DESIGN OF OPTIMUM STRUCTURES FOR DYNAMIC LOADS

V. B. Venkayya**
N. S. Khot*
V. A. Tischler**
R. F. Taylor***

Air Force Flight Dynamics Laboratory
Wright-Patterson Air Force Base, Ohio 45433

An optimization method for the minimum weight design of structures subjected to dynamic loads is presented. The method is called designing in the dynamic mode. The dynamic mode may be a single natural mode of the structure or a linear combination of a set of natural modes depending on the spatial distribution and dynamic characteristics of the forcing function. An optimality criterion for minimum weight structures in the dynamic mode is derived. It is based on the strain energy and kinetic energy of the individual elements. A recursion relation for attaining the optimality criterion is also derived. The natural frequencies are obtained by using the Sturm Sequence property in conjunction with a bisection procedure. The normal modes are determined by inverse iteration. A step by step procedure for the optimum design under dynamic response constraints is presented. It includes a method for determining the modes that predominate in the dynamic response. Some examples of design are presented to illustrate the method.

1. INTRODUCTION

Optimization and the automated design of structures is the subject of intense study in recent years. The result of these studies is the emergence of a number of successful programs for the design of practical structures. In 1968, a combined approach based on an optimality criterion and a numerical search was presented for the optimal design of structures subjected to static loads. (Reference 1). This method demonstrated the feasibility of optimizing structures of several hundred degrees of freedom and design variables. Design conditions included multiple loading conditions and constraints on stresses, displacements and sizes of the elements. A more detailed study of the convergence characteristics of this method and an extension to frame and plate elements were presented in Reference 2. This reference also contained a sparse matrix scheme for the solution and response gradient calculations. Subsequently an alternate method for static load, also based on an optimality criterion, was developed by Gellatly and Berke in Reference 3. A number of bar and plate structures were successfully designed by this approach. Dwyer, Emerton and Ojalvo developed a modified stress ratio method coupled with a numerical search procedure for the automated design of realistic airframe components in Reference 4. Their effort included the development of a large scale computer program for the optimum design of structures subjected to static loads.

* Aerospace Engineers, Structural Synthesis Group
** Mathematician, Structural Synthesis Group
*** Aerospace Engineer, Aeroelasticity Group

The basis for the foregoing developments was the vast body of literature available in the area of both static and dynamic structural optimization. The excellent papers by Maxwell were the earliest to consider the problems of structural optimization (Reference 5). He discussed the fully stressed design of statically determinate and indeterminate bar structures. Later, several investigators examined the limitations of a fully stressed design when applied to different configurations and design conditions. The celebrated paper by Michell dealt with the development of an optimum structural configuration for a given loading (Reference 6). This configuration consisted of an orthogonal network of tension and compression members. Later Hemp (Reference 7), Cox (Reference 8) and Chan (Reference 9) applied the concept of Michell structures to many problems of structural optimization. The more recent papers by Prager and his associates dealt extensively with the derivation of optimality criteria for a variety of design conditions (References 10, 11 and 12). These papers firmly established a place for the optimality criteria approaches in the design of optimum structures. In the same vein, Turner (Reference 13), Taylor (Reference 14), Masur (Reference 15), McIntosh and Eastep (Reference 16) and others investigated specific problems of structural optimization under dynamic and stability constraints. Barnett and Hermen formulated an interesting optimality criterion based on virtual work for displacement contrained problems (Reference 17). Melosh presented an extensive discussion of the validity of the fully stressed design for the design of indeterminate structures (Reference 18). Young and Christiansen used the fully stressed design criterion for designing in the lowest normal mode (Reference 19).

A parallel but a very significant development in the area of structural optimization was initiated by Schmit and Klein (References 20 and 21). Their formulation involved the application of numerical search methods to structural optimization problems. This approach provided a better understanding of structural optimization under different design conditions. Gellatly and Gallagher (Reference 22) and Gellatly (Reference 23) have further developed these ideas and applied them to the design of practical aerospace structures. Fox and Kapoor (Reference 24), Rubin (Reference 25), McCart, Haug and Streeter (Reference 26) have successfully applied search procedures to the design of structures under dynamic load conditions. The recent dissertations of Salinas (Reference 27), Loomis (Reference 28) and Arora (Reference 29) are excellent contributions to the further understanding of structural optimization.

The list of references cited in the foregoing discussion is at best incomplete. A more detailed survey of contributions to structural optimization can be found in papers by Wasiutynski and Brandt (Reference 30), Sheu and Prager (Reference 31), and Pope and Schmit (Reference 32).

The purpose of this paper is to present an efficient method based on an optimality criterion and a numerical search for the design of structures subjected to dynamic loads. In this paper the effect of damping is not considered, but the authors hope to discuss this problem at a later date.

2. ENERGIES OF MOTION AND EQUATIONS OF DYNAMIC ANALYSIS

The interest of the present paper is primarily in built-up structures and the continuum is approximated by a discretized finite element model. The selection of the displacement method of finite element analysis is prompted by the intent to automate the analysis block. However, the basic principles derived in this paper are generally valid for a continuum as well as other discretized models. In an iterative approach a large part of the effort is expended in the repeated analysis of the structure. The basic equations of analysis are presented here as a ready reference for the development that follows. For details the reader is referred to References 33 and 34.

In a finite element scheme the force displacement relations of the individual elements are derived by an energy formulation with the assumption of exact or approximate displacement functions. The energy expressions for the i^{th} element in motion may be written as

$$u_i = \tfrac{1}{2}\int_{V_i} \boldsymbol{\sigma}_i^t \, \mathbf{e}_i \, dV_i \tag{1}$$

$$\tau_i = \tfrac{1}{2}\int_{V_i} \gamma_i \, \dot{\mathbf{w}}_i^t \, \dot{\mathbf{w}}_i \, dV_i \tag{2}$$

where u_i is the strain energy and τ_i the kinetic energy of the element, $\boldsymbol{\sigma}_i$ and $\mathbf{e}_i$ are the stress and strain matrices, respectively, $\mathbf{w}_i$ is the displacement vector, and γ_i is the mass density. The dot $(\dot{\ })$ represents a time derivative throughout this development.

The displacement vector may be represented by a set of discrete generalized coordinates of the element in the following form,

$$\mathbf{w}_i = \boldsymbol{\phi}_i \, \mathbf{v}_i \tag{3}$$

where $\mathbf{v}_i$ is the vector of generalized coordinates for the element and $\boldsymbol{\phi}_i$ is a rectangular matrix whose elements are functions of the spatial coordinates. The strain in the i^{th} element may be written as

$$\mathbf{e}_i = \boldsymbol{\phi}'_i \, \mathbf{v}_i \tag{4}$$

The prime in the above equation indicates a derivative with respect to spatial coordinates. For an element made of a linearly elastic material, stresses and strains are related by

$$\boldsymbol{\sigma}_i = \mathbf{G}_i \, \mathbf{e}_i \tag{5}$$

Substituting Equations 3, 4 and 5 in 1 and 2, the expressions for the strain and kinetic energies of the element are written as

$$u_i = \tfrac{1}{2}\int_{V_i} \mathbf{v}_i^t \, \boldsymbol{\phi}'^t \, \mathbf{G}_i \, \boldsymbol{\phi}'_i \, \mathbf{v}_i \, dV_i \tag{6}$$

$$\tau_i = \tfrac{1}{2}\int_{V_i} \gamma_i \, \dot{\mathbf{v}}_i^t \, \boldsymbol{\phi}_i^t \, \boldsymbol{\phi}_i \, \dot{\mathbf{v}}_i \, dV_i \tag{7}$$

Since the generalized coordinates vector $\mathbf{v}_i$ is independent of the spatial coordinates, it can be taken out of the integration and the energy expressions may be written as

$$u_i = \tfrac{1}{2}\mathbf{v}_i^t \, \mathbf{k}_i \, \mathbf{v}_i \tag{8}$$

$$\tau_i = \tfrac{1}{2}\mathbf{v}_i^t \, \mathbf{m}_i \, \mathbf{v}_i \tag{9}$$

Where $\mathbf{k}_i$ and $\mathbf{m}_i$ are the element generalized stiffness and mass matrices respectively, and they are given by

$$\mathbf{k}_i = \int_{V_i} \boldsymbol{\phi}'^t_i \, \mathbf{G}_i \, \boldsymbol{\phi}'_i \, dV_i \tag{10}$$

$$\mathbf{m}_i = \int_{V_i} \gamma_i \, \boldsymbol{\phi}^t_i \, \boldsymbol{\phi}_i \, dV_i \tag{11}$$

The expressions for the total energies of the structure are obtained by summing the component energies,

$$U = \tfrac{1}{2}\sum_{i=1}^{m} \mathbf{v}_i^t \, \mathbf{k}_i \, \mathbf{v}_i \tag{12}$$

$$T = \tfrac{1}{2}\sum_{i=1}^{m} \mathbf{v}_i^t \, \mathbf{m}_i \, \mathbf{v}_i \tag{13}$$

where the scalar m represents the total number of elements in the structure.

The generalized coordinates of the elements and the structure are related by

$$\mathbf{v}_i = \mathbf{a}_i \, \mathbf{r} \tag{14}$$

where $\mathbf{r}$ is the vector of system generalized coordinates, and $\mathbf{a}_i$ is the compatibility matrix. The invariance of the energy with a coordinate transformation permits writing the energy expressions in the following form,

$$U = \tfrac{1}{2}\mathbf{r}^t \, \mathbf{K} \, \mathbf{r} \tag{15}$$

$$T = \tfrac{1}{2}\dot{\mathbf{r}}^t \, \mathbf{M} \, \dot{\mathbf{r}} \tag{16}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the generalized stiffness and mass matrices respectively of the system and are given by

$$\mathbf{K} = \sum_{i=1}^{m} \mathbf{a}_i^t \, \mathbf{k}_i \, \mathbf{a}_i \tag{17}$$

$$\mathbf{M} = \sum_{i=1}^{m} \mathbf{a}_i^t \, \mathbf{m}_i \, \mathbf{a}_i \tag{18}$$

The j^{th} Lagrange's equation of the system is given by

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{r}_j} - \frac{\partial T}{\partial r_j} + \frac{\partial U}{\partial r_j} = R_j \tag{19}$$

Substituting Equations 15 and 16 into 19 gives the equations of motion of the system in the following form

$$\mathbf{M}\ddot{\mathbf{r}} + \mathbf{K}\mathbf{r} = \mathbf{R} \tag{20}$$

where **R** is the generalized force matrix which includes the externally applied dynamic and static forces. These forces may be concentrated or distributed. In the case of distributed and concentrated forces applied at points and directions other than those of the generalized coordinates, **R** can be derived by the principle of virtual work. **R** can also include damping and other internal elastic forces.

In deriving Equation 20 only structural mass is considered. However, any non-structural mass such as fuel tanks in an airplane wing can be included in the generalized mass matrix by superposition. The magnitude of the nonstructural mass is added to the diagonal elements of **M** corresponding to their points of application. If they are lumped masses without specific geometric dimensions, only the translational degrees of freedom will be effected. When these masses are not located at the points of the generalized coordinates, then the equivalent effect on the generalized mass can be derived by D'Alembert's principle and the principle of virtual work

$$\mathbf{M} = \mathbf{M}_{\text{structural}} + \mathbf{M}_{\text{nonstructural}} \tag{21}$$

When the system is at rest and the rate of loading is sufficiently slow, Equation 20 reduces to the familiar force displacement relation of the static case

$$\mathbf{R} = \mathbf{K}\mathbf{r} \tag{22}$$

In such a case the internal forces and stresses of the elements can be determined by

$$s_i = \mathbf{k}_i \, \mathbf{v}_i \tag{23}$$

$$\boldsymbol{\sigma}_i = \boldsymbol{\beta}_i \, \mathbf{s}_i \tag{24}$$

In the dynamic case the solution of Equation 20 consists of a homogeneous part and a particular solution. The homogeneous part corresponds to free vibration and is obtained by solving

$$\omega^2 \mathbf{M}\mathbf{r} = \mathbf{K}\mathbf{r} \tag{25}$$

where ω represents the frequency of vibration.

