AFFDL-TR-68-150

FINITE ELEMENT ANALYSIS OF SANDWICH PLATES AND CYLINDRICAL SHELLS WITH LAMINATED FACES

G. R. Monforton**

and

L. A. Schmit, Jr.***

Case Western Reserve University
Cleveland, Ohio

A linear finite element capability for predicting displacements, stresses, and natural frequencies of sandwich plates and cylindrical shells with unbalanced laminated faces is reported. The geometric admissibility conditions of the principle of minimum total potential energy are conveniently satisfied by representing the displacement variables in terms of assumed displacement patterns formed by the sum of products of one-dimensional Hermite interpolation polynomials. Stiffness and consistent mass matrices for small displacements are presented in terms of the element geometry, the stiffnesses of the faces (membrane, bending, and coupling) and the transverse shear stiffnesses of the orthotropic core. Specialization for the analysis of thin laminated plates and cylindrical shells is achieved by simply considering one face of the sandwich. Several numerical examples are presented and comparison is made with existing theoretical and experimental results.

---

*This research was sponsored by the Air Force Flight Dynamics Laboratory under Contract AF 33 (615)-3432.

**Graduate Assistant; now Assistant Professor, Civil Engineering Department, University of Windsor, Windsor, Ontario, Canada.

***Professor of Engineering.

SECTION I

INTRODUCTION

Recently, composite materials have been developed and design methods and concepts are evolving in order to incorporate these new materials efficiently in structural designs. It appears that these new high-strength and low-density materials have great potential in both solid laminate and sandwich construction. A particularly interesting phenomenon that arises in unbalanced laminated construction* is coupling between extensional and flexural action. This type of behavior necessitates a greater emphasis on anisotropic and transversely heterogeneous structural analyses as a means of predicting the behavior of these new structural types.

For the lightweight structures such as the sandwich and laminated structures now encountered in aeronautical design practice, finite element methods should prove to be valuable analysis tools. The accuracy of a finite element method is directly dependent on the ability of the element deformation patterns to approach the actual deformation state of the structural system. In a potential energy formulation it has become generally accepted that the element displacement functions should be such that they can satisfy the geometric admissibility conditions. A numerical solution based on admissible displacement states yields an upper bound to the true minimum of the potential energy. Furthermore, if the finite element modeling is sequentially refined using elements of the same type and the refinement contains the previous modeling, convergence of the sequence of upper bounds is monotonic. It is important that the sequence of upper bounds converges to the exact minimum of the potential energy in the limit; this condition is assured if it can be shown that the sequence of displacement functions generated by successive refinements results in a ''complete'' sequence. The problem of demonstrating completeness has been dealt with by various authors (References 1 and 2), and is currently an active research area. It is also pointed out that the element displacement states should be able to represent the rigid body modes of the structural system, in the sense that rigid body displacements produce very little strain energy, even for relatively coarse modelings.

*The term ''unbalanced laminated construction'' is used to describe a section where the lamina are placed unsymmetrically (elastically and/or geometrically) about the middle surface.

A rectangular thin plate element which satisfies all the above mentioned criteria was developed in Reference 3; the formulation was achieved using displacement patterns formed by the sum of products of one-dimensional first-order Hermite interpolation polynomials and undetermined nodal coefficients. Subsequently the method was extended to a cylindrical shell element (Reference 4) and to a skew plate element (Reference 5). Following the same approach, this paper reports on a finite element capability for sandwich plates and cylindrical shells. The faces are considered to be thin shells and may be composed of an arbitrary number of bonded layers, each of which may have different thickness, linear elastic anisotropic material properties, and orientation of elastic axes. The orthotropic core considered is typical of that used in honeycomb sandwich construction. It is assumed that displacements are small and that the transverse deflection is uniform through the thickness of the sandwich. The strain energy of the composite sandwich system is taken to be a collection of the following:

(a) the strain energy due to membrane action of the faces in their reference planes,

(b) the strain energy due to bending of the faces,

(c) the strain energy due to linear coupling between membrane and bending action in the faces,

(d) the strain energy due to transverse shearing of the core.

The strain energies due to transverse shearing of the faces and due to face-parallel deformations in the core are considered negligible.

The finite element method reported is formulated in terms of the shell geometry and the stiffnesses of the faces and core. The displacement behavior is described by the four face-parallel displacements of the skins and the transverse displacement w. This choice of displacement variables admits transverse shear deformations in the core and allows for flexibility in selecting realistic boundary conditions. For example, it is possible to impose membrane displacement boundary conditions on one face while allowing the other face to satisfy natural or imposed force boundary conditions. Furthermore, this choice of displacement variables is such that the analysis of thin anisotropic and transversely heterogeneous plates and cylindrical shells is a special case obtained by simply considering one face of the sandwich.

Using the principle of minimum potential energy as a base, element stiffness and mass matrices (including the effects of rotary inertia) are developed and several numerical examples are given. Solutions are obtained by standard numerical methods or alternatively by direct energy minimization using a scaled conjugate gradient (References 6 and 7) method. Finally, the energy search concept is adapted to predicting the response of structural systems subject to the influence of destabilizing loads; it is noted that this adaptation is analogous to the ''incremental stiffness matrix method'' and that the buckling load associated with a linear eigenvalue formulation is approached asymptotically by the load-displacement curve.

SECTION II

ENERGY FORMULATION

The discussion in this section is focused on presenting expressions for the strain energy, the kinetic energy and the potential of the applied loads for sandwich plates and cylindrical shells. The structural model consists of two anisotropic faces which may be formed from an arbitrary number of bonded layers and an orthotropic core representative of honeycomb sandwich construction. A portion of a cylindrical sandwich shell is shown in Figure 1. The subscripts 1 and 2 are used to denote quantities associated with the inner face and outer face respectively; the subscript c is used to identify quantities pertaining to the core. A general subscript s is used to denote quantities that refer to all three layers (i.e. inner face, outer face and core) while the subscript f is used when attention is limited to the faces. The individual reference surfaces of the skins are located arbitrarily at distances $d_f (f = 1,2)$ from the interfaces between the core and skins. The reference surface of the core is taken to coincide with its middle surface. Thicknesses and radii of the faces and core are t_s and $R_s (s = 1,2,c)$, respectively. Note that distances measured on the individual reference surfaces in the θ direction are represented by $y_s = R_s \theta$ in the cylindrical shell.

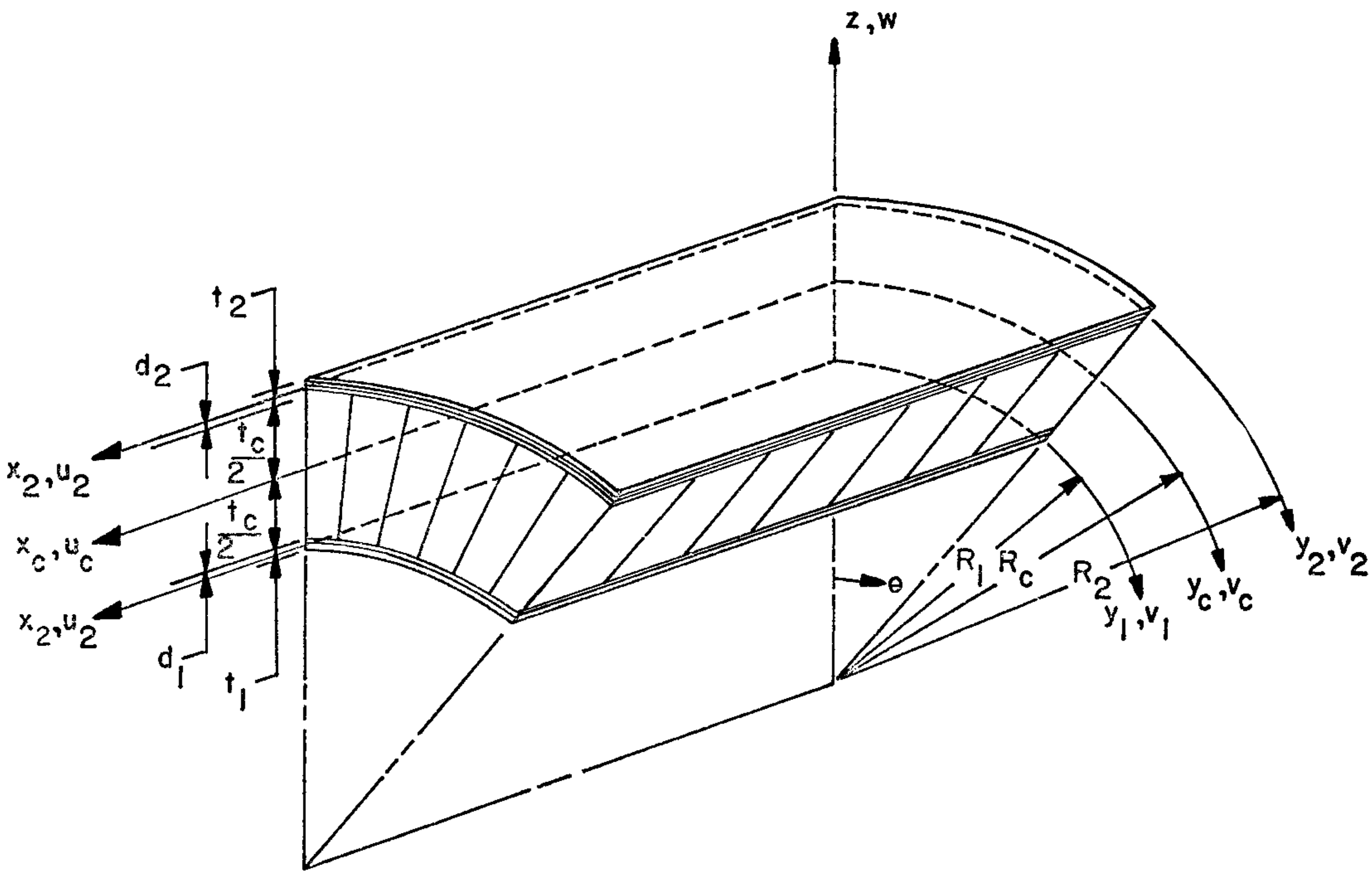


Figure 1. Laminated Sandwich Cylindrical Shell Segment

FACE CONSIDERATIONS

The faces of the sandwich system are considered to be thin anisotropic shells; the usual assumptions of linear shell theory, including the Kirchhoff-Love hypothesis, are retained. The strain-displacement relations for the faces of a sandwich cylinder are represented by

$$\begin{bmatrix} \epsilon_{fx} \\ \epsilon_{fy} \\ \gamma_{fxy} \end{bmatrix} = \begin{bmatrix} \epsilon_{fx}^{o} \\ \epsilon_{fy}^{o} \\ \gamma_{fxy}^{o} \end{bmatrix} + z \begin{bmatrix} \kappa_{fx} \\ \kappa_{fy} \\ \kappa_{fxy} \end{bmatrix} \tag{1}$$

where ϵ_{fx}^{o}, ϵ_{fy}^{o}, γ_{fxy}^{o} are the reference surface strains and κ_{fx}, κ_{fy}, κ_{fxy} are the changes in curvature. Note that z_f is measured along a normal from the reference surface. The expressions for the reference surface strains and curvatures are related to the displacements by

$$\epsilon_{fx}^{o} = u_{fx} \quad ; \quad \epsilon_{fy}^{o} = v_{fy} + \frac{w_f}{R_f} \quad ; \quad \gamma_{fxy}^{o} = v_{fx} + u_{fy} \tag{2}$$

$$\kappa_{fx} = -w_{fxx} \quad ; \quad \kappa_{fy} = -w_{fyy} + \frac{1}{R_f} v_{fy} \; ; \; \kappa_{fxy} = -2\left(w_{fxy} - \frac{1}{R_f} v_{fx}\right) \tag{3}$$

In the above expressions, the notations

$$u_{fx} = \frac{\partial u_f}{\partial x} , \; u_{fy} = \frac{\partial u_f}{\partial y_f} = \frac{1}{R_f} \frac{\partial u_f}{\partial \theta} \quad \text{etc.}$$

has been adopted for convenience. The corresponding strain-displacement relations for the faces of a sandwich plate are obtained from Equations 2 and 3 by setting $y_f = y$ and $\frac{1}{R_f} = 0$.

As previously mentioned, the faces may be of laminated construction. Each ply in the face may have different homogeneous anisotropic material properties, orientation of elastic axes and thickness. For example, such faces are typical of filamentary composite construction in which the piles are assumed to be homogeneous and orthotropic. The stress-strain law for an individual lamina within a face is represented by

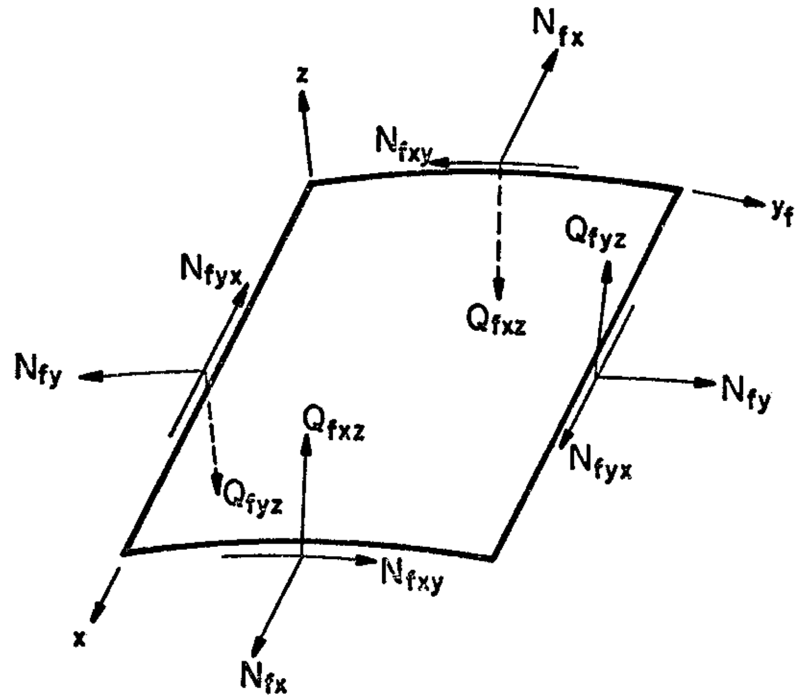
$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{16} \\ c_{21} & c_{22} & c_{26} \\ c_{61} & c_{62} & c_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{4}$$

The elastic coefficients $c_{ij} = c_{ji}$ (i, j = 1, 2, 6) in Equation 4 depend on both the elastic properties of the lamina referred to a set of material axes and the orientation of these axes with respect to the reference coordinates of the sandwich system. In general, the elastic coefficients will be different for each lamina making up the face; therefore, the face is only piecewise homogeneous through the thickness.

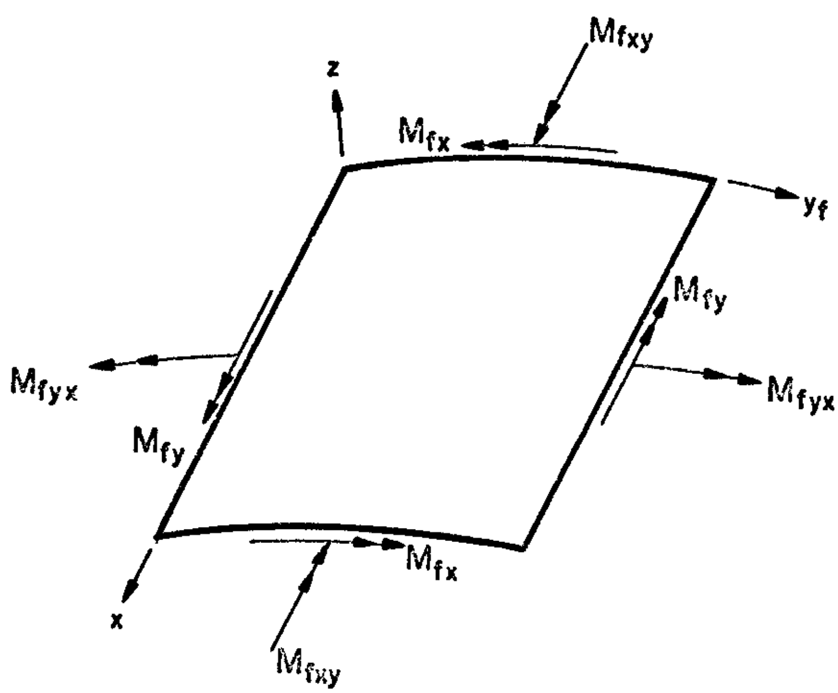
The force-deformation relations for the faces can be expressed (Figure 2).

$$\begin{bmatrix} N_{fx} \\ N_{fy} \\ N_{fxy} = N_{fyx} \\ \hline M_{fx} \\ M_{fy} \\ M_{fxy} = M_{fyx} \end{bmatrix} = \left[\begin{array}{ccc|ccc} A_{f11} & A_{f12} & A_{f16} & B_{f11} & B_{f12} & B_{f16} \\ & A_{f22} & A_{f26} & B_{f12} & B_{f22} & B_{f26} \\ & & A_{f66} & B_{f16} & B_{f26} & B_{f66} \\ \hline & & & D_{f11} & D_{f12} & D_{f16} \\ & \text{(SYMMETRIC)} & & & D_{f22} & D_{f26} \\ & & & & & D_{f66} \end{array}\right] \begin{bmatrix} \epsilon^o_{fx} \\ \epsilon^o_{fy} \\ \gamma^o_{fxy} \\ \hline \kappa_{fx} \\ \kappa_{fy} \\ \kappa_{fxy} \end{bmatrix} \quad (5)$$

The elements A_{fij}, B_{fij}, D_{fij}($f = 1,2; i,j = 1,2,6$) in the force-deformation equations represent the membrane, coupling and bending stiffnesses of the laminated faces. Nonzero values of the coupling stiffnesses are characteristic of unbalanced laminated construction; the B_{fij} vanish when the lamina are placed symmetrically about the middle surface and $d_f = \frac{1}{2} t_f$. Various methods have been proposed for calculating the stiffnesses for laminated structures. The discussions that follow are based on the premise that the stiffnesses of the faces can be obtained by theoretical (Reference 8), semi-emperical (Reference 9) or experimental (Reference 10) methods. In this way the formulation presented herein is not bound to any specific micromechanics theory, although it is bound to the Kirchoff-Love conditions.



