ANALYTICAL AND EXPERIMENTAL MODAL ANALYSIS OF A TWO-TIERED STRUCTURE

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Abstract

A test structure, namely a two-tiered structure (TTS), was used as a physical model for obtaining modal parameters of flexible structures. These parameters, which were determined analytically and experimentally, are natural frequencies, mode shapes, and damping ratios. In the analytical portion, finite element method (FEM) was used with MSC/NASTRAN, MSC/pal 2, and MSC/MOD. In the experimental part, the structure was excited by random noise, and frequency-response function (FRF) plot and modal parameters were obtained. Both the HP 3566A/3567A (Hewlett Packard Spectrum/Network Analyzer) and STAR System (Structural Measurement Systems software) were utilized. An exact model-reduction technique was used to obtain a complete mathematical model of a reduced-order system, which includes the reduced-order *physical* mass, stiffness, and damping matrices.

Introduction

Analytical and experimental modal analysis can be used to determine dynamic properties or modal parameters of flexible structures. These modal parameters are natural frequencies, mode shapes, and damping ratios. The characteristics of flexible structures are low natural frequencies, low damping, and some of the modes are closely spaced.

In finite element analysis, it is necessary that the structure under consideration is modeled using a large number of degrees of freedom (DOF's) for accurate results. But if the number of DOF's is large, the results from finite element program become unmanageable for the purpose of design and analysis of vibration control or for subsequent studies. A model reduction technique [1] can be used to reduce a large-DOF model to a small-DOF model which exactly represents the first few modes.

In experimental modal analysis [2-4], also called *modal testing*, natural frequencies and damping ratios can be obtained from the frequency response function (FRF) plot. In modal testing it is important to have high-quality test setup, testing craftsmanship, and data processing, etc.

The equations of motion of a structure can be written in the *configuration-space form* [5] as

$$\mathbf{m}\ddot{\mathbf{x}} + \mathbf{c}\dot{\mathbf{x}} + \mathbf{k}\mathbf{x} = \mathbf{f} \qquad (1)$$

where **m**, **k**, and **c** are the *physical* mass, stiffness, and damping matrices, respectively; and **f** is the applied forcing vector.

The physical mass and stiffness matrices of a structure under consideration can be obtained analytically by the given physical properties, dimensions, and boundary conditions; however, the physical damping matrix must be determined experimentally.

A complete mathematical model of a reduced-order system will be determined, which includes the reduced-order *physical* mass, stiffness, and damping matrices.

The Test Structure

The test structure, a two-tiered structure (TTS), shown in Fig. 1, is chosen so it can be used for studying vibration characteristics of flexible structures. It was designed specifically to possess the following characteristics:

- low natural frequencies, light damping, and intuitive mode shapes for the first few modes
- inexpensive and easy to build
- instructive for analytical analysis and computer simulation, and experimental modal analysis

Finite Element Model

A finite element model of the structure is created using MSC/MOD (Fig. 2). The floors and the columns are modeled by plate elements and bar elements, respectively. The brackets connecting the plates and columns are modeled by concentrated mass elements. The model has 136 elements, 146 nodal points and 790 (active) degrees of freedoms ($n = 790$). It may appear that the number of plate elements is more than adequate; however, in this study, the model is relatively small and simple so that mesh optimization is ignored.

The finite element model is then analyzed by using two commercially available finite element analysis packages: MSC/NASTRAN (on main-frame computer), and MSC/MOD and MSC/pal 2 (on MS-DOS machine) [6]. The undamped natural frequencies obtained, using these packages, are given in Table 1, and the corresponding mode shapes (from MSC/pal 2) are shown in Figs. 3-8.

Exact Model Reduction

For the undamped free vibration or eigenvalue problem, Eq. (1) reduces to

$$\mathbf{m}\ddot{\mathbf{x}} + \mathbf{k}\mathbf{x} = 0 \tag{2}$$

When the structure vibrates in its natural modes, we have

$$\left(\mathbf{k} - \omega_r^2 \mathbf{m}\right)\phi_r = 0 \qquad r = 1, 2, \ldots, n \tag{3}$$

where ω_r and ϕ_r are the undamped natural frequencies and the corresponding mode-shape vectors, respectively.