Equation 25 represents a standard eigenvalue problem and its solution amounts to obtaining eigenvalues, ω_κ (κ=1,2,...,n) and their corresponding eigenvectors. The matrices **M** and **K** possess symmetry and bandedness properties. The details of a method for obtaining the eigenvalues and eigenvectors of a banded symmetric matrix are presented in Section 5.

There are several approximate methods in the literature for obtaining the particular solution of the matrix Equation 20 . One of these is the normal mode method which is discussed in Section 7.

3. ENERGY CRITERION FOR AN OPTIMUM DESIGN IN FREE VIBRATION

The response of a structure is largely governed by its dynamic characteristics and the nature of the forcing function. The natural frequencies and modes of free vibration are the primary dynamic characteristics of the structure. A design method should have the ability to manipulate the dynamic characteristics to obtain an optimum response for the given forcing function. An optimality criterion for a minimum weight structure in free vibration is derived here in terms of discrete variables. This derivation is similar to the optimality criterion established in References 1 and 2 for static loading conditions.

The dynamic equation governing the free vibration of the system is given by the integral

$$I = \int_{t_1}^{t_2} (T - U)\, dt = \int_{t_1}^{t_2} L\, dt \tag{26}$$

where L is the Lagrangian. The expressions for the strain energy and kinetic energy of the system, U and T are given by Equations 12 and 13 . From Hamilton's principle the natural mode of vibration is the one for which the integral I attains a stationary value in the time interval t_1 to t_2.

Suppose A and A' are two designs in the neighborhood of the minimum weight design. The weights of the structure corresponding to the two designs are proportional to W and W' which are defined as

$$W = \sum_{i=1}^{m} A_i\, l_i \tag{27}$$

$$W' = \sum_{i=1}^{m} A_i'\, l_i \tag{28}$$

For one dimensional elements A_i and l_i are areas and lengths respectively. For plate elements these parameters are defined in Reference 2.

The difference in the strain energy and kinetic energy of the i^{th} element in free vibration in a finite time interval is given by

$$\mu_i = ½[\mathbf{s}_i^t\, \mathbf{k}_i\, \mathbf{s}_i - \omega^2 \mathbf{s}_i^t\, \mathbf{m}_i\, \mathbf{s}_i] \tag{29}$$

$$\mu_i' = ½[\mathbf{s}_i'^t\, \mathbf{k}_i'\, \mathbf{s}_i' - \omega'^2\, \mathbf{s}_i'^t\, \mathbf{m}_i'\, \mathbf{s}_i'] \tag{30}$$

where the vector $\mathbf{s}_i$ is the displacement of the i^{th} element when the structure is in the natural mode of vibration. Here prime refers to the second design. If ρ_i and ρ_i' are defined as μ_i and μ_i', per unit volume (density), then

$$V = \sum_{i=1}^{m} \mu_i = \sum_{i=1}^{m} A_i \, l_i \, \rho_i \tag{31}$$

$$V' = \sum_{i=1}^{m} \mu'_i = \sum_{i=1}^{m} A'_i \, l_i \, \rho'_i \tag{32}$$

where V and V′ are the difference in the strain energy and the kinetic energy of the total system in the two cases.

Since the two designs are assumed to be in the neighborhood of each other, a limiting condition can be written in the following form

$$\lim_{\Delta A \to 0} \left[\sum_{i=1}^{m} A_i \, l_i \, \rho_i - \sum_{i=1}^{m} A'_i \, l_i \, \rho'_i \right] = 0 \tag{33}$$

Let **S** and **S′** be the respective eigenmodes of the two designs. Since the geometric configuration of the structure is the same in both cases, eigenvector **S** of the first design is kinematically admissible for the second design and vice versa. If the second design is forced to vibrate in the mode **S** with eigenvalue ω, then from Raleigh's principle one can write

$$\sum_{i=1}^{m} A'_i \, l_i \, \rho_i \geqslant \sum_{i=1}^{m} A'_i \, l_i \, \rho'_i \tag{34}$$

The quantity ρ_i on the left side of the inequality is valid for the following reasons. In the case of bar structures, ρ_i depends on the mode shape only and not on the sizes of the elements. In the case of beam elements, ρ_i is independent of the areas of the elements, provided the radius of gyration of each element in the first design is the same as that in the second design. It is not necessary that all elements have the same radius of gyration. Since the two designs are assumed to be in the neighborhood of each other, this assumption is not unreasonable.

Invoking the limiting condition stated in Equation 33 , the inequality 34 can be written as

$$\sum_{i=1}^{m} (A'_i - A_i) \, l_i \, \rho_i \geqslant 0 \tag{35}$$

If the first design has constant ρ_i for all its elements, then the weight of the second design, in which this condition is not satisfied, is greater than the first design, i.e.,

$$\sum_{i=1}^{m} A'_i \, l_i \geqslant \sum_{i=1}^{m} A_i \, l_i \tag{36}$$

or

$$W \geqslant W \tag{37}$$

Now the optimality criterion is stated as follows:

A structure with a given natural frequency will be of minimum weight design when the difference in strain energy density and kinetic energy density is a constant for all its elements while vibrating in its natural mode. For the static case, the above optimality criterion is identical to the one stated in References 1 and 2.

An iterative algorithm that yields a structure satisfying the optimality criterion is derived in the next section.

4. RECURSION RELATION BASED ON OPTIMALITY CRITERION

The optimality criterion stated in the previous section can be attained by iteration only. This requires a recursion relation for iteration and an assurance that iteration using this relation converges to a design satisfying the optimality criterion. The recursion relation derived in this section is similar to the one given in References 1 and 2 except for the modification necessary to account for the kinetic energy.

The volume of the i^{th} element may be written as

$$V_i = \alpha_i \Lambda l_i \tag{38}$$

where the quantity l_i is defined as

$$l_i = \frac{V_i}{\Lambda \alpha_i} \tag{39}$$

α_i is the relative design variable, and Λ is an arbitrary normalization parameter for the design vector. In the case of bar elements l_i is simply the length of the element, and $\Lambda \alpha_i$ is the area of the element. In the case of elements in bending and torsion, appropriate variable definitions are given in Section 6 of Reference 2. When the system is vibrating in the normal mode, the difference in strain energy and kinetic energy of the i^{th} element is a finite time interval is given by

$$\mu_i = \frac{1}{2\Lambda} [\mathbf{s}_i^t \mathbf{k}_i \mathbf{s}'_i - \omega_p^2 \mathbf{s}_i^t \mathbf{m}_i \mathbf{s}'_i] \tag{40}$$

where ω_p is the frequency of vibration in the p^{th} mode, and $\mathbf{s}'_i$ is given by

$$\mathbf{s}'_i = \Lambda \mathbf{s}_i \tag{41}$$

If the quantity u_i per unit volume is assumed to be constant for each element, the following relation can be written

$$\Lambda^2 = c^2 \frac{\mu'_i}{V'_i} \tag{42}$$

where c is the constant of proportionality, and μ'_i and V'_i are given by

$$\mu'_i = ½[\mathbf{s}_i^t \mathbf{k}_i \mathbf{s}'_i - \omega_p^2 \mathbf{s}_i^t \mathbf{m}_i \mathbf{s}'_i] \tag{43}$$

$$V'_i = \alpha_i l_i \tag{44}$$

Multiplying both sides of Equation 42 by α_i^2 and taking the square root yields

$$\alpha_i \Lambda = c\,\alpha_i \frac{\mu'_i}{V'_i} \tag{45}$$

where $\alpha_i \Lambda$ is the i^{th} design variable which is expressed as a function of α_i, the relative variable. The form of Equation 45 suggests the following recursion relation for determining the design variable in each cycle.

$$(\alpha_i \Lambda)_{\nu+1} = c(\alpha_i)_\nu \left(\frac{\mu'_i}{V'_i}\right)_\nu \tag{46}$$

where ν refers to the cycle of iteration.

The procedure for using Equation 46 is as follows:

1. With an assumed relative design vector (such as $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 1.0$) the desired normal mode and the corresponding eigenvalue ω_p are determined.

2. Then the relative strain energy and kinetic energy of each element in this mode are determined.

3. Each design variable for the next iteration is determined by substituting the relative energies in Equation 46 .

4. The design vector is now normalized with respect to the largest variable. This normalization eliminates the need for determining the constant of proportionality.

5. The procedure is repeated until the optimality criterion is attained or the desired frequency is obtained.

The optimality criterion and the iterative algorithm are derived in the context of designing the structure in one of the natural modes. However, the same procedure will be used for designing the structure in the dynamic mode. The dynamic mode may consist of a single natural mode or a linear combination of a set of natural modes depending on the spatial distribution and dynamic characteristics of the forcing function. A more detailed discussion is given in Section 7.

5. EIGENVALUES AND EIGENVECTORS OF BANDED SYMMETRIC MATRICES

The algorithm derived in the last section presupposes the existence of an economical means of determining the eigenvalues and eigenvectors of the equation

$$\omega^2 \mathbf{Mr} = \mathbf{Kr} \tag{47}$$

The generalized mass and stiffness matrices **M** and **K** are both symmetric and are, in general, positive definite. In addition, when **M** and **K** are derived by a displacement formulation (Equations 17 and 18), they are sparsely populated and have similar patterns of nonzero element distribution. For the successful application of the optimization algorithm derived in Section 4, the method of solution of Equation 47 should possess the following properties:

1. It should allow for the determination of any desired frequencies and the corresponding normal modes without determining any of the remaining frequencies and modes.

2. The method should have a provision for taking advantage of the sparseness characteristics of the mass and stiffness matrices.

A method of solution that possess these two properties was described by Gupta (Reference 35) and Peters and Wilkinson (Reference 36). The method is based on the Sturm Sequence property for the equation

$$\mathbf{AX} = \lambda \mathbf{IX} \tag{48}$$

where λ is the eigenvalue, $\mathbf{X}$ is the eigenvector, and $\mathbf{I}$ is the identity matrix. When $\mathbf{A}$ is real and symmetric, the leading principal minors of $(\mathbf{A} - \lambda \mathbf{I})$ form a Sturm sequence; i.e., the number of eigenvalues greater than λ is equal to the number of agreements in sign between consecutive members of the sequence p_r $(r = 0, 1, \ldots, n)$ which is given by

$$p_r = \det(\mathbf{A}_r - \lambda \mathbf{I}) \tag{49}$$

where $\mathbf{A}_r$ is the r^{th} leading principal submatrix of $\mathbf{A}$. The $\det(\mathbf{A}_0 - \lambda \mathbf{I})$ is assumed to be equal to 1.

Peters and Wilkinson have shown (Reference 36) that the Sturm Sequence property is valid for the problem

$$\mathbf{AX} = \lambda \mathbf{BX} \tag{50}$$

when the matrix $\mathbf{B}$ is symmetric and positive definite. Under these conditions the sign of $\det(\mathbf{A}_r - \lambda \mathbf{B}_r)$ is the same as that of $\det(\mathbf{A}_r - \lambda \mathbf{I})$.

The Sturm Sequence property in conjunction with a simple bisection procedure allows for the determination of any eigenvalue, without determining any of the remaining eigenvalues, provided there is a means of evaluating the signs of the leading principal minors of $(\mathbf{A} - \lambda \mathbf{B})$.