(a) Force Resutants



(b) Moment Resultants

Figure 2. Face Force and Moment Resultants

The strain energy of a face can be expressed in terms of the displacement components and the various stiffnesses as

$$
\begin{aligned}
U_f = \frac{1}{2}\int_{S_f} \Big\{ \Big[& A_{f11}\, u_{fx}^2 + A_{f22}\, v_{fy}^2 + A_{f66}\,(u_{fy}+v_{fx})^2 + 2A_{f12}\, u_{fx}\, v_{fy} \\
& + 2A_{f16}\,(u_{fy}+v_{fx})\,u_{fx} + 2A_{f26}\,(u_{fy}+v_{fx})\,v_{fy} \Big] \\
& - 2\Big[B_{f11}\, u_{fx}\, w_{fxx} + B_{f22}\, v_{fy}\, w_{fyy} + 2B_{f66}\,(u_{fy}+v_{fx})\, w_{fxy} \\
& + B_{f12}\,(v_{fy}\, w_{fxx} + u_{fx}\, w_{fyy}) + B_{f16}\,(u_{fy}+v_{fx})\, w_{fxx} \\
& + 2B_{f16}\, u_{fx}\, w_{fxy} + B_{f26}\,(u_{fy}+v_{fx})\, w_{fyy} + 2B_{f26}\, v_{fy}\, w_{fxy} \Big] \\
& + \Big[D_{f11}\, w_{fxx}^2 + D_{f22}\, w_{fyy}^2 + 4D_{f66}\, w_{fxy}^2 + 2D_{f12}\, w_{fxx}\, w_{fyy} \\
& + 4D_{f16}\, w_{fxx}\, w_{fxy} + 4D_{f26}\, w_{fyy}\, w_{fxy} \Big] \\
& + \frac{1}{R_f}\Big[\frac{A_{f22}}{R_f}\, w_f^2 + 2\Big(A_{f22} + \frac{B_{f22}}{R_f}\Big) w_f\, v_{fy} \\
& + \Big(\frac{D_{f22}}{R_f} + 2B_{f22}\Big) v_{fy}^2 - 2B_{f22}\, w_f\, w_{fyy} - 2D_{f22}\, v_{fy}\, w_{fyy} \\
& + 4\Big(\frac{D_{f66}}{R_f} + B_{f66}\Big) v_{fx}^2 + 4B_{f66}\, v_{fx}\, u_{fy} - 8D_{f66}\, v_{fx}\, w_{fxy} \\
& + 2A_{f12}\, w_f\, u_{fx} - 2B_{f12}\, w_f\, w_{fxx} + 2B_{f12}\, u_{fx}\, v_{fy} - 2D_{f12}\, v_{fy}\, w_{fxx} \\
& + 4B_{f16}\, u_{fx}\, v_{fx} - 4D_{f16}\, v_{fx}\, w_{fxx} + 2A_{f26}\, w_f\, u_{fy} - 4B_{f26}\, w_f\, w_{fxy} \\
& + 2\Big(A_{f26} + 2\frac{B_{f26}}{R_f}\Big) w_f\, v_{fx} + 2B_{f26}\, u_{fy}\, v_{fy} - 4D_{f26}\, v_{fy}\, w_{fxy} \\
& + 2\Big(2\frac{D_{f26}}{R_f} + 3B_{f26}\Big) v_{fx}\, v_{fy} - 4D_{f26}\, v_{fx}\, w_{fyy} \Big]\Big\}\, dS_f
\end{aligned}
\tag{6}
$$

where S_f is the reference surface area of the face. Note that the strain energy expression for the faces of a sandwich plate are obtained directly from Equation 5 by setting $y_f = y$ and $\frac{1}{R_f} = 0$.

The kinetic energy of a face can be expressed as

$$T_f = \frac{1}{2}\int_{S_f}\left\{Q_f\left[\dot{u}_f^2 + \dot{v}_f^2 + \dot{w}_f^2\right] - 2J_f\left[\dot{u}_f\ \dot{w}_{fx} - \dot{v}_f\left(\dot{w}_{fy} - \frac{\dot{v}_f}{R_f}\right)\right]\right. \tag{7}$$

$$\left. + I_f\left[\dot{w}_{fx}^2 + \left(\dot{w}_{fy} - \frac{\dot{v}_{fy}}{R_f}\right)^2\right]\right\} d\,S_f$$

where differentiation with respect to time is represented by a dot. In Equation 7, Q_f, J_f, and I_f are inertia constants defined by

$$\left\{Q_f,\ J_f,\ I_f\right\} = \int_{-d_{f1}}^{d_{f2}} \left\{1,\ z_f,\ z_f^2\right\} \rho_f\,(z_f)\,d z_f \tag{8}$$

where $d_{11} = t_1 - d_1$, $d_{12} = d_1$ for the inner face (f = 1)

$d_{21} = d_2$, $d_{22} = t_2 - d_2$ for the outer face (f = 2)

$\rho_f(z_f)$ is the mass density of the face and may be a step function through the thickness in order to account for the transverse heterogeneity of the face. Note that Q_f is associated with the translatory inertia terms and I_f with the rotary inertia contributions; J_f is associated with coupling between translatory and rotary inertia and vanishes for balanced laminated faces for which $d_f = \frac{1}{2}\ t_f$.

The potential of the applied loads W_f acting on a face is represented by

$$W_f = \int_{S_f}\left[\bar{p}_{fx}\ u_f + \bar{p}_{fy}\ v_f + \bar{p}_{fz}\ w_f\right] d\,S_f$$

$$+ \oint_{y_f}\left[\bar{N}_{fx}\ u_f + \bar{N}_{fxy}\ v_f + \bar{Q}_{fxz}\ w_f - \bar{M}_{fx}\ w_{fx} - \bar{M}_{fxy}\left(w_{fy} - \frac{v_f}{R_f}\right)\right] dy_f$$

$$-\oint_{x}\left[\bar{N}_{fyx}\ u_f + \bar{N}_{fy}\ v_f + \bar{Q}_{fyz}\ w_f - \bar{M}_{fyx}\ w_{fx} - \bar{M}_{fy}\left(w_{fy} - \frac{v_f}{R_f}\right)\right] dx \tag{9}$$

where $\bar{P}_{fx}$, $\bar{P}_{fy}$, $\bar{P}_{fz}$ are applied reference surface tractions in the x, y_f and z directions respectively. The applied forces $\bar{N}_{fx}$, $\bar{Q}_{fxz}$ etc. and moments $\bar{M}_{fx}$, $\bar{M}_{fy}$ etc. are the components acting on the edges of the faces and are positive when they act in the same direction as the force and moment resultants shown in Figure 2.

CORE CONSIDERATIONS

The filler which separates the faces is considered to be relatively thick and typical of honeycomb sandwich cores. It is assumed that

(a) the core is incompressible in the transverse direction

(b) the face-parallel shear and extensional stiffnesses are negligible compared to the transverse shear stiffnesses.

(c) the face-parallel displacements vary linearly across the thickness of the core.

Furthermore it is assumed that the design of the sandwich system is such that bond failure does not occur at the interfaces between the filler and skins. Based on the foregoing assumptions, the displacement components of the core can be expressed as

$$\bar{u}_c\,(x, y_c, z_c) = u_c + z_c\,\phi_c \tag{10}$$

$$\bar{v}_c\,(x, y_c, z_c) = v_c + z_c\,\psi_c \tag{11}$$

$$\bar{w}_c\,(x, y_c, z_c) = w_c = w_1 = w_2 \equiv w \tag{12}$$

where

$$u_c = \frac{1}{2}\left[u_1 + u_2 - (d_1 - d_2)\,w_x\right] \tag{13}$$

$$v_c = \frac{1}{2}\left[v_1 + v_2 - (h_1 - h_2)\,w_{cy}\right] \tag{14}$$

are the middle surface displacements of the core and

$$\phi_c = -\frac{1}{t_c}\left[u_1 - u_2 - (d_1 + d_2)\,w_x\right] \tag{15}$$

$$\psi_c = -\frac{1}{t_c}\left[v_1 - v_2 - (h_1 + h_2)\,w_{cy}\right] \tag{16}$$

are the rotations of the normals to the middle surface of the core. In the above expressions

$$h_1 = d_1\frac{R_c}{R_1}\,, \quad h_2 = d_2\frac{R_c}{R_2}$$
$$\text{and } w_{cy} = \frac{\partial w}{\partial y_c} = \frac{1}{R_c}\frac{\partial w}{\partial \theta} \tag{17}$$

The strain-displacement equations for the core are expressed as

$$\gamma_{cxz} = \gamma^{o}_{cxz} = \frac{1}{t_c}\left[u_2 - u_1 + (d_1 + d_2 + t_c)\, w_x\right] \tag{18}$$

$$\gamma_{cyz} = \gamma^{o}_{cyz} / \left(1 + \frac{z_c}{R_c}\right) = \frac{1}{t_c}\left[e_2 v_2 - e_1 v_1 + e_3 w_{cy}\right] / \left(1 + \frac{z_c}{R_c}\right) \tag{19}$$

where

$$e_1 = 1 + \frac{t_c}{2R_c}\ , \quad e_2 = 1 - \frac{t_c}{2R_c}\ , \quad e_3 = d_1\left(\frac{R_c}{R_1} + \frac{t_c}{2R_1}\right) + d_2\left(\frac{R_c}{R_2} - \frac{t_c}{2R_2}\right) + t_c \tag{20}$$

The corresponding strain-displacement equations for the core of a sandwich plate are obtained from Equations 18 and 19 by setting $y_c = y$,

$$\frac{R_c}{R_1} = \frac{R_c}{R_2} = 1 \quad \text{and} \quad \frac{1}{R_1} = \frac{1}{R_2} = \frac{1}{R_c} = 0.$$

The constitutive equations for the core under consideration are represented appropriately by

$$\begin{bmatrix} \tau_{cyz} \\ \tau_{cxz} \end{bmatrix} = \begin{bmatrix} c_{44}(z_c) & 0 \\ 0 & c_{55}(z_c) \end{bmatrix} \begin{bmatrix} \gamma_{cyz} \\ \gamma_{cxz} \end{bmatrix} \tag{21}$$

The shear moduli c_{44} and c_{55} are allowed to vary through the thickness in order to accommodate honeycomb cores that may not have a constant core-cell area throughout the thickness. For example, some relatively thick cylindrical shell cores are formed by rolling an initially flat core medium into a cylindrical surface. In such a case,

$c_{44} = G_{44} / \left(1 + \frac{z_c}{R_c}\right)$, $c_{55} = G_{55}$ where G_{44} and G_{55} are the shear moduli for the flat core.

The force deformation equations for the core can be represented by

$$\begin{bmatrix} Q_{cyz} \\ Q_{cxz} \end{bmatrix} = \begin{bmatrix} B_{44} & 0 \\ 0 & B_{55} \end{bmatrix} \begin{bmatrix} \gamma^{o}_{cyz} \\ \gamma^{o}_{cxz} \end{bmatrix} \tag{22}$$

In Equation 22 the transverse shear rigidities of the core are defined by

$$B_{44} = \int_{-\frac{t_c}{2}}^{\frac{t_c}{2}} c_{44}(z_c)\, dz_c \quad ; \quad B_{55} = \int_{-\frac{t_c}{2}}^{\frac{t_c}{2}} c_{55}(z_c) / \left(1 + \frac{z_c}{R_c}\right) dz_c \tag{23}$$

It is emphasized that B_{44} and B_{55} may be determined by theoretical or experimental means; correspondingly, the strain energy of the core is expressed in terms of the stiffnesses so that the formulation presented herein is independent of the method used to obtain B_{44} and B_{55}.

The strain energy of the core is limited to the contributions due to transverse shear. Therefore

$$U_c = \frac{1}{2}\int_{S_c} \left\{ \frac{B_{44}}{t_c^2}\left[e_2 v_2 - e_1 v_1 + e_3 w_{cy}\right]^2 + \frac{B_{55}}{t_c^2}\left[u_2 - u_1 + (d_1 + d_2 + t_c)\, w_x\right]^2 \right\} dS_c \tag{24}$$

where S_c is the middle surface area of the core.

The kinetic energy of the core is expressed as

$$\begin{aligned} T_c = \frac{1}{2}\int_{S_c} \Bigg\{ & \frac{Q_c}{4}\left\langle \left[\dot u_1 + \dot u_2 - (d_1 - d_2)\dot w_x\right]^2 + \left[\dot v_1 + \dot v_2 - (h_1 - h_2)\dot w_{cy}\right]^2 + 4\dot w^2 \right\rangle \\ & - \frac{J_c}{t_c}\left\langle \left[\dot u_1 + \dot u_2 - (d_1 - d_2)\dot w_x\right]\left[\dot u_1 - \dot u_2 - (d_1 + d_2)\dot w_x\right] \right. \\ & \left. + \left[\dot v_1 + \dot v_2 - (h_1 - h_2)\dot w_{cy}\right]\left[\dot v_1 - \dot v_2 - (h_1 + h_2)\dot w_{cy}\right] \right\rangle \\ & + \frac{I_c}{t_c^2}\left\langle \left[\dot u_1 - \dot u_2 - (d_1 + d_2)\dot w_{cy}\right]^2 + \left[\dot v_1 - \dot v_2 - (h_1 + h_2)\dot w_{cy}\right]^2 \right\rangle \Bigg\} dS_c \end{aligned} \tag{25}$$

In Equation 25 the inertia constants are defined by

$$\left\{Q_c\,,\ J_c\,,\ I_c\right\} = \int_{-\frac{t_c}{2}}^{\frac{t_c}{2}} \left\{1,\ z_c\,,\ z_c^2\right\} \rho_c\,(z_c)\, dz_c \tag{26}$$

where $\rho_c(z_c)$ is the mass density of the core. In the case of sandwich plate and most thin cores, J_c would be zero or negligible.

SECTION III

DISCRETIZATION

The sandwich plate and cylindrical shell elements are shown in Figure 3. The displacement components u_1, u_2, v_1, v_2 and w are approximated by the assumed displacement patterns suggested in Reference 3. These assumed displacement patterns are represented by the sum of products of one-dimensional first-order Hermite interpolation polynomials and undetermined nodal coefficients. The reference surface displacement u_f of a sandwich cylindrical shell element is represented by

$$u_f(x, y_f) = \sum_{i=1}^{2} \sum_{j=1}^{2} \Big[H_{oi}^{(1)}(x) H_{oj}^{(1)}(y_f) u_{fij} + H_{1i}^{(1)}(x) H_{oj}^{(1)}(y_f) u_{fxij}$$

$$+ H_{oi}^{(1)}(x) H_{1j}^{(1)}(y_f) u_{fyij} + H_{1i}^{(1)}(x) H_{1j}^{(1)}(y_f) u_{fxyij} \Big] \tag{27}$$

where, for example $u_{fij} = u_f(x = x_i, y_f = y_{fj} = R_f \theta_j)$ are the nodal displacements and

$$u_{fyij} = \frac{\partial u_f}{\partial y_f}(x = x_i, y_f = y_{fj} = R_f \theta_j) = \frac{1}{R_f} \frac{\partial u_f}{\partial \theta}(x = x_i, \theta = \theta_j)$$

are the derivatives of u_f in the circumferential direction at the node points. Similar expressions are used for u_2, v_1, v_2 and w.