The orthogonality properties are *mass normalized* so that the *modal* mass and stiffness matrices are given as

$$\begin{aligned} \mathbf{M} &= \Phi^T \mathbf{m} \Phi = \mathbf{I} \\ \mathbf{K} &= \Phi^T \mathbf{k} \Phi = diag\left[\omega_r^2\right] \qquad r = 1, 2, \ldots, n \end{aligned} \tag{4}$$

where the full-order (mass-normalized) mode-shape matrix is given as

$$\Phi = \begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_n \end{bmatrix} \tag{5}$$

The full-order physical mass and stiffness matrices can be written, from Eq. (4), as

$$\begin{aligned} \mathbf{m} &= \Phi^{-T} \Phi^{-1} \\ \mathbf{k} &= \Phi^{-T} diag\left[\omega_r^2\right] \Phi^{-1} \qquad r = 1, 2, \ldots, n \end{aligned} \tag{6}$$

The 790-DOF full-order model (n = 790) is reduced to a 6-DOF reduced-order model (m = 6) which exactly represents the first six modes using [1]. The reduced-order model is obtained by selecting the four translational DOF's located at the centers of the floors for the first four bending modes. For each of the first two torsional modes, the angular DOF is defined by a set of any two translational DOF's of a given floor. Using the numerical values of the full-order mode-shape matrix (from MSC/NASTRAN, not shown), the reduced-order mode-shape matrix can be selected as

$\Phi_R =$

$$\left[\begin{Bmatrix} 4.175227 \\ 0 \\ 0 \\ 6.724909 \\ 0 \\ 0 \end{Bmatrix}_1 \begin{Bmatrix} 0 \\ 4.195279 \\ 0 \\ 0 \\ 6.710377 \\ 0 \end{Bmatrix}_2 \begin{Bmatrix} 0 \\ 0 \\ -0.500879 \\ 0 \\ 0 \\ -0.711376 \end{Bmatrix}_3 \begin{Bmatrix} 6.274968 \\ 0 \\ 0 \\ -4.559574 \\ 0 \\ 0 \end{Bmatrix}_4 \begin{Bmatrix} 0 \\ -6.261847 \\ 0 \\ 0 \\ 4.580626 \\ 0 \end{Bmatrix}_5 \begin{Bmatrix} 0 \\ 0 \\ 0.657553 \\ 0 \\ 0 \\ -0.561635 \end{Bmatrix}_6\right] \tag{7}$$

where the subscript R denotes reduced-order model.

The differential equations for undamped free vibration of the reduced-order model are given as

$$\mathbf{m}_R \ddot{\mathbf{x}}_R + \mathbf{k}_R \mathbf{x}_R = 0 \tag{8}$$

where

$$\begin{aligned} \mathbf{m}_R &= \Phi_R^{-T} \Phi_R^{-1} \\ \mathbf{k}_R &= \Phi_R^{-T} diag\left[\omega_r^2\right] \Phi_R^{-1} \qquad r = 1, 2, \dots, m \end{aligned} \tag{9}$$

$$\mathbf{x}_R = \{x_1 \quad y_1 \quad 1 \cdot \theta_1 \quad x_2 \quad y_2 \quad 1 \cdot \theta_2\}^T \tag{10}$$

The numerical value 1 in Eq. (10) has dimension of length so $\mathbf{x}_R$ is dimensionally homogeneous, and the subscripts 1 and 2 denote the middle floor and the top floor, respectively (Fig. 1).

Introducing Φ_R from Eq. (7) and ω_r from Table 1 (MSC/NASTRAN) into Eq. (9), we have

