AFFDL-TR-65-93

EVALUATION TESTS FOR STATISTICAL ANALYZERS

R. D. KELLY

FOREWORD

This report was prepared by the Measurement Analysis Corporation of Los Angeles, California, for the Aerospace Dynamics Branch, Vehicle Dynamics Division, AF Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio, under Contract AF 33(615)-1418. The research performed is part of a continuing effort to provide advanced techniques in the application of random process theory and statistics to vibration problems under the Research and Technology Division, Air Force System Command's exploratory development program. The Project No. is 1370 "Dynamic Problems in Flight Vehicles" and the Task No. is 137005, "Prediction and Control of Structural Vibration." Mr. R. G. Merkle of the Vehicle Dynamics Division was the project engineer.

This report covers work conducted from July 1965 to May 1965. This manuscript was released by the author June 1965 for publication as a Technical Report.

The author gratefully expresses his appreciation for the assistance given by personnel from the Aerospace Dynamics Branch of the AF Flight Dynamics Laboratory, and by J. S. Bendat, L. D. Enochson, A. G. Piersol, and G. P. Thrall of the Measurement Analysis Corporation.

This technical report has been reviewed and is approved.

Walter J. Mykytow
WALTER J. MYKYTOW
Asst. for Research & Technology
Vehicle Dynamics Division

ABSTRACT

This report describes a series of tests designed to evaluate the performance of statistical analyzers. The types of analyses that these analyzers typically perform and that must be evaluated are:

1. Instantaneous Amplitude Probability Density
2. Instantaneous Amplitude Probability Distribution
3. Negative Instantaneous Amplitude Probability Distribution
4. Peak Value Probability Density
5. Expected Number of Maxima per Unit Time
6. Expected Number (Total, Positive, or Negative) of Threshold Crossings per Unit Time
7. Joint Instantaneous Amplitude Probability Density
8. Joint Instantaneous Amplitude Probability Distribution
9. Extreme Value Density
10. Extreme Value Distribution

Tests with both periodic (sinusoidal and triangular) and random (broad-band Gaussian, narrow band Gaussian, and clipped Gaussian) signal inputs are delineated for each of the above analysis modes. Tolerances on the output wave shapes of the periodic signal generators are described so that generators whose outputs will not contribute significantly to the measurement errors can be selected. It is suggested that the random test signals be recorded on magnetic tape so that the identical signals can be analyzed by the statistical analyzer and a digital computer. The digital computer analysis will accurately define the statistical properties of the actual test signal so that the problems associated with imperfections in the random noise generator and statistical uncertainty flucutations can be avoided. The analytical derivation of all of the above statistical functions for sinusoidal input signals are included to illustrate the operating principles of this analyzer.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

GLOSSARY

a = the ratio of the zero to peak value of a sine wave to the rms value of the noise

A = the zero to peak amplitude of a periodic wave

B. = the bandwidth of the data being analyzed

e = voltage

E = the maximum error

f = the frequency in cycles per second

$g(\alpha)\,d\alpha$ = the expected number of positive maxima per unit time occurring in a narrow amplitude window

$h_n(z)$ = the probability density function of extreme values

$H_n(z)$ = the probability distribution function of extreme values

= the total number of positive maxima per unit time

M_α = the expected number of positive maxima per unit time in excess of the level α

n = the number of independent samples

N = the number of cycles

N_α = the expected number of threshold crossings per unit time

$p(\alpha)$ = the probability density function of instantaneous amplitudes

$p_p(\alpha)$ = the peak value probability density function

$p(\alpha, \beta)$ = a joint probability density function

$p(\beta/\alpha)$ = a conditional probability density function

$P(\alpha)$ = the positive cumulative probability distribution function of instantaneous amplitudes

$P_p(\alpha)$ = the peak value probability distribution function for positive peaks below the level α

$P(\alpha, \beta)$ = a joint cumulative probability distribution function of instantaneous amplitudes

$P(\beta, \alpha)$ = a conditional cumulative probability distribution function of instantaneous amplitudes

$Q(\alpha)$ = the negative cumulative probability distribution function of instantaneous amplitudes

$Q_p(\alpha)$ = the peak value probability distribution function for positive peaks in excess of the level α

RC = the time constant of a simple RC integrator

t = time

T = a specific value of time

x_0 = a specific amplitude value

z = the indicated average time in the window width Δx

α = a specific value of the normalized amplitude

$\delta(\beta-\alpha)$ = the Dirac delta function at $\beta = \alpha$

Δt = the time inside of each Δx amplitude window

Δx_0 = a small amplitude increment

$\Delta\alpha$ = the width of the analyzer window

λ = percent error

ξ = the average time spent in the amplitude window width when averaging is performed over an integral number of cycles

1. INTRODUCTION

The purpose of this report is to formulate a series of tests for the thorough evaluation of statistical analyzers. Statistical analyzers are those machines used to study the amplitude characteristics of data signals. The tests included are designed to evaluate the magnitude of the intrinsic machine errors of the above analysis system. Errors from other sources should be insignificant as:

1. It is assumed that these tests will be carefully performed so that human errors will not occur.
2. Determination of the accuracy when random signals are used as test signals is handled in a manner that permits one to neglect the statistical errors that occur from taking a finite sample from a theoretically infinite random process.
3. These tests will be conducted under near ideal conditions in a laboratory environment so that the usage and environmentally related machine errors will be negligible.

The tests can be segregated into four categories. These are:

1. tests with periodic wave shapes
2. tests with random wave shapes
3. tests of stability
4. tests of miscellaneous features

In the first category, the basic operating modes of the statistical analyzer can be checked quite accurately because the input signals are completely deterministic and their appropriate probability functions can be easily calculated and compared against the output of the analyzer. The periodic signals are also much simpler to generate in an undistorted form than are the random signals.

Sinusoidal and triangular waves are suggested for the tests with periodic signals since these two wave shapes are relatively easy to obtain and they have markedly different probability density and cumulative distribution functions.

For the evaluation tests to be made with random signal inputs, broadband Gaussian noise, narrow band Gaussian noise, and clipped broadband Gaussian noise are suggested for inputs. These inputs are designed to provide a thorough check on the various analysis modes of this statistical analyzer, and still be relatively easy to generate. (All the equipment that is required to generate the above inputs is a broadband Gaussian noise generator, a bandpass filter, and a clipping circuit.) These random inputs will provide contrasting instantaneous amplitude, peak, and extreme value density and distribution functions, as well as different level crossing rates.

The stability test recommended utilizes a simple periodic triangular wave plus DC as an input. By repeating this test over an extended period of time, the stability of the analyzer, including the mean value detector, can be determined.

Under the miscellaneous category, tests are recommended to evaluate the performance and/or effects of the following items:

1. normalizing meter
2. frequency of periodic inputs
3. sweep rate of the amplitude aperture
4. mean value detector
5. differentiator and integrator

For convenience and easy reference, all figures have been consolidated at the end of the report in numerical order.

2. SIGNAL GENERATOR REQUIREMENTS

The distortion and noise levels of all periodic signal generators must be carefully measured before they are approved for these evaluation tests. The performance of the analyzer will not be accurately evaluated if the density function of the test signal deviates significantly from its theoretical form. It is recommended that the waveform on the output of the signal generator be required to fall within $\pm$ 1% of the theoretical waveform at all points, and the derivative of the signal be compared to the derivative of the theoretical waveform. Determination of the allowable signal-to-noise ratio and percent distortion on the signal generator output for a given percent error in the probability density function is quite complicated. For the case of a sinusoidal waveform, the calculations for the effects of distortion on the density function result in an integral that cannot be integrated in closed form.

The allowable signal-to-noise ratio for a sinusoidal input can be determined by the techniques described in Reference 1. Since the probability density function of a sine wave is infinite where the ratio of the instantaneous value to the rms value is equal to $\pm \sqrt{2}$, a compromise must be made when this type of signal is used to evaluate the performance of a probability density function analyzer. It is entirely reasonable to expect that the analyzer will have no more than its advertised error if computations are performed at

(1) more than one amplitude window width away from the above ratio

(2) a value of probability density less than the analyzers full scale rating (usually 1.0)

The probability density function of a sine wave with zero mean value, (see Reference 2 or Reference 3 for development) is

$$p(\alpha) = \frac{1}{\pi\sqrt{2 - (\alpha)^2}} \qquad \text{when} \qquad -\sqrt{2} \le \alpha \le +\sqrt{2} \tag{1}$$

$$= \quad 0 \qquad\qquad \text{otherwise}$$

where

$p(\alpha)$ = the probability density of instantaneous amplitudes (dimensionless)

$\alpha = \left(\frac{x_0}{\sigma}\right)$ = a specific value of the normalized instantaneous amplitude

σ = the standard deviation of the underlying process $x(t) = \sqrt{2}\ \sigma \sin \omega t$

x_0 = a specific value of $x(t)$

Assume that the amplitude window width of the analyzer is equal to 0.1σ, the full scale density is 1.0, and that $x(t)$ has a zero mean value. Therefore, the density function at α equal to $\pm$ 1.3 should be properly analyzed if its magnitude is less than 1.0 because $\alpha = 1.3$ is slightly more than one window width away from $\alpha = \sqrt{2}$. Using Eq. (1), the value of density function is found as follows. (Because this density function is symmetrical, calculations will be performed only on positive values of α.)

$$p(1.3) = \frac{1}{\pi\sqrt{2 - (1.3)^2}}$$

$$= .571$$

Next, the averaging effect of a finite window width should be checked to see if the rapidly changing slope of the density function near $\sqrt{2}$ will cause an appreciable error. (The analyzer operation assumes that the slope of $p(x)$ is linear in the window.) To do this, the density is integrated over the window width and divided by the magnitude of the window width.

$$p(1.3) \text{ indicated} = \frac{1}{0.1} \int_{1.25}^{1.35} \frac{1}{\pi\sqrt{2 - u^2}} \, du$$

$$= \frac{1}{0.1\pi} \left[\sin^{-1} \frac{u}{\sqrt{2}} \right]_{1.25}^{1.35}$$

$$= .585$$

Comparing this indicated value to the true value of a sine wave, it can be seen that

$$\begin{array}{r} .585 \\ -.571 \\ \hline .014 \end{array}$$

The indicated density function is high by about 1.4% of full scale $\left[p(x) \text{ full scale} = 1.0 \right]$. This factor could be accounted for, but to simplify the book-keeping a point is chosen that is just slightly more than two window widths away from the point where $\alpha = \sqrt{2}$. The true value of the probability density function at $\alpha = 1.2$ is 0.425; the indicated value will be 0.428. Therefore, the averaging error at $\alpha = 1.2$ can be neglected for all practical purposes.

The maximum allowable signal-to-noise ratio on the output of the sinusoidal signal generator for a given error can be found from Reference 1. Assume that a 1% of full scale error from generator noise is permissible, and that the full scale density is 1.0. Then one percent of full scale error is equal to an error in the density of 0.010. Therefore, let the sum of a sine wave plus random noise have a density function value of $p(1.2) + 0.010 = 0.425 + 0.010 = 0.435$ at $\alpha = 1.2$. From Reference 1, an approximation to the density function of a sine wave plus noise, when the ratio of the zero to peak amplitude of the sine wave to the rms value of the noise is large, is given by

$$p(\alpha)_{\substack{\text{sine +}\\ \text{noise}}} \approx \frac{\left(\sqrt{1 + \frac{a^2}{2}}\right)\left[F(y - a)\right]}{\sqrt{a}} \qquad (2)$$

where

$$y = \alpha \left(\sqrt{1 + \frac{a^2}{2}}\right) \qquad (3)$$

a = the ratio of the zero to peak value of the sine wave to the rms value of the noise

F(y - a) = a function graphed in Figure 3 of Reference 1 which permits the approximation of $(\alpha)_{\substack{\text{sine +}\\ \text{noise}}}$ when a is large.

Because a is large, the above equations can be further simplified to the following form:

$$p(\alpha)_{\substack{\text{sine +}\\ \text{noise}}} \approx \frac{\left(\sqrt{a}\right)\left[F(y - a)\right]}{\sqrt{2}} \qquad (2a)$$

$$y \approx (\alpha)\frac{a}{\sqrt{2}} \qquad (3a)$$

It is now desired to solve Eq. (2a) for the value of a such that $p(1.2)_{\substack{\text{sine +}\\ \text{noise}}} = 0.435$.

An iterative procedure can be used. Over most of the range graphed, F(y - a) is between 0.1 and 0.23. As a first attempt, assume that F(y - a) is equal to 0.1. This value occurs only at (y - a) = +0.8.

$$0.435 = \left(\sqrt{a}\right)\left(\frac{0.1}{\sqrt{2}}\right)$$

$$= \quad 37.8$$

$$y = \frac{(1.2)}{\sqrt{2}}\ (37.8)$$

$$= 32.2$$

$$y - a = (32.2 - 37.8) = -5.6$$

which differs from the value of (y - a) where F(y - a) = 0.1. After several iterations of the above steps, one finds that $a \approx 20$.