The $H_{ki}^{(1)}(x)$ are the one-dimensional first-order Hermite interpolation polynomials given by

$$H_{01}^{(1)}(x) = (2x^3 - 3ax^2 + a^3)/a^3 \quad ; \quad H_{02}^{(1)}(x) = -(2x^3 - 3ax^2)/a^3 \tag{28}$$

$$H_{11}^{(1)}(x) = (x^3 - 2ax^2 + a^2x)/a^2 \quad ; \quad H_{12}^{(1)}(x) = (x^3 - ax^2)/a^2 \tag{29}$$

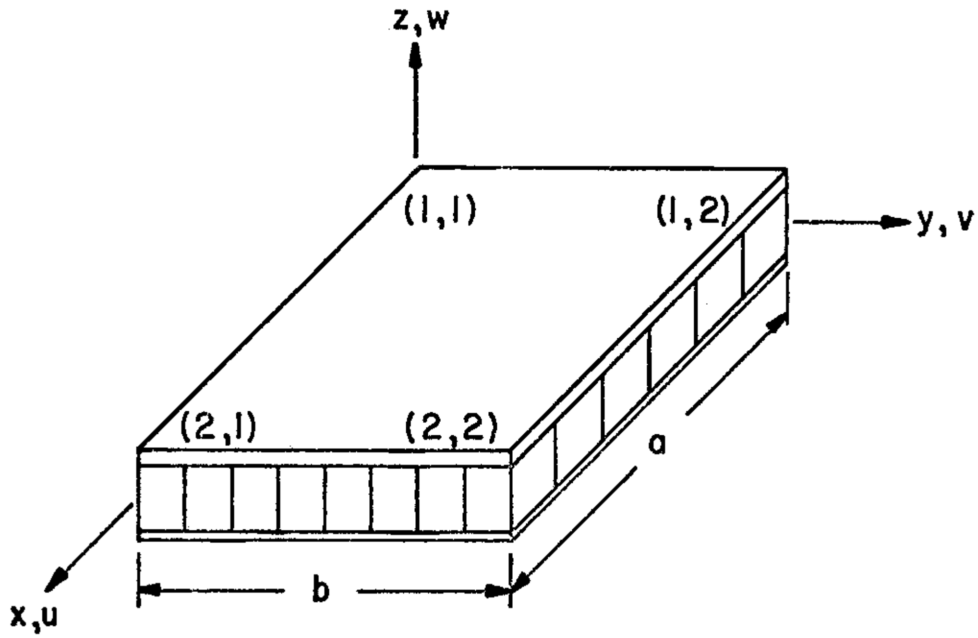
where a is the length of the element in the x direction. Corresponding expressions for the y direction are obtained by replacing x by y_s and a by b_s where

$$y_s = y \quad , \quad b_s = b \qquad (s = 1, 2, c) \text{ for the plate}$$

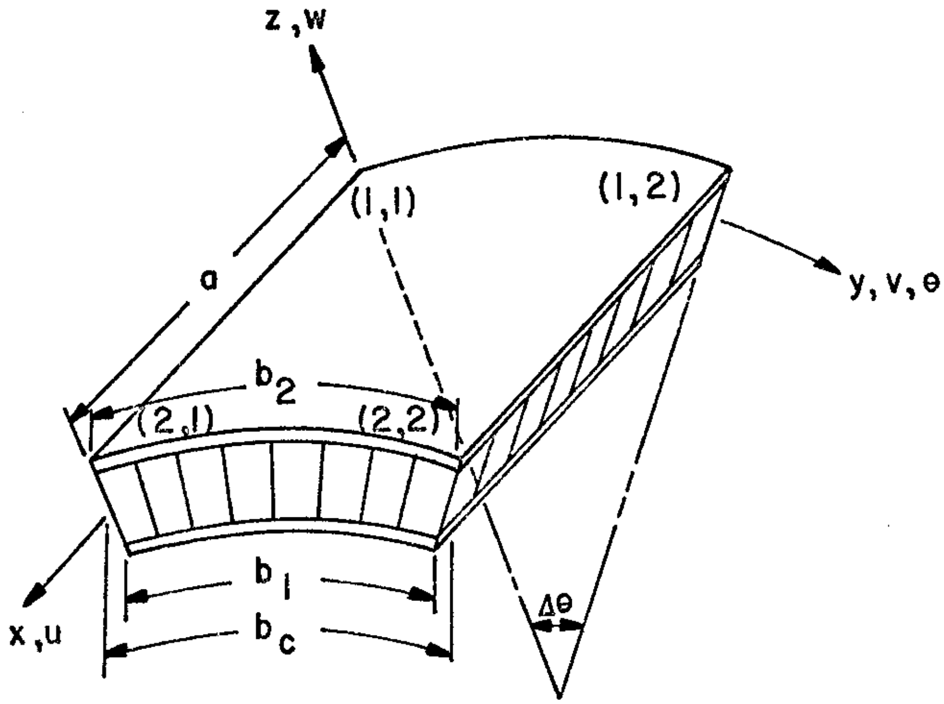
and

$$y_s = R_s \theta \; , \quad b_s = R_s(\Delta\theta) \quad (s = 1, 2, c) \text{ for the cylinder}$$

The subscript s indicates that the lengths are measured on the appropriate reference surface.



(a) Sandwich Plate Element



(b) Sandwich Cylindrical Shell Element

Figure 3. Anisotropic Sandwich Elements

Assumed displacement patterns in the form of Equation 27 readily permit satisfaction of the geometric admissibility conditions for the sandwich system.* It is pointed out here that by virtue of Equations 18 and 19, satisfaction of the admissibility conditions automatically imposes continuity of the transverse shear strains in the core. In the presence of various types of concentrated loads, the imposition of continuous transverse shear strains in the core is not appropriate; however in view of the standard design practice of reinforcing the core in the vicinity of concentrated loads, this difficulty is not a practical limitation.

It is also realized that zero-order (bilinear) interpolation polynomials would be sufficient to generate admissible displacement patterns for the face-parallel displacements. However, it was decided to express u_f and v_f by displacement modes in the form of Equation 27 for the following reasons:

(a) an accurate representation of the elemental strain energy is achieved. Bilinear interpolation may not produce an accurate description of the internal energy when the strain distribution is relatively complicated within an element. Also it was reported in Reference 4 that bilinear interpolation did not adequately represent the rigid body modes for a cylindrical shell element; however, very little strain energy is associated with rigid body displacement when the membrane displacements are represented by the displacement patterns of the form of Equation 27.

(b) a one-to-one linking of degrees of freedom can impose strain continuity in the reference surfaces. If the faces are typical of those used in sandwich construction, the bending rigidities of the faces are small, and imposing strain continuity should result in accurate stress predictions.

(c) Employing displacement functions of the form of Equation 27 makes it possible to model structures using elements joined at arbitrary angles. The interelement admissibility conditions for such structures can be satisfied only if all the displacement states are represented by interpolation polynomials that are of the same order.

*The geometric admissibility conditions require: (1) satisfaction of the imposed displacement boundary conditions, (2) displacement continuity within and between adjacent elements, and (3) continuous first derivatives of the transverse deflection within elements and on the common edges of adjacent elements.

Using displacement patterns of the type described result in sandwich finite elements which incorporate a total of 80 degrees of freedom (16 for each of u_1, u_2, v_1, v_2, and w). At a typical interior node where four elements meet tangentially, the geometric admissibility conditions reduce the number of independent degrees of freedom from 80 to 40; imposing strain continuity further reduces the number of independent degrees of freedom to 20. In other cases, (e.g. pure bending), the number of independent degrees of freedom at a typical interior node can be reduced to 12 by setting $u_1 = cu_2$ and $v_1 = cv_2$ (c = constant).

When the assumed displacement modes are substituted into the expressions characterizing the strain energies of the faces and core (Equations 6 and 24) and then the integrations are carried over the indicated reference surfaces, the strain energy of the element can be expressed as

$$U = U_c + \sum_{f=1}^{2} U_f = \frac{1}{2} X^T K X \tag{30}$$

where X is a vector containing the 80 nodal variables of the element. In this work X is ordered as follows:

$$\underset{1 \times 80}{X^T} = \left\{ X_{u_1}^T, X_{u_2}^T, X_{v_1}^T, X_{v_2}^T, X_w^T \right\} \tag{31}$$

where the X_Δ ($\Delta = u_1, u_2, v_1, v_2, w$) are vectors containing the 16 nodal coefficients associated with each of the Δ's. For example, X_{u_1} is ordered as

$$\begin{aligned} \underset{1 \times 16}{X_{u_1}^T} = \{ & u_{111}, u_{1x11}, u_{1y11}, u_{1xy11}, u_{112}, u_{1x12}, u_{1y12}, u_{1xy12}, \\ & u_{122}, u_{1x22}, u_{1y22}, u_{1xy22}, u_{121}, u_{1x21}, u_{1y21}, u_{1xy21} \} \end{aligned} \tag{32}$$

where $u_{1yij} = \frac{1}{R_1}\frac{\partial u_1}{\partial \theta}(x = x_i,\ \theta = \theta_j)$ etc. for cylinders. The element stiffness matrix K is partitioned as

$$\underset{80\times 80}{K} = \begin{bmatrix} K^{(u_1 u_1)} & K^{(u_1 u_2)} & K^{(u_1 v_1)} & 0 & K^{(u_1 w)} \\ & K^{(u_2 u_2)} & 0 & K^{(u_2 v_2)} & K^{(u_2 w)} \\ & & K^{(v_1 v_1)} & K^{(v_1 v_2)} & K^{(v_1 w)} \\ & \text{(SYMMETRIC)} & & K^{(v_2 v_2)} & K^{(v_2 w)} \\ & & & & K^{(ww)} \end{bmatrix} \qquad (33)$$

where all the submatrices are 16x16 dimensional arrays. The submatrices on the diagonal are symmetric and contain contributions that do not involve coupling between the various displacement components. Generally the off-diagonal arrays are not symmetric and account for the coupling terms in the strain energy; the superscripts indicate which displacements are coupled. Formulas for the elements of the submatrices are given in Appendix A.

The potential of the applied loads (Equation 9) is dealt with on a work equivalent basis. Substitution of the assumed displacement patterns into Equation 9 and then performing the indicated integrations gives

$$W_f = P^T X \qquad (34)$$

where P is a vector containing the 80 work equivalent loads, 80 x 1 associated with the corresponding nodal degrees of freedom in X.

The kinetic energy of the sandwich elements can be discretized by substituting the assumed displacement patterns into the kinetic energy for the faces and core (Equations 7 and 25). Making the assumption that the displacements are sinusoidal functions of time with frequency ω and integrating over the reference surfaces, the kinetic energy can be represented by

$$T = T_c + \sum_{f=1}^{2} T_f = \frac{1}{2}\omega^2 X^T M X \qquad (35)$$

where X is the vector of nodal coefficients (Equation 31). The consistent mass matrix M is partitioned as

$$\underset{80 \times 80}{M} = \left[\begin{array}{c|c|c|c|c} M^{(u_1 u_1)} & M^{(u_1 u_2)} & 0 & 0 & M^{(u_1 w)} \\ \hline & M^{(u_2 u_2)} & 0 & 0 & M^{(u_2 w)} \\ \hline & & M^{(v_1 v_1)} & M^{(v_1 v_2)} & M^{(v_1 w)} \\ \hline & \text{(SYMMETRIC)} & & M^{(v_2 v_2)} & M^{(v_2 w)} \\ \hline & & & & M^{(ww)} \end{array}\right] \qquad (36)$$

Again each of the submatrices in M are 16 x 16 dimensional arrays; those on the diagonal are symmetric while the off-diagonal submatrices account for the coupling between variables in the kinetic energy and, in general, are not symmetric. Formulas for the elements of the submatrices in M are also given in Appendix A.

SECTION IV

NUMERICAL EVALUATION

The numerical results given in this section are an attempt to test and evaluate the finite element method presented herein. In all problems, the admissibility conditions are satisfied by linking and dismissing degrees of freedom; also, additional linking conditions which impose continuity of the reference surface strains are applied in all cases. The face stiffnesses given are based on calculation which take $d_f = 1/2\ t_f$. Problem A was solved using a standard Gaussian elimination procedure while a Householder eigenvalue algorithm was employed in problem D. The other solutions were obtained by direct minimization of the total potential energy. Since the total potential energy of the structural system is approximated by the sum of the potential energies of the individual finite elements, a minimum of computer storage is required when the energy search concept is adopted. Such an approach is appealing provided that an efficient algorithm is available for minimizing the potential energy functional. It was found that the conjugate gradient algorithm of Reference 6 was very efficient when the coordinate scaling technique reported in Reference 7 was used. All problems were solved on the UNIVAC 1108 digital computer using the FORTRAN IV compiler and the execution times include the time required to form stiffness and mass matrices as well as other pertinent calculations.

THIN LAMINATED PLATE IN TENSION (FIGURE 4)

A 2 in. x 2 in. plate constructed of two orthotropic layers each 0.05 in. thick is considered. The axes of elastic symmetry (x′,y′) of the upper and lower layers make an angle of -45 and +45 degrees respectively with the (x,y) axes of the plate system. Only the lower face of the sandwich system was considered and the face stiffnesses were taken as:

$$A_{111} = A_{122} = 2A_{166} = 6.0 \times 10^4 \text{ lb/in.}\ ;\quad A_{112} = A_{116} = A_{126} = 0$$

$$B_{111} = B_{122} = B_{112} = B_{166} = 0\ ;\quad B_{116} = B_{126} = 5.0 \times 10^2 \text{ lb}$$

$$D_{111} = D_{122} = 2D_{166} = 50.0 \text{ lb-in.}\ ;\quad D_{112} = D_{116} = D_{126} = 0$$

A uniform tension $\bar{N}_x = 1.0$ lb/in. was applied along the edges $x = \pm 1$ and the plate was modeled using four 1 in. x 1 in. elements. The restraints, $u = v = w = w_x = w_y = u_y - v_x = 0$ at $x = 0$, $y = 0$, were imposed at the center of the plate in order to preclude rigid body displacements. The discretized formulation involves 102 degrees of freedom and the results are in complete agreement with those of Reference 11. For example, at the corner, $x = 1$, $y = 1$,

$$w = 2.5 \times 10^{-4} \text{ in.}, \quad \epsilon_x^o = 2.0833 \times 10^{-5} \text{ in./in.}, \quad \epsilon_y^o = 0.4167 \times 10^{-5} \text{ in./in.}$$

$$N_x = 1.0 \text{ lb/in.}, \qquad N_y = N_{xy} = M_x = M_{xy} = 0.$$

These "exact" results are not unexpected since the plate deforms into a hyperbolic paraboloid which can be represented exactly by the assumed displacement functions used.

SIMPLY SUPPORTED SANDWICH PLATE UNDER UNIFORM PRESSURE (FIGURE 5)

A 20 in. x 20 in. sandwich plate consisting of a 1 in. thick orthotropic aluminum honeycomb core and 0.020 in. thick aluminum alloy faces is considered. The stiffnesses of the faces are (F = 1,2)

$$A_{f11} = A_{f22} = 2.1978 \times 10^5 \text{ lb/in.}; \qquad A_{f12} = 0.6593 \times 10^5 \text{ lb/in.}$$

$$A_{f16} = A_{f26} = 0 \;; \qquad A_{f66} = 0.7700 \times 10^5 \text{ lb/in.}$$

$$B_{fij} = 0, \quad (i, j = 1, 2, 6)$$

$$D_{f11} = D_{f22} = 7.3260 \text{ lb-in.} \;; \qquad D_{f12} = 2.1978 \text{ lb-in.}$$

$$D_{f16} = D_{f26} = 0 \;; \qquad D_{f66} = 2.5667 \text{ lb-in.}$$

For the core

$$B_{44} = 7.5200 \times 10^4 \text{ lb/in.} \;; \qquad B_{55} = 3.2900 \times 10^4 \text{ lb/in.}$$

It is assumed that the transverse shear strains in the core parallel to the boundaries are prevented by the presence of edge reinforcement. The simply supported displacement boundary conditions are:

$$w = 0, \quad v_1 = v_2 = 0 \quad \text{along the boundaries} \quad x = \text{constant}$$

$$w = 0, \quad u_1 = u_2 = 0 \quad \text{along the boundaries} \quad y = \text{constant}$$

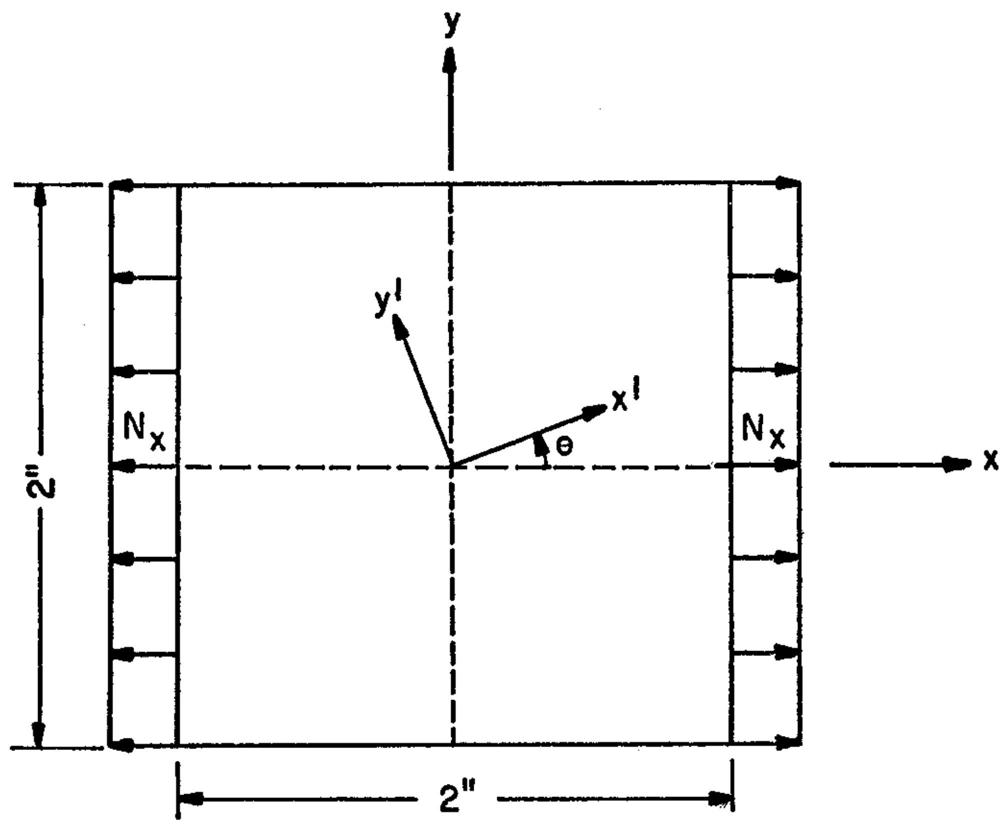


Figure 4. Thin Laminated Plate

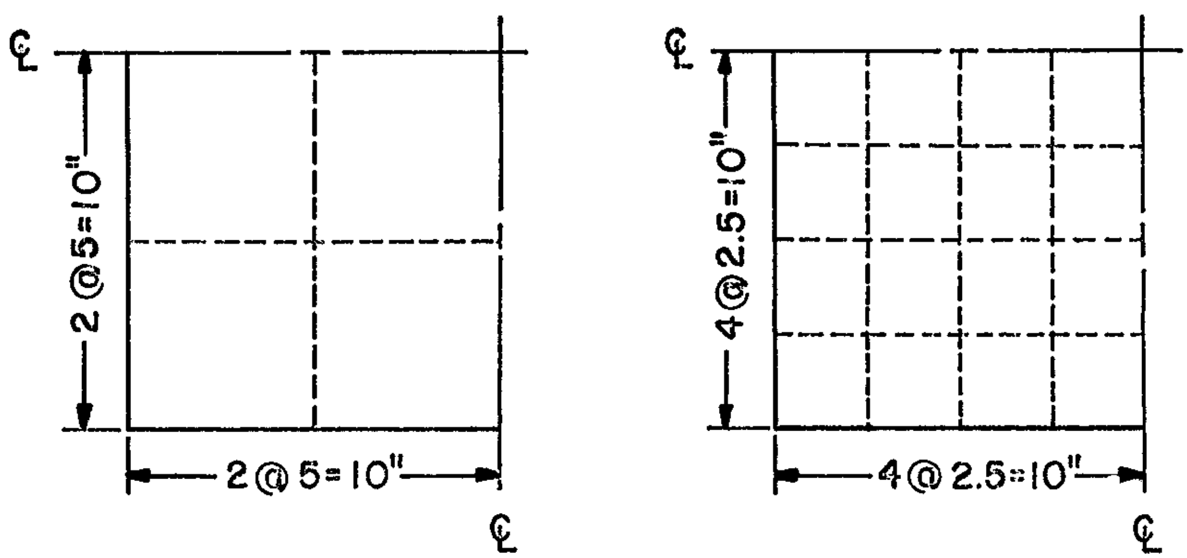


Figure 5. Simply Supported Sandwich Plate Modelings

A uniform transverse pressure of one psi is applied to the system. For the plate and loading system described, $u_1 = -u_2$ and $v_1 = -v_2$. Taking symmetry into account, one quadrant of the plate was modeled using four and 16 elements (56 and 208 degrees of freedom respectively). The two modelings are illustrated in Figure 5 and the corresponding results are given in Table 1. These results apply to locations where the values are maximum: thus w_{max}, $\epsilon^o_{x,max}$, $\epsilon^o_{y,max}$ refer to the deflection and strains in the faces at the center of the panel while $\gamma_{cxz,max}$ and $\gamma_{cyz,max}$ refer to the core shear strains at the center of the sides. Also, the results are compared with the theoretical and experimental results given in Reference 12. The results are in close agreement for w_{max}, $\epsilon^o_{x,max}$, and $\epsilon^o_{y,max}$. As reported in Reference 12, the discrepancy between experimental and theoretical transverse shear results "was probably due to method of measurement rather than error in theory".* The computer-run times required to solve the four element and 16 element cases were 11.7 sec and 56.9 sec respectively.