$$\begin{aligned} \mathbf{m}_R &= 10^{-2} \begin{bmatrix} 1.7605 & 0 & 0 & 0.0142 & 0 & 0 \\ 0 & 1.7604 & 0 & 0 & 0.0142 & 0 \\ 0 & 0 & \frac{146.40}{1^2} & 0 & 0 & \frac{2.3153}{1^2} \\ 0.0142 & 0 & 0 & 1.5149 & 0 & 0 \\ 0 & 0.0142 & 0 & 0 & 1.5150 & 0 \\ 0 & 0 & \frac{2.3153}{1^2} & 0 & 0 & \frac{121.77}{1^2} \end{bmatrix} \frac{lb_f s^2}{in} \\ \mathbf{k}_R &= 10^{3} \begin{bmatrix} 0.0171 & 0 & 0 & -0.0084 & 0 & 0 \\ 0 & 0.0171 & 0 & 0 & -0.0085 & 0 \\ 0 & 0 & \frac{2.0839}{1^2} & 0 & 0 & \frac{-1.1283}{1^2} \\ -0.0084 & 0 & 0 & 0.0082 & 0 & 0 \\ 0 & -0.0085 & 0 & 0 & 0.0083 & 0 \\ 0 & 0 & \frac{-1.1283}{1^2} & 0 & 0 & \frac{1.1921}{1^2} \end{bmatrix} \frac{lb_f}{in} \end{aligned} \tag{11}$$

The numerical value 1^2 in Eq. (11) has dimension of length squared so the elements of $\mathbf{m}_R$ and $\mathbf{k}_R$ have proper dimensions.

Modal Testing

The experimental setup is shown in Fig. 9. Continuous random signal and Hanning window were used to obtain the FRF plot (Fig. 10). This plot includes the first six modes of the structure, of which the natural frequencies can be read directly. In the process of obtaining the *modal* damping ratios, the FRF data was first converted from the HP 3566A/3567A format to the STAR System format, then curve fitting methods were used. For widely spaced modes, the determination of damping ratios by curve fitting is straight forward; because in these modes, the structure behaves as if it were single degree of freedom (SDOF). However, for closely spaced modes, the damping ratios are difficult to obtained with great accuracy. The experimental results for natural frequencies ω_r and damping ratios ζ_r are given in Table 2.

Physical Damping Matrix

A physical damping matrix can be determined as

$$\mathbf{c}_R = \Phi_R^{-T} diag[2\varsigma_r\omega_r]\,\Phi_R^{-1} \qquad r = 1, 2, \ldots, m \tag{12}$$

The modal damping matrix is given as

$$(\mathbf{C}_R)_e = diag[2\varsigma_r\omega_r] = diag[0.4008 \quad 0.3314 \quad 0.1633 \quad 0.9068 \quad 0.8671 \quad 0.2771]\frac{rad}{s} \tag{13}$$

or

$$(\mathbf{C}_R)_a = diag[2\varsigma_r\omega_r] = diag[0.4032 \quad 0.3191 \quad 0.1184 \quad 0.9392 \quad 0.8767 \quad 0.2205]\frac{rad}{s} \tag{14}$$

where ω_r in Eqs. (13) and (14) are the experimental and analytical (MSC/NASTRAN, Table 1) natural frequencies, respectively.

Introducing Eqs. (7, 13) into Eq. (12), we have

$$\mathbf{c}_R = \begin{bmatrix} 0.0132 & 0 & 0 & -0.0037 & 0 & 0 \\ 0 & 0.0123 & 0 & 0 & -0.0040 & 0 \\ 0 & 0 & \frac{0.3417}{1^2} & 0 & 0 & \frac{-0.0685}{1^2} \\ -0.0037 & 0 & 0 & 0.0084 & 0 & 0 \\ 0 & -0.0040 & 0 & 0 & 0.0075 & 0 \\ 0 & 0 & \frac{-0.0685}{1^2} & 0 & 0 & \frac{0.2497}{1^2} \end{bmatrix} \frac{lb_f s}{in} \tag{15}$$

The numerical value 1^2 in Eq. (15) has dimension of length squared so $\mathbf{c}_R$ is dimensionally homogeneous.

Concluding Remarks

A complete mathematical model of the reduced-order system has been determined, as given by

$$\mathbf{m}_R\ddot{\mathbf{x}}_R + \mathbf{c}_R\dot{\mathbf{x}}_R + \mathbf{k}_R\mathbf{x}_R = \mathbf{0} \tag{16}$$

It should be noted that the physical damping matrix can be obtained using the experimental damping ratios and experimental/analytical natural frequencies and mode-shape matrix. If the physical damping matrix is *proportional*, the modal damping matrix is diagonal, or if the off-diagonal elements of the modal damping matrix are negligible, then the physical damping matrix can be approximated as proportional. Modal analysis can, then, be performed since the equations of motion can be decoupled via orthogonality properties [7].

