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OPTIMIZATION OF STRUCTURES BASED ON THE STUDY OF ENERGY DISTRIBUTION

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An automated procedure is presented for minimum weight design of structures. It is an iterative procedure in which the design for the next cycle is determined by the study of the strain energy distribution in the present cycle. Displacement method of analysis is used in developing the method. Any other method of analysis which has the capability to determine strain energy in various parts of the structure should be applicable. Designs in the presence of stress constraints, and stress and displacement constraints are also considered. Where there are only stress constraints, a simple iteration based on the study of energy distribution is adequate. In the presence of displacement constraints, the design is carried in two stages. The first stage of iteration is similar to that in stress constraint problems and the second stage is based on a search procedure. Examples of two and three dimensional bar structures are presented to illustrate the effectiveness of the method. It proved to be extremely efficient in arriving at minimum weight structures.

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SECTION I

INTRODUCTION

Traditionally, structural design has been a trial procedure depending heavily on semi-empirical rules and the judgment of the designer. The empirical rules are being replaced gradually by sophisticated analysis methods made possible by the modern computer. The judgment of the designer still plays an important part in formulating intelligent design criteria. Because of the interdependence of the response and the design variables (particularly in case of indeterminate structures) the structural design will always be a trial procedure. The trial procedure can be developed into iterative procedures, and these procedures are amenable for automation.

In an ideal situation an automated design should include the development of an optimum configuration as well as efficient proportioning of structural components to achieve the desired objective. Utility, appearance, feasibility, and economy are some of the important considerations in developing an optimum configuration for the structure. Since many of these qualities are subjective and are not quantifiable, automation of configuration optimization is a difficult problem. For these reasons the present report is concerned only with minimum weight design of structures of predetermined geometry. An iterative method in which the resizing of the elements is based on the study of the strain energy distribution is presented in this paper.

The present structural design procedures can be classified, arbitrarily into four categories:

1. Geometrical optimization.

2. Design based on simultaneous failure modes (includes fully stressed design).

3. Optimization using search procedures based on linear and nonlinear programming techniques.

4. Methods using combination of the above approaches.

Under geometrical optimization come the development of the whole class of Michell Structures (References 1, 2, and 3). This development is based on the premise that the members of a minimum weight structure should follow the direction of principal strains. In other words the tension and compression members of a minimum weight structure form a mutually orthogonal network. In addition, such a structure has greater stiffness than any other structure of the same weight. It is evident from the principal strain requirement that the members of Michell Structures can be only axial force members. For a number of reasons Michell Structures are impractical for realistic structures. For one thing they are not relevant in the design of complex structures in the environment of multiplicity of loading conditions. In addition, it is difficult to incorporate geometrical constraints and stability considerations. But in spite of these limitations, Michell Structures provide a qualitative understanding to the designer and a norm for comparison of the minimum weight structures obtained by other methods.

The second approach in the list is more practical and frequently used in the minimum weight design. Much of the earlier research on structural optimization is based on the axiom that all components of a minumum weight structure reach their limiting capacity at the same time, with respect to either buckling or allowable stresses. The works of Shanley, (Reference 4), Gerard (Reference 5), and many others are based on the concept of simultaneous failure modes design or fully stressed design. In this approach, design starts with an initial design variable vector and the structure is analyzed for internal forces and displacements. Each element of the structure is then resized with the object of attaining a fully stressed design. The structure is reanalyzed with the new design variable vector and the members are resized and the process is repeated until there is no reduction in weight of the structure.

Simplicity and speed of convergence are the attractive features of the fully stressed design approach in the relevant cases. In the case of determinate structures subjected to single loading condition and with stress constraints only, the fully stressed design can be shown to yield a minimum weight structure. In case of design for multiple loading conditions, the concept of simultaneous failure mode design should be modified to read that every component of the structure should be stressed to its limit at least under one loading condition in which case the resulting design would be of minimum weight. In the case of indeterminate structures with constraints on the sizes and in the presence of displacement constraints it is impossible to satisfy this criteria and the method becomes irrelevant. Some interesting discussions on fully stressed design and its limitations can be found in References 6 and 7.

The third approach for structural optimization has come into prominence in recent years. References 8, 9, 10, 11, and 12 are some of the prominent papers on minimum weight design using this approach. In this approach minimum weight design is treated as a problem of mathematical extremization of an objective function in "m" dimensional design variable space. Then the search for extremum is carried out by methods of linear and nonlinear programming techniques. The feasibility of this approach for optimization of realistic structures is established in References 8, 10, and 11.

The reliability of the search methods and the speed of convergence of the earlier two approaches can be combined for optimization of structures with large number of design variables. The method presented here falls into fourth category. However, it does not use simultaneous failure modes or principal strain requirements as the design criteria. Resizing of members is based on the study of distribution of strain energy in the structure. If there are only stress constraints the optimal design can be reached quite rapidly by simple iteration. If there are displacement constraints however, some sort of a search procedure is necessary in the final stages of the design.

SECTION II

STRAIN ENERGY EXPRESSIONS AND EQUATIONS OF ANALYSIS

In all optimization methods a large part of the effort is expended in the repeated analysis of the structure with different design variable vectors. Only after analysis it is possible to determine whether a given design is acceptable and its position in the design space relative to the constraint surface. The direction of future travel is also determined from the results of the present analysis. The necessary equations for the displacement method of analysis and the expressions for strain energy of the structure and the elements are presented here.

In a finite element scheme the total structure is represented by a group of structural elements. The force displacement relations of the individual elements are derived by energy formulation with the assumption of exact or approximate displacement distributions. The strain energy u_i of i^{th} such element may be written as

$$u_i = \frac{1}{2} \int_{V_i} \sigma_i^t \quad e_i \, dv \tag{1}$$

σ_i and e_i are the internal stress and strain matrices respectively of the i^{th} element and V_i is its volume. For an element made of linearly elastic material the stress and strains are related by

$$\sigma_i = G_i \; e_i \tag{2}$$

where G_i is the matrix of elastic constants for the i^{th} element. The size and form of matrix G_i depends on the type of element. Substitution of Equation 2 into 1 gives the expression for strain energy in the form

$$u_i = \frac{1}{2} \int_{V_i} e_i^t \; G_i \; e_i \; dv \tag{3}$$

So far no approximations regarding displacement variations are involved in Equations 1 to 3. To derive the strain displacement relations the knowledge of the displacement variations in the element is necessary. In most cases the displacements in the element are assumed to vary linearly or in some other fashion depending on the type of element and the accuracy requirements. Reference 13 lists some of the expressions for displacement variations used in various element stiffness formulations. For an assumed displacement distribution the strain at any point in the structural element may be expressed as a function of a set of discrete generalized coordinates in the form

$$e_i = \phi_i \; \nu_i \tag{4}$$

where e_i is the strain matrix as defined before. $\left[\nu_1, \nu_2 \cdots \nu_p\right]$ are a set of "p" discrete generalized coordinates of the element as shown in Figure 1 for various structural elements. These are in the present case displacements of the nodal points that connect the element to the structure. ϕ_i is a rectangular matrix whose elements are in general, functions of the spatial coordinates expressed in element coordinate system. From Equations 3 and 4 the strain energy of the element may be written as

$$u_i = \frac{1}{2} \int_{V_i} \nu_i^t \; \phi_i^t \; G_i \; \phi_i \; \nu_i \; dv \tag{5}$$

ν_i is the displacement vector of the discrete coordinates and is independent of the spatial coordinates. Since the integration is over volume the matrix, ν_i does not participate in the integration. Then Equation 5 may be written as

