geometrically nonlinear problems, extending over the pre-buckling and initial post-buckling regimes, snap-through buckling, and accounting for initial imperfections, is described. The computation of the nonlinear pre-buckling path is accomplished by direct iterative solution. The bifurcation point is established by interpolation of solution points of the pre-buckling and immediate post-buckling analyses. A static perturbation method is then developed for determination of the post-buckling path of the bifurcating structure or the limit point of a structure with initial imperfections. Three numerical examples, involving an arch, flat plate and shallow shell, are presented in illustration of the procedure and in comparison with alternative approaches.

INTRODUCTION

The analysis of instability phenomena of complicated thin shells has drawn intensified interest due, in part, to the development of finite element analysis procedures for such structures.(1) Structures of this class may collapse at load levels which are less than those predicted by linear instability theory because of the role played by initial imperfections and geometric nonlinearities. The extensive efforts in the development of theories to cope with this problem have been surveyed by Hutchinson and Koiter,(2) Haftka, et al(3) and Bienek(4).

Although the various types of instability phenomena which might occur in the complete range of load-displacement behavior

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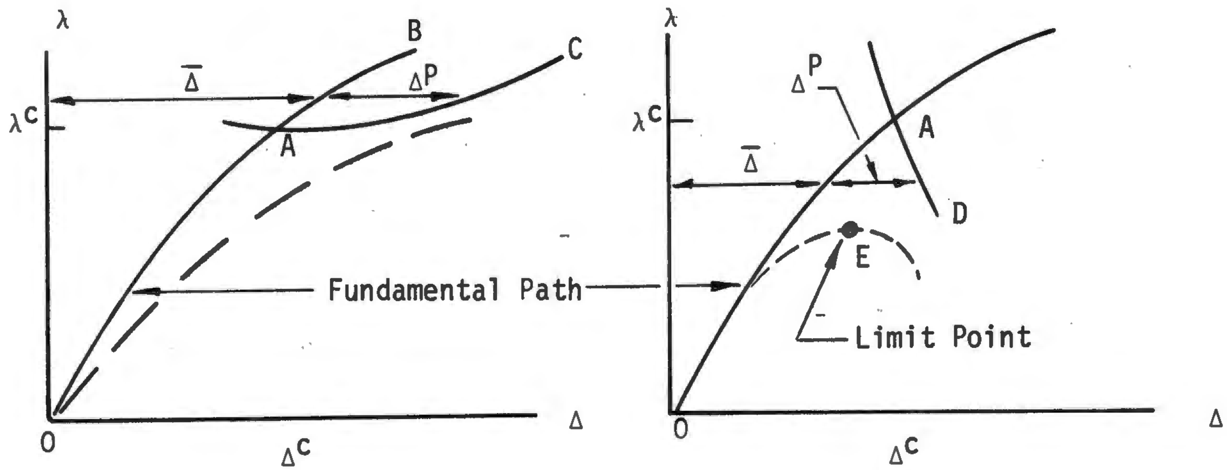
prior to final collapse are not as yet fully understood, certain forms are known and are of considerable practical importance, especially those which occur in the earliest stages of loading. Figure 1a applies to "perfect" structures and represents the case in which the structure first displaces along the path defined by OAB (the fundamental path) and bifurcates (or branches) at the Point A to another path, AC. In contrast to a rising post-buckling path, as AC, a descending path AD (as pictured in Figure 1b) may be encountered.

When the structure possesses fabricational imperfections, the load-displacement behavior follows the paths indicated by dotted lines. The structure with a rising post-buckling path will have strength exceeding the bifurcation load. The strength of an imperfect structure with a descending post-buckling path in the perfect state will not achieve strengths as high as the bifurcation load unless the load-displacement path again rises at larger displacements. Such structures, under the appropriate load condition, are termed "imperfection sensitive" and the maximum load attained (Point E) is termed the "limit point".

A non-bifurcating load-displacement behavior may also occur for a structure assumed to be devoid of imperfections and may take the form similar in shape to the curve OE (Figure 1b) of the imperfection-sensitive structure. For this case the buckling phenomenon is of the 'snap-through' type.

A landmark development of procedures for establishing the shape of the post-buckling path and for determining the limit point for imperfection-sensitive structures is due to Koiter.[5] This approach uses the concept of perturbations from the bifurcation point and enables an efficient definition of load-displacement behavior in the immediate post-buckling range. Further contributions or alternative forms of these concepts have been presented by Budiansky and Hutchinson[6], Sewell[7], and Thompson[8,9].

Extensions of Koiter's procedure to the format of finite element analysis, as well as other finite element approaches to the same physical problem, have recently appeared [10-12]. Morin[10] applies a predictor-corrector scheme in calculation of



(a) (b)

FIGURE 1. FORMS OF ELASTIC INSTABILITY

non-linear prebuckling behavior, in which a perturbation approach is employed as the predictor and Newton-Raphson iteration is employed as the corrector. The perturbation approach, in both the pre- and post-buckling computational phases, reflects earlier work by Thompson[8]. Thompson[9] has also advocated a new perturbation approach. Haftka, et al[3] propose the definition of an "equivalent structure", one in which the non-linear terms are treated as initial imperfections, in order to exploit the concepts derived by Koiter for imperfect structures. Dupuis, et al[11], attack the solution of the nonlinear equations in an incremental-iterative manner. The work by Lang[12] is a direct adaptation of Koiter's concepts, including retention of the condition of a linear prebuckling state.

Recent analyses have shown[13] that the assumption of a linear prebuckling state may lead to inaccurate results. One of the principal aspects of the work described in this paper is the method of determination of the load and displacement state on the fundamental path, at the bifurcation point, following upon a nonlinear prebuckling state. The calculated information furnishes the necessary ingredients for a perturbation analysis of the post-buckling or limit point behavior.

The starting point of the present small strain-finite displacement formulation is the definition of element stiffness equations in the Lagrangian frame of reference. The element stiffness matrices extend to both first- and second-degree geometric nonlinearities in the element displacement parameters. Then, direct iteration is used for solution of the nonlinear algebraic equations in the prebuckling range. Unlike many widely used and seemingly more efficient procedures, direct iteration permits calculation of the fundamental path beyond the bifurcation point and definition of the latter by interpolation of the determinants of such solutions through the zero point.

For post-buckling, and for snap-through buckling for initial-imperfection situations, both displacements and loads are expanded about the bifurcation point of the perfect structure in power series in a single parameter which is related to the amplitude of the eigen-function in the deflected shape of the structure.

Upon determination of the series coefficients, the solution is a parametric representation of load vs. displacement.

