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FOREWORD

This report describes the research done during the period April 1, 1960 to December 31, 1960. This report was prepared by The University of Michigan under USAF Contract No. AF 33(616)-6041. The contract was initiated under Project No. 7021, "Solid States Research and Properties of Matter," Task No. 73653, "Mechanisms of Flow and Fracture of Metallic and Non-Metallic Crystalline Substances." The work was administered under the direction of the Materials Central, Directorate of Advanced Systems Technology, Wright Air Development Division, with Dr. A. J. Herzog acting as project engineer.

WADD TR 60-869

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ABSTRACT

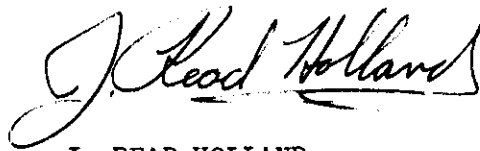
The results of the research indicate that thin tubular specimens of cast Zamak-3 zinc alloy are brittle for some states of combined tension and torsion load and ductile for others. A testing temperature in the range -8°F to 80°F appears to be permissible.

General characteristics of a yield function in accord with plasticity theory are discussed. The experimental data for yielding and fracture are compared with the predictions of theory. The large degree of scatter in the data make theoretical correlation difficult, but it appears that all the needed data can be obtained by use of the present apparatus with modifications. All data for yielding and failure under combined tension and torsion loading were obtained at a test temperature of 32°F .

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:



J. READ HOLLAND
Actg. Chief, Advanced Metallurgical Studies Branch
Metals & Ceramics Laboratory
Materials Central

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INTRODUCTION

A group at The University of Michigan has been investigating the conditions for failure of metals at various temperatures and under complex stress states, particular attention being given to the circumstances under which ductile and brittle fractures occur.

Operating under Air Force Contract No. AF 33(616)-6041, the group has constructed and proof-tested a unique machine for the simultaneous application of tension, torsion, and internal pressure to tubular specimens. This machine is equipped with a hydraulic feedback loop so that stresses set up in the tubular specimen by the applied loads will remain in constant ratio with one another from the beginning of loading to fracture of the specimen, under all conditions where large deformations do not occur. The machine is described in a technical report.¹

A series of tests has been completed using Zamak-3 (zinc base alloy) cast tubular specimens. These tests indicate that, at temperatures at which the metal is extremely ductile in pure torsion, so much so that fracture cannot be achieved, brittle behavior occurs at all other stress combinations. The description and analysis of these tests are expected to be incorporated in a Ph.D. thesis by one of the investigators (D.R.J.). A preliminary account of this work forms the body of this technical report.

A theoretical study has been made² to pinpoint the range of yield conditions possible while retaining the concept of stability of the material in the sense of plastic theory. This formed part of the introductory material used by one of the investigators (R.M.H.) in a recent talk.³

A study has been initiated, and will be completed shortly, to delineate the predictions of various hardening theories in the case of the tube test.⁴ This is necessary background work for the analysis of the test data.

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THE INVESTIGATION OF ZAMAK-3 ALLOY

The objective of the research program during the past year has been to investigate experimentally and theoretically the conditions of failure in a cast zinc-alloy material. As has been discussed in detail in a previous report,¹ fracture (i.e., rupture or separation of the material) is of primary interest; two major fracture modes are considered:

1. ductile fracture in which appreciable plastic deformation occurs before rupture; and
2. brittle fracture in which rupture is preceded by very little plastic deformation.

Attention has been concentrated on the macroscopic behavior in terms of stress combinations associated with the various phenomena. However, a failure-fracture criterion in terms of the macroscopic stresses must be based on carefully executed increments of research which permit a thorough understanding of the variables involved. Since both ductile and brittle fracture are preceded by yielding on crystalline materials,⁵ it seemed appropriate to investigate at first states of combined stress which produce yielding followed by both brittle and ductile fracture. This was done by selecting a testing temperature which would cause the material to exhibit ductile behavior for some states of stress and brittle behavior for others. The purpose of this work is to determine whether theoretical failure surfaces in stress space will conform to both types of result.

MATERIAL AND SPECIMEN

Of the several materials considered in this research, the Zamak-3 zinc alloy was selected. This was based on a series of preliminary tests in tension and torsion which indicated that brittle behavior in tension occurred at temperatures below 80°F, while brittle behavior in torsion was exhibited at temperatures below -8°F. Thus by selecting a testing temperature intermediate between -8°F and 80°F, it was possible to produce brittle failure in tension and ductile failure in torsion without resorting to extremes of temperature.

In no case does the zinc alloy give the great ductility of, say, mild steel. However, a marked change in the amount of plastic deformation preceding fracture is observed at the "brittle point." This is demonstrated in Table I for tests in torsion. To emphasize this point, it should be noted that the biaxial loading machine¹ had insufficient range of deformation to induce fracture in torsion for temperatures of 30°F to 80°F. The data for strain to fracture in torsion at 70°F were obtained in a conventional torsion-testing machine.

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TABLE I

PROPERTIES OF ZAMAK-3 AT VARIOUS TEMPERATURES
(Tubular Specimen Unless Otherwise Noted)

Type of Test	Temperature	Principal* Strain to Fracture
Torsion (0.500-in. Diam. Solid Specimen)	70°F	0.049
Torsion (0.500-in. Diam. Solid Specimen)	70°F	0.044
Torsion	75°F	0.034
Torsion	-8°F	0.007
Tension	80°F	0.008
Tension	37°F	0.008
Tension	19°F	0.007
Tension	32°F	0.007

$$* \epsilon_{\max} = \frac{1}{2} \gamma_{\max}$$

The data for tension tests listed in Table I indicate brittle behavior for temperatures below 80°F with the relatively small strain to fracture corresponding to that for the brittle torsion test. Although not presented here, earlier tests on small solid specimens indicated that the testing temperature must be raised to at least 200°F before ductile behavior is observed in pure tension.