$$u_i = \frac{1}{2} \; \nu_i^t \; k_i \; \nu_i \tag{6}$$

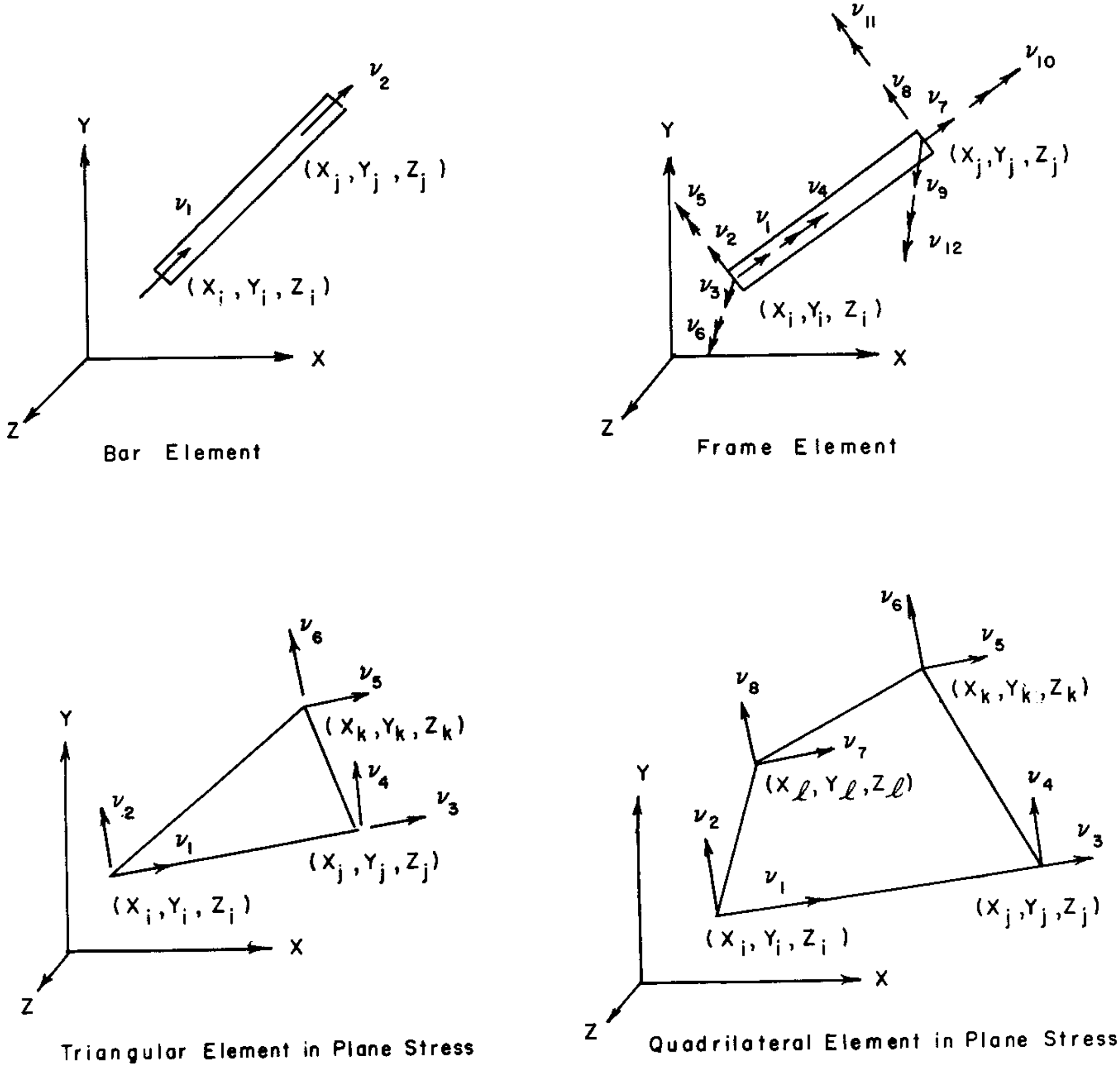


Figure 1. Structural Elements and their Coordinates

where $\mathbf{k}_i$ is the i^{th} element stiffness matrix with respect to the displacement coordinates $\boldsymbol{\nu}_i$ and is given by

$$\mathbf{k}_i = \int_{V_i} \boldsymbol{\phi}_i^t \; G_i \; \boldsymbol{\phi}_i \; dv \tag{7}$$

The stiffness matrices of bar, frame, plate and shell elements are derived by expressions similar to 7. From Castigliano's first theorem the element generalized force matrix $\mathbf{S}_i$ corresponding to the displacement matrix $\boldsymbol{\nu}_i$ may be written as

$$\mathbf{S}_i = \left\{ \frac{\partial u_i}{\partial \nu_j} \right\} = \mathbf{k}_i \; \boldsymbol{\nu}_i \tag{8}$$

The element strain energy may be written also in terms of its forces and displacements

$$u_i = \frac{1}{2} \; \mathbf{S}_i^t \; \boldsymbol{\nu}_i \tag{9}$$

The total strain energy U of the structure is obtained by summing the strain energies of its components

$$U = \frac{1}{2} \sum_{i=1}^{m} \boldsymbol{\nu}_i^t \; \mathbf{k}_i \; \boldsymbol{\nu}_i \tag{10}$$

where m represents the total number of elements in the structure. If r_1, r_2 ----- r_n are the generalized displacement coordinates of the structure, they can be related to the element coordinates ν_1, ν_2 ---- ν_p, by the following relation

$$\boldsymbol{\nu}_i = \mathbf{a}_i \; \mathbf{r} \tag{11}$$

"n" is the number of degrees of freedom of the structure. $\boldsymbol{\nu}_i$ are the displacement coordinates of i^{th} element. The matrix $\mathbf{a}_i$ may be called compatibility matrix and its elements can be obtained by kinematic reasoning alone. From Equations 10 and 11 the strain energy of the total structure is written as

$$U = \frac{1}{2} \sum_{i=1}^{m} \mathbf{r}^t \; \mathbf{a}_i^t \; \mathbf{k}_i \; \mathbf{a}_i \; \mathbf{r} \tag{12}$$

If the total structure stiffness, $\mathbf{K}$ is defined as

$$\mathbf{K} = \sum_{i=1}^{m} \mathbf{a}_i^t \; \mathbf{k}_i \; \mathbf{a}_i \tag{13}$$

The total strain energy of the structure can be written as

$$U = \frac{1}{2} \mathbf{r}^t \mathbf{K} \mathbf{r} \tag{14}$$

The generalized force matrix **R** of the structure corresponding to displacement matrix **r** is given by

$$\mathbf{R} = \left\{ \frac{\partial u}{\partial r_j} \right\} = \mathbf{K} \mathbf{r} \tag{15}$$

The strain energy of the structure can also be written as

$$U = \frac{1}{2} \mathbf{R}^t \mathbf{r} \tag{16}$$

Equations 1 to 16 are all that are necessary for analysis (by displacement method) and determination of strain energy. Usually, programming for computer analysis starts from Equation 13. This equation indicates the procedure for assembling the structure stiffness matrix. The stiffness matrices **k** for bar elements, frame elements, triangular and quadrilateral plate elements, and so forth, are well documented in the literature (for example, References 13 and 14).

The matrix **a** serves two purposes. It transforms the element stiffness matrix from the local coordinate system to the structure coordinate system and indicates the nodes to which the element is connected. The detailed description of the stiffness matrix assembly procedure from Equation 13 may be found in References 13 or 14 or any other reference on matrix structural analysis.

For a given load matrix **R** the displacement matrix **r** can be determined from Equation 15. The element displacements and forces can be determined by equations 11 and 8, respectively. From internal forces the internal stresses in the element can be determined by

$$\boldsymbol{\sigma}_i = \boldsymbol{\beta}_i \, \mathbf{S}_i \tag{17}$$

The elements of matrix $\boldsymbol{\beta}_i$ are functions of the geometrical properties and the coordinates of the points of the i^{th} structural element expressed in the local coordinate system.

If the internal forces and the displacements of the elements are known, their strain energy can be determined by Equations 6 or 9. Equations 14 or 16 give the strain energy of the total structure.

SECTION III

ITERATION FOR OPTIMUM DESIGN VARIABLE VECTOR

A structure deforms under a given loading, inducing strain energy in various elements. The strain energy in an element is a measure of its participation in resisting the applied loads. Intuitively, it is evident that an efficient structure is obtained by distributing the material among various components in proportion to the strain energy in the elements.

To establish a measure of quantitativeness to this statement the term average strain energy density is defined as the energy per unit volume of the element. Ideally the minimum weight structure is the one that has the same average strain energy density in various parts of the structure. The implications of the statement will be discussed further in the next section. The design of hybrid structures and variable stress limits can be handled more effectively by replacing the concept of strain energy density by energy capacity. The minimum weight structure is the one in which strain energy in various elements is equal to their energy capacity. The energy capacity of the element is defined as the total strain energy stored in the element if the entire element is stressed to its limiting stress. This condition is impossible to meet except possibly in determinate bar structures. But the objective can be modified to state that the ratio of the actual strain energy to the energy capacity should be made constant in various parts of the structure. Once again it should be pointed out that this is the objective to strive for and may not be possible to attain in view of the various constraints on the structure. Except with determinate structures the relative distribution of strain energy depends on the distribution of material. Because of the interdependence of the material distribution and strain energy, an iterative procedure can be used in obtaining an efficient structure.

The expression for energy capacity, τ_i of the i^{th} element is given by

$$\tau_i = \frac{1}{2} \sigma_i^{(u)} \epsilon_i^{(u)} V_i \tag{18}$$

$\sigma_i^{(u)}$ and $\epsilon_i^{(u)}$ are the limiting normal stress and strain respectively in the element. V_i is the volume of the element. The relation between the limiting stress and strain is assumed to be

$$\sigma_i^{(u)} = E_i \epsilon_i^{(u)} \tag{19}$$

Then, the energy capacity of the element can be written as

$$\tau_i = \frac{1}{2} \left(\epsilon_i^{(u)} \right)^2 \Lambda \, \alpha_i \, \ell_i \tag{20}$$

The quantity ℓ_i is defined as

$$\ell_i = \frac{V_i \; E_i}{\Lambda \alpha_i} \tag{21}$$

The scalar Λ is the base parameter for all the elements.

α_i is the relative value of the i^{th} design variable. In case of bar elements ℓ_i is simply the length of the element and $\Lambda \alpha_i$ is the product of area and modulus of elasticity of the element ($\Lambda \alpha_i = A_i E_i$). The actual design variable vector may be written as $\Lambda \boldsymbol{\alpha}$. The vector $\boldsymbol{\alpha}$ alone will be referred to as relative design variable vector. The relative response of the structure depends upon the vector $\boldsymbol{\alpha}$. The absolute response can be manipulated simply by changing the scalar Λ. By altering Λ every feasible design can be brought to the constraint surface.

The energy capacity Γ of the total structure may be written as

$$\Gamma = \sum_{i=1}^{m} \tau_i = \frac{\Lambda}{2} \sum_{i=1}^{m} \left(\epsilon_i^{(u)} \right)^2 \alpha_i \ell_i \tag{22}$$

If Γ' is defined as

$$\Gamma' = \frac{1}{2} \sum_{i=1}^{m} \left(\epsilon_i^{(u)} \right)^2 \alpha_i \, \ell_i \tag{23}$$

then Γ can be written as

$$\Gamma = \Lambda \Gamma' \tag{24}$$

From Equation 16 the strain energy of the structure (in terms of base parameter) is given by

$$U = \frac{1}{2\Lambda} \mathbf{R}^t \mathbf{r}' = \frac{U'}{\Lambda} \tag{25}$$

The vector $\mathbf{r}'$ is the relative displacement vector. The relation between the relative and absolute displacement vector is given by

$$\mathbf{r} = \frac{1}{\Lambda} \mathbf{r}' \tag{26}$$

The relation between U and U' is obvious from Equation 25.

