ON THE DERIVATION OF STIFFNESS MATRICES FOR THE ANALYSIS OF LARGE DEFLECTION AND STABILITY PROBLEMS

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The derivation of stiffness matrices necessary for analyzing large deflection and stability problems is developed from basic nonlinear theory. Detailed derivations are given for the stringer, beam-column, and arbitrary thin triangle in plane stress. A suggestion is offered for representing large deflection behavior of the triangular element in bending. A brief account of previous contributions is included.

INTRODUCTION

Development and application of the direct stiffness method to geometrically nonlinear problems has been underway since 1958. Only a portion of the early work has appeared in the general technical literature. Furthermore, the derivations which have been given for stiffness matrices required for the large deflection problem have often been confusing and, in some instances, have led to incorrect conclusions.

Therefore, in this present paper, several goals will be attempted. The first is that of presenting an account of previous work in the subject – from 1958 to 1965. The second is to present a basic and unified approach to deriving the desired stiffness matrices. The suggested theoretical procedure will then be illustrated by application to the axial force member, the beam-column, and the arbitrary triangle in plane stress. Finally, it will be pointed out that currently available theoretical information is sufficient for handling large deflections of the triangular element in bending.

HISTORICAL BACKGROUND

The original work in the subject was presented in 1959 and appeared the next year as Reference 1. This first paper contributed the following: (1) showed that a new class of stiffness matrices had to be introduced if large deflection and stability calculations were to be undertaken; (2) presented derivations for this new stiffness matrix for the axial force member and the arbitrary triangle in plane stress; (3) described the concept of using linear, incremental steps for numerically approximating large deflection behavior.

The new stiffness matrix $\mathbf{K}^{(1)}$ depends on the state of stress existing in the element prior to the imposition of an additional disturbance. Hence, it is termed the "initial stress stiffness matrix." The conventional stiffness matrix is then termed $\mathbf{K}^{(0)}$. Superimposing $\mathbf{K}^{(0)}$ and $\mathbf{K}^{(1)}$ furnishes the total stiffness matrix $\mathbf{K}$.

References 2, 3 and 4 represent some of the work carried out during 1959-60 on the beam-column. Reference 2 is particularly noteworthy. It arrives at the correct $\mathbf{K}^{(1)}$ matrix for the beam-column. Several later publications have given an over-simplified form for this matrix. References 3 and 4 contain some of the early attempts at carrying out large deflection calculations by the direct stiffness method.

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In 1962, a solution was obtained for $K^{(1)}$ for the thin, rectangular element in bending (Reference 5). In this reference it is also shown that, using the stiffness method, excellent results could be obtained for critical compressive stresses on thin plates. It is interesting to note that no corresponding development had appeared for the triangular element. This is not surprising, of course, since derivation of a satisfactory $K^{(0)}$ stiffness matrix for the triangle has been a much more difficult problem than for the case of the rectangle. Basically, the problem is that of choosing a displacement function which will permit a solution to be obtained while, at the same time, satisfying continuity conditions along common edges of adjacent triangles.

Reference 6 was presented at the Agard Structures and Materials Panel in Paris, France, 1962. One of the four main sections in this paper was devoted to large deflection problems. It made clear, for example, how well-known matrix techniques could be applied to the calculation of critical loads once $K^{(0)}$ and $K^{(1)}$ had been assembled. In addition, it suggested a new procedure for deriving $K^{(1)}$ matrices. This new approach was essentially geometric in nature and, in Reference 6, was applied to right triangular element in plane stress. The result for $K^{(1)}$ is not in agreement with that obtained from Reference 1.

In a technical note, Reference 7 takes up the question of column stability from the stiffness method point-of-view. Although details are necessarily kept to a minimum in this note, it does present the same $K^{(1)}$ matrix as found in Reference 2. The derivation, however, is quite different. Reference 7 also shows that excellent results are obtained for the buckling load of a uniform, pin-ended column even when a minimum idealization is used to represent the actual structure.

Reference 8 also discusses structural stability and gives energy expressions for calculating $K^{(1)}$ for the beam-column. Explicit final results are, however, not given in this reference.

Further work on the beam-column is presented in Reference 9. Based on the geometric point-of-view a derivation is given for $K^{(1)}$ for the beam-column. The result differs from that found in References 2 and 7. In fact, this work arrives at a $K^{(1)}$ which is identical with that previously found for the stringer (Reference 1). Reference 9 goes on to show that, by using a number of elements to represent a nonuniform column, excellent buckling loads can be calculated by using the stringer $K^{(1)}$ stiffness matrix. Similar conclusions are reported in Reference 10.

It now becomes evident that a problem of considerable interest has arisen. Two different $K^{(1)}$ matrices have been derived for the beam-column and each has given satisfactory numerical results when applied to column stability problems. Is there then a basis for selecting a $K^{(1)}$ matrix as being correct? Also, what is the significance of the fact that different matrices have led to the same result for column buckling loads?

A useful step toward eventually resolving some of these questions arose during the course of the work reported in Reference 11. Although this reference is primarily concerned with the application of computer programs to nonlinear vibration analysis, it also contains a discussion of basic nonlinear theory as found in Reference 12. In particular, it suggests that fundamental theory be taken into account when basic derivations of $K^{(1)}$ matrices are to be undertaken.

More or less concurrently with the developments leading to Reference 11, a Master's Thesis at the University of Washington undertook the derivation of $K^{(1)}$ for a thin, right triangular element in bending (Reference 13). The results given in this reference are probably of questionable value since a modified cubic equation was selected for representing the bending displacement. Nevertheless, this reference approaches the derivation of $K^{(1)}$ by utilizing certain fundamental expressions from the nonlinear theory of elasticity.

