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ON INFLUENCE FUNCTIONS IN THE THEORY OF FORCED VIBRATIONS OF MEMBRANES

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FOREWORD

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ABSTRACT

In this report a general method is outlined for the calculation of the response of membranes with arbitrary boundaries to arbitrary loadings. It is assumed that, by projecting the area of the given membrane on the surface of an unbounded membrane and then applying the given loading to this projection, the application of a suitable load distribution around the boundary of the projection will enable us to satisfy the boundary conditions appropriate to the given membrane. An attempt to find the distribution in question leads to a logarithmically singular integral equation of an unusual type. A few solutions are outlined.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:

W. J. Trapp
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LIST OF SYMBOLS

σ	Superficial density of membrane
T	Tension per unit length of membrane
$c = (T/\sigma)^{1/2}$	Velocity of wave propagation in membrane
p	Pressure on surface
P	Point load or pole
ω	Circular frequency
s	Distance along boundary
$k = \omega/c$	Wave number
$r,\ r^*$	Radial co-ordinates
$\theta,\ \phi,$	Angular co-ordinates
$\Delta = r/R,\ \Delta^* = r^*/R$	Dimensionless radialco-ordinates for circular membrane
∇^2	Laplace's operator
R	Radius of circular membrane
a	Characteristic dimension for general membrane
W	Transverse displacement of membrane
$J_0(x),\ Y_0(x)$	Zero order Bessel functions of the first and second kinds respectively
$\xi = 2kR$ or $2ka$	(According to context)

$$\Delta_n(\xi) = \int_0^{\pi} Y_0(\xi \sin \tfrac{1}{2}\phi)\,d\phi = \pi J_0(\tfrac{1}{2}\xi)Y_0(\tfrac{1}{2}\xi)$$

$$\Theta_0(\xi,\Delta) = Y_0(\tfrac{1}{2}\xi\Delta)J_0(\tfrac{1}{2}\xi) - J_0(\tfrac{1}{2}\xi\Delta)Y_0(\tfrac{1}{2}\xi)$$

$$\ell(\Delta,\Delta^*,\theta) = \sqrt{\Delta^2 - 2\Delta\Delta^*\cos\theta + \Delta^{*2}}$$

I. INTRODUCTION

The theory of vibrations is a branch of modern science that has received considerable attention since the 17th Century. At the present time one of the problems which has attracted attention is that of calculating the response of a stiff metal structure to random pressure fluctuations such as may occur, for example, in the vicinity of a jet engine exhaust. The problem of obtaining the characteristics of the pressure field is itself extremely complex. In addition to this, the analysis of modern airframes is difficult and some degree of idealisation is unavoidable at some stage. For some purposes, for example, we may be in a position to focus attention on a single panel and regard it as a flat plate. Even the analysis of flat plates is by no means simple and no approach can safely be overlooked. The two methods most familiar to workers in this field are the Rayleigh - Ritz or Lagrange method, with all its modifications, and the method of separation of variables i.e. the normal mode method.

The idea of arranging a distribution of forces and couples around the projection of the boundary of the plate in question on to an unbounded plate with the same physical characteristics and then choosing proper forms for these distributions so as to satisfy the boundary conditions appropriate to the finite plate does not seem to have been exploited to any extent in the literature available in this country or elsewhere. Much of the basic work has been done by H. and L. Cremer[1] and by E. Reissner[2] but they have made no detailed analysis of boundary value problems using their results. In order to develop the basic concepts of such an approach, it is neither necessary nor wise to begin with a difficult problem. It is therefore the aim of this report to discuss briefly the application of the method to the well known and relatively simple problem of the uniformly stretched membrane in free and forced vibrations. For this particular problem many exact solutions are known for a number of boundary configurations but the

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'influence function' method, which has been briefly described above, can be used with advantage in cases where the boundaries are not simple.

II. THE INFLUENCE FUNCTION FOR AN UNBOUNDED MEMBRANE

We may make the concept of the 'influence function' method a little more clear if, instead of the membrane, we consider a finite stretched string. Then if a load P $\exp(i\omega t)$ is applied at any point, the particular integral may readily be found and the addition of the two general solutions of the appropriate homogeneous equation to this gives two arbitrary parameters to be determined. These are found by satisfying the boundary conditions at both ends of the string. Alternatively, however, we could apply the same load to an infinite string and apply two, at present unknown, loads at the points which would be fixed on the finite string in question. A suitable choice of these unknown loads would then allow us to satisfy the boundary conditions at those points so that the portion of string inside the 'boundary loads' would behave in the same way as the finite string in question. The problem of the vibrating string has been studied by several investigators and the work of Professor Lyon[4] is of particular interest in connection with the modern interest in the vibrations of systems subjected to random loads.

The same approach may be adopted in the case of the finite membrane. The end loads are now replaced by a load distribution around the boundary of the projected area of the given membrane on to an unbounded membrane. An attempt to satisfy the boundary condition then leads to a logarithmically singular integral equation for the boundary distribution. For a flat plate we must use two influence functions, corresponding to a point load and a point couple, and a pair of simultaneous integral equations for the two distributions is obtained.

The motion of a uniform membrane of superficial density σ and tension T is given by the well known equation:-

$$\nabla^2 W - (1/c^2)\,\partial^2 W/\partial t^2 = p(x,y,t)/T \qquad (2.1)$$

where p(x,y,t) is the pressure distribution over the surface and c is the velocity of propagation of small transverse disturbances. If an oscillating point load or pole P exp($i\omega t$) is applied at the origin of co-ordinates of such a membrane, assumed to be unbounded, then the singular part of the solution must be:-

$$W(r) = (P/4T)\,Y_0(kr)\exp(i\omega t) \qquad (2.2)$$

with $k = \omega/c$. For a circular membrane with a point load at the centre we now add the non-singular solution of the homogenous equation, namely A $J_0(kr)$, and make use of the boundary condition to determine A. When the membrane is unbounded, this cannot be done, and for all finite time after the first application of the load an outward travelling wave must result far from the origin. We are not here interested in this aspect of the matter, as we shall always be seeking stationary wave solutions for finite membranes and (2.2) will be used merely as a particular integral or "influence function".

