THE APPLICATION OF THE MATRIX DISPLACEMENT METHOD IN PLANE ELASTO-PLASTIC PROBLEMS*

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A procedure is developed for the analysis of elasto-plastic plane stress and plane strain problems when the stress-strain relations are of the incremental type. The stressed region is divided for purposes of analysis into triangular elements and the deformation is analysed on a step-by-step basis, using the matrix displacement method. The deformational characteristics of a typical yielded element when the loading is increased by a small increment are expressed in terms of simple basic matrices describing the elastic properties, the derivatives of the yield surface with respect to the stress components, the work-hardening properties, and the geometry of the element. Simple applications are described.

I. INTRODUCTION

Plane elasto-plastic problems are analysed in this paper by a procedure in which the stressed region is represented approximately by an assembly of triangular elements between the vertices of which the displacement components are assumed to vary linearly; the compatibility conditions are thus satisfied exactly. In this idealisation, which has been used extensively in linearly elastic problems (References 1 and 2), the stresses within each element are necessarily uniform; consequently there are no complications in the elasto-plastic analysis due to partially yielded elements. The deformation, which is, in general, nonlinear, is analysed by varying the loading in small increments and by performing a linear analysis of the deformation due to each load increment (References 2 and 3), using the matrix displacement method (Reference 4); the relevant properties of the idealisation are then modified before the next load increment is added. In practical applications it is, of course, advisable to repeat the analysis using different sizes of element and of load increment to verify that the required accuracy has been achieved.

Attention is here restricted, for simplicity, to isothermal problems where the displacements are infinitesimal, and where the deformation is stable. The specification of infinitesimal displacements is itself sufficient to confine applications to stable problems when there is no yielding, and when the elastic properties are linear. When the material properties are nonlinear, however, it is also necessary to specify that the stress-strain relations are, themselves, stable. Drucker (Reference 5) demonstrates that, under isothermal conditions, the plastic strain increments in a time-independent material that is stable in this sense are governed by the usual stress-strain relations of the incremental theory of plasticity, which are given in Section 3. The available experimental evidence suggests that the deformation of most metals used for structural purposes is reasonably well represented by such stress-strain relations over the range of strain that is likely to occur in problems of contained plastic deformation. A rather embarrassing exception is provided, however, by the slightly unstable behaviour of mild steel immediately after yielding. Approximate stress-strain relations which are stable for mild steel are, however, sufficiently accurate in many practical applications.

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The relative magnitudes of the plastic strain increments within each yielded element are assumed in this step-by-step analysis to be specified by the appropriate derivatives of the yield surface at the beginning of the corresponding load increment. Furthermore the deformation during each load increment is assumed to be linear. The effect of these assumptions when the stress increments in a typical element are not in the same ratio as the stresses is illustrated in the Appendix. When the stress increments are in the same ratio as the stresses, the first assumption has no effect on the analysis, and the influence of the second assumption may be demonstrated by the following example. Consider a yielded linearly work-hardening element in which plastic flow is governed by the von Mises criterion (see Equation 1). Suppose that the plastic strain components in this element are all increased in the same ratio until some specified state is reached. The stresses after this increase in strain can obviously be found directly from the work-hardening properties of the element. Alternatively the straining process may be divided into equal increments, so that the stresses at the end of each strain increment may be computed from the stresses at the beginning of the increment, using the linearised relationship employed in this paper. If this approximate technique is used it can be demonstrated that the final error in the second invariant of the deviatoric stress components is inversely proportional to the number of strain increments; furthermore it can easily be seen that this error vanishes in the limiting case when the element is perfectly plastic. On the basis of this example and of the results presented in the Appendix it may reasonably be inferred that the approximate linearization technique employed in this paper converges to the appropriate solution if the load increment size is progressively reduced. In practical applications it is, of course, also necessary to verify that the convergence is not affected by rounding errors in the computations.

Triangular elements have been employed previously in plane elasto-plastic problems by Padlog, Huff and Holloway (Reference 6) who, in common with the formulation described here, assume that the relative magnitudes of the plastic strain increments within each element are specified by the appropriate derivatives of the yield surface at the beginning of the corresponding load increment. In their analysis of the deformation due to a small load increment, however, they first obtain a hypothetical stress distribution by assuming that the additional deformation is purely elastic. Assuming then that the corresponding total strain is the true total strain they deduce the size of the plastic strain increments from the work-hardening properties. It can easily be seen that the error per load increment in this process gets less as the load increments are made smaller, but it has not been established that the accumulated error necessarily decreases consistently.

2 NOTATION

x_i , u_i	Cartesian coordinates and the corresponding displacement components
σ_{ij}	stress components
σ'_{ij}	deviatoric components of stress
ϵ_{ij}	strain components
δ_{ij}	Kronecker delta
f	yield function
E , ν	Young's modulus and Poisson's ratio
h	thickness of a typical element

λ		see Equation 2
N_q		see Equation 10
L_q		see Equation 24
$\{\ \}$		column matrix
S		see Equation 5
v		see Equation 9
k		see Equation 28
J		see Equation 29
σ_q	=	$\{\sigma_{11q}\ \sigma_{22q}\ \sigma_{12q}\}$
ϵ_q	=	$\{\epsilon_{11q}\ \epsilon_{22q}\ \epsilon_{12q}\}$
T_q		see Equation 8
M_q		see Equations 15 and 16
m_q, p_q		see Equation 26
Φ_q		see Equation 18
Ψ_q		see Equation 21
I		unit diagonal matrix

Superscripts

*	denotes that the components in the x_3 direction are incorporated
e	elastic components
p	plastic component
t	transpose of matrix

Subscripts

q	scalars and matrices referring to a typical triangular element q
o	values before current load increment was applied

3 STRESS-STRAIN RELATIONS

The properties of the incremental stress-strain relations of an elasto-plastic solid are now briefly described. The reader is referred to Hill (Reference 7) for a general account of the theory of plasticity and to Green and Naghdi (Reference 8) for an account of more recent developments in the theory of elasto-plastic solids; Hill also describes the suffix notation and summation convention used in this Section.

The state of stress in an infinitesimal element in a continuous medium may be specified by a nine component stress tensor σ_{ij} and may thus be represented by a point P in a nine dimensional stress space, the coordinates of which are given by the stress components. It should, however, be noted that the stress tensor is symmetrical (i.e. $\sigma_{ij} = \sigma_{ji}$) and that consequently only six of the stress components are independent.

The yield condition may be represented in the nine-dimensional space by a yield surface which is convex and which is given by

$$f(\sigma_{ij}) = 0$$

where the function f is symmetrical with respect to σ_{ij} and σ_{ji} and is, in general, a function of the loading history. It is assumed here, for simplicity, that the yield surface is everywhere smooth.