Next, one should check to see that errors greater than 1% of full scale do not occur at values of $|\alpha| \leq 1.2$. From Figure 1 in Reference 1, this appears to be a reasonable assumption. The graphed values of F(y - a) permit a check only where α is in the range between 1.13 and 1.56 when a is equal to 20. A check will now be made on the magnitude of the error due to a signal-to-noise ratio of 20 at $\alpha = 1.15$.

$$y = (1.15)\left(\sqrt{1 + \frac{20^2}{2}}\right)$$

$$= 16.3$$

$$y - a = 16.3 - 20 = -3.7$$

$$F(-3.7) = .124$$

$$p(\alpha)_{\substack{\text{sine +}\\ \text{noise}}} = \left(\sqrt{1 + \frac{20^2}{2}}\right)\frac{(.123)}{\sqrt{20}}$$

$$= .390$$

The value of the probability density of a pure sine wave at $\alpha = 1.15$ is

$$p(\alpha)_{\text{sine}} = \frac{1}{\pi\sqrt{2 + (1.15)^2}}$$

$$= .386$$

The percent of full scale error is

$$\frac{(.390 - .386)}{1.00} \times 100 = 0.4\%$$

This is less than the 1% error at $\alpha = 1.2$.

Further check of the error at $\alpha = 1.3$ reveals that it is about a 1.3% error. So it is a reasonable assumption that the error is continually increasing from $\alpha = 0$ to beyond $|\alpha| = 1.2$ when $a = 20$.

3. MISCELLANEOUS TESTS

This category of tests should be performed first since these tests check some of the basic operating features that must be used to perform all other tests. If any of the following tests reveal malperformance of the analyzer, all testing should be discontinued until the performance can be corrected.

For most of these tests, a triangular wave superimposed upon a DC voltage will be used as the input. This particular input is chosen because it has a uniform probability density function with a nonzero mean value. Figure 1 depicts the input wave shape and Figure 2 shows the resulting probability density of instantaneous amplitudes. That the probability density function of a triangular wave is uniform can be seen from inspection of the wave shape. Because both the positive and negative slopes of the wave are linear, the probability that the signal will fall between $x_0 + \Delta x_0$ where

x_0 = an arbitrary voltage amplitude

Δx_0 = a small voltage increment

is independent of the magnitude of x as long as $-A \leq x_0 \leq A$ where A is the zero to peak amplitude of the triangular wave, and is equal to 1/2A. The probability is zero that x_0 is outside of these bounds. Note that the positive and negative slopes of the triangular wave do not need to be equal.

Before starting any of the following tests, the triangular wave shape should be carefully checked as described in Section 1.

3.1 TEST OF NORMALIZING METER ACCURACY

Almost all commercial statistical analyzers have a voltage level normalizing circuit on their input. This normalizing circuit maintains the rms voltage level into the main analyzer sections constant. There are three basic reasons for inclusion of this circuit. First, it permits the dynamic range of the main analyzer sections to be independent of the level of the data signal. Second, it greatly simplifies calibration of the relation of the amplitude window to the standard deviation of the signal, and calibration of the readout. Third, it provides a measure of the rms level of the data signal if the normalizing controls are appropriately designed.

The purpose of the following test is threefold. First, the amplitude linearity of the statistical analyzer is determined; secondly, the accuracy of the normalizing meter is measured; and thirdly, the accuracy of the long term averager is measured. The test signal is a triangular wave voltage superimposed upon a DC voltage. The peak to peak or rms voltages can be measured. In either case, the input voltage should be measured to an accuracy of .01% of the reading or better.

The rms value of a triangular wave can be calculated from Figure 1. Assume for simplicity that the DC value is zero so that the amplitude of the triangular wave varies with -A to +A. Then the expression for the triangular wave can be written as follows

$$f(t) = \begin{cases} -A + \left(\dfrac{2A}{T_1}\right) t & ; \; 0 \leq t \leq T_1 \\ A - \left(\dfrac{2A}{T_2 - T_1}\right)(t - T_1) ; & T_1 \leq t \leq T_2 \end{cases} \tag{4}$$

The rms value is simply

$$e_{rms} = \sqrt{\frac{1}{T_2}\int_0^{T_2}\left[f(t)\right]^2\,dt}$$

$$= \sqrt{\frac{1}{T_2}\left[\int_0^{T_1}\left[-A + \left(\frac{2A}{T_1}\right)t\right]^2 dt + \int_{T_1}^{T_2}\left[A - \left(\frac{2A}{T_2 - T_1}\right)(t - T_1)\right]^2 dt\right]}$$

$$= A\sqrt{\frac{1}{3}} \tag{5}$$

$$= (.57735)A$$

Note that this result is independent of the slopes of the triangular wave. If there is a DC voltage (e_{DC}) present, then the total rms voltage simply is

$$e_{rms} = \sqrt{\frac{A^2}{3} + (e_{DC})^2} \tag{6}$$

Compute the probability density functions and record the DC level indicated by the mean value detector over the entire permissible input voltage range as shown below for an analyzer that has a -5 volt to +5 volt input range. Then compare the results to the theoretical values. (Set the triangular wave frequency at about one octave above the lowest advertised operating frequency.)

test	zero to peak triangular wave voltage	DC voltage
(1)	5.000	0
(2)	0.500	0
(3)	0.050	0
(4)	0.005	0
(5)	2.5	+2.5
(6)	2.5	-2.5
(7)	0.500	+4.500
(8)	0.050	+4.950
(9)	0.005	+4.995
(10)	0.005	-4.995
(11)	4.500	+0.500
(12)	4.950	+0.050
(13)	4.995	+0.005
(14)	4.995	-0.005

All of the above tests should result in probability density functions that fall within the advertised accuracy of the analyzer.

3.2 TEST OF FREQUENCY RESPONSE

For this series of tests use a test signal composed of a positive DC voltage equal to one-half of the full scale voltage range superimposed upon a triangular wave whose peak to peak voltage level is also equal to one-half of the full scale voltage range. Compute the probability density function at a number of frequencies over the advertised frequency range, or ranges, of the analyzer. Also record the DC voltage indicated by the mean value detector. Compare the computed values to the theoretical values (density function and mean) and determine if the advertised accuracy is obtained. The following example is from a series of tests designed to measure the frequency response of an analyzer that had an advertised upper operating frequency of 20 KC, and an advertised lower operating frequency of 4, 1, or 0.1 cps, depending upon the time constant setting of the mean value detector (MVD) circuit (this circuit provides high pass filtering of the data signal in addition to measuring the mean value).

test	M. V. D. time constant	input frequency	test	M. V. D. time constant	input frequency
(1)	(short)	4	(12)		1,000
(2)		8	(13)		10,000
(3)		10	(14)		20,000
(4)		100	(15)	(long)	0.1
(5)		1,000	(16)		0.2
(6)		10,000	(17)		1
(7)		20,000	(18)		10
(8)	(medium)	1	(19)		100
(9)		2	(20)		1,000
(10)		10	(21)		10,000
(11)		100	(22)		20,000

3.3 TEST OF SWEEP RATE

The probability density function of a periodic signal should be "in theory" independent of the sweep rate. But, in fact, the indicated density function will be dependent upon two factors.

First, in any analyzer that uses RC averaging instead of true integration, the allowable sweep rate is controlled by the magnitude of the time constant of the averager. The error caused by failure of the output of the RC circuit to accurately track the probability density function as the amplitude window is scanned is called the smoothing error. The faster the window is scanned over a given amplitude range for a given RC time constant, the greater will be the magnitude of the smoothing error. The probability density function of periodic signals or truncated random signals may have infinite slopes at their end points. One method of determining the allowable sweep rate for these types of density functions is to assume that the RC circuit has a step input of voltage applied. This voltage step is proportional to a step change in the probability density function. The time required for the smoothing error to diminish to an allowable value is then computed as follows.

The response of an RC circuit to a step input voltage of unit amplitude is

$$e_{out} = 1 - e^{-t/RC} \tag{7a}$$

where

e_{out} = the voltage at the output of the RC circuit

RC = the time constant of the circuit

The percent error by which the output fails to track the input step function is

$$\% \text{ smoothing error} = 100\, e^{-t/RC} \tag{7b}$$

If the analyzer is to be accurate to 1%, the time required for the analyzer to reach 99% of the final value can be found from Eq. (7a) or (7b) to be 4.6 time constants. Thus, the sweep rate used should be set so that 5 or more time constants are required to sweep through one amplitude window width. If the data being analyzed has an exactly Gaussian probability density function, digital computer studies have shown that 2 or more time constants per window width are required for the smoothing error to be 1% of full scale or less. Since the tests in this report are designed to accurately evaluate the performance of the analyzer, a sweep rate should be selected so that the resulting error is about an order of magnitude below the advertised error figure for the analyzer.

The second practical consideration limiting the sweep rate is the number of cycles of the periodic function over which averaging is performed. If this average were always to be taken over an integral number of cycles, then there would be no problem. However, if the averaging is over a fraction of a cycle or n integral cycles plus a fraction, then an error in the averaging occurs. Obviously, the greater the number of integral cycles, the lower the percentage error for a given fraction of a cycle of a particular periodic wave. This error can be computed from Figure 3 for a triangular wave.

To simplify computations, the triangular wave will be assumed to have complete symmetry and only the bounds on the maximum error will be computed. When Δx is located very near to the positive peak of the triangular wave in Figure 3, the maximum positive error occurs, and when Δx is located very near to the negative peak of the triangular wave, the maximum negative error occurs. The average time that the triangular wave is in the interval Δx is $2\Delta t/T$ when the averaging is performed over an integral number of cycles. The indicated average value reads high when the Δx window is near to the top of the triangular wave. Assume that Δx is very small and that the average has been carried out over N cycles plus Δt. Then the indicated average is

$$Z = \frac{2N\Delta t + \Delta t}{NT + \Delta t} \approx \frac{(N + 1/2)\,2\Delta t}{NT} \quad \text{since } \Delta t << T \qquad (8)$$

where

Z = the indicated average time in the window width Δx (Probability density function when $\Delta x = 0$)

N = the integral number of cycles over which averaging is performed

Δt = the time inside each Δx amplitude window

T = the period of the triangular wave

Equation (8) can be modified slightly to read

$$Z = \xi \left(\frac{N + 1/2}{N} \right)$$

where

$\xi = \frac{2\Delta t}{T}$ = the average time spent in the window width Δx when averaging is performed over an integral number of cycles.

The maximum positive error in percent is

$$\begin{aligned} E_1 &= \left(\frac{Z - \xi}{\xi} \right) 100 \\ &= \left[\frac{N + 1/2}{N} - 1 \right] 100 \\ &= \frac{50}{N} \end{aligned} \tag{9}$$

Thus, for the probability density function to read 1% higher than the theoretical value, N must be 50 cycles.

The indicated average time reads low when the Δx window is near to the negative peak. To determine the maximum negative error, assume that an average is taken over N cycles plus one-half of a cycle minus Δt. If we assume Δx is very small, then Δt becomes negligible with respect to $T/2$ and the indicated average value of time is

$$Z = \frac{2N\Delta t}{NT + \frac{T}{2}}$$

$$= \xi\left(\frac{N}{N + 1/2}\right) \tag{10}$$

The maximum negative error in percent is

$$E_2 = \left(\frac{Z - \xi}{\xi}\right) 100$$

$$= \left(\frac{N}{N + 1/2} - 1\right) 100 \tag{11}$$

$$= \frac{-100}{2N + 1}$$

Hence, the positive error is the controlling error.

Although the above development is for the special case of a triangular wave, note that in reality the result in Eq. (9) applies for any periodic wave shape. To reach this general result, replace the Δt in Eq. (8) by Δt_i, the time inside the Δx amplitude window when it is centered at the ith amplitude level. (For the triangular wave Δt_i is independent of i.) However, in the development of Eq. (9), it can be seen that $\xi_i = 2\Delta t_i/T$ and that the Δt_i's divide out of the final result.

In terms of the allowable percent error for a nonintegral number of cycles, the total sweep time for a triangular wave should be

$$T_s > \left(\frac{1}{f}\right)\left(\frac{\alpha_2 - \alpha_1}{\Delta\alpha}\right)\left(\frac{50}{E_1}\right) \tag{12a}$$

where

T_s = the time in seconds for a complete amplitude scan

f = the frequency of the triangular wave in cycles per second

α_1 = the amplitude value at which the scan is started

α_2 = the amplitude value at which the scan is ended

$\Delta\alpha$ = the width of the analyzer window

For an analyzer that has a window width of 0.1, an error of 0.25%, and scans only between equal starting and ending amplitude values, the scan time is

$$T_s > \left(\frac{4000}{f}\right) \alpha_1$$

The above discussion also illustrates the necessity for high pass filtering of the data before it is applied to the main analyzer sections. Rearranging Eq. (12a) in terms of the time per amplitude window

$$t/\text{window} = \left[\frac{T_s}{\left(\frac{\alpha_2 - \alpha_1}{\Delta\alpha}\right)}\right] = \frac{50}{fE_1} \tag{12b}$$

or in terms of the lowest frequency permissible for a given scan time

$$f = \left(\frac{1}{T_s}\right)\left(\frac{\alpha_2 - \alpha_1}{\Delta\alpha}\right)\left(\frac{50}{E_1}\right) \tag{12c}$$

To numerically illustrate this result, assume that 1000 seconds has been chosen for a complete scan between -6σ and $+6\sigma$ on an analyzer with a 0.1σ window width. Further assume that only 0.25% error from a non-integral number of cycles is permissible. The lowest frequency permissible in the data is

$$f = \left(\frac{1}{1000}\right)\left(\frac{12}{0.1}\right)\left(\frac{50}{0.25}\right) = 24 \text{ cps} \tag{13}$$

To test the scan rate circuitry, the amplitude level voltage should be recorded on an oscillograph along with an accurate timing signal. The scanned voltage level versus time should be recorded for a number of scan rates between, and including the maximum and minimum scan rates. Deviations of the voltage versus time curve from linearity should be measured and compared against the advertised linearity for the scan rate.