For membranes which are of shape other than circular, it is difficult, if not impossible, to write down a general solution which can then be specialized by introducing the boundary conditions. We must therefore seek some means of finding the appropriate special solution directly. The method used has already been briefly described and will be discussed in more detail in the following sections.

It is also of some interest to consider the effects of damping. If the damping is assumed to oppose the motion in such a way that it is always proportional to the velocity at a point, then the equation of motion becomes:-

$$\partial^2 W/\partial t^2 = c^2\,\nabla^2 W + p(x,y,t)/\sigma - \beta\,\partial W/\partial t \qquad (2.3)$$

where β is a constant. If a harmonically varying solution of the form $W = W(r) \exp(i\omega t)$ is assumed to apply for the case of forced vibrations and $p(x,y,t)$ is of the form $p(x,y) \exp(i\omega t)$ then the equation of motion becomes:-

$$(\nabla^2 + k_o^2)W = p(x,y)/\sigma \tag{2.4}$$

where

$$k_o = k(1 - i\omega\beta)^{\frac{1}{2}} \tag{2.5}$$

This equation is identical with that obtained in the absence of damping provided that we are prepared to make use of solutions involving Bessel functions with complex arguments. If $\beta\omega$ is very small we may replace equation (2.5) by the approximate expression:-

$$k_o = k(1 - i\mu) \tag{2.6}$$

where $\omega\beta = 2\mu$. The incorporation of damping of the type postulated is therefore simply a matter of replacing k by k_o in any results that we may derive later.

III. GENERAL INTEGRAL EQUATION FOR MEMBRANE WITH ARBITRARY BOUNDARIES

In Fig. 1 let C be any curve, of class C_2, which represents the projection of the boundary of a given finite membrane on to an infinite membrane of identical physical characteristics.

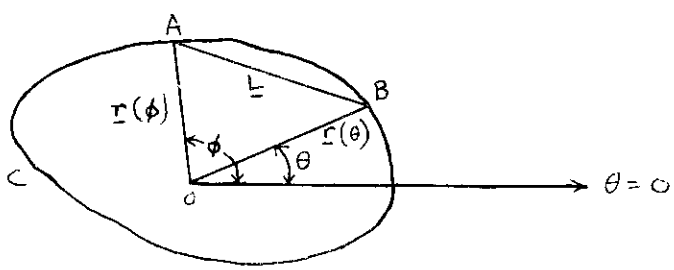

Fig. 1

The contribution to the transverse displacement W at any point of C due to the applied pressure field is know and is denoted by $W_1(\theta)$. Consider a distribution on $P(\phi)$ of loads around C, where ϕ is an auxiliary co-ordinate (see Fig. 1).

We note that W_i (θ) must have a single discrete frequency (ω) from the point of view of this analysis but the generalisation would be fairly simple if it were required. Then the total contribution of $P(\phi)$, in the absence of damping, to the response at any point θ of C is:-

$$W_f(\theta) = (1/4T)\int_0^{2\pi} P(\phi)\, Y_0\left\{kL(\theta,\phi)\right\}\frac{ds}{d\phi}(\phi)\, d\phi \qquad (3.1)$$

$$L(\theta,\phi) = \left|\underline{r}(\phi) - \underline{r}(\theta)\right| \qquad (3.2)$$

as in Figure 1 . The vectors $\underline{r}(\phi)$ and $\underline{r}(\theta)$ are the position vectors of the points A and B respectively. The integral equation which $P(\phi)$ must satisfy can now be derived by making use of the fact that the sum of W (θ) is zero at all points of the boundary i.e.

$$\int_0^{2\pi} P(\phi)\, Y_0\left\{kL(\theta,\phi)\right\}\frac{ds}{d\phi}(\phi)\, d\phi = -\,4T\,W_i(\theta) \qquad (3.3)$$

This equation represents the most general formulation of the membrane vibration problem. Once a solution has been obtained, it is a relatively simple matter to combine the effects of the loading and the boundary distribution to obtain the response at any point. The real problem is to solve equation (3.3) which is an integral equation of a type that does not seem to have been studied previously, although it bears a resemblance to some standard forms. We note that the integral equation is logarithmically singular since $Y_0\left\{kL(\theta,\phi)\right\}$ becomes logarithmically infinite as $\theta \to \phi$.

If we now suppose that a solution has been obtained in some way, so that $P(\phi)$ is known, then the response at any point is found in the following manner:-

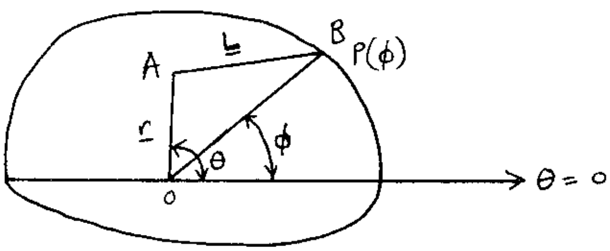

Fig. 2

Consider the response at the point $A(r,\theta)$ of Fig. 2 and let $\underline{L}$ be the vector $\underline{AB}$. The contribution of the loading is known and is denoted by $W_i(r,\theta)$. The contribution of $P(\phi)$ to the response at A is, evidently:-

$$W_1(r,\theta) = (1/4T)\int_0^{2\pi} P(\phi)\, Y_0\{kL(r,\theta,\phi)\}\, \frac{ds}{d\phi}(\phi)\, d\phi \qquad (3.4)$$

where L is the length of AB. The total response at A is therefore:-

$$W(r,\theta) = W_i(r,\theta) + (1/4T)\int_0^{2\pi} P(\phi)\, Y_0\{kL(r,\theta,\phi)\}\, \frac{ds}{d\phi}(\phi)\, d\phi \qquad (3.5)$$

Once $P(\phi)$ has been determined, therefore, the calculation of the response at any point is simply a matter of substituting $P(\phi)$ into equation (3.5). We shall next consider as an example the simple problem of the membrane with a circular boundary. This will enable us to make a comparison between the present solution and the known solution obtained by the method of separation of variables.