Only a limited amount of information is available on the yield surfaces of practical structural materials. A reasonable approximation to the behaviour of many metals is, however, obtained by assuming that yielding is independent of the hydrostatic component of stress, and is governed by the second invariant of the deviatoric stress components in such a way that the yield surface is given by

$$f = \frac{1}{2}\sigma'_{ij}\sigma'_{ij} - k^2 = 0 \tag{1}$$

where k is a function of the strain history and where the deviatoric stress components σ'_{ij} are given by

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

where

$$\delta_{ij} = 1,\ i = j;\quad \delta_{ij} = 0,\ i \neq j\ .$$

Equation 1 is generally known as the von Mises yield criterion.

Returning now to the deformation of a solid with an unspecified yield surface, let us again consider the stressed state represented by a point P in stress space. When P lies within the region bounded by the yield surface f = 0, the strain increment due to any infinitesimal increment of stress is purely elastic. When, however, the point P lies on the yield surface, the strain increment is purely elastic only if the stress increment $d\sigma_{ij}$ has a positive component in the direction of the inward normal to the yield surface; when $d\sigma_{ij}$ does not have a positive increment in this direction the solid yields and the plastic strain increments are given by

$$d\epsilon^{p}_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}}\ ; \tag{2}$$

the total strain components are defined by

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) \tag{3}$$

where u_i and x_i denote respectively the displacement components and the corresponding Cartesian coordinates. The factor λ is independent of all the components of $d\sigma_{ij}$ except for the component in the direction normal to the yield surface. In a perfectly plastic solid the yield surface is independent of the load history; the normal component of $d\sigma_{ij}$ is then necessarily zero and λ is not specified by the stress-strain relations.

It is assumed in this paper that changes in the yield surface during deformation depend on the plastic strain history only. If the yield condition in a work-hardening solid of this type is given at some specific instant by $f = 0$, the yield condition after a further infinitesimal increment of plastic strain is given by

$$f + t_{ij}\, d\epsilon^{p}_{ij} + \frac{\partial f}{\partial \sigma_{ij}}\, d\sigma_{ij} = 0$$

where the function t_{ij} describes the work-hardening properties.

Hence

$$t_{ij}\, d\epsilon^{p}_{ij} = - \frac{\partial f}{\partial \sigma_{ij}}\, d\sigma_{ij} \; .$$

The functions f and t_{ij} depend not only on the current values of the stress and plastic strain components but also on the loading process by which the current state was reached.

It should be noted that plastic deformation may influence the elastic stress-strain relations. For example, cold working may introduce anisotropic elastic properties in a material which is isotropic in the annealed state. The above stress-strain relations are, however, only synonymous with stability when the elastic properties are independent of the plastic strain.

4 ANALYSIS

4.1 Outline of method

An analysis of elasto-plastic deformation must necessarily start at some instant at which the material properties, the stresses and the strains are known within each of the finite elements used to represent the stressed region. Local plastic deformation occurs when the load system is modified in such a way that positive work is done on at least one element in which the stress components satisfy the yield condition. When the region has nonlinear properties due either to plastic or nonlinear elastic behaviour, the loading process is divided into a series of increments over each of which the deformation is assumed to be linear. It is usually advisable to verify that these increments are sufficiently small by repeating the analysis with different increment sizes. Now the common boundaries of the elastic and plastic regions cannot move continuously during the loading process because the stresses are uniform in each of the elements. The deformation due to a typical load increment is therefore analysed tentatively on the assumption that there is no corresponding movement of these boundaries. If it is then found that the plastic strain increment factor λ (Equation 2) is negative in any element which has been assumed to yield, the analysis is repeated with this element added to the purely elastic region. If, on the other hand, the mean stresses over the load increment in a supposedly unyielded element correspond to a point in stress space either on or outside the yield surface, the analysis is repeated with this element permitted to deform plastically. If, however, this point lies inside the yield surface and the corresponding point at the end of the load increment lies either on or just outside the yield surface, the analysis is left unaltered but the element is allowed to deform plastically if further positive load increments are added. Now if two elements yield during a single load increment, it is always possible that the inaccurate representation of the element that yields first is the sole reason for the yielding of the second element; the size of the load increments should therefore be adjusted, if possible, in such a way that one element at most yields during each increment. Similar precautions are also advisable when the loading is so modified that some but not all of the yielded elements cease to deform plastically.

4.2 Deformational Properties of a Typical Triangular Element

A derivation is now given of the deformational properties of a typical triangular element in a state of plane stress or plane strain. The material behaviour may be nonlinear due either to yielding or to nonlinear elastic properties, or to both simultaneously; the elastic constants may themselves be functions of the plastic strain history. A uniform state of stress and strain is enforced within the element by the linear variation of the displacement components which is prescribed along the edges, and hence the compatibility and the internal equilibrium conditions are satisfied exactly. The deformational properties may thus be derived directly from these conditions using a generalisation of the derivation given by Turner et al (Reference 1) for linearly elastic elements.

A typical triangular element q lies in the (x_1, x_2) plane and has vertices at the points (x_{1a}, x_{2a}), (x_{1b}, x_{2b}) and (x_{1c}, x_{2c}). The forces acting on the element are expressed in terms of concentrated force components at the vertices, parallel to the reference axes, as shown in Figure 2. These force components are represented in the analysis by the following column matrix

$$\mathbf{S}_q = \left\{ S_{q1a}\, S_{q2a}\, S_{q1b}\, S_{q2b}\, S_{q1c}\, S_{q2c} \right\}. \tag{5}$$

The stresses are expressed in terms of the forces applied at the vertices by assuming that these forces are in equilibrium with the stress resultants on the sides of the triangle obtained by joining the mid-points of the sides of the element.

The forces on the vertices may thus be related to the stresses in the element by the following matrix equation:

$$\mathbf{S}_q = \frac{h}{2}\, \mathbf{T}_q\, \boldsymbol{\sigma}_q \tag{6}$$

where h is the thickness of the element,

$$\boldsymbol{\sigma}_q = \left\{ \sigma_{11q}\, \sigma_{22q}\, \sigma_{12q} \right\} \tag{7}$$

and

$$\mathbf{T}_q = \begin{bmatrix} x_{2bc} & 0 & -x_{1bc} \\ 0 & -x_{1bc} & x_{2bc} \\ x_{2ca} & 0 & -x_{1ca} \\ 0 & -x_{1ca} & x_{2ca} \\ x_{2ab} & 0 & -x_{1ab} \\ 0 & -x_{1ab} & x_{2ab} \end{bmatrix} \tag{8}$$

where

$$x_{1bc} = x_{1b} - x_{1c}, \text{ etc.}$$

It should be noted that Equation 6 satisfied implicitly the overall equilibrium conditions on the element.