Wilkinson presented (Reference 37) a simple method of evaluating the leading principal minors of a square matrix. It is a variation of Gaussian elimination with partial pivoting. The method consists of $(n - 1)$ major steps, and each major step, r, consists of $(n - r)$ minor steps, where n is the order of the matrix. The r^{th} major step in the elimination only involves up to $(r + 1)$ rows of the matrix and consists of the following (p238, Reference 37):

For each value of i from 1 to r

a. Compare a_{ii} and $a_{r+1,i}$ i.e. If $|a_{r+1,i}| \geqslant |a_{ii}|$, interchange $a_{r+1,j}$ and a_{ij} $(j = 1, \ldots, n)$. When an interchange takes place, it should be recorded.

b. Compute $m_{r+1,i} = \dfrac{a_{r+1,i}}{a_{ii}}$ and overwrite on $a_{r+1,i}$

c. For each value of j from (i + 1) to n: Compute $a_{r+1,j} - m_{r+1,i}a_{ij}$ and overwrite on $a_{r+1,j}$. If the cumulative sum of the number of interchanges from the beginning to the end of the r^{th} major step is k, then the principal minor p_{r+1} is given by

$$p_{r+1} = (-1)^k a_{11}\ a_{22} \cdots a_{r+1,\,r+1} \tag{51}$$

It should be noted that only the sign of p_{r+1} is of interest and not its magnitude. When the matrix $(\mathbf{A} - \lambda\mathbf{B})$ is banded, evaluation of the principal minor involves only (r − q) to r rows, i.e., a total of (q + 1) rows, where (2q + 1) is the maximum band width of $(\mathbf{A} - \lambda\mathbf{B})$. This procedure requires only (q + 1). (2q + 1) storage locations, in addition to the storage required for each of the matrices **A** and **B**. Some discussion of the storage requirements of the mass and stiffness matrices of the displacement method is presented at the end of this section.

After determining the eigenvalues by the above procedure, the eigenvectors can be determined by inverse iteration using the following recurrence relation (Reference 37).

$$(\mathbf{A} - \lambda_i\mathbf{B})\ \mathbf{X}_i^{\nu+1} = \mathbf{B}\mathbf{X}_i^{\nu} \tag{52}$$

where the subscript i stands for the number of the eigenvalue and mode of vibration, and ν refers to the cycle of iteration. Only two or three cycles of iteration are necessary for obtaining the eigenvector.

The selection of the initial vector $\mathbf{X}_i^o$ can normally be quite arbitrary. However, in some special circumstances, the iteration may not converge to the desired vector. An example of this is when the mass and stiffness of a beam include the degrees of freedom corresponding to the axial and flexural deformation. The inverse iteration may fail to converge to the flexural modes when the initial vector contains nonzero terms only in the degrees of freedom corresponding to axial deformation. This happens because the linear theory of displacement formulation does not include the coupling terms between axial and flexural deformations. To avoid such a condition it is recommended that all the elements of the initial vector be set equal to 1.

Although Gaussian elimination with partial pivoting was used in solving for the eigenvalues as described in Reference 37, storage limitations necessitated the use of direct Gaussian elimination in the inverse iteration for determining the eigenvectors. Though Wilkinson strongly recommends partial pivoting to avoid an instability in the decomposition, our results were found to be quite satisfactory.

The following comments on the pattern of distribution of the nonzero elements of the mass and stiffness matrices will help in the efficient use of the above method (Reference 38). In a displacement formulation the mass and stiffness matrices generally have the same pattern of distribution of the nonzero elements. Thus, the comments made in reference to the stiffness matrix are valid for the mass matrix.

The distribution of the nonzero elements is dependent upon the way the nodes of the finite element model are numbered. Because of the symmetry of the stiffness matrix only the lower or upper triangular matrix is considered. For the purpose of this discussion definitions of the following terms are in order. The gross population (P_{gross}) of the stiffness matrix is defined as the total number of elements in the upper triangle of the matrix. The net population (P_{net}) is the total number of nonzero elements in the upper triangle. Zeros resulting from transformations are not excluded from the net population. The apparent population ($P_{apparent}$) is the actual number of elements considered as nonzeros by a given solution scheme. From these definitions

$$P_{net} \leqslant P_{apparent} \leqslant P_{gross} \tag{53}$$

For a given structure P_{gross} and P_{net} are invariant and are given by

$$P_{gross} = \frac{N\ (N+1)}{2} \tag{54}$$

and

$$P_{net} = \frac{n\ (n+1)}{2}\ (\text{number of nodes}) + \sum_{i=1}^{m} \frac{n^2[k_i(k_i-1)]}{2} - n^2\,(NR) \tag{55}$$

where N is the total number of degrees of freedom of the structure, n is the number of degrees of freedom of each node (all the nodes are assumed to have the same number of degrees of freedom; when this is not true the necessary modification is simple), k_i is the number of nodes to which the i^{th} element is connected, and m is the number of elements in the structure. The quantity NR is given by

$$NR = \sum_{i=1}^{p} (a_i - 1) \tag{56}$$

where a_i is the number of elements connecting the same pair of nodes and p is the total number of pairs of directly connected nodes. If the structure consists of bar and/or beam elements only, NR is always zero. For the example shown in Figure 1, the value of NR is 3.

The quantity $P_{apparent}$ is dependent on the nature of the solution scheme used. For Gaussian elimination with no pivoting (LDL^T), $P_{apparent}$ may be defined as

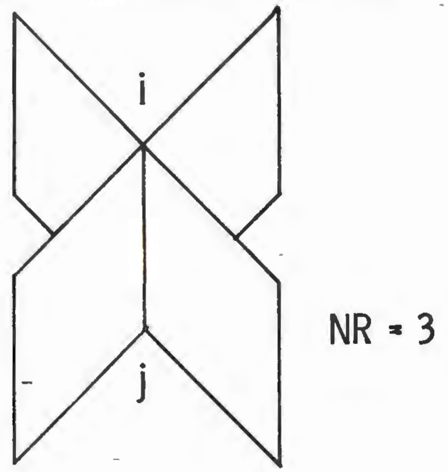
$$P_{apparent} = \sum_{i=1}^{n} Q_j \tag{57}$$

where $Q_j = j - R_j + 1$ and where R_j is the row number of the first nonzero element in the j^{th} column. The solution scheme is most efficient when $P_{apparent} = P_{net}$. However, in large practical structures this condition is difficult to attain.

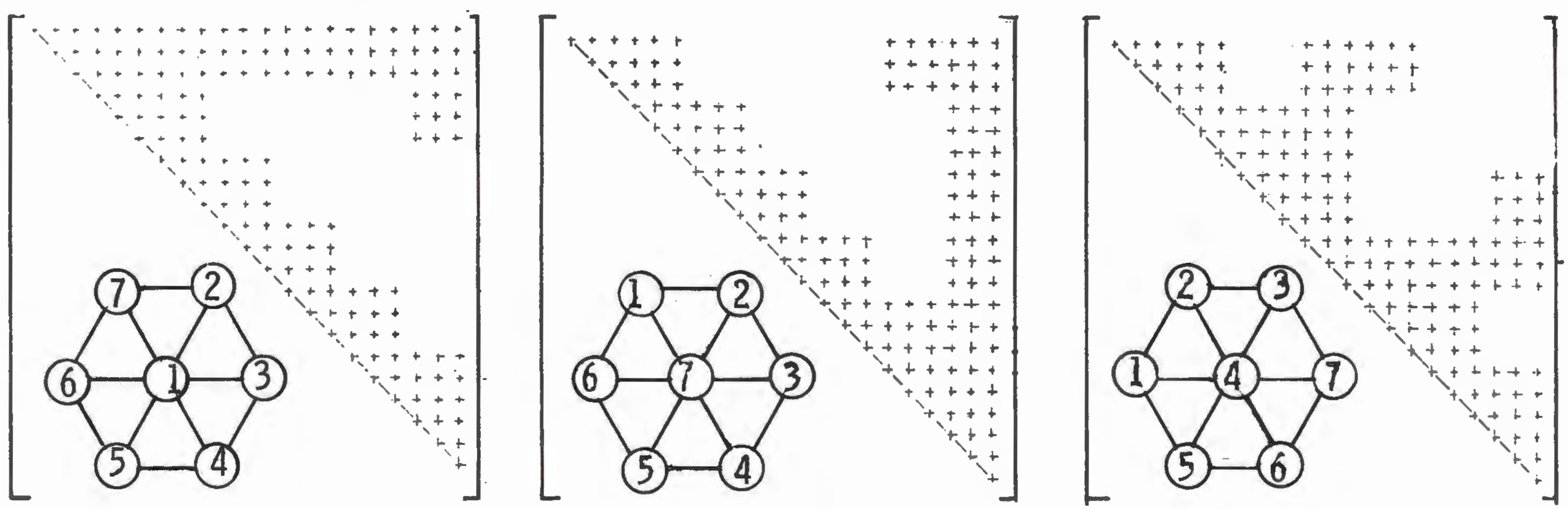
The value of $P_{apparent}$ changes with the node numbering scheme of the finite element model. The example shown in Figure 2 illustrates this point. A seven node three dimensional bar structure (n = 3) is numbered in three different ways and the resulting effect on the respective stiffness matrices is shown. The non-zero elements are marked by (+). The populations for the three cases are also given in the same figure.

$P_{apparent}$ represents the number of storage locations required for the stiffness matrix and an equal number is required for the mass matrix. This is the procedure used for storing the stiffness and mass matrices for all the problems discussed in Section 8.

FIGURE 1. INTERSECTING PLATES



SCHEME NO.	P_{GROSS}	P_{NET}	$P_{APPARENT}$
1	231	150	231
2	231	150	177
3	231	150	177



SCHEME 1 | SCHEME 2 | SCHEME 3

FIGURE 2: DISTRIBUTION OF NONZERO ELEMENTS IN THE STIFFNESS MATRIX

6. FREQUENCY GRADIENTS AND NUMERICAL SEARCH

The recursion relation derived in Section 4, Equation 46, is extremely rapid in effecting changes in the frequencies of the structure. If the object is simply to obtain a minimum weight structure with maximum stiffness, then Equation 46 is adequate and there is no need for determining frequency gradients. However, when the situation demands that less than a maximum stiffness structure is required, then finer adjustments can be made by a numerical search using frequency gradients.

The expressions for the frequency gradients are derived here with the aid of the Raleigh quotient. Similar expressions are derived earlier in a number of papers (References 24, 26, 28, etc). The i^{th} frequency of the structure may be written as a function of the design variables

$$\lambda_i = \omega_i^2 = F_i\,(A_1, A_2, \ldots, A_m) \tag{58}$$

where λ_i is the i^{th} eigenvalue and $A_1, A_2, \cdots, A_m$ are the m design variables. The function F_i is actually the Raleigh quotient corresponding to the i^{th} natural mode, and it may be written as (Reference 34)

$$\lambda_i = F_i\,(A_1, A_2, \ldots, A_m) = \frac{\mathbf{X}_i^t\,\mathbf{K}\,\mathbf{X}_i}{\mathbf{X}_i^t\,\mathbf{M}\,\mathbf{X}_i} \tag{59}$$

where $\mathbf{X}_i$ is the i^{th} natural mode of the structure, and $\mathbf{K}$ and $\mathbf{M}$ are the generalized stiffness and mass matrices, respectively. If the change in λ_i due to a change in the variable A_j is $d\lambda_{ij}$, then its value can be determined by the Taylor's series expansion

$$d\lambda_{\omega} = \frac{\partial F_i}{\partial A_j}\,dA_j + \frac{1}{2!}\,\frac{\partial^2 F_i}{\partial A_j^2}\,dA_j^2 + \ldots \tag{60}$$

In the following derivation the terms beyond the first are neglected. From Equation 59

$$\frac{\partial F_i}{\partial A_j} = \frac{2\,\frac{\partial}{\partial A_j}\,(\mathbf{X}_i^t)\,[\mathbf{K} - \lambda_i \mathbf{M}]\,\mathbf{X}_i + [\mathbf{X}_i^t\,\Delta\mathbf{K}_j\,\mathbf{X}_i - \lambda_i \mathbf{X}_i^t\,\Delta\mathbf{M}_j\mathbf{X}_i]}{\mathbf{X}_i^t\,\mathbf{M}\mathbf{X}_i} \tag{61}$$

where $\Delta\mathbf{K}_j$ and $\Delta\mathbf{M}_j$ are the change in the stiffness and mass matrices of the structure due to a change in the variable A_j. Since a change in the size of the element effects only the stiffness and mass matrices of that particular element, $\Delta\mathbf{K}_j$ and $\Delta\mathbf{M}_j$ are the j^{th} element matrices expressed in the general structure coordinate system.