An additional application of the geometric approach to obtaining $K^{(1)}$ is given in Reference 14. The problem undertaken is that of the tetrahedron – a useful element for idealizing solid bodies. Assuming $K^{(1)}$ for the tetrahedron to be necessary, it would appear worthwhile now to re-examine the basis for its derivation. This is indicated by the restricted result obtained for the beam-column when $K^{(1)}$ is obtained from essentially geometric arguments.

Reference 15 again takes up the problem of the beam-column. The end result is the same as given in References 2 and 7. However, Reference 15 refers to $K^{(1)}$ as the "stability coefficient matrix." This term seems unnecessarily restrictive in that it fixes attention on the role of $K^{(1)}$ in stability analyses, but completely overlooks its much larger role in describing the detailed behavior of large deflection problems. Of interest to the structural engineer is a comparison given in Reference 15 for critical column loads as calculated by the direct stiffness method and by finite differences. The superiority of the stiffness method, in terms of accuracy of results based on equivalent representations, is clearly brought out in this reference.

Finally Reference 16 presents an alternative solution to that found in Reference 5 for the case of the rectangular element in bending. A large number of stability calculations are reported in Reference 5. For a wide variety of applied loadings and boundary conditions, it is shown that stiffness method results compare very favorably with theoretically calculated data.

We, therefore, see that a considerable backlog of useful information already exists for applying the stiffness method to large deflection problems. At the same time questions remain which should be answered. The most important of these at this time would seem to be connected with the derivation of the initial stress stiffness matrices. In particular, a straightforward and consistent procedure is desired, which rests on basic theory and which can be applied to any structural element.

THEORETICAL BACKGROUND

No attempt will be made to discuss the stiffness method, or the fundamental equations of the nonlinear theory of elasticity. However, those concepts which are useful in applying the basic theory will be briefly outlined at this time.

First, the large deflection problem is intrinsically different from the small deflection problem. This is so, not because large deflections necessarily occur in a literal sense, but rather because stresses exist which, in the presence of certain displacements, exert a significant influence on structural stiffness. The beam-column illustrates this typical, large deflection behavior. Existence of axial loading in the presence of bending displacement does affect the stiffness of the member. In fact, if the loading is compressive and approaches the critical value, the bending stiffness tends toward zero. Consequently, the need for an "initial stress stiffness matrix" becomes evident.

Two sources of nonlinearity exist for the large deflection problem. The first is connected with the strain-displacement equations. Even if strains remain small in the conventional sense, rotation of the element adds nonlinear terms to the strain-displacement equation. As will be seen in the simple case of the stringer, if these nonlinear, rotational terms are omitted, the derivation becomes incapable of yielding $K^{(1)}$

The second source of nonlinearity exists with respect to the equilibrium equations. It is necessary to keep the deformed geometry in mind when writing the equilibrium equations. This, in turn, causes these equations to become nonlinear. In the stiffness method this is taken into account by the incremental step procedure. The deformed geometry is taken into account at the start of each step. In this manner a close approximation to the actual behavior can be maintained.

It is therefore seen that the stiffness method accounts for both sources of nonlinearity in the large deflection problem. That entering into the derivation of the stiffness matrices through the strain-displacement equations is sufficient for stability analyses. By using the stiffness matrices so derived, in conjunction with the incremental step procedure, corrections in the equilibrium equations due to structural deformation can be taken into account. This makes it possible to carry out a detailed analysis of the large deflection problem.

A particularly convenient procedure for deriving stiffness matrices is to write the strain energy U in terms of nodal displacements u_i . An application of Castigliano's first theorem then gives the stiffness coefficient k_{ij} as

$$k_{ij} = \frac{\partial^2 U}{\partial u_i \quad \partial u_j} \tag{1}$$

Equation 1 need not be carried out in detail provided U can be expressed in quadratic form as follows:

$$U = \frac{1}{2} \mathbf{u}^T (\mathbf{A}^T \ \mathbf{B} \ \mathbf{A}) \ \mathbf{u} \tag{2}$$

where, $\mathbf{u}^T$ = transpose of $\mathbf{u}$, which is the column of nodal displacements. The stiffness matrix $\mathbf{K}$, whose elements are k_{ij} , is then given by the triple matrix product

$$\mathbf{K} = \mathbf{A}^T \ \mathbf{B} \ \mathbf{A}$$

This procedure will be found to yield the desired stiffness matrices with a minimum of mathematical effort.

AXIAL FORCE MEMBER

We first consider the case of the uniform, constant stress, truss member or stringer. It is assumed to have a cross-sectional area A , modulus of elasticity E , and length L . Nodes 1 and 2 lie at opposite ends of the member.

We concern ourselves with the behavior which takes place during an incremental step. Let ϵ^o be the strain present at the start of the step. As additional deformation takes place, additional strain ϵ^a develops. The total strain ϵ is then simply

$$\epsilon = \epsilon^o + \epsilon^a \tag{3}$$

Total strain energy U may be expressed as

$$\begin{aligned} U &= \frac{1}{2} \iiint \sigma \epsilon \ dx \ dy \ dz \\ &= \frac{1}{2} \ AE \int_0^L \epsilon^2 \ dx \end{aligned}$$

On substituting from Equation 3 we find that

$$\begin{aligned} U &= \frac{1}{2} AEL \ (\epsilon^o)^2 + AE \epsilon^o \int_0^L \epsilon^a \ dx + \frac{1}{2} AE \int_0^L (\epsilon^a)^2 \ dx \\ &= U_0 + U_1 + U_2 \end{aligned} \tag{4}$$

The first term on the right side of Equation 4, U_0, is simply the strain energy present prior to imposition of the additional disturbance. The second term, U_1, depends on the initial stress; hence, it must yield $K^{(1)}$. The third term, U_2, depends on the additional strain; therefore, it is no different from the conventional, small deflection, elastic case. It must yield $K^{(0)}$.