Table 1 shows that the results obtained from the finite-element model agree very well with the experimental results in bending modes but not so well in torsional modes. Some explanation for these discrepancies is currently being sought.

References

1. Hallauer, W. and S. Lamberson, "Experimental Active Vibration Damping of a Plane Truss Using Hybrid Actuation," *Proceedings of the AIAA/ASME/ASCE/AHS 30th Structures, Structure Dynamics, and Material Conference*, 1989, pp. 80-90.
2. Ewins, D. J., *Modal Testing: Theory and Practice*. New York: John Wiley & Sons, 1988.
3. Inman, D. J., *Vibration with Control, Measurement, and Stability*. Englewood, NJ: Prentice-Hall, Inc., 1989.
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5. Meirovitch, L., *Dynamics and Control of Structures*. New York: John Wiley & Sons, 1990.
6. MSC/pal 2 and MSC/NASTRAN *User's Manuals*, The MacNeal-Schwendler Corporation, Los Angeles.
7. Meirovitch, L., *Elements of Vibration Analysis*. McGraw-Hill Book Company, 1986.

Table 1 Comparison of Experimental and Analytical Natural Frequencies

Mode Number	Experimental (Hz)	MSC/pal 2 (Hz) (% Diff.)	MSC/NASTRAN (Hz) (% Diff.)
1	2.125	2.184 (+2.78)	2.228 (+4.85)
2	2.334	2.225 (-4.76)	2.247 (-3.73)
3	3.938	2.796 (-29.00)	2.854 (-27.53)
4	5.594	5.746 (+2.72)	5.794 (+2.77)
5	5.750	5.789 (+0.68)	5.814 (+1.11)
6	9.188	7.224 (-21.38)	7.311 (-20.43)

Table 2 Experimental Natural Frequencies and Damping Ratios

Mode Number	Frequency (Hz)	Damping Ratio (%)
1	2.125	1.44
2	2.334	1.13
3	3.938	0.33
4	5.594	1.29
5	5.750	1.20
6	9.188	0.24

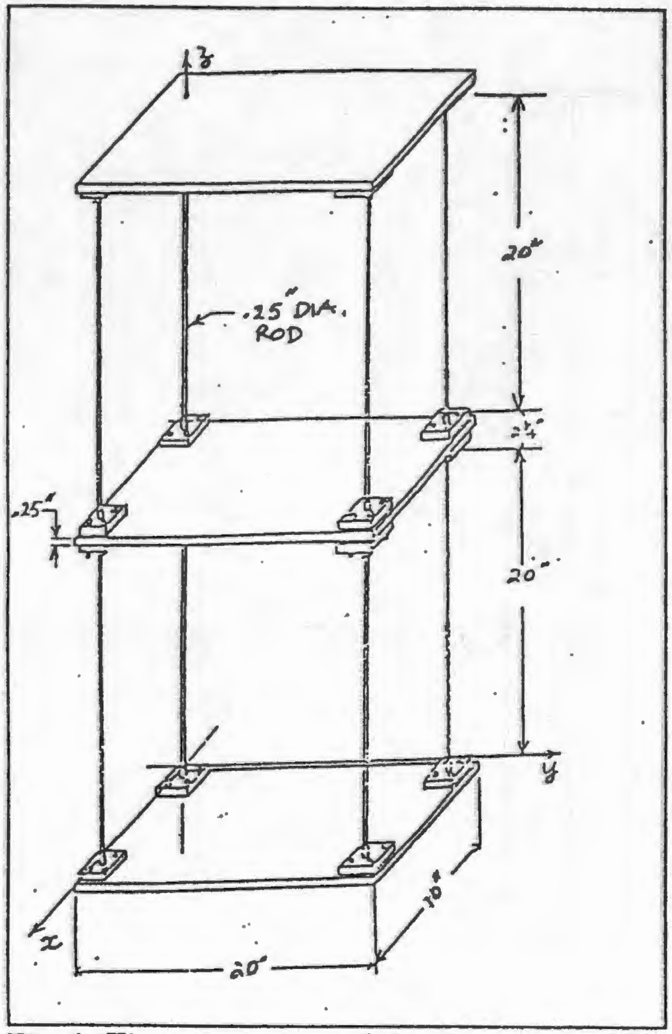

Fig. 1 The test structure (Aluminum 6061-T6)

