FINITE ELEMENT GALERKIN METHOD SOLUTIONS TO SELECTED ELLIPTIC AND PARABOLIC DIFFERENTIAL EQUATIONS

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In this paper a discussion of the use of variational analysis versus Galerkin formulation in developing integral finite element equations is presented. In view of this discussion finite element Galerkin type equations are developed for selected elliptic and parabolic equations. In developing these equations special emphasis is given to the inter element boundary conditions. Two examples are included in the area of fluid mechanics for illustrative purposes.

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SECTION I

INTRODUCTION

The finite element technique appears to have had its origin in the area of structural analysis, beginning, at least for most engineers in the U.S. and in the U.K. with the paper by Turner, Clough, Martin, and Topp [1] [1]. For several years afterwards, the necessary stiffness or flexibility matrices were derived from basic concepts such as the principle of virtual work or Castiglianos's Theorem as applied to the individual finite elements. The so-called global or overall matrix was formed by using familiar ideas from the analysis of discrete structures such as trusses and frames.

A paper by Melosh [2] appears to have been one of the first to demonstrate that the finite element displacement method can be viewed as an application of the principle of stationary potential energy. After the finite element method was given this firm basis on energy principles, applications began to appear in areas other than solid mechanics and structural analysis [3], [4], [5], [6]. It was recognized that any problem whose solution could be associated with a stationary value of some functional could be solved by the finite element technique. Unfortunately, this seeming dependence on the existence of a variational principle led many investigators to formulate statements which were not, in reality, true variational principles [7], [8], [9]. Although these quasivariational principles and restricted variational principles give useful results, the fact remains that they do not possess a rigorous foundation in the classical calculus of variations. Therefore, it would appear that their use should be avoided whenever possible, especially when a sound alternative is available which gives exactly the same results.

There have been at least two earlier formulations of the finite element technique without recourse to a stationary principle. Oden and Kross [10] developed the equations of coupled thermoelasticity by means of what they described as energy balance equations for each element. Szabo and Lee [11] discussed the development of a plate bending element by Galerkin's method.

The procedure illustrated in this paper is the method of weighted residuals, primarily the Galerkin method, extended to systems in which the external and internal boundary continuity conditions are not identically satisfied by the trial functions. The procedure requires only a knowledge of the differential equations and boundary conditions which must be satisfied in a given domain and boundary. If a true, quasi, or restricted variational principle exists the Galerkin method can be made to give exactly the same result; and in the case of quasi or restricted principles, this result will have a more firm theoretical basis.

This paper presents formulations in two dimensional domains for the following differential equations with appropriate boundary conditions:

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[1] Numbers in brackets refer to the list of references.

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial \phi}{\partial y}\right)=0 \qquad (1)$$

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial \phi}{\partial y}\right)-K_t\frac{\partial \phi}{\partial t}=0 \qquad (2)$$

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial \phi}{\partial y}\right)-u\frac{\partial \phi}{\partial x}-v\frac{\partial \phi}{\partial y}-\frac{\partial \phi}{\partial t}=0 \qquad (3)$$

Equations (1) is an elliptic differential equation describing, for example, steady incompressible potential flow, or steady fluid seepage through a anisotropic, non-homogeneous medium, Zienkiewicz, Mayer, and Cheung[3]. This equation has also been considered by Martin[5] through a true variational principle; it is presented again only to establish the correspondence between Galerkin's method and the variational principle.

Equation (2) is a parabolic differential equation which might represent unsteady heat conduction[4]. The finite element solution can be compared to exact solutions for simple geometries, thereby demonstrating the accuracy of the numerical integration scheme used to solve the differential equations in time.

Equation(3) is a parabolic equation which describes the unsteady diffusion of the concentration of a substance in a fluid flow of specified velocities(u,v).

SECTION II

TRUE, QUASI, AND RESTRICTED VARIATIONAL PRINCIPLES

This section presents a brief description of the meaning of the terms true, quasi, and restricted variational principles. The ideas are drawn very heavily from a most outstanding series of papers by Finlayson and Scriven[7], [8], [9], and anyone interested in a more thorough discussion is encouraged to read the indicated references.

It is appropriate to begin with the concept of a true variational principle as defined within the framework of the classical calculus of variations. First there must exist a functional which associates a definite real number with every function (or set of functions) belonging to the class of admissible comparison functions. Next it must be realized that if two admissible comparison functions are different, then their partial derivatives with respect to all independent variables will also be different. Finally, the variational principle states that if the functional is made stationary by the techniques of the calculus of variations, then the resulting Euler equations and natural boundary conditions will indeed describe the physical behavior of the system under consideration.

As an example of the three ideas associated with the true variational principle, consider the following functional:

$$J[\phi] = \int_0^L \frac{1}{2} K(x) \left[\frac{d\phi(x)}{dx} \right]^2 dx \qquad (4)$$

The class of admissible comparison functions might be specified as containing all functions defined on the closed interval $0 \leq x \leq L$ which are continuous have continuous first derivatives, and have specified values at $x = 0$ and $x = L$. If ϕ_1 and ϕ_2 are related by

$$\phi_1(x) = \phi_2(x) + \eta(x) \qquad (5)$$

where $\eta(x)$ is simply the difference between $\phi_1(x)$ and $\phi_2(x)$, then the derivatives with respect to the independent variable (x) will be related by

$$\frac{d\phi_1}{dx} = \frac{d\phi_2}{dx} + \frac{d\eta}{dx} \qquad (6)$$

This means that if $\eta(x)$ is not identically equal to zero, then $d\eta/dx$ cannot be identically equal to zero. Finally, if this functional is made stationary, the resulting Euler equation will be

$$\frac{d}{dx}\left(K\frac{d\phi}{dx}\right) = 0 \tag{7}$$

which does describe, for example, one dimensional steady state heat transfer for a nonhomogeneous material.

If a physical system is supposedly described by a variational principle which lacks any of the three requirements of a true variational principle, then ipso facto, that principle can not be a true variational principle. In this case it is useful to have terminology which indicates that the so-called "principle" is not exactly what one might think.

As an example, suppose the problem is one dimensional steady state heat transfer for a nonhomogeneous material with temperature dependent conductivity. Further assume that the temperature has specified values at x = 0 and x = L. The differential equation is

$$\frac{d}{dx}\left[K(\phi,x)\frac{d\phi(x)}{dx}\right] = 0 \tag{8}$$

One could now define the quantity δJ as follows

$$\delta J \equiv \int_0^L \frac{d}{dx}\left[K(\phi,x)\frac{d\phi(x)}{dx}\right]\delta\phi(x)\,dx \tag{9}$$

and enunciate the "variational principle" that

"Among all functions $\phi(x)$ which are defined on the closed interval $0 \leq x \leq L$, are continuous, have continuous second derivatives, and have specified values at x = 0 and x = L, the correct (exact) function ϕ will make $\delta J = 0$ for arbitrary $\delta\phi$."