On the basis of these data, a testing temperature of 32°F was selected. This temperature was attained conveniently by immersing the specimen and grips in a tank or jacket containing an ice-water mixture. In this way a uniform specimen temperature could be maintained for fairly long periods of time (1-2 hours) and the conditions could be reproduced from test to test without difficulty. Of course, the ice point temperature is not especially important in itself. It is expected that other testing temperatures should be employed.

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Since comparatively little is known about the mechanical properties of Zamak-3, a number of tensile tests were performed on 1/4-inch-diameter solid specimens in a conventional testing machine. Of interest here are the form of the stress-strain curve and the influence of various strain rates on the mechanical properties. Figure 1 presents results from five tensile tests at 70°F. Note that tests 5-B, 6-B, 9-B, and 13-B were performed at a strain rate of 0.001 inches per inch per minute, while test 10-B was performed at a strain rate of 0.01 inches per inch per minute. These data indicate first that Zamak-3 is a material with a limited range of linearity between stress and strain. The second observation to be made is that increasing the strain rate by a factor of ten has no appreciable influence on the results. The degree of scatter between tests 5-B, 6-B, 9-B, and 13-B may not be characteristic of the material due to the small number of tests. However, a variation in strain rate of one order of magnitude in this range is characteristic of later results.

All the specimens used in obtaining the results presented in this report were machined from solid cylinders about 1-1/8 inch in diameter of cast Zamak-3 zinc alloy. A certified analysis of the material is listed below:

<u>Element</u>	<u>Per Cent of Total</u>
Aluminum	3.99
Iron	Trace
Magnesium	0.040
Copper	0.084
Lead	0.0016
Tin	0.0013
Cadmium	0.001
Zinc	Balance

It will be noted that the material is within the composition limits of ASTM Specification B-86.⁶ The solid cylinders were cast in two equal batches, totaling 154 altogether. The casting procedure was to heat the melt to 1100°F in an induction furnace and to pour in graphite split-molds. The molds were machined from solid blocks of graphite and consisted of two double molds and one single mold so that five cylinders could be cast in each pour. No effort was made to preheat the molds.

To check the thermodynamic stability of the alloy, photomicrographs of an area in the region of the specimen cross section from one of these cylinders were made at intervals. The cylinder from which this sample was taken was cast June 16, 1960. Photomicrographs were made on July 5, 1960, July 20, 1960, October 3, 1960, and November 25, 1960. These are presented in Figure 2. Of course, the necessity to polish and etch the specimen for each examination causes some change in the area observed. However, there appear to have been no radical changes in the microstructure during the period of the research reported here.