From Equation 25 the first term in the numerator of Equation 61 is zero. Then Equation 61 becomes

$$\frac{\partial F_i}{\partial A_j} = \frac{\mathbf{X}_i^t\,\Delta\mathbf{K}_j\mathbf{X}_i - \lambda_i \mathbf{X}_i^t\,\Delta\mathbf{M}_j\,\mathbf{X}_i}{\mathbf{X}_i^t\,\mathbf{M}\,\mathbf{X}_i} \tag{62}$$

Thus, to the first order of approximation, $d\lambda_{ij}$ can be written as

$$d\lambda_{ij} = \frac{[\mathbf{s}_{ij}^t \mathbf{k}_j \mathbf{s}_{ij} - \lambda_i \mathbf{s}_{ij}^t \mathbf{m}_j \mathbf{s}_{ij}]}{\mathbf{X}_i^t \mathbf{M} \mathbf{X}_i} \tag{63}$$

where $\mathbf{s}_{ij}$ is the displacement vector of the j^{th} element when the structure is vibrating in the i^{th} mode, and $\mathbf{k}_j$ and $\mathbf{m}_j$ are the stiffness and mass matrices of the j^{th} element corresponding to the change in size dA_j, the value of dA_j is assumed to be equal to A_j in determining $d\lambda_{ij}$. The numerator in Equation 63 represents the difference in the strain energy and the kinetic energy of the i^{th} element in free vibration in a finite time interval (Equation 29).. As can be seen from Equation 63 determination of frequency gradients is computationally inexpensive when once the frequencies and modes are determined. This situation is quite contrary to the calculation of displacement gradients in the static case (Reference 2).

The numerical search algorithm using frequency gradients for the adjustment of frequencies is represented by the following recursion relation.

$$\alpha^{\nu+1} = \alpha^{\nu} \pm \Delta \mathbf{D}^{\nu} \tag{64}$$

where α is the design variable vector, ν refers to the cycle of iteration and Δ represents the scalar step size. Each element D_j of vector $\mathbf{D}$ is assumed to be directly proportional to $d\lambda_{ij}$ and inversely proportional to the length l_j

$$D_j = c \frac{d\lambda_{ij}}{l_j} \tag{65}$$

The change in λ_i corresponding to the change D_j is

$$\delta\lambda_{ij} = \frac{d\lambda_{ij}}{\alpha_j} D_j \tag{66}$$

The total change in λ_i due to a change in the size of all the elements may be written as

$$\delta\lambda_i = \sum_{j=1}^{m} \delta\lambda_{ij} = c \sum_{j=1}^{m} \frac{1}{\alpha_j} \frac{(d\lambda_{ij})^2}{l_j} \tag{67}$$

Thus, the proportionality constant c is given by

$$c = \frac{\delta\lambda_i}{\sum_{j=1}^{m} \frac{1}{\alpha_j} \frac{(d\lambda_{ij})^2}{l_j}} \tag{68}$$

Then the element D_j is given by

$$D_j = \frac{\delta\lambda_i}{\sum_{p=1}^{m} \frac{1}{\alpha_p} \frac{(d\lambda_{ip})^2}{l_p}} \quad \frac{d\lambda_{ij}}{l_j} \tag{69}$$

where $\delta\lambda_i$ is the desired change in the i^{th} frequency of the structure. A similar search algorithm is derived in References 1 and 2 for displacement and stress-constraint problems.

The value of Δ is set equal to 1.0. By adjusting Δ the rate of change can be further adjusted.

7. DYNAMIC RESPONSE AND OPTIMUM DESIGN

The response of a structural system subjected to a time dependent forcing function is governed by (Equation 20)

$$\mathbf{M}\ddot{\mathbf{r}} + \mathbf{K}\mathbf{r} = \mathbf{R}_o F(t) \tag{70}$$

where $\mathbf{R}_o$ is the generalized force matrix whose elements are assumed to be functions of the spatial coordinates only. It is further assumed that the time function F(t) is the same for all the generalized forces.

If the number of degrees of freedom of the system is n, then Equation 70 represents n coupled second order differential equations.

The Sturm Sequence property in conjunction with a bisection procedure and inverse iteration permit the determination of the frequencies and normal modes economically at any range of the frequency spectrum. Then the normal mode method can be used for finding the response of the system.

In the normal mode method the response vector is approximated by a finite number of normal coordinates in the following form:

$$\mathbf{r} = \mathbf{\Psi}\,\mathbf{q} \tag{71}$$

where each column of the matrix $\mathbf{\Psi}$ is a normal mode and $\mathbf{q}$ represents the vector of normal coordinates. If all the normal modes are included in $\mathbf{\Psi}$, then Equation 71 represents the exact response of the system. However, in practice only a small number of normal modes contribute significantly to the response of the system and the remaining modes need not be included in $\mathbf{\Psi}$. Which modes are significant depends on the nature of the forcing function. A procedure for determining the significant modes is discussed later in this section. Thus, the number of normal modes included in $\mathbf{\Psi}$ is p ($p \leqslant n$). Substitution of Equation 71 into 70 and premultiplication by $\mathbf{\Psi}^t$ gives

$$\mathbf{\Psi}^t\mathbf{M}\,\mathbf{\Psi}\ddot{\mathbf{q}} + \mathbf{\Psi}^t\mathbf{K}\,\mathbf{\Psi}\mathbf{q} = \mathbf{\Psi}^t\mathbf{R}_o F(t) \tag{72}$$

Equation 72 represents a set of uncoupled second order differential equations in normal coordinates and can be written as

$$\mathbb{M}\ddot{\mathbf{q}} + \mathbb{K}\mathbf{q} = \mathbf{\Psi}^t \mathbf{R}_o F(t) \tag{73}$$

where $\mathbb{M}$ and $\mathbb{K}$ are diagonal matrices and their elements are given by

$$\mathbb{M}_i = \mathbf{\Psi}_i^t \mathbf{M} \mathbf{\Psi}_i \tag{74}$$

$$\mathbb{K}_i = \mathbf{\Psi}_i^t \mathbf{K} \mathbf{\Psi}_i \tag{75}$$

Now the number of uncoupled equations is equal to the number of normal modes (p) selected to approximate the response. It should be pointed out that the subscript i in Equations 74 and 75 does not refer to the i^{th} column of the nxn normal mode matrix, but instead it refers to the i^{th} column of the nxp normal mode matrix selected to express the dynamic response.

The i^{th} uncoupled equation may be written as

$$\ddot{q}_i + \omega_i^2 q_i = \frac{1}{\mathbb{M}_i} \mathbf{\Psi}_i^t \mathbf{R}_o F(t) \tag{76}$$

where ω_i is the natural frequency corresponding to the i^{th} normal mode selected for expressing the response. With the use of the convolution integral the solution of Equation 76 can be written in the following form (Reference 34)

$$q_i(t) = \frac{1}{\omega_i^2 \mathbb{M}_i} \mathbf{\Psi}_i^t \mathbf{R}_o D_i(t) \tag{77}$$

where $D_i(t)$ is the dynamic load factor or magnification factor and in the absence of damping is given by

$$D_i(t) = \int^t \omega_i \sin \omega_i (t - \tau) F(\tau) d\tau \tag{78}$$

where t represents the time at which the response is desired and τ is the intermediate time variable.

The complexity of evaluating the integral in Equation 78 depends on the nature of the forcing function F(t). When F(t) is periodic or aperiodic and satisfies the Dirichlet conditions in any finite interval, it can be expanded into a Fourier integral and an approximate solution to the integral can be obtained. If F(t) is obtained by experimental or other empirical data, the integral can be evaluated by any of the numerical integration schemes.

Substitution of Equation 77 in 71 gives the response of the system in the following form

$$\mathbf{r} = \boldsymbol{\Psi}\boldsymbol{\Omega}\,\mathbf{D}\,\boldsymbol{\Psi}^t\,\mathbf{R}_o \tag{79}$$

where $\boldsymbol{\Omega}$ and $\mathbf{D}$ are diagonal matrices with elements given by

$$\Omega_i = \frac{1}{\omega_i^2\,\mathbb{M}_i} \tag{80}$$

$$D_i = D_i(t) \tag{81}$$

The dynamic response of the system in generalized coordinates is given by Equation 79. The element generalized coordinates, forces and stresses can be determined by substituting Equation 79 into Equations 14, 23, and 24.

If the structure is subjected to a static load vector $\mathbf{R}_o$, then the dynamic load factor matrix $\mathbf{D}$ becomes an identity matrix and Equation 79 can be written as

$$\mathbf{r}_o = \boldsymbol{\Psi}\boldsymbol{\Omega}\,\boldsymbol{\Psi}^t\,\mathbf{R}_o \tag{82}$$

In the static case the generalized masses are unity, and ω_i and $\boldsymbol{\Psi}$ are the eigenvalues and eigenvectors of the stiffness matrix.

A comparison of Equations 22 and 82 reveals that

$$\mathbf{K}^{-1} \simeq \boldsymbol{\Psi}\,\boldsymbol{\Omega}\,\boldsymbol{\Psi}^t \tag{83}$$

assuming $\mathbf{K}$ is non-singular. Equation 82 represents the static response of the structure. When the loads are time dependent, the static response is magnified by the dynamic load factor matrix $\mathbf{D}$.

The dynamic response as given by Equation 79 can also be written as a linear combination of natural modes of the structure

$$\mathbf{r} = \sum_{i=1}^{p} c_i\,\boldsymbol{\Psi}_i \tag{84}$$

From Equations 79 and 84 c_i may be written as

$$c_i = \frac{D_i}{\omega_i \mathbb{M}_i}\,\boldsymbol{\Psi}_i^t\,\mathbf{R}_o \tag{85}$$

where the dynamic load factor D_i is given by Equation 78.

Each term in Equation 84 represents the contribution of one mode to the dynamic response. If all the modes of the system are included in the summation, the response given by Equation 84 would be exact to the degree of approximation expected of a discretized model. When the forcing function is in the low frequency range compared to the actual frequencies of the structure, two or three modes at the lower end of the frequency spectrum are adequate for satisfactory representation of the dynamic response. In high frequency vibrations, however, the number of active terms are expected to be larger.

It should be recognized that the nature of the time dependence as well as the spatial distribution of the forcing function determine which of the modes predominate in the dynamic response. The design procedures that simply stiffen the structure to raise the fundamental frequency are inadequate or even produce grossly non-optimum designs for some types of forcing functions. The procedure outlined in this paper does not suffer from this deficiency, because it takes into consideration the nature of the dynamic response as represented by Equation 84 .