The nonlinear strain-displacement equation is represented by

$$\epsilon^a = \frac{du}{dx} + \frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 \tag{5}$$

When strains are small compared to unity ϵ^a can be taken as the true physical strains. Furthermore we can omit $\left(\frac{du}{dx}\right)^2$ compared to $\left(\frac{du}{dx}\right)$. However, we cannot similarily discard $(dv/dx)^2$. If we do we are omitting the contribution of rotation to ϵ^a and this is precisely the term which must be retained. As mentioned previously, the lowest order rotational term appears as a nonlinear contribution to the strain-displacement equation. In view of these considerations we retain Equation 5 in the simplified form

$$\epsilon^a = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 \tag{6}$$

For this member u_1, v_1, u_2, v_2 represent the relevant nodal displacement components. We choose $u(x)$ and $v(x)$ as linear functions or

$$u(x) = a_0 + a_1 x \qquad v(x) = b_0 + b_1 x \tag{7}$$

This choice provides the necessary constant strain along the length of the member and, at the same time, furnishes as many constants as there are nodal degrees of freedom. The need for the first condition is obvious; the second condition is necessary if the strain energy is to be expressible in terms of nodal displacements.

Writing u and v at each node (Node 1 at $x = y = 0$, Node 2 at $x = L, y = 0$) gives

$$\begin{aligned} a_0 &= u_1 & b_0 &= v_1 \\ a_1 &= \frac{u_2 - u_1}{L} & b &= \frac{v_2 - v_1}{L} \end{aligned} \tag{8}$$

Equation 6 may then be written as

$$\epsilon^a = a_1 + \frac{1}{2} b_1^2 = \frac{u_2 - u_1}{L} + \frac{1}{2}\left(\frac{v_2 - v_1}{L}\right)^2 \tag{9}$$

Substituting Equation 9 into Equation 4 gives the following form for the strain energy:

$$\begin{aligned} U = U_0 &+ AE\epsilon^o \int_0^L \left[\frac{u_2 - u_1}{L} + \frac{1}{2}\left(\frac{v_2 - v_1}{L}\right)^2\right] dx \\ &+ \frac{1}{2} AE \int_0^L \left[\frac{u_2 - u_1}{L} + \frac{1}{2}\left(\frac{v_2 - v_1}{L}\right)^2\right]^2 dx \end{aligned}$$

This last equation can be rewritten. First, we note that the integrands are constant. Second, we use the fact that $AE\epsilon^o = A\sigma^o = P^o$, the initial loading. Third, we only retain quadratic terms in the displacements. Lower order terms do not contribute to **K**, (see Equation 1) and higher terms give **K** as a function of displacements. These terms are dropped, as is also the case in the classical nonlinear theory. We, therefore, retain U as

$$
\begin{aligned}
U &= \frac{1}{2}\,\frac{P^{0}}{L}\,(v_2 - v_1)^2 + \frac{1}{2}\,\frac{AE}{L}\,(u_2 - u_1)^2 \\
&= \frac{1}{2}\,\frac{P^{0}}{L}\begin{bmatrix} v_1 & v_2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \frac{1}{2}\,\frac{AE}{L}\begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\end{aligned}
\tag{10}
$$

Equation 10 conforms to Equation 2. Hence, we have obtained

$$
K^{(1)} = \frac{P^{0}}{L}\begin{array}{c}\begin{array}{cc} v_1 & v_2 \end{array}\\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\end{array} \qquad K^{(0)} = \frac{AE}{L}\begin{array}{c}\begin{array}{cc} u_1 & u_2 \end{array}\\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\end{array}
\tag{11}
$$

It is now a simple matter to expand these stiffness matrices to order 4 x 4. This is done by adding columns and corresponding rows of zeros for the added displacements (as u_1 and u_2 for the $K^{(1)}$ matrix). Finally, a simple matrix transformation will give the stiffness matrix for the element when it is arbitrarily oriented in the xy plane. The final result for $K^{(1)}$ is then the same as that given in Reference 1. As expected, $K^{(0)}$ turns out to be the conventional stiffness matrix for the axial force member. We also observe that if the term $(dv/dx)^2/2$ is omitted in ϵ^{a}, Equation 6, we will never obtain $K^{(1)}$ from the above derivation.

The derivation given above for the stringer illustrates all the features which arise when more complex structural elements are being considered. It therefore represents a useful guide for investigating other, more difficult cases.

Initial stress stiffness matrix $K^{(1)}$, as obtained above for the stringer is now sufficient for analyzing several large deflection problems. Among these are: (1) critical loading for the arbitrary pin-jointed truss; (2) deflection and self-equilibrating internal forces for the self-strained truss; (3) deflection and stiffness of the pretensioned string.

As an example, we consider the stability of the truss shown in Figure 1. A detailed solution along classical lines may be found on page 147 in Reference 17. The critical value for load P is found to be

$$
P_{crit} = \frac{A_d\, E \sin\alpha \cos^2\alpha}{1 + \frac{A_d}{A_v}\sin^3\alpha}
\tag{12}
$$

The notation used in Equation 12 is that of Reference 17.

The stiffness method of solution will now be applied to this same problem. The applicable stiffness matrices, $K^{(0)}$ and $K^{(1)}$, are derivable from Equation 11. For the truss of Figure 1 we obtain

$$
K^{(0)}_{1-3} = \frac{A_d E}{d}\begin{array}{c}\begin{array}{cc} u_3 & v_3 \end{array}\\ \begin{bmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix}\end{array} \qquad K^{(1)}_{1-3} = 0
$$

$$
K^{(0)}_{2-3} = \frac{A_v E}{\ell}\begin{array}{c}\begin{array}{cc} u_3 & v_3 \end{array}\\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\end{array} \qquad K^{(1)}_{2-3} = -\frac{P}{\ell}\begin{array}{c}\begin{array}{cc} u_3 & v_3 \end{array}\\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\end{array}
$$

In writing the above stiffness matrices, boundary conditions have been imposed; hence, only u_3 and v_3 are retained as displacement components. The above matrices may now be superimposed to give the total stiffness matrix **K** . The critical condition is obtained by putting the determinant of **K** equal to zero. This leads, quickly and simply, to the result expressed by Equation 12. The reader should note the simplicity of the stiffness solution compared with that given in Reference 17. More important, the opportunity for readily applying the stiffness procedure to complex problems should be appreciated.