The displacements of the vertices are represented by a column matrix $\mathbf{v}_q$ corresponding to the colume matrix $\mathbf{S}_q$ of the applied forces, and are related to the total strain in the element by the following equations:

$$\boldsymbol{\epsilon}_q = \left\{ \epsilon_{11q} \; \epsilon_{22q} \; 2\epsilon_{12q} \right\} = N_q \, T_q^t \, v_q \tag{9}$$

where

$$N_q = \frac{1}{x_{1a} x_{2b} + x_{1b} x_{2c} + x_{1c} x_{2a} - (x_{2a} x_{1b} + x_{2b} x_{1c} + x_{2c} x_{1a})} \tag{10}$$

and where shear strain is defined by Equation 3. The notation of the subsequent analysis is simplified by including twice the shear strain in the matrices of the various strain components.

We denote by the suffix o symbols relating to some initial loaded state in which the stresses and strains are everywhere known. The total strain after this initial state has been modified by a small load increment is given, in general, by

$$\boldsymbol{\epsilon}_q^* = \boldsymbol{\epsilon}_{qo}^* + \delta \boldsymbol{\epsilon}_q^{*e} + \delta \boldsymbol{\epsilon}_q^{*p} \tag{11}$$

where

$$\boldsymbol{\epsilon}_q^* = \left\{ \boldsymbol{\epsilon}_{11q} \; \boldsymbol{\epsilon}_{22q} \; 2\boldsymbol{\epsilon}_{12q} \; \boldsymbol{\epsilon}_{33q} \right\} \tag{12}$$

and where $\delta \boldsymbol{\epsilon}_q^{*e}$ and $\delta \boldsymbol{\epsilon}_q^{*p}$ are corresponding matrices of the elastic and plastic strain increments.

Provided that there is no significant change in the elastic constants during the load increment, the elastic strain increments are given by

$$\boldsymbol{\sigma}_q^* - \boldsymbol{\sigma}_{qo}^* = \mathbf{M}_{qo} \, \delta \boldsymbol{\epsilon}_q^{*e} \tag{13}$$

where

$$\boldsymbol{\sigma}_{qo}^* = \left\{ \sigma_{11q} \; \sigma_{22q} \; \sigma_{12q} \; \sigma_{33q} \right\} . \tag{14}$$

When the element is in a state of plane stress, the $\mathbf{M}_{qo}$ matrix is written in the form which prescribes σ_{33} zero. Thus, when the elastic behaviour is isotropic, this matrix is given by

$$\mathbf{M}_{qo} = \frac{E_o}{1-\nu_o^2}\begin{bmatrix} 1 & \nu_o & 0 & 0 \\ \nu_o & 1 & 0 & 0 \\ 0 & 0 & \frac{1-\nu_o}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

When, however, the element is in a state of plane strain and plastic flow occurs, the $\mathbf{M}_{qo}$ matrix must be written in the more general form which does not prescribe the stress or strain components normal to the strained plane. When the elastic behaviour is isotropic this more general matrix is given by

$$\mathbf{M}_{qo} = E_o\begin{bmatrix} 1 & -\nu_o & 0 & -\nu_o \\ -\nu_o & 1 & 0 & -\nu_o \\ 0 & 0 & 2(1+\nu_o) & 0 \\ -\nu_o & -\nu_o & 0 & 1 \end{bmatrix}^{-1} \tag{16}$$

If the load increment is such that plastic flow takes place in element q, the plastic strain increments are governed by Equation 2 and are given approximately by

$$\delta\epsilon_q^{*p} = \lambda_q \Phi_{qo} \tag{17}$$

where

$$\Phi_q = \left\{ \frac{\partial f_q}{\partial\sigma_{11q}} \; \frac{\partial f_q}{\partial\sigma_{22q}} \; 2\frac{\partial f_q}{\partial\sigma_{12q}} \; \frac{\partial f_q}{\partial\sigma_{33q}} \right\}. \tag{18}$$

Thus, for example, when yielding is governed by the von Mises criterion, Equation 1, we have

$$\Phi_q = \left\{ \sigma'_{11q} \; \sigma'_{22q} \; 2\sigma'_{12q} \; \sigma'_{33q} \right\}. \tag{19}$$

Provided that the stress increments are small compared with the stresses themselves, the following linearized form of Equation 4 may be used to obtain an approximate relationship between the yield conditions before and after the load increment is added:

$$\Psi_{qo}\,\delta\epsilon_q^{*p} = -\Phi_{qo}^t\,(\sigma_q^* - \sigma_{qo}^*) \tag{20}$$

where

$$\Psi_q = \left[t_{11q} \; t_{22q} \; t_{12q} \; t_{33q} \right] . \qquad (21)$$

It should be noted that the terms in the matrices $\delta\epsilon_q^{*p}$ and Φ_{qo} corresponding to the strain component ϵ_{12q} are each multiplied by a factor of two which is not included in the other terms of these matrices; this factor compensates for the omission from Equation 20 of terms corresponding to the complementary shear strain ϵ_{21q}.

Substituting Equations 11 and 17 in Equation 13 it may be shown that

$$\sigma_q^* = M_{qo}\left(\epsilon_q^* - \epsilon_{qo}^* - \lambda_q \Phi_{qo}\right) + \sigma_{qo}^* . \qquad (22)$$

The following equation for λ_q may now be obtained by substituting Equations 17 and 22 in Equation 20

$$\lambda_q = L_q \Phi_{qo}^t M_{qo}\left(\epsilon_q^* - \epsilon_{qo}^*\right) \qquad (23)$$

where L_q is a scalar given by

$$L_q^{-1} = \Phi_{qo}^t M_{qo} \Phi_{qo} - \Psi_{qo} \Phi_{qo} . \qquad (24)$$

Substituting Equation 23 in Equation 22, we obtain

$$\sigma_q^* = M_{qo}\left(I - L_q \Phi_{qo} \Phi_{qo}^t M_{qo}\right)\left(\epsilon_q^* - \epsilon_{qo}^*\right) + \sigma_{qo}^* \qquad (25)$$

where I denotes a unit diagonal matrix.

We now consider elements in plane stress and plane strain separately to eliminate the stress components σ_{33} and σ_{33o} and the strain components ϵ_{33} and ϵ_{33o} from Equation 25.

Consider first an element in plane stress. The form of the matrix M_{qo} ensures that the final row and the final column of the matrix $M_{qo} \Phi_{qo} \Phi_{qo}^t M_{qo}$ are both zero. We may thus write

$$\sigma_q = \left(m_{qo} - L_q \, p_{qo}\right)\left(\epsilon_q - \epsilon_{qo}\right) + \sigma_{qo} \qquad (26)$$

where m_{qo} and p_{qo} are square matrices which consist of the first three rows and columns of M_{qo} and $M_{qo} \Phi_{qo} \Phi_{qo}^t M_{qo}$ respectively. A further consequence of the form of the matrix M_{qo} is that the derivative of the yield surface with respect to σ_{33q} only enters Equation 26 through the work-hardening term in the constant L_q. Hence, if the element is perfectly plastic, the stress and strain components parallel to the x_3 axis may be omitted from the analysis.