This is clearly a correct statement, but not very profound and really not worthy of dignifying as a "variational principle". With this example, the point is that because k is a function of ϕ there is no functional $J(\phi)$ whose first variation has the form given in Equation (9). This type of formulation was called a quasi-variational principle by Finlayson and Scriven [7] because the first requirement of a true variational principle is obviously missing.

As another example, consider the problem of one dimensional transient diffusion described by the following differential equation, initial condition, and boundary conditions:

$$K \frac{\partial^2 \phi(x,t)}{\partial x^2} = \frac{\partial \phi(x,t)}{\partial t} \quad , \quad 0 < x < L \ , t > 0 \qquad (10)$$

$$\phi(x,0) = 0 \quad , \quad 0 \leq x \leq L \ , \quad t = 0 \qquad (11)$$

$$\phi(0,t) = \phi_0 (1 - e^{-\alpha t}) , \quad x = 0 \ , \quad t > 0 \qquad (12)$$

$$\phi(L,t) = \phi_L (1 - e^{-\alpha t}) , \quad x = L \ , \quad t > 0 \qquad (13)$$

The boundary conditions have been taken as shown in order to provide continuity with the initial condition at t = 0, x = 0, and t = 0, x = L. If a functional J is defined as follows

$$J[\phi,\psi] = \int_0^t \int_0^L \left[\frac{1}{2} K \left(\frac{\partial \phi}{\partial x} \right)^2 - \psi \phi \right] dx \, dt \qquad (14)$$

then one can state the "variational principle" that

"Among all functions $\emptyset(x,t)$ which are defined in the domain $0 \leq x \leq L$ and $t \geq 0$, are continuous, have continuous first derivatives with respect to x, and satisfy Equations (11) through (13), the correct function $\emptyset$ will make $\delta J = 0$ for arbitrary $\delta\emptyset$ and fixed ψ subject to the condition that $\psi(x,t) = \partial\emptyset(x,t)/\partial t$ after the variation if performed."

In this example there is a functional (for a given value of t), and a superficial resemblance to a stationary condition leading to an Euler equation which does indeed describe the physical behavior of the system. However, the variational process is not carried out according to the classical calculus of variations, with the result that the functional is not stationary for the exact $\emptyset$. While the function $\emptyset$ is allowed to vary, the derivative $\partial\emptyset/\partial t$ is held fixed. This rather unclassical maneuver is accomplished by introducing the "alias variable" [7] $\psi(x,t)$ as an independent function during the variation and not identifying ψ as equal to $\partial\emptyset/\partial t$ until after the variation. This type of variational principle has been called a <u>restricted variational principle</u> because of the restriction imposed upon the allowable variation.

Although there are demonstrated deficiencies in rigor associated with both quasi and restricted variational principles, these formulations do give useful results, particularly in the area of developing approximate solutions for difficult problems for which no true variational principle exists. For this reason, any analyst would be justifiably inclined toward using these principles, especially if no other procedure were available. However, the fact is that another technique is available. As shown by Finlayson and Scriven [7] through the consideration of many examples, a straightforward application of the general method of weighted residuals (or, in particular, Galerkin's method) is completely equivalent to the application of a quasi or restricted principle. Furthermore, the method of weighted residuals provides the answer without all the internal inconsistencies which are intrinsic to quasi and restricted variational principles. Using the closing words of Reference [7];

> "When approximate solutions are in order the applied scientist and engineer are better advised to turn immediately to direct approximation methods for their problems, rather than search for or try to understand quasi-variational formulations and restricted variational principles."

SECTION III

FINITE ELEMENT GALERKIN METHOD

In the preceding section it was shown that the method of weighted residuals should be preferred over quasi or restricted variational principles. However, the conventional application of the method of weighted residuals involves trial functions which are continuous over the entire domain, resulting in a formulation not strictly applicable to the finite element technique in which discontinuities may exist. There have been many papers in the past which have successfullly considered the problem of developing approximate solutions with certain discontinuities in the trial functions [12], [13]. However, the attention has been given to problems for which a true variational principle exists; and the previous papers have been primarily directed toward demonstrating how to modify the variational principle so as to expand the class of admissible functions to include functions with the specified discontinuities.

Reference [14] shows that the method of weighed residuals can also be generalized to allow for discontinuities in the assumed solution functions. This then completely frees the finite element method from any dependence on a variational principle and allows the rigorous use of the finite element concepts for problems with no true variational principle.

For a simple example of Galerkin's method extended to allow for discontinuities, consider the following problem:

$$\text{for} \quad 0 < x < L, \quad \frac{d^2\phi}{dx^2} = 0 \tag{15}$$

$$\text{at} \quad x = 0, \quad \phi = \phi_o \tag{16}$$

$$\text{at} \quad x = L, \quad \frac{d\phi}{dx} = q_L \tag{17}$$

where ϕ_o and q_1 are specified constants. This set of equations could describe one dimensional steady state heat conduction based on the Fourier law of conduction for a homogeneous material with temperature independent conductivity. At x = 0, the temperature is specified; at x = L, the heat flux is specified.

If this problem is to be solved by the finite element Galerkin method (certainly not recommended in this case), then it is necessary to recognize that the final solution must be continuous and posses continuous first derivatives in the domain. These additional requirements are established either from a knowledge of the physics of the problem or from a knowledge of the character of solutions for the given differential equation. Therefore, at the i^{th} internal node, with location denoted by x_i, it will be necessary to require

$$\phi^{i+} - \phi^{i-} = 0 \tag{18}$$

$$\left.\frac{d\phi}{dx}\right|^{i+} - \left.\frac{d\phi}{dx}\right|^{i-} = 0 \tag{19}$$

where

$$\phi^{i+} = \phi(x_i^+) \quad , \quad \phi^{i-} = \phi(x_i^-) \tag{20}$$

and

$$x_i^+ = x_i + \epsilon \;, \quad x_i^- = x_i - \epsilon \;, \qquad 0 < \epsilon << 1 \tag{21}$$

After dividing the domain into N subdomains (or finite elements), the next step in this example solution is to assume ϕ in the form

$$\phi(x) = \sum_{r=1}^{N} {}_r\phi(x) \tag{22}$$

where

$${}_r\phi(x) = \left\{ \begin{array}{ll} \sum_i {}_ra_i \; {}_r\phi_i(x) \; , & \text{if } x \text{ is in the } r^{th} \text{ finite element} \\ \\ 0 \; , & \text{if } x \text{ is not in the } r^{th} \text{ finite element} \end{array} \right\} \tag{23}$$

In Equation (22), the summation over r represents a summation over the N finite elements which make up the domain. In Equation (23), the ${}_r\phi_i$ are completely specified functions of x, and the ${}_ra_i$ represents the as-yet unknown generalized coordinates which are to be determined by the solution technique.