Three problems are solved in verification of the present procedure and for the purpose of comparison with other methods. The first problem consists of a uniformly loaded shallow circular arch. This case, for which an exact solution is available, illustrates the calculation of the bifurcation point following upon a nonlinear fundamental path and demonstrates prediction of behavior for both perfect and imperfect structural forms. The second problem concerns the familiar case of post-buckling behavior of a perfectly flat, simply-supported rectangular plate under uniaxial compression. Finally, a solution is obtained for a hypothetical cylindrical shell structure which has been employed as the basis for numerical verification of alternative analysis methods.

A. ELEMENT AND SYSTEM FORMULATIONS

The purpose of this section is to define the general algebraic form of finite element stiffness equations for the present geometrically nonlinear analysis. This form, which applies to any given type of element, is subsequently employed in the development of procedures for pre- and post-buckling analysis of the complete structure. Presentation of the equations for specific elements is beyond the scope of this paper, although the formulative bases of certain arch, flat plate, and shell elements are outlined in Section E.

It has been shown[14] that the element stiffness equations of a "perfect" structure (no initial displacements) for small strain non-incremental finite displacement analysis, for conservative loading and a Lagrangian frame of reference, are of the general form.

$$[k]\{\Delta\} + [n_1(\Delta)]\{\Delta\} + [n_2(\Delta)]\{\Delta\} = \{F\} \qquad (1)$$

where $\{F\}$ and $\{\Delta\}$ are the element joint forces and corresponding displacements (degrees-of-freedom) respectively.

$[k]$ is the linear (small displacement theory) stiffness matrix.

$[n_1(\Delta)]$ is the first-order ("geometric") stiffness matrix, where the individual terms are linear functions of the degrees-of-freedom $\{\Delta\}$. A simplified form of this matrix permits linear stability analysis, as in Euler buckling.

$[n_2(\Delta)]$ is the second-order ("geometric") stiffness matrix, with individual terms a quadratic function of the degrees-of-freedom $\{\Delta\}$. These terms arise from the components of strain energy which are the first derivatives of w with respect to the spatial variables raised to the fourth power.

Upon assembly of the element relationships defined by Eq. (1) to form a representation of the complete structure, (global representation) the following equations are obtained.

$$[K]\{\Delta\} + [N_1(\Delta)]\{\Delta\} + [N_2(\Delta)]\{\Delta\} - \lambda\{P\} = 0 \qquad (2)$$

where the definitions of K, N_1, and N_2 for the global representation correspond to those given above for k, n_1, and n_2 for the respective elements. The "normalized" load vector $\{P\}$ represents the relative magnitude of the loads corresponding to the respective degrees-of-freedom $\{\Delta\}$; thus, the joint loads are applied in fixed proportion to one another. λ, the loading parameter is a scalar which can be adjusted to define a desired intensity of loading.

In indicial notation, Equation (2) becomes

$$K_{ij}\Delta_j + N_{ijk}\Delta_j\Delta_k + N_{ijkl}\Delta_j\Delta_k\Delta_l - \lambda P_i = 0 \qquad (2a)$$

Indicial notation is especially useful in nonlinear finite element analysis since the constants of the problem (N_{ijk} and N_{ijkl}) are readily identified and can be stored permanently, in contrast to the matrix format where $[N_1(\Delta)]$ and $[N_2(\Delta)]$ are dependent on the displacements and change continually during the numerical analysis process. The matrix (Eq. 2) and indicial (Eq. 2a) notations will be employed interchangeably throughout.

The "perfect" structure, to which the above equations apply, constitutes an analytical reference base for the study of the behavior of imperfect structures or for structures for which the applied loads deviate slightly from those which produce bifurcation, by use of the perturbation method. Analysis of such problems requires an extension of Equations (1) and (2) to include the influence of initial displacements and the above-cited load deviations. To account for the former, we assume that the initial displacements are distributed throughout the structure in a form identical to the elastic displacements; hence, the initial displacements are properly described by joint values Δ_j^i. Also, we designate the total displacements by Δ_j^T. The system equilibrium equations now become (see Ref. 15 for a representative detailed development)

$$K_{ij}(\Delta_j^T - \Delta_j^i) + N_{ijk}\Delta_j^T\Delta_k^T - N_{ijk}\Delta_j^i\Delta_k^i + N_{ijkl}\Delta_j^T\Delta_k^T\Delta_l^T$$
$$-N_{ijkl}\Delta_j^i\Delta_k^i\Delta_l^T - \lambda P_i = 0 \qquad (3)$$

Noting now that the net displacements are $\Delta_j = \Delta_i^T - \Delta_j^i$, substituting this in Eq. (3) and collecting terms, we have

$$K_{ij}\Delta_j + N_{ijk}\Delta_j\Delta_k + N_{ijkl}\Delta_j\Delta_k\Delta_l$$
$$+ 2N_{ijk}\Delta_j^i\Delta_k + 3N_{ijkl}\Delta_j^i\Delta_k\Delta_l - \lambda P_i = 0 \qquad (4)$$

At this juncture two assumptions are made, consistent with Koiter's original development,[5] which simplify considerably the above equation. First, the term $3N_{ijkl}\Delta_j^i\Delta_k\Delta_l$ is assumed to be negligible and is discarded. Secondly, the evaluation of the term $2N_{ijk}\Delta_j^i\Delta_k$ is to be based on a <u>linearized</u> pre-buckling solution for Δ_k. Thus, we can write Δ_k as $\lambda(\Delta_k^o)^1$ where λ is the load parameter and (Δ_k^o) is the slope of the linear pre-buckling load-displacement relationship, and

$$2N_{ijk}\Delta_j^i\Delta_k = \lambda(2N_{ijk}\Delta_j^i(\Delta_k^o)^1) = -\lambda\gamma I_i \qquad (5)$$

where I_i is the component of a "load" vector representative of initial imperfections and γ is a parameter which can be adjusted to define the severity of the initial imperfections, Eq. 4 becomes

$$K_{ij}\Delta_j + N_{ijk}\Delta_j\Delta_k + N_{ijkl}\Delta_j\Delta_k\Delta_l - \lambda P_i - \lambda\gamma I_i = 0 \qquad (6)$$

It is clear that one may deal with loads which deviate slightly from those which produce bifurcation by assigning such load values directly to the vector {I}. It should also be noted that the restriction of Δ_k^o above to the linearized prebuckling state does not exclude consideration of a nonlinear prebuckling state in the total problem; the latter is included in all subsequent operations.