Consider now an element in plane strain. The final columns of the matrices M_{qo} and $M_{qo} \Phi_{qo} \Phi_{qo}^t M_{qo}$ may be omitted from Equation 25 because they are multiplied by $\epsilon_{33q} - \epsilon_{33qo}$ which is zero by definition. Equation 25 may thus be contracted to Equation 26 in the same way as when the element is in plane stress. When, however, the element is in plane strain the derivative of the yield surface with respect to σ_{33q} enters Equation 26, in general, even when the yielded behaviour is perfectly plastic.

The displacements are related to the applied forces by the following Equation which may be derived by substituting Equations 6 and 9 in Equation 26:

$$S_q = \frac{h}{2} T_q \left[\left(m_{qo} - L_q \, P_{qo} \right) \left(N_q \, T_q^t \, v_q - \epsilon_{qo} \right) + \sigma_{qo} \right] . \tag{27}$$

Thus the deformation of the element due to a small increment of loading is equivalent to that of an elastic element with stiffness given by

$$k_q = \frac{hN_q}{2} T_q \left(m_{qo} - L_q \, P_{qo} \right) T_q^t \; . \tag{28}$$

The forces at the vertices necessary to make this hypothetical linearly elastic element conform to its nominal dimensions are then given by

$$J_q = -\frac{h}{2} T_q \left[\left(m_{qo} - L_q \, P_{qo} \right) \epsilon_{qo} - \sigma_{qo} \right] . \tag{29}$$

It should be noted that the matrices k_q and J_q must be recomputed after each load increment.

When the deformation is purely elastic, Equations 28 and 29 reduce to

$$k_q = \frac{hN_q}{2} T_q \, m_{qo} \, T_q^t \, , \qquad J_q = -\frac{h}{2} T_q \left[m_{qo} \, \epsilon_{qo} - \sigma_{qo} \right] .$$

When, in addition, the elastic behaviour is linear,

$$J_q = -\frac{h}{2} T_q \, m_q \, \epsilon_{qo}^p \; .$$

When the yielded behaviour is perfectly plastic,

$$L_q^{-1} = \Phi_{qo}^t \, M_{qo} \, \Phi_{qo}$$

and it may be shown that the matrix

$$\left(m_q - L_q \, P_{qo} \right)$$

is singular. Consequently Equation 25 does not uniquely define the total strains in terms of the stresses in this limiting case, and an isolated element thus reduces, in effect, to a mechanism. This non-unique relationship between stress and total strain in perfectly plastic elements has some interesting consequences. Consider, for example, a singly-connected perfectly plastic region which is represented by a grid of triangular elements. If a very coarse grid is used in which all the nodal points lie on the boundaries, the stresses within the region can be expressed in terms of the external forces applied at the nodal points without reference to the compatibility conditions. The region is thus, in effect, statically determinate and the compatible system of plastic strain increments within the region may have an arbitrary positive amplitude, provided that no restrictions are imposed on the displacements at the boundaries. If, however, the elements forming this grid are subdivided into smaller triangular elements to form a grid which includes nodal points within the plastic region, the idealisation ceases to be statically determinate and the amplitude of the plastic strain increments is established by the additional internal compatibility conditions. This finer grid necessarily gives a better approximation to a region of contained plastic deformation, because the work done in applying a given system of displacements at the boundaries is reduced by any relaxation in the imposed displacement pattern. Now it can be shown that the stress field in a perfectly plastic region is statically determinate in certain circumstances. An adequate finite element idealisation of such a region must, however, include internal nodal points and the consequent redundant properties because, in general, the lines of slip cannot be represented adequately by the very crude statically determinate idealisation which may be obtained fortuitously in such an analysis.

5 ILLUSTRATIVE EXAMPLES

The practicability of the approximate technique described in Section 4 has been demonstrated by analysing a rectangular panel with the geometry shown in Figure 2, under two loading sequences in which yielding occurs. The panel consists of an isotropic sheet with a length/width ratio of two, with uniform edge members intimately attached along the longer sides; the shorter sides are free. The edge members contribute half the cross-sectional area of the panel normal to the longer sides, and are assumed to have negligible stiffness in bending. The elastic behaviour of the panel is linear and the edge members and the sheet have the same Young's modulus, but the initial yield stress of the edge members is 10 percent higher than that of the sheet in uniaxial stress. The yielding of the sheet at any instant is governed by the von Mises criterion, and the work-hardening properties are linear and are such that the yield surface is given by

$$f = \frac{1}{2}\sigma'_{ij}\,\sigma'_{ij} - \frac{1}{3}\sigma_f^2 - \frac{2}{3}\left(\frac{1}{E_t} - \frac{1}{E}\right)^{-1}\int_0^{\epsilon^p_{ij}} \sigma_{ij}\, d\epsilon^p_{ij} = 0 \qquad (30)$$

where σ_f is the initial uniaxial yield stress and where E_t is the tangent modulus for an increase in uniaxial stress after yielding. The sheet thus has elastic-(perfectly plastic) stress-strain relations in the limiting case when $E_t = 0$.

The two load sequences considered are both symmetrical about the centre-lines of the panel, and thus only one quadrant of the panel need be considered in the analyses. The coarse and fine grids of elements which are used, in turn, to represent this quadrant, and which are shown in Figure 3, are both orientated in such a way that the loaded corner is not positioned at the common vertices of two triangular elements. Argyris et al (Reference 2) demonstrate with a simple example that this grid orientation is preferable with the kind of loadings to be discussed here.

The first and more important load sequence consists of equal and monotonically increasing end loads Q applied to the edge members. The distribution of longitudinal stress under this loading when the panel is purely elastic was calculated using both the fine and the coarse grid of elements; the results are shown in Figure 4. The stresses shown in this figure, which correspond to an end load Q_1, say, applied to each edge member, are such that the triangular element at the loaded corner in the fine grid analysis is on the point of yielding. Comparisons with the results of the coarse grid analysis and with those of an approximate analysis in terms of continuous functions, which is given by Mansfield (Reference 9), suggest that the results of the fine grid analysis are sufficiently accurate to give an adequate overall picture of the distribution of longitudinal stress. Now it is obvious that the accuracy of the results could be further improved by employing an even finer grid of elements, so that the higher gradients of stress at the ends of the panel could be represented more accurately. Mansfield has shown, however, that the "exact" elastic stresses at the corners of the panel, which may be deduced on the assumption that all deformations are infinitesimal, correspond, in fact, to a rotation of the corners through an angle of R/2; it is thus evident that no analysis which is based on infinitesimal deformations can be valid in the immediate vicinity of the corners. A discrete element analysis using a very fine grid and incorporating finite displacements on a step-by-step basis would be possible, but the results would be of academic interest only in this application where, in practice, the stress distribution near the corners is likely to be modified by local constructional details.