IV. THE RESPONSE OF CIRCULAR MEMBRANE

For a circular membrane the integral equation (3.3) may be simplified, for:-

$$ds/d\phi = R \quad \text{and} \quad L = 2R\,\mathrm{Sin}\,\tfrac{1}{2}|\phi-\theta|$$

i.e. equation (3.3) takes the form

$$(R/4T)\int_0^{2\pi} P(\phi)\, Y_0\{\xi\,\mathrm{Sin}\,\tfrac{1}{2}|\phi-\theta|\}\, d\phi + W_i(\theta) = 0 \qquad (4.1)$$

where $\xi = 2kR$. Let $W_i(\theta)$ be caused be a pressure field $p(r,{}^*\psi)$ over the membrane surface. Then, if we regard the elementary load $p(r,{}^*\psi)\,r^*.dr^*.d\psi$ as a discrete point load and then integrate over the surface we may use equation (2.2) to obtain:-

$$W_i(\Delta,\theta) = (R^2/4T)\int_0^1\int_0^{2\pi} p(\Delta^*,\psi)\, Y_0\{\tfrac{1}{2}\xi\,\ell(\Delta,\Delta^*,\psi-\theta)\}\,\Delta^* d\Delta^* d\psi \qquad (4.2)$$

where Δ^* $(= r^*/R)$ and ψ are auxiliary angular co-ordinates and radial co-ordinates.

This expression for $W_i(\Delta,\theta)$ with $\Delta = 1$ may be expressed as the Fourier Series:-

$$W_i(\theta) = W_{oo} + \sum_{n=1}^{\infty}\left(W_{on}\,\mathrm{Cos}\,n\theta + W_{1n}\,\mathrm{Sin}\,n\theta\right) \qquad (4.3)$$

where the coefficients are given by:-

$$\left.\begin{aligned}
W_{oo} &= (R^2/8\pi T)\int_0^{2\pi}\int_0^{2\pi}\int_0^1 p(\Delta^*,\psi)\, Y_0\{\tfrac{1}{2}\xi\,\ell(1,\Delta^*,\psi-\theta)\}\,\Delta^* d\Delta^* d\psi\, d\theta\\
W_{on} &= (R^2/4\pi T)\int_0^{2\pi}\int_0^{2\pi}\int_0^1 p(\Delta^*,\psi)\, Y_0\{\tfrac{1}{2}\xi\,\ell(1,\Delta^*,\psi-\theta)\}\,\mathrm{Cos}\,n\theta\,\Delta^* d\Delta^* d\psi\, d\theta\\
W_{1n} &= (R^2/4\pi T)\int_0^{2\pi}\int_0^{2\pi}\int_0^1 p(\Delta^*,\psi)\, Y_0\{\tfrac{1}{2}\xi\,\ell(1,\Delta^*,\psi-\theta)\}\,\mathrm{Sin}\,n\theta\,\Delta^* d\Delta^* d\psi\, d\theta
\end{aligned}\right\} \qquad (4.4)$$

These triple integrals are complex in form but it may be shown that one of the integrations can be carried out irrespective of the form of $p(\Delta^*,\psi)$

Using some results given in Ref. 3 it may, in fact, be shown that:-

$$\left.\begin{aligned}
W_{oo} &= (R^2/4T)\, Y_0(\tfrac{1}{2}\xi)\int_0^1\int_0^{2\pi} p(\Delta^*,\psi)\, J_0(\tfrac{1}{2}\xi\Delta^*)\,\Delta^* d\Delta^* d\psi\\
W_{on} &= (R^2/4T)\, Y_n(\tfrac{1}{2}\xi)\int_0^1\int_0^{2\pi} p(\Delta^*,\psi)\, J_n(\tfrac{1}{2}\xi\Delta^*)\,\mathrm{Cos}\,n\psi\,\Delta^* d\Delta^* d\psi\\
W_{1n} &= (R^2/4T)\, Y_n(\tfrac{1}{2}\xi)\int_0^1\int_0^{2\pi} p(\Delta^*,\psi)\, J_n(\tfrac{1}{2}\xi\Delta^*)\,\mathrm{Sin}\,n\psi\,\Delta^* d\Delta^* d\psi
\end{aligned}\right\} \qquad (4.5)$$

In the same way, we may expand $P(\phi)$ as a Fourier Series in ϕ:-

$$P(\phi) = P_{oo} + \sum_{n=1}^{\infty}\left(P_{on}\,\mathrm{Cos}\,n\phi + P_{1n}\,\mathrm{Sin}\,n\phi\right) \qquad (4.6)$$

After a little simplification, equation (4.1) then reduces to:-

$$(RP_{oo}/2T)\int_0^{\pi} Y_0(\xi\,\mathrm{Sin}\,\tfrac{1}{2}\phi)\,d\phi + (R/2T)\sum_{n=1}^{\infty}(P_{on}\,\mathrm{Cos}\,n\theta + P_{1n}\,\mathrm{Sin}\,n\theta)\int_0^{\pi} Y_0(\xi\,\mathrm{Sin}\,\tfrac{1}{2}\phi)\,\mathrm{Cos}\,n\phi\, d\phi +$$
$$+ W_{oo} + \sum_{n=1}^{\infty}\left(W_{on}\,\mathrm{Cos}\,n\theta + W_{1n}\,\mathrm{Sin}\,n\theta\right) = 0$$

We now equate to zero the coefficients of Sin $n\theta$ and Cos $n\theta$ in this equation to obtain:-

$$\left.\begin{aligned}
P_{oo} &= -\frac{2T W_{oo}}{R\Delta_0(\xi)}\\
P_{on} &= -\frac{2T W_{on}}{R\Delta_n(\xi)}\\
P_{1n} &= -\frac{2T W_{1n}}{R\Delta_n(\xi)}
\end{aligned}\right\} \qquad (4.7)$$

where $$\Delta_n(\xi) = \int_0^{\pi} Y_0(\xi \sin \tfrac{1}{2}\phi) \cos n\phi \, d\phi \qquad (4.8)$$

It may be shown, again using some results given in Ref. 3, that:-

$$\int_0^{\pi} Y_0\{\tfrac{1}{2}\xi \, \ell(\Delta,1,\phi)\} \cos n\phi \, d\phi = \pi J_n(\tfrac{1}{2}\xi\Delta) Y_n(\tfrac{1}{2}\xi) \qquad (4.9)$$

where $\Delta \leq 1$. It now remains to calculate the response at any point. In Figure 3 below we consider the contribution of a load element $P(\phi)$ ds at the boundary to the response at an interior point (r,θ) of the membrane:-