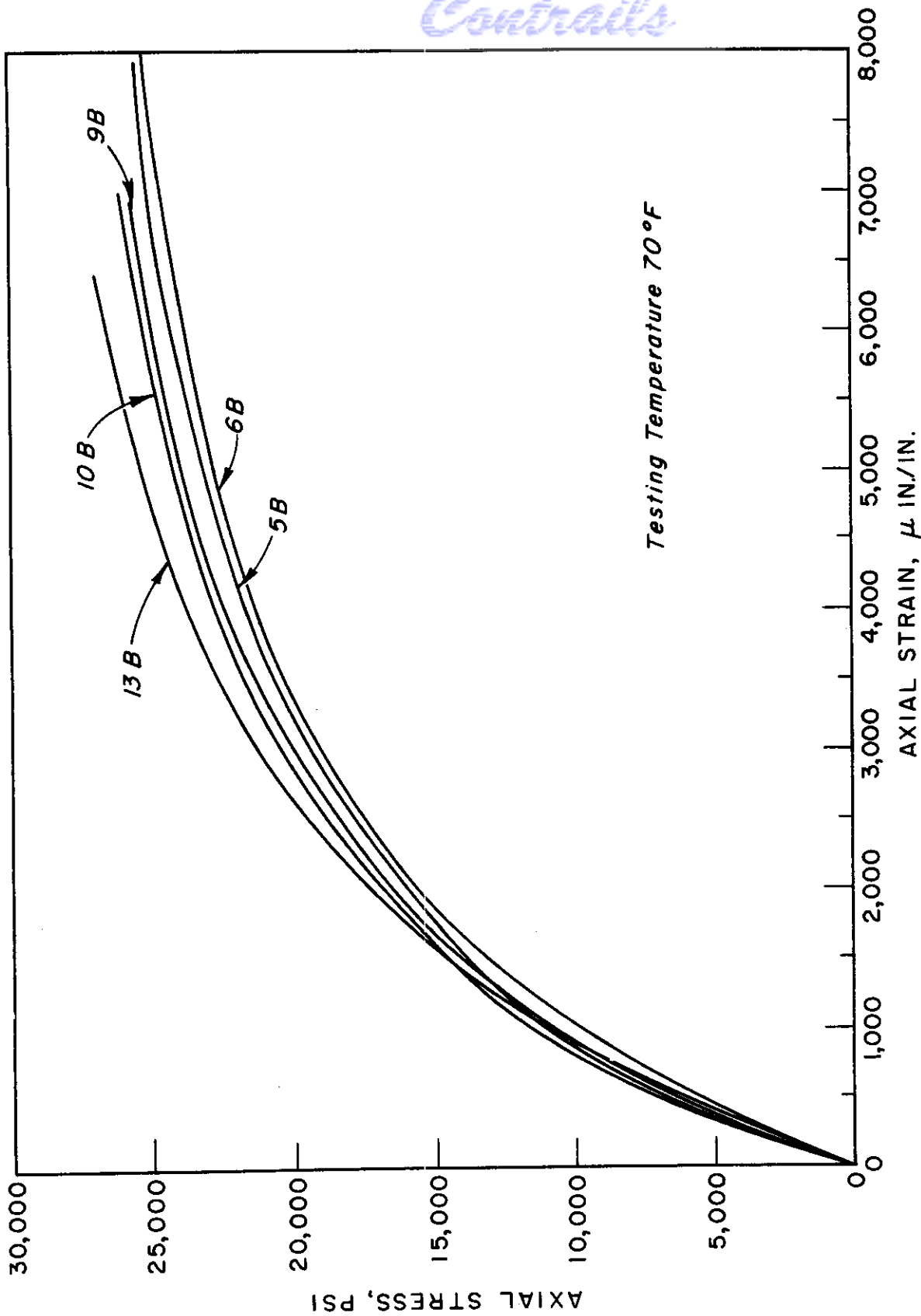
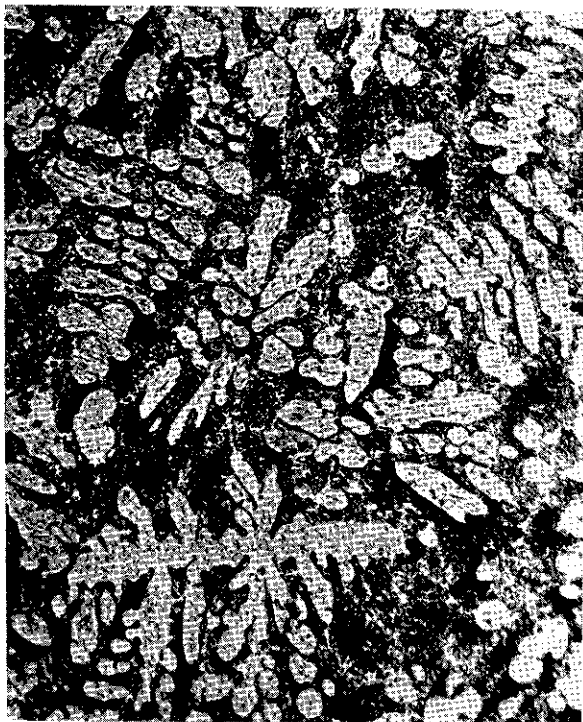
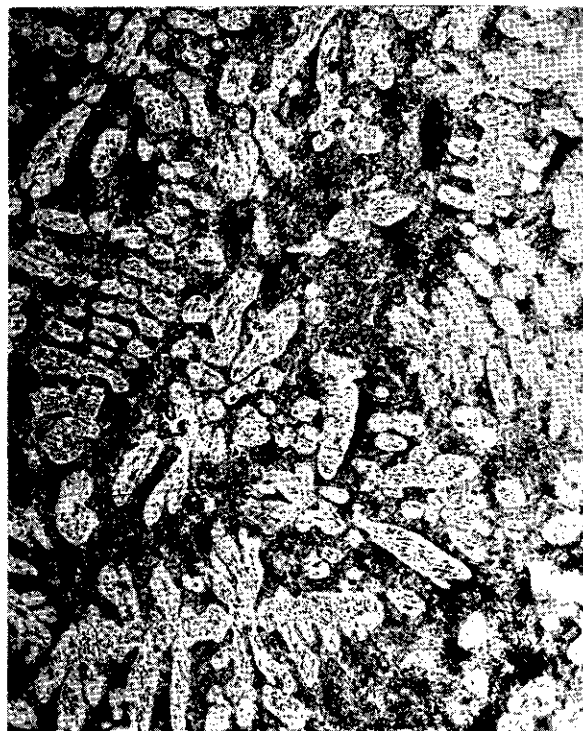


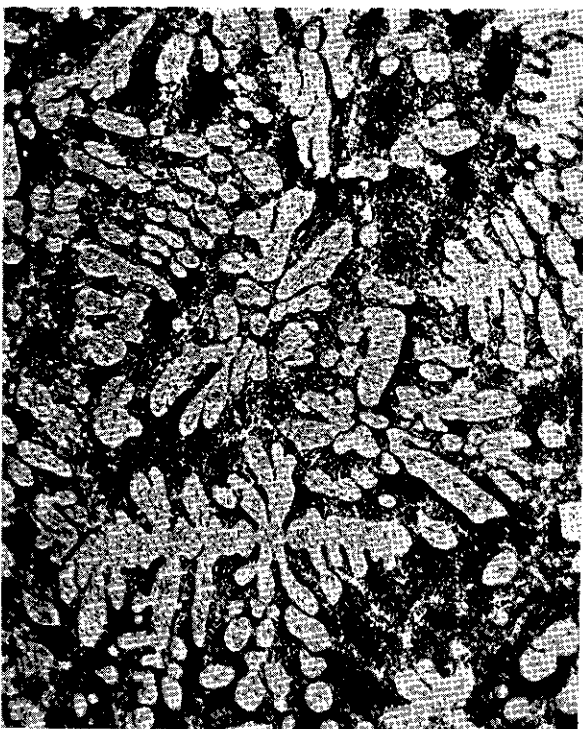
Figure 1. Stress-Strain Curves for Zamak-5 (1/16-inch-diameter solid specimens).