3.4 TEST OF THE INTEGRATOR AND DIFFERENTIATOR

An auxiliary feature of some statistical analyzers is that a portion of the circuitry required to perform special analyses can be used to provide integration or differentiation of the data signal when the circuitry is not required for the special analyses. Because these are strictly auxiliary features, only relatively simple tests are recommended.

3.4.1 Integrator Test

The integrator output should theoretically be

$$e_{out}(t) = \int e_{in}(t)\, dt \tag{14}$$

Actually, the output voltage is dependent upon the frequency of $e_{in}(t)$ since all practical integrators have upper and lower frequency limits. If the input voltage is sinusoidal, the output voltage should be a cosine wave of the same frequency as the input and whose amplitude is inversely proportional to the frequency since

$$\int A \sin 2\pi ft\, dt = \frac{-u}{2\pi f} \cos 2\pi ft$$

where u = the integrator gain times the amplitude of the sine wave.

Use the above equation in conjunction with the gain and output voltage characteristics of the integrator under test to calculate the theoretical integrator output voltage as a function of the frequency of a constant amplitude sinusoidal input voltage. Apply test voltages at the frequencies used in these calculations. Compare the measured and theoretical results and determine if the advertised accuracy for the integrator is met. The phase shift should also be monitored. This should be 90° or 270°, depending upon the measurement points. From the measurements above, determine the frequency where the signal to noise ratio $\left(\frac{\text{rms output volts}}{\text{rms noise volts}}\right)$ is equal to ten, and the frequency where this ratio is equal to one.

As a further check, apply a one-volt rms square wave voltage to the integrator. The output voltage should be a triangular wave. Vary the frequency of the square wave and determine the low frequency where the wave shape on the output of the integrator departs from the shape of a theoretical triangular wave by 5%. Also determine the high frequency where there is 5% distortion on the triangular wave. This may be due to a low signal to noise ratio in this case. (Note that the square wave source should be checked on a high quality integrator to assure that source distortion is low enough to permit the above measurements.)

The test conditions for an integrator with an advertised frequency range of 2 cps to 20 KC and a 7.07 volts rms output voltage for a 1.0 volt rms 2 cps sinusoidal input are listed below:

frequency	theoretical output voltage (rms)
2	7.07
3	4.72
4	3.54
6	2.36
8	1.77
10	1.41
20	.707
40	.354
80	.177
100	.141
200	.0707
1000	.0141
2000	.00707
10,000	.00141
20,000	.000707

3.4.2 Differentiator Test

Theoretically, the output of the differentiator should be

$$e_{out}(t) = \frac{d\left[e_{in}(t)\right]}{dt} \tag{15}$$

Actually, the upper frequency of the input is limited in all practical differentiators. If the input voltage is sinusoidal, the output voltage will be a cosine wave of the same frequency as the input and whose amplitude is directly proportional to the frequency since

$$\frac{d(A \sin 2\pi ft)}{dt} = 2\pi fu \cos 2\pi ft$$

where u = the differentiator gain times the amplitude of the sine wave. Apply a full scale sinusoidal signal at the upper cutoff frequency. The output voltage should be the maximum permissible. Maintain this full scale input to the differentiator and measure the output voltage and phase shift at a number of frequencies over the frequency range of the differentiator. Compare these measured voltages to the theoretical voltages computed by use of the above equation and determine if the advertised accuracy is attained. Also monitor the phase shift between the input and output. It should be 90^o or 270^o depending upon the measurement points. Also measure the rms noise level on the output of the differentiator when the input to the analyzer is shorted. From the above measurements, determine the frequencies where the rms signal to rms noise ratio equals ten and equals one.

As an additional check of the differentiator, apply a triangular wave of one volt rms. The output should be a square wave voltage. Vary the frequency of the triangular wave and measure the upper and lower (if possible) frequencies where the square wave has 5% distortion in its wave shape. (Note that the triangular wave generator must be carefully checked for distortion prior to this test.)

The test conditions for the differentiator with three different advertised frequency ranges of 200 cps, 2 KC, and 20 KC are listed below. This particular one has a 7.07 volt rms output whenever a one volt rms sinusoid of the highest rated frequency is applied.

frequency in cps			theoretical output voltage (rms)
200 cps range	2000 cps range	20 KC range	
200	2000	20 KC	7.07
170	1700	17 KC	6.01
140	1400	14 KC	4.95
110	1100	11 KC	3.88
85	850	8.5 KC	3.00
55	550	5.5 KC	1.95
30	300	3 KC	1.06
15	150	1.5 KC	0.531
10	100	1 KC	0.354
5	50	500	0.177
2	20	200	0.0707
---	10	100	0.0354
---	5	50	0.0177
---	2	20	0.00707
---	---	10	0.00354
---	---	5	0.00177
---	---	2	0.000707

4. TESTS WITH PERIODIC INPUTS

Periodic waves represent a convenient method to check out all of the analysis modes of a statistical analyzer because their probability parameters are well defined and the analyses have zero statistical uncertainty fluctuations associated with them. Therefore, it is recommended that these tests with periodic inputs be performed before the tests with random inputs. Specifically, sine waves and triangular waves should be used as inputs.

This section is divided into three parts. The first part discusses the meaning of the various probability functions that can be measured with this analyzer, and analytically derives the theoretical results for the analysis of sinusoidal data. The second part lists tests with sinusoidal inputs that should be conducted to evaluate the performance of the analyzer. The third part lists tests with triangular wave inputs that should be conducted to evaluate the performance of the analyzer.

4.1 DERIVATION OF PROBABILITY FUNCTIONS

To illustrate how these analyzers compute all of the various probability functions, the corresponding theoretical probability functions of a sine wave are analytically derived.

4.1.1 Instantaneous Amplitude Probability Density Function

The instantaneous amplitude probability density function of a sine wave is stated in Eq. (1) to be

$$p(\alpha) = \frac{1}{\pi\sqrt{2 - (\alpha)^2}} \quad \text{when} \quad -\sqrt{2} \leq \alpha \leq +\sqrt{2}$$

$$= 0 \qquad ; \quad \text{otherwise}$$

To more thoroughly illustrate how this is derived, assume that

$$x = A \sin \theta \tag{16}$$

where

$A =$ the zero to peak amplitude of the sine wave

Therefore,

$$\theta = \sin^{-1}\left(\frac{x}{A}\right) \tag{17}$$

If the function x is sampled randomly in time, any value of the angle θ is equally likely because it varies linearly between limits with time. To make the function θ single valued, limit it to be between $-\pi/2$ and $\pi/2$. Because the area under the probability density function must equal unity, the density,

$$\begin{aligned} p(\theta) &= \frac{1}{\pi} \quad ; \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ &= 0 \quad ; \quad \text{elsewhere} \end{aligned} \tag{18}$$

To find the density of x_0, we set

$$p(x_0)\,dx = p(\theta)\,d\theta \tag{19}$$

which can be done because the probability of being in the interval $(y, y+\Delta y)$ is equal to being in the interval $f(y, y+\Delta y)$ where $f(y)$ is a function of y with only fairly general restrictions. Rearranging

$$p(x_0) = p(\theta)\,\frac{d\theta}{dx}$$

$$\frac{d\theta}{dx} = \left(\frac{1}{A}\right)\left(\frac{1}{\sqrt{1-\left(\frac{x_0}{A}\right)^2}}\right)$$

$$p(x_0) = \left(\frac{1}{\pi}\right)\left[\left(\frac{1}{A}\right)\left(\frac{1}{\sqrt{1-\left(\frac{x_0}{A}\right)^2}}\right)\right] \tag{20}$$

To get into the normalized form of most analyzer inputs, we want $p(x_0/\sigma)$ where σ is the standard deviation of x. Let $\alpha = (x_0/\sigma)$, then as before we can say

$$p(\alpha)\, d\alpha = p(x_0)\, dx$$

$$p(\alpha) = p(x_0)\, \frac{dx}{d\alpha}$$

$$x = \sigma \alpha$$

$$\frac{dx}{d\alpha} = \sigma = \frac{A}{\sqrt{2}}$$

$$p(\alpha) = \left(\frac{1}{\pi}\right)\left(\frac{1}{A}\right)\left(\frac{1}{\sqrt{1 - \left(\frac{x_0}{\sigma\sqrt{2}}\right)^2}}\right)\left(\frac{A}{\sqrt{2}}\right)$$

$$= \left(\frac{1}{\pi}\right)\left(\frac{1}{\sqrt{2 - 2\left(\frac{\alpha}{\sqrt{2}}\right)^2}}\right)$$

$$= \left(\frac{1}{\pi}\right)\left(\frac{1}{\sqrt{2 - \alpha^2}}\right) \quad ; \quad |\alpha| \leq \sqrt{2} \qquad (21)$$

This function is plotted in Figure 4.

4.1.2 Instantaneous Amplitude Probability Distribution Functions

The positive cumulative distribution function for the instantaneous amplitudes is the probability that the random variable will be less than some given amplitude and is simply the integral of the density function. For a sine wave

$$P(\alpha) = \int_{-\infty}^{\alpha} p(u)\, d(u) \qquad (22)$$

Thus,

$$P(\alpha) = \int_{-\sqrt{2}}^{\alpha} \left(\frac{1}{\pi}\right)\left(\frac{1}{\sqrt{2-(u)^2}}\right) du$$

$$= \left(\frac{1}{\pi}\right)\left[\sin^{-1}\left(\frac{\alpha}{\sqrt{2}}\right) + \frac{\pi}{2}\right] \quad ; \quad -\sqrt{2} < \alpha < +\sqrt{2} \tag{23}$$

$$= 0 \quad ; \quad \alpha < -\sqrt{2}$$

$$= 1 \quad ; \quad \alpha > \sqrt{2}$$

This function is plotted in Figure 5.

The negative cumulative distribution is the probability that some random variable exceeds a given amplitude and is simply equal to one minus the positive cumulative distribution function

$$Q(\alpha) = 1 - P(\alpha) \tag{24}$$

For a sine wave

$$Q(\alpha) = 1 \quad ; \quad \alpha < -\sqrt{2}$$

$$= 0.5 - \left(\frac{1}{\pi}\right)\left[\sin^{-1}\left(\frac{\alpha}{\sqrt{2}}\right)\right] \quad ; \quad -\sqrt{2} \le \alpha \le +\sqrt{2} \tag{25}$$

$$= 0 \quad ; \quad \alpha > +\sqrt{2}$$

This function is plotted in Figure 6.

4.1.3 Expected Number of Threshold Crossings per Unit Time

The expected number of threshold crossings per unit time, N_α, describes the mean number of times per unit time that the process $x(t)$ crosses through a narrow amplitude window centered at α. Note that this is neither a probability density or distribution function,

$$\int_{-\infty}^{\infty} (N_\alpha)\, d\alpha \neq 1 \quad \text{and} \quad N_\infty \neq 1$$

For a sine wave, this function is zero at α less than $-\sqrt{2}$ and at α greater than $+\sqrt{2}$. For $-\sqrt{2} \leq \alpha \leq +\sqrt{2}$, N_α is independent of the value of α and is proportional to the frequency of the sine wave. (See Figure 7.) For example, the magnitude of N_α for a 200 cps sine wave is four times the magnitude of N_α for a 50 cps sine wave. In addition to measuring the total expected number of times per unit time that the process crosses through the amplitude window, the analyzer can measure the mean number of times per unit time that the process crosses through the amplitude window with positive slope only, or with negative slope only. The crossings with positive slope are designated N_α^+ and can be thought of as the mean number of times per unit time that the level α is exceeded. The crossings with negative slope are designated as N_α^-. Both N_α^+ and N_α^- are equal to one-half the total crossing rate.

4.1.4 Expected Number of Maxima per Unit Time

(a) In a Narrow Amplitude Window

The expected number of positive maxima per unit time occurring in a narrow amplitude window, $g(\alpha)\, d\alpha$, is determined by measuring the average number of times that the process $y(t)$ falls inside this amplitude window while simultaneously having the first derivative equal to zero, and the second derivative negative. In equation form (see Reference 3),

$$g(\alpha)\, d\alpha = d\alpha \int_0^{-\infty} p(\alpha, 0, \nu)\, \nu\, d\nu \qquad (26)$$

where

$$p(\alpha, \beta, \nu) = \frac{\text{Prob}\left[\alpha < y(t) \leq \alpha + d\alpha,\ \beta \leq y(t) < \beta + d\beta,\ \nu < y(t) \leq \nu + d\nu\right]}{d\alpha\, d\beta\, d\nu}$$

For a sine wave, positive maxima clearly occur only at $\alpha = +\sqrt{2}$. Therefore, $g(\alpha)\, d\alpha$ is equal to zero except for one amplitude window width centered at $\alpha = +\sqrt{2}$. (See Figure 8.) Here, the magnitude of $g(\alpha)\, d\alpha$ is directly proportional to the frequency of the sine wave. For example, a 50 cps sine wave has 50 maxima per second falling inside an amplitude window centered at $\alpha = +\sqrt{2}$ while a 200 cps sine wave has 200 maxima per second in this window. Therefore, the magnitude of $g(\alpha)\, d\alpha$ for the 50 cps sine wave is one-fourth that of the 200 cps sine wave.