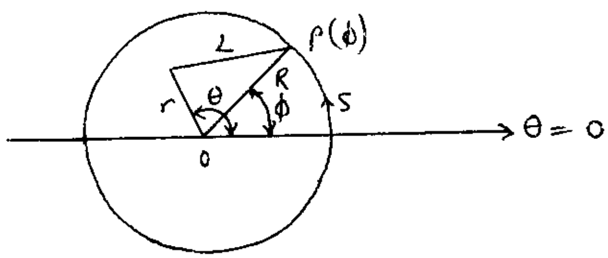

Fig. 3

Then it may readily be shown that:-

$$W(\Delta,\theta) = (R^2/4T) \int_0^1 \int_0^{2\pi} p(\Delta^*,\psi) \, Y_0\{\tfrac{1}{2}\xi \, \ell(\Delta,\Delta^*,\psi-\theta)\} \Delta^* d\Delta^* d\psi -$$
$$- \frac{W_{00}}{\Delta_0(\xi)} \int_0^{\pi} Y_0\{\tfrac{1}{2}\xi \, \ell(\Delta,1,\phi)\} \, d\phi -$$
$$- \sum_{n=1}^{\infty} \left\{ \frac{W_{0n}\cos n\theta + W_{1n}\sin n\theta}{\Delta_n(\xi)} \right\} \int_0^{\pi} Y_0\{\tfrac{1}{2}\xi \, \ell(\Delta,1,\phi)\} \cos n\phi \, d\phi \qquad (4.10)$$

Using equation (4.9) we may reduce this to the form:-

$$W(\Delta,\theta) = (R^2/4T) \int_0^1 \int_0^{2\pi} p(\Delta^*,\psi) \, Y_0\{\tfrac{1}{2}\xi \, \ell(\Delta,\Delta^*,\psi-\theta)\} \Delta^* d\Delta^* d\psi -$$
$$- \frac{W_{00} J_0(\tfrac{1}{2}\xi\Delta)}{J_0(\tfrac{1}{2}\xi)} - \sum_{n=1}^{\infty} \left\{ \frac{W_{0n}\cos n\theta + W_{1n}\sin n\theta}{J_n(\tfrac{1}{2}\xi)} \right\} J_n(\tfrac{1}{2}\xi\Delta) \qquad (4.11)$$

It will be noted that this is exactly the result obtained for the same problem by the normal mode method. The condition for resonance is evidently that $J_n(\tfrac{1}{2}\xi) = 0$. The equation for the response is still rather complicated but a very simple expression can be found for the response at the centre of the membrane, as many terms in (4.11) will then vanish. We denote the response at this particular point by W(0) and it may be shown that:-

$$W(0) = \frac{R^2}{4T\, J_0(\frac{1}{2}\xi)} \int_0^1 \int_0^{2\pi} p(\Delta^*, \phi)\, \Theta_0(\xi, \Delta^*)\, \Delta^*\, d\Delta^*\, d\phi \qquad (4.12)$$

where

$$\Theta_0(\xi, \Delta^*) = Y_0(\tfrac{1}{2}\xi\Delta^*)\, J_0(\tfrac{1}{2}\xi) - J_0(\tfrac{1}{2}\xi\Delta^*)\, Y_0(\tfrac{1}{2}\xi) \qquad (4.13)$$

when $p(\Delta^*, \phi)$ is independent of ϕ i.e. when the membrane is being subjected to an axi-symmetric oscillating pressure field, this expression for W(0) reduces to a particularly simple form and the results are illustrated in Fig. 9 for the case of a uniform pressure field. For points other than the origin this simplicity is lost even in the case where $p(\Delta^*, \phi)$ is idependent of ϕ. If we consider the axi-symmetric case then, for points other than the origin, two situations arise according as $\Delta \leq \Delta^*$ or $\Delta \geq \Delta^*$ and the range of integration must be split. If this is done we have the final solution:-

$$\left.\begin{aligned} W(\Delta) = {} & \frac{\pi R^2\, \Theta_0(\xi, \Delta)}{2T\, J_0(\frac{1}{2}\xi)} \int_0^{\Delta} p(\Delta^*)\, J_0(\tfrac{1}{2}\xi\Delta^*)\, \Delta^*\, d\Delta^* \; + \\ & + \frac{\pi R^2\, J_0(\frac{1}{2}\xi\Delta)}{2T\, J_0(\frac{1}{2}\xi)} \int_{\Delta}^{1} p(\Delta^*)\, \Theta_0(\xi, \Delta^*)\, \Delta^*\, d\Delta^* \end{aligned}\right\} \qquad (4.14)$$

A similar but far more complicated expression may be derived for the case where p depends on both Δ^* and ϕ. This problem is well known, however (see, for example, Refs 5 and 6), and will not be discussed further. For more complex boundary shapes exact solutions are not always available, however. For example, in the case of the elliptical membrane, a separable solution can be obtained in terms of the Mathieu functions but the analysis is far more complex than for the circular membrane and the 'influence function' method offers some advantage. For unusual shapes such as the cardioid, lemniscate etc., it offers the only practical method apart from integration of the equation of motion by purely numerical methods. Such shapes are mainly of theoretical interest, of course.

V. THE ELLIPTIC MEMBRANE

Consider an ellipse of semi-major axis 'a' and eccentricity 'e'. Take the centre as origin and the semi-major axis as initial line. A single load element (denoted by +) can be separated into a linear combination of unit symmetric and anti-symmetric components as in Figure 4.