SANDWICH CYLINDER WITH AXISYMMETRIC TEMPERATURE LOAD (FIGURE 6)

A long sandwich cylinder constructed of isotropic temperature independent materials is considered. The stiffnesses of the faces are

$$A_{f11} = A_{f22} = 1.0989 \times 10^4 \text{ lb/in.} \quad ; \quad A_{f12} = 0.3297 \times 10^4 \text{ lb/in.}$$

$$A_{f16} = A_{f26} = 0 \quad ; \quad A_{f66} = 0.3846 \times 10^4 \text{ lb/in.}$$

$$B_{fij} = 0 \quad (i, j = 1, 2, 6)$$

$$D_{f11} = D_{f22} = 9.1575 \times 10^{-4} \text{ in.-lb} \quad ; \quad D_{f12} = 2.7473 \times 10^{-4} \text{ in.-lb}$$

$$D_{f16} = D_{f26} = 0 \quad ; \quad D_{f66} = 3.2051 \times 10^{-4} \text{ in.-lb}$$

For the core

$$B_{44} = B_{55} = 5.4945 \times 10^3 \text{ lb/in.}$$

*Reference 12, Page 7 (Panel No. 6).

Both faces of the sandwich are subjected to a uniform axisymmetric temperature load T = 250 degrees along a length $a_T = 2R_c$. The strain energy U_{fT} associated with the temperature load in a face is

$$U_{fT} = -\frac{E\alpha t T}{1-\nu} \int_{S_f} \left[u_{fx} + \frac{w}{R_f} \right] dS_f \tag{37}$$

where $E = 10 \times 10^6$ psi, $\nu = 0.3$, t = 0.001 in., T = 250 degrees, and $\alpha = 1 \times 10^{-5}$ in/degree. Substitution of the assumed displacement patterns into Equation 30 and integrating over the middle surface area of the face produces the discretized strain energy due to the temperature load. Since U_{fT} is a linear function in the displacement variables, it is treated as an equivalent work term.

The cylinder was modeled using 10 sandwich cylindrical shell finite elements as shown in Figure 6 (the temperature load was maintained on five elements). The conditions $u_f = 0$ and $w_x = 0$ were imposed in the circumferential direction at x = 0. By virtue of the axisymmetric response, the idealized system incorporates a total of 63 degrees of freedom (21 associated with each of u_1, u_2, and w). Solution time was 25.7 seconds. The transverse deflection at points along the length is given in Table 2 and the results are compared with those due to Oberndorfer (Reference 13). The results are in very close agreement; the discrepancy in the results at $x/R_c = 1.5$ and 2.0 can be attributed to the fact that the finite element results are based on the assumption that the circumference is free at $x = 2R_c$.

NATURAL FREQUENCIES OF A SIMPLY SUPPORTED SANDWICH PLATE (FIGURE 7)

A 72 in. x 48 in. sandwich plate has two identical aluminum facings of thickness 0.016 in. and an orthotropic aluminum honeycomb core with $t_c = 0.25$ in. The stiffnesses and inertia constants of the faces are:

$$A_{f11} = A_{f22} = 1.7582 \times 10^5 \text{ lb/in.} \quad ; \quad A_{f12} = 0.5275 \times 10^5 \text{ lb/in.}$$

$$A_{f16} = A_{f26} = 0 \quad ; \quad A_{f66} = 0.6160 \times 10^5 \text{ lb/in.}$$

$$B_{fij} = 0 \quad (i, j = 1, 2, 6)$$

TABLE I SIMPLY SUPPORTED SANDWICH PLATE RESULTS

Solution Method	w_{max} (in)	$\epsilon^{o}_{x,max}$ (in/in)	$\epsilon^{o}_{y,max}$ (in/in)	$\gamma_{cxz,max}$ (in/in)	$\gamma_{cyz,max}$ (in/in)
Experiment (Ref. 12)	6.1×10^{-3}	$+6.2 \times 10^{-5}$ -6.4×10^{-5}	$+7.4 \times 10^{-5}$ -7.8×10^{-5}	4.3×10^{-4}	3.9×10^{-4}
Theory (Ref. 12)	6.3×10^{-3}	$\pm 6.3 \times 10^{-5}$	$\pm 6.7 \times 10^{-5}$	2.0×10^{-4}	0.9×10^{-4}
Discrete Element 4 Elements 16 Elements	6.30×10^{-3} 6.30×10^{-3}	$\pm 6.29 \times 10^{-5}$ $\pm 6.28 \times 10^{-5}$	$\pm 6.90 \times 10^{-5}$ $\pm 6.86 \times 10^{-5}$	2.06×10^{-4} 2.01×10^{-4}	0.95×10^{-4} 0.90×10^{-4}

TABLE II TRANSVERSE DISPLACEMENT OF AXISYMMETRIC HEATED SANDWICH CYLINDER ($w \times 10^2$ in.)

x/R_c	0	0.1	0.3	0.5	0.7	0.9	1.0	1.1	1.2	1.5	2.0
Ref. 13	2.498	2.497	2.498	2.546	2.578	2.016	1.250	0.484	0.057	0.038	0.001
Present	2.497	2.497	2.498	2.548	2.577	2.013	1.250	0.488	0.053	-0.036	0.008

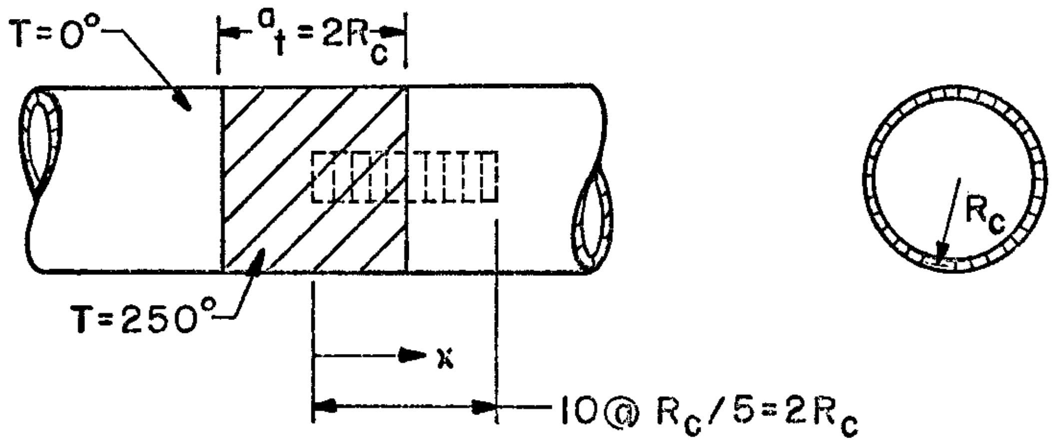


Figure 6. Heated Sandwich Cylinder

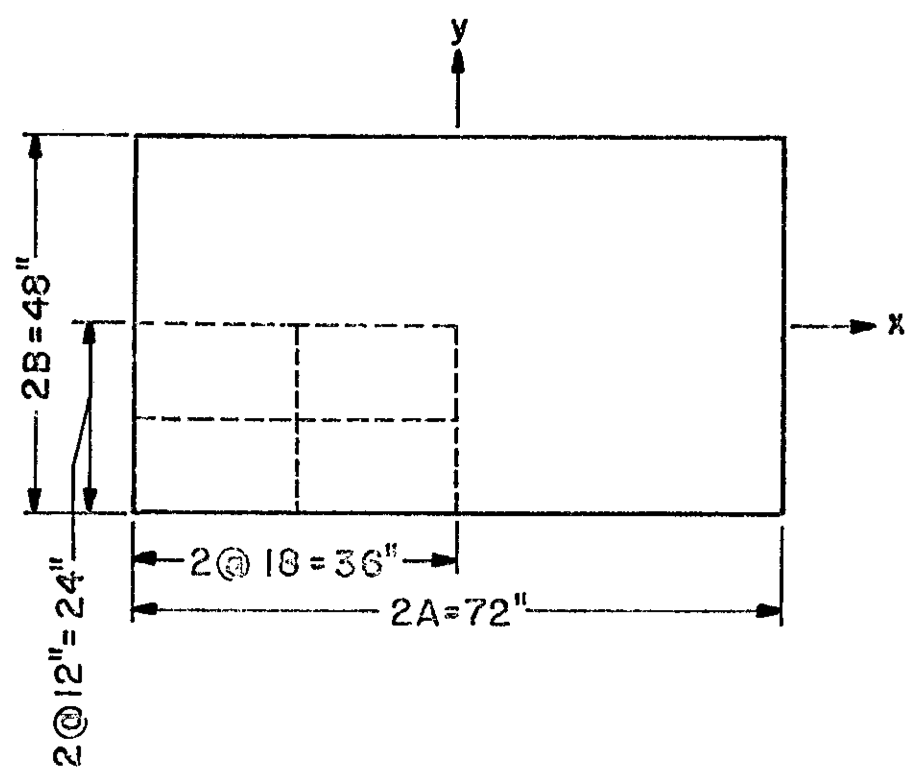


Figure 7. Simply Supported Sandwich Plate

$D_{f11} = D_{f22} = 3.7509$ in.-lb ; $D_{f12} = 1.1253$ in.-lb

$D_{f16} = D_{f26} = 0$; $D_{f66} = 1.3141$ in.-lb

$Q_f = 4.144 \times 10^{-6}$ lb-sec^2/in^3 ; $J_f = 0$

$I_f = 8.8405 \times 10^{-11}$ lb-sec^2/in.

For the core

$B_{44} = 4.875 \times 10^3$ lb/in. ; $B_{55} = 1.875 \times 10^3$ lb/in.

$Q_c = 2.850 \times 10^{-6}$ lb-sec^2/in^3 ; $J_c = 0$

$I_c = 1.484 \times 10^{-8}$ lb-sec^2/in.

Due to pure bending $u_1 = -u_2$, $v_1 = -v_2$ and the prescribed simply supported boundary conditions are taken as

$$w = 0 \quad , \quad v_1 = v_2 = 0 \quad \text{at} \quad x = \pm A$$

$$w = 0 \quad , \quad u_1 = u_2 = 0 \quad \text{at} \quad y = \pm B$$

Let m and n represent the mode numbers along the length and width directions respectively. In order to obtain solutions efficiently only one quadrant of the plate was modeled with four elements and four separate problems were solved. Each of the four possible combinations of m and n require different conditions on the displacement variables along the lines x = 0 y = 0; these conditions are summarized below:

1. m and n odd (symmetric-symmetric modes; 57 degrees of freedom)

$$w_x = 0; \quad u_{fy} - v_{fx} = 0 \quad \text{at} \quad x = 0$$

$$w_y = 0; \quad u_{fy} - v_{fx} = 0 \quad \text{at} \quad y = 0$$

2. m odd and n even (symmetric-antisymmetric modes; 65 degrees of freedom)

$$w_x = 0; \quad u_{fy} - v_{fx} = 0 \quad \text{at} \quad x = 0$$

$$w = 0; \quad \text{at} \quad y = 0$$

3. m even and n odd (antisymmetric-symmetric modes; 65 degrees of freedom)

$$w = 0 \quad \text{at} \quad x = 0$$

$$w_y = 0; \quad u_{fy} - v_{fx} = 0 \quad \text{at} \quad y = 0$$

4. m even and n even (antisymmetric-antisymmetric modes; 76 degrees of freedom)

$$w = 0 \quad \text{at} \quad x = 0$$
$$w = 0 \quad \text{at} \quad y = 0$$

Experimental and theoretical values for the natural frequencies of vibration of the panel described above have been presented in Reference 14. The approximate finite element results for the first ten natural frequencies are given in Table 3 where comparison is also made with the results reported in Reference 14. Inspection of Table 3 indicates that the finite element method is capable of rendering good frequency predictions with relatively coarse modeling.

INSTABILITY ANALYSIS BY ENERGY SEARCH (FIGURE 8)

An interesting adaptation of the energy search concept involves the prediction of the response of structural systems subject to the influence of destabilizing loads. The approach is valid when the membrane and bending behaviors can be uncoupled in a linear formulation. The buckling analysis is based on the premise that the distribution of membrane displacements due to membrane loads is a predetermined linear function of the applied membrane loads.

In order to illustrate the energy search approach, a simply supported sandwich plate is considered. The geometric and elastic properties of the faces and core as well as the loading conditions are such that the membrane and bending responses are completely uncoupled in a linear formulation. For simplicity, it is assumed that the faces of the sandwich are identical and that the membrane loads acting on the individual faces are equal.*

Let N_O represent a set of reference membrane loads. The resulting distribution of in-plane displacements for a representative discrete element of the idealized system is of the form (Equation 31)

$$x_o^T = \left\{ x_{u_o}^T , x_{u_o}^T , x_{v_o}^T , x_{v_o}^T , 0 \right\} \tag{39}$$

*In general the requirement that the plate remains flat under the action of membrane forces is satisfied if the total axial load is distributed on the individual faces in direct proportion to their axial stiffnesses.

TABLE III NATURAL FREQUENCIES OF SIMPLY SUPPORTED SANDWICH PLATE (CPS)

Modal Numbers		Experiment (Ref. 14)	Theoretical (Ref. 14)	Discrete Element
m	n			
1	1	----	23	23
2	1	45	45	44
1	2	69	71	70
3	1	78	80	80
2	2	92	91	90
3	2	129	126	125
4	1	133	129	139
1	3	152	146	145
2	3	169	165	164
4	2	177	174	179

In Equation 39 X_{u_o} and X_{v_o} are vectors containing the 16 nodal coefficients associated with the assumed displacement patterns for u_o and v_o respectively. Correspondingly, when the membrane load level is $N_m = \lambda_m N_o$ the membrane solution is

$$X_m = \lambda_m X_o \tag{40}$$

Similarly, due to pure bending, the typical elemental solution assumes the form

$$X_b^T = \left\{ X_{u_b}, -X_{u_b}, X_{v_b}, -X_{v_b}, X_w \right\} \tag{41}$$

and the total solution vector is

$$X = \lambda_m X_o + X_b \tag{42}$$

Again it is noted that X_o is a predetermined quantity while X_b is unknown. The middle-surface strains of the faces are taken as