(b) In Excess of a Specific Level

The expected number of positive maxima per unit time exceeding the level α is simply

$$M_\alpha = \int_\alpha^\infty g(u)\, du \qquad (27)$$

For a sine wave, it can be seen that $M_\alpha = 0$ for $\alpha > +\sqrt{2}$ as a sine wave vs. no peaks greater than the $\sqrt{2}$. Also, M_α will be constant from $-\infty$ to $+\sqrt{2}$ and will be directly proportional to the frequency of the sine wave (see Figure 9).

(c) Total

The total number of positive maxima per unit time, $\mathcal{M}$, is

$$\mathcal{M} = \int_{-\infty}^{\infty} g(u)\, du \qquad (28)$$

Only a single value is obtained through implementation of Eq. (28). This value is a direct measure of the frequency if the signal is periodic.

4.1.5 Peak Value (or Maxima) Probability Density Functions

(a) General

There are two peak probability distribution functions of interest. The peak value probability distribution function, $Q_p(\alpha)$, describes the probability of positive peaks exceeding the level α. This function is closely related to the expected number of maxima functions discussed in the preceding section. In fact, from Eqs. (27) and (28),

$$Q_p(\alpha) = \frac{M_\alpha}{\mathfrak{M}} \tag{29}$$

The peak value probability distribution function, $P_p(\alpha)$, describes the probability of positive peaks less than the level α occurring.

$$P_p(\alpha) = 1 - Q_p(\alpha) = 1 - \left(\frac{M_\alpha}{\mathfrak{M}}\right) \tag{30}$$

The peak probability distribution $Q_p(\alpha)$ of a sine wave has a shape identical to that for the expected number of maxima above a specified level, except for the scale factor $\mathfrak{M}$ (see Figure 10). $Q_p(\alpha)$ is independent of frequency and is zero below $\alpha = \sqrt{2}$ and equal to one above $\alpha = \sqrt{2}$. $P_p(\alpha)$, on the other hand, is equal to one below $\alpha = \sqrt{2}$ and equal to zero above $\alpha = \sqrt{2}$ (see Figure 11).

The peak value probability density function $p_p(\alpha)$ is defined by

$$p_p(\alpha) = \lim_{\Delta\alpha \to 0} \frac{P_p(\alpha + \Delta\alpha) - P_p(\alpha)}{\Delta\alpha} \tag{31a}$$

$$= \lim_{\Delta\alpha \to 0} \left(\frac{1}{\Delta\alpha}\right)\left(\frac{1}{\mathfrak{M}}\right)\left[M_\alpha - M_{(\alpha+\Delta\alpha)}\right] \tag{31b}$$

$$= \lim_{\Delta\alpha \to 0} \left(\frac{1}{\Delta\alpha}\right)\left(\frac{1}{\mathfrak{M}}\right)\left[g(\alpha)\ \Delta\alpha\right] \tag{31c}$$

In practice, the peak value probability distribution and density functions are usually measured by a counting process. This is equivalent to multiplying both the numerator and denominator of Eqs. (31b) and (31c) by a time interval T. Also, the limit as $\Delta\alpha$ approaches zero is not taken so that one obtains estimates of the quantities defined by

$$P_p(\alpha) = \frac{\mathfrak{M}T - M_\alpha T}{\mathfrak{M}T} \tag{32a}$$

$$p_p(\alpha) = \left(\frac{1}{\Delta\alpha}\right)\left(\frac{1}{\mathfrak{M}T}\right)\left[M_\alpha T - M_{(\alpha+\Delta\alpha)} T\right] \tag{32b}$$

or

$$p_p(\alpha) = \left(\frac{1}{\Delta\alpha}\right)\left(\frac{1}{\mathfrak{M}T}\right)\left[T\,g(\alpha)\ \Delta\alpha\right] \tag{32c}$$

To analyze the peak value probability distribution function both the total expected number of positive peaks and the expected number of positive peaks exceeding the level α are measured. The difference between these two values is divided by the total expected number of positive peaks as shown in Eq. (32a). The operation in Eq. (32b) is implemented by subtracting the expected number of positive peaks that exceed the level $\alpha + \Delta\alpha$ from the expected number of positive peaks that exceed the level α and dividing this difference by the product of the total expected number of positive peaks and the amplitude window width. Equation (32c) is similarly implemented except that the expected number of peaks occurring inside the amplitude window width is measured directly.

For sine waves, the theoretical peak value probability density function $p_p(\alpha)$ is a delta function at $\alpha = \sqrt{2}$. For actual analyzers, the peak value probability density function will be constant over one amplitude window width centered at $\alpha = \sqrt{2}$. The magnitude of the density function will be equal to $1/\Delta\alpha$ since $T\,g(\alpha)\,\Delta\alpha/\mathcal{M}T = 1$ for sine waves. (See Figure 9.)

(b) <u>For Narrowband Signals Only</u>

If the data being analyzed is restricted to having a bandwidth that is small compared to the center frequency of the bandwidth, the peak value probability distribution and density functions can be measured with simpler circuitry. In equation form, the restriction is

$$\left(f_c + \frac{B}{2}\right) - \left(f_c - \frac{B}{2}\right) << f_c \qquad (33)$$

where

B = the bandwidth of the data being analyzed

f_c = the center frequency of the above bandwidth

The virtue of the above restriction is that it means that there is one and only one positive peak, greater than the level α, associated with each crossing of the level α with positive slope. Thus, the expected number of maxima (or positive peaks) per unit time can be determined by measuring the expected number of level crossings with positive slope per unit time, N_α^+ . For very narrowband processes,

$$N_\alpha^+ = M_\alpha \qquad (34a)$$

and

$$N_0^+ = \mathcal{M} \qquad (34b)$$

Hence, the peak value probability distribution function, $Q_p(\alpha)$, (the probability of positive peaks exceeding the level α) can be found for narrowband processes as follows:

$$Q_p(\alpha) = \frac{N_\alpha^+}{N_0^+} \tag{35}$$

Similarly, the peak value probability distribution function, $P_p(\alpha)$, (the probability of positive peaks below the level α) for narrowband processes is:

$$P_p(\alpha) = 1 - \frac{N_\alpha^+}{N_0^+} \tag{36}$$

These functions are also usually measured by a counting operation.

$$Q_p(\alpha) = \frac{N_\alpha^+ T}{N_0^+ T} \tag{37a}$$

$$P_p(\alpha) = \frac{N_0^+ T - N_\alpha^+ T}{N_0^+ T} \tag{37b}$$

where

$N_\alpha^+ T$ = the expected number of crossings of the level α with positive slope in the time interval T

$N_0^+ T$ = the expected number of zero crossings with positive slope in the time interval T

The peak value probability density function for narrowband signals can be implemented by substituting Eqs. (34a) and (34b) into Eq. (32b).

$$p_p(\alpha) = \left(\frac{1}{\Delta\alpha}\right)\left(\frac{1}{N_0^+ T}\right)\left[N_\alpha^+ T - N_{(\alpha+\Delta\alpha)}^+ T\right] \tag{38}$$

Thus, the peak value probability density function of narrowband signals can be computed by dividing the difference between the expected number of crossing of the levels α and $(\alpha + \Delta\alpha)$ by the product of the analyzer's window width and the total expected number of zero crossings with positive slope. By inspection of Eq. (38), it can be seen that any difference between the expected number of threshold crossing with positive slope at the levels of α and $(\alpha + \Delta\alpha)$ must be equal to the expected number of positive peaks occurring in the amplitude window between α and $(\alpha + \Delta\alpha)$. (The narrowband assumption excludes negative peaks for $\alpha > 0$.)

It should be noted that no assumptions have been made on the shape of the instantaneous amplitude probability density function. It has only been assumed that the bandwidth of the data is small compared to the center frequency of the data so that there is only one positive peak corresponding to each positive zero crossing. For example, a sine wave meets the above restriction since it has zero bandwidth. If one were to compute the peak probability density function of a 100 cps sine wave at $\alpha = 1.35$ with an amplitude window width of 0.1 over a 5-second time interval,

$$N_{1.35}^+ = 100 \text{ per second}$$

$$N_{(1.35+.1)}^+ = 0 \text{ per second}$$

$$N_0^+ = 1000 \text{ per second}$$

From Eq. (38),

$$p_p(1.35) = \frac{(5)(100) - 0}{(5)(100)(.1)} = 10$$

When the data being analyzed has a finite bandwidth, the output of this analyzer will have an error associated with it that is a function of the bandwidth.

This occurs because the basic assumption of only one positive peak per positive zero crossing is violated when $B \neq 0$. Evaluation of the magnitude of this error is quite complicated because it is a function of the instantaneous amplitude probability density function and the spectral shape of the data. However, for illustrative purposes, consider the error occurring when the data signal being analyzed is stationary Gaussian noise with a zero mean value and a variance of unity. Rearranging Eq. (38),

$$p_p(\alpha) = \left(\frac{1}{\Delta\alpha}\right)\left[\left(\frac{N_\alpha^+}{N_0^+}\right) - \left(\frac{N_{(\alpha+\Delta\alpha)}^+}{N_0^+}\right)\right] \tag{39}$$

From Reference 4, Equation 13,

$$\frac{N_\alpha^+}{N_0^+} = \frac{p(\alpha)}{p(0)} \tag{40}$$

Equation (40) requires only that the random process being analyzed and its derivative be statistically independent as is the case for the Gaussian distribution assumed in this example. By substituting Eq. (40) into Eq. (39), one obtains

$$p_p(\alpha) = \left(\frac{1}{\Delta\alpha}\right) \frac{1}{p(0)} \left[p(\alpha) - p(\alpha+\Delta\alpha)\right] \tag{41a}$$

Notice that Eq.(41a) is completely independent of bandwidth. This means that one would obtain the same peak probability density function for any Gaussian process analyzed on this type of a machine. Obviously, this is not desirable. To estimate the magnitude of this error as a function of the bandwidth of the signal, a few calculations follow. Equation (41a) can be rewritten as

$$p_p(\alpha) = \left(\frac{1}{p(0)}\right)\left(\frac{1}{\Delta\alpha}\right)\left\{\exp\left[-\frac{(\alpha)^2}{2}\right] - \exp\left[-\frac{(\alpha+\Delta\alpha)^2}{2}\right]\right\}\left(\frac{1}{\sqrt{2\pi}}\right) \tag{41b}$$

because for a Gaussian process with zero mean value,

$$p(\alpha) = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-(\alpha)^2}{2}\right] \tag{42}$$

Equation (41b) becomes

$$p_p(\alpha) = \left(\frac{1}{p(0)}\right)\left(\frac{1}{\sqrt{2\pi}}\right) (\alpha) \exp\left[\frac{-(\alpha)^2}{2}\right] \tag{43}$$

if one takes the limit as $\Delta\alpha$ approaches zero. Equation (43) is essentially Eq. (41b) rewritten to remove the error due to a finite amplitude aperature width so that the error in the analyzer due to the data bandwidth can be examined separately. Notice that Eq. (43) has the form of a true Rayleigh probability density function.

The true peak probability density function for a Gaussian signal, from Reference 4, is

$$p_p(\alpha) = \frac{K_1}{\sqrt{2\pi}} \exp\left[\frac{-(\alpha)^2}{2K_1^2}\right] + \left(\frac{N_0}{2\mathcal{M}}\right) (\alpha) \exp\left[\frac{-(\alpha)^2}{2}\right]\left[1 - P_n\left(\frac{\alpha}{K_2}\right)\right] \tag{44}$$

where

$$K_1 = \sqrt{1 - \frac{N_0^2}{2\mathcal{M}}}$$

$$K_2 = \frac{K_1}{(N_0/2\mathcal{M})}$$

$2\mathcal{M}$ = the total number of positive and negative peaks per unit time

$$P_n\left(\frac{\alpha}{K_2}\right) = \frac{1}{\sqrt{2\pi}} \int_{\alpha/K_2}^{\infty} e^{-y^2/2}\, dy$$

When one has a process with zero bandwidth, the total number of zero crossings becomes equal to the total number of peaks. Hence,

$$\frac{N_0}{2\mathcal{M}} = 1$$

and Eq. (36) becomes

$$p_p(\alpha) = (\alpha) \exp\left[\frac{-(\alpha)^2}{2}\right] \tag{45}$$

which is identical to Eq. (43) except for the scale factor. Thus, it can be seen this analyzer operates properly on Gaussian signals as long as the bandwidth of the signal is infinitesimally small.