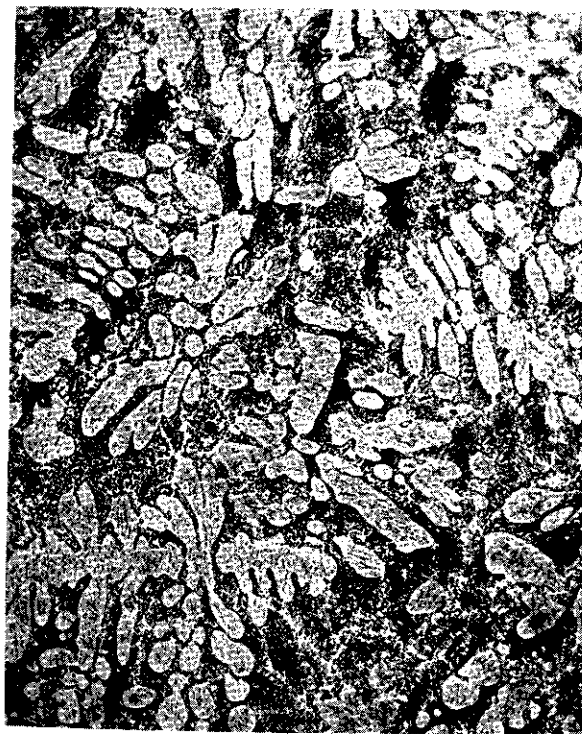
July 5, 1960



July 20, 1960



October 3, 1960



November 28, 1960

Figure 2. Photomicrographs of Zamak-3.

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The lower strength of Zamak-3 as compared to Bessemer steel necessitated some changes in the configuration of the specimen described previously.¹ This involved increasing the diameter and wall thickness in the test section so that axial forces and torques would be large enough for easy observation. Figure 3 is a sketch of this new specimen. Note that the diameter-to-thickness ratio is about 16 or essentially that of the previous specimen. For these dimensions the shear stress due to torsion is uniform to a good approximation and the tensile stress is uniformly distributed in the reduced section so that a large volume of the specimen is in a state of uniform stress.

THEORETICAL PREDICTIONS

Any theory of failure that might result from this research program would be more useful if it could be fitted into the framework of the theory of plasticity, for then a substantial amount of existing theory could be drawn upon for the purpose of solving boundary value problems. In this light it is possible to make several observations concerning the general character of the yield condition. A general statement of the yield condition or yield function as

$$F(\sigma_{ij}) = k^2$$

implies that, for stress combinations satisfying the equation, plastic deformation occurs, and for stress combinations in which the left side of the equation is less than the right, only elastic deformation may occur. As a result of the "fundamental postulate" introduced by D. C. Drucker,⁷ it can be shown that the yield surface must be convex and that the strain increment vector must be normal to the yield surface. The yield surface would be a plot of the yield function, say in principal stress space. The gradient of the yield function $F(\sigma_{ij})$ is normal to the yield surface and this leads to the flow rule.

$$\dot{\epsilon}_{ij} = \lambda \frac{\partial F}{\partial \sigma_{ij}}$$

of von Mises.⁸

Now particular cases must be considered. Assuming that the material in question is initially isotropic and is stable, i.e., conforms to the fundamental postulate, it follows that the yield function or yield condition must depend only on the magnitudes of the principal stresses and not on their directions. Thus

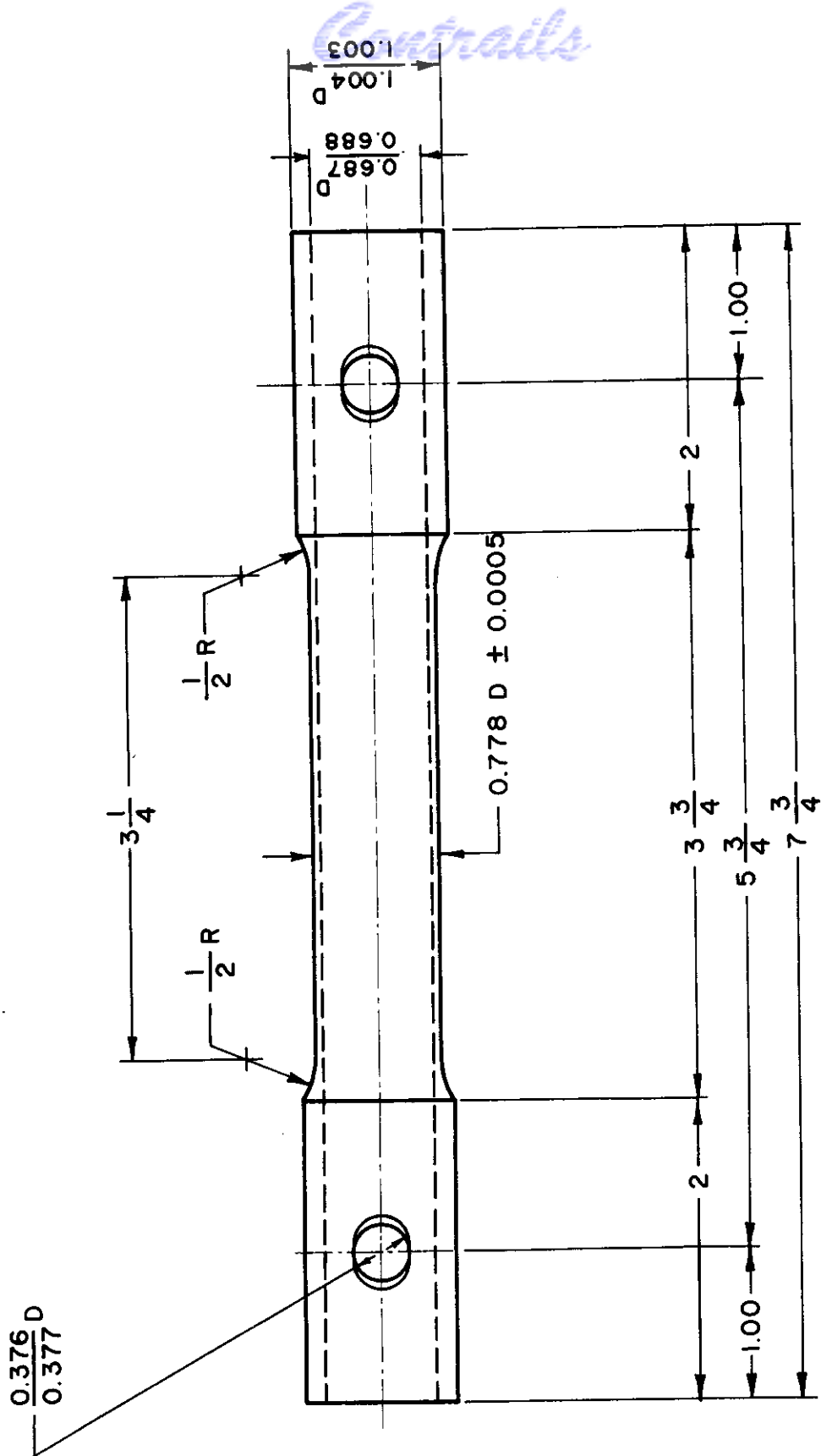


Figure 3. Sketch of Specimen Used for Zamak-3.