Consider now the applied loading increased until the ends of the edge members are on the point of yielding; the end load is then 21 percent greater than that when the first element yielded in the fine grid representation of the sheet. The distribution of longitudinal stress after this further increase in load is shown in Figure 5 for elastic-(perfectly plastic) stress-strain relations (i.e. $E_t = 0$). At this stage element I of the coarse grid and elements 1, 2 and 3 of the fine grid have yielded. The saw-tooth shape of the yielded region in the fine grid analysis indicates that it would be necessary to employ much smaller elements to portray the growth of the plastic region adequately. The good overall agreement between the results of the fine and coarse grid analyses suggests, however, that the fine grid analysis gives a reasonable estimate of the effect of plastic deformation on the overall distribution of longitudinal stress.

The influence of work-hardening has been investigated by repeating the fine grid analysis with stress-strain relations in the sheet such that $E_t = E/3$. The resulting distribution of longitudinal stress in the sheet when the ends of the edge members are on the point of yielding is virtually identical with that obtained with elastic-(perfectly plastic) stress-strain relations everywhere except in elements 1 and 2, where the longitudinal stress is increased by 7 and 3-1/2 percent respectively.

The plastic deformation in the above examples was analysed several times using different sizes of load increment; it was found that the results of the fine grid analysis are virtually independent of the size of the load increments provided that the increments applied to each edge member are not larger than about $0.01\ Q_1$. Moreover this maximum increment size is not restricted primarily by the linearized form of the stress-strain relations in the yielded elements, but by the limited number of discrete values of the end load at which yielding can commence in the elements in which the deformation was previously purely elastic.

Now in the foregoing examples the ratios of the stress components in each yielded element vary very little during the loading process. The effect of more significant changes in the stress field in the yielded elements, which is illustrated by the results of the Appendix, has been investigated further by analysing the deformation of the panel due to the following somewhat artificial loading sequence, using the fine grid of elements and assuming elastic-(perfectly plastic) properties in the sheet. End loads Q_1 are first applied to the edge members, as in the previous loading sequence, so that element 1 at the loaded corner is on the point of yielding. These end loads are then held constant while a normal uniformly distributed load $2Q_1/5b$ is applied to the longer sides over a length 3b/4 at each end of the panel, as shown in Figure 6, where 2b denotes the width of the panel. Element 1 continues to deform plastically as this distributed loading is applied, but the stress pattern within it changes to such an extent that the longitudinal stress reduces to about 65 percent of the value when yielding first occurred; element 2 yields when the distributed load has reached about 80 percent of the final value. The results show that, provided this distributed loading is divided into more than about five increments, the final stressed state is virtually independent of the number of increments.

APPENDIX

VERIFICATION OF THE ACCURACY OF THE APPROXIMATE STRESS-STRAIN RELATIONS

The plastic strain increments in a typical element due to a small load increment are assumed in this paper to be governed by the following approximate equations, which are based on Equations 2 and 4:

$$\delta\epsilon^p_{ij} = \lambda\left(\frac{\partial f}{\partial\sigma_{ij}}\right)_o , \tag{31}$$

$$t_{ijo}\,\delta\epsilon^p_{ij} = -\left(\frac{\partial f}{\partial\sigma_{ij}}\right)_o \delta\sigma_{ij} \tag{32}$$

where the suffix o indicates values at the beginning of the load increment. The rate of convergence of the resulting stress-strain relations as the increment size is reduced may be illustrated by considering an element in which the yield surface is given by

$$f = \frac{1}{2}\sigma'_{ij}\,\sigma'_{ij} - \frac{1}{3}\sigma_f^2 - \frac{2}{3}\left(\frac{1}{E_t} - \frac{1}{E}\right)^{-1}\int_0^{\epsilon^p_{ij}} \sigma_{ij}\, d\epsilon^p_{ij} = 0 ; \tag{30}$$

This form of yield surface is discussed in Section 5.

Suppose now that a uniform stress σ_f is applied in the x_1 direction before any work-hardening has taken place, so that the material is on the point of yielding. The direct stress is then maintained constant while a monotonically increasing shear stress σ_{12} is applied to the element. This loading sequence puts a severe test on the approximate stress-strain relations because the stress increments are far from proportional to the total stresses. It may easily be deduced from Equations 30, 31 and 32 that the plastic strain increments in the stressed directions due to a small increment $\delta\sigma_{12}$ in the shear stress are given by

$$\delta\epsilon^p_{11} = \frac{2\sigma_{120}\,\delta\sigma_{12}}{\mu\left(2\sigma_f^2 + 3\sigma_{120}^2\right)} , \qquad \delta\epsilon^p_{12} = \frac{3\sigma_{120}^2\,\delta\sigma_{12}}{\mu\sigma_f\left(2\sigma_f^2 + 3\sigma_{120}^2\right)} ,$$

where

$$\mu\sigma_f = \frac{2}{3}\left(\frac{1}{E_t} - \frac{1}{E}\right)^{-1}.$$

The convergence of the approximate stress-strain relations as the increment size is reduced is illustrated in Figure 7 where the plastic strain components ϵ^p_{11} and ϵ^p_{12} are plotted as functions of the shear stress σ_{12} for a series of different sizes of stress increment.

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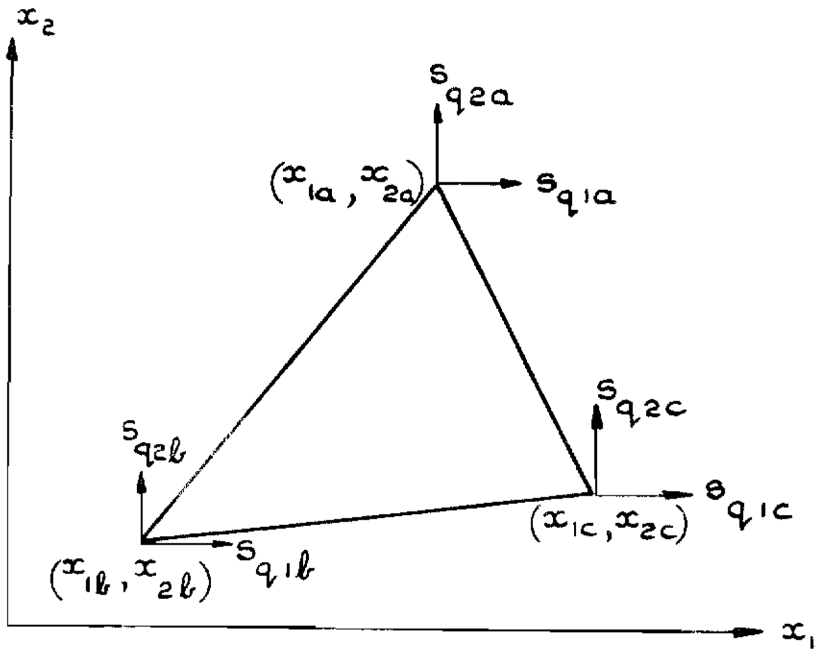