If one has a signal with a spectral density, G(f), such that

$$\begin{aligned} G(f) &= \text{constant} \quad ; \quad f_a \leq f \leq f_b \\ &= 0 \qquad\qquad\qquad \text{elsewhere} \end{aligned}$$

(white noise passed through an ideal bandpass filter), then from Reference 2,

$$N_0 = \frac{2}{\sqrt{3}} \left[\frac{f_b^3 - f_a^3}{f_b - f_a}\right]^{\frac{1}{2}} \tag{46}$$

$$2\mathcal{M} = \frac{2\sqrt{3}}{\sqrt{5}} \left[\frac{f_b^5 - f_a^5}{f_b^3 - f_a^3}\right]^{\frac{1}{2}} \tag{47}$$

By letting

$$f_c = \frac{f_b - f_a}{2} + f_a \quad , \qquad \text{(center frequency of the band)}$$

and

$$K = \frac{f_b - f_a}{f_c} \quad , \qquad \text{(ratio of the bandwidth to its center frequency)}$$

one can combine Eqs. (46) and (47) and use the binomial expansion on them to obtain

$$\frac{N_0}{2\mathfrak{M}} = \frac{1 + \left(\frac{K^2}{12}\right)}{\left(1 + \frac{K^2}{2} + \frac{K^4}{80}\right)^{\frac{1}{2}}} \tag{48}$$

which is an expression for the ratio of the total number of zero crossings per unit time to the total number of peaks per unit time for white noise passed through an ideal bandpass filter. Note that Eq. (48) is written as a function of the fractional bandwidth (K).

If one substitutes Eq. (48) into Eq. (44), one can determine what the true peak probability density function should be and how it varies with the fractional bandwidth. One should have the entire peak density function plotted at different fractional bandwidths to thoroughly study the effects of bandwidth, but for simplicity in calculation, the effect of bandwidth will be considered at only one point, $\alpha = 0$. Table 1 shows the true value of the peak probability density function of a white Gaussian signal that has been passed through an ideal bandpass filter as a function of the fractional bandwidth of the filter.

Since from Eq. (43), the output of an analyzer using this narrowband approximation should always be zero at $\alpha = 0$ for any truly Gaussian process, and since the full scale setting of such analyzers are generally equal to a density of 1.0, the values in the righthand column of Table 1 can be converted to percentage of full scale errors due to finite bandwidth simply by multiplying these values by 100. Thus, it can be seen from Table 1 that a 5% of full scale error occurs at $\alpha = 0$ when this type of analyzer is used to compute the peak density function of a white Gaussian signal whose bandwidth is equal to 20% of its center frequency. It should be emphasized that those error values only apply at $\alpha = 0$ for the special case analyzed. Other α values and other spectral density shapes should result in entirely different error values. In fact, practical bandpass filters are quite likely to yield greater errors at $\alpha = 0$ because the frequency components above $f = f_b$ are not completely removed as is the case for the ideal filter. Hence, more peaks per zero crossing are possible which reduces the $N_0/2\mathfrak{M}$ ratio and increases the error for a given fractional bandwidth.

Fractional Bandwidth (K)	Peak Probability Density at $\alpha = 0$ $p_p(0)$
0	0.0
.01	.0023
.02	.0046
.05	.0116
.10	.0230
.20	.0456
.30	.0785
.40	.0887
.50	.1088
.60	.1269
.70	.1440
.80	.1597
.90	.1739
1.00	.1868
2.00	.2659

Table 1. Peak Probability Density of White Gaussian Noise at $\alpha = 0$ as a Function of the Fractional Bandwidth of an Ideal Bandpass Filter. (Also, the Fractional Error in the Peak Probability Density Function at $\alpha = 0$ for a Narrowband Analyzer with $p(x)_{F.S.} = 1.0$)

In addition, some analyzers compute the peak value probability density function for narrowband signals implementing the equation formed from substituting Eq. (34b) into Eq. (32c).

$$p_p(\alpha) = \left(\frac{1}{\Delta\alpha}\right)\left(\frac{1}{N_0^+ T}\right)\left(T\, g(\alpha)\, \Delta\alpha\right) \tag{49}$$

The results are very similar to those obtained by implementing Eq. (38), but more sophisticated circuitry is required.

4.1.6 Joint Instantaneous Amplitude Probability Density Function

The joint probability functions measure the probability of multiple events occurring simultaneously. The joint probability density function of the instantaneous amplitudes of two harmonically related sine waves is quite interesting. The density function becomes an infinitely thin shell whose locus in the (α, β) plane becomes the Lissajous pattern associated with the two sine waves. To illustrate this, the locus of the joint probability density function for a fundamental and a third harmonic shifted in phase 45° will be computed.

$$x(t) = \sqrt{2}\ \sigma_x \cos(\omega t)$$

$$y(t) = \sqrt{2}\ \sigma_y \cos\left(3\omega t + \frac{\pi}{4}\right)$$

($\sigma_x = \sigma_y$, at least after normalizing in the analyzer)

The time histories of these two functions are shown in Figure 12 for one cycle of the fundamental. When the amplitude window is set at some value α of the function $x(t)$, it can be seen from Figure 12 that there are only one or two values of the height β, (y_0/σ_y), of the amplitude window in the function $y(t)$ which will permit the joint density function to be other than zero.

For example, when

$$\alpha = \frac{x}{\sigma_x} = .500 \quad , \quad \text{then } \beta = \frac{y}{\sigma_y} \text{ must equal } -.707$$

For example, if

$$\alpha = .500 \quad \text{and} \quad \beta = .600$$

there is no possibility of $x(t)$ being in the interval $(.500 + \Delta)$ while β is in the interval $(.6 + \Delta)$, where Δ is a small amplitude increment, so the joint probability density function is zero. Expressed in another way, it can be said that harmonically related sine waves are not independent random variables because their joint probability density is not equal to the product of their individual density functions.

$$p(\alpha, \beta) \neq p(\alpha)\, p(\beta) \tag{50}$$

which is a necessary condition for independence. The joint density function for dependent variables is

$$p(\alpha, \beta) = p(\alpha)\, p(\beta \mid \alpha) \tag{51}$$

where

$p(\beta \mid \alpha) =$ the conditional density function, or the density function of β given that α has occurred

In the above example, the conditional density function has nonzero values only along the locus in the (α, β) plane, as shown in Figure 13. In fact, in the above example $p(\alpha \mid \beta) = .5$, except at $\alpha/\sqrt{2} = \pm 0.5$ where $p(\beta \mid \pm 0.5) = 1.0$ and $\left| \alpha/\sqrt{2} \right| > 1.0$ where $p(\beta \mid \alpha) = 0$.

Now consider the case where the frequency of one sine wave is not an integer multiple of the frequency of the other sine wave, but both sine waves have frequencies which are integer multiples of a common frequency (a fundamental that is not present). The joint density function in each amplitude window width should be computed over exactly one period of the periodicity that would result if the two sine waves were summed. (In actual practice one could not compute over exactly one period, so the computation should be over several hundred periods to minimize the error due to not having an exactly integer multiple of periods.) The period of the sum of two sine waves is equal to the reciprocal of the highest common factor in the ratio of two frequencies. For example, let

$$x(t) = \sin 2\pi f_1 t$$

$$y(t) = \sin 2\pi f_2 t$$

Arbitrarily select $f_1 < f_2$. Let $f_1 = 1000$ and $f_2 = 1200$. Then

$$\frac{f_2}{f_1} = \frac{1200}{1000}$$

The highest common factor is 200.

$$\frac{(1200/200)}{(1000/200)} = \frac{6}{5}$$

and the duration of the periodicity is $1/200 = 5$ milliseconds. Note that there will be 5 cycles of f_1 and 6 cycles of f_2 in each period. As a second example, let $f_1 = 1000$ and $f_2 = 1001$. Then the highest common factor is one, so it takes one second for the wave shape to repeat exactly. Note that there will be 1000 cycles of f_1 and 1001 cycles of f_2 in each period.

For every cycle of f_1 there are two values of β corresponding to each value of α, (see Figure 14 for an example), that result in nonzero joint density functions. Thus, the joint density function of two sine waves can have a very complicated locus in the (α, β) plane, but the two sine waves will be dependent functions because

$$p(\beta \mid \alpha) \neq p(\beta)$$

(The case where the ratio f_2/f_1 is an irrational number is an exception because it would result in a nonperiodic process.)

4.1.7 Joint Instantaneous Amplitude Probability Distribution Functions

The joint cumulative distribution function of instantaneous amplitudes measures the probability that the process $x(t)$ is between $-\infty$ and α simultaneously with the process $y(t)$ being between $-\infty$ and β.

$$P(\alpha, \beta) = P\left(-\infty \leq x(t) \leq \alpha, \quad -\infty \leq y(t) \leq \beta\right) \tag{52}$$

Analogously to the joint density function

$$P(\alpha, \beta) = P(\alpha)\, P(\beta \mid \alpha) \tag{53}$$

where $P(\beta \mid \alpha)$ is the conditional distribution function of β given that α has occurred. In the special case where $x(t)$ and $y(t)$ are independent random variables

$$P(\alpha, \beta) = P(\alpha) \cdot P(\beta) \tag{54}$$

Also,

$$P(\alpha, \beta) = \int_{-\infty}^{\beta} \int_{-\infty}^{\alpha} p(u, v)\, du\, dv \tag{55}$$

As an illustration of a simple sinusoidal joint cumulative distribution function, consider the case where the joint distribution is computed for two identical (same frequency and same phase angle) cosine waves

$$x(t) = y(t) = \sqrt{2}\ \sigma \cos \omega t$$

The time history for one period is shown in Figure 15. From Eq. (53),

$$P(\alpha, \beta) = P(\alpha)\, P(\beta \,|\, \alpha)$$

From Eq. (23),

$$\begin{aligned} P(\alpha) &= \frac{1}{\pi}\left[\sin^{-1}\left(\frac{\alpha}{\sqrt{2}}\right) + \frac{\pi}{2}\right] &&; \quad -\sqrt{2} < \alpha < +\sqrt{2} \\ &= 0 &&; \quad -\sqrt{2} > \alpha \\ &= 1 &&; \quad +\sqrt{2} < \alpha \end{aligned}$$

It can be seen from Figure 16 that the conditional probability

$$\begin{aligned} P(\beta \,|\, \alpha) &= 1 \quad ; &&\beta = \alpha \\ &= 0 &&\text{elsewhere} \end{aligned}$$

Thus,

$$\begin{aligned} P(\alpha, \beta) &= \frac{1}{\pi}\left[\sin^{-1}\left(\frac{\alpha}{2}\right) + \frac{\pi}{2}\right]; &&-\sqrt{2} < \alpha < +\sqrt{2} \text{ and } \alpha = \beta \\ &= 1 &&; \quad \alpha > +\sqrt{2} \text{ and } \alpha = \beta \\ &= 0 &&\text{elsewhere} \end{aligned}$$

Next solve the same example for the joint distribution by finding the joint density function and integrating. From Eq. (51),

$$p(\alpha, \beta) = p(\alpha)\, p(\beta \,|\, \alpha)$$

Likewise, from Eq. (21)

$$\begin{aligned} p(\alpha) &= \frac{1}{\pi\sqrt{2 - \alpha^2}} &&; \quad -\sqrt{2} < \alpha < +\sqrt{2} \\ &= 0 &&\text{elsewhere} \end{aligned}$$

From Figure 15, it can be seen that when the amplitude window in process $x(t)$ is set at α, that the amplitude window in process $y(t)$ must be set at $\beta = \alpha$ for the joint probability density function to be anything but zero. When $\beta = \alpha$, the conditional density function has the form

$$p(\beta \mid \alpha) = \delta(\beta - \alpha) \tag{56}$$

where $\delta(\beta - \alpha)$ = the Dirac delta function and has a value of zero except when $\beta = \alpha$, and then has infinite amplitude, zero width, but a finite area of unity.

Therefore,

$$\begin{aligned} p(\alpha, \beta) &= \frac{1}{\pi\sqrt{2 - \alpha^2}} \, \delta(\beta - \alpha) \qquad ; \quad -\sqrt{2} \leq \alpha \leq +\sqrt{2} \\ &= 0 \qquad\qquad \text{elsewhere} \end{aligned} \tag{57}$$

Because the analyzer does not have an infinitesimally small amplitude window width, the measured joint density function will have both a finite width and a finite amplitude, not the zero width and infinite amplitude of the theoretical density function. The total volume enclosed by the surface of the joint probability density function must equal one. [The probability of $x(t)$ and $y(t)$ occurring simultaneously is one when computed over all possible values of α and β].