The modes that predominate in the dynamic response are called the critical modes, and a trial procedure for filtering these modes is presented here. It is based on a study of the virtual work of the peak dynamic forces when subjected to the dynamic response as given by Equation 84 . The virtual work of the peak forces is defined as

$$V_w = \mathbf{R}_o^t \, \mathbf{r} \tag{86}$$

Figure 3 illustrates two possible cases of the variation of the virtual work, V_w, with the number of modes included in the dynamic response. In the first case, the five modes at the lower end of the spectrum are significant. In the second case modes 7 – 11 are significant, and the remaining modes can be left out.

The recommended procedure for determining the critical modes is as follows: For a given design the dynamic response is determined by adding the effect of one mode at a time in Equation 84 . After each addition the virtual work, V_w, is determined from Equation 86 . A plot of the virtual work against the number of modes would reach a plateau when an adequate number of modes are included in the response. In general this filtering procedure need not be repeated in every optimization cycle. It would be adequate to go through this procedure for the initial design.

Thus, the step by step procedure for the minimum weight design of a dynamically loaded structure is as follows:

1. For an initial design the critical modes are determined by the procedure outlined in the foregoing discussion.

2. The peak dynamic response of the structure is determined by including all the critical modes in Equation 84 .

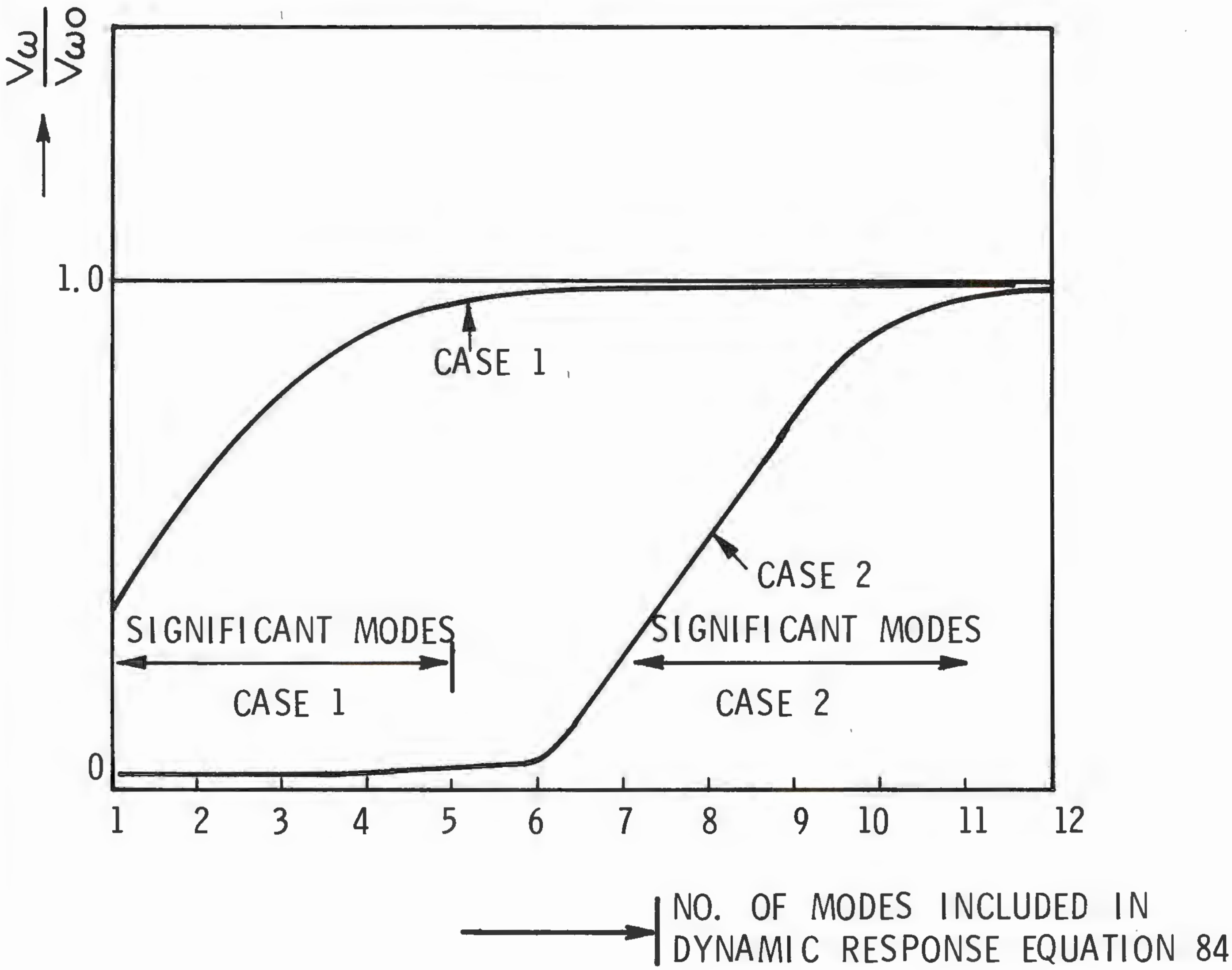
3. The dynamic stresses corresponding to the peak response are determined by Equations 23 and 24 .

4. By scaling the design to the constraint surface, the feasible design and its weight are determined. (See Reference 2 for the scaling procedure and other details).

5. The frequency of the forced vibration of the structure is determined by substituting the dynamic displacement vector $\mathbf{r}$ (Equation 84) into the Raleigh quotient

$$\omega^2 = \frac{\mathbf{r}^t \mathbf{K} \, \mathbf{r}}{\mathbf{r}^t \mathbf{M} \, \mathbf{r}} \tag{87}$$

6. The structure is resized by substituting the dynamic displacement vector $\mathbf{r}$ (Equation 84) and ω^2 (determined in the last step) into the recurrence relation based on the optimality criterion, Equation (46).



Vω/Vω0
1.0
CASE 1
CASE 2
SIGNIFICANT MODES
CASE 1
SIGNIFICANT MODES
CASE 2
0
1
2
3
4
5
6
7
8
9
10
11
12
NO. OF MODES INCLUDED IN
DYNAMIC RESPONSE EQUATION 84

FIGURE 3.

7. Steps 2 thru 6 are repeated as long as there is an improvement in the design.

The above procedure is valid for both stress and displacement constrained problems. However, in the case of displacement constraints the design can be further improved by a numerical search at the end as in the static case, Reference 2.

In the foregoing discussion it was tacitly assumed that the forcing function could be expressed by a finite number of harmonics.

8. DESIGN EXAMPLES

The design examples presented in this section may be grouped into two categories. The first group of four examples are for the purpose of studying the variation of the natural frequencies and material distribution when the structures are designed in their natural modes. Designing in one of the natural modes is tantamount to designing in the actual dynamic mode if the given forcing function activates only that natural mode. The scaling of the design to the constraint surface is the only additional step required to obtain the actual design. For instance, if a loading condition activates only the fundamental mode, then the relative distribution of the material obtained by designing in the fundamental mode would be the same as that obtained by designing in the actual dynamic mode.

The recursion relation based on the optimality criterion derived in Section 4 gives the stiffest structures when they are designed in their fundamental mode. The stiffness of the structure is measured by the ratio of the fundamental frequency to the relative weight.

In the next group of two examples the structures are designed for dynamic loads. The first example in this group is a rectangular frame with beam elements. This frame is designed for three different loading cases. These loading cases are devised to show how the spatial distribution of the forcing function activates different modes even though the dynamic characteristics are the same in all cases. The second example in this group is a wing structure idealized by bar elements. This structure is designed for a periodic forcing function. It represents a medium size problem in terms of degrees of freedom and number of elements.

In the examples involving beam elements the principal moment of inertia is assumed to be the design variable. It is further assumed for the problems solved in this paper that the radius of gyration of the beam section is the same for all the elements. A sandwich beam with constant depth and variable face sheet thicknesses would conform to this assumption. The elastic constants, mass properties, and other design data are given in Table 1 for all the examples.

Example 1: Cantilever Beam — Design in the First Mode.

The cantilever beam shown in Figure 4 is designed in its fundamental mode. The design started with the assumption of equal sizes for all the elements. The elements are resized by the iterative algorithm based on the optimality criterion (Equation 46). The first four natural frequencies, the relative weight, and the ratio of the fundamental frequency to the relative weight are given in Table 2 for

Table 1: Design Data for all Examples

Example	1,2,3	4	5	6
Elastic Modulus (10^6 Lbs/In^2)	29.000	29.000	29.000	10.000
Radius of Gyration (In)	2.646	3.426	8.731	
Mass Density (10^{-2} Slugs/In^3)	.884	.884	.888	.313
Stress Limit (10^3 Lbs/In^2)			29.000	25.000

Table 2: Cantilever Beam – Design in First Mode

$$\lambda_i = \omega_i^2 \; 10^6 / E$$

Cycle No.	λ_1	λ_2	λ_3	λ_4	Rel. Wt.	λ_1 / Wt.
1	.316	12.312	97.680	377.734	16.966	.019
2	2.353	31.270	150.098	448.047	7.413	.317
3	5.043	46.914	179.102	483.203	5.885	.857
4	7.398	58.164	200.195	504.297	5.280	1.401
5	9.086	65.195	210.742	511.328	4.991	1.820
6	10.264	70.117	221.289	518.359	4.840	2.121
7	10.967	72.578	224.805	525.391	4.754	2.307
8	11.494	75.391	228.320	525.391	4.704	2.443
9	11.846	76.797	231.836	532.422	4.673	2.535
10	12.022	76.797	231.836	532.422	4.653	2.584

Sizes of the Elements in the Final Design

Element No.	1	2	3	4	5	6	7
Rel. Size	.0002	.003	.018	.082	.250	.566	1.00

each iteration. This table also contains relative sizes of the elements in the final design. The distribution of the moment of inertia along the length of the beam is shown in Figure 4.

Example 2: Simply Supported Beam: Design in the First Mode

The simply supported beam shown in Figure 5 is designed in its fundamental mode. The first four natural frequencies, the relative weight of the beam, and the ratio of the fundamental frequency to the relative weight are given in Table 3 for each iteration. It is interesting to note that the values of the natural frequencies are reduced slightly but the reduction in weight is much more pronounced. The relative sizes of the elements in the final design are given in Table 3. The graphic representation of the material distribution is shown in Figure 5.

Example 3: Fixed – Fixed Beam: Design in the First Mode

The fixed–fixed beam shown in Figure 6 is designed in its fundamental mode. The results of this design are given in Table 4 and the graphic representation of the material distribution is shown in Figure 6.

Example 4: Rectangular Frame: Design in the First Four Modes

The rectangular frame shown in Figure 7 is designed independently in the first four modes. The natural frequencies and the relative sizes are given in Table 5. The natural modes and the material distribution of the final designs are shown qualitatively in Figure 7.

Example 5: Rectangular Frame: Design for Dynamic Loads

The rectangular frame shown in Figure 7 is designed for three different cases of dynamic loads. In all three cases, the forcing functions are periodic and have the same circular frequency. The spatial distribution of the forcing functions are shown for the three cases in Figure 8.