Figure 1. Reference Axes

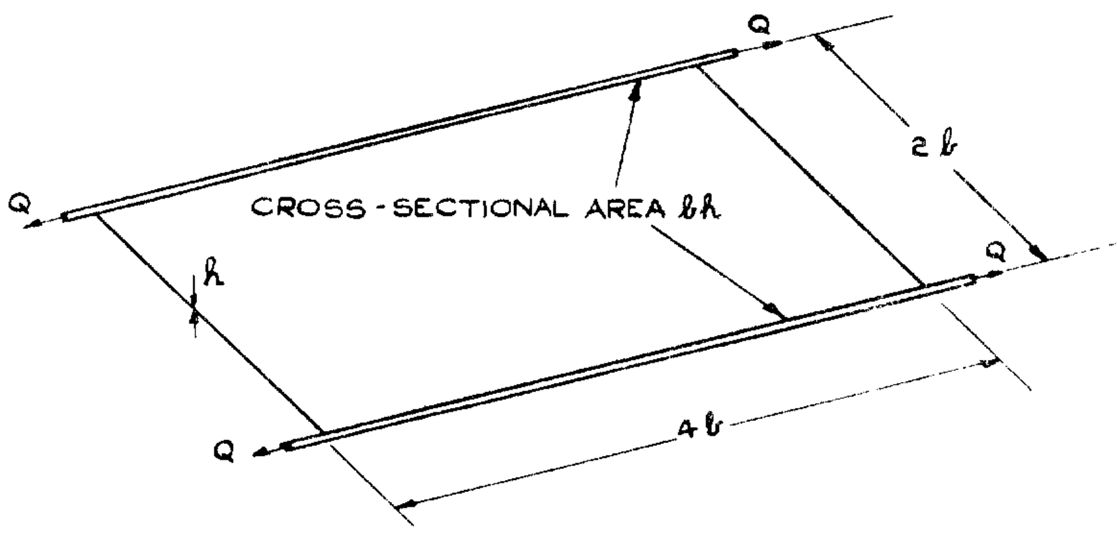
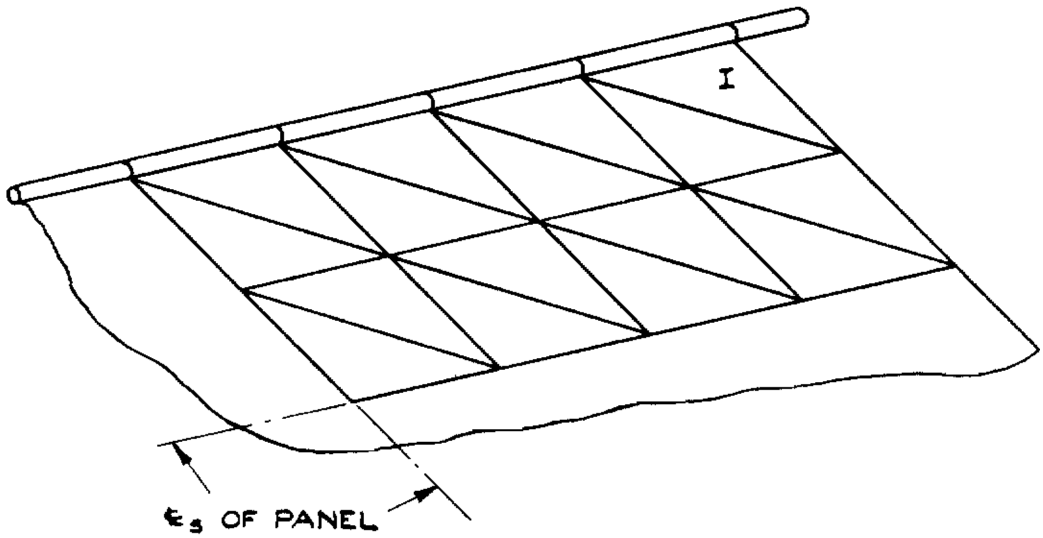
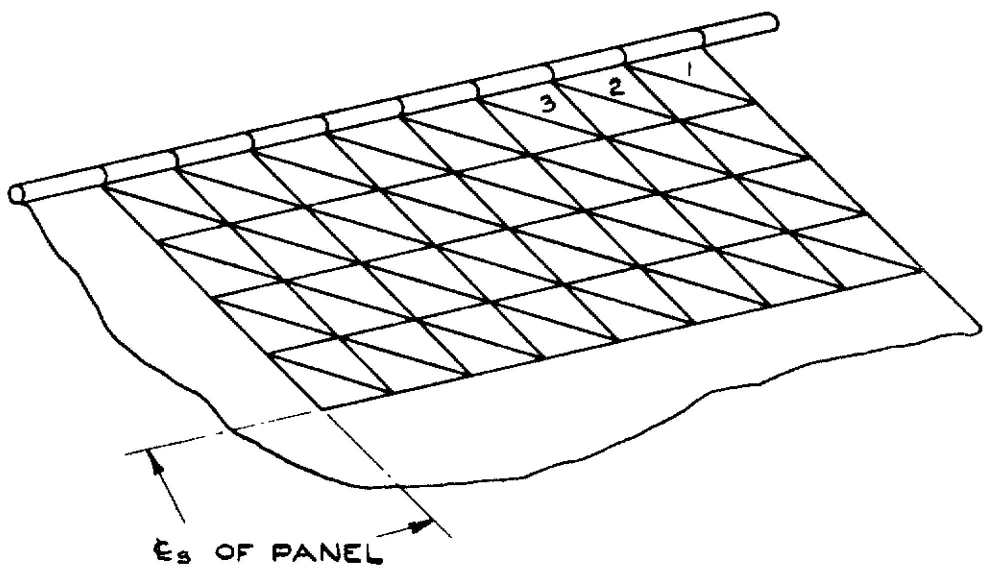