It will be assumed in what follows that by proper choice of the functions ${}_r\phi_i$ and by proper identification of the ${}_ra_i$ it is possible to identically satisfy Equations (16) and (18). Then, according to Reference [14], the complete finite element Galerkin method equation will be

$$\sum_{r=1}^{N} \int_{{}_rL} \Big[\sum_i {}_ra_i \frac{d^2{}_r\phi_i}{dx^2}\Big]\Big[\sum_j \delta_r a_j\, {}_r\phi_j\Big] dx$$

$$- \Big\{\Big[\sum_i {}_Na_i \frac{d_N\phi_i}{dx}\Big|_{x=L}\Big] - q_L\Big\}\Big\{\sum_j \delta_N a_j\, {}_N\phi_j(L)\Big\} \qquad (24)$$

$$+ \sum_{n=1}^{N-1}\Big[\sum_i {}_pa_i \frac{d_p\phi_i}{dx}\Big|_{x_n^+} - \sum_i {}_ma_i \frac{d_m a_i}{dx}\Big|_{x_n^-}\Big]\Big[\sum_j \delta_p a_j\, {}_p\phi_j(x_n^+) = \sum_j \delta_m a_j\, {}_m\phi_j(x_n^-)\Big] = 0$$

The first term represents a summation over all the finite elements, with the r^{th} element having length ${}_rL$. The $\delta_r a_i$ are variations of those generalized coordinates which are unspecified. The third term represents a summation over all interior nodes; the left subscript p represents the "plus" side of the node, and the left subscript m represents the "minus" side of the node. It is also noted that

$$\sum_j \delta_p a_j\, {}_p\phi_j(x_n^+) = \sum_j \delta_m a_j\, {}_m\phi_j(x_n^-) \qquad (25)$$

since, by assumption, Equation (18) is satisfied at each interior node.

If desired, the solution could proceed directly from Equation (24) by requiring that the equation be satisfied for arbitrary values of the $\delta_r a_i$. However, for this particular problem, the equation can be put in a more simple form by integrating the first term by parts. The results will be

$$\sum_{r=1}^{N} \int_{{}_rL} \Big[\sum_i {}_ra_i \frac{d_r\phi_i}{dx}\Big]\Big[\sum_j \delta_r a_j \frac{d_r\phi_j}{dx}\Big] dx - q_L\Big[\sum_j \delta_N a_j\, {}_N\phi_j(L)\Big] = 0 \qquad (26)$$

Equation (26) is immediately recognized as the true first variation of the familiar functional

$$V = \frac{1}{2} \int_L \left(\frac{d\phi}{dx}\right)^2 dx - q_L \, \phi(L) \tag{27}$$

when ϕ is specified by Equation (22) and Equations (16) and (18) are satisfied. Since a true variational principle does exist for this problem, there is really no reason to introduce the Galerkin method formulation. The point is, however, that for problems with no true variational principle, the Galerkin procedure (or, more generally, the method of weighted residuals as presented in reference [14]) still remains rigorously valid.

The principle idea which has been illustrated with this example is the need for explicitly including interior boundary continuity equations in the Galerkin formulation. In other words, it is not enough to develop a solution which approximately satisfies only the differential equation in each finite element and the exterior boundary conditions. Also certain conditions must be satisfied across the boundaries between adjacent elements, and these conditions should be included in the approximation formula.

Suppose that the ${}_r\phi_i$ functions were chosen as linear functions of x, which is exactly the type of approximation utilized for the more complicated examples presented later in this paper. Then the differential equation is satisfied identically in each element; and if Equation (24) is written without the interior node terms, there is no way to establish the values of the ${}_ra_i$. However, the correct form of the Galerkin method equation, as given in Equation (24) or equivalently in Equation (26), does provide a set of equations which can be used to evaluate ${}_ra_i$.

SECTION IV

BASIC ANALYSIS

For each of the differential equations presented in Equations (1) through (3), the two dimensional domain is divided into triangular subdomains. For each subdomain the origin of the local axes which defines the local coordinates of the element nodes is placed at the centroid of the element and the principle axes are inclined in the direction of local anisotropy. Considering the triangular element shown in Figure 1, the variation of the dependent variable Ø in the r^{th} element is approximated by a linear polynimial of the form,

$$ {}_r\phi = \sum_i {}_rN_i \, \phi_i \qquad (28) $$

where

$$ {}_rN_i = {}_ra_i + {}_rb_i \, {}_rx + {}_rc_i \, {}_ry \qquad (29) $$

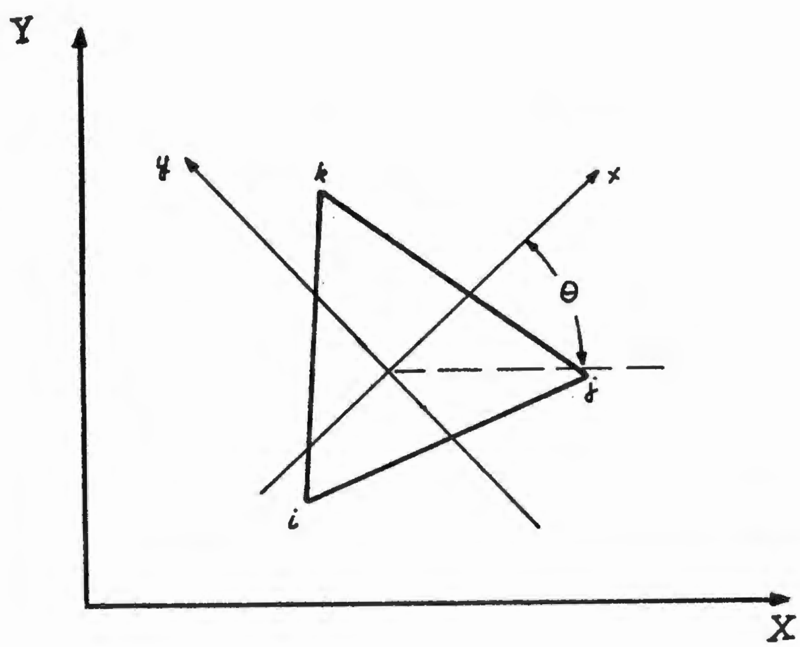

Figure 1. A Triangular Element

and

$$ {}_r a_i = \left({}_r x_j \, {}_r y_k - {}_r x_k \, {}_r y_j \right) \frac{1}{2 \, {}_r A} \tag{30} $$

$$ {}_r b_i = \left({}_r y_j - {}_r y_k \right) \frac{1}{2 \, {}_r A} \tag{31} $$

$$ {}_r c_i = \left({}_r x_k - {}_r x_j \right) \frac{1}{2 \, {}_r A} \tag{32} $$

where ${}_r A$ is the area of the r^{th} triangular element (see reference [15] for details). Similarly a linear variation over each subdomain is assumed for the weighting functions, with a modification of the weighting function for Equation (3), Aral [16]. Then the method of weighted residuals is applied to the subdivided domain, paying particular attention to the continuity requirements which must be satisfied across the boundaries between the elements. The result is a set of algebraic equations in the case of Equation (1), or a set of ordinary differential equations in time, in the case of Equations (2) and (3), which are then solved for the system response.