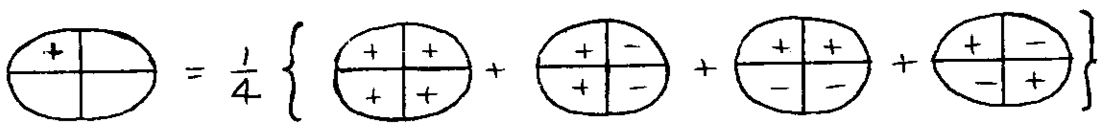

Fig. 4

For the sake of illustration we shall consider only the first of these cases, namely that where the loading is symmetric about both axes. This is the most important case as the first eigenvalue corresponds to a mode with this type of symmetry. Let $W_i(\theta)$ be expressible in the form:-

$$W_i(\theta) = \sum_{m=0}^{\infty} W_m \cos 2m\theta \qquad (5.1)$$

It can readily be shown that the integral equation for the load distribution around the boundary now takes the form:-

$$\int_0^{2\pi} P(\phi)\, Y_0\{\xi G(e,\theta,\phi)\} \sqrt{1-e^2\cos^2\phi}\, d\phi = -(4T/a)\, W_i(\theta) \qquad (5.2)$$

where
$$G(e,\theta,\phi) = \sqrt{1-e^2\cos^2\tfrac{1}{2}(\theta+\phi)}\ \sin\tfrac{1}{2}|\phi-\theta| \qquad (5.3)$$

We now expand $P(\phi)$ as a Fourier Series in ϕ (see equation 4.6) and after some simplification equation (5.2) reduced to the form:-

$$\sum_{m=0}^{\infty} W_m \cos 2m\theta + (a/4T)\sum_{n=0}^{\infty} P_n \int_0^{2\pi} Y_0\{\xi G(e,\theta,\phi)\} \sqrt{1-e^2\cos^2\phi}\, \cos 2n\phi\, d\phi = 0 \qquad (5.4)$$

where terms in Cos2nϕ only have been retained in the Fourier series for P(ϕ).

We now expand the integrals involved in this equation in the form of Fourier Series in θ:-

$$\int_0^{2\pi} Y_0\{\xi G(e,\theta,\phi)\} \sqrt{1-e^2\cos^2\phi}\, \cos 2n\phi\, d\phi = \sum_{m=0}^{\infty} A_{mn} \cos 2m\theta \qquad (5.5)$$

where the coefficients A_{mn} are given by:-

$$A_{mn} = \frac{1}{\pi(1+\delta_0^m)} \int_0^{2\pi}\int_0^{2\pi} Y_0\{\xi G(e,\theta,\phi)\}\sqrt{1-e^2\cos^2\phi}\, \cos 2n\phi \cos 2m\theta\, d\theta\, d\phi \qquad (5.6)$$

with the symbol δ_0^m representing a particular form of the Kronecker delta.

We may further write (5.4) in the following form, if use is made of equation (6.6):-

$$\sum_{m=0}^{\infty} \left\{ W_m + (a/4T) \sum_{n=0}^{\infty} P_n A_{mn} \right\} \cos 2m\theta = 0$$

$$\text{i.e.} \sum_{n=0}^{\infty} P_n A_{mn} = -(4T/a) W_m \qquad (5.7)$$

This infinite array of algebraic equations may be solved, in principle, for all P_n. In practice, the array must be curtailed after a finite number of terms. The cut off point depends on the degree of accuracy required and on the values of the parameters ξ and e. The coefficients A_{mn} may be obtained by numerical integration or, if e is small, the functions involved in the kernel of A_{mn} may be expanded as Taylor series in e^2 up to a specified number of terms, and the double integrals can then be evaluated in terms of known functions. Moreover, the array of equations now terminates automatically, this being ensured by limiting the number of terms in the Taylor series.

For resonance in a mode of the type under consideration we have the infinite determinantal equation:-

$$|A_{mn}| = 0 \qquad (5.8)$$

and from this may be determined the allowable values of ξ for free vibration. If e^2 is so small that e^4 may be ignored, it may be shown that the first resonance condition is given by the first root of the equation:-

$$J_0\left\{\tfrac{1}{2}\xi\left(1-\tfrac{1}{4}e^2\right)\right\} = 0$$

$$\text{i.e.} \qquad \xi = 4.81\left(1+\tfrac{1}{4}e^2\right) \tag{5.9}$$

If terms in e^4 are incorporated, the determinantal equation is far more cumbersome and will not be included. It is sometimes useful to consider a different eigenvalue ξ' corresponding to the radius R of a circle with area equal to that of the given figure. In this case:-

$$\begin{aligned}\xi' &= 2kR = 2ka\,(R/a)\\ &\doteq 4.81\end{aligned} \tag{5.10}$$

if terms in e^4 are ignored. This means that the first eigenvalue, defined in this way, is very insensitive to quite large changes of shape from the circular form. This fact was first explicitly stated by Rayleigh[5] and allows us to obtain an approximation to the first eigenvalue of any shape of membrane.

VI. CONCLUSIONS

The fundamental influence function for membrane vibrations is given and the result is sufficient to enable us to solve the most general problems of infinite and finite membranes subjected to arbitrary loadings. The foundation has therefore been laid for the solution of particular problems of academic and practical interest. Particularly simple expressions have been derived for the response of a circular membrane to an axi-symmetric pressure field and these could be used as a basis for the investigation of the response of a circular membrane to an axi-symmetric pressure field with a random time dependence.

REFERENCES

1. Cremer, H. and L., "Theorie der Entstehung des Klopfschalls",

   Z. f. Schwingungs - und Schwachstromtechnik 2, 61 - 72 (1948).

2. Reissner, E. "On the axi-symmetric vibrations of shallow spherical shells,"

   Journal of Appl. Maths., 13, 279-290 (1955).

3. Whittaker, E.T. and Watson, G.N. "A course of Modern Analysis",

   Cambridge University Press, 380-383 (1927).

4. Lyon, R.H. "The excitation of continuous systems by random noise fields",

   Ph.D. Thesis, Massachusetts Institute of Technology (1955).

5. Lord Rayleigh, "The Theory of Sound",

   Volumes 1 and 2, Second Edition by Dover Publications, New York (1945).

6. Kinsler, L.E., and Frey, A.R. "Fundamentals of Acoustics",

   Wiley and Sons Inc.,New York (1950).