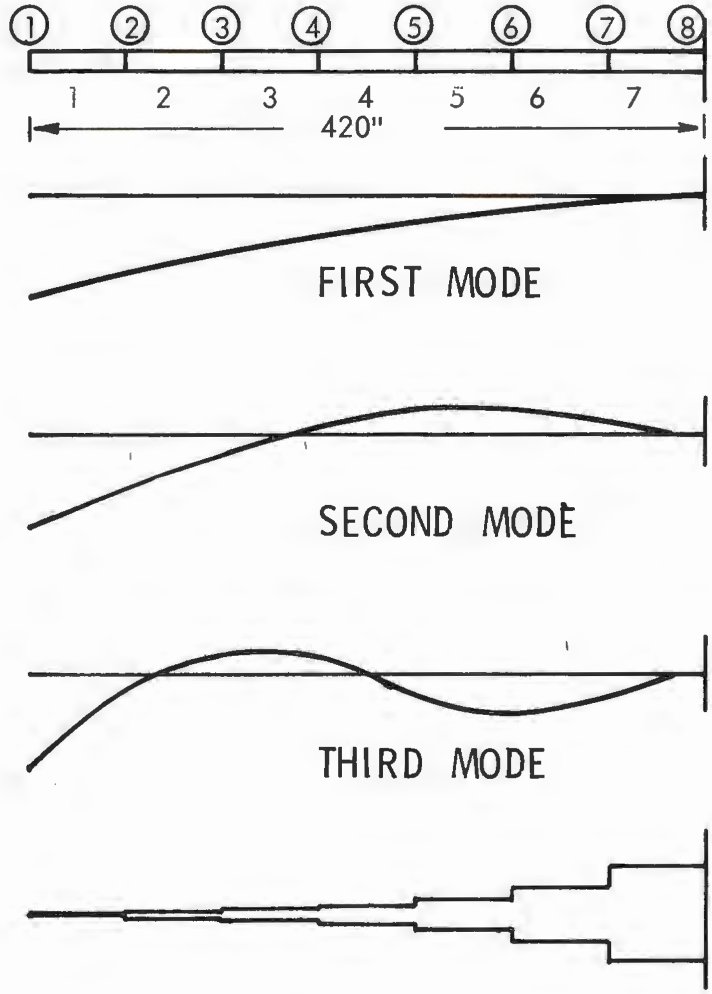
The dynamic load factor, $D_r(t)$, corresponding to the periodic forcing function after integration (Equation (78)) is given by

$$D_r(t) = \frac{1}{\left[1-\left(\frac{p}{\omega_r}\right)^2\right]} \left(\sin pt - \frac{p}{\omega_r} \sin \omega_r t\right)$$

A plot of the dynamic load factors corresponding to the first four natural frequencies of the final design in Case 1 is given in Figure 9. The peak values of $D_r(t)$ are used in the dynamic load factor matrix in evaluating response (Equation 84) even though they do not occur at the same time. This approximation, in general, produces conservative designs. A more accurate procedure would be to evaluate the dynamic response at sufficiently small time intervals and to select the response at which the dynamic stresses are a maximum.

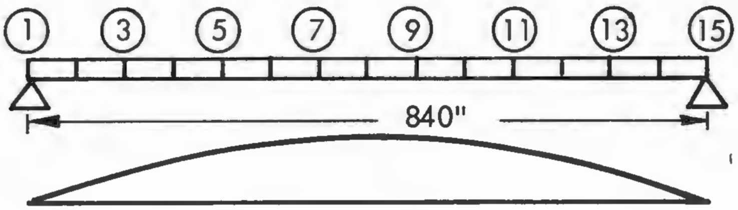
Case 1: Dynamic Lateral Force

In the first case the frame is subjected to a periodic lateral force. The design started with the assumption of equal sizes for all the elements. For this design a plot of the virtual work, V_w, against the number of modes in the response is given

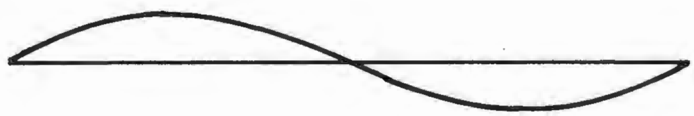


DESIGN IN FIRST MODE

FIGURE 4. CANTILEVER BEAM



FIRST MODE


SECOND MODE


THIRD MODE


DESIGN IN FIRST MODE

FIGURE 5. SIMPLY SUPPORTED BEAM

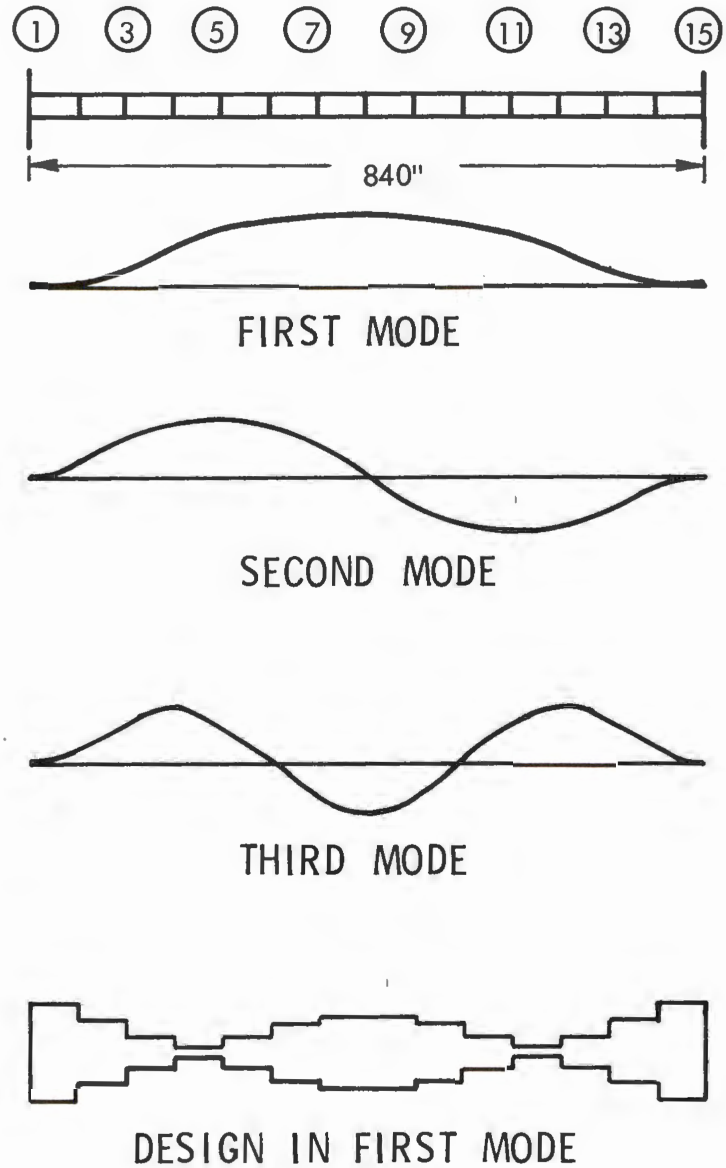


FIGURE 6. FIXED - FIXED BEAM

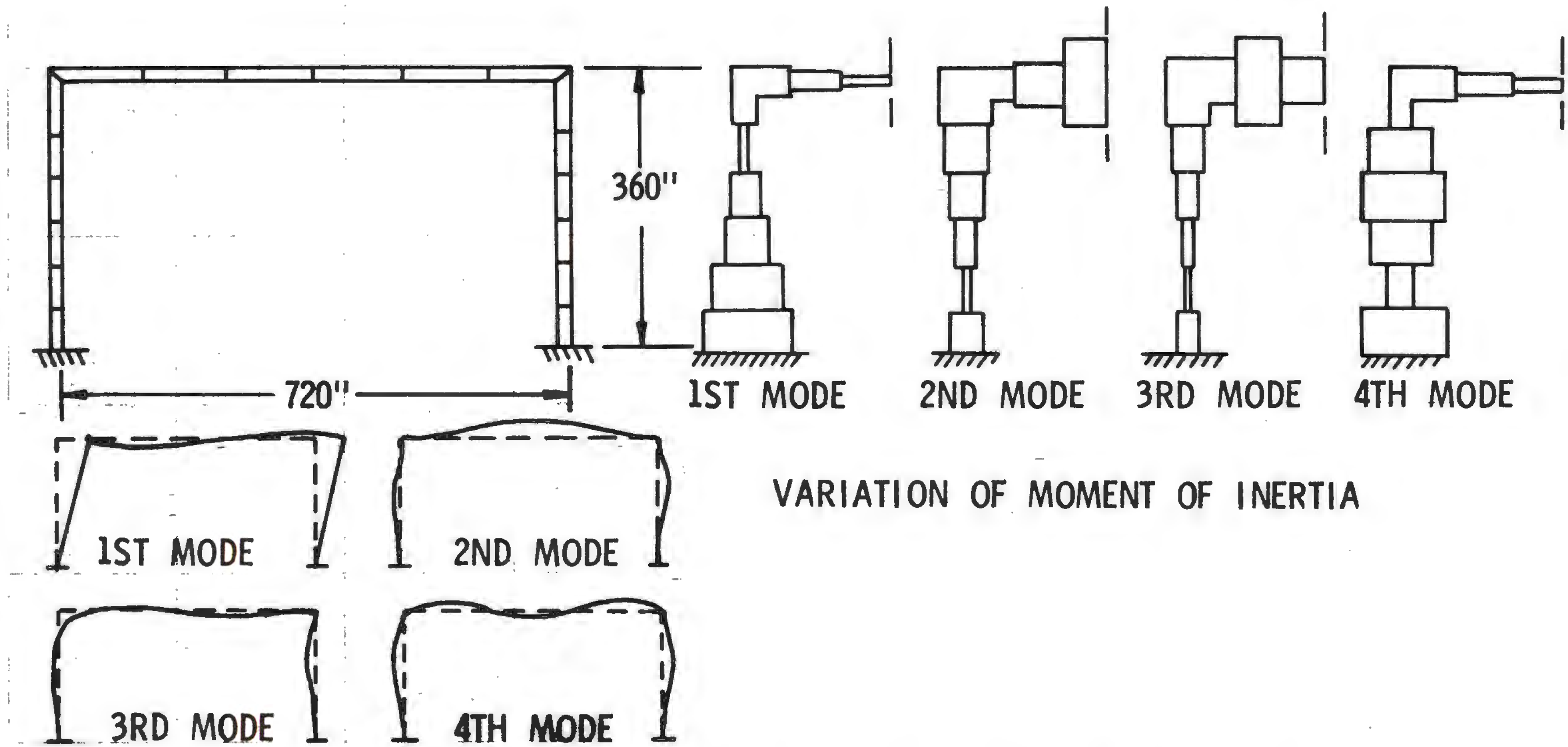
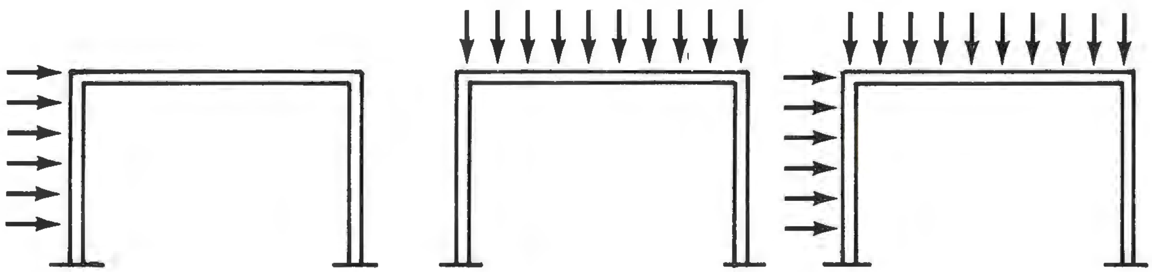


FIGURE 7 RECTANGULAR FRAME: DESIGNS IN NATURAL MODES



CASE 1 CASE 2 CASE 3

$$\underset{\sim}{R} = \underset{\sim}{R}_0 \text{ Sin } pt$$

*Note: $p/\omega_r \ll 1/10$ in all cases

PEAK FORCES

CASE 1	1000 LBS AT NODES 2 TO 7 (X-DIRECTION)
CASE 2	1000 LBS AT NODES 7 TO 13 (Y-DIRECTION)
CASE 3	COMBINATION OF CASES 1 AND 2