B. PREBUCKLING ANALYSIS

The perturbation method is based upon an expansion about the bifurcation point of the perfect structure. In the present development it is necessary to trace the fundamental path of the perfect structure, represented by the solution of Equation (2) to points beyond bifurcation. The method chosen here is that of direct iteration.

In the basic form of the direct iterative method, assume that the solution is to be obtained for a load intensity designated by λ^q. Also, assume that solution data from a prior load level (say λ^{q-1}) is available and is designated as $\{\Delta\}^o$. Thus, the matrices $[N_1]$ and $[N_2]$ may be formed using $\{\Delta\}^o$ and we may solve Equation (2) to yield

$$\{\Delta\}^1 = [K]^{-1} \{\lambda^q \{P\} - [N_1(\Delta)^o] \{\Delta^o\} - [N_2(\Delta)^o]\{\Delta\}^o\} \quad (7)$$

where the superscript 1 on $\{\Delta\}^1$ denotes the first iteration in the solution at λ^q. We then re-form $[N_1]$ and $[N_2]$ on the basis of $\{\Delta\}^1$, so that

$$\{\Delta\}^2 = [K]^{-1} \{\lambda^q \{P\} - [N_1(\Delta)^1] \{\Delta\}^1 - [N_2(\Delta)^1]\{\Delta\}^1\} \quad (8)$$

which is now solved for $\{\Delta\}^2$. In the general, j^{th}, iterative solution

$$\{\Delta\}^j = [K]^{-1} \{\lambda^q \{P\} - [N_1(\Delta)^{j-1}] \{\Delta\}^{j-1} - [N_2(\Delta)^{j-1}] \{\Delta\}^{j-1} \quad (9)$$

The iterative sequence continues until $\{\Delta\}^j$ is within $\{\Delta\}^{j-1}$ to a specified tolerance. Note that direct iteration requires only the inversion of the linear stiffness matrix and continued re-formation of $[N_1]$ and $[N_2]$.

The knowledge of a nearby solution, as for $\{\Delta\}^o$ in Equation 7, enhances the efficiency of the iterative process. Hence, the analysis is performed at various load levels, extending from a level close to zero load through to a level somewhat beyond the bifurcation load.

Convergence difficulties are encountered when the non-linearities are severe. Such difficulties are often manifested by continued iteration in a loop about the convergent solution. In such cases an improved procedure is to employ a higher-order iterative scheme as described in Reference 16.

C. DETERMINATION OF BIFURCATION

To determine the first branching from the fundamental path (the bifurcation point) we invoke the familiar stability condition that the second variation of the potential energy be zero at such a point. The equilibrium equation (Equation 2) represents the first variation of the potential energy and by applying the second variation one obtains the following condition at $\lambda = \lambda^c$

$$\text{Det} = \left| [K] + 2\,[N_1(\Delta)] + 3\,[N_2(\Delta)] \right| = 0 \qquad (10)$$

where Det symbolizes the determinant of the indicated matrix. (The factors 2 and 3 on $[N_1]$ and $[N_2]$ arise from imposition of the second variation; see Equation 2a).

To illustrate the manner in which the above condition is employed in identification of the bifurcation point Figure 2a shows a representative load-parameter-displacement (λ-Δ) plot while Figure 2b shows the corresponding variation of Det with λ. Thus, Det > 0 for $0 < \lambda < \lambda^c$ and Det < 0 for $\lambda > \lambda^c$. By establishing m solution points to either side of λ^c and by Lagrange interpolation we have

$$\lambda = \sum_{i=1}^{m} \left(\prod_{j=1,\ j=i+1}^{i-1,\ m} \frac{\text{Det}-\text{Det}_j}{\text{Det}_j-\text{Det}_i} \right) \lambda_i \qquad (11)$$

where Det_i and λ_i denote the corresponding values at the ith load level. From Eq. (11), the bifurcation load λ^c is calculated by setting Det = 0, i.e.

$$\lambda^c = \lambda \big|_{\text{Det} = 0}$$

Since the displacements and their derivatives at λ^c are needed for determination of the postbuckling path, they are also calculated by interpolation.

$$\Delta_i(\lambda) = \sum_{j=1}^{m} \left(\prod_{k=1,\ k=j+1}^{j-1,\ m} \frac{\lambda-\lambda_k}{\lambda_k-\lambda_j} \right) \Delta_i^j \qquad (12)$$

where Δ_i^j denotes the value of Δ_i at the jth load level λ_j. Then the desired displacements Δ_i^c are found by setting $\lambda=\lambda^c$. The

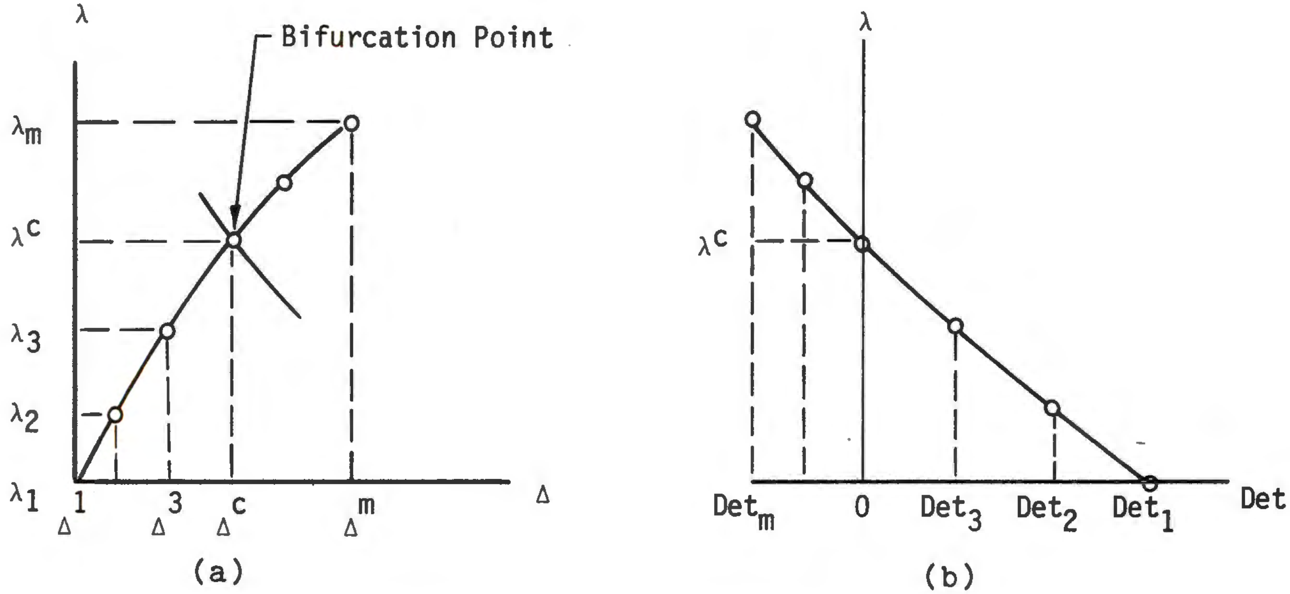


FIGURE 2. DETERMINATION OF BIFURCATION
POINT VIA INTERPOLATION

derivatives of Δ_i are found by direct differentiation of Eq. (12) with respect to λ and evaluation at λ^c.