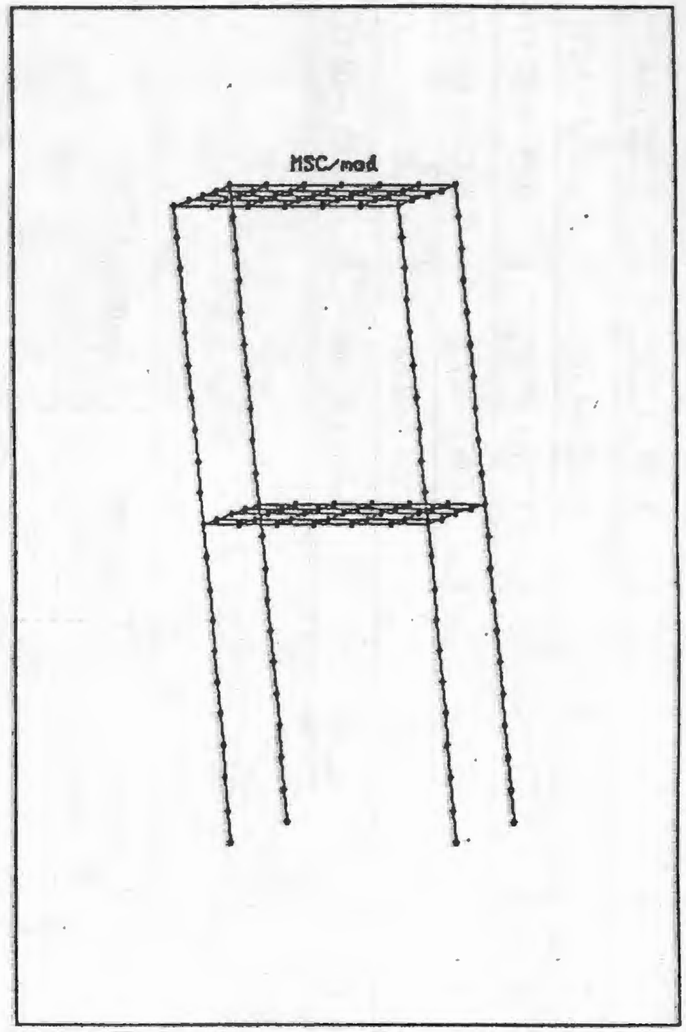
Fig. 2 Finite element model

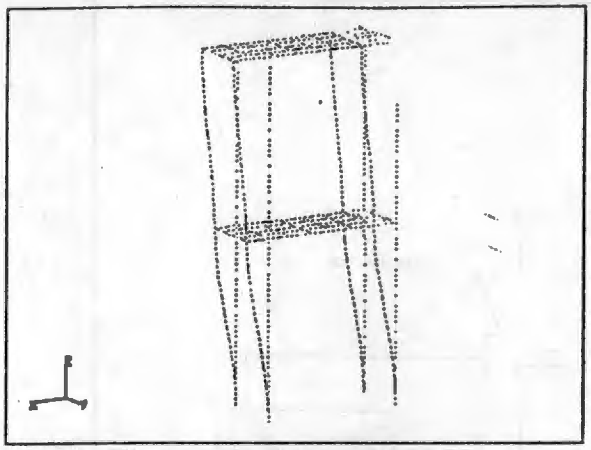

Fig. 3 First x-z bending, 2.184 Hz

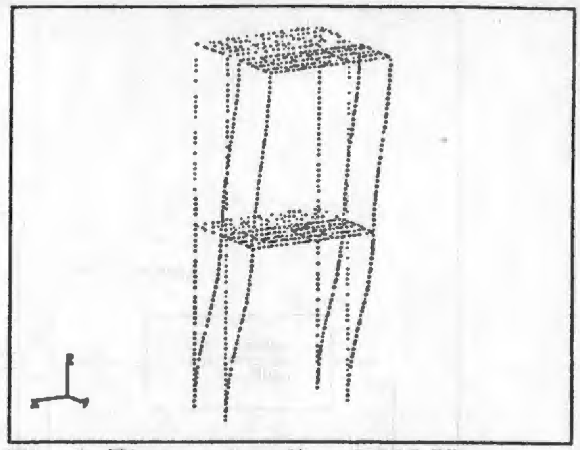