First consider Equation (1). The complete problem is assumed as follows:

$$ \frac{\partial}{\partial x}\left(K_x \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial \phi}{\partial y} \right) = 0 \,, \quad \text{in the domain } A \tag{33} $$

$$ \phi = \bar{\phi}(s) \,, \quad \text{on the boundary denoted by } S_D \tag{34} $$

$$ K_n \frac{d\phi}{dn} = \bar{q}(s) \,, \quad \text{on the boundary denoted by } S_N \tag{35} $$

Also, across any internal boundaries, there must be continuity in values of ϕ and values of $K_n \, d\phi/dn$. In Equation (34), $\bar{\phi}(s)$ is a specified function of arc length parameter s on boundaries with a Dirichlet type boundary condition. In Equation (35), n denotes the outer normal direction and $\bar{q}(s)$ is a specified function on boundaries with a Neumann type boundary condition. Across interior boundaries, n denotes an arbitrarily directed normal direction; by definition this normal direction is an outward normal for the negative side of the inner boundary and an inward normal for the positive side of the boundary.

The choice of the ${}_r\phi$ as given by Equations (28) through (32) makes it possible to easily provide continuity in values of ϕ across internal boundaries. Furthermore, it will be assumed that on S_D the specified $\bar{\bar{\phi}}(s)$ is such that Equation (34) can be identically satisfied. Therefore, the complete finite element Galerkin method equation will be

$$\sum_r \iint_{{}_rA} \left\{ \left[\frac{\partial}{\partial x}\left({}_rK_x \frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left({}_rK_y \frac{\partial}{\partial y}\right)\right]\left[\sum_\ell {}_rN_\ell \phi_\ell\right]\right\}\left\{\sum_m {}_rN_m \delta\phi_m\right\} d_rA$$

$$-\sum_b \int_{{}_bS_N} \left[{}_bK_n \frac{d}{dn}\left(\sum_\ell {}_bN_\ell \phi_\ell\right) - \bar{q}\right]\left[\sum_m {}_bN_m \delta\phi_m\right] d_bs \tag{36}$$

$$+\sum_{m+p} \int_{S_{int}} \left[{}_pK_n \frac{d}{dn}\left(\sum_\ell {}_pN_\ell \phi_\ell\right) - {}_mK_n \frac{d}{dn}\left(\sum_\ell {}_mN_\ell \phi_\ell\right)\right]\left[\sum_i {}_pN_i \delta\phi_i = \sum_i {}_mN_i \delta\phi_i\right] ds = 0$$

The first term represents a summation over all finite elements. The second term represents a summation over all those finite elements which are adjacent to the S_N boundary. The last term is a summation over all interior boundaries, with the left subscript p denoting the plus side and the left subscript m denoting the minus side of the boundary.

Green's theorem can be applied to the first term in Equation (36), resulting in the following completely equivalent form of the Galerkin equation:

$$\sum_r \iint_{{}_rA} \left\{ \left[{}_rK_x \sum_\ell \frac{\partial_r N_\ell}{\partial x} \phi_\ell\right]\left[\sum_m \frac{\partial_r N_m}{\partial x} \delta\phi_m\right] + \left[{}_rK_y \sum_\ell \frac{\partial_r N_\ell}{\partial y} \phi_\ell\right]\left[\sum_m \frac{\partial_r N_m}{\partial y} \delta\phi_m\right]\right\} d_rA \tag{37}$$

$$-\sum_b \int_{{}_bS_N} \bar{q}\left[\sum_m {}_bN_m \delta\phi_m\right] d_bs = 0$$

When going from Equation (36) to Equation (37), it is necessary to recall that Green's theorem involves outward normal directions, while the normals on interior boundaries are arbitrarily inward or outward depending on whether the region is positive or negative with respect to that inner boundary. Also it has been necessary to note that

$$K_n \frac{d\phi}{dn} = K_x \frac{\partial \phi}{\partial x} n_x + K_y \frac{\partial \phi}{\partial y} n_y \tag{38}$$

where n_x and n_y are the components of a unit vector in the direction specified by a vector $\underline{n}$.

Equation (37) is exactly what would result from a finite element evaluation of the true variational principle for this problem. This furnishes one more demonstration of the equivalence between the complete Galerkin equation, Equation (36), and a true variational principle.

Requiring that Equation (37) be satisfied for arbitrary $\delta\phi_m$ leads to the usual matrix formulation of the form

$$[S]\,\{\phi\} = \{F\} \tag{39}$$

where (S) is the global matrix of coefficients (analogous to the stiffness matrix) which incorporates the properties of the materials in the domain and the geometry, $\{\phi\}$ is the vector of unknown ϕ's at the nodes, and $\{F\}$ is the global "load" vector. The details of the development of Equation (39) and possible solution techniques are conventional and will not be discussed in this paper.

The same type of formulation can be carried out for the parabolic partial differential equation described by Equation (2), a problem for which no true variational principle exists. A complete statement of the problem is assumed to be

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial \phi}{\partial y}\right) - K_t \frac{\partial \phi}{\partial t} = 0 \;, \quad \text{in } A \;, \quad t > 0 \tag{40}$$

$$\phi = \bar{\phi}(s,t) \;, \quad \text{on } S_D \;, \quad t > 0 \tag{41}$$

$$K_n \frac{d\phi}{dn} = \bar{q}(s,t) \;, \quad \text{on } S_N \;, \quad t > 0 \tag{42}$$

$$\phi = \phi_0(x,y) \;, \quad \text{in } A \;, \quad t = 0 \tag{43}$$

with no great concern regarding continuity of boundary and initial data. The interior continuity requirements associated with Equations (33) through (35) are applicable again. In addition, $\emptyset$ should be a continuous function of time t.