D. POSTBUCKLING ANALYSIS

Figures 1.a and 1.b illustrate, in a representative λ-Δ space, the single post-buckling path of a perfect structure emanating from the bifurcation point. We now consider the use of the perturbation method in establishment of an analytical description of this path and, simultaneously, of the path of imperfect (or differently loaded) forms of the same structure. Thus, we direct attention to the more general equilibrium relationship (Equation 6) which accounts for these imperfections and/or load components.

In initiating the analytical description of these paths, we describe the displacement state in terms of "sliding coordinates"[8]

$$\{\Delta\} = \{\bar{\Delta}\} + \{\Delta^p\} \tag{13}$$

where now $\{\bar{\Delta}\}$ describes displacements on the fundamental path of the perfect structure and $\{\Delta^p\}$ gives the displacements on the postbuckling path with the fundamental path as a reference base. Thus, a mapping of the postbuckling behavior in λ - $\{\Delta^p\}$ space can be effected with $\{\bar{\Delta}\} = 0$.

To obtain the equilibrium equation in terms of the new coordinates, we substitute Equation (13) into Equation (6). It is convenient to revert to indicial notation with $\{\Delta\} = \Delta_j$, $\{\bar{\Delta}\} = \bar{\Delta}_j$ and $\{\Delta^p\} = \Delta^p_j$, since in effecting the products $\Delta_j\Delta_k$ and $\Delta_j\Delta_k\Delta_l$ the term $\Delta_j = (\bar{\Delta}_j + \Delta^p_j)$ can be treated as a binomial in conventional manner. Thus, we have

$$\begin{aligned} K_{ij}\bar{\Delta}_j + K_{ij}\Delta^p_j &+ N_{ijk}(\bar{\Delta}_j + \Delta^p_j)(\bar{\Delta}_k + \Delta^p_k) \\ &+ N_{ijkl}\,(\bar{\Delta}_j + \Delta^p_j)(\bar{\Delta}_k + \Delta^p_k)(\bar{\Delta}_l + \Delta^p_l) = \lambda P_i + \lambda\gamma I_i \end{aligned} \tag{14}$$

and, expanding and collecting multipliers of Δ^p_j and noting that a group of the resulting terms satisfies Equation (2), we obtain the equilibrium equation in the new coordinates

$$(K_{ij} + 2N_{ijk}\,\overline{\Delta}_k + N_{ijkl}\,\overline{\Delta}_k\,\overline{\Delta}_l)\Delta_j^p$$

$$+ (N_{ijk} + 3N_{ijkl}\,\overline{\Delta}_l)\,\Delta_j^p\,\Delta_k^p + N_{ijkl}\,\Delta_j^p\,\Delta_k^p\,\Delta_l^p = \lambda\gamma I_i \qquad (15)$$

An analytical representation of the fundamental path must now be established. Thus, by Taylor series expansion about the bifurcation point

$$\overline{\Delta} = \{\overline{\Delta}^c\} + (\lambda-\lambda^c)\{\overline{\Delta}'^c\} + \frac{1}{2}(\lambda-\lambda^c)^2\{\overline{\Delta}''^c\} + \ldots \qquad (16)$$

and, by substitution into Equation (15)

$$[(K_{ij} + 2N_{ijk}\,\overline{\Delta}_k^c + 3N_{ijkl}\,\overline{\Delta}_k^c\,\Delta_l^c) + (2N_{ijk}\,\overline{\Delta}_k'^c + 6N_{ijkl}\,\overline{\Delta}_k'^c\overline{\Delta}_l^c)(\lambda-\lambda^c)$$
$$+ (N_{ijk}\,\overline{\Delta}_k''^c + 3N_{ijkl}(\overline{\Delta}_k'^c\,\overline{\Delta}_l'^c + \overline{\Delta}_k^c\,\overline{\Delta}_l''^c)\,(\lambda-\lambda^c)^2]\Delta_j^p$$
$$+ [(N_{ijk} + 3N_{ijkl}\,\overline{\Delta}^c{}_l)+(3N_{ijkl}\overline{\Delta}_l^c(\lambda-\lambda^c))]\Delta_j^p\,\Delta_k^p + N_{ijkl}\Delta_j^p\,\Delta_k^p\,\Delta_l^p = \lambda\gamma I_i \qquad (17)$$

where the indicated result is obtained by truncation of the series (Equation 16) at the third term and discard of terms higher than third order in Δ_i^p or in the product Δ_i^p and $(\lambda-\lambda^c)$.

It is assumed that the postbuckling path can be described by the series

$$\begin{Bmatrix} \Delta_1^p \\ \Delta_2^p \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \Delta_n^p \end{Bmatrix} = \Delta_1^p \begin{Bmatrix} 1 \\ q_{2_1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ q_{n_1} \end{Bmatrix} + (\Delta_1^p)^2 \begin{Bmatrix} 0 \\ q_{2_2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ q_{n_2} \end{Bmatrix} + (\Delta_1^p)^3 \begin{Bmatrix} 0 \\ q_{2_3} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ q_{n_3} \end{Bmatrix} + \ldots \qquad (18)$$

i.e., $\{\Delta^p\} = \Delta_1^p\,\{q_1\} + (\Delta_1^p)^2\,\{q_2\} + (\Delta_1^p)^3\{q_3\} + \ldots \qquad (19)$

or, in indicial form

$$\Delta_i^p = \Delta_1^p\,q_{i_1} + (\Delta_1^p)^2\,q_{i_2} + (\Delta_1^p)^3\,q_{i_3} + \ldots \qquad (19a)$$

where each vector ($\{q_1\}$,$\{q_2\}$, etc.) is a mode of displacement to be determined as described below, and Δ_1^p, the "path parameter", is a pre-selected degree-of-freedom. Equation (18) designates this parameter as the first-listed degree-of-freedom. This can be done without loss of generality since it is always possible to arrange the equations so that a chosen degree-of-freedom (generally, the displacement at a prominent point) appears in the first location.