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$$F(\sigma_{ij}) = G(I_1, I_2, I_3)$$

where I_1 , I_2 , and I_3 are the invariants of the stress tensor. It is implied here that the sense of the stress is immaterial so that the yield stress in tension and compression would be the same. Since the principal stresses are not ordered and any symbol may be assigned to a particular principal stress, the yield function must be symmetrical in the principal stresses. A further assumption that the yield process is independent of the hydrostatic component of stress is often made. In the cases where this is valid, the yield function will depend only on the stress deviation or

$$F(\sigma_{ij}) = G(J_2, J_3)$$

where J_2 and J_3 are the invariants of the stress deviator tensor.

First, consider the case of an ideally plastic material. In this case, yielding would occur for stress states lying on the yield surface and plastic flow would continue without increase in stress. The yield function then is invariable. Both the yield criterion of von Mises⁹ and that of Tresca¹⁰ are applicable. Note that no Bauschinger effect is included in these yield criteria and all the above requirements and assumptions are satisfied.

Second, consider the strain-hardening material. This is essential when working with a material such as zinc since it does, in fact, strain-harden. Currently there are three types of strain-hardening theories which may be useful. These are:

1. Isotropic hardening theories;
2. Kinematic hardening theories; and
3. Theories using piecewise linear loading functions.

The application of these theories to the plastic behavior of a thin tube subjected to tension and torsion will be the subject of a forthcoming paper⁴ by two of the authors of this report (R.M.H. and D.R.J.). Thus this subject will not be treated in detail here. However, certain general observations may be useful.

In isotropic strain-hardening¹¹ the yield surface expands during plastic flow with the surface maintaining a symmetrical position with respect to the origin. It is usually assumed that the form of the yield surface is unchanged. Thus the initial yield surface would be of the above form and subsequent yield surfaces

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after hardening has occurred would be

$$F(\sigma_{ij}) = C \gg k^2$$

The flow rule becomes

$$\dot{\epsilon}_{ij} = Q \cdot \frac{\partial F}{\partial \sigma_{ij}} \cdot \frac{\partial F}{\partial \sigma_{kl}} \cdot \dot{\sigma}_{kl}$$

Note that this may be regarded as a general form for the stress strain law. The above equation is equivalent to the assumption of a linear relation between stress increments and strain increments.

The kinematic hardening theory proposed by Prager^{12,13} states that the yield surface after plastic flow becomes

$$F(\sigma_{ij} - \alpha_{ij}) = k^2$$

where α_{ij} represents the translation of the yield surface in stress space. It is further assumed that the yield surface moves in the direction of the strain increment vector in stress space. Here the flow rule becomes¹³

$$\dot{\epsilon}_{ij} = \frac{1}{c} \cdot \frac{\frac{\partial F}{\partial \sigma_{ij}} \cdot \frac{\partial F}{\partial \sigma_{kl}}}{\frac{\partial F}{\partial \sigma_{mn}} \cdot \frac{\partial F}{\partial \sigma_{mn}}} \dot{\sigma}_{kl}$$

Again there is linearity between stress increments and strain increments.

Here we are primarily interested in the initial yield function. Since all stresses are zero except σ_z (axial stress) and $\tau_{\theta z}$ (shearing stress due to torsion) in the thin tubular specimen under axial force and torque about the tube axis, the von Mises⁹ yield criterion becomes

$$\sigma_z^2 + 3\tau_{\theta z}^2 = \sigma_0^2$$

and the Tresca¹⁰ yield criterion becomes

$$\sigma_z^2 + 4\tau_{\theta z}^2 = \sigma_0^2$$

In both cases, σ_0 is the yield stress in pure tension loading. Thus if the experimental results are plotted as shearing stress against axial normal stress, a comparison with the theoretical predictions can be made.

To make the above comparison, it is necessary to establish a definition of yielding for tests in combined tension (or compression) and torsion. The two definitions used here are based on the so-called "effective" stress—"effective" strain curves. "Effective" stress is defined as

$$\bar{\sigma} = \sqrt{\frac{3}{2} S_{ij} \cdot S_{ij}}$$

where S_{ij} is the stress deviator tensor. For the tube under tension and torsion

$$\bar{\sigma} = \sqrt{\sigma_z^2 + 3\tau_{\theta z}^2}$$

The definition of "effective" strain is

$$\bar{\epsilon} = \sqrt{\frac{2}{3} \epsilon_{ij} \cdot \epsilon_{ij}}$$

in general and for the tube is

$$\bar{\epsilon} = \sqrt{\epsilon_z^2 + \frac{1}{3} \gamma_{\theta z}^2}$$

The "effective" stress-strain curve derives from another strain-hardening hypothesis as follows:¹¹

$$\bar{\sigma} = H \int d\bar{\epsilon}^p$$

where $d\bar{\epsilon}^p$ is the "effective" plastic strain increment

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$$\bar{d}\epsilon^p = \sqrt{\frac{2}{3}} d\epsilon_{ij}^p \cdot d\epsilon_{ij}^p$$

and H is a function depending on the material. The integral represents the "effective" strain over the actual strain path. Now if the principal axes of strain do not rotate and if the components of the strain increment remain in constant ratio, then

$$\int \bar{d}\epsilon^p = \bar{\epsilon}^p \approx \bar{\epsilon}$$

Strictly speaking, this equation holds only for plastic strains. With the above restrictions

$$\bar{\sigma} = H(\bar{\epsilon}) .$$

Note that the definition of $\bar{\sigma}$ is directly related to the von Mises⁹ yield condition. Thus, if all the experimental data plotted on a single "effective" stress—"effective" strain curve, the conclusion would be that the von Mises yield condition was applicable.