FIGURE 8: DYNAMIC LOAD DISTRIBUTION

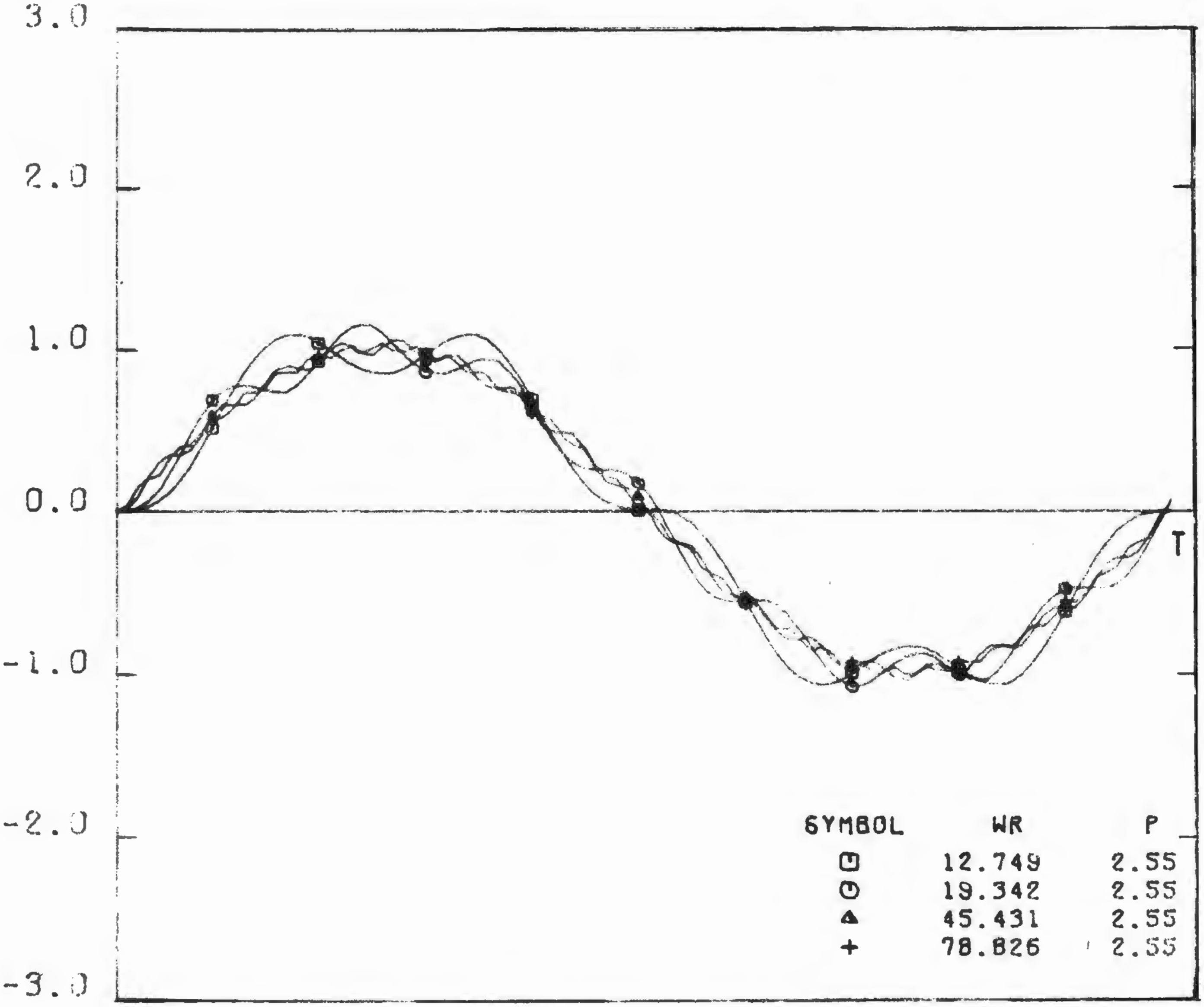


Figure 9: Variation of Elements of Dynamic Load Factor Matrix

Table 3: Simply Supported Beam — Design in First Mode

$\lambda_i = \omega_i^2 \, 10^6 / E$

Cycle No.	λ_1	λ_2	λ_3	λ_4	Rel. Wt.	λ_1 / Wt.
1	.155	2.494	12.549	39.883	33.932	.005
2	.145	2.248	11.846	38.477	21.921	.007
3	.143	2.213	11.846	38.477	21.402	.007
4	.143	2.213	11.846	38.477	21.365	.007
5	.143	2.213	11.846	38.477	21.362	.007

Sizes of the Elements in the Final Design

Element No.	1	2	3	4	5	6	7
Rel. Size	.119	.313	.508	.687	.836	.944	1.00

TABLE 4: FIXED – FIXED BEAM – DESIGN IN FIRST MODE

$\lambda_i = \omega_i^2 \; 10^6/E$

Cycle No.	λ_1	λ_2	λ_3	λ_4	Rel. Wt.	λ_1/Wt.
1	.796	6.027	23.183	63.789	33.933	.023
2	.993	8.242	33.379	76.797	18.034	.055
3	1.009	8.102	31.973	72.578	18.231	.055
4	1.044	8.102	31.621	70.117	17.879	.058
5	1.062	8.242	31.621	69.414	17.691	.060
6	1.062	8.242	31.621	69.414	17.597	.060
7	1.062	8.242	31.621	69.414	17.553	.061
8	1.062	8.242	31.270	68.711	17.530	.061
9	1.062	8.242	31.270	68.711	17.518	.061
10	1.062	8.242	31.270	68.711	17.512	.061

SIZES OF THE ELEMENTS IN THE FINAL DESIGN

Element No.	1	2	3	4	5	6	7
Rel. Size	1.00	.657	.330	.089	.307	.545	.684

TABLE 5: RECTANGULAR FRAME (DESIGNS IN FOUR MODES)

$\lambda_i = \omega_i^2 \; 10^6/E$

		λ_1	λ_2	λ_3	λ_4	Rel. Wt.
Initial Design		.413	1.343	10.440	24.590	34.700
Final Design	1st. Mode	1.062	1.642	9.508	31.270	11.227
	2nd. Mode	.175	.979	9.367	23.184	16.198
	3rd. Mode	.164	.852	8.242	26.699	15.026
	4th. Mode	.589	1.896	11.494	22.832	14.866

SIZES OF THE ELEMENTS IN THE FINAL DESIGN

Element No.		1	2	3	4	5	6	7	8	9
Rel. Size	1st. Mode	1.000	.733	.475	.234	.066	.236	.292	.198	.079
	2nd. Mode	.209	.062	.154	.316	.473	.620	.366	.518	1.000
	3rd. Mode	.183	.060	.125	.231	.288	.293	.433	1.000	.574
	4th. Mode	1.000	.267	.621	.947	.722	.219	.334	.180	.169

SYMMETRICAL ABOUT CENTER LINE

in Figure 10. This plot shows that the first mode is the most significant one, and the modes beyond the fourth have little contribution. The first four natural frequencies of the initial and the final designs for this loading case are given in Table 6. Table 6 also contains the sizes of the elements in the final design. The graphic representation of the material distribution is shown in Figure 11.

Case 2: Dynamic Vertical Force.

The plot of the virtual work, V_w, for this Case (Figure 10) shows that only the second mode contributes significantly to the dynamic response. All other modes including the first have very little contribution. The final design and the first four natural frequencies are given in Table 6. The material distribution is shown in Figure 11.

Case 3: Combination of Lateral and Vertical Forces.

For this loading case the first two are the significant modes (see Figure 10). The modes beyond the fifth have little contribution. The final design and the natural frequencies are given in Table 6.

It is evident from the results of this example that the spatial distribution of the forcing function determines primarily the modes that participate in the dynamic response.

Example 6: Wing Structure Subjected to Periodic Forces

Figure 12 shows a typical transport wing idealized as a three dimensional bar structure. It has two hundred forty degrees of freedom and four hundred forty elements. The wing is subjected to a sinusoidal forcing function. The final design and the first four natural frequencies are given in Table 7.

9. SUMMARY AND CONCLUSIONS

The optimization method presented in this paper may be called designing in the dynamic mode. The dynamic mode may be a single natural mode of the structure or a linear combination of a set of natural modes depending on the spatial distribution and dynamic characteristics of the forcing function. There are three basic steps in this optimization procedure. The first step consists of determining the number of significant modes necessary to represent the dynamic mode. In the second step, the given design is scaled to satisfy the dynamic stress and displacement constraints. After this step the weight of the feasible design can be determined for comparison with the designs in the previous iterations. In the third step, the elements of the structures are resized by the algorithm based on either the optimality criterion or the numerical search. Only the last two steps are repeated in each iteration for obtaining the optimum structure.

Each of the three major steps consist of several substeps. The details of these substeps are pointed out throughout the body of this paper. For instance, Section 2 contains a summary of the equations of dynamic analysis. They are presented in the context of the displacement method of finite element analysis. An optimality criterion for minimum weight structures is derived in Section 3. It is based on the study of the strain energy and the kinetic energy of the elements in the dynamic mode. An iterative algorithm for attaining the optimality criterion is derived in Section 4. This section also contains a step by step procedure for using the algorithm in the design of minimum weight structures. In Section 5 a procedure for the evaluation of eigenvalues and eigenvectors of banded symmetric matrices is discussed. The eigenvalues