$$\epsilon'_{fx} = \epsilon^o_{fx} + \frac{1}{2} w_x^2 \tag{43}$$

$$\epsilon'_{fy} = \epsilon^o_{fy} + \frac{1}{2} w_y^2 \tag{44}$$

$$\gamma'_{fxy} = \gamma^o_{fxy} + w_x w_y \tag{45}$$

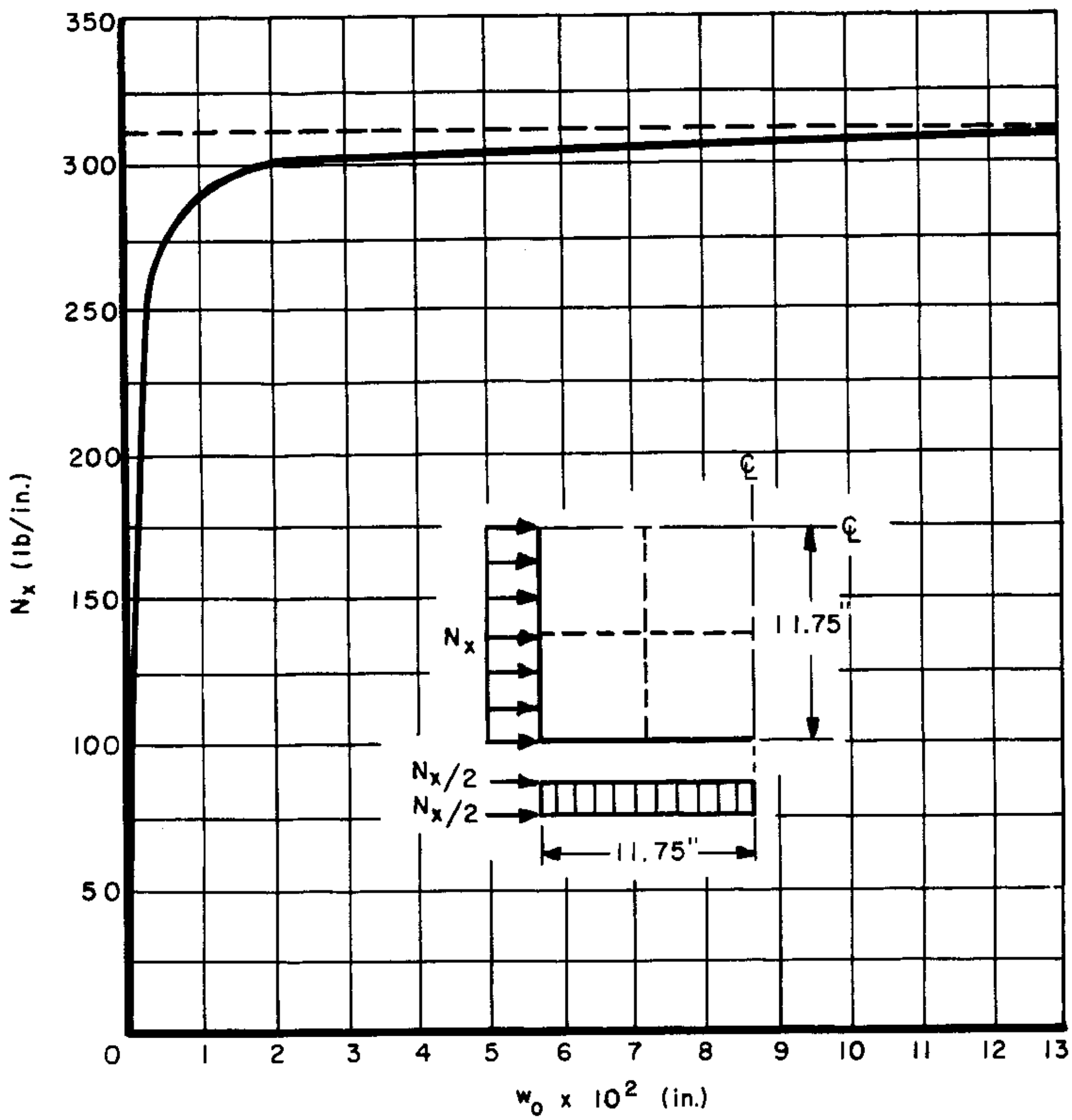


Figure 8. Load Deflection Curve for Axially Loaded Sandwich Plate

where ϵ^o_{fx}, ϵ^o_{fy}, and γ^o_{fxy} are defined by Equation 2. Neglecting fourth-order contributions, the strain energy of the finite element can be written as

$$U_e = U_m + U_b + U_b' \tag{46}$$

where

(a) $U_m = \frac{1}{2} X_m^T K X_m = \frac{1}{2} \lambda_m^2 X_o^T K X_o$ is the quadratic elemental strain energy associated with the membrane loading system N_m . K is the stiffness matrix represented by Equation 33. Since X_o is a known vector, U_m is a known constant for any level of the membrane loading.

(b) $U_b = \frac{1}{2} X_b^T K X_b$ is the quadratic elemental strain energy associated with a pure bending state and its numerical value depends on the unknown values of the displacement vectors X_{u_b}, X_{v_b}, and X_w.

(c)

$$U_b' = \frac{1}{2} \sum_{f=1}^{2} \int_{S_f} \Big\{ A_{f11} \left[u_{mx} w_x^2 \right] + A_{f22} \left[v_{my} w_y^2 \right]$$

$$+ 2A_{f66} \left[(u_{my} + v_{mx}) w_x w_y \right] + A_{f12} \left[u_{mx} w_x^2 + v_{my} w_x^2 \right]$$

$$+ A_{f16} \left[2u_{mx} w_x w_y + (u_{my} + v_{mx}) w_x^2 \right]$$

$$+ A_{f26} \left[2v_{my} w_x w_y + (u_{my} + v_{mx}) w_y^2 \right] \Big\} dS_f \tag{47}$$

where

$$u_{mx} = \frac{\partial u_m}{\partial x} = \lambda_m \frac{\partial u_o}{\partial x} \quad , \quad u_{my} = \frac{\partial u_m}{\partial y} = \lambda_m \frac{\partial u_o}{\partial y} \quad ,$$

etc. Since X_o is a known quantity, U_b' is a quadratic function of the transverse displacement only. Therefore U_b' can be written as

$$U_b' = \frac{1}{2} \lambda_m X_w^T K_1^{(ww)} X_w \tag{48}$$

It is apparent that the symmetric matrix $K_1^{(ww)}$ is analogous to the "incremental stiffness matrix" (Reference 15) and its numerical definition is based on a knowledge of the membrane solution X_o.

The strain energy for the sandwich element can be represented by

$$U_e = U_m + \frac{1}{2} X_b^T \left[K + \lambda_m K_1 \right] X_b \tag{49}$$

where the matrix K_1 is partitioned as

$$\underset{80 \times 80}{K_1} = \left[\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & K_1^{(ww)} \end{array} \right]$$

The potential of the applied loads acting on an element is represented by

$$W_e = P_m^T X_m + P_b^T X_b = \lambda_m^2 P_o^T X_o + P_b^T X_b \tag{50}$$

where

$$P_o^T = \left\{ P_{u_o}^T \; , \; P_{u_o}^T \; , \; P_{v_o}^T \; , \; P_{v_o}^T \; , \; 0 \right\} \tag{51}$$

and

$$P_b^T = \left\{ 0 \; , \; 0 \; , \; 0 \; , \; 0 \; , \; P_w^T \right\} \tag{52}$$

The vector P_o is the discretized form of N_o which produces the membrane solution X_o. The vector P_b is the descritized transverse loading system which produces the pure bending solution X_b. The total elemental potential energy is given by

$$\pi_{pe} = \pi_{pm} + \pi_{pb} \tag{53}$$

where

$$\pi_{pm} = U_m - P_m^T X_m \tag{54}$$

and

$$\pi_{pb} = U_b + U_b' - P_b^T X_b \tag{55}$$

It is noted that π_{pm} is a known constant for a particular membrane displacement state X_m. Thus the incremental elemental potential energy at a predetermined load level is defined by

$$\pi_{pe}' = \pi_{pe} - \pi_{pm}$$

As usual, the total effective potential energy Π_p' for an assemblage of elements is equal to the sum of the π_{pe}' of the individual elements. Minimization of Π_p' results in the pure bending solution which can be superimposed on the membrane solution in order to obtain the complete description of the behavior of the loaded system.

As an example problem consider a 23.5 in. x 23.5 in. simply supported sandwich plate with identical isotropic facings. The membrane and bending stiffnesses of the faces are (f = 1,2)

$$A_{f11} = A_{f22} = 2.1923 \times 10^5 \text{ lb/in.}; \quad A_{f12} = 0.6577 \times 10^5 \text{ lb/in.}$$

$$A_{f16} = A_{f26} = 0 \quad ; \quad A_{f66} = 0.7673 \times 10^5 \text{ lb/in.}$$

$$D_{f11} = D_{f22} = 8.0567 \text{ in.-lb} \quad ; \quad D_{f12} = 2.4170 \text{ in.-lb}$$

$$D_{f16} = D_{f26} = 0 \quad ; \qquad D_{f66} = 2.8200 \text{ lb-in.}$$

The transverse shear stiffnesses for the core are

$$B_{44} = B_{55} = 3.439 \times 10^3 \text{ lb/in}$$

Assuming symmetry, only one quadrant of the plate was considered and the quadrant was modeled with four square elements (Figure 8). Initially, a compressive membrane load of N_o = -10.0 lb/in. (5.0 lb/in. on each face) was applied to the system and the corresponding membrane solution was obtained; this solution established X_o for each element and was used to calculate the K_1 for the individual elements. Subsequently, a constant lateral load of 0.1 lb. was applied at the center of the plate and the membrane load N_x was incremented until the search for the minimum of the potential energy became unbounded. It is important to realize that the load incrementation was simulated by simply multiplying K_1 for each element by varying values of λ_m. A plot of the membrane load N_x versus the center deflection w_o is shown in Figure 8. The buckling load for the system is obtained from the asymptote of the load-displacement curve. For the problem under study, bounded potential energy solutions were obtained for $N_x \leq 309$ lb/in.; at a load of $N_x = 310$ lb/in. The modified potential energy was unbounded indicating that the system was unstable. It can therefore be concluded that the approximate buckling load N_{CR} for the system is given by

$$309 \leq N_{CR} \leq 310 \text{ lb/in.}$$

The critical load found by the energy search procedure is comparable to the value of 303 lb/in. reported by Hoff in Reference 16 and a value of 308 lb/in. using the approach outlined in Reference 17. Tests at the Forest Products Laboratories reported experimental values ranging from 266 to 300 lb/in. on four specimens (Reference 18).

It is noted that the energy search approach described above is analogous to the "incremental stiffness matrix method", alternately labeled the "initial stress matrix method" (Reference 19 for a brief account of previous work). The incremental stiffness matrix method is characterized by a work term that is quadratic in the transverse displacement which reduces the effective bending stiffness. In the energy search adaptation given here, geometric nonlinearities are considered in the strain energy to the extent necessary to produce a corresponding reduction in the effective stiffness of the structural system. A situation analogous to the vanishing of the effective bending stiffness occurs; instability is detected when the strain energy of the structural system vanishes and the total potential energy becomes unbounded. It is pointed out that although cubic strain energy contributions are included in the formulation, the description of the total potential energy functional presumes the knowledge of the uncoupled membrane displacement state. As such it will be noted that the minimization process is carried out on a potential energy functional which is, in fact, a quadratic function.

SECTION V

CONCLUSIONS

A finite element capability for the analysis of sandwich plates and cylindrical shells with unbalanced laminated faces has been presented. Using Hermite interpolation polynomials, the structural behavior was described by the membrane displacements of the individual faces and the transverse displacement of the sandwich system; this set of displacement variables admits transverse shear deformations in the core and provides a rather wide choice of boundary conditions. The formulation was presented in terms of the various stiffnesses of the faces and core; in this way the analysis method is not bound to any one of several micromechanics theories available for laminated structures.

The stiffness and consistent mass matrices which were generated incorporate eighty nodal degrees of freedom; the corresponding matrices for thin anisotropic and transversely heterogeneous plates and cylindrical shells can be obtained by simply considering one face of the sandwich and involves 48 degrees of freedom.

Although they do not provide a complete evaluation of the potential of the elements reported, the numerical examples presented indicate that relatively few sandwich elements are required in order to predict accurately displacements, stresses and natural frequencies of sandwich systems. Also, it appears that the elements will be very useful for the analysis of thin composite-type structures for which only a few "closed form" solutions now exist. Finally, the energy search concept was extended to predicting the response of structures in the presence of destabilizing loads. Although the method is not generally recommended as a substitute for calculating buckling loads by more traditional methods, it is interesting in that it exhibits the energy search equivalent of the incremental stiffness or initial stress matrix method.

SECTION VI

REFERENCES

1. Key, S. W., A Convergence Investigation of the Direct Stiffness Method, Ph. D. thesis, Department of Aeronautics and Astronautics, University of Washington, Seattle, Washington (1966).

2. McLay, R. W., "Completeness and Convergence Properties of Finite Element Dispacement Functions--A General Treatment," presented at the AIAA 5th Aerospace Sciences Meeting, New York, Jan. 23-26, (1967).

3. Bogner, F. K., Fox, R. L., and Schmit, L. A., "The Generation of Interelement-Compatible Stiffness and Mass Matrices by the Use of Interpolation Formulas," Proc. of Conf. on Matrix Methods in Structural Mechanics, Wright-Patterson Air Force Base, AFFDL-TR-66-80, pp. 397-443, (1966).

4. Bogner, F. K., Fox, R. L., and Schmit, L. A., "A Cylindrical Shell Discrete Element," AIAA Journal, Vol. 5, No. 4, pp. 745-750, (1967).

5. Monforton, G. R., and Schmit, L. A., "Finite Element Analysis of Skew Plates in Bending," AIAA Journal, Vol. 6, No. 6, pp. 1150-1153, (1968).

6. Fletcher, R., and Reeves, C. M., "Function Minimization by Conjugate Gradients", Computer Journal, Vol. 7, pp. 149-154 (1964).

7. Fox, R. L., and Stanton, E. L., "Developments in Structural Analysis by Direct Energy Minimization," AIAA Journal, Vol. 6, No. 6, pp. 1036-1042, (1968).

8. Dong, S. B., Matthiesien, R. B., Pister, K. S., and Taylor, R. L., Analysis of Structural Laminates, ARL 76, Aeronautical Research Lab., Office of Aerospace Research, USAF, Wright-Patterson Air Force Base, Sept., (1961).

9. Chamis, C. C., Design Oriented Analysis and Structural Synthesis of Multilayered Filamentary Composites, Ph.D. thesis, Case Western Reserve University, Cleveland, Ohio, (1967).

10. Azzi, V. D., and Tsai, S. W., "Elastic Moduli of Laminated Anisotropic Composites," presented at SESA Annual Meeting, Cleveland, Ohio, Oct. 28-30, (1964).

11. Reissner, E., and Stavsky, Y., "Bending and Stretching of Certain Types of Heterogeneous Aeolotropic Elastic Plates," J. Appl. Mech., pp. 402-408, Sept. (1961).

12. Lewis, W. C., "Supplement to: Deflection and Stresses in A Uniformly Loaded, Simply Supported Rectangular Sandwich Plate--Experimental Verification of Theory," FPL Report No. 1847-A, Dec., (1956).

13. Oberndorfer, W. J., Equations for Thermoelastic and Viscoelastic Cylindrical Sandwich Shells, NASA CR-645, Nov. (1966).

14. Raville, M. E., and Ueng, C. E. S., "Determination of Natural Frequencies of Vibration of a Sandwich Plate," Exp. Mech., pp. 490-493, Nov., (1967).

REFERENCES (CONT)

15. Gallagher, R. H., and Padlog, J., "Discrete Element Approach to Structural Instability Analysis," AIAA J., Vol. 1, No. 6, pp. 1437-1439, (1963).

16. Hoff, N. J., "Bending and Buckling of Rectangular Sandwich Plates", NACA TN 2225, pp. 20-21, (1950).

17. Plantema, F. J., Sandwich Construction, John Wiley & Sons, Inc., New York, (1966).

18. Boller, K. H., Buckling Loads of Flat Sandwich Panels in Compression: The Buckling of Flat Sandwich Panels with Edges Simply Supported, FPL Report No. 1525-A, Feb. (1947).

19. Martin, H. C., "On the Derivation of Stiffness Matrices for the Analysis of Large Deflection and Stability Problems," Conference on Matrix Methods in Structural Mechanics, Wright-Patterson Air Force Base, AFFDL-TR-66-80, (1966).

APPENDIX

STIFFNESS AND MASS MATRIX FORMULAS

The elements of the submatrices which make up the stiffness matrix K and the mass matrix M, Equations 33 and 36 are given by the formulas that follow. For i,j = 1,16 the constants $q_{ij}^{(k)}$, k = 1,2,...,13 are symmetric (i.e. $q_{ij}^{(k)} = q_{ji}^{(k)}$) and appear in Table IV. The constants $q_{ij}^{(k)}$, k = 14,15,...,25 are nonsymmetric and appear in Table V. In addition the following definitions are made:

$$L_{cij} = a^{\ell_{ij}} \, b_c^{\eta_{ij}}/(420)^2 \; ; \; L_{fij} = a^{\ell_{ij}} \, b_f^{\eta_{ij}}/(420)^2 \tag{57}$$

$$N_{cij} = a \, b_c \, L_{cij} \qquad ; \; N_{fij} = a \, b_f \, L_{fij} \; , (f = 1,2) \tag{58}$$

where ℓ_{ij} and η_{ij} are also given in Tables IV and V. The remaining terms appearing in the formulas have been defined in the test. The formulas are valid for both plate

$$\left(\frac{1}{R_1} = \frac{1}{R_2} = \frac{1}{R_c} = 0, \quad \frac{R_c}{R_1} = \frac{R_c}{R_2} \equiv 1, = b_c = b_f \equiv b \right)$$

and cylindrical shell elements.