The necessary scale factor to account for the finite window width is obtained in the following way. The equation for $p(\alpha, \beta)$, the joint density of two sine waves has been shown to be

$$p(\alpha, \beta) = \begin{cases} \dfrac{1}{\pi\sqrt{2 - \alpha^2}} \, \delta(\beta - \alpha) & , \; -\sqrt{2} \leq \alpha \leq +\sqrt{2} \\ 0 & , \; |\alpha| > \sqrt{2} \end{cases}$$

Now assume one is to estimate $p(\alpha, \beta)$ in the rectangle bounded by the interval

$$\left[\alpha - \frac{\Delta\alpha}{2}, \ \alpha + \frac{\Delta\alpha}{2}\right] \quad \text{and} \quad \left[\beta - \frac{\Delta\beta}{2}, \ \beta + \frac{\Delta\beta}{2}\right]$$

One now integrates $p(\alpha, \beta)$ over these limits and divides by $\Delta\alpha\Delta\beta$ to obtain an estimate $p(\alpha, \beta)$, namely

$$\begin{aligned} p(\alpha, \beta) &= \frac{1}{\Delta\alpha\Delta\beta} \int_{\alpha-(\Delta\alpha/2)}^{\alpha+(\Delta\alpha/2)} \int_{\beta-(\Delta\beta/2)}^{\beta+(\Delta\beta/2)} \left(\frac{1}{\pi}\right)\left(\frac{1}{\sqrt{2 - \alpha^2}}\right) \delta(\beta - \alpha)\, d\beta\, d\alpha \\ &= \frac{1}{\Delta\alpha\Delta\beta} \int_{\alpha-(\Delta\alpha/2)}^{\alpha+(\Delta\alpha/2)} \left(\frac{1}{\pi}\right)\left(\frac{1}{\sqrt{2 - \alpha^2}}\right) d\alpha \qquad (58) \\ &= \frac{1}{\Delta\alpha\Delta\beta} \left[P\left(\alpha + \frac{\Delta\alpha}{2}\right) - P\left(\alpha - \frac{\Delta\alpha}{2}\right)\right] \end{aligned}$$

where $P(\alpha)$ is the one-dimensional distribution function of a sinusoid. One now notes that the term in brackets on the right side of Eq. (58) is approximately $p(\alpha)$ (the densitv function) when divided bv $\Delta\alpha$. In equation form,

$$\begin{aligned} p(\alpha, \beta) &= \frac{1}{\Delta\alpha\Delta\beta} \ P\left[\left(\alpha + \frac{\Delta\alpha}{2}\right) - P\left(\alpha - \frac{\Delta\alpha}{2}\right)\right] \\ &= \frac{1}{\Delta\beta}\left[\frac{P\left(\alpha + \frac{\Delta\alpha}{2}\right) - P\ \alpha - \frac{\Delta\alpha}{2}}{\Delta\alpha}\right] \\ &= \frac{1}{\Delta\beta}\ p(\alpha) \qquad (59) \end{aligned}$$

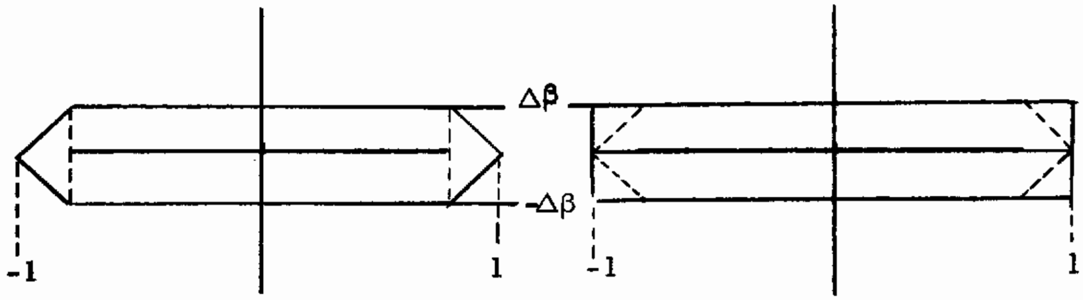
The term $P\left[\left(\alpha + (\Delta\alpha/2)\right) - P\left(\alpha - (\Delta\alpha/2)\right)\right] \Big/ \Delta\alpha$ is approximately $p(\alpha)$ and is the estimate obtained when measuring the one-dimensional density function of a sine wave.

One can see that $p(\alpha, \beta)$ integrates approximately to a volume of unity as should be the case. Thus, by noting that the coordinate system may be rotated so that the integration is performed along the α axis instead of a 45° line, the integration is

$$\int_{-1}^{1}\int_{-\Delta\beta/2}^{\Delta\beta/2} p(\alpha,\beta)\,d\beta\,d\alpha = \int_{-1}^{1}\int_{-\Delta\beta/2}^{\Delta\beta/2} \frac{1}{\Delta\beta}\,p(\alpha)\,d\beta\,d\alpha$$

$$= \int_{-1}^{1} p(\alpha)\,d\alpha \approx 1.0$$

This is only an approximate result since a portion of the region is neglected. The region along the 45° line is rotated to form the region in the left part of the sketch below, but the region actually integrated over is indicated in the right half of the sketch. For small $\Delta\alpha$, this error is not important.



In the actual operation of some analyzers, complete scaling is not accounted for. Consider the above example where identical sine waves are fed into the two inputs of the joint density function analyzer. If the α and β amplitude levels are not set equal, there will be no output. However, if the amplitude levels are set equal, the output will be the same as the ordinary first-order probability density function for these analyzers. When the magnitude of the function $x(t)$ passes into the amplitude window around α, a gate is opened and pulses from an internal clock are fed out of the first section of the analyzer and into one side of an AND gate in the second section of the analyzer. Simultaneously, the magnitude of $y(t)$ passes into the

the amplitude window around β and opens the AND gate in the second section of the analyzer so the pulses from the first section of the analyzer pass into an averaging circuit. When the magnitudes of $x(t)$ and $y(t)$ pass out of the amplitude windows, the pulses are gated off. Hence, it can be seen that in this case the second section of the analyzer does not influence the results when $\alpha = \beta$ so that the output is identical to the first-order density function. It is quite important that this scale factor be remembered and accounted for in the final data presentation.

4.1.8 Extreme Value Distribution Function

The distribution of extreme values mode of analysis is useful for measuring the probability that the maximum magnitude obtained in n independent samples of a random process will be less than some specific amplitude value. In almost all ordinary random processes, as the number of these samples is increased, the probability of finding at least one sample whose magnitude exceeds some specific amplitude also increases. Hence, the probability that none of the samples will exceed this specific amplitude decreases as the number of independent samples increases.

In Reference 5, it is shown that

$$H_n(z) = \left[P(z)\right]^n \tag{60}$$

where

$H_n(z)$ = the distribution of extreme values

n = the number of independent samples (It should be noted that independence of the samples is required for the above equation to be applicable.)

$P(z)$ = the cumulative distribution function of the instantaneous amplitudes of the random process

z = a specific amplitude value

The cumulative distribution function for a sine wave is, from Eq. (23)

$$P(\alpha) = \frac{1}{\pi}\left[\sin^{-1}\left(\frac{\alpha}{\sqrt{2}}\right) + \frac{\pi}{2}\right] \qquad ; \quad -\sqrt{2} \le \alpha \le +\sqrt{2}$$

$$= 0 \qquad ; \quad \alpha < -\sqrt{2}$$

$$= 1 \qquad ; \quad \alpha > +\sqrt{2}$$

Let $z = \alpha$. Then the distribution of extreme values for n samples is

$$H_n(z) = \left\{\left(\frac{1}{\pi}\right)\left[\sin^{-1}\left(\frac{z}{\sqrt{2}}\right) + \frac{\pi}{2}\right]\right\}^n \qquad ; \quad -\sqrt{2} \le z \le +\sqrt{2}$$

$$= 0 \qquad ; \quad z < -\sqrt{2} \qquad (61)$$

$$= 1 \qquad ; \quad z > +\sqrt{2}$$

As an example, consider the probability that the maximum amplitude in n samples is less than $\alpha = 1$ for (a) n = 2 samples, (b) n = 10 samples, and (c), n = 100 samples.

(a) $$H_2(1) = \left\{\frac{1}{\pi}\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi}{2}\right]\right\}^2$$

$$= .563$$

(b) $$H_{10}(1) = .056$$

(c) $$H_{100}(1) = 32 \times 10^{-14}$$

As might be expected, the probability of finding n samples, none of whose amplitudes exceed the specified value, decreases rapidly as n, the number of samples, is increased. Figure 17 plots the extreme value distribution function of a sine wave for several values of n.

It does not appear practical to perform this type of an analysis on analog equipment for a large number of samples because the accuracy required in the underlying distribution function becomes inordinately high. To illustrate this, consider the percent error resulting in the extreme value due to an error in the underlying distribution function.

$$\% \text{ error} = \left\{ \frac{\left[P(x) + \frac{\lambda P(x)}{100}\right]^n - \left[P(x)\right]^n}{\left[P(x)\right]^n} \right\} \left[100\right] \tag{62}$$

where

$P(x)$ = the instantaneous amplitude distribution function

λ = the percent error in measuring $P(x)$

Rewritten, the above equation is

$$\% \text{ error} = 100\left[\left(1 + \frac{\lambda}{100}\right)^n - (1)^n\right]$$

From the binomial expansion

$$\left[1 + \frac{\lambda}{100}\right]^n = 1 + n\frac{\lambda}{100} + \frac{n(n-1)}{2!}\left(\frac{\lambda}{100}\right)^2 \cdots \left(\frac{\lambda}{100}\right)^n$$

$$\approx 1 + n\frac{\lambda}{100}, \text{ where } \frac{n\lambda}{100} < 1$$

Therefore

$$\% \text{ error} \approx n\lambda \quad ; \quad \text{where } n\lambda < 100 \tag{63}$$

Where $n\lambda \geq 100$, the remaining terms in the expansion cannot be neglected (assuming that the % error is equal to $n\lambda$ is optimistic -- the real error is much greater).

The probability density of extreme values $h_n(z)$ is simply the derivative of the distribution function

$$\begin{aligned} h_n(z) &= \frac{d\left[H(z)\right]}{dz} \\ &= \frac{d\left\{\left[P(z)\right]^n\right\}}{dz} \\ &= n\left[P(z)\right]^{n-1} \frac{d\left[P(z)\right]}{dz} \\ &= n\left[P(z)\right]^{n-1} p(z) \end{aligned} \tag{64}$$

4.1.9 Extreme Value Density Function

The extreme value density function of a sine wave is

$$\begin{aligned} h_n(z) &= n\left\{\left(\frac{1}{\pi}\right)\left[\sin^{-1}\left(\frac{z}{\sqrt{2}}\right) + \frac{\pi}{2}\right]\right\}^{n-1}\left[\left(\frac{1}{\pi}\right)\left(\frac{1}{\sqrt{2 - z^2}}\right)\right]; \quad -\sqrt{2} \leq z \leq +\sqrt{2} \\ &= 0 \quad ; \text{ elsewhere} \end{aligned} \tag{65}$$

The relation of the extreme value density function to the extreme value distribution function is analogous to the relation of the instantaneous amplitude density function to the instantaneous amplitude distribution function. The extreme value density function measure $1/\Delta z$ times the probability that the maximum amplitude observed in n samples will fall within the interval $z + \Delta z$ where Δz is a small amplitude increment

$$h_n(z) = \lim_{\Delta z \to 0} \left(\frac{1}{\Delta z}\right) \text{(probability that the maximum value of } n \text{ samples occurs in the interval } z + \Delta z)$$

In Figure 18, the extreme value density function of a sine wave is plotted for several values of n. Notice how increasing the number of samples increases the density near the positive peak of the sine wave.

4.2 EVALUATION TESTS - SINUSOIDAL INPUTS

It is assumed that the analyzer will have successfully passed all of the tests in Section 3 before the following tests are performed so that no more than normal caution need be exercised in the selection of such things as signal amplitude, signal frequency, scan rate, etc. It is further assumed that the sinusoidal signal generator used to perform these tests will have been examined and found to have met all of the requirements of Section 2, and that the analyzer has been calibrated per the manufacturer's recommendations.

In the following paragraphs a list of test conditions are presented along with the corresponding theoretical values in either graphic or tabular form.

4.2.1 Instantaneous Amplitude Probability Density Function

Apply a full scale sine wave to the input of the analyzer. Set the frequency of the sine wave near to the middle of the rated frequency range of the analyzer. In usage, the next step would normally be to select a time constant that satisfies uncertainty error requirements. However, there are none of these errors associated with a sine wave. Hence, the time constant is chosen to satisfy the requirements of Eq. (12b), that averaging be performed over a large number of cycles of the sine wave. Next, the scan rate is selected by using the smoothing error criterion in Eq. (7b), if the analyzer uses RC averaging. If the analyzer uses true integration, the scan rate is completely determined by Eq. (12b).

The following example is to illustrate how these test conditions are selected. First assume that the analyzer has a frequency range of 1 cps to 3 kc. Next assume that the analyzer uses RC smoothing and that a 1% error from a nonintegral number of cycles and a 1% smoothing error are permissible. The required averaging time is:

$$RC = \left(\frac{1}{f}\right)\left(\frac{50}{1}\right) = \left(\frac{1}{2000}\right)\left(\frac{50}{1}\right) = \frac{1}{40} \text{ sec.} = .025 \text{ sec.}$$

For 1% smoothing error, $\frac{t}{RC} \approx 5$

$$t = (5)(.025) = .125 \text{ sec. (per window)}$$

Now if the amplitude window has a width of 0.1σ and is swept from -3σ to $+3\sigma$, the total can time is

$$T_s = (.125)(6/.1) = 7.5 \text{ seconds}$$

If the analyzer has discrete settings instead of continuously variable settings for the time constant and/or the scan time, use the next longer setting. However, if the time constant is increased over the minimum permissible, the scan time must also be increased proportionately. Assume in the above example that the shortest time constant setting in the analyzer is 0.1 seconds. This position would be selected, but then the scan time becomes 0.5 seconds/window or 30 seconds for a scan between -3 and $+3\sigma$.

Perform the test and compare the results obtained to the theoretical curve in Figure 4. The maximum difference between the two should not be greater than the rated error of the analyzer as long as full scale on the analyzer output is not exceeded. The theoretical density function of a sine exceeds 1.0 when $1.378 \leq |\alpha| \leq \sqrt{2}$. If the full scale analyzer output corresponds to a density of 1.0, then the accuracy should not be measured beyond one-half of an amplitude window of the above values. For example, if the analyzer window width is 0.1σ, ignore readings when $1.328 \leq |\alpha| \leq 1.464$.

4.2.2 Instantaneous Amplitude Probability Distribution Functions

Repeat the test conditions of Section 4.2.1. Compute and record the cumulative distribution function for instantaneous amplitudes. Compare the measured results to the theoretical curve in Figure 5. The maximum difference should not exceed the advertised error of the analyzer.