A corresponding calculation, utilizing the incremental step procedure for obtaining a nonlinear force-displacement curve will not be given here. Such calculations are given in Reference 11.

BEAM-COLUMN

This problem has already received considerable attention as pointed out in the historical introduction. We examine it here with the following three purposes in mind: (1) to see whether the derivation procedure used for the stringer will again lead to the end result with a minimum of confusion and calculation detail; (2) to see what $\mathbf{K}^{(1)}$ is obtained and to compare it with results from previous derivations; (3) to explain, if possible, the reasons for the different results which have been found for this structural element.

The physical picture is the same as that for the stringer with the following additions: we add θ_1, and θ_2 as nodal bending slopes and introduce EI as the flexural stiffness factor. The member again lies in the xy plane both before and after deformation takes place. Initially it is oriented along the X-axis.

Equation 3 applies once again. However, Equation 6 must be augmented by including the contribution of bending to the strain expression. This term is known from elementary beam theory. As a result we replace Equation 6 with

$$\epsilon^{a} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 - y\,\frac{d^2 v}{dx^2} \tag{13}$$

The negative sign is used since positive y (measured from the neutral axis in the xz plane) and positive curvature correspond to fibers undergoing compressive bending strain.

The next step is crucial. It consists of selecting displacement functions $u(x)$ and $v(x)$. For bending $v(x)$ must be a cubic. This is necessary since the third derivative (the shear) is then a constant, which is consistent with the nodal force pattern assumed for beam elements (shear and bending moment at the nodes represent the beam loading permitted the beam element). This also gives a second derivative which is linear in x . Hence, the term $y\left(\frac{d^2 v}{dx^2}\right)$ in Equation 13 is seen to fulfill the usual requirements of beam theory for the case of uniform shear along the member. The contribution of $u(x)$ to ϵ^{o} must be the same as for the stringer. On this basis we choose

$$u(x) = a_0 + a_1 x \qquad v(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 \tag{14}$$

As a final check on Equations 14 we now note that they contain a total of six constants. This agrees with the total nodal degrees of freedom (u , v , and θ at each node). Consequently, the problem of expressing U in terms of nodal displacements is now mathematically possible. Hence, having justified our choice of $v(x)$, we are forced to accept $u(x)$ as a linear function.

Strain energy U is now given by a slightly modified form of Equation 4. Since we cannot integrate immediately, we write

$$\begin{aligned} U &= U_0 + U_1 + U_2 \\ &= U_0 + E\epsilon^0 \iiint \epsilon^a \, dx\, dy\, dz + \frac{1}{2} E \iiint (\epsilon^a)^2 \, dx\, dy\, dz \end{aligned} \tag{15}$$

Since U_0 does not contribute to K we drop it at this point. Substituting Equation 13 into U_1 gives

$$\begin{aligned} U_1 &= E\epsilon^0 \iiint \left(\frac{du}{dx} - y \frac{d^2 v}{dx^2} \right) dx\, dy\, dz \\ &\quad + \frac{1}{2} E\epsilon^0 \iiint \left(\frac{dv}{dx} \right)^2 dx\, dy\, dz \end{aligned} \tag{16}$$

In the same manner we find U_2 to be given by

$$\begin{aligned} U_2 &= \frac{1}{2} E \iiint \left[\left(\frac{du}{dx}\right)^2 - 2y \frac{du}{dx} \frac{d^2 v}{dx^2} + y^2 \left(\frac{d^2 v}{dx^2}\right)^2 \right] dx\, dy\, dz \\ &+ \frac{1}{2} E \iiint \left[\frac{1}{4} \left(\frac{dv}{dx}\right)^4 + \frac{du}{dx} \left(\frac{dv}{dx}\right)^2 - y \left(\frac{dv}{dx}\right)^2 \left(\frac{d^2 v}{dx^2}\right) \right] dx\, dy\, dz \end{aligned} \tag{17}$$

These equations must now be rewritten in terms of nodal displacements. However, it is useful to substitute first from Equations 14 as follows:

$$\frac{du}{dx} = a_1 \qquad \frac{dv}{dx} = b_1 + 2b_2 x + 3b_3 x^2$$
$$\frac{d^2 v}{dx^2} = 2b_2 + 6b_3 x \tag{18}$$

With these substitutions U_1 and U_2 may be expressed as follows:

$$\begin{aligned} U_1 &= E\epsilon^0 \iiint \left[a_1 - y \left(2b_2 + 6b_3 x \right) \right] dx\, dy\, dz \\ &+ \frac{1}{2} P^0 \int_0^L \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} 1 & & \text{sym} \\ 2x & 4x^2 & \\ 3x^2 & 6x^3 & 9x^4 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} dx \end{aligned} \tag{19}$$

$$\begin{aligned} U_2 &= \tfrac{1}{2}E\iiint \left[a_1\ b_2\ b_3\right]\begin{bmatrix}1 & & \underline{\text{sym}}\\ -2y & 4y^2 & \\ -6xy & 12xy^2 & 36x^2y^2\end{bmatrix}\begin{bmatrix}a_1\\ b_2\\ b_3\end{bmatrix} dx\ dy\ dz \\ &+ \tfrac{1}{2}E\iiint\Big[\tfrac{1}{4}(b_1+2b_2x+3b_3x^2)^4 + a_1(b_1+2b_2x+3b_3x^2)^2 \\ &- y(b_1+2b_2x+3b_3x^2)^2(2b_2+6b_3x)\Big]\,dx\ dy\ dz \end{aligned} \tag{20}$$

Equations 19 and 20 are interesting. The first integral in Equation 19 will be discovered to contain only linear terms in the nodal displacements; hence, it does not contribute to **K** . The second integral in Equation 20 will be discovered to contain only cubic and higher terms in the displacement components. These are dropped for the same reasons as given for the stringer. Hence, only the quadratic terms in Equations 19 and 20 need be retained.