An expansion of the load parameter in the post-buckling regime is also required and is given in terms of the path parameter.

$$\lambda-\lambda^c = \Gamma_1 (\Delta_1^p) + \Gamma_2 (\Delta_1^p)^2 + \Gamma_3 (\Delta_1^p)^3 + \ldots \tag{20}$$

where the load parameters Γ_1, Γ_2, Γ_3, .. are as yet unknown. We next substitute (19a) and (20) into (17), write the imperfection term $\lambda\gamma I_1$ as $\frac{\lambda\gamma I_1}{(\Delta_1^p)^2} \times (\Delta_1^p)^2$, and collect terms in like powers of Δ_1^p. We obtain, for $\Delta_1^p \neq 0$

$$\bar{K}_{ij}\, q_{j_1} = 0 \tag{21a}$$

$$(\bar{K}_{ij}\, q_{j_2} + \bar{K}'_{ij}\, \Gamma_1\, q_{j_1} + \bar{N}_{ijk}\, q_{j_1}\, q_{k_1}) = \frac{\lambda\gamma I_i}{(\Delta_1^p)^2} \tag{21b}$$

$$(\bar{K}_{ij}\, q_{j_3} + \bar{K}'_{ij}(\Gamma_2\, q_{j_1} + \Gamma_1\, q_{j_2}) + \bar{K}''_{ij}\, \Gamma_1^2\, q_{j_1} + 2\bar{N}_{ijk} q_{j_1}\, q_{k_2}$$
$$+ \bar{N}'_{ijk}\, \Gamma_1\, q_{j_1}\, q_{k_1} + N_{ijkl}\, q_{j_1}\, q_{k_1}\, q_{l_1}) = 0 \tag{21c}$$

The initial imperfection term is accounted for in the manner indicated since it would have no effect if represented as a constant and would disallow the overall procedure of definition of the postbuckling path if represented as linear in Δ_1^p. Thus, a quadratic representation in Δ_1^p is chosen.

Now, from (21a) it is clear that $\{q_1\}$ is the eigenvector of $[\bar{K}_{ij}]$, normalized on the term corresponding to Δ_1^p. The value of Γ_1 is obtained by solution of an equation resulting from the premultiplication of Eq. (21b) by $\{q_1\}$, and then $\{q_2\}$ is obtained from Eq. (21b) after back-substitution of the expression for Γ_1. The values of Γ_2 and $\{q_3\}$ are determined by similar operations on Eq. (21c). If additional terms are taken in the above series the corresponding load parameters and postbuckling displacement vectors are also obtained in this manner.

Space limitations do not allow here the presentation of details of this procedure, or of the specific form of the results in terms of the basic quantities $\overline{K}_{ij}$, etc. This information is given in Reference 16. It may be of interest, however, to cite a single term of the series Eq. (20), the post-buckling load-displacement parameter relationship. Thus, for Γ_1 it is found that

$$\Gamma_1 = \frac{\overline{N}_{ijk}\; q_{i_1} q_{j_1} q_{k_1}\; (\Delta_1^p)^2 + \lambda\gamma I_i\; q_{i_1}}{\overline{K}'_{ij}\; q_{i_1} q_{j_1}\; (\Delta_1^p)^2} \tag{22}$$

Note that when the initial imperfection is zero (represented by $\gamma = 0$), a nonzero value of Γ_1 remains, defining the postbuckling path of the perfect structure.

E. ILLUSTRATIVE EXAMPLES

The present section is devoted to a general description of numerical results for three illustrative examples, involving a shallow arch, a plate, and a curved thin shell structure, respectively. Although the problems solved are elementary from the standpoint of finite element representation, they delineate all features of the more complex situations and are among the few cases which have been studied thoroughly and for which comparison solutions or test data are available. Such comparisons are essential to a study addressed to a class of problems for which a multitude of alternative procedures have only recently been proposed.

1. Clamped Thin Shallow Circular Arch

The problem of instability of the thin shallow circular arch with clamped ends has drawn much attention in the literature of geometrically nonlinear and postbuckling analysis because it is perhaps the most sophisticated structure for which "exact" solutions have been obtained(13,17). The objectives of this illustrative example are to verify accuracy in determination of the bifurcation point following upon a nonlinear fundamental path and to demonstrate prediction of behavior for both perfect and imperfect forms of the arch by use of the present perturbation method.

The geometry of the arch (see Figure 3) is characterized by the parameter $R\Theta_o^2/h$ and this parameter also governs in part the