STIFFNESS MATRIX ELEMENTS

$$k_{ij}^{(u_f u_f)} = N_{cij} \frac{B_{55}}{t_c^2} q_{ij}^{(1)} + L_{fij} \left[A_{f11} \left(\frac{b_f}{a} \right) q_{ij}^{(2)} + A_{f66} \left(\frac{a}{b_f} \right) q_{ij}^{(3)} + A_{f16} \, q_{ij}^{(4)} \right] \tag{59}$$

$$k_{ij}^{(v_f v_f)} = N_{cij} \frac{B_{44}}{t_c^2} e_f^2 \, q_{ij}^{(1)} + L_{fij} \left[\left(A_{f66} + 4 \frac{D_{f66}}{R_f^2} + 4 \frac{B_{f66}}{R_f} \right) \left(\frac{b_f}{a} \right) q_{ij}^{(2)} \right.$$
$$\left. + \left(A_{f22} + \frac{D_{f22}}{R_f^2} + 2 \frac{B_{f22}}{R_f} \right) \left(\frac{a}{b_f} \right) q_{ij}^{(3)} + \left(A_{f26} + 2 \frac{D_{f26}}{R_f^2} + 3 \frac{B_{f26}}{R_f} \right) q_{ij}^{(4)} \right] \tag{60}$$

$$k_{ij}^{(ww)} = \frac{L_{cij}}{t_c^2} \left[(d_1 + d_2 + t_c)^2 \; B_{55} \left(\frac{b_c}{a} \right) q_{ij}^{(2)} + e_3^2 \; B_{44} \left(\frac{a}{b_f} \right) q_{ij}^{(3)} \right]$$

$$+ \sum_{f=1}^{2} \left\{ L_{fij} \left[\frac{A_{f22}}{R_f^2} (a b_f) \, q_{ij}^{(1)} + D_{f11} \left(\frac{b_f}{a^3} \right) q_{ij}^{(5)} + D_{f22} \left(\frac{a}{b_f^3} \right) q_{ij}^{(6)} \right. \right.$$

$$+ D_{f12} \left(\frac{1}{a b_f} \right) q_{ij}^{(7)} + 2D_{f16} \left(\frac{1}{a^2} \right) q_{ij}^{(8)} + 2D_{f26} \left(\frac{1}{b_f^2} \right) q_{ij}^{(9)}$$

$$+ 4D_{f66} \left(\frac{1}{a b_f} \right) q_{ij}^{(10)} - \frac{B_{f22}}{R_f} \left(\frac{a}{b_f} \right) q_{ij}^{(11)}$$

$$\left. \left. - \frac{B_{f12}}{R_f} \left(\frac{b_f}{a} \right) q_{ij}^{(12)} - 2 \, \frac{B_{f26}}{R_f} \, q_{ij}^{(13)} \right] \right\} \tag{61}$$

$$k_{ij}^{(u_1 u_2)} = - N_{cij} \, \frac{B_{55}}{t_c^2} \, q_{ij}^{(1)} \tag{62}$$

$$k_{ij}^{(v_1 v_2)} = - N_{cij} \, \frac{B_{44}}{t_c^2} \, e_1 e_2 \, q_{ij}^{(1)} \tag{63}$$

$$k_{ij}^{(u_f v_f)} = L_{fij} \left[\left(A_{f16} + 2 \frac{B_{f16}}{R_f} \right) \left(\frac{b_f}{a} \right) q_{ij}^{(2)} + \left(A_{f26} + \frac{B_{f26}}{R_f} \right) \left(\frac{a}{b_f} \right) q_{ij}^{(3)} \right.$$

$$\left. + \left(A_{f66} + 2 \frac{B_{f66}}{R_f} \right) q_{ij}^{(14)} + \left(A_{f12} + \frac{B_{f12}}{R_f} \right) q_{ij}^{(15)} \right] \tag{64}$$

$$k_{ij}^{(u_f w)} = (-1)^f L_{cij} \left[(d_1 + d_2 + t_c) \frac{B_{55}}{t_c^2} (b_c) q_{ij}^{(16)} \right]$$

$$- L_{fij} \left[B_{f11} \left(\frac{b_f}{a^2} \right) q_{ij}^{(18)} + 2 B_{f66} \left(\frac{1}{b_f} \right) q_{ij}^{(19)} + B_{f12} \left(\frac{1}{b_f} \right) q_{ij}^{(20)} \right.$$

$$+ B_{f16} \left(\frac{1}{a} \right) \left(q_{ij}^{(21)} + 2 q_{ij}^{(22)} \right) + B_{f26} \left(\frac{a}{b_f^2} \right) q_{ij}^{(23)}$$

$$\left. - \frac{A_{f12}}{R_f} (b_f) q_{ij}^{(24)} - \frac{A_{f26}}{R_f} (a) q_{ij}^{(25)} \right] \qquad (65)$$

$$k_{ij}^{(v_f w)} = (-1)^f L_{cij} \left[\left(\frac{e_f e_3}{t_c^2} \right) B_{44} (a) q_{ij}^{(17)} \right] - L_{fij} \left[\left(B_{f16} + 2 \frac{D_{f16}}{R_f} \right) \left(\frac{b_f}{a^2} \right) q_{ij}^{(18)} \right.$$

$$+ 2 \left(B_{f26} + \frac{D_{f26}}{R_f} \right) \left(\frac{1}{b_f} \right) q_{ij}^{(19)} + \left(B_{f26} + 2 \frac{D_{f26}}{R_f} \right) \left(\frac{1}{b_f} \right) q_{ij}^{(20)}$$

$$+ \left(B_{f12} + \frac{D_{f12}}{R_f} \right) \left(\frac{1}{a} \right) q_{ij}^{(21)} + 2 \left(B_{f66} + 2 \frac{D_{f66}}{R_f} \right) \left(\frac{1}{a} \right) q_{ij}^{(22)}$$

$$+ \left(B_{f22} + \frac{D_{f22}}{R_f} \right) \left(\frac{a}{b_f^2} \right) q_{ij}^{(23)} - \left(\frac{A_{f26}}{R_f} + 2 \frac{B_{f26}}{R_f^2} \right) (b_f) q_{ij}^{(24)}$$

$$\left. - \left(\frac{A_{f22}}{R_f} + \frac{B_{f22}}{R_f^2} \right) (a) q_{ij}^{(25)} \right] \qquad (66)$$

MASS MATRIX ELEMENTS

$$m_{ij}^{(u_f u_f)} = \left\{ N_{fij} Q_f + \frac{N_{cij}}{t_c^2} \left[\frac{Q_c}{4} + (-1)^f \frac{J_c}{t_c} + \frac{I_c}{t_c^2} \right] \right\} q_{ij}^{(1)} \qquad (67)$$

$$m_{ij}^{(v_f v_f)} = \left\{ N_{fij} \left[Q_f - 2 \frac{J_f}{R_f} + \frac{I_f}{R_f^2} \right] + \frac{N_{cij}}{t_c^2} \left[\frac{Q_c}{4} + (-1) f \frac{J_c}{t_c} + \frac{I_c}{t_c^2} \right] \right\} q_{ij}^{(1)} \qquad (68)$$

$$m_{ij}^{(ww)} = \sum_{f=1}^{2} L_{fij} \left[Q_f (ab_f)\, q_{ij}^{(1)} + I_f \left(\frac{b_f}{a}\right) q_{ij}^{(2)} + I_f \left(\frac{a}{b_f}\right) q_{ij}^{(3)} \right]$$

$$+ L_{cij} \left\{ Q_c (ab_c)\, q_{ij}^{(1)} + \left[\frac{Q_c}{4}(d_1 - d_2)^2 - \frac{J_c}{t_c}(d_1^2 - d_2^2) + \frac{I_c}{t_c^2}(d_1 + d_2)^2\right]\left(\frac{b_c}{a}\right) q_{ij}^{(2)} \right.$$

$$\left. + \left[\frac{Q_c}{4}(h_1 - h_2)^2 - \frac{J_c}{t_c}(h_1^2 - h_2^2) + \frac{I_c}{t_c^2}(h_1 + h_2)^2\right]\left(\frac{a}{b_c}\right) q_{ij}^{(3)} \right\} \tag{69}$$

$$m_{ij}^{(u_1 u_2)} = N_{cij} \left[\frac{Q_c}{4} - \frac{I_c}{t_c^2}\right] q_{ij}^{(1)} = m_{ji}^{(u_1 u_2)} \tag{70}$$

$$m_{ij}^{(v_1 v_2)} = N_{cij} \left[\frac{Q_c}{4} - \frac{I_c}{t_c^2}\right] q_{ij}^{(1)} = m_{ji}^{(v_1 v_2)} \tag{71}$$

$$m_{ij}^{(u_f w)} = -\left\{ L_{fij}\, J_f\, b_f + L_{cij}\, b_c \left[\frac{Q_c}{4}(d_1 - d_2) - \frac{J_c}{t_c}\, d_f \right.\right.$$

$$\left.\left. - (-1)^f \frac{I_c}{t_c^2}(d_1 + d_2)\right]\right\} q_{ij}^{(16)} \tag{72}$$

$$m_{ij}^{(v_f w)} = -\left\{ L_{fij} \left[J_f - \frac{I_f}{R_f}\right] + L_{cij} \left[\frac{Q_c}{4}(h_1 - h_2) - \frac{J_c}{t_c}\frac{R_c}{R_f}\, d_f \right.\right.$$

$$\left.\left. - (-1)^f \frac{I_c}{t_c^2}(h_1 + h_2)\right]\right\} a\, q_{ij}^{(17)} \tag{73}$$

TABLE IV SYMMETRIC CONSTANTS AND EXPONENTS FOR SUBMATRICES

i	j	$q_{ij}^{(1)}$	$q_{ij}^{(2)}$	$q_{ij}^{(3)}$	$q_{ij}^{(4)}$	$q_{ij}^{(5)}$	$q_{ij}^{(6)}$	$q_{ij}^{(7)}$	$q_{ij}^{(8)}$	$q_{ij}^{(9)}$	$q_{ij}^{(10)}$	$q_{ij}^{(11)}$	$q_{ij}^{(12)}$	$q_{ij}^{(13)}$	ℓ_{ij}	n_{ij}
1	1	24336	78624	78624	88200	786240	786240	508032	0	0	254016	-157248	-157248	88200	0	0
1	2	3432	6552	11088	0	393120	110880	254016	0	0	21168	-22176	-78624	0	1	0
1	3	3432	11088	6552	0	110880	393120	254016	0	0	21168	-78624	-22176	0	0	1
1	4	484	924	924	-3528	55440	55440	38808	-35280	-35280	1764	-11088	-11088	3528	1	1
1	5	8424	27216	-78624	0	272160	-786240	-508032	0	0	-254016	157248	-54432	0	0	0
1	6	1188	2268	-11088	-17640	136080	-110880	-254016	-176400	0	-21168	22176	-27216	17640	1	0
1	7	-2028	-6552	6552	0	-65520	393120	42336	0	0	21168	-13104	13104	0	0	1
1	8	-286	-546	924	3528	-32760	55440	21168	35280	35280	1764	-1848	6552	-3528	1	1
1	9	2916	-27216	-27216	-88200	-272160	-272160	508032	0	0	254016	54432	54432	88200	0	0
1	10	-702	2268	6552	17640	136080	65520	-42336	176400	0	-21168	-13104	-4536	-17640	1	0
1	11	-702	6552	2268	17640	65520	136080	-42336	0	176400	-21168	-4536	-13104	-17640	0	1
1	12	169	-546	-546	-3528	-32760	-32760	3528	-35280	-35280	1764	1092	1092	3528	1	1
1	13	8424	-78624	27216	0	-786240	272160	-508032	0	0	-254016	-54432	157248	0	0	0
1	14	-2028	6552	-6552	0	393120	-65520	42336	0	0	21168	13104	-13104	0	1	0
1	15	1188	-11088	2268	-17640	-110880	136080	-254016	0	-176400	-21168	-27216	22176	17640	0	1
1	16	-286	924	-546	3528	55440	-32760	21168	35280	35280	1764	6552	-1848	-3528	1	1
2	2	624	8736	2016	0	262080	20160	56448	88200	0	28224	-4032	-17472	0	2	0
2	3	484	924	924	3528	55440	55440	215208	35280	35280	1764	-11088	-11088	-3528	1	1
2	4	88	1232	168	0	36960	10080	28224	0	0	2352	-2016	-2464	0	2	1
2	5	1188	2268	-11088	17640	136080	-110880	-254016	176400	0	-21168	22176	-27216	-17640	1	0
2	6	216	3024	-2016	0	90720	-20160	-56448	0	0	-28224	4032	-6048	0	2	0
2	7	-286	-546	924	-3528	-32760	55440	21168	-35280	-35280	1764	-1848	6552	3528	1	1
2	8	-52	-728	168	0	-21840	10080	4704	0	0	2352	-336	1456	0	2	1
2	9	702	-2268	-6552	-17640	-136080	-65520	42336	-176400	0	21168	13104	4536	17640	1	0
2	10	-162	-756	1512	2940	45360	15120	14112	88200	0	7056	-3024	1512	-2940	2	0
2	11	-169	546	546	3528	32760	32760	-3528	35280	35280	-1764	-1092	-1092	-3528	1	1
2	12	39	182	-126	-588	-10920	-7560	-1176	-17640	-5880	-588	252	-364	588	2	1
2	13	2028	-6552	6552	0	-393120	65520	-42336	0	0	-21168	-13104	13104	0	1	0
2	14	-468	-2184	-1512	0	131040	-15120	-14112	0	0	-7056	3024	4368	0	2	0
2	15	286	-924	546	-3528	-55440	32760	-21168	-35280	-35280	-1764	-6552	1848	3528	1	1
2	16	-66	-308	-126	588	18480	-7560	-7056	17640	5880	-588	1512	616	-588	2	1
3	3	624	2016	8736	0	20160	262080	56448	0	88200	28224	-17472	-4032	0	0	2
3	4	88	168	1232	0	10080	36960	28224	0	0	2352	-2464	-2016	0	1	2
3	5	2028	6552	-6552	0	65520	-393120	-42336	0	0	-21168	13104	-13104	0	0	1
3	6	286	546	-924	-3528	32760	-55440	-21168	-35280	-35280	-1764	1848	-6552	3528	1	1
3	7	-468	-1512	-2184	0	-15120	131040	-14112	0	0	-7056	4368	3024	0	0	2
3	8	-66	-126	-308	588	-7560	18480	-7056	5880	17640	-588	616	1512	-588	1	2
3	9	702	-6552	-2268	-17640	-65520	-136080	42336	0	-176400	21168	4536	13104	17640	0	1
3	10	-169	546	546	3528	32760	32760	-3528	35280	35280	-1764	-1092	-1092	-3528	1	1
3	11	-162	1512	-756	2940	15120	45360	14112	0	88200	7056	1512	-3024	-2940	0	2
3	12	39	-126	182	-588	-7560	-10920	-1176	-5880	-17640	-588	-364	252	588	1	2
3	13	1188	-11088	2268	17640	-110880	136080	-254016	0	176400	-21168	-27216	22176	-17640	0	1
3	14	-286	924	-546	-3528	55440	-32760	21168	-35280	-35280	1764	6552	-1848	3528	1	1
3	15	216	-2016	3024	0	-20160	90720	-56448	0	0	-28224	-6048	4032	0	0	2
3	16	-52	168	-726	0	10080	-21840	4704	0	0	2352	1456	-336	0	1	2
4	4	16	224	224	0	6720	6720	6272	0	0	3136	-448	-448	0	2	2
4	5	286	546	-924	3528	32760	-55440	-21168	35280	35280	-1764	1848	-6552	-3528	1	1
4	6	52	728	-168	0	21840	-10080	-4704	0	0	-2352	336	-1456	0	2	1
4	7	-66	-126	-308	-588	-7560	18480	-7056	-5880	-17640	-588	616	1512	588	1	2
4	8	-12	-168	-56	0	-5040	3360	-1568	0	0	-784	112	336	0	2	2
4	9	169	-546	-546	-3528	-32760	-32760	3528	-35280	-35280	1764	1092	1092	3528	1	1
4	10	-39	-182	126	588	10920	7560	1176	17640	5880	588	-252	364	-588	2	1
4	11	-39	126	-182	588	7560	10920	1176	5880	17640	588	364	-252	-588	1	2
4	12	9	42	42	-98	-2520	-2520	392	-2940	-2940	196	-84	-84	98	2	2
4	13	286	-924	546	3528	-55440	32760	-21168	35280	35280	-1764	-6552	1848	-3528	1	1
4	14	-66	-308	-126	-588	18480	-7560	-7056	-17640	-5880	-588	1512	616	588	2	1
4	15	52	-168	726	0	-10080	21840	-4704	0	0	-2352	-1456	336	0	1	2
4	16	-12	-56	-168	0	3360	-5040	-1568	0	0	-784	336	112	0	2	2
5	5	24336	78624	78624	-88200	786240	786240	508032	0	0	254016	-157248	-157248	-88200	0	0
5	6	3432	6552	11088	0	393120	110880	254016	0	0	21168	-22176	-78624	0	1	0
5	7	-3432	-11088	-6552	0	-110880	-393120	-254016	0	0	-21168	78624	22176	0	0	1
5	8	-484	-924	-924	-3528	-55440	-55440	-38808	-35280	-35280	-1764	11088	11088	3528	1	1
5	9	8424	-78624	27216	0	-786240	272160	-508032	0	0	-254016	-54432	157248	0	0	0
5	10	-2028	6552	-6552	0	393120	-65520	42336	0	0	21168	13104	-13104	0	1	0
5	11	-1188	11088	-2268	-17640	110880	-136080	254016	0	-176400	21168	27216	-22176	17640	0	1
5	12	286	-924	546	3528	-55440	32760	-21168	35280	35280	-1764	-6552	1848	-3528	1	1
5	13	2916	-27216	-27216	88200	-272160	-272160	508032	0	0	254016	54432	54432	-88200	0	0
5	14	-702	2268	6552	-17640	136080	65520	-42336	-176400	0	-21168	-13104	-4536	17640	1	0
5	15	702	-6552	-2268	17640	-65520	-136080	42336	0	176400	21168	4536	13104	-17640	0	1
5	16	-169	546	546	-3528	32760	32760	-3528	-35280	-35280	-1764	-1092	-1092	3528	1	1
6	6	624	8736	2016	0	262080	20160	56448	-88200	0	28224	-4032	-17472	0	2	0
6	7	-484	-924	-924	3528	-55440	-55440	-215208	35280	35280	-1764	11088	11088	-3528	1	1
6	8	-88	-1232	-168	0	-36960	-10080	-28224	0	0	-2352	2016	2464	0	2	1
6	9	2028	-6552	6552	0	-393120	65520	-42336	0	0	-21168	-13104	13104	0	1	0
6	10	-468	-2184	-1512	0	131040	-15120	-14112	0	0	-7056	3024	4368	0	2	0
6	11	-286	924	-546	-3528	55440	-32760	21168	-35280	-35280	1764	6552	-1848	3528	1	1
6	12	66	308	126	588	-18480	7560	7056	17640	5880	588	-1512	-616	-588	2	1
6	13	702	-2268	-6552	17640	-136080	-65520	42336	176400	0	21168	13104	4536	-17640	1	0
6	14	-162	-756	1512	-2940	45360	15120	14112	-88200	0	7056	-3024	1512	2940	2	0
6	15	169	-546	-546	3528	-32760	-32760	3528	35280	35280	1764	1092	1092	-3528	1	1
6	16	-39	-182	126	-588	10920	7560	1176	-17640	-5880	588	-252	364	588	2	1
7	7	624	2016	8736	0	20160	262080	56448	0	-88200	28224	-17472	-4032	0	0	2
7	8	88	168	1232	0	10080	36960	28224	0	0	2352	-2464	-2016	0	1	2
7	9	-1188	11088	-2268	17640	110880	-136080	254016	0	176400	21168	27216	-22176	-17640	0	1
7	10	286	-924	546	-3528	-55440	32760	-21168	-35280	-35280	-1764	-6552	1848	3528	1	1
7	11	216	-2016	3024	0	-20160	90720	-56448	0	0	-28224	-6048	4032	0	0	2
7	12	-52	168	-726	0	10080	-21840	4704	0	0	2352	1456	-336	0	1	2
7	13	-702	6552	2268	-17640	65520	136080	-42336	0	-176400	-21168	-4536	-13104	17640	0	1
7	14	169	-546	-546	3528	-32760	-32760	3528	35280	35280	1764	1092	1092	-3528	1	1
7	15	-162	1512	-756	-2940	15120	45360	14112	0	-88200	7056	1512	-3024	2940	0	2
7	16	39	-126	182	588	-7560	-10920	-1176	5880	17640	-588	-364	252	-588	1	2
8	8	16	224	224	0	6720	6720	6272	0	0	3136	-448	-448	0	2	2
8	9	-286	924	-546	3528	55440	-32760	21168	35280	35280	1764	6552	-1848	-3528	1	1
8	10	66	308	126	-588	-18480	7560	7056	-17640	-5880	588	-1512	-616	588	2	1
8	11	52	-168	726	0	-10080	21840	-4704	0	0	-2352	-1456	336	0	1	2
8	12	-12	-56	-168	0	3360	-5040	-1568	0	0	-784	336	112	0	2	2
8	13	-169	546	546	-3528	32760	32760	-3528	-35280	-35280	-1764	-1092	-1092	3528	1	1
8	14	39	182	-126	588	-10920	-7560	-1176	17640	5880	-588	252	-364	-588	2	1
8	15	-39	126	-182	-588	7560	10920	1176	-5880	-17640	588	364	-252	588	1	2
8	16	9	42	42	98	-2520	-2520	392	2940	2940	196	-84	-84	-98	2	2
9	9	24336	78624	78624	88200	786240	786240	508032	0	0	254016	-157248	-157248	88200	0	0
9	10	-3432	-6552	-11088	0	-393120	-110880	-254016	0	0	-21168	22176	78624	0	1	0