It is now necessary to write u , v , and $\theta = dv/dx$ at each node from Equations 14. Doing this and algebraically solving for the constants in Equations 14 provides:

$$\begin{aligned} a_0 &= u_1 & b_0 &= v_1 \\ a_1 &= \frac{u_2-u_1}{L} & b_1 &= \frac{v_2-v_1}{L} \\ & & b_2 &= \frac{3}{L^2}(v_2-v_1) - \frac{1}{L}(2\theta_1+\theta_2) \\ & & b_3 &= -\frac{2}{L^3}(v_2-v_1) + \frac{1}{L^2}(\theta_1+\theta_2) \end{aligned} \tag{21}$$

From these last equations we can now check to see why only the matrix expressions in Equations 19 and 20 need be retained.

Equations 21 enable us to write the quadratic part of U_1 (termed U_1') as,

$$U_1' = \tfrac{1}{2}P^0\int_0^L \left[v,\theta\right]\left[L_1\right]^T\left[x\right]\left[L_1\right]\left\{v,\theta\right\} dx \tag{22}$$

where

$$\left[v,\theta\right] = \left[v_1 \; \theta_1 \; v_2 \; \theta_2\right]$$

$$\left[L_1\right]^T = \begin{bmatrix} 0 & -3/L^2 & 2/L^3 \\ 1 & -2/L & 1/L^2 \\ 0 & 3/L^2 & -2/L^3 \\ 0 & -1/L & 1/L^2 \end{bmatrix}$$

$\left[x\right]$ = square matrix of Equation 19

The conformity of Equation 22 with Equation 2 permits us to immediately write the initial stress stiffness matrix for the beam-column as,

$$K^{(1)} = P^0 \left[L_1\right]^T \left(\int_0^L \left[x\right] dx \right) \left[L_1\right]$$

$$= P^0 \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \\ \hline \end{array} \begin{bmatrix} 6/5L & & & \underline{\text{sym}} \\ 1/10 & 2L/15 & & \\ -6/5L & -1/10 & 6/5L & \\ 1/10 & -L/30 & -1/10 & 2L/15 \end{bmatrix} \tag{23}$$

Equation 23 gives the same form for $K^{(1)}$ as originally found in Reference 2, and later reported in References 7 and 15. It differs from $K^{(1)}$ as given for the beam-column in References 9 and 10. These latter sources give $K^{(1)}$ for the beam-column identical with that found for the stringer. This point will be discussed subsequently.

The quadratic form for U_2 (termed U_2') may similarly be written as

$$U_2' = \frac{1}{2} E \iiint \left[u\,v\,\theta\right] \left[L_2\right]^T \left[x,y\right] \left[L_2\right] \left\{u\,v\,\theta\right\} dx\,dy\,dz \tag{22a}$$

where

$$\left[u\, v\, \theta \right] = \left[u_1 \; v_1 \; \theta_1 \; u_2 \; v_2 \, \theta_2 \right]$$

$$\left[L_2 \right]^T = \begin{bmatrix} -1/L & 0 & 0 \\ 0 & -3/L^2 & 2/L^3 \\ 0 & -2/L & 1/L^2 \\ 1/L & 0 & 0 \\ 0 & 3/L^2 & -2/L^3 \\ 0 & -1/L & 1/L^2 \end{bmatrix}$$

$\left[x, y \right]$ = square matrix of Equation 20

From Equation 22a we then have

$$K^{(0)} = E \left[L_2 \right]^T \left(\iiint \left[x,y \right] dx\ dy\ dz \right) \left[L_2 \right] \tag{24}$$

It is a straightforward calculation to now show that the well-known result for $K^{(0)}$ for a beam element (including uncoupled axial force stiffness for the element) comes directly out of Equation 24. Details are not given here.

At this point we can examine the basis by which $K^{(1)}$ for the beam-column turns out to be identical with that obtained for the stringer. This result occurs when the first two terms in ϵ^a, Equation 13, are based on Equations 7, while the last term is calculated from $v(x)$ as given by Equation 14. The net effect is to represent dv/dx in Equation 13 by a constant, rather than by a quadratic as used in the derivation given above. This change will then produce $K^{(1)}$ which is the same as that for the stringer.

It is also interesting to note that U_2' does not include a term in dv/dx . As a result $K^{(0)}$ will turn out to be the same, no matter which choice is followed for determining $K^{(1)}$.