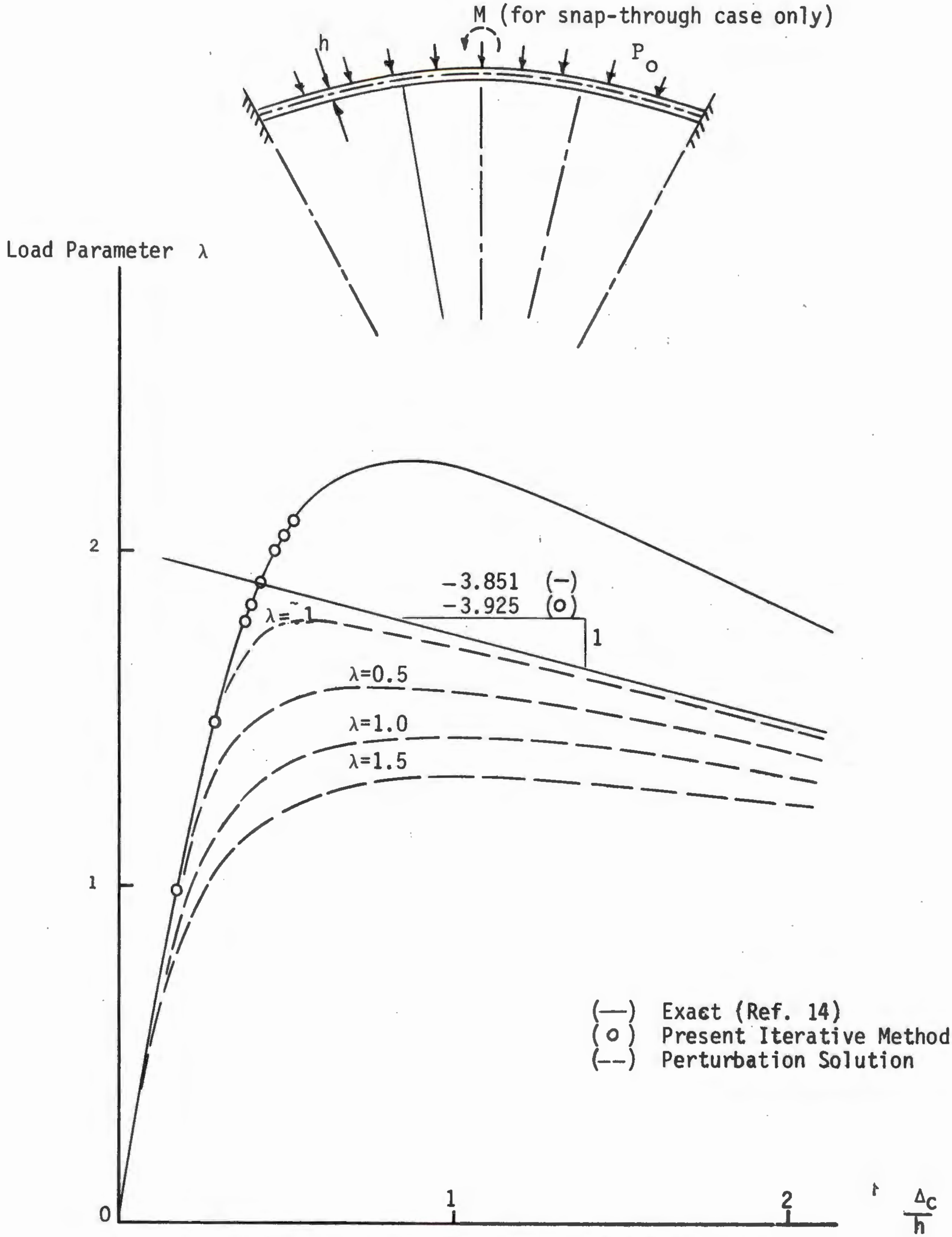


FIGURE 3. LOAD VS. CENTRAL DEFLECTION FOR PRE BUCKLING, POST BUCKLING AND IMPERFECT BEHAVIORS

form of buckling, i.e., snap-through or bifurcation. The dimensions chosen here yield a value of 10.0, the same value employed in Reference 13. The finite element representation consists of eight equally-spaced arch elements.

In the case of uniform radial loading of intensity p_o, bifurcation occurs prior to snap-through as illustrated in Figure 3 by solid lines for the classical solution. The direct iterative scheme discussed in Section C is used to yield the solution points on the fundamental path as given by the circled points. Lagrangian interpolation gives the bifurcation load $\lambda^c = 1.9075$, which is within 0.2% of the exact value. The numbers of iterative cycles to achieve convergence at each load level for various specified convergence criteria are plotted in Fig. 4. It is of interest to note that near the bifurcation point the number of cycles increases sharply, but, that monotonic convergence is observed at all load levels above or below the bifurcation point.

2. Flat Plate Post-Buckling

The next example refers to the post-buckling behavior of a flat, simply-supported rectangular plate under uniform axial compression. The element employed in these calculations is doubly-curved shell element portrayed in Figure 5, whose properties are based upon a 16-term (bicubic) expansion of each of the displacement components u, v and w. This element is a generalization of the cylindrical shell element introduced by Bogner, et al[18]. Formulation of the present representation is described in Reference 19. For this problem the principal radii of curvature are set equal to infinity.

The problem data are shown in Figure 6; due to symmetry, the analysis is performed with a single element in one quadrant. It should be noted that the analytical model permits freedom of inplane displacement; the usual restriction of classical solutions[20] to linear edge displacement states, etc. are not invoked.

The bifurcation point, calculated in a linear stability analysis (the pre-buckling state is zero in the transverse displacement) is found to be 36.3 lb./in. compared to the classical result[21] of 36.1 lb./in. (0.5% error). The postbuckling path calculated by the present perturbation method is shown via a dashed line in

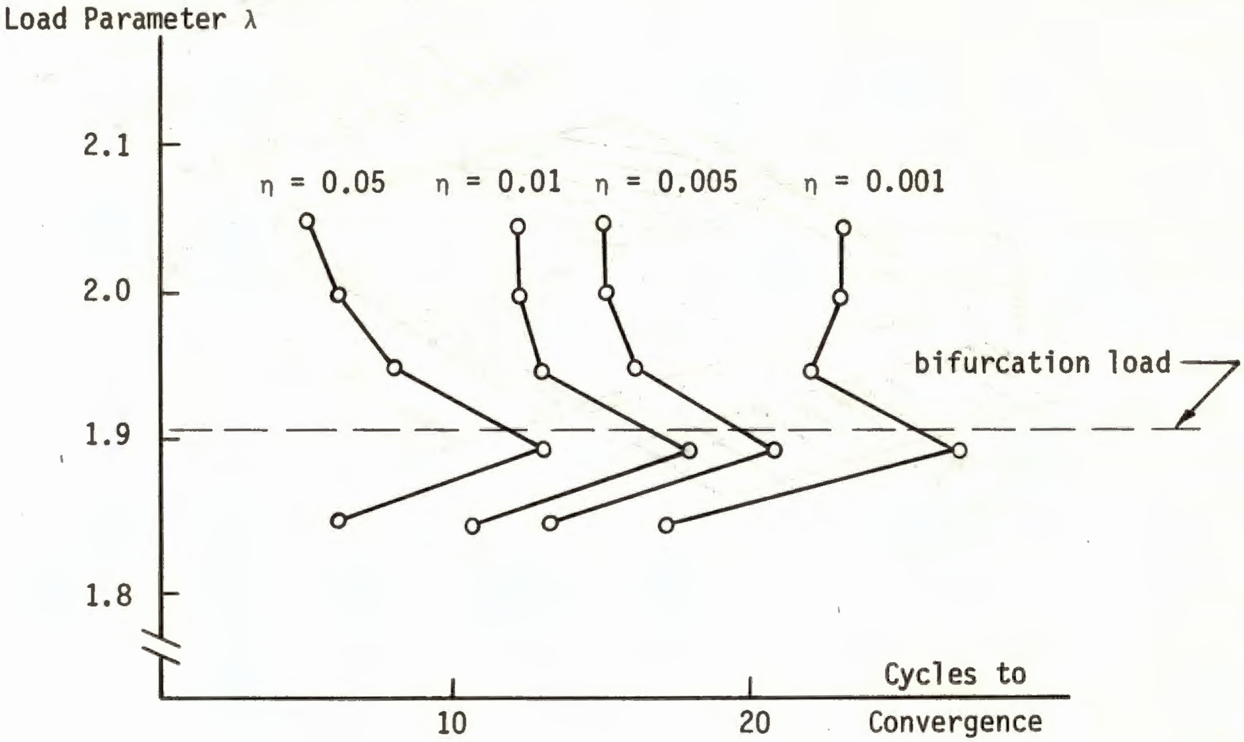


FIGURE 4. NUMBER OF ITERATIONS FOR CONVERGENCE VS. LOAD LEVEL FOR VARIOUS CONVERGENCE CRITERIA (η)