FIG. 5

THE FUNCTION $\Delta_0(\xi)$

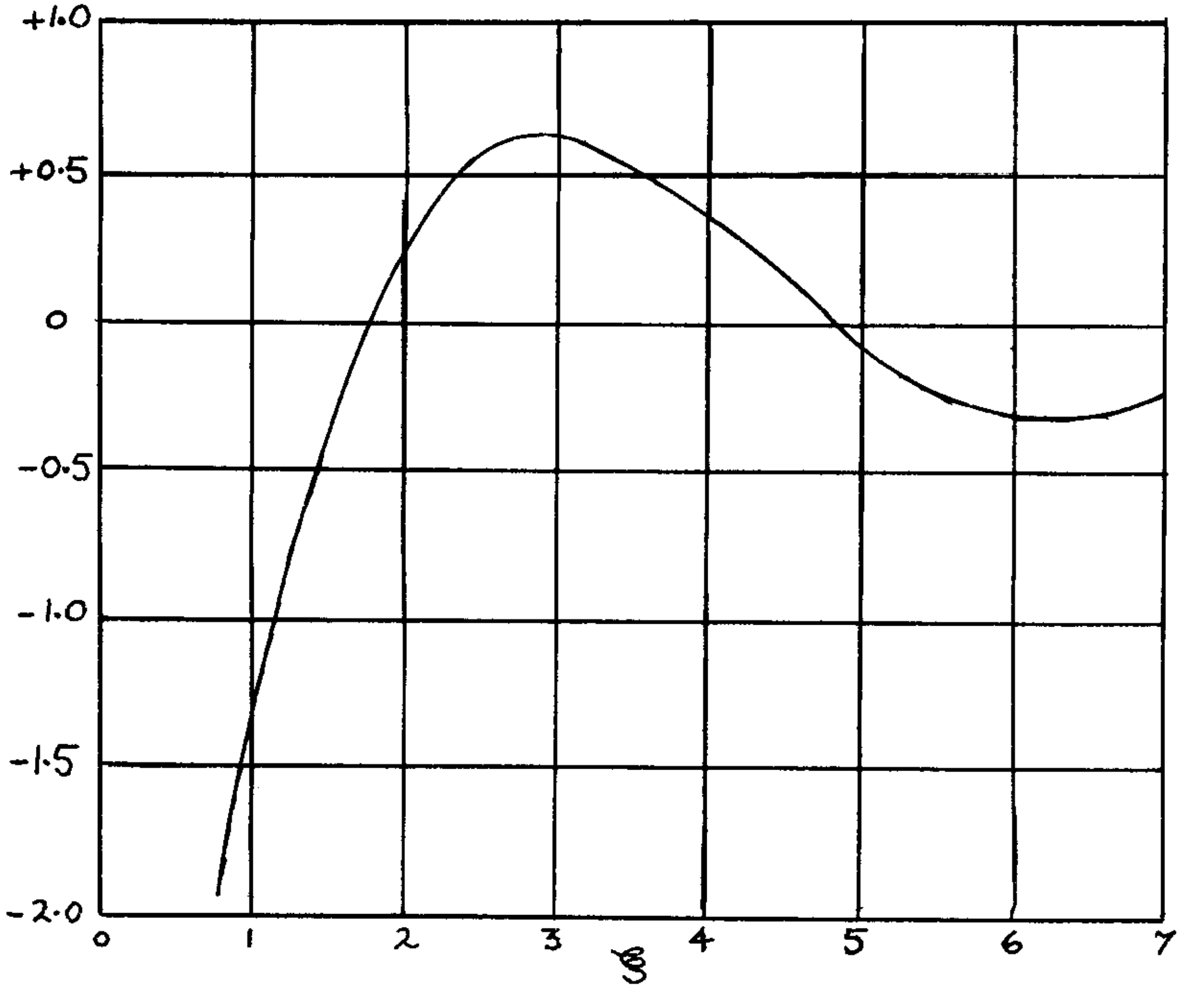

FIG. 6

THE FUNCTION $\Delta \Theta_0(\xi, \Delta)$

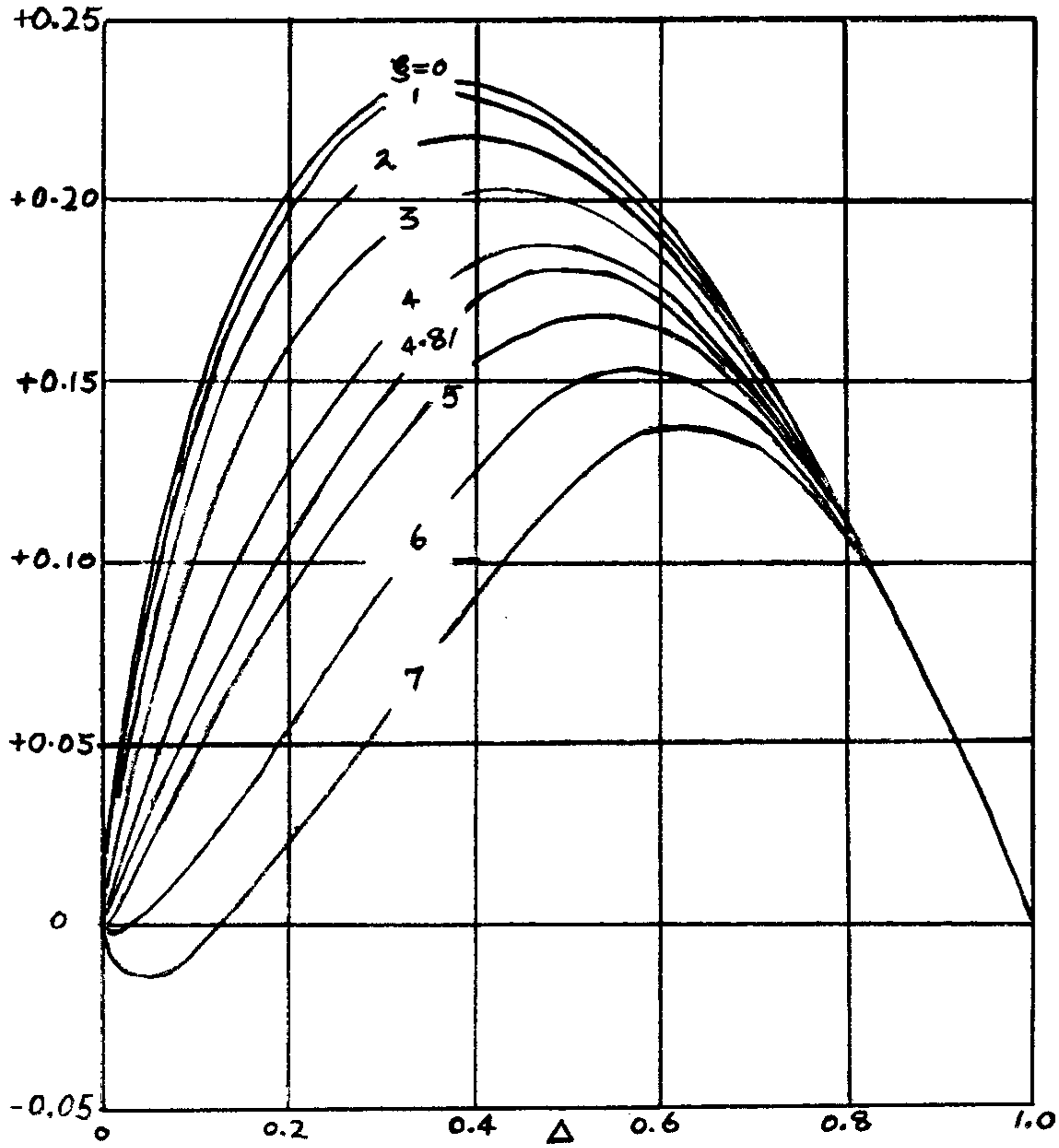

FIG. 7