The above makes clear that a consistent use of terms leads to $K^{(1)}$ for the beam-column as expressed by Equation 23. This therefore may be regarded as the correct result. On the other hand, it is interesting to realize that the simpler form for $K^{(1)}$, namely that of the stringer, permits a solution to be found for large deflection beam problems. The reason for this is explainable on physical grounds. The beam-column can be viewed as a member having distinct bending and axial force stiffnesses. The axial force stiffness is essentially that of the equivalent stringer. Furthermore, it is an initial axial loading which has a significant influence on the overall stiffness against subsequent transverse loading. This point-of-view would indicate that $K^{(1)}$ might well be taken as that of the stringer. Numerical results verify this conclusion. On the other hand additional elements are needed to secure the same order of accuracy as can be obtained from the correct $K^{(1)}$ matrix. This can be demonstrated by means of a simple example, Figure 2. For this problem the theoretical solution as found from the differential equation is

$$P^{o}_{crit.} = 4\pi^2 \frac{EI}{\ell^2} = 39.44 \frac{EI}{\ell^2}$$

For the first stiffness solution we use $\mathbf{K}^{(1)}$ for the stringer, Equation 11, together with $\mathbf{K}^{(0)}$ for combined, but uncoupled, bending and axial loading. Due to boundary conditions and assumed symmetry of displacement only u_1 and v_2 need be retained as displacements in the total stiffness equation. It is then easy to show that $\mathbf{K}$ is given by

$$\mathbf{K} = \begin{array}{c} \begin{array}{cc} u_1 & v_2 \end{array} \\ \left[\begin{array}{cc} \frac{AE}{L} & 0 \\ 0 & 2\left(\frac{12EI}{L^3} + \frac{P^o}{L}\right) \end{array} \right] \end{array}$$

The critical condition results when the determinate of $\mathbf{K}$ vanishes. This gives

$$P^{o}_{crit.} = -48 \frac{EI}{\ell^2}$$

The negative sign in the stiffness solution indicates that P^o is compressive.

On the other hand we can alternatively choose to represent $\mathbf{K}^{(1)}$ by the beam-column initial stress stiffness matrix as given by Equation 23. We then obtain

$$\mathbf{K} = \begin{array}{c} \begin{array}{cc} u_1 & v_2 \end{array} \\ \left[\begin{array}{cc} \frac{AE}{L} & 0 \\ 0 & 2\left(\frac{12EI}{L^3} + \frac{6}{5}\frac{P^o}{L}\right) \end{array} \right] \end{array}$$

Again setting the determinant of $\mathbf{K}$ equal to zero now gives

$$P^{o}_{crit.} = -40 \frac{EI}{\ell^2}$$

We therefore see that, based on a two element idealization and using the correct form for $\mathbf{K}^{(1)}$, a surprisingly accurate result is obtained for the column critical load. Furthermore, the stiffness method of solution in this case is surprisingly simple to carry out.

At the same time we should keep in mind that by using the stringer matrix, and a finer idealization for representing the beam-column, any order of accuracy (as far as we know) can be obtained for the critical load.

THIN TRIANGULAR ELEMENT IN PLANE STRESS

This problem is essentially a generalization of that already given for the stringer. In its details it is simpler to carry out than the previous case of the beam-column. This is perhaps surprising, particularly when viewed in the light of the previous derivations as found in References 1 and 6.

We start by assuming the triangle to be initially in the xy plane. Nodal locations are specified in Figure 3. In this initial position the element carries initial stresses σ_x^o, σ_y^o, and τ_{xy}^o. As in the case of the stringer these stresses have no influence on subsequent displacements of the element within the initial plane. They do, however, have a significant effect on displacements of the element out of its initial plane. Consequently, the rotational terms in the strain-displacement equations must be taken as those associated with a w - component of displacement.

Starting from the initial position, subsequent deformation of the element is therefore assumed to take place such that

$$\epsilon_x = \epsilon_x^o + \epsilon_x^a \qquad \epsilon_y = \epsilon_y^o + \epsilon_y^a \qquad \gamma_{xy} = \gamma_{xy}^o + \gamma_{xy}^a$$

where

$$\begin{aligned} \epsilon_x^a &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \epsilon_y^a &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \gamma_{xy}^a &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{aligned} \tag{25}$$

It is seen that in Equations 25 the additional strains include the terms representing rotation of the triangle out of its initial position in the xy plane. Incidentally, the derivation is simplest when carried out in these terms. Later, a simple matrix transformation can be carried out to give the result for the case of arbitrary initial orientation in the xyz coordinate system. Also, it should be observed that Equations 25 do not include rotation of the element in the initial plane. This effect is also dropped in developing the classical nonlinear plate equations.

Strain energy U is given by

$$U = \frac{1}{2}\iiint(\sigma_x\,\epsilon_x + \sigma_y\,\epsilon_y + \tau_{xy}\,\gamma_{xy})\,dx\;dy\;dz \tag{26}$$

At this point a choice is available for writing the stress-strain relations. In the calculations which follow the conventional Hooke's law expressions are used. In matrix form we have

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \lambda_1\end{bmatrix}\begin{bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{bmatrix} \tag{27}$$

where $\lambda_1 = (1-\nu)/2$ and ν = Poisson's ratio.

Alternatively we might have chosen an orthotropic material defined by

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & & \underline{\text{sym.}} \\ C_{12} & C_{22} & \\ 0 & 0 & C_{33} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{28}$$

Elements C_{ij} then represent the actual properties of the orthotropic material. No additional difficulty is encountered if Equation 28 is used in place of Equation 27 in the calculations which follow.

Substituting Equation 27 into the expression for U results once again in three terms as follows:

$$U = U_0 + U_1 + U_2$$

where,

$$U_0 = \frac{Et}{2(1-\nu^2)} \iint \left[(\epsilon_x^0)^2 + 2\nu\, \epsilon_x^0\, \epsilon_y^0 + (\epsilon_y^0)^2 + \lambda_1 (\gamma_{xy}^0)^2 \right] dx\, dy$$

$$U_1 = \frac{Et}{1-\nu^2} \iint \left[\epsilon_x^o\, \epsilon_x^a + \nu(\epsilon_x^o \epsilon_y^a + \epsilon_y^o\, \epsilon_y^a) + \epsilon_y^o \epsilon_y^a + \lambda_1\, \gamma_{xy}^o\, \gamma_{xy}^a \right] dx\, dy \tag{29}$$

$$U_2 = \frac{Et}{2(1-\nu^2)} \iint \left[(\epsilon_x^a)^2 + 2\nu\, \epsilon_x^a \epsilon_y^a + (\epsilon_y^a)^2 + \lambda_1 (\gamma_{xy}^a)^2 \right] dx\, dy \tag{30}$$

The similarity between these expressions and those arising in the two previous derivations can now be noted. As before, we omit U_0 in further calculations.