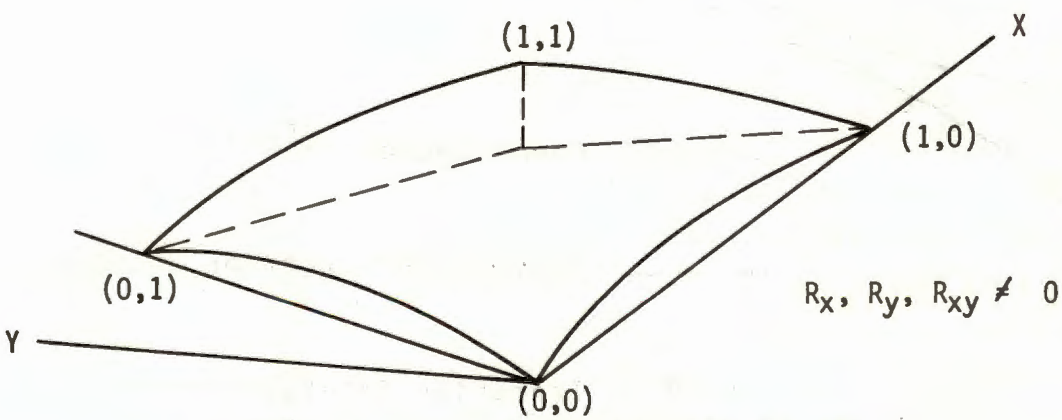


FIGURE 5. GENERAL QUADRILATERAL SHELL ELEMENT (DEFINED IN ISOPARAMETRIC COORDINATES)

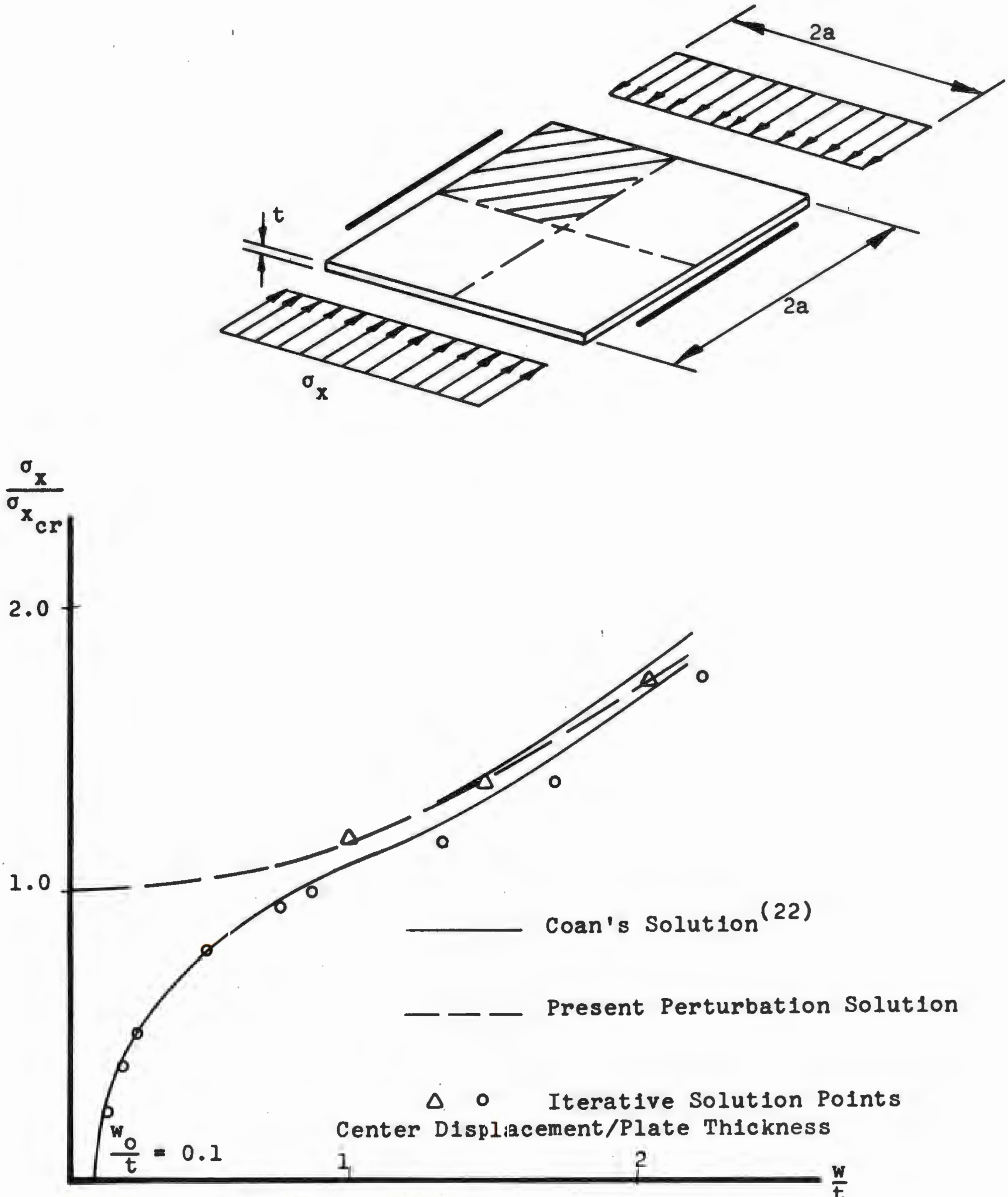


FIGURE 6. FLAT PLATE POSTBUCKLING BEHAVIOR

Fig. 6 and is compared with the solution by Coan[22] who also permits unrestricted inplane displacement. The two solutions are in close agreement. These results were checked by performance of an iterative solution of Equation 2 at applied load levels $\sigma_x/\sigma_{x_{cr}}$ = 1.193, 1.385, and 1.748. These solutions, designated by triangular symbols in Fig. 6, confirm the range of validity of the perturbation solution.

The analysis of the plate with initial geometric imperfections of the form $w^1 = w^o \sin \pi\xi \sin \pi\eta$, with w^o the central displacement and $\xi = x/a$, $\eta = y/b$, was also attempted. For this case it is convenient to effect an iterative solution of Eq. 3, rather than to apply perturbation concepts. The solution points of this procedure are shown circled in Figure 6. The solution by Coan[22] for the same data is compared and good agreement is again indicated.