The following relation between the strain energy and the energy capacity is assumed

$$\Gamma = C^2 U \tag{27}$$

From the above relation the value of the base parameter may be written in the form

$$\Lambda = C \sqrt{\frac{U'}{\Gamma'}} \tag{28}$$

The constant C equal to 1.0 represents the case when the strain energy and energy capacity are equal. In most cases this is an impossible condition to obtain without exceeding the stress constraints. However, the value of C can be adjusted to make the given design acceptable.

For optimal condition the ratio of the strain energy in the element to its energy capacity should be same throughout the structure. Since the design starts with trial variable vector this condition can be attained only by iteration. The necessary iteration equation is derived on the basis of the distribution of strain energy among the elements in relation to their energy capacity. Equation 9 gives the strain energy in the element

$$u_i = \frac{1}{2\Lambda} \mathbf{S}_i^t \boldsymbol{\nu}_i' = \frac{u_i'}{\Lambda} \tag{29}$$

Again the relation between the absolute and relative displacement vectors of the element $\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$ is given by

$$\boldsymbol{\nu}_i = \frac{1}{\Lambda} \boldsymbol{\nu}_i' \tag{30}$$

The energy capacity of the element is given by Equation 20

$$\tau_i = \frac{1}{2} (\epsilon_i^{(u)})^2 \Lambda \alpha_i \ell_i = \Lambda \tau_i' \tag{31}$$

For an optimum structure the ratio of strain energy to energy capacity should be constant throughout the structure. From this condition the following relation can be written.

$$\frac{u_i'}{\Lambda} = \frac{1}{C_i^2} \Lambda \tau_i' \tag{32}$$

or

$$\Lambda^2 = C_i^2 \frac{u_i'}{\tau_i'} \tag{33}$$

$1/C_1$ is the constant of proportionality. Since the object is to find the value of the design variable and not the base parameter both sides of Equation 33 are multiplied by α_i^2

$$(\alpha_i \Lambda)^2 = C_1^2 \alpha_i^2 \left(\frac{u_i'}{\tau_i'}\right) \qquad (34)$$

The Equation 34 is the basis for writing the recurrence formula for iteration of design variable vector,

$$\left(\alpha_i \Lambda\right)_{\nu+1} = C_1 (\alpha_i)_\nu \sqrt{\left(\frac{u_i'}{\tau_i'}\right)_\nu} \qquad (35)$$

The subscripts $\nu+1$ and ν refer to the cycles of iteration. The constant C_1 need not be determined because the relative design variable vector is obtained by dividing the actual design variable vector by one of its components. The condition for optimality can be obtained by iteration using Equation 35 if the design is for a single loading condition and there are no constraints on displacements and sizes of the elements.

SECTION IV

OPTIMUM DESIGN VARIABLE VECTOR

The statement of optimality condition in the last section is based on purely intuitive reasoning. The iteration formula, Equation 35, is derived on the premise that an optimum structure with respect to weight, is the one that has ratio of strain energy to its energy capacity constant throughout the structure. For a structure made of same material and stress limits this statement is equivalent to the saying that strain energy density is constant throughout an optimum structure. Strain energy density is defined here as strain energy per unit volume. Theoretically strain energy density (of a non-optimal structure) may vary from point to point in the structure. Since an actual structure will be replaced by a discrete model the term strain energy density will be replaced by average strain energy density. The average strain energy density of an element is the ratio of its total strain energy to its volume.

Under certain conditions a measure of validity can be established for the stated optimal condition. In Reference 15 Sheu and Prager presented an interesting reasoning for establishing the optimality condition for a single degree of freedom portal frame. A similar reasoning is presented here in establishing the optimality condition for a multi-degree of freedom system. The presentation is in terms of generalized displacement coordinates and avoids the use of such terms as curvature which are particular cases of generalized displacements.

A structure is subjected to a load vector $\mathbf{R}$ and the problem is to obtain optimum sizes for its elements so that the structure weight will be a minimum. Suppose A and A' are two designs proposed for the same structure to carry the same load vector. The weights of the two designs are proportional to W and W' which are defined as

$$W = \sum^{m} A_i \ell_i \tag{36}$$

$$W' = \sum_{i=1}^{m} A'_i \ell_i \tag{37}$$

If the geometry of the structure is fixed, ℓ_i are same in both cases. For one dimensional elements, A and ℓ are the area and length of the members respectively. For other elements interpretations similar to that given in Equation 21 are more appropriate. Let $\rho_i(u)$ and $\rho_i'(u)$ be the average strain energy densities in the elements corresponding to the two designs. By this definition

$$\rho_i(u) = \frac{u_i}{V_i}$$

$$\rho_i'(u) = \frac{u_i'}{V_i} \qquad (38)$$

u_i and u_i' are the strain energies (defined by Equation 6) and v_i and v_i' are volumes of the i th element in two cases. Also let $\{r\}$ and $\{r'\}$ (r' and u' do not carry the same meaning as in the previous section) be the displacement vectors corresponding to designs one and two respectively. If the potential of the external forces is assumed to be the same in both cases (this does not necessarily imply equality of the displacement vectors, $\{r\}$ and $\{r\}$) then it can be shown

$$\sum_{i=1}^{m} A_i \ell_i \rho_i(u) = \sum_{i=1}^{m} A_i' \ell_i \rho_i'(u) \qquad (39)$$

The displacement vector $\{r'\}$ is the natural displacement vector for the second design. However, the displacement vector $\{r\}$ is kinematically admissible for the second design. If the second design is forced to have the displacement configuration represented by vector $\{r\}$ then, from the principle of minimum potential energy, the following inequality can be written

$$\sum_{i=1}^{m} A_i' \ell_i \rho_i(u) \geq \sum_{i=1}^{m} A_i' \ell_i \rho_i'(u) \qquad (40)$$

Use of $\rho_i(u)$ in the left side of the inequality is justified because the strain energy density depends only on the displacement configuration and not on the sizes of the elements. The validity of this statement can easily be established from Equations 6, 11, and the definition of strain energy density. From Equation 39 the Inequality 40 may be witten as

$$\sum_{i=1}^{m} A_i' \ell_i \rho_i(u) - \sum_{i=1}^{m} A_i \ell_i \rho_i(u) \geq 0 \qquad (41)$$

The same inequality may be written also as

$$\sum_{i=1}^{m} (A_i' - A_i) \ell_i \rho_i(u) \geq 0 \qquad (42)$$

If the strain energy densities, $\rho_i(u)$ of the first design are assumed to be same throughout the structure i·e $\rho_1(u) = \rho_2(u) \cdots \rho_m(u) = \rho(u)$ the inequality 42 becomes

$$\sum_{i=1}^{m} A_i' \ell_i \geq \sum_{i=1}^{m} A_i \ell_i \qquad (43)$$

From Equation 36, 37 and Inequality 43 it is evident that

$$W' \geq W \tag{44}$$

This means that the design which has the constant average strain energy density throughout is a lower weight design than the one in which this condition is not satisfied.

The quantity average strain energy density is called mean square curvature in Reference 15. However, the term mean square curvature is restrictive since it applies only when the deformation of the members is due mainly to bending. In Reference 16 it is called specific energy or the difference between certain specific energies per unit specific stiffness.

SECTION V

DESIGN FOR MULTIPLE LOADING CONDITIONS

In practice most structures have to be designed for more than one loading condition. Selection of proper design loading conditions is one of the important aspects of design criteria. Each loading condition consists of a set of loads and each set acts independently of the others. The reliability and efficiency of the structure can be improved by intelligent selection of design loading conditions.

However, design for multiple loading conditions presents difficulties that can at best be overcome by approximate means. For example, the optimality condition of constant average strain energy density throughout the structure is impossible to fulfill in case of design for multiple loading conditions. The distribution of strain energy in the structure is different for different loading conditions and an optimality condition, for one loading condition may not be quite optimal for other loading conditions.

In case of multiple design loading conditions the optimality requirement can be modified to state that the largest average energy density in each element is the same throughout the structure. This largest strain energy density is not caused by the same loading condition in all the elements. Under these conditions the recurrence formula (Equation 35) for iteration can be modified to read

$$(\alpha_i \ \Lambda)_{\nu+1} = C_1 \ (\alpha_i)_\nu \sqrt{\left(\frac{u'_{i\,max}}{\tau'_i}\right)_\nu} \qquad (45)$$

$u'_{i\,max}$ is a measure of the maximum strain energy in the i^{th} element due to any of the design loading conditions. In the absence of displacement constraints, iteration using Equation 45 is very effective in arriving at minimum weight design. Even in the presence of displacement constraints this iteration is quite effective in the initial design stages. For completion however, iteration using a search procedure is necessary in the final stages of the design. A search procedure is presented for this purpose in the next section.

SECTION VI

DISPLACEMENT LIMITS AND A SEARCH PROCEDURE

In a practical design the constraints on displacements may be required for a variety of reasons. Some of these reasons are requirements of tolerances in fabrication to assure the overall stability, and sometimes to provide certain stiffness characteristics to the structure.

In the presence of displacement constraints the proposed method handles the problem in two stages. In the first stage iteration is carried out as in stress constraint design, using recurrence Formula 45. The first stage ends when further reduction in weight is not achieved by this approach. The second stage, which is referred here as a search procedure, is outlined with the aid of necessary equations.