Fig. 4 First y-z bending, 2.225 Hz

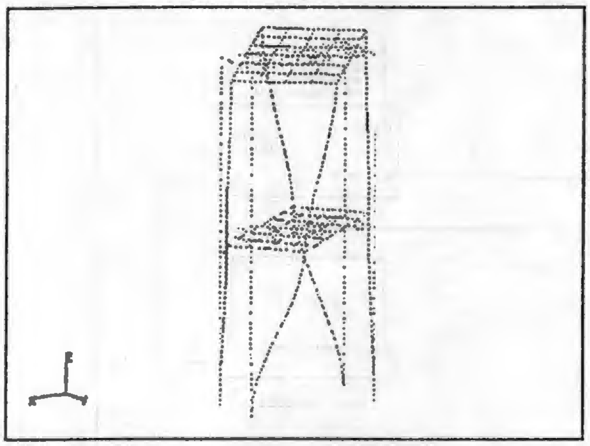

Fig. 5 First z torsion, 2.796 Hz

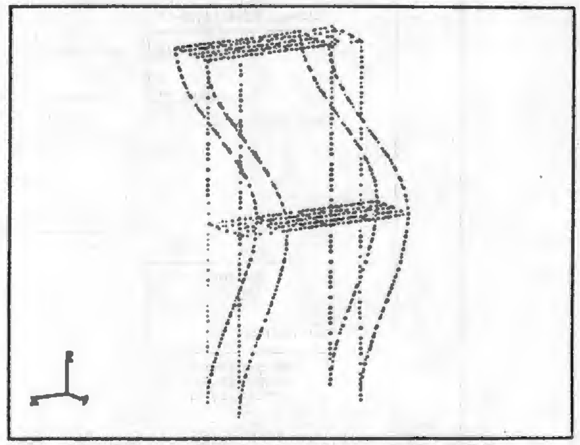