TABLE IV (CONT)

i	j	$q_{ij}^{(1)}$	$q_{ij}^{(2)}$	$q_{ij}^{(3)}$	$q_{ij}^{(4)}$	$q_{ij}^{(5)}$	$q_{ij}^{(6)}$	$q_{ij}^{(7)}$	$q_{ij}^{(8)}$	$q_{ij}^{(9)}$	$q_{ij}^{(10)}$	$q_{ij}^{(11)}$	$q_{ij}^{(12)}$	$q_{ij}^{(13)}$	ℓ_{ij}	n_{ij}
9	11	-3432	-11088	-6552	0	-110880	-393120	-254016	0	0	-21168	78624	22176	0	0	1
9	12	484	924	924	-3528	55440	55440	38808	-35280	-35280	1764	-11088	-11088	3528	1	1
9	13	8424	27216	-78624	0	272160	-786240	-508032	0	0	-254016	157248	-54432	0	0	0
9	14	-1188	-2268	11088	17640	-136080	110880	254016	176400	0	21168	-22176	27216	-17640	1	0
9	15	2028	6552	-6552	0	65520	-393120	-42336	0	0	-21168	13104	-13104	0	0	1
9	16	-286	-546	924	3528	-32760	55440	21168	35280	35280	1764	-1848	6552	-3528	1	1
10	10	624	8736	2016	0	262080	20160	56448	88200	0	28224	-4032	-17472	0	2	0
10	11	484	924	924	3528	55440	55440	215208	35280	35280	1764	-11088	-11088	-3528	1	1
10	12	-88	-1232	-168	0	-36960	-10080	-28224	0	0	-2352	2016	2464	0	2	1
10	13	-1188	-2268	11088	-17640	-136080	110880	254016	-176400	0	21168	-22176	27216	17640	1	0
10	14	216	3024	-2016	0	90720	-20160	-56448	0	0	-28224	4032	-6048	0	2	0
10	15	-286	-546	924	-3528	-32760	55440	21168	-35280	-35280	1764	-1848	6552	3528	1	1
10	16	52	728	-168	0	21840	-10080	-4704	0	0	-2352	336	-1456	0	2	1
11	11	624	2016	8736	0	20160	262080	56448	0	88200	28224	-17472	-4032	0	0	2
11	12	-88	-168	-1232	0	-10080	-36960	-28224	0	0	-2352	2464	2016	0	1	2
11	13	-2028	-6552	6552	0	-65520	393120	42336	0	0	21168	-13104	13104	0	0	1
11	14	286	546	-924	-3528	32760	-55440	-21168	-35280	-35280	-1764	1848	-6552	3528	1	1
11	15	-468	-1512	-2184	0	-15120	131040	-14112	0	0	-7056	4368	3024	0	0	2
11	16	66	126	308	-588	7560	-18480	7056	-5880	-17640	588	-616	-1512	588	1	2
12	12	16	224	224	0	6720	6720	6272	0	0	3136	-448	-448	0	2	2
12	13	286	546	-924	3528	32760	-55440	-21168	35280	35280	-1764	1848	-6552	-3528	1	1
12	14	-52	-728	168	0	-21840	10080	4704	0	0	2352	-336	1456	0	2	1
12	15	66	126	308	588	7560	-18480	7056	5880	17640	588	-616	-1512	-588	1	2
12	16	-12	-168	-56	0	-5040	3360	-1568	0	0	-784	112	336	0	2	2
13	13	24336	78624	78624	-88200	786240	786240	508032	0	0	254016	-157248	-157248	-88200	0	0
13	14	-3432	-6552	-11088	0	-393120	-110880	-254016	0	0	-21168	22176	78624	0	1	0
13	15	3432	11088	6552	0	110880	393120	254016	0	0	21168	-78624	-22176	0	0	1
13	16	-484	-924	-924	-3528	-55440	-55440	-38808	-35280	-35280	-1764	11088	11088	3528	1	1
14	14	624	8736	2016	0	262080	20160	56448	-88200	0	28224	-4032	-17472	0	2	0
14	15	-484	-924	-924	3528	-55440	-55440	-215208	35280	35280	-1764	11088	11088	-3528	1	1
14	16	88	1232	168	0	36960	10080	28224	0	0	2352	-2016	-2464	0	2	1
15	15	624	2016	8736	0	20160	262080	56448	0	-88200	28224	-17472	-4032	0	0	2
15	16	-88	-168	-1232	0	-10080	-36960	-28224	0	0	-2352	2464	2016	0	1	2
16	16	16	224	224	0	6720	6720	6272	0	0	3136	-448	-448	0	2	2

TABLE V UNSYMMETRIC CONSTANTS AND EXPONENTS FOR SUBMATRICES

i	j	$q_{ij}^{(14)}$	$q_{ij}^{(15)}$	$q_{ij}^{(16)}$	$q_{ij}^{(17)}$	$q_{ij}^{(18)}$	$q_{ij}^{(19)}$	$q_{ij}^{(20)}$	$q_{ij}^{(21)}$	$q_{ij}^{(22)}$	$q_{ij}^{(23)}$	$q_{ij}^{(24)}$	$q_{ij}^{(25)}$	ℓ_{ij}	n_{ij}
1	1	44100	44100	-32760	-32760	0	-105840	105840	105840	-105840	0	-32760	-32760	0	0
1	2	-8820	8820	6552	-4620	65520	21168	21168	97020	-8820	0	-6552	-4620	1	0
1	3	8820	-8820	-4620	6552	0	-8820	97020	21168	21168	65520	-4620	-6552	0	1
1	4	-1764	-1764	924	924	9240	1764	19404	19404	1764	9240	-924	-924	1	1
1	5	44100	-44100	-11340	32760	0	105840	-105840	105840	105840	0	-11340	-32760	0	0
1	6	-8820	-8820	2268	4620	22680	-21168	-21168	97020	8820	0	-2268	-4620	1	0
1	7	-8820	8820	2730	-6552	0	-8820	8820	-21168	-21168	-65520	2730	6552	0	1
1	8	1764	1764	-546	-924	-5460	1764	1764	-19404	-1764	-9240	546	924	1	1
1	9	-44100	-44100	11340	11340	0	-105840	-105840	-105840	-105840	0	-11340	-11340	0	0
1	10	8820	8820	-2268	-2730	-22680	21168	21168	8820	8820	0	2268	2730	1	0
1	11	8820	8020	-2730	-2268	0	8820	8820	21168	21168	-22680	2730	2268	0	1
1	12	-1764	-1764	546	546	5460	-1764	-1764	-1764	-1764	5460	-546	-546	1	1
1	13	-44100	44100	32760	-11340	0	105840	105840	-105840	105840	0	-32760	-11340	0	0
1	14	8820	-8820	-6552	2730	-65520	-21168	-21168	8820	-8820	0	6552	2730	1	0
1	15	-8820	-8820	4620	2268	0	8820	97020	-21168	-21168	22680	-4620	-2268	0	1
1	16	1764	1764	-924	-546	-9240	-1764	-19404	1764	1764	-5460	924	546	1	1
2	1	8820	-8820	-6552	-4620	-65520	-21168	-21168	8820	-8820	0	6552	-4620	1	0
2	2	0	0	0	-840	-32760	0	0	11760	-11760	0	0	-840	2	0
2	3	1764	1764	-924	924	-9240	-1764	-19404	1764	1764	9240	924	-924	1	1
2	4	0	0	0	168	-4620	0	0	2352	2352	1680	0	-168	2	1
2	5	8820	8820	-2268	4620	-22680	21168	21168	8820	8820	0	2268	-4620	1	0
2	6	0	0	0	840	-11340	0	0	11760	11760	0	0	-840	2	0
2	7	-1764	-1764	546	-924	5460	-1764	-1764	-1764	-1764	-9240	-546	924	1	1
2	8	0	0	0	-168	2730	0	0	-2352	-2352	-1680	0	168	2	1
2	9	-8820	-8820	2268	2730	22680	-21168	-21168	-8820	-8820	0	-2268	-2730	1	0
2	10	1470	1470	-378	-630	-11340	3528	3528	-2940	-2940	0	378	630	2	0
2	11	1764	1764	-546	-546	-5460	1764	1764	1764	1764	-5460	546	546	1	1
2	12	-294	-294	91	126	2730	-294	-294	588	588	1260	-91	-126	2	1
2	13	-8820	8820	6552	-2730	65520	21168	21168	-8820	8820	0	-6552	-2730	1	0
2	14	1470	-1470	-1092	630	-32760	-3528	-3528	-2940	2940	0	1092	630	2	0
2	15	-1764	-1764	924	546	9240	1764	19404	-1764	-1764	5460	-924	-546	1	1
2	16	294	294	-154	-126	-4620	-294	-3234	-588	-588	-1260	154	126	2	1
3	1	-8820	8820	-4620	-6552	0	-8820	8820	-21168	-21168	-65520	-4620	6552	0	1
3	2	1764	1764	924	-924	9240	1764	1764	-19404	-1764	-9240	-924	924	1	1
3	3	0	0	-840	0	0	-11760	11760	0	0	-32760	-840	0	0	2
3	4	0	0	168	0	1680	2352	2352	0	0	-4620	-168	0	1	2
3	5	8820	-8820	-2730	6552	0	8820	-8820	21168	21168	65520	-2730	-6552	0	1
3	6	-1764	-1764	546	924	5460	-1764	-1764	19404	1764	9240	-546	-924	1	1
3	7	-1470	1470	630	-1092	0	2940	-2940	-3528	-3528	-32760	630	1092	0	2
3	8	294	294	-126	-154	-1260	-588	-588	-3234	-294	-4620	126	154	1	2
3	9	-8820	-8820	2730	2268	0	-8820	-8820	-21168	-21168	22680	-2730	-2268	0	1
3	10	1764	1764	-546	-546	-5460	1764	1764	1764	1764	-5460	546	546	1	1
3	11	1470	1470	-630	-378	0	-2940	-2940	3528	3528	-11340	630	378	0	2
3	12	-294	-294	126	91	1260	588	588	-294	-294	2730	-126	-91	1	2
3	13	8820	8820	4620	-2268	0	8820	8820	21168	21168	-22680	-4620	2268	0	1
3	14	-1764	-1764	-924	546	-9240	-1764	-1764	-1764	-1764	5460	924	-546	1	1
3	15	0	0	840	0	0	11760	11760	0	0	-11340	-840	0	0	2
3	16	0	0	-168	0	-1680	-2352	-2352	0	0	2730	168	0	1	2
4	1	-1764	-1764	-924	-924	-9240	-1764	-1764	-1764	-1764	-9240	924	924	1	1
4	2	0	0	0	-168	-4620	0	0	-2352	-2352	-1680	0	168	2	1
4	3	0	0	-168	0	-1680	-2352	-2352	0	0	-4620	168	0	1	2
4	4	0	0	0	0	-840	0	0	0	0	-840	0	0	2	2

TABLE V (CONT)