The force displacement relation, in terms of the generalized coordinates, is given by Equation 15 and is repeated here as Equation 46

$$\mathbf{R} = \mathbf{K}\,\mathbf{r} \tag{46}$$

Any changes in the sizes of the elements reflect change in stiffness matrix $\mathbf{K}$ and the displacement matrix, $\mathbf{r}$. If $\Delta\mathbf{K}$ is the change in stiffness matrix, the force displacement relation may be written as

$$\mathbf{R} = \left[\mathbf{K} + \Delta\mathbf{K}\right]\left\{\mathbf{r} + \Delta\,\mathbf{r}\right\} \tag{47}$$

Since the external force matrix is same, the left hand side of Equation 46 remains unaltered. Solving for $\Delta\mathbf{r}$ the following relation is obtained:

$$\Delta\mathbf{r} = -\,\mathbf{K}^{-1}\,\Delta\mathbf{K}\,\mathbf{r} - \mathbf{K}^{-1}\,\Delta\mathbf{K}\,\Delta\mathbf{r} \tag{48}$$

If the change in stiffness $\Delta\mathbf{K}$ is very small, then the last term on the right, containing the second order term, $\Delta\mathbf{K}\;\Delta\mathbf{r}$, can be ignored. Alternately, the following iteration formula may be used for determining $\Delta\mathbf{r}$:

$$\Delta\mathbf{r}_{\nu+1} = -\,\mathbf{K}^{-1}\,\Delta\mathbf{K}\left\{\mathbf{r}_{\nu} + \Delta\mathbf{r}_{\nu}\right\} \tag{49}$$

ν and $\nu+1$ refer to the cycles of iteration in a given design cycle. Use of iteration Formula 49 permits slightly larger changes in the stiffness matrix. Only two or three cycles are necessary to obtain a reasonable approximation when the changes are not too large.

The matrix $\mathbf{K}^{-1}$ is available from the last design cycle and need not be calculated again. If $\Delta\mathbf{K}$ is considered to be due to change in size of one of the elements, it is also available since it was determined during the assembly of structure stiffness matrix. The $\Delta\mathbf{r}$ matrices corresponding to unit change in the size of each element can be determined by Equation 49.

The entire $\Delta\mathbf{r}$ matrix corresponding to each element change is not of interest and need not be stored. If "j" is the direction corresponding to the active displacement constraint, then only the elements of $\Delta\mathbf{r}$ corresponding to "j" direction need to be stored. From $\Delta\mathbf{r}$ calculations the influence of all the members on active displacement constraints can be determined. The sizes of the elements that have negligible effect (or negative effect) on active displacement constraints can be reduced without exceeding the limits. (If the increase in size of the element increases the displacement in the constrained direction it is called negative influence here). By this procedure, weight can be reduced further but this reduction in some elements may activate constraints at other locations. This creates a problem of determining the magnitude of change. The following procedure is adopted to overcome this problem:

The displacements at the active constraints are allowed to exceed by a certain percentage (say 10-20%). This can be accomplished easily by changing the parameter Λ. The constraint displacements are then brought to their limiting values by increasing the sizes of the elements in proportion to their influence on the exceeded displacements. This influence is determined by $\Delta\mathbf{r}$ calculations (Equation 49).

If "j" is the direction of active displacement constraint and $\Delta r_j^{(i)}$ is the influence of of the i^{th} element on r_j, then the required change in the size of the i^{th} element is determined by the relation

$$\Delta A_i = C\,\Delta r_j^{(i)} / \ell_i \tag{50}$$

Since the increase in size of an element with large length parameter penalizes the weight, the ΔA_i, is assumed to be inversely proportional to ℓ_i. C is the constant of proportionality. The $\{\Delta r\}$ calculations are due to unit changes in sizes of the elements and the change $\delta r_j^{(i)}$ due to ΔA_i. is obtained by

$$\delta r_j^{(i)} = C\,(\Delta r_j^{(i)})^2 / \ell_i \tag{51}$$

The constant of proportionality, C may be determined from the condition that the displacement r_j should be brought to its limiting value by changing the sizes of the elements according to Equation 50. If ψ_j is the magnitude of the displacement exceeding the limiting value then

$$C = \frac{\Psi_j}{\sum_{i=1}^{m} \frac{\left[\Delta r_j^{(i)}\right]^2}{\ell_i}} \tag{52}$$

In the summation, only the elements that have positive influence are included. The expression for ΔA_i becomes

$$\Delta A_i = \frac{\Psi_j}{\sum_{p=1}^{m} \frac{\left[\Delta r^{(p)}\right]^2}{\ell_p}} \left[\frac{\Delta r_j^{(i)}}{\ell_i} \right] \tag{53}$$

If more than one displacement exceeded the limit, the necessary change in each element size is determined separately for each constraint, using Equation 53. The largest ΔA_i is used for actual change in the size of the i^{th} element. Use of largest increase for each element is conservative and the displacements after the increase are usually lower than the limiting values. Since the procedure will be repeated more than once the final displacements can be brought to the limiting values without penalizing the final weight. This procedure was tried on number of examples of bar structures and proved to be very satisfactory. The examples in the next section attest to the validity of this statement.

The major disadvantage of this search procedure is that calculations by Equation 49 require the inverse of the total stiffness matrix. This requirement imposes some restrictions on the sizes of the structures that can be optimized in the presence of displacement constraints. If there are no displacement constraints, structures with a large number of degrees of freedom can be optimized with the aid of iterative analysis methods such as the one proposed in Reference 17. With displacement constraints however, problems with degrees of freedom larger than 185 to 190, will require extensive use of peripheral storage. For larger systems search procedures which do not require inverse of the total stiffness matrix are more advantageous. The complete method is presented in the form of a flow-chart in Figure 2. The details of the method in a form suitable for computer programming can be found in Reference 18.

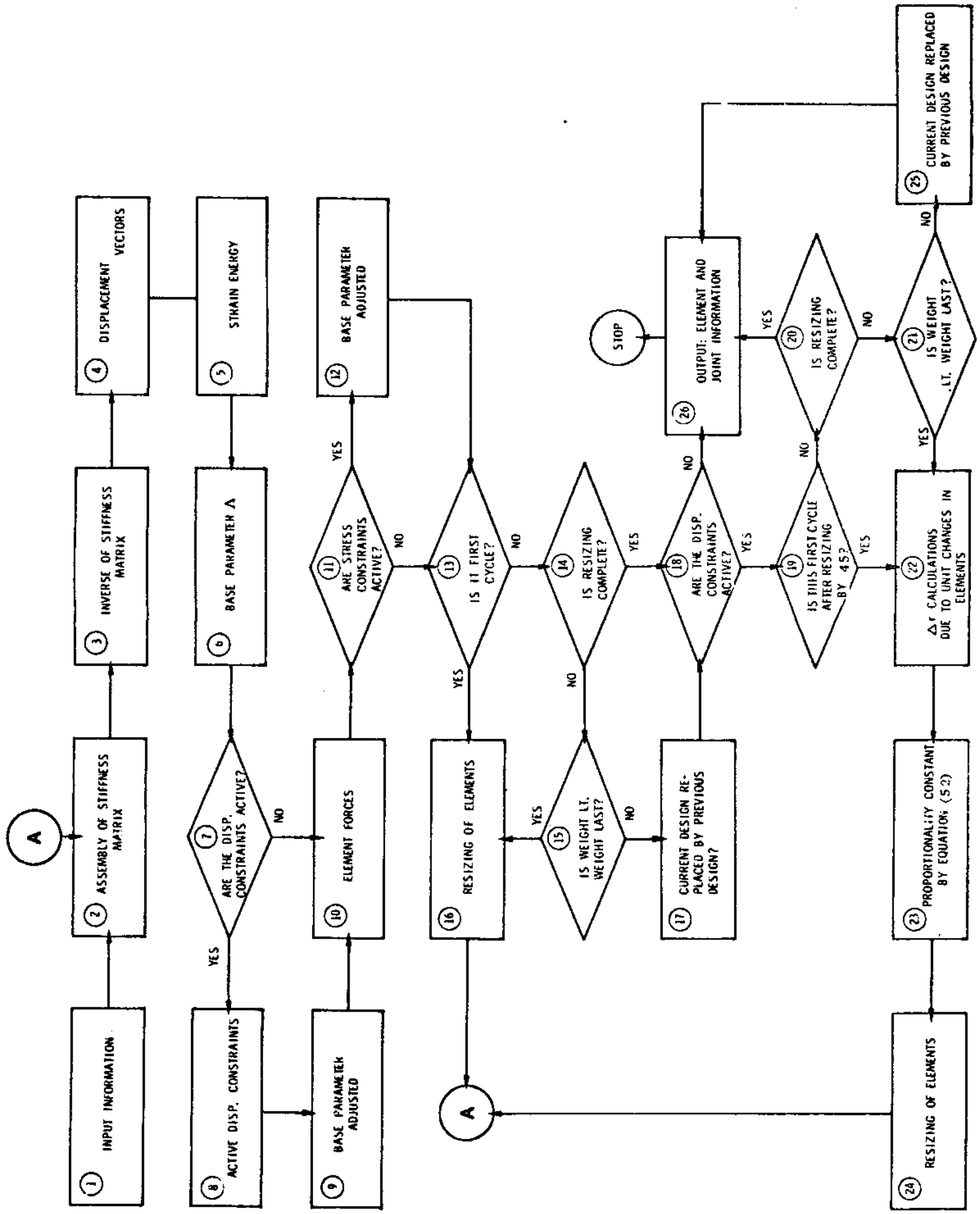


Figure 2. Flow Chart for Optimization

SECTION VII
EXAMPLES - BAR STRUCTURES - CONCLUSIONS

Five bar structures are designed by applying the method proposed in the paper. It is hoped that these examples highlight the important features of the method and also provide basis for comparison with the existing methods. A program based on flow-chart shown in Figure 2 is written in FORTRAN IV. The reported computational times are for IBM 7094-II-7044-DCS. The stiffness matrix is inverted by the Cholesky scheme modified for symmetrical matrices. No attempt is made to reduce the computational times for analysis by using the techniques of banded matrix inversion or any such procedures.