Fig. 6 Second x-z bending, 5.746 Hz

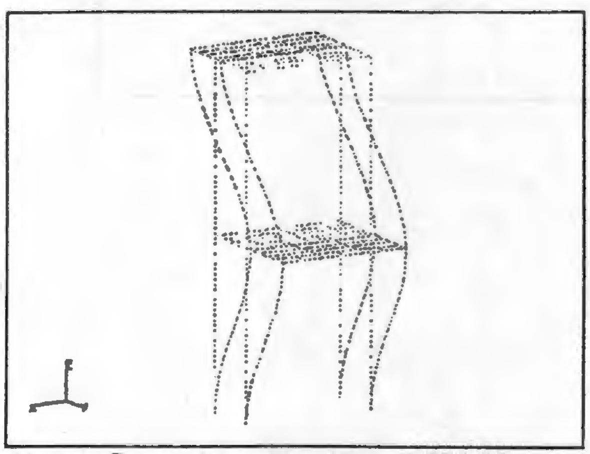

Fig. 7 Second y-z bending, 5.789 Hz

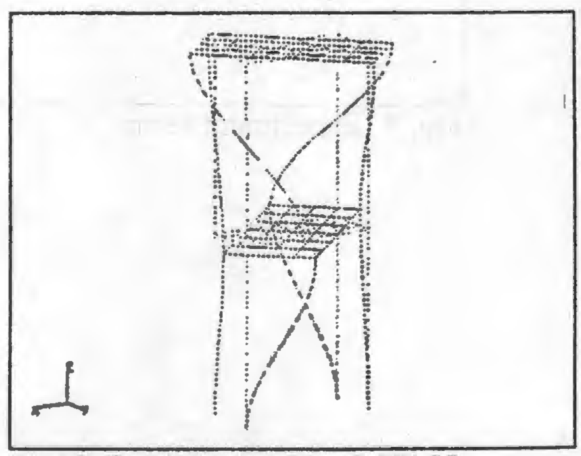

Fig. 8 Second z torsion, 7.224 Hz