Since the solution $\emptyset$ is a function of both space and time the approximate solution within the r^{th} element is now taken in the form

$$ {}_r\phi(x,y,t) = \sum_i {}_rN_i(x,y)\,\phi_i(t) \tag{44} $$

It will be assumed that Equations (41) and (43) and internal continuity of $\emptyset$ in both space and time are identically satisfied. Then the finite element Galerkin equation will be

$$ \begin{aligned} \int_0^t \Bigg\langle & \sum_r \iint_{{}_rA} \left\{\left[\frac{\partial}{\partial x}\left({}_rK_x\frac{\partial}{\partial x}\right)+\frac{\partial}{\partial y}\left({}_rK_y\frac{\partial}{\partial y}\right)-{}_rK_t\frac{\partial}{\partial t}\right]\left[\sum_\ell {}_rN_\ell\,\phi_\ell\right]\right\}\left\{\sum_m {}_rN_m\,\delta\phi_m\right\} d_rA \\ & - \sum_b \int_{{}_bS_N}\left[{}_bK_n\frac{d}{dn}\left(\sum_\ell {}_bN_\ell\,\phi_\ell\right)-\bar{q}\right]\left[\sum_m {}_bN_m\,\delta\phi_m\right] d_bs \\ & + \sum_{m-p}\int_{S_{int}}\left[{}_pK_n\frac{d}{dn}\left(\sum_\ell {}_pN_\ell\,\phi_\ell\right)-{}_mK_n\frac{d}{dn}\left(\sum_\ell {}_mN_\ell\,\phi_\ell\right)\right] \\ & \qquad\qquad \left[\sum_i {}_pN_i\,\delta\phi_i = \sum_i {}_mN_i\,\delta\phi_i\right] ds \Bigg\rangle\, dt = 0 \end{aligned} \tag{45} $$

After application of Green's theorem, the Galerkin equation will be

$$ \begin{aligned} \int_0^t \Bigg\langle & \sum_r \iint_{{}_rA}\Bigg\{\left[{}_rK_x\sum_\ell \frac{\partial_rN_\ell}{\partial x}\phi_\ell\right]\left[\sum_m \frac{\partial_rN_m}{\partial x}\,\delta\phi_m\right] \\ & + \left[{}_rK_y\sum_\ell \frac{\partial_rN_\ell}{\partial y}\phi_\ell\right]\left[\sum_m \frac{\partial_rN_m}{\partial y}\,\delta\phi_m\right]\Bigg\} d_rA \\ & + \sum_r \iint_{{}_rA}\left[{}_rK_t\sum_\ell {}_rN_\ell\frac{\partial\phi_\ell}{\partial t}\right]\left[\sum_m {}_rN_m\,\delta\phi_m\right] d_rA \\ & - \sum_b \int_{{}_bS_N}\bar{q}\left[\sum_m {}_bN_m\,\delta\phi_m\right] d_bs \Bigg\rangle\, dt = 0 \end{aligned} \tag{46} $$

Requiring that Equation (46) be satisfied for arbitrary $\delta\phi_m$ leads to a system of differential equations which can be written in the matrix form

$$[S]\{\phi\} + [P]\left\{\frac{\partial\phi}{\partial t}\right\} = \{F\} \tag{47}$$

To proceed with the transient part of the solution, it will be assumed that $\partial\phi/\partial t$ associated with each degree of freedom of discrete system varies linearly within a time increment (Δt), as first suggested by Clough and Wilson [17]. Thus, from a direct integration over the time interval, Δt for all nodal points, the following equation for ϕ at the end of a time interval can be obtained.

$$\{\phi\}_t = \{\phi\}_{t-\Delta t} + \left(\left\{\frac{\partial\phi}{\partial t}\right\}_{t-\Delta t} + \left\{\frac{\partial\phi}{\partial t}\right\}_t\right)\frac{\Delta t}{2} \tag{48}$$

Thus, if the initial values of ϕ are known, Equations (47) and (48) can be solved simultaneously to obtain the values of ϕ at the time ($t + \Delta t$).

This simultaneous time and space solution process can be formulated neatly for computer applications, thus decreasing the algebra, computer storage and time. At time t substituting Equation (48) into Equation (47), one can write

$$\left([P]\frac{2}{\Delta t} + [S]\right)\{\phi\}_t = [P]\left(\left\{\frac{\partial\phi}{\partial t}\right\}_{t-\Delta t} + \frac{2}{\Delta t}\{\phi\}_{t-\Delta t}\right) + \{F\} \tag{49}$$

Again substituting Equation (47) into Equation (49), this time at time ($t - \Delta t$), one can write

$$\left([P]\frac{2}{\Delta t} + [S]\right)\{\phi\}_t = \left([P]\frac{2}{\Delta t} - [S]\right)\{\phi\}_{t-\Delta t} + 2\{F\} \tag{50}$$

If one defines

$$[S]^* = [P]\frac{2}{\Delta t} + [S] \tag{51}$$

and

$$\{F\}^* = \{F\} + \frac{2}{\Delta t}[P]\{\phi\}_{t-\Delta t} \qquad (52)$$

then from Equation (53) one can solve for the unknowns $\{\emptyset\}^*$

$$[S]^*\{\phi\}^* = \{F\}^* \qquad (53)$$

Once the $\{\emptyset\}^*$'s are determined, the problem reduces to solving Equation (54) for the values of $\emptyset_t$ which are the nodal values of the function sought at time t.

$$\{\phi\}_t = 2\{\phi\}^* - \{\phi\}_{t-\Delta t} \qquad (54)$$

Incrementing by Δt and repeating the same process, continuous solutions can be obtained in time and space coordinates for unsteady problems.

The solution process for the convective diffusion problem, Equation (3), is slightly modified based on a true variational principle for a simplified problem. For the case of unsteady diffusion with uniform flow in a homogeneous medium, reference [16] shows that there exists a true variational principle with the functional

$$I = \iint_A \left[\frac{1}{2}K_x\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{2}K_y\left(\frac{\partial\phi}{\partial y}\right)^2\right]\exp\left(-\frac{ux}{K_x} - \frac{vy}{K_y}\right)dA \qquad (55)$$

The physical interpretation for the magnitude of I is not clear, but the fact remains that $\delta I = 0$ does supply the governing equations for a system with specified $\emptyset$ on the boundary. It should be noted that in the term $\exp(-ux/K_x - vy/K_y)$, the x and y values are evaluated in a global or overall coordinate system.

For the case of transient diffusion with uniform flow in a homogeneous medium, the differential equation will be

$$K_x\frac{\partial^2\phi}{\partial x^2} + K_y\frac{\partial^2\phi}{\partial y^2} - u\frac{\partial\phi}{\partial x} - v\frac{\partial\phi}{\partial y} - \frac{\partial\phi}{\partial t} = 0 \quad, in\ A\ ,\ t>0 \qquad (56)$$

with boundary conditions given by Equations (41) through (43) and continuity requirements in space and time as specified for the previous example. Once again it will be assumed that Equations (41) through (43) and internal continuity of $\emptyset$ in both space and time are identically satisfied. Now the finite element Galerkin type equation will be taken as follows