Figure 2. Geometry of Panel



COARSE GRID



FINE GRID

Figure 3. Finite Grid Representations

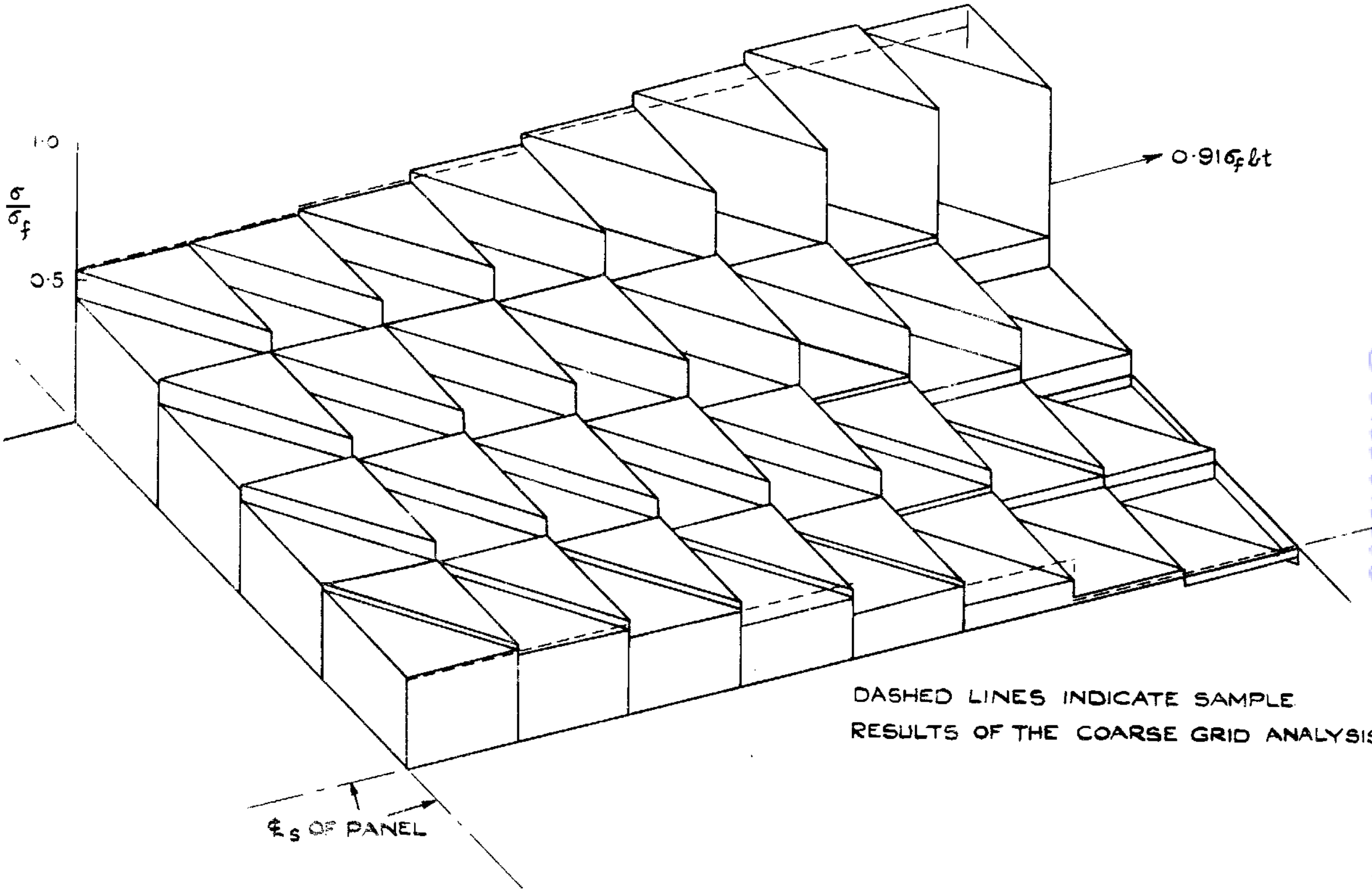


Figure [illegible] Distribution of Longitudinal Stress in Panel When [illegible] Point of Yielding

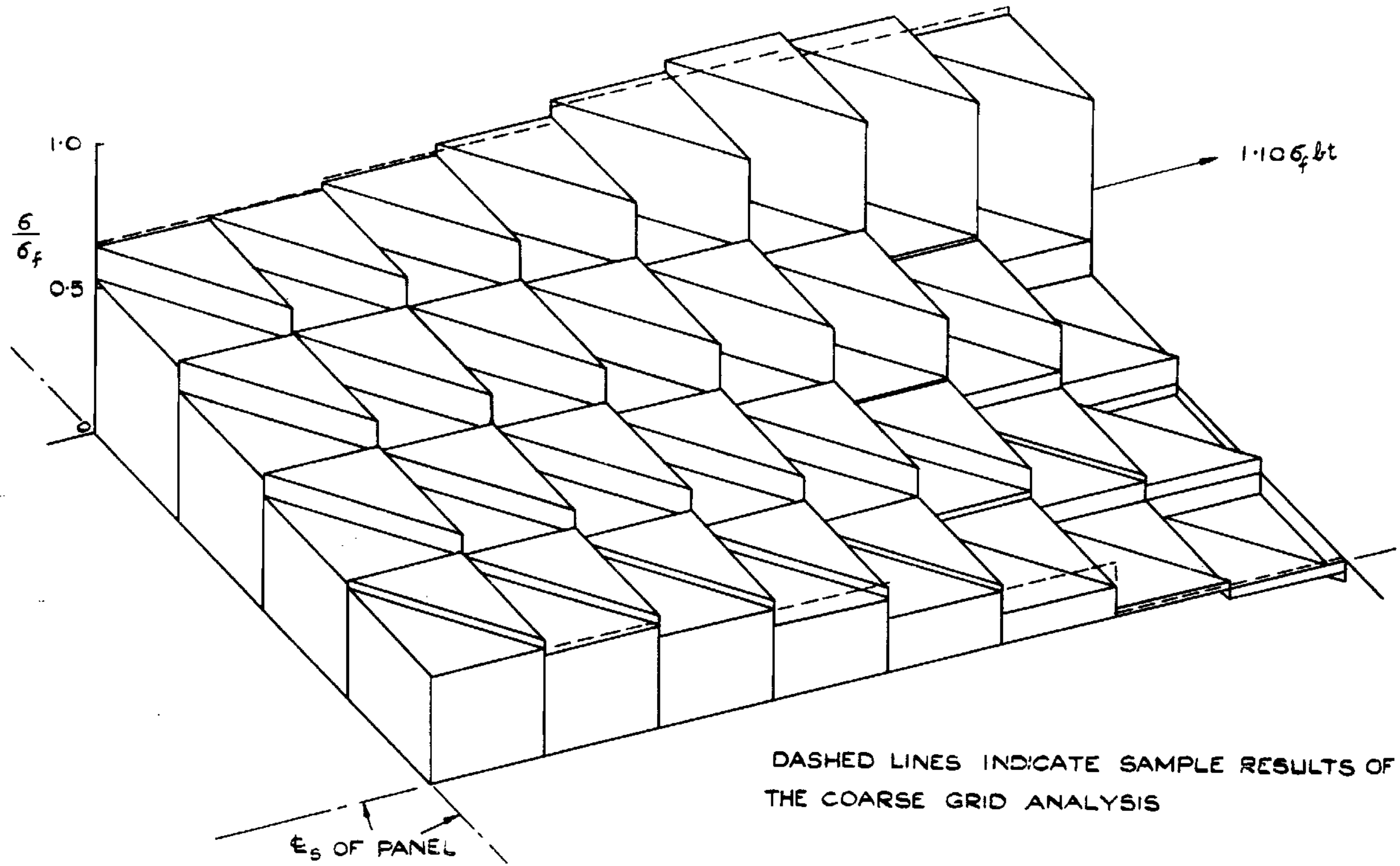


Figure 5. Distribution of Longitudinal Stress in Elastic-(Perfectly Plastic) Panel When Stress in Ends of Edge Member is $1.10\,\sigma_f$