One definition of yielding is the stress at 0.002 plastic strain as determined from the "effective" stress—"effective" strain curve. The second definition is that yielding or further flow occurs at a given ratio of stress increment to strain increment. On the $\bar{\sigma} - \bar{\epsilon}$ curves this represents a given value of tangent modulus.

In the preceding it was assumed that the hydrostatic component of the state of stress (i.e., the mean normal stress) has no influence on yielding. However, in those cases where yielding is followed by a brittle fracture, there may, in fact, be a discernible influence of the hydrostatic component of stress, and yield conditions such as that of von Mises⁹ or of Tresca¹⁰ may give poor predictions.

Theories of fracture, in contrast to yielding, were discussed in the previous technical report.¹ Although macroscopic theories of fracture are not so well defined as are those for yielding, it does seem reasonable to expect that brittle fracture may occur when the maximum normal stress reaches a limiting value. The previous discussion also indicated that the ratio of maximum normal stress to maximum shearing stress could influence the limiting value. This criterion of fracture, ignoring the influence of normal stress to shearing stress ratio, is

$$\sigma_1 = \sigma_R$$

where σ_R is the stress at rupture (brittle) in, say, a pure tension test. For the thin tube subjected to axial force and torque, this becomes

$$\sigma_1 = \frac{1}{2} \sigma_z + \sqrt{\frac{1}{4} \sigma_z^2 + \tau_{\theta z}^2} = \sigma_R$$

or

$$\sigma_R \sigma_z + \tau_{\theta z}^2 - \sigma_R^2 = 0$$

Experimental fracture stresses may be compared to the predictions of this theory on a plot of shearing stress against normal stress.

EXPERIMENTAL RESULTS AND CONCLUSIONS

Figure 4 presents the results for yielding and fracture under combined tension and torsion. The circles represent combinations of normal stress and shearing stress which correspond to the effective stress at a plastic strain of 0.002 for that test. The triangles represent combinations of normal stress and shearing stress corresponding to the effective stress at a "tangent modulus" of 2×10^6 psi. "Effective" stress—"effective" strain curves, as defined in the previous section, were used in obtaining these data. The crosses refer to stress combinations representing brittle fracture in all cases. All these data are for a temperature of 32°F.

Consider first the data for fracture. In all these fractures there is no large localized deformation in the immediate proximity of the rupture and, in fact, there is no discernible reduction in thickness of the tube at the edge of the rupture. Thus, the "necking" phenomenon is nonexistent in this material. It is then quite easy to compute stresses from the measured loads at fracture, and it is expected that a reliable fracture stress has been found, at least within the accuracy of the measured data.

Figure 4 presents curves representing the maximum normal stress criterion for a tensile rupture stress of 27,500 psi and 22,000 psi. The first of these ($\sigma_R = 27,500$ psi) is based on the average rupture stress from two pure tension tests. It will be noted that the theoretical curve lies considerably above the combined stress fracture points. The second curve ($\sigma_R = 22,000$ psi) was arbitrarily drawn to fit at least some of the experimental points. Although in any case the agreement with the theory is only fair, it does appear that a greater deviation occurs as the ratio of shearing stress to normal stress increases. Thus, as observed ear-