$$
\begin{aligned}
&\int_0^t \Big\langle \sum_r \iint_{{}_rA} \Big\{ \Big(K_x \frac{\partial^2}{\partial x^2} + K_y \frac{\partial^2}{\partial y^2} - u\frac{\partial}{\partial x} - v\frac{\partial}{\partial y} - \frac{\partial}{\partial t} \Big) \Big(\sum_\ell {}_rN_\ell \phi_\ell \Big) \Big\} \\
&\quad \otimes \Big\{ \sum_m {}_rN_m \delta\phi_m \exp\Big[-\frac{u}{K_x}({}_rx + {}_ra) - \frac{v}{K_y}({}_ry + {}_rb) \Big] \Big\} d_rA \\
&\quad - \sum_b \int_{{}_bS_N} \Big\{ K_n \frac{d}{dn} \Big(\sum_\ell {}_bN_\ell \phi_\ell \Big) - \bar{q} \Big\} \\
&\quad \otimes \Big\{ \sum_m {}_bN_m \delta\phi_m \exp\Big[-\frac{u}{K_x}({}_bx + {}_ba) - \frac{v}{K_y}({}_by + {}_bb) \Big] \Big\} d_bs \\
&\quad + \sum_{m-p} \int_{S_{int}} \Big\{ K_n \frac{d}{dn} \Big(\sum_\ell {}_pN_\ell \phi_\ell \Big) - K_n \frac{d}{dn} \Big(\sum_\ell {}_mN_\ell \phi_\ell \Big) \Big\} \\
&\quad \otimes \Big\{ \sum_i {}_pN_i \delta\phi_i \exp\Big[-\frac{u}{K_x}({}_px + {}_pa) - \frac{v}{K_y}({}_py + {}_pb) \Big] \Big\} ds \Big\rangle dt = 0
\end{aligned}
\tag{57}
$$

In Equation (57), the integral over the r^{th} finite element or the b^{th} exterior boundary or a given interior boundary is normally expressed in terms of the appropriate local coordinate systems. Therefore it is necessary to transform the global coordinates to local coordinates by the following equations which account for the translation:

$$
x = {}_rx + {}_ra \quad , \qquad y = {}_ry + {}_rb \tag{58}
$$

Note that there need be no coordinate rotations for this case of homogeneous material.

After application of Green's theorem, the Galerkin-type equation will be

$$\int_0^t \Big\langle \sum_r \iint_A \Big[\Big(K_x \sum_\ell \frac{\partial\, {}_rN_\ell}{\partial x}\, \phi_\ell \Big) \Big(\sum_m \frac{\partial\, {}_rN_m}{\partial x}\, \delta\phi_m \Big)$$

$$+ \Big(K_y \sum_\ell \frac{\partial\, {}_rN_\ell}{\partial y}\, \phi_\ell \Big) \Big(\sum_m \frac{\partial\, {}_rN_m}{\partial y}\, \delta\phi_m \Big) \Big] \exp\Big[-\frac{u}{K_x} ({}_rx + {}_ra) - \frac{v}{K_y} ({}_ry + {}_rb) \Big] d_rA \tag{59}$$

$$+ \sum_r \iint_{{}_rA} \Big(\sum_\ell {}_rN_\ell \frac{\partial \phi_\ell}{\partial t} \Big) \sum_m {}_rN_m\, \delta\phi_m \exp\Big[-\frac{u}{K_x} ({}_rx + {}_ra) - \frac{v}{K_y} ({}_ry + {}_rb) \Big] d_rA$$

$$- \sum_b \bar{q} \Big\{ \sum_m {}_bN_m\, \delta\phi_m \exp\Big[-\frac{u}{K_x} ({}_bx + {}_ba) - \frac{v}{K_y} ({}_by + {}_bb) \Big] \Big\} d_bs \Big\rangle dt$$

Requiring that Equation (58) be satisfied for arbitrary $\delta\phi_m$ leads to system of differential equations which can be written in the form given by Equation (47). If the flow is steady, then the $[S]$ and $[P]$ matrices are constants, and the solution proceeds as described earlier.

The numerical results presented in Example 2 of the next section are based on Equation (59) with the simplifying assumption that ${}_ra = 0$ and ${}_rb = 0$; that is the translation between local and global coordinates has been ignored. This means that the interior boundary continuity requirement for $K_n\, d\phi/dn$ is not satisfied. However, the results agree very well with the known finite difference solution.

SECTION V

NUMERICAL EXAMPLES

In this section, numerical examples to the differential equations treated in detail in earlier sections will be given. These examples are chosen in the field of fluid mechanics. The variety of possibilities for the application of the computer program written for the solution of these problems is obvious [16]. Also obviously, the appropriate employment of the mathematical form is by no means limited to the problem types specifically treated in this study. Indeed, any physical phenomena properly modelled by elliptic and parabolic differential equations, accompanied by appropriate boundary conditions can be investigated using the techniques developed in this study.

Example One: An engineering application of the finite element method developed in this paper can be the study of ground water seepage under a dam as demonstrated earlier in a paper by Zienkiewicz, Mayer and Cheung [3]. If the earth material under this dam is layered and anisotropic, it becomes very difficult for an engineer to estimate the quantity of seepage passing under this dam.

In this problem an arbitrarily conceptualized dam is located on a layered and anisotrpic media. The problem may be the determination of equipotential lines under a constant head. A schematic description of this problem is given in Figure 2.

Within each layer the mathematical model describing this problem is

$$K_x \frac{\partial^2 \phi}{\partial x^2} + K_y \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{60}$$

where K_x and K_y are permeabilities of the medium in the principle local directions, respectively. ϕ in this model represents the equipotential function. As the boundary conditions of the problem, ϕ is assumed to vary from ten on the upstream side of the dam to zero on the downstream side of the dam. Under the dam and at the impervious boundary the normal derivative of ϕ is equal to zero since there is no flux across these boundaries.

Finite element results for this problem can be seen on Figure 3. In this problem, the continuum analyzed was idealized by 297 elements and the computer execution time was 15 seconds.

Example Two: Another application of the finite element method can be the study of dispersion in a porous medium. This phenomenon can be modelled by the convective dispersion equation presented in the earlier section. The two-dimensional form of this equation in cartesian coordinates is

$$D_x \frac{\partial^2 \phi}{\partial x^2} + D_y \frac{\partial^2 \phi}{\partial y^2} - u \frac{\partial \phi}{\partial x} - v \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial t} \tag{61}$$