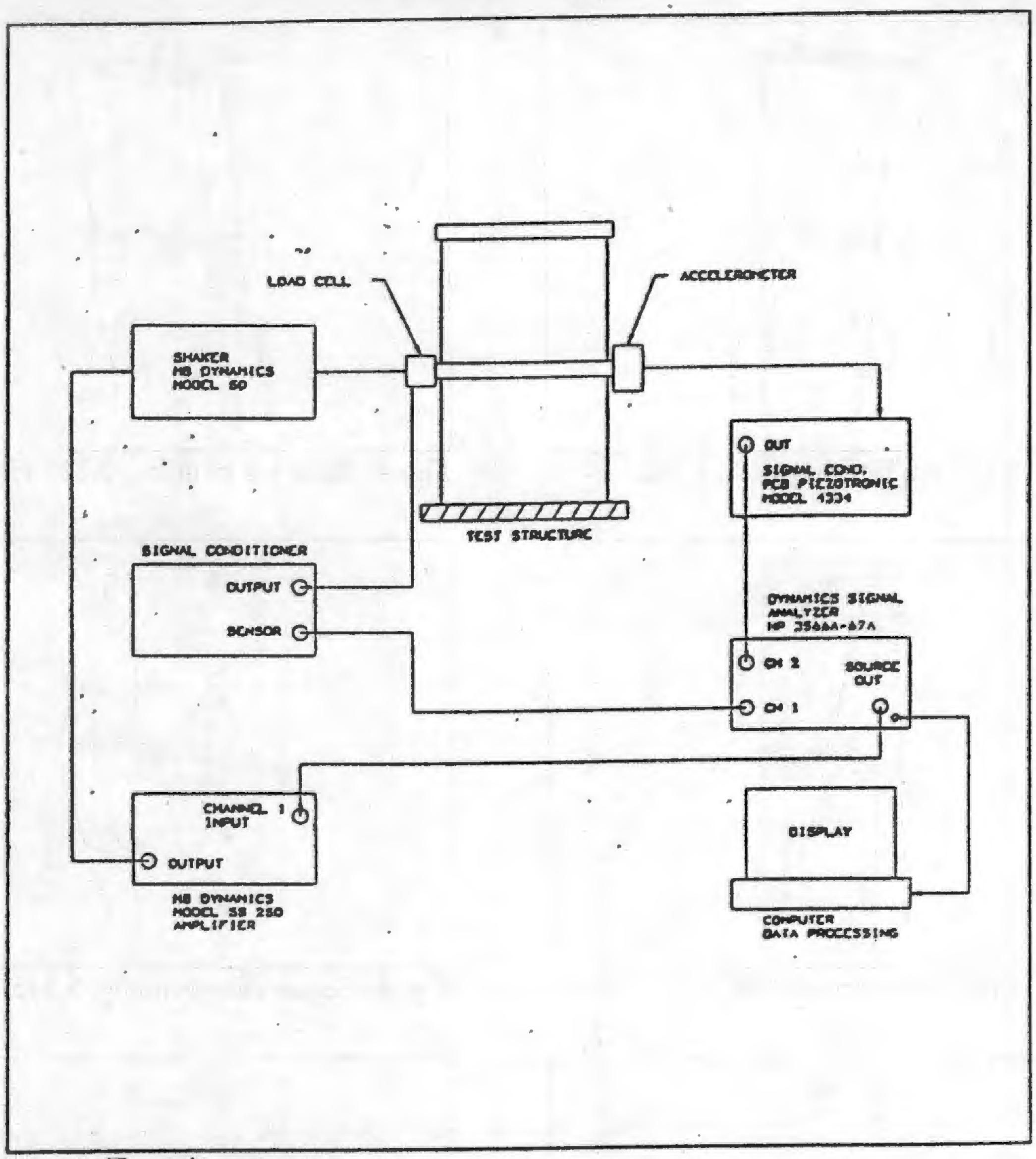

Fig. 9 Experiment setup

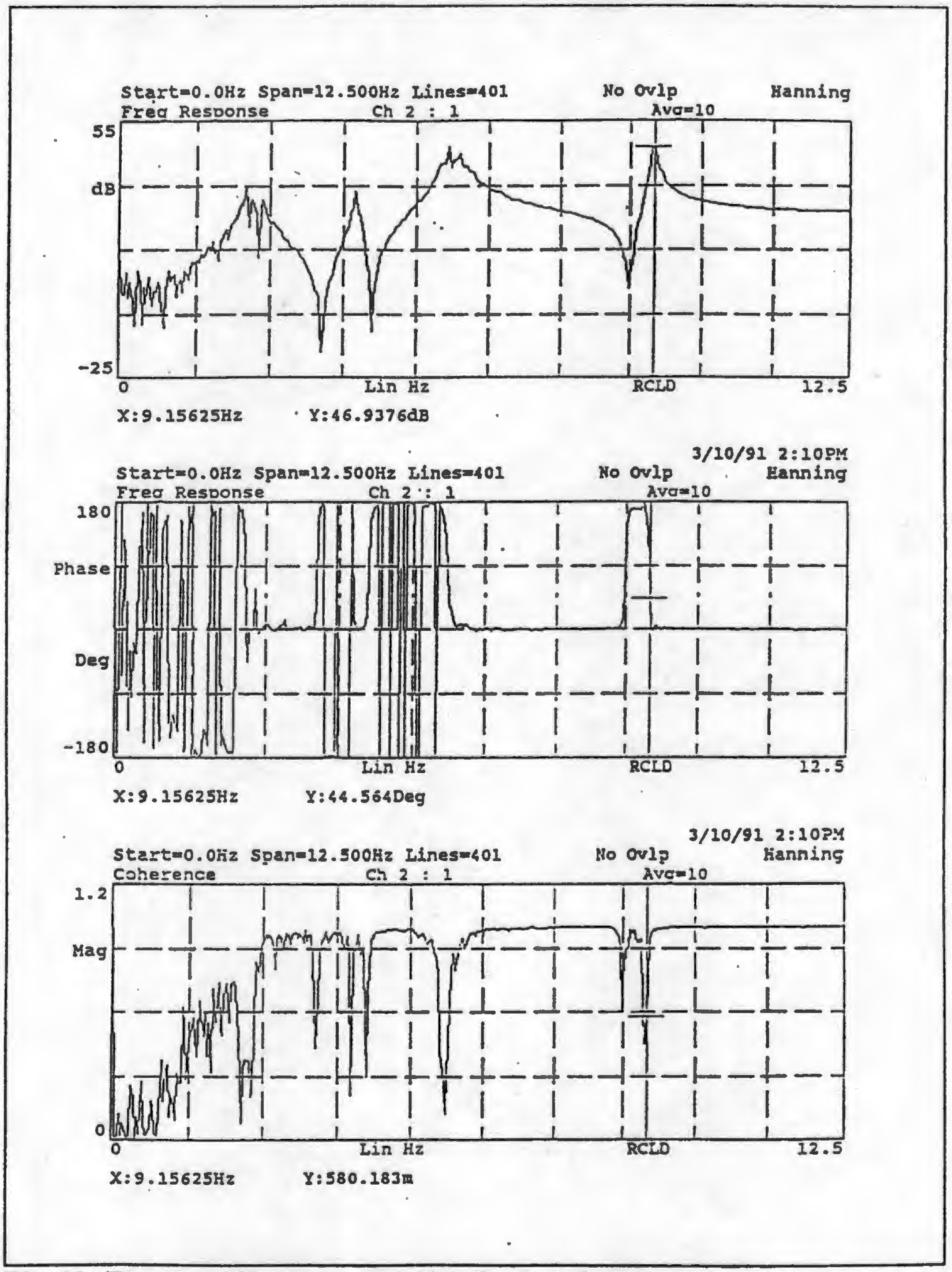

Fig. 10 Frequency Response spanning the first 6 modes