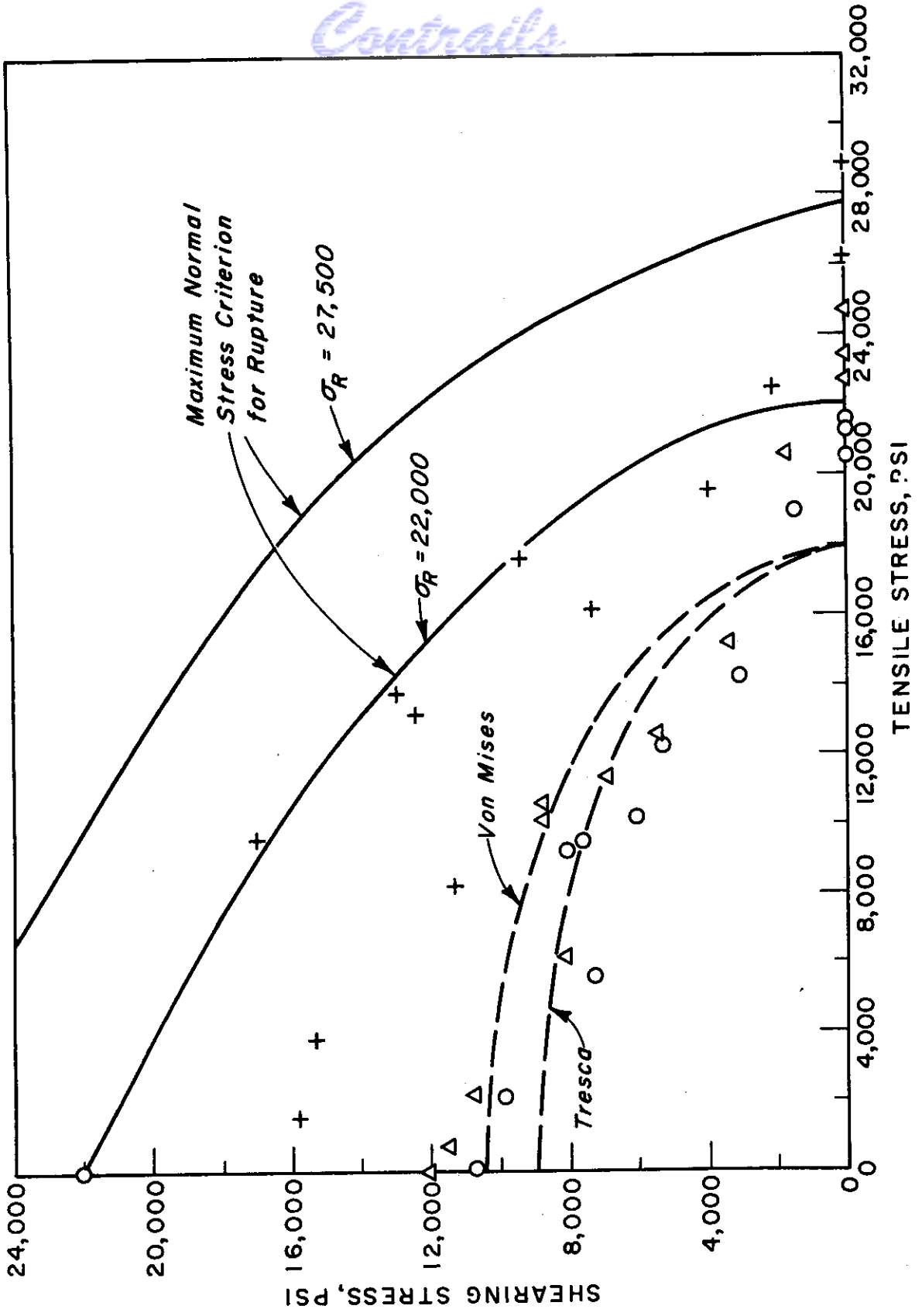


Figure 4. Yielding and Fracture Under Combined Tension and Torsion.

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lier, some correction to the maximum normal stress criterion based on the ratio of maximum shearing stress to maximum normal stress might improve the theoretical predictions. The large degree of scatter in the experimental results perhaps masks the real variation in these quantities.

There are several factors which may contribute to the scatter in the fracture stresses and to the apparent disagreement between combined loading results and the simple tension results. The first factor is, of course, variation in the material itself. However, the variation here appears to be greater than that indicated in Figure 1. A second factor is inaccuracy in measuring loads. Such inaccuracy might be introduced by friction in the bearing which lies between the test specimen and the dynamometer bar of the biaxial loading machine.¹ A third factor might be variation in strain rate at fracture. The rate of load application is controlled in the biaxial loading machine so that the stress rates are nearly uniform, but strain rates in the plastic range are not uniform. In fact, the strain rate increases during a test as the slope of the stress-strain curve decreases.

Figure 5 presents the typical fracture appearance of a pure tension specimen and several combined loading specimens. Note that, in all cases of combined loading, there are two active fracture paths. One of these is essentially circumferential and the other follows a helical path. It would be expected that the helical path might at all times be perpendicular to the direction of maximum normal (principal) stress. No effort has been made to determine whether fracture initiated along one path or along both simultaneously.

Now consider the data for yielding. Both von Mises and Tresca curves are shown for a tensile yield stress of 18,000 psi. It will be noted that the tensile yield stress observed by test is not in agreement with this assumed value. The observed yield stress is in the range 20,500 psi to 21,600 psi for the 0.002 plastic strain definition and is in the range of from 22,700 psi to 24,700 psi for the tangent modulus equaling 2×10^6 psi definition. Obviously von Mises and Tresca yield curves for, say, $\sigma_0 = 21,000$ psi would lie entirely outside the points for combined loading. The degree of scatter is again sufficiently large to obviate any conclusion regarding agreement with a particular theory of yielding.

As discussed above, no doubt several factors contribute to the scatter in yielding results and to the disagreement between combined loading results and the simple tension results. The first two factors may be the same as before, i.e., variation in material and inaccuracy in measuring loads. However, the range of strain-rates observed in the region of yielding is fairly limited, the "effective" strain rate ranging from 0.003 inches per inch per minute to 0.0003 inches per inch per minute. The data of Figure 1 would suggest that this amount of variation has little effect on the response of the material.

Similarly, the "effective" stress—"effective" strain curves for the tests under combined loading did not coincide but diverged as the "effective" strain