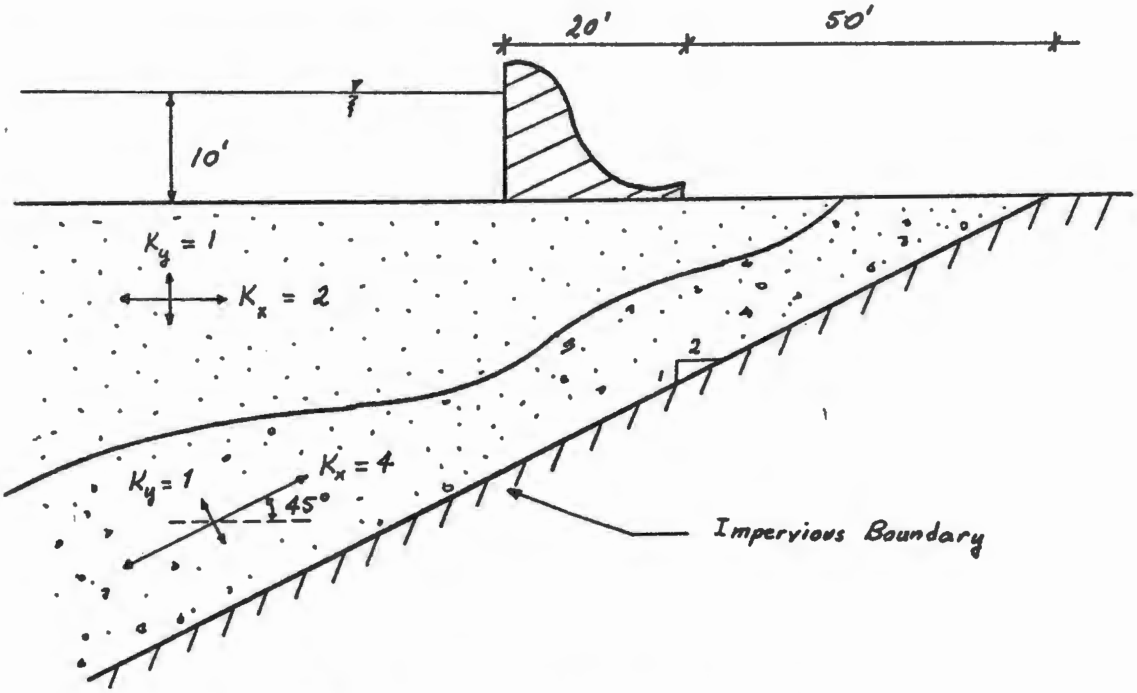

Figure 2. Concrete Dam on Non-Homogeneous and Anisotropic Media

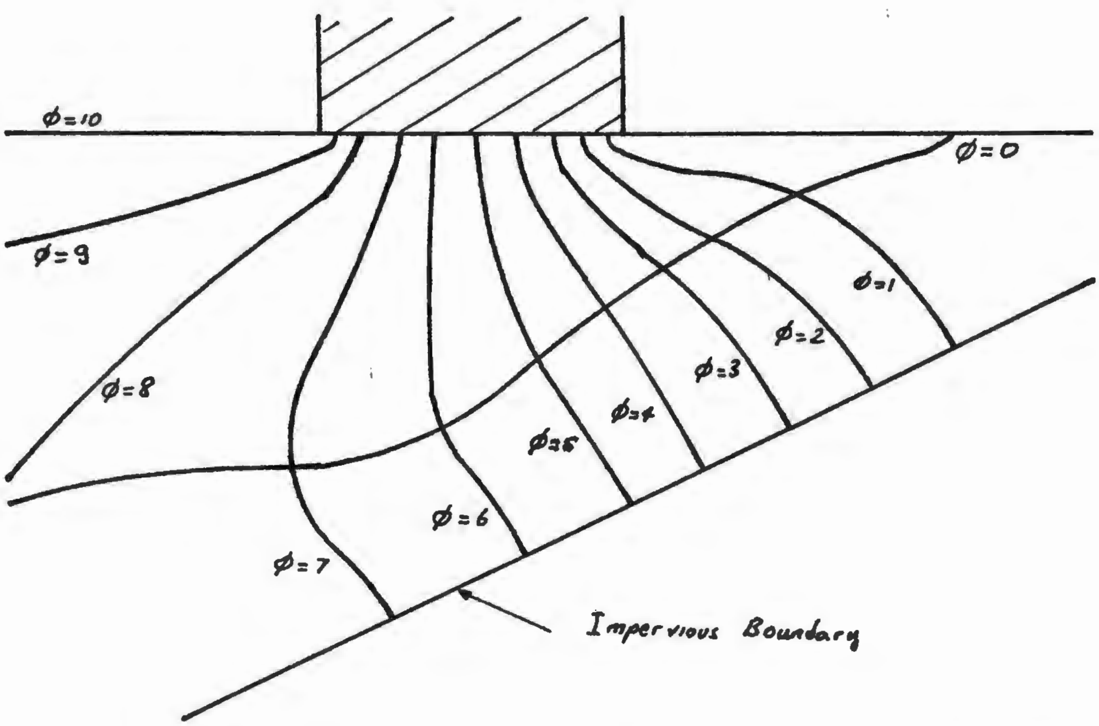

Figure 3. Equipotential Line Configuration Under a Dam on Non-Homogeneous and Anisotropic Media

where ϕ is the concentration of the dispersing mass, D_x and D_y are the dispersion coefficients and u and v are the seepage velocities in (x) and (y) directions respectively. Shamir and Harlemen [18] studied steady and unsteady problems of this type in detail and presented a numerical scheme for solving such problems. In this example, a typical unsteady problem studied in their report was solved by the finite element method.

The problem is to determine the one-dimensional dispersion of a tracer (concentration) introduced to the porous medium at a constant rate at x = 0 cm. There is a constant rate of seepage in the x-direction. The velocity distribution is assumed to be uniform in y-direction. Initially, the distribution of this tracer in the porous medium is zero. A schematic description of this problem is given in Figure 4.

The dimensionless form of the unsteady convective diffusion equation in one-dimensional studies is

$$\frac{\partial^2 \phi}{\partial \xi^2} - \lambda \frac{\partial \phi}{\partial \xi} = \frac{\partial \phi}{\partial \tau} \tag{62}$$

where

$$\phi = \frac{\phi}{\phi_0}, \quad \xi = \frac{x}{L}, \quad \lambda = \frac{uL}{D}, \quad \tau = \frac{Dt}{L^2} \tag{63}$$

where L is the length of the medium, t is the time coordinate and ϕ_o is the amount of concentration at x = 0 which is kept constant throughout the analysis. In applying the numerical solution the following data, which are taken from the above report, are used.

$$0 \leq x \leq 10 \text{ cm}$$

$$u = 0.1 \text{ cm/sec} \tag{64}$$

$$D = 0.01 \text{ cm}^2/\text{sec}$$

Thus the convective diffusion equation to be solved and the transformed boundary conditions take the form

$$\frac{\partial^2\phi}{\partial\xi^2} - 100\frac{\partial\phi}{\partial\xi} = \frac{\partial\phi}{\partial\tau} \tag{65}$$

$$\text{I.C.:} \quad \text{at} \quad \tau=0 \quad \phi=0 \tag{66}$$

$$\text{B.C.1:} \quad \text{at} \quad \xi=0 \quad \phi=1 \tag{67}$$

$$\text{B.C.2:} \quad \text{at} \quad \xi=1 \quad \frac{\partial\phi}{\partial\xi}=0 \tag{68}$$

Numerical results and a comparison with the exact solution [18] are given in Figure 5.