THE FUNCTION $\int_{\Delta}^{1} \Theta_0(\xi,\Delta^*)\Delta^* d\Delta^*$

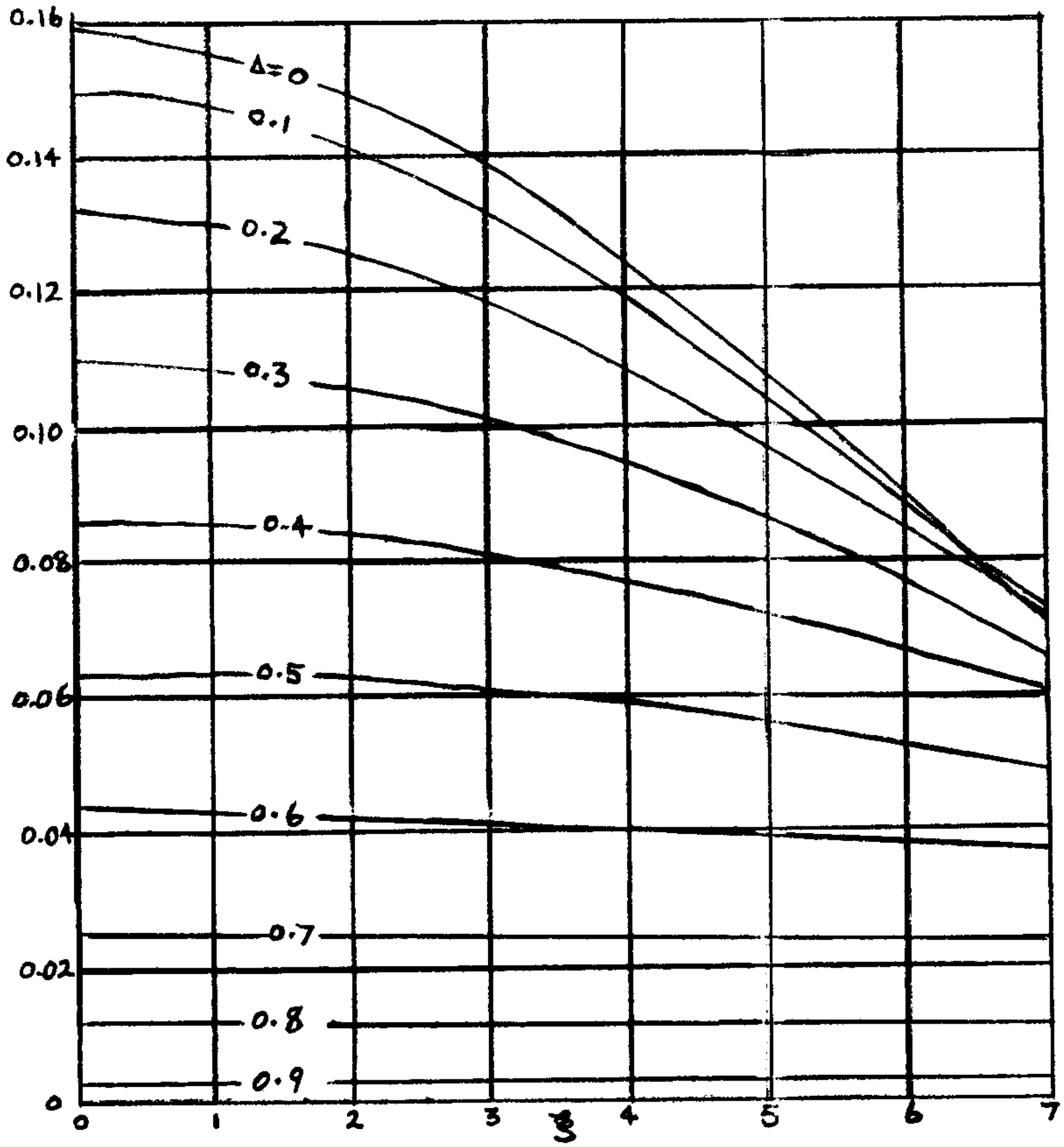

FIG. 8

THE FUNCTION $\int_0^{\Delta} J_0(\frac{1}{2}\xi\Delta^*)\Delta^* d\Delta^*$