From Equation 29, the strain expressions of Equations 25 and Equation 27, we obtain

$$\begin{aligned} U_1 = {} & t \iint \left[\sigma_x^o \frac{\partial u}{\partial x} + \sigma_y^o \frac{\partial v}{\partial y} + \tau_{xy}^o \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] dx\, dy \\ & + \frac{1}{2} t \iint \left[\frac{\partial w}{\partial x} \quad \frac{\partial w}{\partial y} \right] \begin{bmatrix} \sigma_x^o & \tau_{xy}^o \\ \tau_{xy}^o & \sigma_y^o \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} dx\, dy \end{aligned} \tag{31}$$

At this point we assume displacement functions for u , v , and w . Guided by the one-dimensional case of the stringer we select linear functions in x and y as follows:

$$\begin{aligned} u(x,y) &= a_0 + a_1 x + a_2 y \\ v(x,y) &= b_0 + b_1 x + b_2 y \\ w(x,y) &= c_0 + c_1 x + c_2 y \end{aligned} \tag{32}$$

Equations 32 contain 9 constants. These agree in number with the total nodal degrees of freedom. Consequently we can determine these constants in terms of nodal displacements. Writing u , v , and w from Equations 32 at each node – see Figure 4 – and solving for the constants $a_0, \cdots, c_2$ gives

$$\begin{array}{lll} a_0 = u_1 & a_1 = (u_2 - u_1)/x_2 & a_2 = (x_{32}u_1 - x_3 u_2 + x_2 u_3)/x_2 y_3 \\ b_0 = v_1 & b_1 = (v_2 - v_1)/x_2 & b_2 = (x_{32}v_1 - x_3 v_2 + x_2 v_3)/x_2 y_3 \\ c_0 = w_1 & c_1 = (w_2 - w_1)/x_2 & c_2 = (x_{32}w_1 - x_3 w_2 + x_2 w_3)/x_2 y_3 \end{array} \tag{33}$$

where $x_{32} = x_3 - x_2$ and $x_2\, y_3 = 2A$, $A =$ area triangle.

In terms of Equations 32 and 33 we see that the first integral in U_1 , Equation 31, contains only linear terms in the nodal displacements. We therefore neglect this term and write the remaining integral as U_1' . We then have

$$\begin{aligned} U_1' &= \frac{1}{2} t \iint \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} \sigma_x^0 & \tau_{xy}^0 \\ \tau_{xy}^0 & \sigma_y^0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} dx\, dy \\ &= \frac{1}{2}\left(\frac{t}{4A}\right) \begin{bmatrix} w \end{bmatrix}^T \begin{bmatrix} x, y \end{bmatrix}^T \begin{bmatrix} \sigma^0, & \tau^0 \end{bmatrix} \begin{bmatrix} x, y \end{bmatrix} \begin{bmatrix} w \end{bmatrix} \end{aligned} \tag{34}$$

where

$$\begin{aligned} \begin{bmatrix} w \end{bmatrix}^T &= \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \\ \begin{bmatrix} x, y \end{bmatrix}^T &= \begin{bmatrix} -y_3 & x_{32} \\ y_3 & -x_3 \\ 0 & x_2 \end{bmatrix} \\ \begin{bmatrix} \sigma^0, \tau^0 \end{bmatrix} &= \text{square matrix from Equation 34} \end{aligned} \tag{35}$$

Following the procedure established in the previous derivations we therefore find that the initial stress matrix for the triangle in plane stress is given by

$$\begin{aligned} K^{(1)} &= \frac{t}{4A}\left[x, y\right]^T \left[\sigma^o, \tau^o\right]\left[x, y\right] \\ &= K^{(1)}(\sigma_x^o) + K^{(1)}(\sigma_y^o) + K^{(1)}(\tau_{xy}^o) \end{aligned} \tag{36}$$

where

$$K^{(1)}(\sigma_x{}^o) = \frac{\sigma_x{}^o t}{4A} \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \left[\begin{array}{ccc} y_3^2 & \underline{\text{sym.}} & \\ -y_3^2 & y_3^2 & \\ 0 & 0 & 0 \end{array}\right] \end{array} \tag{37a}$$

$$K^{(1)}(\sigma_y{}^o) = \frac{\sigma_y{}^o t}{4A} \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \left[\begin{array}{ccc} x_{32}^2 & \underline{\text{sym.}} & \\ -x_3 x_{32} & x_3^2 & \\ x_2 x_{32} & -x_2 x_3 & x_2^2 \end{array}\right] \end{array} \tag{37b}$$

$$K^{(1)}(\tau_{xy}{}^o) = \frac{\tau_{xy}{}^o t}{4A} \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \left[\begin{array}{ccc} -2 y_3 x_{32} & \underline{\text{sym.}} & \\ (x_3 + x_{32}) y_3 & -2 x_3 y_3 & \\ -x_2 y_3 & x_2 y_3 & 0 \end{array}\right] \end{array} \tag{37c}$$

It is now a simple matter to enlarge the $K^{(1)}$ matrices of Equations 37a, b, c to order 9 x 9. In doing this we note that columns and corresponding rows of zeros must be introduced for all u and v components of nodal displacement. The solution then agrees in form with the 4 x 4 $K^{(1)}$ matrix obtained for the stringer.

The solution, as expressed by Equations 37 is significantly simpler in form than that previously reported in References 1 and 6. In Reference 1 additional terms to those appearing in Equations 25 were used in writing the strain-displacement equations. These additional terms are unnecessary. Retaining them does not lead to an incorrect result; however, neither is the result in its simplest form. The initial stress matrix obtained in Reference 6 for the triangle also contains fewer zero elements than found above. Hence, in this case also, the geometric approach has led to a result different from that given by a straightforward application of basic theory.