In this example, 200 elements were used in idealizing the continuum given in Figure 4. The computer execution time was 0.16 seconds per time step. Although the finite element results are very accurate for this particular example, extreme care should be given to the coefficients appearing in the exponential terms in the solution of this type of problems. Very large numbers appearing in the exponent create truncation and roundoff errors in computer computations which may be significant and may result in unstable solutions.

This example was actually a laboratory model of flow of a certain concentration in a confined aquifer. A practical application of this example could be the study of the flow of some concentration in a confined aquifer between two rivers, of course, the appropriate constants of the problem may have to be changed for a specific application.

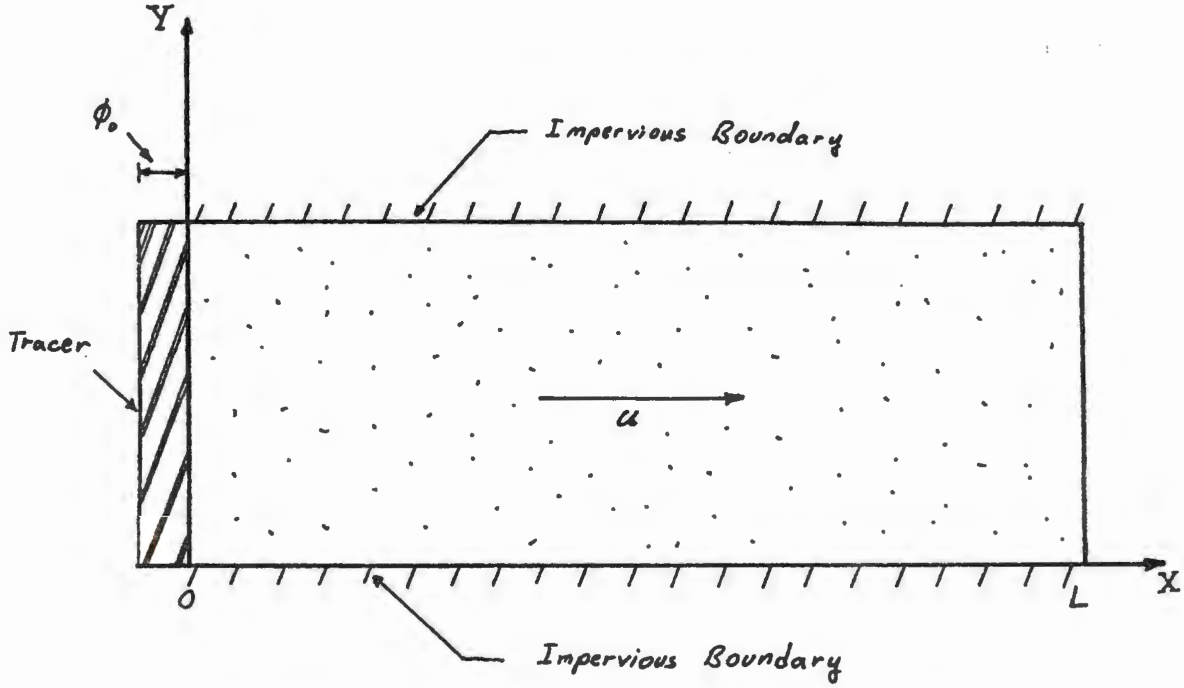

Figure 4. Longitudunal Dispersion Model

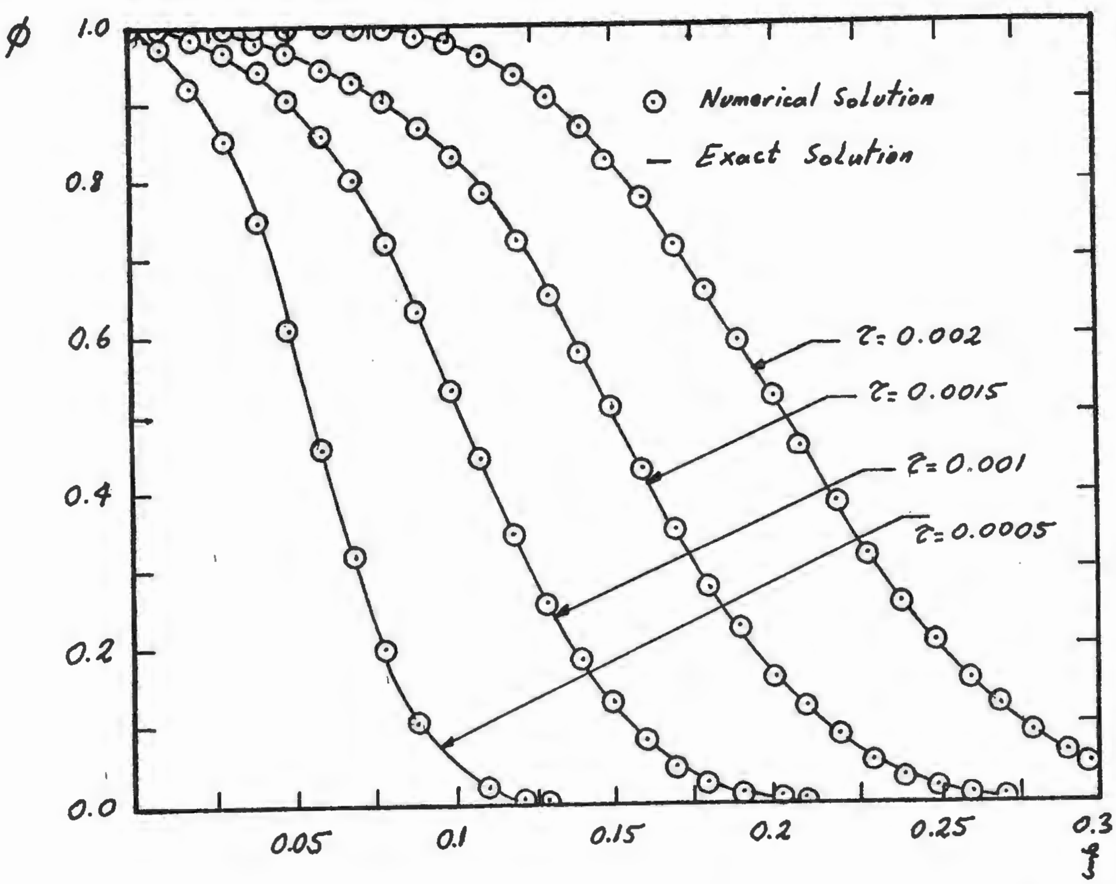

Figure 5. Convective Diffusion in Porous Media

SECTION VI

CONCLUDING REMARKS

In this paper a technique has been presented for the formulation of approximate solution equations for systems with no true variational principle. While the results are not necessarily any more accurate than earlier presented solutions, the formulations possess a rigor which is lacking in those solutions based on quasi or restricted variational principles. It is now clear that the finite element method can be used to provide solutions to many significant problems in fluid mechanics, heat transfer, and other nonstructural areas.

SECTION VII

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