All the structures, except the geodesic dome, are designed with stress and displacement constraints as well as stress constraints only. The geodesic dome is designed for the case of stress and displacement constraints only. All of them have lower limits on the areas of elements and no upper limits. The material is assumed to be aluminum in all cases except the plane truss, Example 5, in which steel is used.

EXAMPLE 1 - TEN NODE TWENTY-FIVE BAR TRANSMISSION TOWER

The schematic diagram of the tower with dimensions is shown in Figure 3. This entire figure is reproduced from Reference 11. The design information for the tower is given in Table 1. The same table contains the output information.

The structure is designed for six loading conditions. Actually only two are independent loading conditions; the other four are meant to maintain the symmetry of the structure. If the program has a symmetry option as in the one reported in Reference 11, only two loading conditions are necessary. The symmetry option is not incorporated in the present program; the structure was designed for six loading conditions. The increase in computational times for additional loading conditions is insignificant and there is no particular disadvantage in omitting symmetry option.

The final design weight is 555.11 lb when there are displacement constraints. Total computational time was 24 seconds. The same tower, also designed without displacement

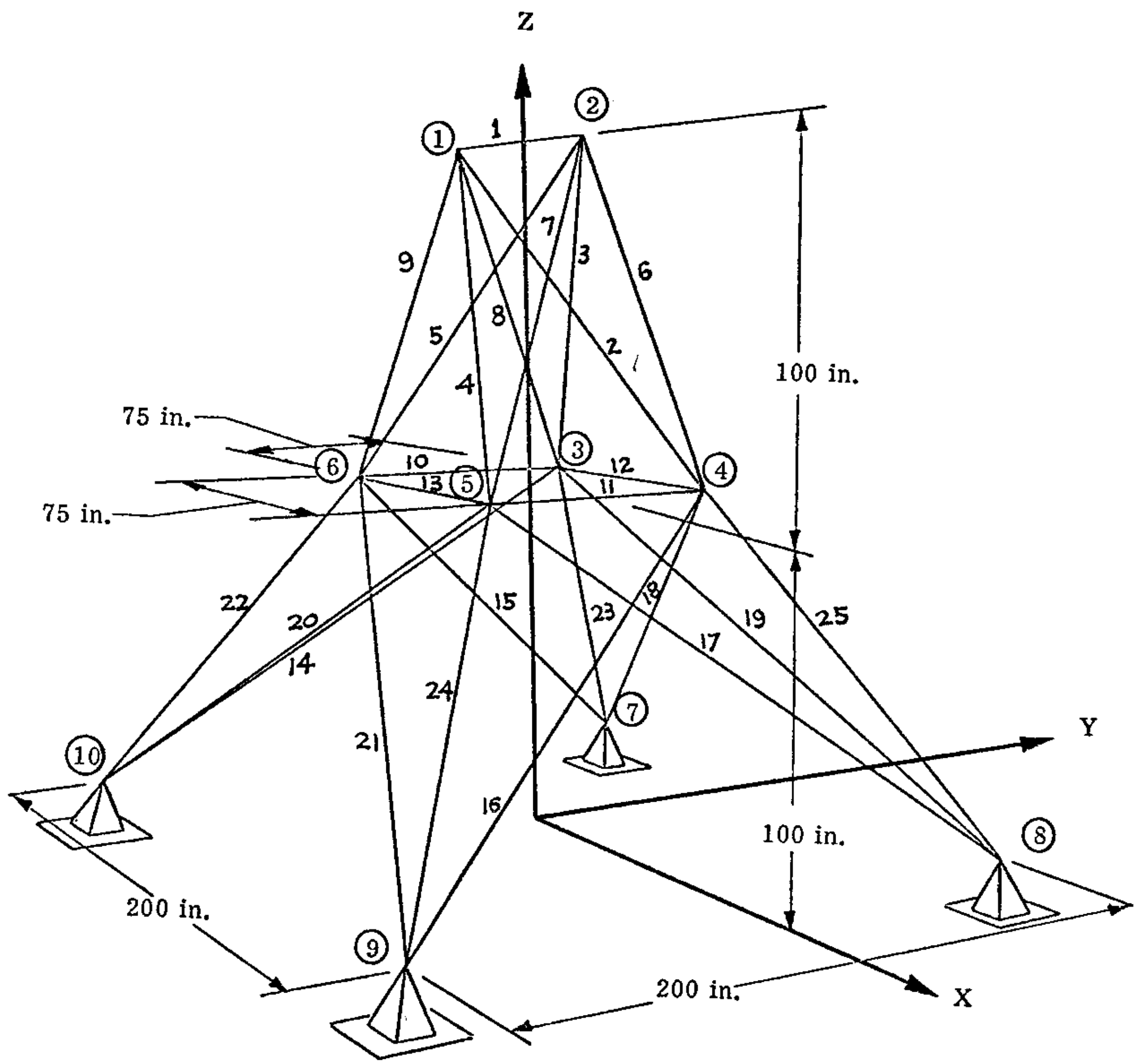


Figure 3. Transmission Tower

TABLE 1

EXAMPLE 1 - TRANSMISSION TOWER

Input: Design Information

Material Aluminum

Stress Limits 40,000 psi

Modulus of Elasticity, $E = 10^7$ psi

Specific Weight = 0.1 lb/cu in.

Lower Limit on Membersizes 0.01 sq inch with displacement limits and 0.1 sq inch without displacement limits

Upper Limits: None

Displacement Limits: 0.35 inch on all nodes and all directions

Number of Loading Conditions: 6 (all loads are in pounds)

Load Condition	Node	Direction of Load		
		x	y	z
1	1	1,000	10,000	- 5,000
	2	0	10,000	- 5,000
	3	500	0	0
	6	500	0	0
2	1	0	10,000	- 5,000
	2	- 1,000	10,000	- 5,000
	4	- 500	0	0
	5	- 500	0	0
3	1	1,000	-10,000	- 5,000
	2	0	-10,000	- 5,000
	3	500	0	0
	6	500	0	0

TABLE 1 (contd)

Loading Condition	Node	Direction of Load		
		x	y	z
4	1	0	-10,000	- 5,000
	2	-1,000	-10,000	- 5,000
	4	- 500	0	0
	5	- 500	0	0
5	1	0	20,000	- 5,000
	2	0	-20,000	- 5,000
6	1	0	-20,000	- 5,000
	2	0	20,000	- 5,000

OUTPUT

With Displacement Limits						Without Displacement Limits					
Final Design Weight in lb, 555.12						Final Design Weight in lb, 91.14					
Computer Time in sec, 24						Computer Time in sec, 9					
El.No	Area	El.No	Area	El.No	Area	El.No	Area	El.No	Area	El.No	Area
1	0.033	10	0.010	19	1.760	1	0.100	10	0.100	19	0.278
2	2.015	11	0.010	20	1.760	2	0.376	11	0.100	20	0.278
3	2.015	12	0.014	21	1.760	3	0.376	12	0.100	21	0.278
4	2.015	13	0.014	22	2.440	4	0.376	13	0.100	22	0.380
5	2.015	14	0.980	23	2.440	5	0.376	14	0.100	23	0.380
6	2.823	15	0.980	24	2.440	6	0.471	15	0.100	24	0.380
7	2.823	16	0.980	25	2.440	7	0.471	16	0.100	25	0.380
8	2.823	17	0.980			8	0.471	17	0.100		
9	2.823	18	1.760			9	0.471	18	0.278		

constraints had a design weight of 91.14 lb and used nine seconds of the computational time. The lower limit on the area of the members is 0.01 square inch when there are displacement constraints and 0.1 square inch when there are no displacement constraints. Both the input and the output data are summarized in Table 6.

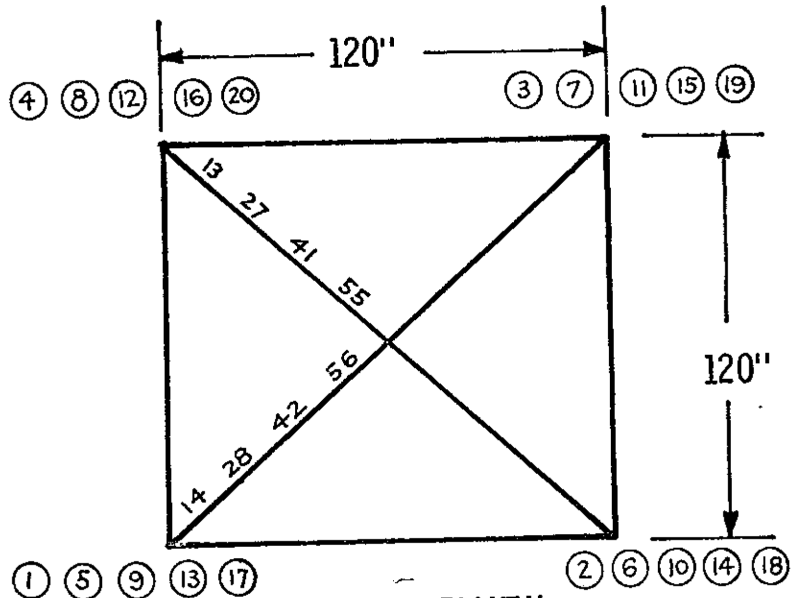
The design of this transmission tower was reported in Reference 19. The reported weight is 570.4 lb. The same tower was also designed by Gellatly in Reference 11 and the final design weight is reported here as 550.68 lb and it was designed in 20 minutes. Later, in Reference 12, Gellatly reports the design for the same tower as weighing 555 lbs and needing 20 minutes of computational time on IBM 7090. In the same reference Gellatly discussed the number of possibilities of reducing computational times. He estimates about 110 seconds as the running time for this problem if all the suggested improvements are incorporated. The computational time of 24 seconds by the present program is lower than the projected time in the above reference by factors of at least 2 to 3. A simple analysis of the design from Reference 11 revealed the displacements to exceed their limits by at least 1%. This is the reason for the low weight of 550.68 lb. The design reported in Reference 12 yields displacements which are closer to the actual displacement limits and it appears to be more realistic.