Next compute and record the negative cumulative distribution function for instantaneous amplitudes and compare the results to Figure 6. Again, the maximum difference should not exceed the advertised error of the analyzer.

4.2.3 Expected Number of Threshold Crossings per Unit Time

In this mode of analysis, the theoretical output should be zero when the level of the amplitude window (actually only a threshold in this case) is below $-\sqrt{2}$ or above $+\sqrt{2}$. The average crossing rate should be constant when the amplitude window is between the above two values. The magnitude of the average total level crossing rate is proportional to the ratio of the input frequency to the calibration frequency used to set the full scale output. The magnitude of the average level crossings with only positive or only negative slope is equal to one-half the magnitude of the average total (both slopes) level crossing rate.

Because there are so many combinations of input and analyzer settings possible in this analysis mode, quite a few tests must be made. Choose several frequencies to span the entire rated frequency range(s). The use of an electronic counter is recommended for accurate setting of the input frequencies. Measure total, positive and negative crossing rates for each frequency. Measure the output and compare to the theoretical voltage determined from the following equation.

$$
\begin{aligned}
e_{out} &= (e_{F.S.})\left(\frac{f_{in}}{f_{F.S.}}\right) && ; \quad -\sqrt{2} \geq \alpha \leq \sqrt{2} \\
&= 0 && ; \quad \text{elsewhere}
\end{aligned}
\tag{66}
$$

where

$e_{F.S}$ = the full scale output voltage

f_{in} = the frequency of the input signal

$f_{F.S.}$ = the frequency of the full scale calibration signal

The maximum difference between the measured and theoretical voltage should not exceed the accuracy rating of the analyzer. Use the guidelines of Section 4.2.1 to determine the proper time constants and scan times.

4.2.4 Expected Number of Maxima per Unit Time

Apply a number of sine waves to cover the entire frequency range of the analyzer. Select the averaging times and the scan times by the method described in Section 4.2.1. For each frequency compute the expected number of maxima per unit time in a narrow amplitude window, $g(\alpha)\,d\alpha$, the expected number of maxima per unit time in excess of a given level, M_α, and the total expected number of maxima per unit time, $\mathfrak{M}$. Compare the $g(\alpha)\,d\alpha$ results to the theoretical curve in Figure 8, and the M_α results to the theoretical curve in Figure 9. The magnitude of these curves and the value of $\mathfrak{M}$ should be proportional to the frequency of the input signal. The theoretical relation is expressed in Eq. (66). Compare the theoretical and measured results. Any difference should be no greater than the advertised error figure for the analyzer.

4.2.5 Peak Value Probability Functions

Repeat the test conditions of Section 4.2.1 and compute the peak value probability distribution function $Q_p(\alpha)$, the peak value distribution function, $P_p(\alpha)$, and the peak value probability density function $p_p(\alpha)$. Compare the results to Figures 10, 11 and 8, respectively. Any difference between the measured and theoretical values should not exceed the accuracy allowance of the analyzer.

4.2.6 Joint Instantaneous Amplitude Probability Density Function

Apply the sine wave used in Section 4.2.1 to the A input of the analyzer. Also, connect the A input in parallel with the B input. Connect the threshold level inputs of both analyzers together electrically. This will cause the analyzer to scan along the line $\alpha = \beta$ in the $\alpha\beta$ plane. As discussed in Section 4.1.1, the joint density for the above input exists only along this line. Set the time constant and scan time equal to the values used in Section 4.2.1. Scan the threshold levels over the desired amplitude range and record the joint probability density output. Compare the output from the analyzer to Figure 4. The output should fall within the rated analyzer accuracy of the theoretical curve. As discussed in Section 4.2.1, the analyzer output will exceed full scale between approximately -1.46σ to -1.33σ and between $+1.33\sigma$ to $+1.46\sigma$ and should be ignored. (This assumes that the full scale density = 1.0.)

4.2.7 Joint Instantaneous Amplitude Probability Distribution Functions

Repeat the test described in Section 4.2.6 except compute the joint distribution function instead of the joint density function. Record the output of the analyzer and compare with the theoretical results in Figure 5. (Note that this is identical to the first-order distribution function.) The maximum difference between the measured and the theoretical joint distribution function should be within the rated accuracy of the analyzer.

4.2.8 Extreme Value Distribution Function

Repeat the test conditions of Section 4.2.1 and compute the extreme value distribution for $n = 2$, 10 and 100. Compare the results to the theoretical curves plotted in Figure 17. The maximum difference should be no greater than the advertised accuracy of the analyzer.

4.2.9 Extreme Value Density Function

Repeat the above test for $n = 2$ and $n = 10$, except compute the extreme value density instead of distribution. Compare the results to the theoretical curves plotted in Figure 18. The maximum difference should not exceed the quoted accuracy of the analyzer.

4.3 EVALUATION TESTS - TRIANGULAR WAVE INPUTS

The following set of tests are given to provide a further evaluation of the performance of this analyzer. While all of the basic analysis modes are checked relatively thoroughly, as far as discrete inputs go, with the sine wave tests in Section 4.2, the tests below provide a measure of the effects of the shape of the density function on the analyzer's performance. The triangular wave is a particularly attractive input to use because it has a uniform density function (see the discussion in Section 3). As with the sine wave tests, it is assumed that both the analyzer and signal generator have passed all of the tests of Section 2 before either is used for the following tests.

4.3.1 Instantaneous Amplitude Probability Density Function

Apply a full scale triangular wave at a frequency near the center of the frequency range of the analyzer. Set the averaging time and the scan period by the method described in Section 4.2.1. Scan over the appropriate amplitude range and record the output of the analyzer. Compare the output to the theoretical curve in Figure 2. The maximum difference should be within the rated accuracy of the analyzer.

4.3.2 Instantaneous Amplitude Probability Distribution Functions

Repeat the tests in the above section except compute the distribution functions (both positive and negative) instead of the density function. Compare the measured positive distribution function to Figure 19 and the measured negative distribution function to Figure 20. The maximum difference between the measured results and the theoretical curves should be no greater than the advertised error figure for the analyzer.

4.3.3 Expected Number of Threshold Crossings per Unit Time

Repeat the test conditions of Section 4.2.3 except apply a triangular wave instead of a sine wave. Record the outputs and compare these to the theoretical curve in Figure 21. The theoretical magnitude as a function of frequency can be from Eq. (66). The measured results should be within the advertised error tolerance of the theoretical values shown on the curve.

4.3.4 Expected Number of Maxima per Unit Time

Repeat the test conditions of Section 4.2.4 except apply a triangular wave instead of a sine wave. Compute the expected number of maxima per unit time in a narrow amplitude window; in excess of a specified level; and total. In the first case, the output of the analyzer should be zero except at $\alpha = \sqrt{3}$. The theoretical curve is shown in Figure 22. The theoretical curve for the expected number of maxima per unit time in excess of a specified level is constant from $-\infty$ up to $\alpha = \sqrt{3}$. The magnitude of this function and the total expected number of maxima per unit time are proportional to the frequency of the applied signal as described in Eq. (66). Compute the expected number of maxima per unit time in excess of a specified level and compare to Figure 23. Compute the total expected number of maxima per unit time and compare to theoretical calculations from Eq. (66).

4.3.5 Peak Value Probability Density Function

Repeat the test conditions of Section 4.3.4 and compute the peak value probability density function. The density function is zero except at $\alpha = \sqrt{3}$. The theoretical peak density function for a triangular wave is shown in Figure 22.

Also compute the negative and positive peak value probability distribution functions and compare to Figures 24 and 25.

4.3.6 Joint Instantaneous Amplitude Probability Density Function

Apply a full scale triangular wave in parallel to both inputs of the analyzer. Connect the threshold level inputs of both channels together electrically, so that the joint density is analyzed along the line $x = y$ in the xy plane. (As with the sine wave test, the density will be zero when $x \neq y$.)

Select the frequency of the test signal and analyzer operating conditions as described in Section 4. 2. 1. Record the output and compare to the theoretical curve in Figure 2.

4. 3. 7 Joint Instantaneous Amplitude Probability Distribution Function

Repeat the test described in Section 4. 3. 6, except compute the joint distribution function instead of the joint density function. Record the output of the analyzer and compare it with the theoretical results in Figure 19. (Note that this is identical with the first-order distribution function.)

4. 3. 8 Extreme Value Distribution Function

Repeat the test described in Section 4. 3. 1 except compute the extreme value distribution function for $n = 2, 10$ and 100. Compare the results to the theoretical curves in Figure 26.

4. 3. 9 Extreme Value Density Function

Repeat the test described in Section 4. 3. 1 except compute the extreme value density function for $n = 2$ and $n = 10$. Compare the results to the theoretical curve in Figure 27.

5. TESTS WITH RANDOM INPUTS

Rather than going into great detail on the results that would be expected from the analysis of certain theoretical types of random signals, only to have the actual results differ from the theoretical results by a wide margin due to imperfections in the noise generator or statistical uncertainty fluctuations, a completely empirical approach is recommended. For each of the three different types of random input test signals, it is recommended that the test signal be recorded on magnetic tape. This tape should then be used as the input to the statistical analyzer for the evaluation test. The same tape (and time portion) that is used for the evaluation test should be digitized and all of the statistical properties measured during the evaluation test should also be computed on the digital computer. The results from the digital computer analysis will serve as an accurate description of the test signal and its appropriate statistical parameters.

The results from the actual evaluation test can be compared to the results from the digital computer analysis to determine the accuracy of the statistical analyzer with random input signals.

Three different test signals are recommended. These are, (a) broadband Gaussian noise, (b) narrowband Gaussian noise, and (c) clipped Gaussian noise. For the first set of tests, connect the output of a Gaussian random noise generator to a bandpass filter. Set the low cutoff frequency and the high cutoff frequency so that most of the operating frequency range of the analyzer is covered. Next record the output of the filter on magnetic tape. Adjust the sensitivity of the tape recorder so that the input voltage that represents the maximum amplitude level of interest is equal to the full scale deviation of the FM carrier. For example, if the tape recorder is adjusted to give a full scale deviation of 40% with a DC input of 1.5 volts, then the rms input voltage would be $1.5/6 = .250$ volts rms if 6σ peaks are to be analyzed. Cut a sample record from this tape. This sample should then be used to provide the test input for all analysis modes except the joint analyses. This same sample should also be digitally analyzed. Set the averaging time constant equal to the duration of the sample record. Adjust the scan period as previously described. Scan over the amplitude range of interest and perform all but the joint probability computations. These computations are

instantaneous amplitude probability density function

instantaneous amplitude probability distribution function

negative instantaneous amplitude probability distribution function

peak value probability density function

peak value probability distribution function

negative peak value probability distribution function

expected number of maxima per unit time in a narrow amplitude interval function

expected number of maxima per unit time above a given level function

total expected number of maxima per unit time function

expected number of total threshold crossings per unit time function

expected number of positive threshold crossings per unit time function

expected number of negative threshold crossings per unit time function

extreme value distribution function

extreme value density function

For the narrowband Gaussian random noise input, connect the output of a Gaussian noise generator to a bandpass filter. Set the bandwidth of the filter to be about 1/20 of the center frequency of the filter. Select the center frequency so that the upper 3 db point of the filter is just slightly below the upper operating frequency of the analyzer. Connect the output of this bandpass filter to the input of a magnetic tape recorder. Adjust the recording sensitivity so that the maximum voltage amplitude of interest (on the output of the filter) represents full scale input to the FM carrier. Record the above signal and then cut a sample from the recording. Digitize this sample and analyze it with a digital computer. In addition, use the statistical analyzer to perform the analyses listed Previously on this sample.

For the evaluation tests with a clipped Gaussian random noise input, connect the output of a Gaussian noise generator to a bandpass filter. Set the lower cutoff frequency and the upper cutoff frequency as in the first random test. Connect the output of the bandpass filter to a relatively "hard" clipping nonlinear circuit. Adjust the clipping level relative to the output voltage of the bandpass filter so that clipping occurs at 1.0σ or just slightly higher. Connect the output of the clipper to the input of the tape recorder and adjust the sensitivity so that full scale carrier deviations of the tape recorder is equal to 1.0 (or $1.0+)\sigma$. Record this clipped signal and select a sample for analysis. Analyze this sample with the statistical analyzer and the digital computer. Perform the same analyses as listed above for the broadband and narrowband Gaussian signals.

Only one test of each of the joint probability functions is recommended because a large amount of time is required for these analyses. To perform these tests, two Gaussian noise generators and two bandpass filters will be required. Adjust the filters so that they have identical passband (gain factor and phase factor) characteristics. The upper frequency should be kept below 200 cps if possible to minimize the interchannel time displacement errors in the recording process. Set the low frequency cutoff as low as is consistent with satisfactory analyzer and/or random noise generator operation. Connect the outputs of the filters to adjacent tape tracks <u>on the same head stack</u> and record at the highest speed available to minimize the phase errors from interchannel time delay errors in the recording process. (Prior to performing this test the static phase shift between the two data channels should be adjusted to be within 1° of each other as measured at the output of the tape recorder.) This phase tolerance should be maintained at least between the 3db frequencies. Set the full scale sensitivity of the tape recorder as described in the preceding paragraphs and record the two signals. Cut a sample record from the tape and analyze this same sample on both the digital computer and the statistical analyzer. Perform both joint probability density and joint probability distribution analyses on this sample.