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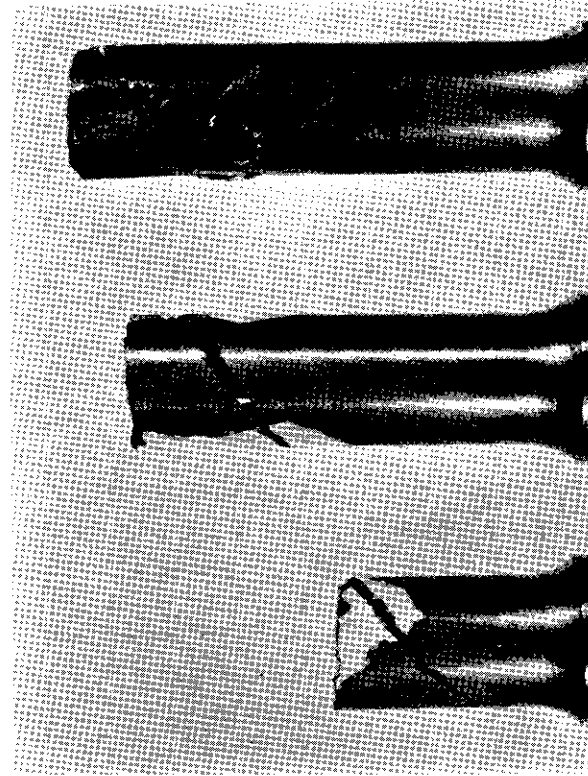
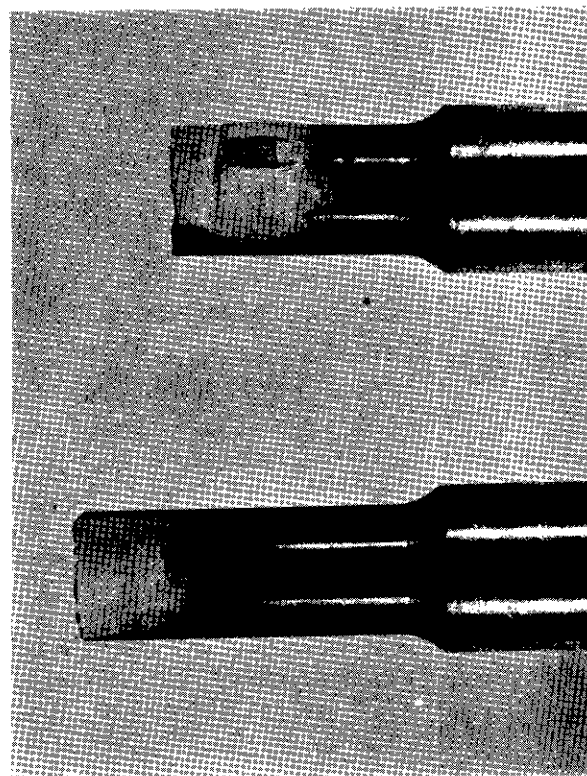
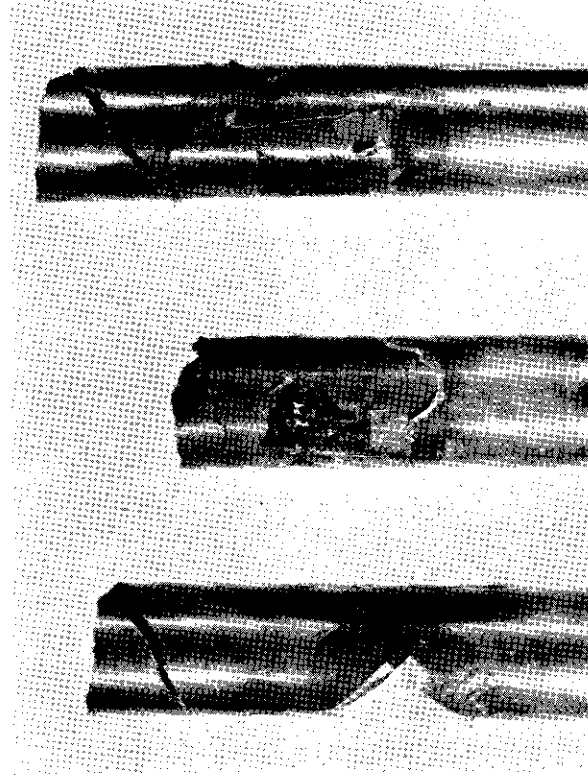
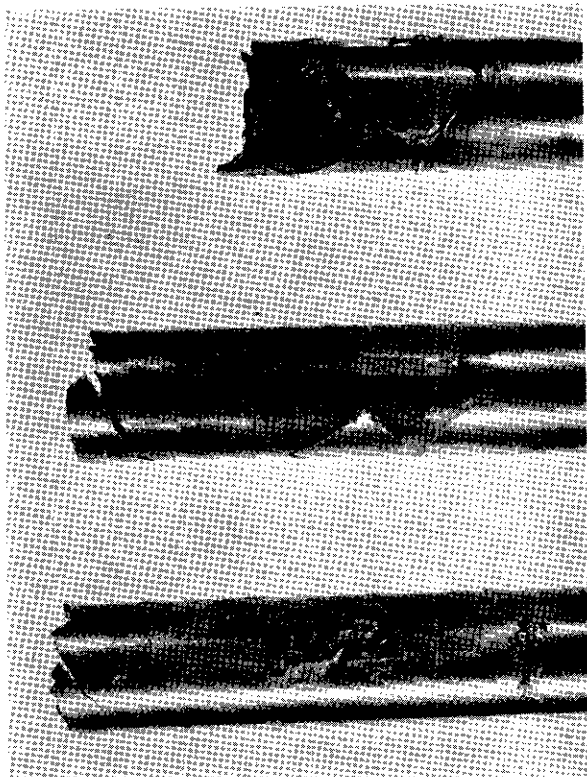


Figure 5. Typical Fractures in Zamak-3 Alloy.

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increased, thus suggesting that the von Mises criterion may not predict yielding in Zamak-3. This divergence depends, in addition to the above, on strain measurement and the attendant assumptions. It was assumed that there was no volume change in the material and that the radial and circumferential strains were equal. With these assumptions, all the desired strain data could be obtained from two measured strains, i.e., an axial strain and one at 45° to the tube axis. The assumption of zero volume change is an integral part of all theory of plasticity and is in accord with most experimental data. One set of experimental data on Zamak-3 indicates no plastic volume change, but more data are needed. The assumption of equal radial and circumferential strains is physically reasonable in view of the flow rule.

The research conducted during the past year appears to support the following conclusions:

1. Zamak-3 exhibits ductile behavior for some stress states and brittle behavior for others at temperatures in the range from -8°F to 80°F .
2. The biaxial loading machine and appurtenant recording equipment appear to furnish the necessary data for the program but some changes, e.g., load sensing, must be made.
3. Although scatter in the results may mask the trends, it appears that simple theories of yielding and fracture may prove to be inaccurate in describing the behavior of a relatively brittle material.

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