EXAMPLE 2 - TWENTY NODE SEVENTY-TWO BAR TRUSS

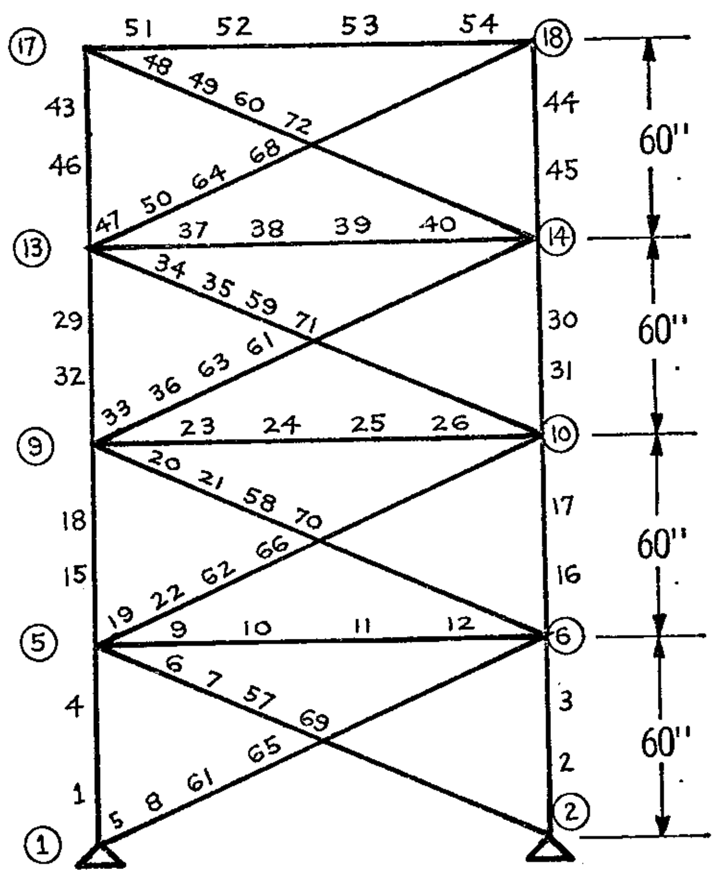
The top and front views of the truss are shown in Figure 4. The design information is given in Table 2. The structure is designed for five loading conditions. The loads are devised to keep the symmetry of the design. The purpose of this example is to study whether the higher degree of indeterminacy has any detrimental effect on the speed of approaching the optimum design.

The final design, in the presence of displacement constraints, weighed 425.8 lb and the computational time was 5 minutes 45 seconds. The design weight of the same tower without displacement constraints was 96.6 lb and the computational time was 59 seconds.

It is evident from these examples that the presence of displacement constraints increase the computational times substantially. Similar experience is reported by Gellatly (Reference 11). The degree of indeterminacy does not appear to have significant effect on the rate of convergence. The input and output data are summarized in Table 6.



TYPICAL FLOORVIEW



FRONT VIEW

Figure 4. Four Level Tower

TABLE 2

EXAMPLE 2 - 72 BAR TRUSS

Input: Design Information

Material Aluminum

Stress - Limits 25,000 psi

Modulus of Elasticity $E = 10^7$ psi

Specific Weight = 0.1 lb/cu in.

Lower Limit on Member Sizes 0.1 square inch

Upper Limits: None

Displacement Limits = 0.25 on all nodes and all directions

NUMBER OF LOADING CONDITIONS 5

Load Condition	Node	Direction of Load		
		x	y	z
1	17	5,000	5,000	- 5,000
2	18	-5,000	5,000	- 5,000
3	19	-5,000	-5,000	- 5,000
4	20	5,000	-5,000	- 5,000
5	17	0	0	- 5,000
	18	0	0	- 5,000
	19	0	0	- 5,000
	20	0	0	- 5,000

TABLE 2 (contd)

OUTPUT

With Displacement Limits		
Final Design Weight in lb 425.8		
Computer Time in Seconds, 345		
El. No	Area	Elements with Same Area
1	1.471	2, 3, 4
5	0.548	6, 7, 8
9	0.090	10, 11, 12, 13, 14
15	1.083	16, 17, 18
19	0.559	20, 21, 22
23	0.090	24, 25, 26, 27, 28
29	0.933	30, 31, 32
33	0.589	34, 35, 36
37	0.137	38, 39, 40
41	0.112	42
43	0.858	44, 45, 46
47	0.674	48, 49, 50
51	0.518	52, 53, 54
55	0.503	56
57	0.548	61, 65, 69
58	0.559	62, 66, 70
59	0.589	63, 67, 71
60	0.674	64, 68, 72

Without Displacement Limits		
Final Design Weight in lb 96.6		
Computer Time in Seconds, 59		
El. No	Area	Members with Same Area
1	0.294	2, 3, 4
15	0.199	16, 17, 18
29	0.191	30, 31, 32
43	0.189	44, 45, 46
All other members are of 0.1 square inch		

EXAMPLE 3 — FIVE NODE - FOUR BAR TRUSS

Figure 5 shows the geometry and the dimensions of the truss. This figure is reproduced directly from Reference 10. Three different cases are considered in designing the truss.

CASE I:	Design for a Single Loading Condition
CASE II:	Design for Three Loading Conditions
CASE III:	Same as CASE I except Vertical Load (z-direction) is doubled

The input and output details of these three cases are given in Table 3. CASES I and II are designed with and without displacement constraints. The third case is optimized with stress constraints only. The purpose of the third case is to establish that iteration using Equation 35 coverges to a structure with constant average strain energy density in absence of displacement constraints and single loading condition. The vertical load is doubled to insure the condition that the sizes of the elements are not governed by minimum area requirements.

Final design weight with displacement constraints in CASE I is 39.88 lb and the computational time, three seconds. For the same case without displacement constraints, the design weight is 34.5 lb and the computational time, four seconds.

In the second case, design with displacement constraints weighs 14.3 lb and the time, 10 seconds. Design without displacement constraints weighs 9.09 lb and the time, three seconds.

The weight of the design in the third case is 65.76 lb and the time, three seconds. The strain energy density is same in all the members.

The first two cases with displacement constraints are also designed in Reference 10. The weights reported there differ slightly from the weights reported here. The definition of displacement constraints in Reference 10 is ambiguous (they seem to be in error) and it is difficult to make any comments on these differences. In any case the differences are not very significant.

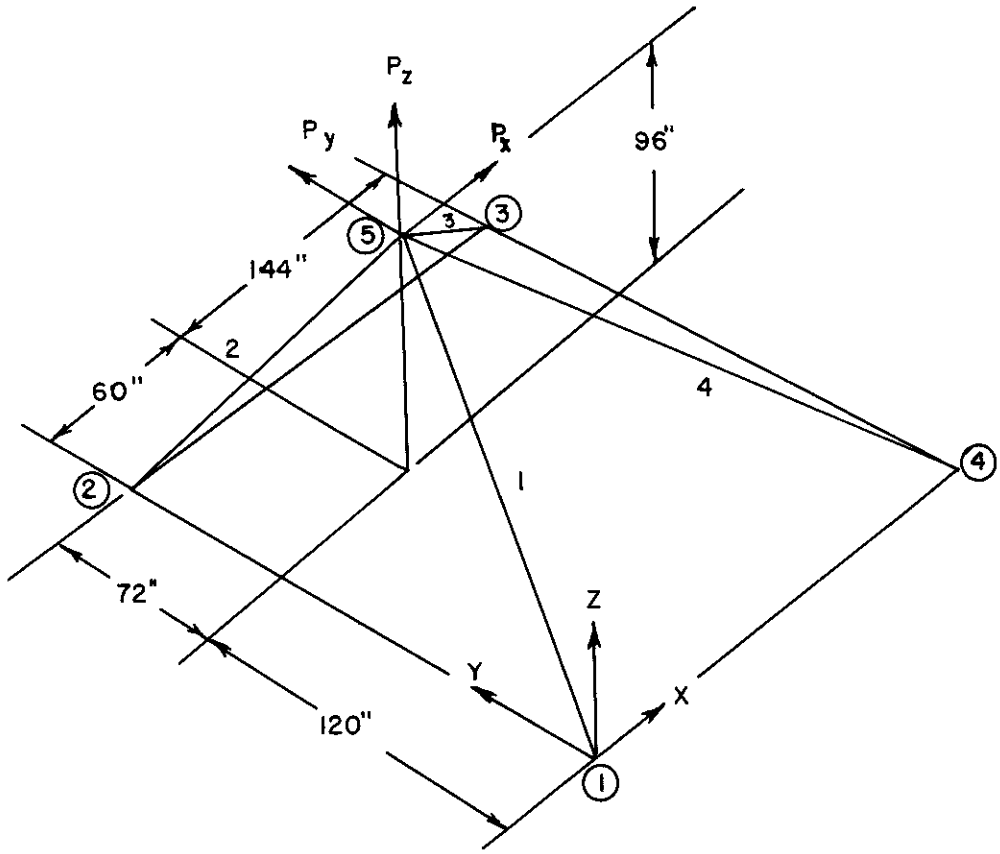


Figure 5. Four Bar Truss

EXAMPLE 4 - SIXTY-ONE NODE, ONE HUNDRED THIRTY-TWO ELEMENT (BARS) GEODESIC DOME

The Geodesic Dome with necessary dimensions is shown in Figure 6. This figure is reproduced from Reference 20 except the height of the dome is doubled (60 inches). Reference 20 is concerned with stability of the dome and not its design. The purpose of this example is to illustrate the application of the proposed method to moderately large size problems. All the input and output information for this problem is given in Table 4. The information is also summarized in Table 6.