If we return to U_2, Equation 30, we discover that it can be written as

$$U_2 = \frac{1}{2}\frac{Et}{4(1-\nu^2)A}\left[u,\ v\right]^T\left[x,y\right]^T\left[\lambda_1,\nu\right]\left[x,y\right]\left\{u,\ v\right\}$$

plus cubic and higher order terms in displacement components

where,

$$\left[u,\ v\right]^T = \left[\begin{matrix} u_1 & u_2 & u_3 & v_1 & v_2 & v_3 \end{matrix}\right]$$

$$\left[\lambda_1,\nu\right] = \left[\begin{matrix} 1 & 0 & 0 & \nu \\ 0 & \lambda_1 & \lambda_1 & 0 \\ 0 & \lambda_1 & \lambda_1 & 0 \\ \nu & 0 & 0 & 1 \end{matrix}\right] \tag{38}$$

$$\left[x,y\right]^T = \left[\begin{matrix} -y_3 & 0 & x_{32} & 0 \\ y_3 & 0 & -x_3 & 0 \\ 0 & 0 & x_2 & 0 \\ 0 & -y_3 & 0 & x_{32} \\ 0 & y_3 & 0 & -x_3 \\ 0 & 0 & 0 & x_2 \end{matrix}\right]$$

Dropping the higher terms in U_2 we then immediately write $K^{(0)}$ as

$$K^{(0)} = \frac{Et}{4(1-\nu^2)A}\left[x,\ y\right]^T\left[\lambda_1,\nu\right]\left[x,\ y\right] \tag{39}$$

Carrying out the triple matrix product yields $K^{(0)}$ as originally given in Reference 18. In comparing solutions congruent nodal locations must be used.

Again it is a relatively simple matter to transform the results obtained for $K^{(0)}$ and $K^{(1)}$ so that they apply to the triangle which is arbitrarily oriented in an x, y, z coordinate system. Reference 18 gives detailed information on such transformations.

TRIANGULAR ELEMENTS IN BENDING

There are various approaches to developing a satisfactory $K^{(0)}$ matrix for an arbitrary triangular element in bending. These will not be discussed here. Additional publications on this subject will undoubtedly appear during the next several years.

Of interest in this present paper is the problem of obtaining a satisfactory $K^{(1)}$ matrix for the triangle in bending. Such a matrix is essential if large deflection and stability problems of thin plates and shells are to be investigated by the stiffness method.

Guided by the discovery that the beam-column can be satisfactorily described by using the stringer $K^{(1)}$ matrix in conjunction with $K^{(0)}$ for the beam in bending we arrive at the following hypothesis: large deflection of the triangle in bending can be satisfactorily described by using $K^{(1)}$ for the triangle in plane stress, plus a $K^{(0)}$ matrix which has been found to be suitable for the case of small deflections. This hypothesis was used in Reference 6 to show that critical loads for a thin plate could be accurately calculated using an idealization based on triangular elements. More complete data on such calculations will be submitted for publication in the near future.

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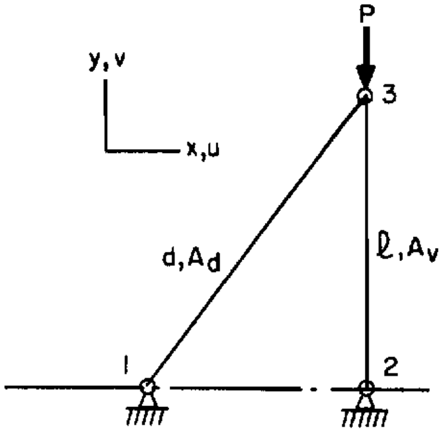

Figure 1. Truss Stability Problem

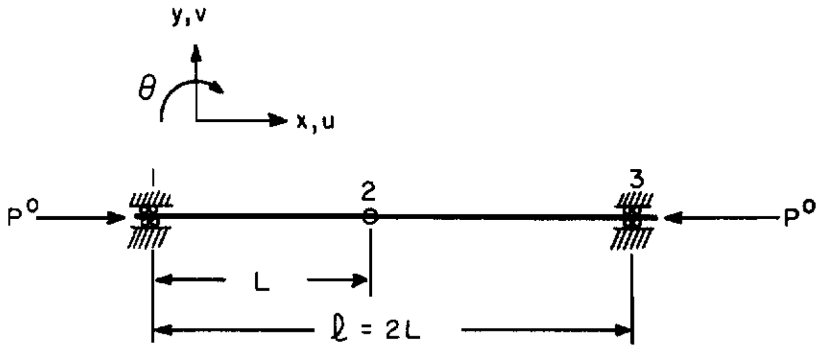

Figure 2. Two Element Idealization of the Uniform Beam-Column

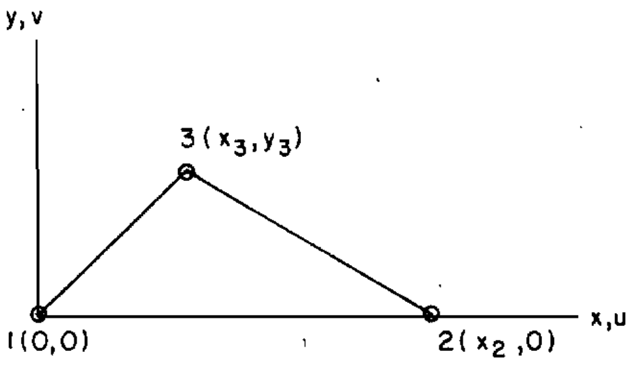

Figure 3. Triangular Element as Initially Located in the XY Plane