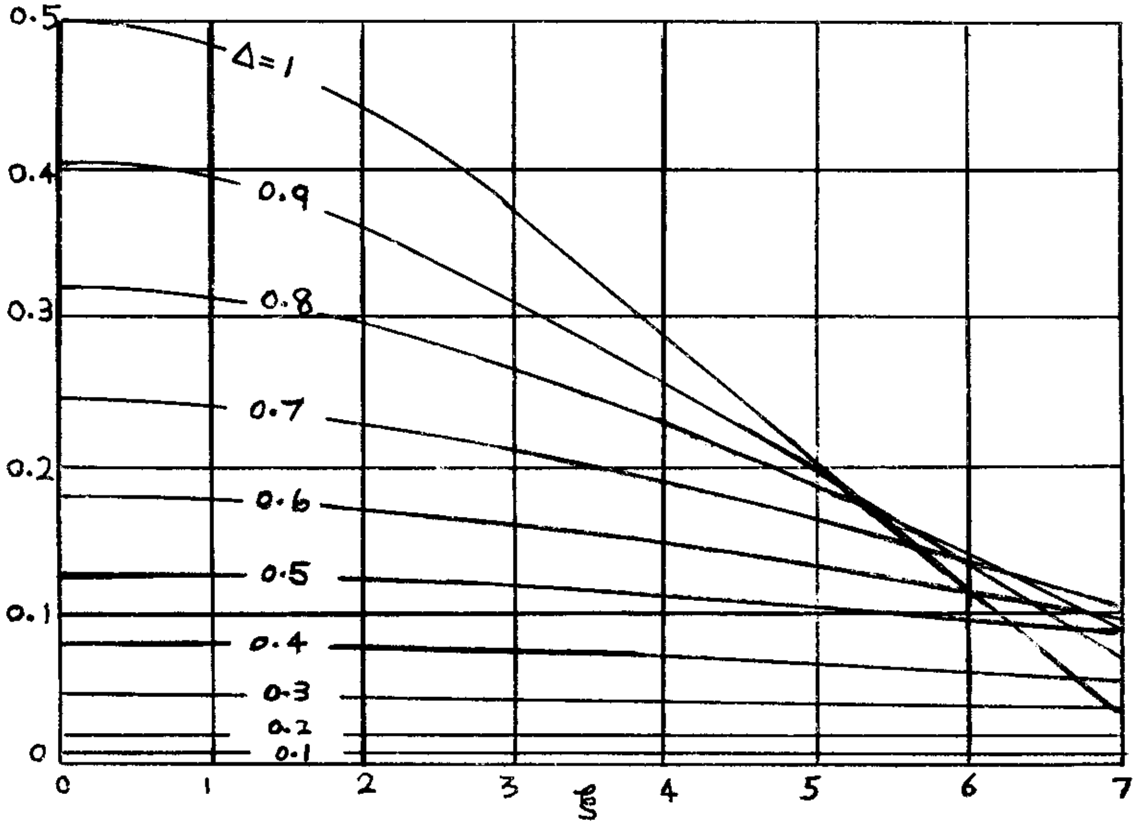

FIG. 9

THE FUNCTION $2T|w(0)|/\pi PR^2$

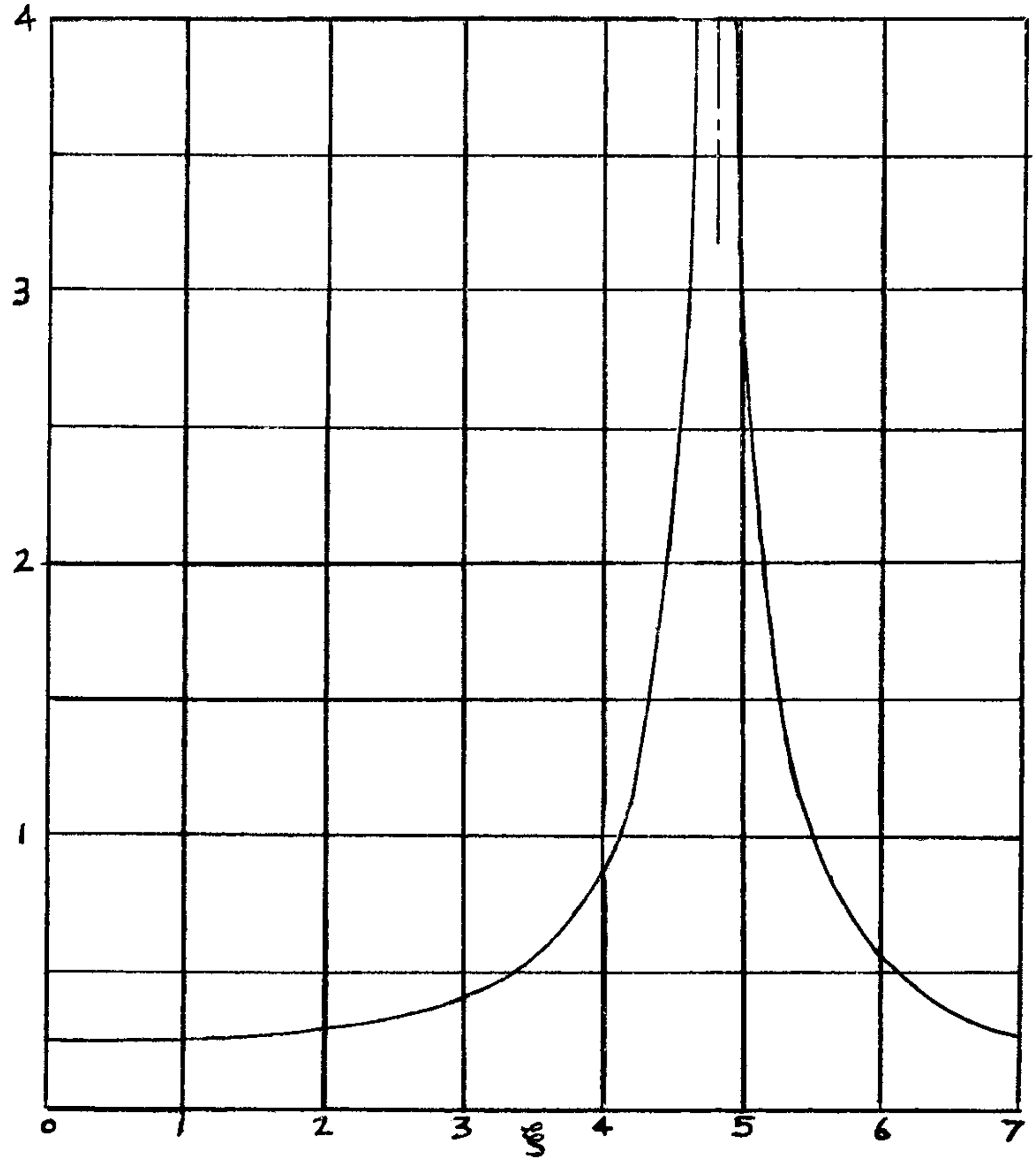

FIG. 10

THE FUNCTION $\Delta \Theta_0(\xi, \Delta)$