The final design of the dome weighs 180.95 lb and the computer time was 68 minutes and 30 seconds. For the size of the structure this time appears to be reasonable. This structure is somewhat unusual in the sense that out of a total of 10 cycles of iteration, eight of them are involved in search procedure and only two cycles used iteration Formula 35. In most of the other structure this proportion is about half or less. The large computational time may be partly attributed to this peculiarity because the computational time for a search cycle is much larger than the cycle using Equation 35.

TABLE 3

EXAMPLE 3 - 4 BAR TRUSS

Input: Design Information

Material: Aluminum

Stress Limits = 25,000 psi

Modulus of Elasticity, $E = 10^7$ psi

Specific Weight = 0.1 lb/cu in.

Lower Limit on Member Sizes = 0.1 square inch

Upper Limits: None

Displacement Limits for Both Examples (3-I and 3-II)

Node	Direction		
	x	y	z
5	None	± 0.3"	± 0.4"

Number of Loading Conditions

Example 3-I 1

Example 3-II 3

Example 3-III 1

Details of Loads: Case I

Load Condition	Node	Direction of Load		
		x	y	z
1	5	10,000	20,000	-30,000

TABLE 3 (contd)

Details of Loads: Case II

Load Condition	Node	Direction of Load		
		x	y	z
1	5	5,000	0	0
2	5	0	5,000	0
3	5	0	0	-7,500

Details of Loads: Case III

Load Condition	Node	Direction of Load		
		x	y	z
1	5	10,000	20,000	- 60,000

OUTPUT: With Displacement Limits

Case	Areas of Elements in Square Inches				Weight in lb
	1	2	3	4	
I	0.112	1.108	0.958	0.249	39.88
II	0.217	0.319	0.186	0.140	14.30

OUTPUT: Without Displacement Limits

Case	Areas of Elements in Square Inches				Weight in lb
	1	2	3	4	
I	0.100	0.890	1.002	0.100	34.48
II	0.132	0.193	0.122	0.097	9.09
III	0.430	1.755	1.258	0.548	65.76

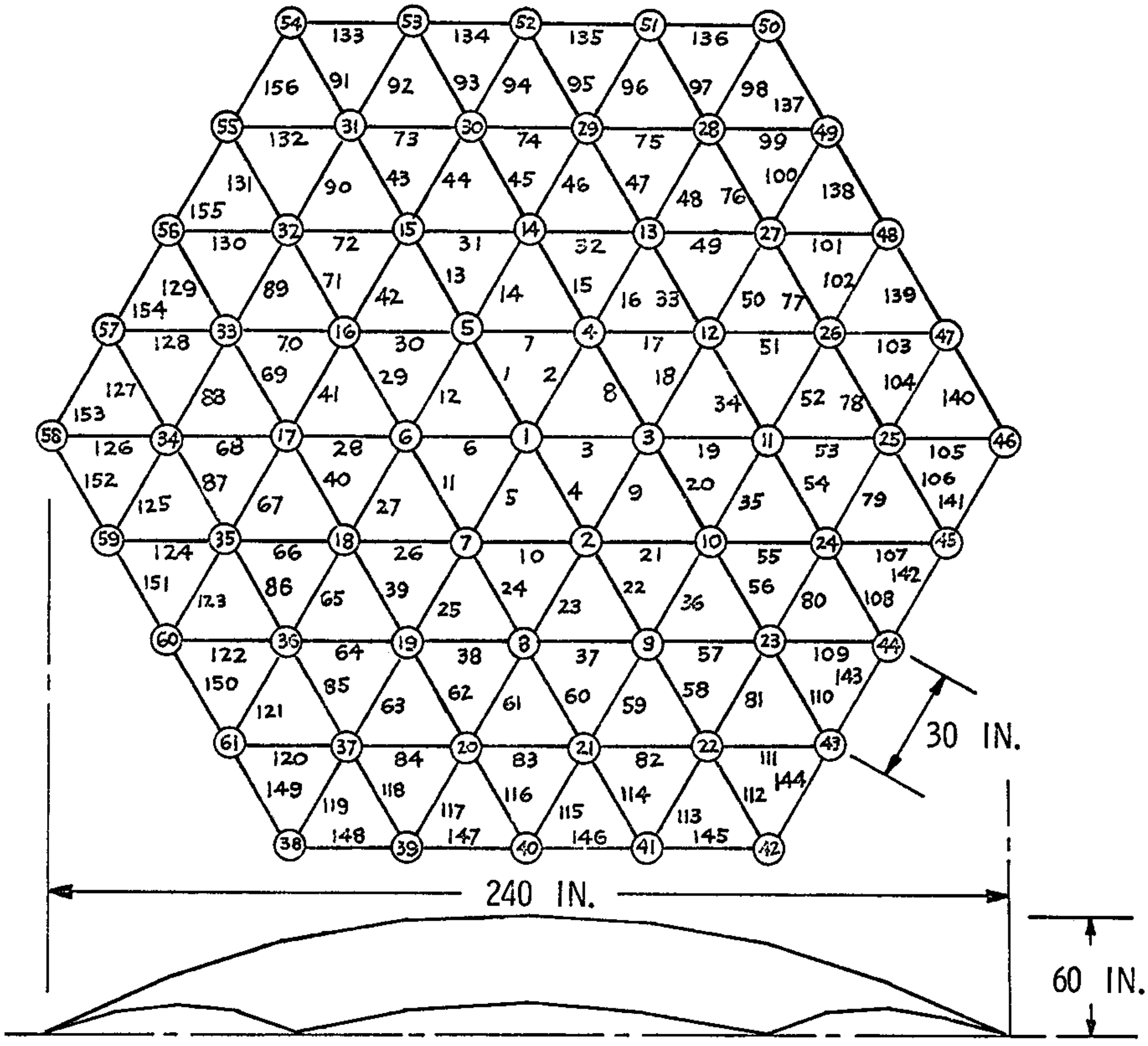


Figure 6. Geodesic Dome

TABLE 4

EXAMPLE 4 - GEODESIC DOME

Input: Design Information

Material Aluminum

Stress Limits 25,000 psi

Modulus of Elasticity, $E = 10^7$ psi

Specific Weight = 0.1 lb/cu in.

Lower Limits on Member Sizes = 0.1 sq in.

Upper Limits: None

Displacement Limits 0.1 in. on all nodes and in all directions

Number of Loading Conditions: 4

All Loads are Acting Down in z - direction and their magnitude is 1000 lb each

Nodes Loaded in each Loading Condition

Loading Condition 1	1
Loading Condition 2	1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 37.
Loading Condition 3	ALL NODES ARE LOADED
Loading Condition 4	1, 4, 5, 6, 7, 13, 14, 15, 16, 17, 18, 19, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37

OUTPUT

Final Design Weight with Displacement Limits = 180.94 lb

Computational Time = 68 minutes

TABLE 4 (contd)

Areas of Elements (in square inches)

There is symmetry about the vertical plane containing nodes 1, 38, and 50. Only the areas of the elements on and to one side of this plane are listed.

El. No	Area	El. No	Area	El. No	Area	El. No	Area	El. No	Area
1	1.015	28	0.437	48	0.503	85	0.305	119	0.273
2	1.060	29	0.407	63	0.503	86	0.245	120	0.515
5	1.060	30	0.407	64	0.383	87	0.164	121	0.423
6	1.015	31	0.354	65	0.516	88	0.251	122	0.319
7	0.819	32	0.391	66	0.379	89	0.276	123	0.388
11	0.819	39	0.391	67	0.397	90	0.251	124	0.348
12	0.828	40	0.354	68	0.360	91	0.267	125	0.292
13	0.437	41	0.511	69	0.274	92	0.292	126	0.267
14	0.542	42	0.511	70	0.429	93	0.348	127	0.232
15	0.449	43	0.360	71	0.429	94	0.388	128	0.365
16	0.646	44	0.397	72	0.274	95	0.319	129	0.323
25	0.646	45	0.379	73	0.164	96	0.423	130	0.323
26	0.449	46	0.516	74	0.245	97	0.515	131	0.365
27	0.542	47	0.383	75	0.305	98	0.273	132	0.232

EXAMPLE 5 - SEVENTY-SEVEN NODE - TWO HUNDRED ELEMENT PLANE TRUSS

The schematic diagram with dimensions is shown in Figure 7. The structure is designed for five loading conditions and the input and output data are given in Table 5. Table 6 gives summary of this data.

It is the only structure designed in steel. The final design weight in the presence of displacement constraints in 31,120 lb. The computational time for this design was 93 minutes. The design weight of the same structure without displacement limits was 22,115 lb and the computational time, 64 minutes.

The five illustrative examples conclusively show that the method of structural optimization based on the study of strain-energy distribution coupled with a search procedure is quite efficient in arriving at minimum weight structures. The formulation of the method is general enough to be applicable to any structure. A preliminary application of the method to frame structures is found to be quite successful and the results of this study will be reported in the near future. The computational times for large structures may be reduced substantially by the use of sparse matrix techniques and banded matrix techniques for inversion of large matrices. The approach is simple and is no more complicated than the analysis itself.