3. Cylindrical Shell Under Uniform Load

As a final example we consider the clamped circular cylindrical panel subjected to uniform lateral load, whose properties are described in Fig. 7. This structure has been analyzed in several studies, but only Morin[10] defines bifurcation and analyzes for higher load levels. The present computation is performed with 4 elements of the type shown in Fig. 5 (2x2 grid in a quadrant). Again, direct iterative solution is employed in preference to the perturbation method. Results are shown in Fig. 7.

The displacements of the linear analysis were chosen as the initial guess for the first load level. Subsequent initial guesses were obtained by multiplying the displacements of the previous load level by the ratio of the current to the prior load levels. It is interesting to note that the lowest determinant in the present analysis occurs at λ = 0.230 p.s.i., compared to the 0.223 p.s.i. calculated by Morin.[10] The present results and those of Refs. 23 and 24 compare well and are collectively in significant disparity with those of Ref. 10.

F. CONCLUDING REMARKS

A procedure for finite element analysis of geometrically nonlinear problems, covering the pre-buckling and initial post-

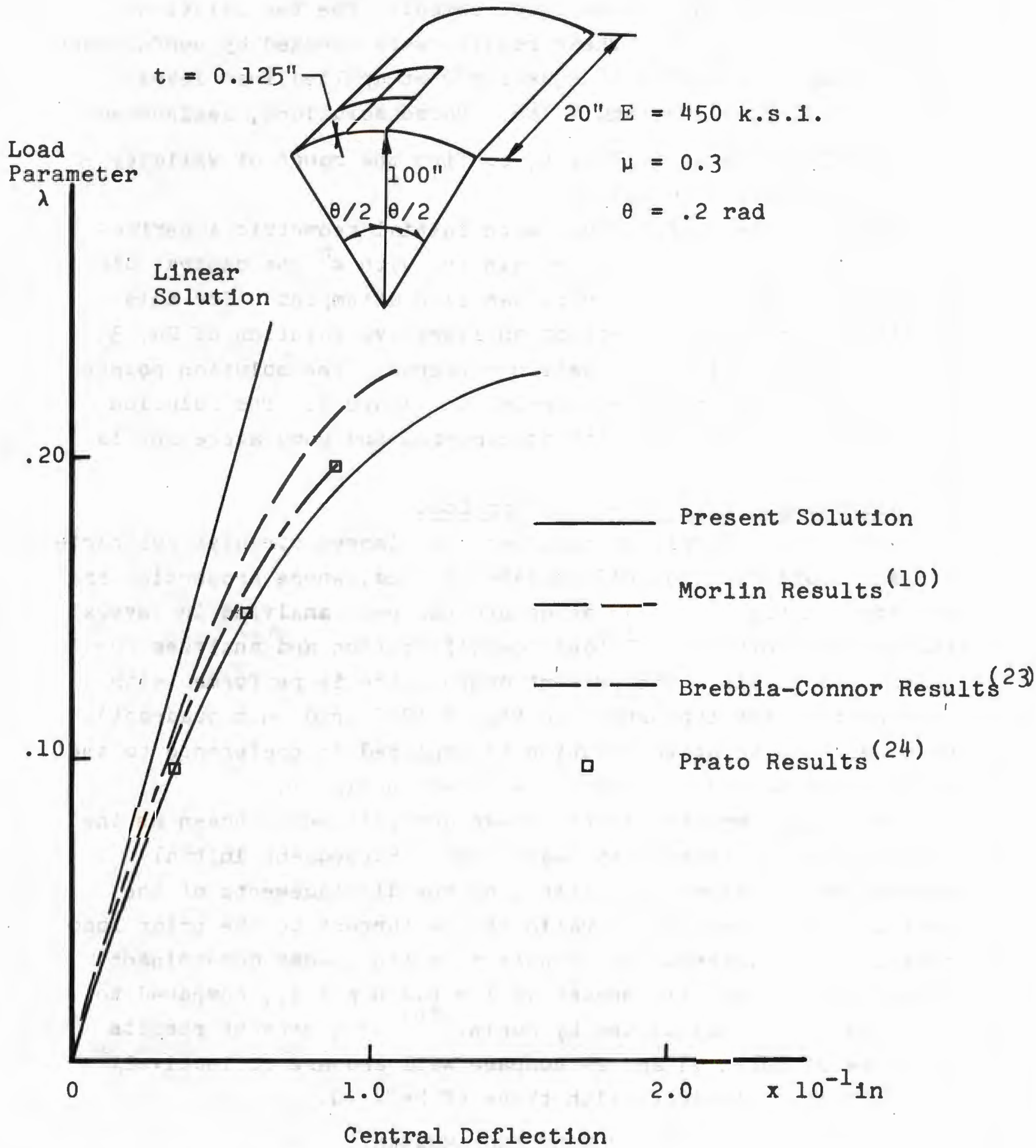


FIGURE 7. LOAD-DEFLECTION RESPONSE. CLAMPED CIRCULAR CYLINDRICAL SHELL UNDER UNIFORM PRESSURE LOAD.

buckling regimes, snap-through buckling, and accounting for initial imperfections, has been presented. The initial post-buckling and snap-through aspects of this procedure correspond closely to Thompson's approach[9], differing in detailed application to the limit points of imperfection-sensitive structures where the present method is believed to be computationally more efficient.

The present development applies only to the case of a single post-buckling path. Certain classical situations are known to possess multiple post-buckling paths; these stem from linear pre-buckling analyses. Even if analysis based on the more realistic representation of a nonlinear pre-buckling state discloses only a single post-buckling path at the first critical point, a serious problem may arise due to the closeness of adjacent critical points. It does not appear that the present approach is capable of dealing with this situation, due to the restriction of its validity, as an asymptotic theory, to behavior in the vicinity of the critical point.

In an alternative approach developed by the writers, and also detailed in Reference 16, none of the limitations of an asymptotic theory are present. This approach is based upon an extrapolation of determinants of the equilibrium equation and applies to limit point problems. The computational costs are higher than for a perturbation approach, however, and the reliability of an extrapolation is always in question.

The relative efficiency and accuracy of the subject procedures and various alternative approaches (e.g., references 3, 9, 10) remain open questions. Since analysts cannot agree upon the optimum procedures in the restricted area of nonlinear pre-buckling analysis (see Reference 25) it is unrealistic to expect the definition of the most appropriate approach to post-buckling analysis at the present time. These measures will be obtained only after significant experience in practical application is recorded.

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