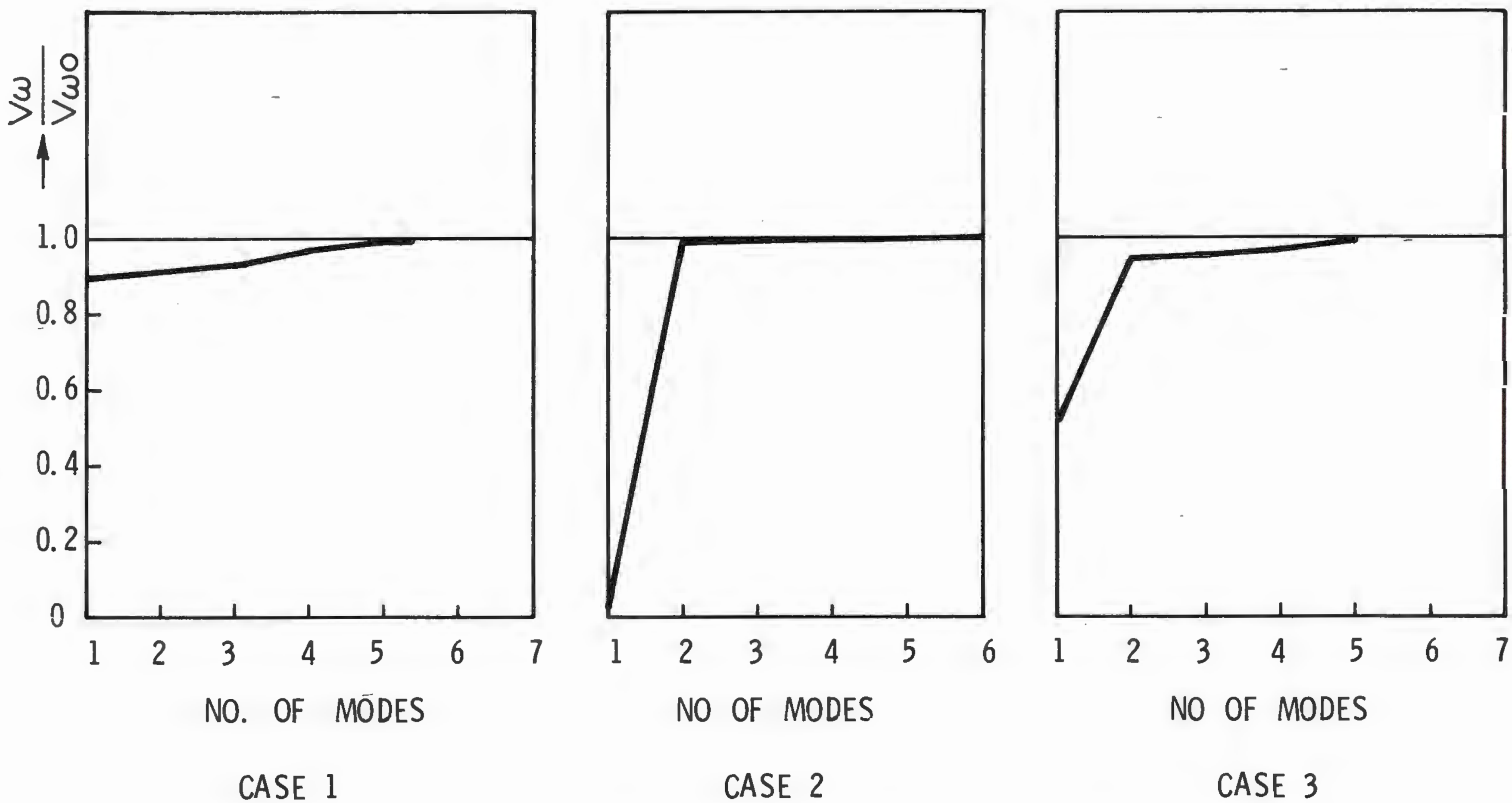
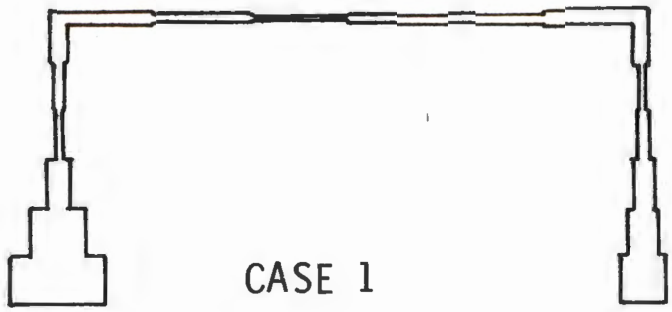
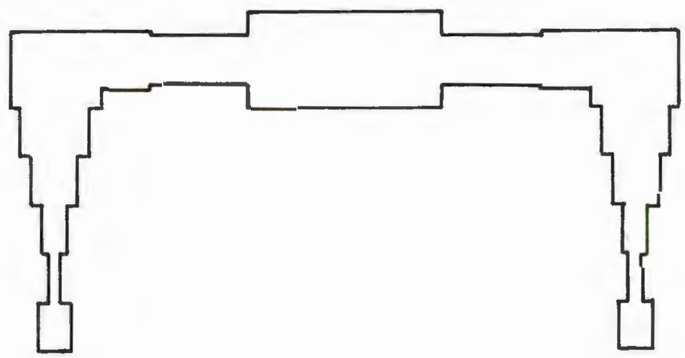


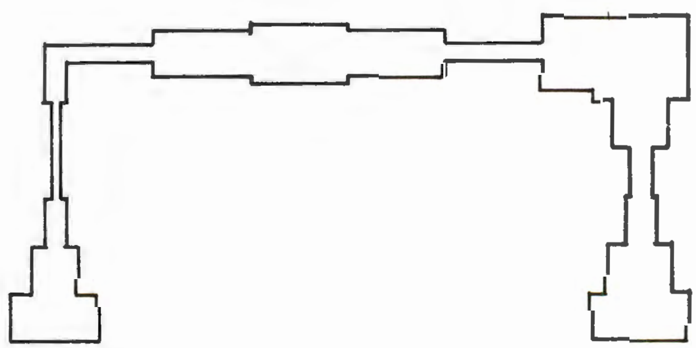
FIGURE 10. PLOT OF VIRTUAL WORK ~ NO. OF MODES IN DYNAMIC RESPONSE



CASE 1



CASE 2



CASE 3

FIGURE 11. MATERIAL DISTRIBUTION OF DYNAMICALLY LOADED FRAME

LOADS PER NODE (POUNDS)

1	2	3
500	1000	1500

RIB DIMENSIONS

RIB AT	AB	CD	EF	GH
X = 0"	5.25"	7.00"	5.25"	3.50"
X=600"	24.00"	32.00"	24.00"	16.00"

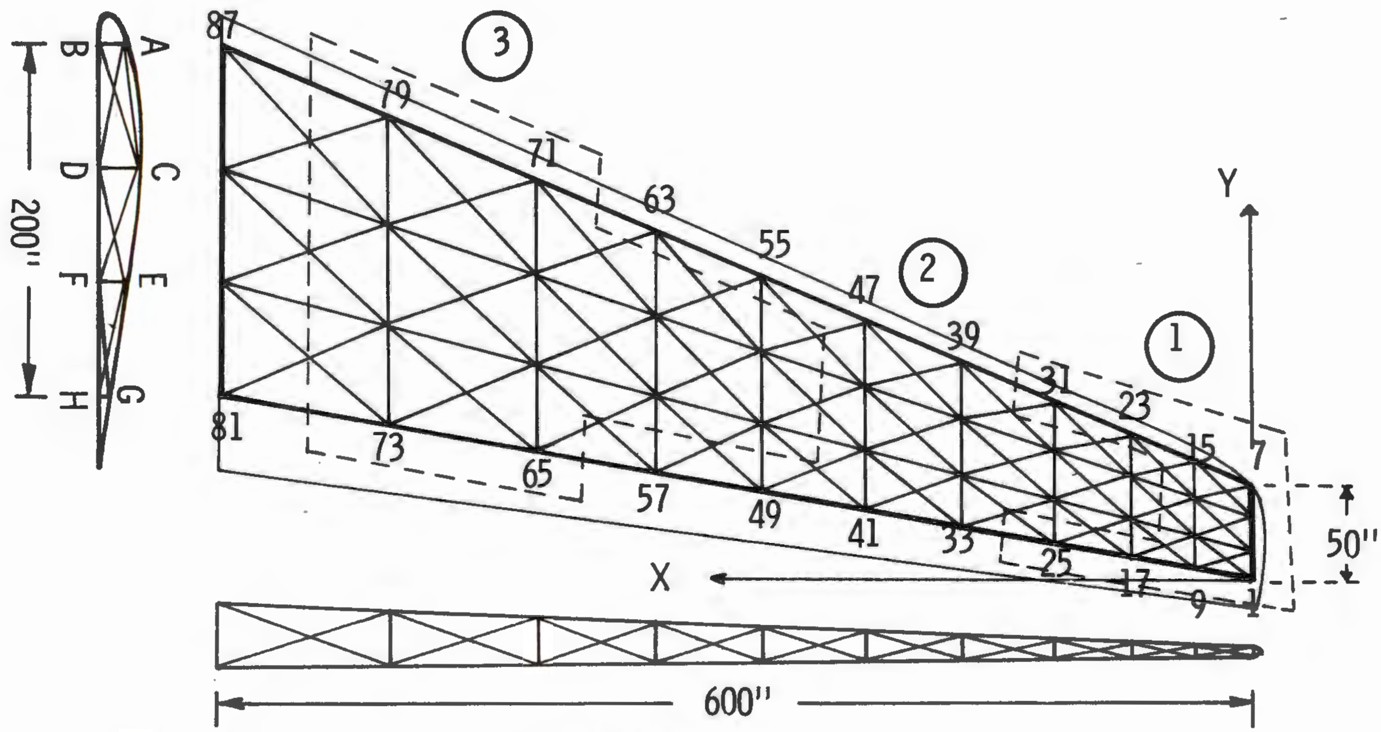


FIGURE 12. WING STRUCTURE

TABLE 6: RECTANGULAR FRAME: DESIGN FOR DYNAMIC LOADS

Case 1: Periodic Lateral Forces

$$\lambda_i = \omega_i^2\ 10^6/E$$

Design	λ_1	λ_2	λ_3	λ_4	Weight
Initial	2.67	8.66	67.30	155.37	1080
Final	6.52	12.73	71.17	214.26	522

Cycle No.	1	2	3	4
Weight	1080	808	528	522

Elem.	1	2	3	4	5	6
Area	6.61	3.75	1.56	0.36	0.99	1.18
Inertia	503.80	285.5l	119.11	27.77	75.43	90.05
Elem.	7	8	9	10	11	12
Area	0.91	0.53	0.19	0.36	0.76	1.16
Inertia	69.64	40.16	14.23	27.64	57.64	88.47
Elem.	13	14	15	16	17	18
Area	0.99	0.30	0.63	1.39	2.15	2.92
Inertia	75.77	22.54	48.20	105.64	163.89	222.61

Case 2: Periodic Vertical Forces

$$\lambda_i = \omega_i^2\ 10^6/E$$

Design	λ_1	λ_2	λ_3	λ_4	Weight
Initial	2.67	8.66	67.31	155.37	563
Final	1.57	7.54	63.79	151.86	506

Cycle No.	1	2	3	4
Weight	563	534	517	506

Elem.	1	2	3	4	5	6
Area	0.77	0.35	0.48	0.95	1.48	2.02
Inertia	58.71	26.40	36.29	72.36	112.61	154.19
Elem.	7	8	9	10	11	12
Area	1.22	1.13	2.06	2.06	1.13	1.22
Inertia	92.68	85.76	157.17	157.17	85.76	92.68
Elem.	13	14	15	16	17	18
Area	2.02	1.48	0.95	0.48	0.35	0.77
Inertia	154.19	112.61	72.36	36.29	26.40	58.71

Case 3: Periodic Combined Forces

$$\lambda_i = \omega_i^2 \, 10^6/E$$

Design	λ_1	λ_2	λ_3	λ_4	Weight
Initial	2.67	8.66	67.31	155.37	940
Final	2.53	8.10	63.09	200.20	718

Cycle No.	1	2
Weight	940	718

Elem.	1	2	3	4	5	6
Area	3.41	1.73	0.65	0.26	0.26	0.82
Inertia	260.18	131.87	49.34	19.70	19.70	62.16
Elem.	7	8	9	10	11	12
Area	0.70	1.79	2.35	1.82	0.69	2.98
Inertia	53.40	136.49	178.89	138.87	52.90	227.35
Elem.	13	14	15	16	17	18
Area	3.83	2.30	0.90	0.98	2.41	3.91
Inertia	292.10	175.65	68.82	74.49	183.49	297.90

TABLE 7: WING STRUCTURE: DESIGN FOR DYNAMIC LOADS

Design	λ_1	λ_2	λ_3	Weight
Initial	23.887	146.582	448.047	9220.312
Final	880.469	1043.95	3337.89	2115.030

Cycle No.	1	2	3	4
Weight	9220.312	3278.977	2389.517	2115.030

are determined by the use of the Sturm Sequence property and a bisection procedure. The eigenvectors are obtained by inverse iteration. A search algorithm for frequency constraint problems is derived in Section 6. It is similar to the one derived for displacement and stress constraint problems in References 1 and 2.

Section 7 contains a step by step procedure for the design of structures with dynamic stress and displacement constraints. This section also describes a procedure for filtering the significant normal modes of a structure subjected to a given dynamic load. It is based on the virtual work of the peak forces in the dynamic mode. This is the most crucial section for the successful application of the method presented in this paper.

Section 8 contains applications of the general procedure presented in this paper to specific problems. In the first group of problems the structures are designed in their natural modes. In the second group, the structures are designed in the dynamic mode when they are subjected to periodic dynamic forces.

It should be pointed out that the method is not limited to the case of periodic forces only. Extensions of the method to the design of structures in dynamic modes such as the ones resulting from aperiodic forces, flutter modes, and static stability modes are contemplated for the future. It is assumed that these dynamic modes can be represented by a linear combination of the natural modes of the structure.

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