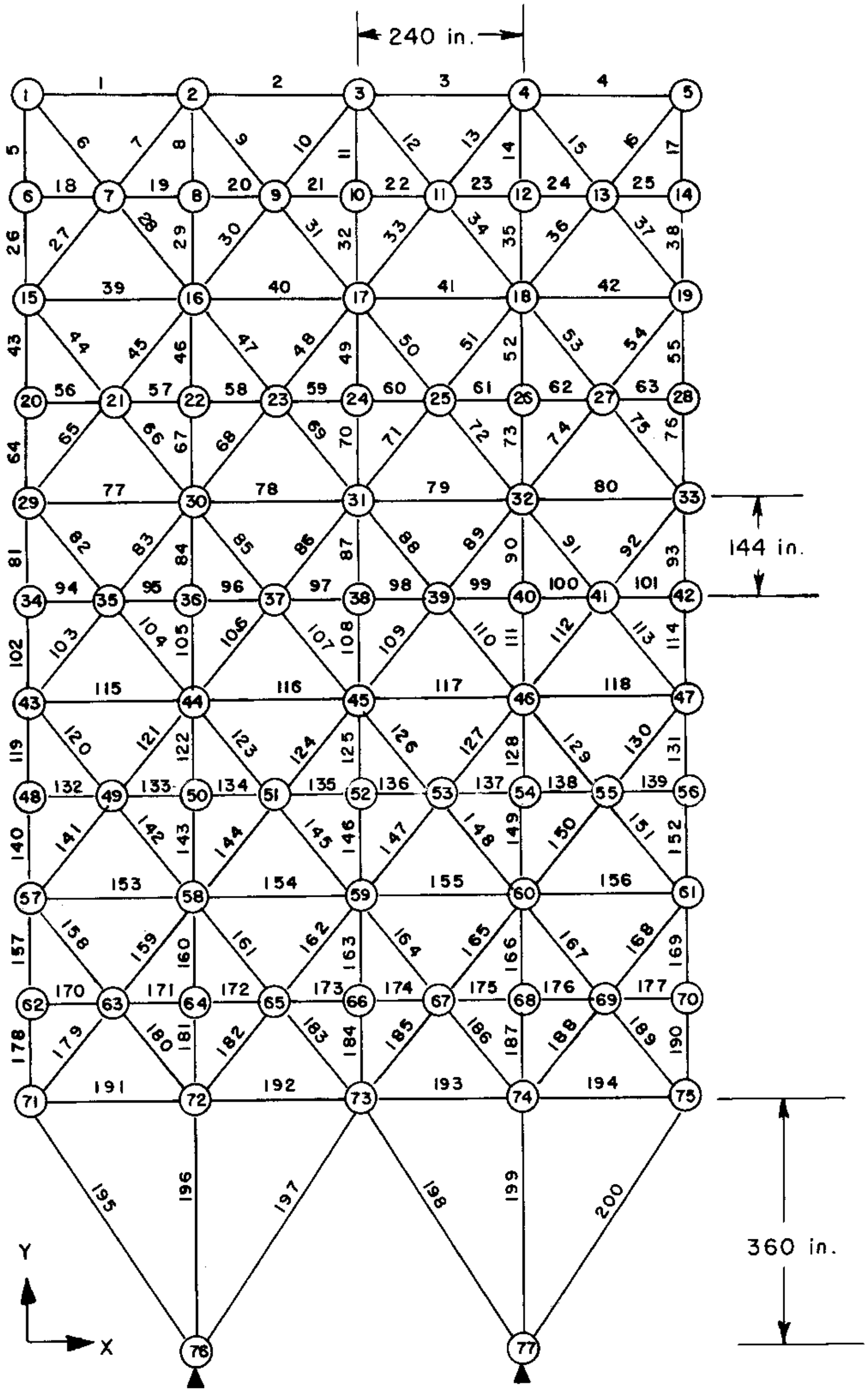


Figure 7. Plane Truss

TABLE 5

EXAMPLE 5 - 77 NODE - 200 BAR TWO DIMENSIONAL FRAME

Input: Design Information

Material Steel

Stress Limits 30,000 psi

Modulus of Elasticity, $E = 30. \times 10^6$ psi

Specific Weight = 0.283 lb/cu in.

Lower Limit on Member Sizes = 0.1 sq in.

Displacement Limits 0.5 in. on all nodes and in all directions

Number of Loading Conditions: 5

Loading Condition 1	1000 lb acting in positive x direction at node points 1, 6, 15, 20, 29, 34, 43, 48, 57, 62, 71
Loading Condition 2	1000 lb acting in negative x direction at node points 5, 14, 19, 28, 33, 42, 47, 56, 61, 70, 75
Loading Condition 3	10,000 lbs acting in negative Y directions at node points 1, 2, 3, 4, 5, 6, 8, 10, 12, 14, 15, 16, 17, 18,19, 20, 22, 24,---------71, 72, 73, 74, 75
Loading Condition 4	Loading Conditions 1 and 3 acting together.
Loading Condition 5	LOading Conditions 2 and 3 acting together.

OUTPUT

Final design weight with displacement limits = 31,020 lb

Computational time = 93 minutes

Final design weight without displacement limits = 7550 lbs

Computational time = 50 minutes

TABLE 5 (contd)

Areas of Elements (in square inches) on one side of symmetrical line with displacement limits.

El. No	Area	El.No	Area	El. No	Area	El.No	Area	El.No	Area
1	1.348	40	0.233	81	5.737	120	2.558	159	0.210
2	1.313	43	4.798	82	1.988	121	0.237	160	14.981
5	3.402	44	1.850	83	0.201	122	10.649	161	1.175
6	1.771	45	0.127	84	7.220	123	0.966	162	1.251
7	0.173	46	4.318	85	0.984	124	0.991	163	9.800
8	1.497	47	0.971	86	0.797	125	7.822	170	0.116
9	0.742	48	0.749	87	5.626	132	0.116	171	0.816
10	0.782	49	3.346	94	0.116	133	0.634	172	0.816
11	1.156	56	0.116	95	0.491	134	0.634	173	0.703
18	0.116	57	0.333	96	0.491	135	0.512	178	6.713
19	0.377	58	0.333	97	0.318	140	7.285	179	0.713
20	0.377	59	0.208	102	6.688	141	0.587	180	4.281
21	0.435	64	5.662	103	0.533	142	2.835	181	16.104
26	4.575	65	0.519	104	2.151	143	11.752	182	1.309
27	0.538	66	1.950	105	8.288	144	1.049	183	1.317
28	1.895	67	5.326	106	0.884	145	1.011	184	10.950
29	2.483	68	0.813	107	0.984	146	8.969	191	5.073
30	0.750	69	0.954	108	6.770	153	2.495	192	3.243
31	0.784	70	4.495	115	1.687	154	1.024	195	8.983
32	2.278	77	1.391	116	0.605	157	5.695	196	20.687
39	1.294	78	0.343	119	6.274	158	3.932	197	9.594

TABLE 5 (contd)

Areas of Elements (in square inches)

Without displacement limits

Areas of the elements to one side of the center line are given

El. No	Area	El. No	Area	El.No	Area	El.No	Area	El.No	Area
1	0.100	40	0.100	81	1.027	120	0.613	159	0.100
2	0.100	43	0.687	82	0.415	121	0.100	160	4.615
5	0.251	44	0.247	83	0.100	122	3.379	161	0.100
6	0.117	45	0.100	84	2.270	123	0.100	162	0.132
7	0.100	46	1.271	85	0.100	124	0.104	163	2.706
8	0.371	47	0.100	86	0.100	125	2.102	170	0.100
9	0.100	48	0.100	87	1.475	132	0.100	171	0.100
10	0.100	49	0.851	94	0.100	133	0.100	172	0.100
11	0.250	56	0.100	95	0.100	134	0.100	173	0.100
18	0.100	57	0.100	96	0.100	135	0.100	178	1.686
19	0.100	58	0.100	97	0.100	140	1.595	179	0.114
20	0.100	59	0.100	102	1.364	141	0.126	180	0.922
21	0.100	64	1.024	103	0.107	142	0.689	181	4.951
26	0.588	65	0.100	104	0.484	143	3.716	182	0.137
27	0.100	66	0.309	105	2.607	144	0.106	183	0.100
28	0.172	67	1.608	106	0.100	145	0.100	184	3.043
29	0.708	68	0.100	107	0.100	146	2.438	191	1.263
30	0.100	69	0.100	108	1.811	153	0.430	192	0.875
31	0.100	70	1.187	115	0.298	154	0.100	195	2.362
32	0.586	77	0.190	116	0.100	157	1.349	196	5.964
39	0.101	78	0.100	119	1.258	158	0.851	197	2.602

TABLE 6

SUMMARY OF INPUT AND OUTPUT INFORMATION FOR FIVE OPTIMIZED STRUCTURES

Structure	Number of Nodes	Number of Elements	Stress Limits (psi)	Number of Loading Conditions	Displacement Limits Inches			Weight in Lb	Computer Time
					x	y	z		
1 (Fig 3)	10	25	40,000	6	0.35	0.35	0.35	555.11	24 Sec
						None		91.14	9 Sec
2 (Fig 4)	20	72	25,000	5	0.25	0.25	0.25	425.77	5.75 Min
						None		96.58	59 Sec
3-I (Fig 5)	5	4	25,000	1		0.3	0.4	39.88	3 Sec
						None		34.48	4 Sec
3-II (Fig 5)	5	4	25,000	3		0.3	0.4	14.30	10 Sec
						None		9.09	3 Sec
4 (Fig 6)	61	132	20,000	4	0.1	0.1	0.1	180.94	68 Min
*5 (Fig 7)	77	200	30,000	5	0.5	0.5		31,020	93 Min
						None		7550	50 Min

*Steel Structure

SECTION VIII

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