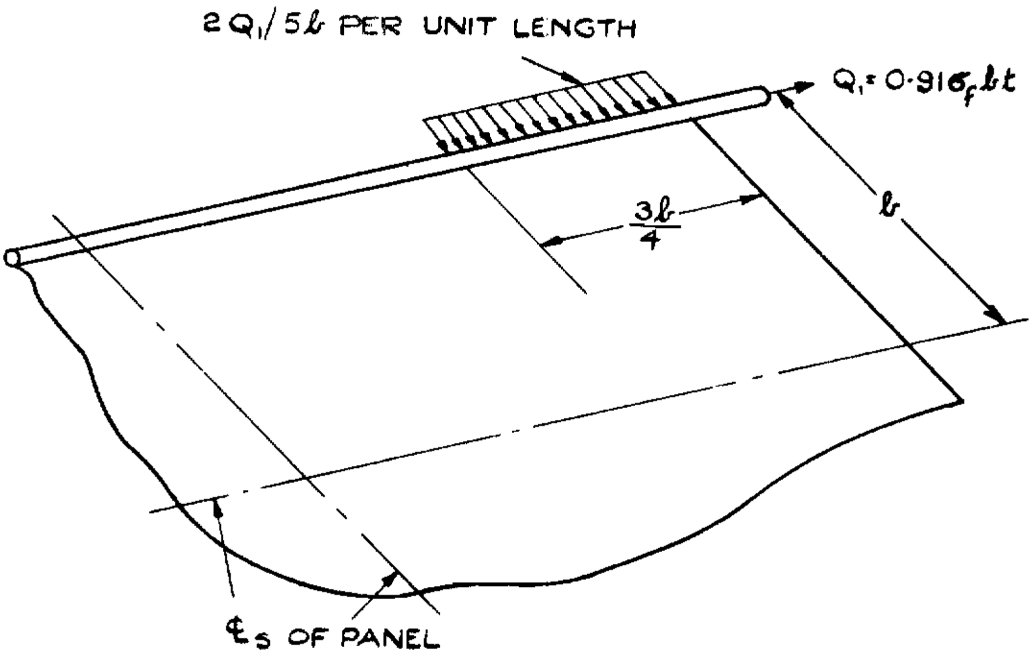


Figure 6. Example to Illustrate the Effect of Nonproportional Loading

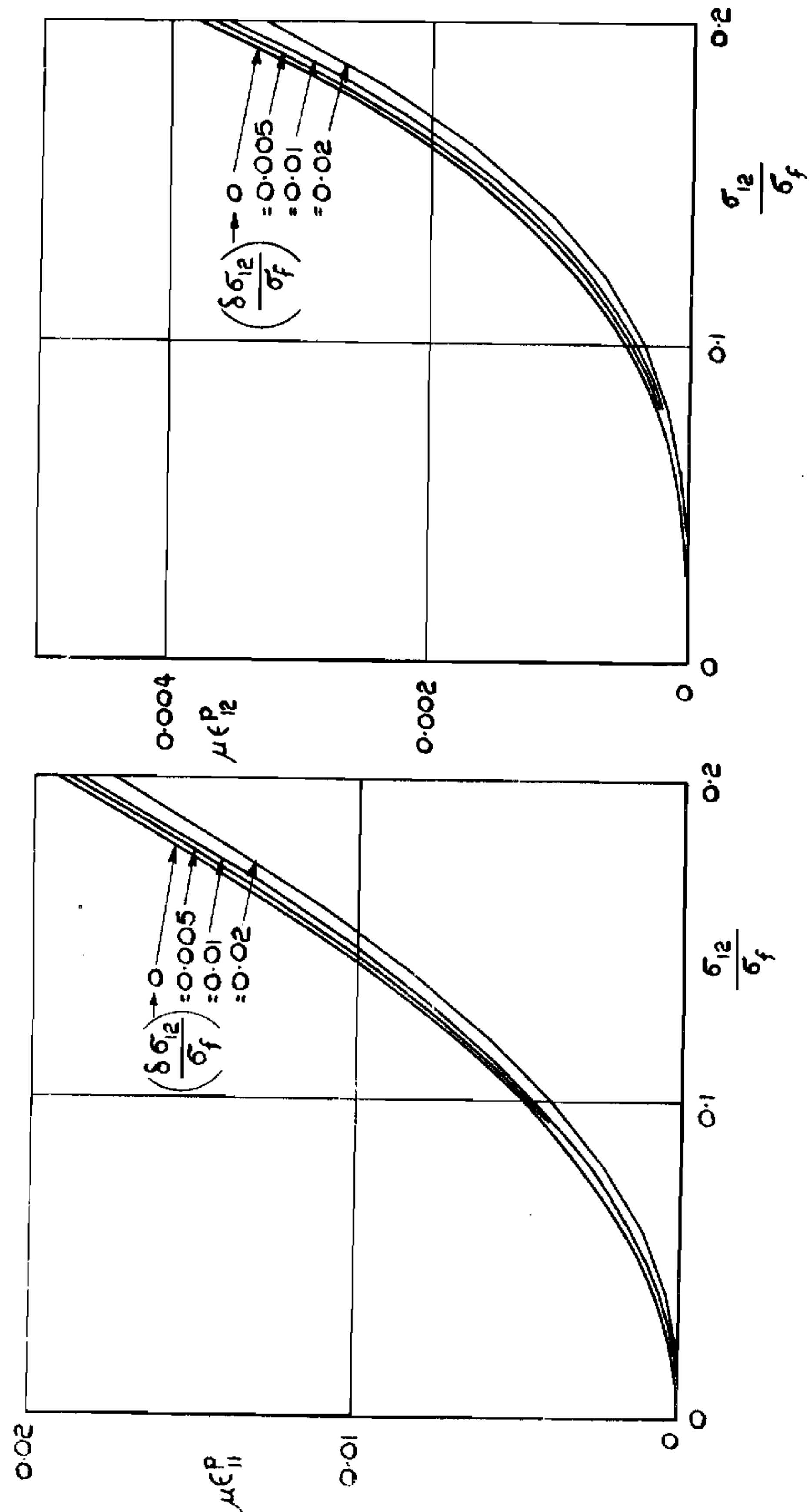


Figure 7. Approximate Stress-(Plastic Strain) Relations