i	j	$q_{ij}^{(14)}$	$q_{ij}^{(15)}$	$q_{ij}^{(16)}$	$q_{ij}^{(17)}$	$q_{ij}^{(18)}$	$q_{ij}^{(19)}$	$q_{ij}^{(20)}$	$q_{ij}^{(21)}$	$q_{ij}^{(22)}$	$q_{ij}^{(23)}$	$q_{ij}^{(24)}$	$q_{ij}^{(25)}$	ε_{ij}	η_{ij}
4	5	1764	1764	-546	924	-5460	1764	1764	1764	1764	9240	546	-924	1	1
4	6	0	0	0	168	-2730	0	0	2352	2352	1680	0	-168	2	1
4	7	-294	-294	126	-154	1260	588	588	-294	-294	-4620	-126	154	1	2
4	8	0	0	0	-28	630	0	0	-392	-392	-840	0	28	2	2
4	9	-1764	-1764	546	546	5460	-1764	-1764	-1764	-1764	5460	-546	-546	1	1
4	10	294	294	-91	-126	-2730	294	294	-588	-588	-1260	91	126	2	1
4	11	294	294	-126	-91	-1260	-588	-588	294	294	-2730	126	91	1	2
4	12	-49	-49	21	21	630	98	98	98	98	630	-21	-21	2	2
4	13	1764	1764	924	-546	9240	1764	1764	1764	1764	-5460	-924	546	1	1
4	14	-294	-294	-154	126	-4620	-294	-294	588	588	1260	154	-126	2	1
4	15	0	0	168	0	1680	2352	2352	0	0	-2730	-168	0	1	2
4	16	0	0	-28	0	-840	-392	-392	0	0	630	28	0	2	2
5	1	-44100	44100	-11340	-32760	0	105840	-105840	-105840	-105840	0	-11340	32760	0	0
5	2	8820	8820	2268	-4620	22680	-21168	-21168	-97020	-8820	0	-2268	4620	1	0
5	3	-8820	8820	-2730	-6552	0	8820	-8820	-21168	-21168	-65520	-2730	6552	0	1
5	4	1764	1764	546	-924	5460	-1764	-1764	-19404	-1764	-9240	-546	924	1	1
5	5	-44100	-44100	-32760	32760	0	-105840	105840	-105840	105840	0	-32760	32760	0	0
5	6	8820	-8820	6552	4620	65520	21168	21168	-97020	8820	0	-6552	4620	1	0
5	7	8820	-8820	4620	6552	0	8820	-97020	21168	21168	65520	4620	-6552	0	1
5	8	-1764	-1764	-924	924	-9240	-1764	-19404	19404	1764	9240	924	-924	1	1
5	9	44100	-44100	32760	11340	0	105840	105840	105840	-105840	0	-32760	11340	0	0
5	10	-8820	8820	-6552	-2730	-65520	-21168	-21168	-8820	8820	0	6552	-2730	1	0
5	11	-8820	-8820	-4620	2268	0	-8820	-97020	-21168	-21168	22680	4620	-2268	0	1
5	12	1764	1764	924	-546	9240	1764	19404	1764	1764	-5460	-924	546	1	1
5	13	44100	44100	11340	-11340	0	-105840	-105840	105840	105840	0	-11340	11340	0	0
5	14	-8820	-8820	-2268	2730	-22680	21168	21168	-8820	-8820	0	2268	-2730	1	0
5	15	8820	8820	2730	-2268	0	-8820	-8820	21168	21168	-22680	-2730	2268	0	1
5	16	-1764	-1764	-546	546	-5460	1764	1764	-1764	-1764	5460	546	-546	1	1
6	1	-8820	-8820	-2268	-4620	-22680	21168	21168	-8820	-8820	0	2268	4620	1	0
6	2	0	0	0	-840	-11340	0	0	-11760	-11760	0	0	840	2	0
6	3	-1764	-1764	-546	-924	-5460	1764	1764	-1764	-1764	-9240	546	924	1	1
6	4	0	0	0	-168	-2730	0	0	-2352	-2352	-1680	0	168	2	1
6	5	-8820	8820	-6552	4620	-65520	-21168	-21168	-8820	8820	0	6552	4620	1	0
6	6	0	0	0	840	-32760	0	0	-11760	11760	0	0	840	2	0
6	7	1764	1764	924	924	9240	1764	19404	1764	1764	9240	-924	-924	1	1
6	8	0	0	0	168	4620	0	0	2352	2352	1680	0	-168	2	1
6	9	8820	-8820	6552	2730	65520	21168	21168	8820	-8820	0	-6552	2730	1	0
6	10	-1470	1470	-1092	-630	-32760	-3528	-3528	2940	-2940	0	1092	-630	2	0
6	11	-1764	-1764	-924	546	-9240	-1764	-19404	-1764	-1764	5460	924	-546	1	1
6	12	294	294	154	-126	4620	294	3234	-588	-588	-1260	-154	126	2	1
6	13	8820	8820	2268	-2730	22680	-21168	-21168	8820	8820	0	-2268	2730	1	0
6	14	-1470	-1470	-378	630	-11340	3528	3528	2940	2940	0	378	-630	2	0
6	15	1764	1764	546	-546	5460	-1764	-1764	1764	1764	-5460	-546	546	1	1
6	16	-294	-294	-91	126	-2730	294	294	588	588	1260	91	-126	2	1
7	1	8820	-8820	2730	6552	0	-8820	8820	21168	21168	65520	2730	-6552	0	1
7	2	-1764	-1764	-546	924	-5460	1764	1764	19404	1764	9240	546	-924	1	1
7	3	1470	-1470	630	1092	0	2940	-2940	3528	3528	32760	630	-1092	0	2
7	4	-294	-294	-126	154	-1260	-588	-588	3234	294	4620	126	-154	1	2
7	5	-8820	8820	4620	-6552	0	8820	-8820	-21168	-21168	-65520	4620	6552	0	1
7	6	1764	1764	-924	-924	-9240	-1764	-1764	-19404	-1764	-9240	924	924	1	1
7	7	0	0	-840	0	0	-11760	11760	0	0	32760	-840	0	0	2
7	8	0	0	168	0	1680	2352	2352	0	0	4620	-168	0	1	2
7	9	8820	8820	-4620	-2268	0	-8820	-8820	21168	21168	-22680	4620	2268	0	1
7	10	-1764	-1764	924	546	9240	1764	1764	-1764	-1764	5460	-924	-546	1	1
7	11	0	0	840	0	0	11760	11760	0	0	11340	-840	0	0	2
7	12	0	0	-168	0	-1680	-2352	-2352	0	0	-2730	168	0	1	2
7	13	-8820	-8820	-2730	2268	0	8820	8820	-21168	-21168	22680	2730	-2268	0	1
7	14	1764	1764	546	-546	5460	-1764	-1764	1764	1764	-5460	-546	546	1	1
7	15	-1470	-1470	-630	378	0	-2940	-2940	-3528	-3528	11340	630	-378	0	2
7	16	294	294	126	-91	1260	588	588	294	294	-2730	-126	91	1	2
8	1	1764	1764	546	924	5460	-1764	-1764	1764	1764	9240	-546	-924	1	1
8	2	0	0	0	168	2730	0	0	2352	2352	1680	0	-168	2	1
8	3	294	294	126	154	1260	588	588	294	294	4620	-126	-154	1	2
8	4	0	0	0	28	630	0	0	392	392	840	0	-28	2	2
8	5	-1764	-1764	924	-924	9240	1764	1764	-1764	-1764	-9240	-924	924	1	1
8	6	0	0	0	-168	4620	0	0	-2352	-2352	-1680	0	168	2	1
8	7	0	0	-168	0	-1680	-2352	-2352	0	0	4620	168	0	1	2
8	8	0	0	0	0	-840	0	0	0	0	840	0	0	2	2
8	9	1764	1764	-924	-546	-9240	-1764	-1764	1764	1764	-5460	924	546	1	1
8	10	-294	-294	154	126	4620	294	294	588	588	1260	-154	-126	2	1
8	11	0	0	168	0	1680	2352	2352	0	0	2730	-168	0	1	2
8	12	0	0	-28	0	-840	-392	-392	0	0	-630	28	0	2	2
8	13	-1764	-1764	-546	546	-5460	1764	1764	-1764	-1764	5460	546	-546	1	1
8	14	294	294	91	-126	2730	-294	-294	-588	-588	-1260	-91	126	2	1
8	15	-294	-294	-126	91	-1260	-588	-588	-294	-294	2730	126	-91	1	2
8	16	49	49	21	-21	630	98	98	-98	-98	-630	-21	21	2	2
9	1	-44100	-44100	-11340	-11340	0	105840	105840	105840	105840	0	11340	11340	0	0
9	2	-8820	-8820	-2268	-2730	-22680	21168	21168	8820	8820	0	2268	2730	1	0
9	3	-8820	-8820	-2730	-2268	0	8820	8820	21168	21168	-22680	2730	2268	0	1
9	4	-1764	-1764	-546	-546	-5460	1764	1764	1764	1764	-5460	546	546	1	1
9	5	-44100	44100	-32760	11340	0	-105840	-105840	105840	-105840	0	32760	11340	0	0
9	6	-8820	8820	-6552	2730	-65520	-21168	-21168	8820	-8820	0	6552	2730	1	0
9	7	8820	8820	4620	2268	0	8820	97020	-21168	-21168	22680	-4620	-2268	0	1
9	8	1764	1764	924	546	9240	1764	19404	-1764	-1764	5460	-924	-546	1	1
9	9	44100	44100	32760	32760	0	105840	-105840	-105840	105840	0	32760	32760	0	0
9	10	8820	-8820	6552	-4620	65520	21168	21168	97020	-8820	0	-6552	-4620	1	0
9	11	-8820	8820	-4620	6552	0	-8820	97020	21168	21168	65520	-4620	-6552	0	1
9	12	-1764	-1764	-924	-924	-9240	-1764	-19404	-19404	-1764	-9240	924	924	1	1
9	13	44100	-44100	11340	-32760	0	-105840	105840	-105840	-105840	0	11340	32760	0	0
9	14	8820	8820	2268	4620	22680	-21168	-21168	97020	8820	0	-2268	-4620	1	0
9	15	8820	-8820	2730	-6552	0	-8820	8820	-21168	-21168	-65520	2730	6552	0	1
9	16	1764	1764	546	924	5460	-1764	-1764	19404	1764	9240	-546	-924	1	1
10	1	8820	8820	2268	2730	22680	-21168	-21168	-8820	-8820	0	-2268	-2730	1	0
10	2	1470	1470	378	630	11340	-3528	-3528	2940	2940	0	-378	-630	2	0
10	3	1764	1764	546	546	5460	-1764	-1764	-1764	-1764	5460	-546	-546	1	1
10	4	294	294	91	126	2730	-294	-294	588	588	1260	-91	-126	2	1
10	5	8820	-8820	6552	-2730	65520	21168	21168	-8820	8820	0	-6552	-2730	1	0
10	6	1470	-1470	1092	-630	32760	3528	3528	2940	-2940	0	-1092	-630	2	0
10	7	-1764	-1764	-924	-546	-9240	-1764	-19404	1764	1764	-5460	924	546	1	1
10	8	-294	-294	-154	-126	-4620	-294	-3234	-588	-588	-1260	154	126	2	1
10	9	-8820	8820	-6552	-4620	-65520	-21168	-21169	8820	-8820	0	6552	-4620	1	0
10	10	0	0	0	840	32760	0	0	-11760	11760	0	0	840	2	0

TABLE V (CONT)

i	j	$q_{ij}^{(14)}$	$q_{ij}^{(15)}$	$q_{ij}^{(16)}$	$q_{ij}^{(17)}$	$q_{ij}^{(18)}$	$q_{ij}^{(19)}$	$q_{ij}^{(20)}$	$q_{ij}^{(21)}$	$q_{ij}^{(22)}$	$q_{ij}^{(23)}$	$q_{ij}^{(24)}$	$q_{ij}^{(25)}$	ℓ_{ij}	n_{ij}
10	11	1764	1764	924	-924	9240	1764	19404	-1764	-1764	-9240	-924	924	1	1
10	12	0	0	0	168	-4620	0	0	2352	2352	1680	0	-168	2	1
10	13	-8820	-8820	-2268	4620	-22680	21168	21168	8820	8820	0	2268	-4620	1	0
10	14	0	0	0	-840	11340	0	0	-11760	-11760	0	0	840	2	0
10	15	-1764	-1764	-546	924	-5460	1764	1764	1764	1764	9240	546	-924	1	1
10	16	0	0	0	-168	2730	0	0	-2352	-2352	-1680	0	168	2	1
11	1	8820	8820	2730	2268	0	-8820	-8820	-21168	-21168	22680	-2730	-2268	0	1
11	2	1764	1764	546	546	5460	-1764	-1764	-1764	-1764	5460	-546	-546	1	1
11	3	1470	1470	630	378	0	2940	2940	-3528	-3528	11340	-630	-378	0	2
11	4	294	294	126	91	1260	588	588	-294	-294	2730	-126	-91	1	2
11	5	-8820	-8820	4620	-2268	0	8820	8820	21168	21168	-22680	-4620	2268	0	1
11	6	-1764	-1764	924	-546	9240	1764	1764	1764	1764	-5460	-924	546	1	1
11	7	0	0	-840	0	0	-11760	-11760	0	0	11340	840	0	0	2
11	8	0	0	-168	0	-1680	-2352	-2352	0	0	2730	168	0	1	2
11	9	8820	-8820	-4620	-6552	0	-8820	8820	-21168	-21168	-65520	-4620	6552	0	1
11	10	1764	1764	-924	924	-9240	-1764	-1764	19404	1764	9240	924	-924	1	1
11	11	0	0	840	0	0	11760	-11760	0	0	32760	840	0	0	2
11	12	0	0	168	0	1680	2352	2352	0	0	-4620	-168	0	1	2
11	13	-8820	8820	-2730	6552	0	8820	-8820	21168	21168	65520	-2730	-6552	0	1
11	14	-1764	-1764	-546	-924	-5460	1764	1764	-19404	-1764	-9240	546	924	1	1
11	15	-1470	1470	-630	1092	0	-2940	2940	3528	3528	32760	-630	-1092	0	2
11	16	-294	-294	-126	-154	-1260	-588	-588	-3234	-294	-4620	126	154	1	2
12	1	-1764	-1764	-546	-546	-5460	1764	1764	1764	1764	-5460	546	546	1	1
12	2	-294	-294	-91	-126	-2730	294	294	-588	-588	-1260	91	126	2	1
12	3	-294	-294	-126	-91	-1260	-588	-588	294	294	-2730	126	91	1	2
12	4	-49	-49	-21	-21	-630	-98	-98	-98	-98	-630	21	21	2	2
12	5	1764	1764	-924	546	-9240	-1764	-1764	-1764	-1764	5460	924	-546	1	1
12	6	294	294	-154	126	-4620	-294	-294	588	588	1260	154	-126	2	1
12	7	0	0	168	0	1680	2352	2352	0	0	-2730	-168	0	1	2
12	8	0	0	28	0	840	392	392	0	0	-630	-28	0	2	2
12	9	-1764	-1764	924	924	9240	1764	1764	1764	1764	9240	-924	-924	1	1
12	10	0	0	0	-168	-4620	0	0	-2352	-2352	-1680	0	168	2	1
12	11	0	0	-168	0	-1680	-2352	-2352	0	0	-4620	168	0	1	2
12	12	0	0	0	0	840	0	0	0	0	840	0	0	2	2
12	13	1764	1764	546	-924	5460	-1764	-1764	-1764	-1764	-9240	-546	924	1	1
12	14	0	0	0	168	-2730	0	0	2352	2352	1680	0	-168	2	1
12	15	294	294	126	-154	1260	588	588	-294	-294	-4620	-126	154	1	2
12	16	0	0	0	28	-630	0	0	392	392	840	0	-28	2	2
13	1	44100	-44100	-32760	-11340	0	-105840	-105840	-105840	105840	0	32760	-11340	0	0
13	2	8820	-8820	-6552	-2730	-65520	-21168	-21168	-8820	8820	0	6552	-2730	1	0
13	3	8820	8820	-4620	2268	0	-8820	-97020	-21168	-21168	22680	4620	-2268	0	1
13	4	1764	1764	-924	546	-9240	-1764	-19404	-1764	-1764	5460	924	-546	1	1
13	5	44100	44100	-11340	11340	0	105840	105840	-105840	-105840	0	11340	-11340	0	0
13	6	8820	8820	-2268	2730	-22680	21168	21168	-8820	-8820	0	2268	-2730	1	0
13	7	-8820	-8820	2730	-2268	0	-8820	-8820	21168	21168	-22680	-2730	2268	0	1
13	8	-1764	-1764	546	-546	5460	-1764	-1764	1764	1764	-5460	-546	546	1	1
13	9	-44100	44100	11340	32760	0	-105840	105840	105840	105840	0	11340	-32760	0	0
13	10	-8820	-8820	2268	-4620	22680	-21168	-21168	-97020	-8820	0	-2268	4620	1	0
13	11	8820	-8820	-2730	-6552	0	8820	-8820	-21168	-21168	-65520	-2730	6552	0	1
13	12	1764	1764	-546	924	-5460	1764	1764	19404	1764	9240	546	-924	1	1
13	13	-44100	-44100	32760	-32760	0	105840	-105840	105840	-105840	0	32760	-32760	0	0
13	14	-8820	8820	6552	4620	65520	21168	21168	-97020	8820	0	-6552	4620	1	0
13	15	-8820	8820	4620	6552	0	8820	-97020	21168	21168	65520	4620	-6552	0	1
13	16	-1764	-1764	924	-924	9240	1764	19404	-19404	-1764	-9240	-924	924	1	1
14	1	-8820	8820	6552	2730	65520	21168	21168	8820	-8820	0	-6552	2730	1	0
14	2	-1470	1470	1092	630	32760	3528	3528	-2940	2940	0	-1092	630	2	0
14	3	-1764	-1764	924	-546	9240	1764	19404	1764	1764	-5460	-924	546	1	1
14	4	-294	-294	154	-126	4620	294	3234	-588	-588	-1260	-154	126	2	1
14	5	-8820	-8820	2268	-2730	22680	-21168	-21168	8820	8820	0	-2268	2730	1	0
14	6	-1470	-1470	378	-630	11340	-3528	-3528	-2940	-2940	0	-378	630	2	0
14	7	1764	1764	-546	546	-5460	1764	1764	-1764	-1764	5460	546	-546	1	1
14	8	294	294	-91	126	-2730	294	294	588	588	1260	91	-126	2	1
14	9	8820	8820	-2268	-4620	-22680	21168	21168	-8820	-8820	0	2268	4620	1	0
14	10	0	0	0	840	11340	0	0	11760	11760	0	0	-840	2	0
14	11	-1764	-1764	546	924	5460	-1764	-1764	1764	1764	9240	-546	-924	1	1
14	12	0	0	0	-168	-2730	0	0	-2352	-2352	-1680	0	168	2	1
14	13	8820	-8820	-6552	4620	-65520	-21168	-21168	-8820	8820	0	6552	4620	1	0
14	14	0	0	0	-840	32760	0	0	11760	-11760	0	0	-840	2	0
14	15	1764	1764	-924	-924	-9240	-1764	-19404	-1764	-1764	-9240	924	924	1	1
14	16	0	0	0	168	4620	0	0	2352	2352	1680	0	-168	2	1
15	1	-8820	-8820	-4620	-2268	0	-8820	-8820	21168	21168	-22680	4620	2268	0	1
15	2	-1764	-1764	-924	-546	-9240	-1764	-1764	1764	1764	-5460	924	546	1	1
15	3	0	0	-840	0	0	-11760	-11760	0	0	-11340	840	0	0	2
15	4	0	0	-168	0	-1680	-2352	-2352	0	0	-2730	168	0	1	2
15	5	8820	8820	-2730	2268	0	8820	8820	-21168	-21168	22680	2730	-2268	0	1
15	6	1764	1764	-546	546	-5460	1764	1764	-1764	-1764	5460	546	-546	1	1
15	7	-1470	-1470	630	-378	0	2940	2940	3528	3528	-11340	-630	378	0	2
15	8	-294	-294	126	-91	1260	588	588	294	294	-2730	-126	91	1	2
15	9	-8820	8820	2730	6552	0	-8820	8820	21168	21168	65520	2730	-6552	0	1
15	10	-1764	-1764	546	-924	5460	-1764	-1764	-19404	-1764	-9240	-546	924	1	1
15	11	1470	-1470	-630	-1092	0	-2940	2940	-3528	-3528	-32760	-630	1092	0	2
15	12	294	294	-126	154	-1260	-588	-588	3234	294	4620	126	-154	1	2
15	13	8820	-8820	4620	-6552	0	8820	-8820	-21168	-21168	-65520	4620	6552	0	1
15	14	1764	1764	924	924	9240	1764	1764	19404	1764	9240	-924	-924	1	1
15	15	0	0	840	0	0	11760	-11760	0	0	-32760	840	0	0	2
15	16	0	0	168	0	1680	2352	2352	0	0	4620	-168	0	1	2
16	1	1764	1764	924	546	9240	1764	1764	-1764	-1764	5460	-924	-546	1	1
16	2	294	294	154	126	4620	294	294	588	588	1260	-154	-126	2	1
16	3	0	0	168	0	1680	2352	2352	0	0	2730	-168	0	1	2
16	4	0	0	28	0	840	392	392	0	0	630	-28	0	2	2
16	5	-1764	-1764	546	-546	5460	-1764	-1764	1764	1764	-5460	-546	546	1	1
16	6	-294	-294	91	-126	2730	-294	-294	-588	-588	-1260	-91	126	2	1
16	7	294	294	-126	91	-1260	-588	-588	-294	-294	2730	126	-91	1	2
16	8	49	49	-21	21	-630	-98	-98	98	98	630	21	-21	2	2
16	9	1764	1764	-546	-924	-5460	1764	1764	-1764	-1764	-9240	546	924	1	1
16	7	294	294	-126	91	-1260	-588	-588	-294	-294	2730	126	-91	1	2
16	8	49	49	-21	21	-630	-98	-98	98	98	630	21	-21	2	2
16	9	1764	1764	-546	-924	-5460	1764	1764	-1764	-1764	-9240	546	924	1	1
16	10	0	0	0	168	2730	0	0	2352	2352	1680	0	-168	2	1
16	11	-294	-294	126	154	1260	588	588	294	294	4620	-126	-154	1	2
16	12	0	0	0	-28	-630	0	0	-392	-392	-840	0	28	2	2
16	13	-1764	-1764	-924	924	-9240	-1764	-1764	1764	1764	9240	924	-924	1	1
16	14	0	0	0	-168	4620	0	0	-2352	-2352	-1680	0	168	2	1
16	15	0	0	-168	0	-1680	-2352	-2352	0	0	4620	168	0	1	2
16	16	0	0	0	0	840	0	0	0	0	-840	0	0	2	2