6. STABILITY TEST

This is a simple test that is designed to measure the time variation (drift) in the sensitivity of the statistical analyzer. From the results of this test, a minimum time between recalibration of the analyzer can be established. (Calibration is used here in the sense of a daily, or more frequent, set up or adjustment of the analyzer.)

After the analyzer has been properly adjusted according to the manufacturer's instructions, including any warm-up time, apply a half of full scale midfrequency range triangular wave plus a half of full scale DC voltage. Compute and record the instantaneous amplitude probability density function and the mean value. Also record the time when the computation started.

Twenty minutes after the first test is started, repeat the above test. Caution must be exercised that the same DC and triangular voltage levels are applied and that no changes are made in any of the analyzer settings (including the normalization controls).

Repeat this test every twenty minutes over a continuous eight-hour period. The error in the long term average reading should be plotted as a function of time. Likewise, the maximum error in the density function should be plotted as a function of time. (Care must be taken in this step. If any anomalies occur in the density function, their causes should be carefully ascertained before deciding whether or not to accept them as a stability error.)

REFERENCES

1. S. O. Rice, "Properties of a Sine Wave Plus Random Noise," Bell Systems Technical Journal, vol. 27, pp. 109-157, January 1948.
2. S. O. Rice, "Mathematical Analysis of Random Noise," Selected Papers on Noise and Stochastic Processes, edited by Nelson Wax, Dover Publications, Inc., New York. 1954. (Also in Bell Systems Technical Journal, vols. 23 and 24.)
3. J. S. Bendat, Principles and Application of Random Noise Theory, John Wiley and Sons, Inc. New York. 1958.
4. J. S. Bendat, "Probability Functions for Random Responses: Prediction of Peaks, Fatigue Damage, and Catastrophic Failures," NASA CR-33, National Aeronautics and Space Administration, Washington, D.C. April 1964.
5. J. S. Bendat, L. D. Enochson, G. H. Klein, and A. G. Piersol, "The Application of Statistics to the Flight Vehicle Vibration Problem," ASD TR 61-123, Aeronautical Systems Division, AFSC, USAF, WPAFB, Ohio. Chapter 4, pp. 4-103 to 4-109. December 1961. (AD 271 913).

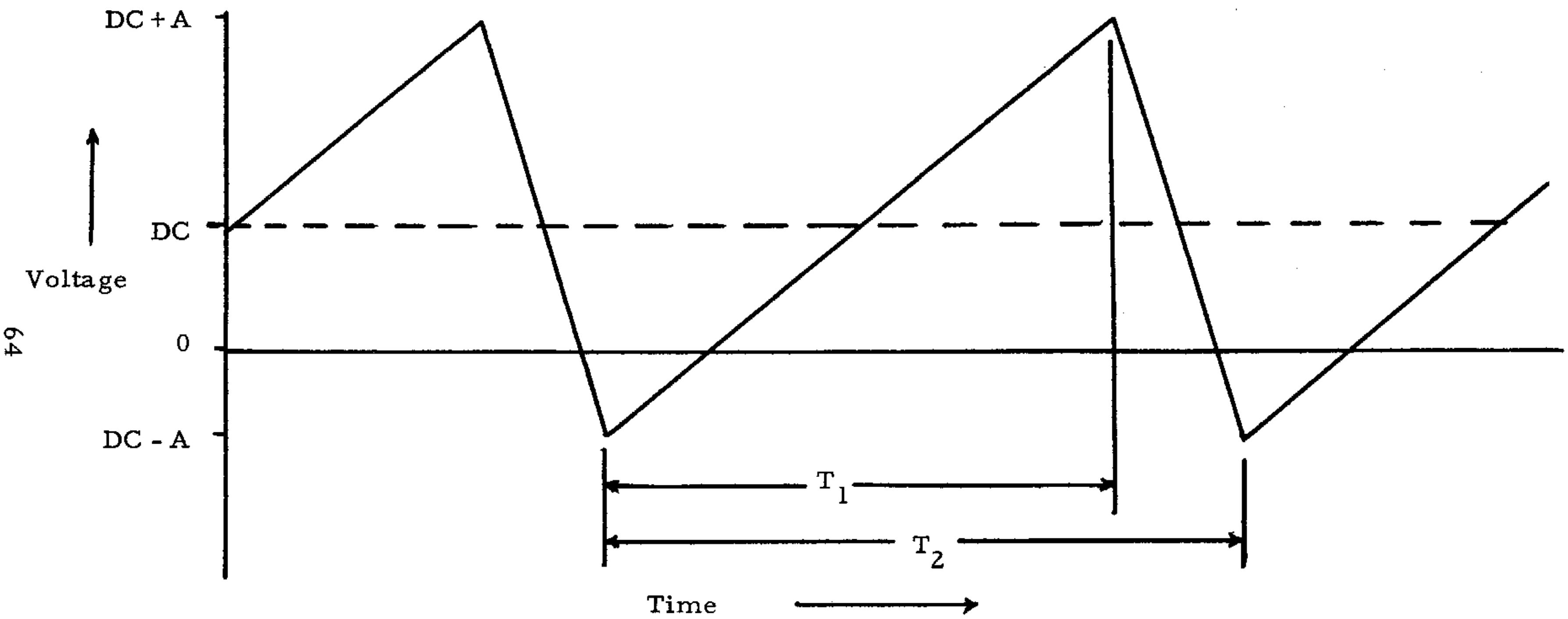


Figure 1. Time History of the Sum of a DC Voltage and a Triangular Voltage

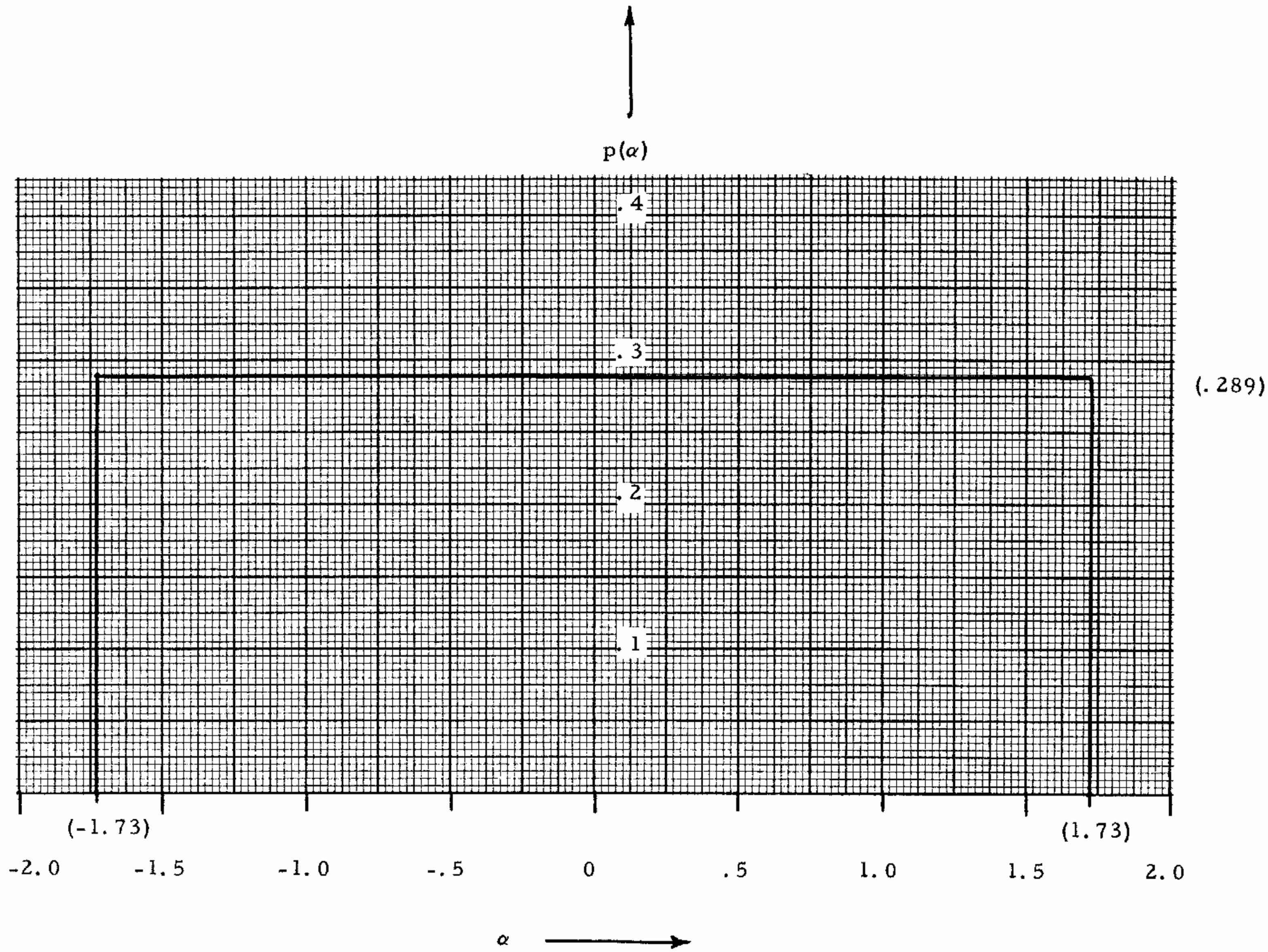


Figure 2. Instantaneous Amplitude Probability Density Function of a Triangular Wave

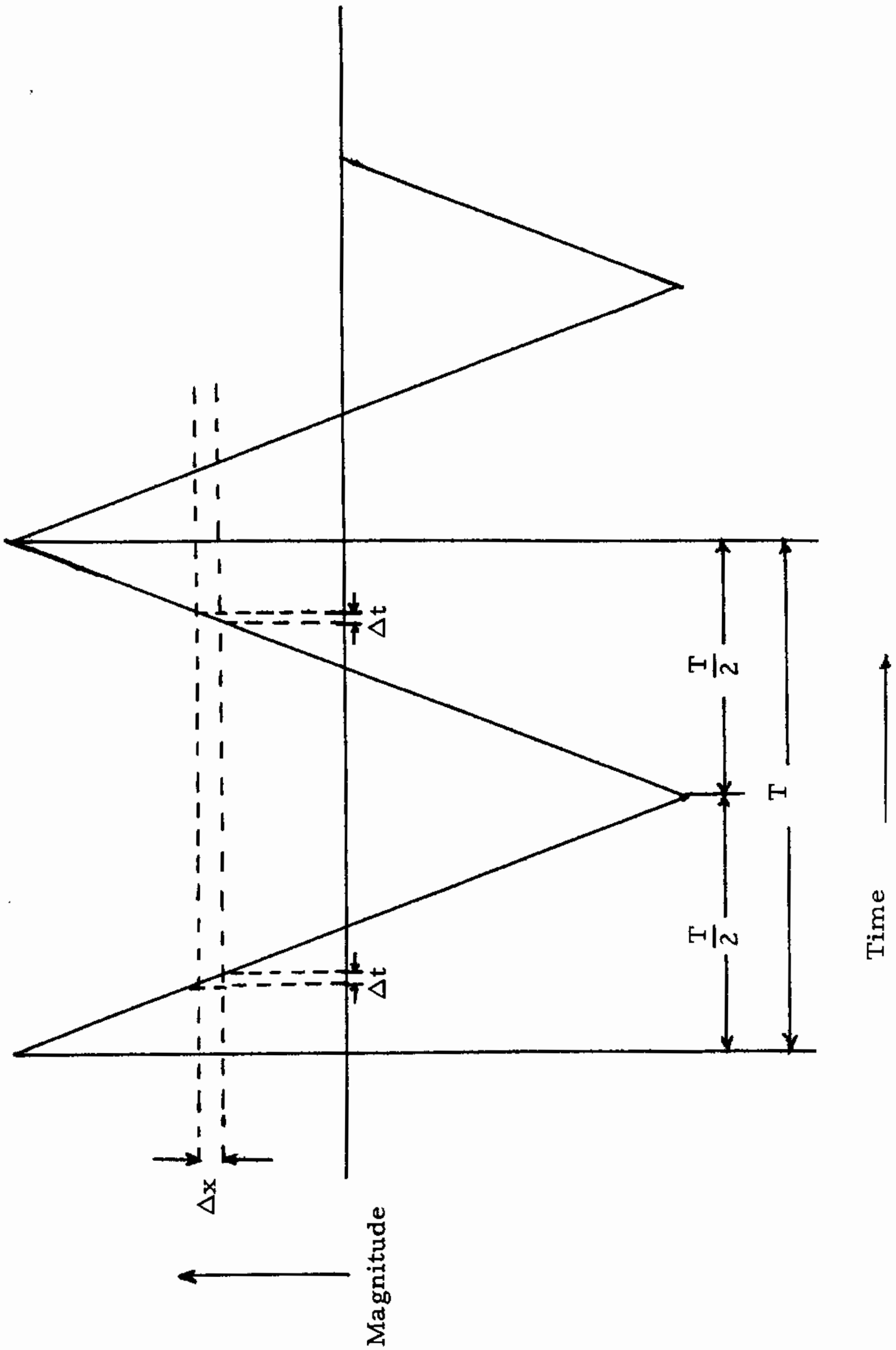


Figure 3. Time History of a Symmetrical Triangular Wave

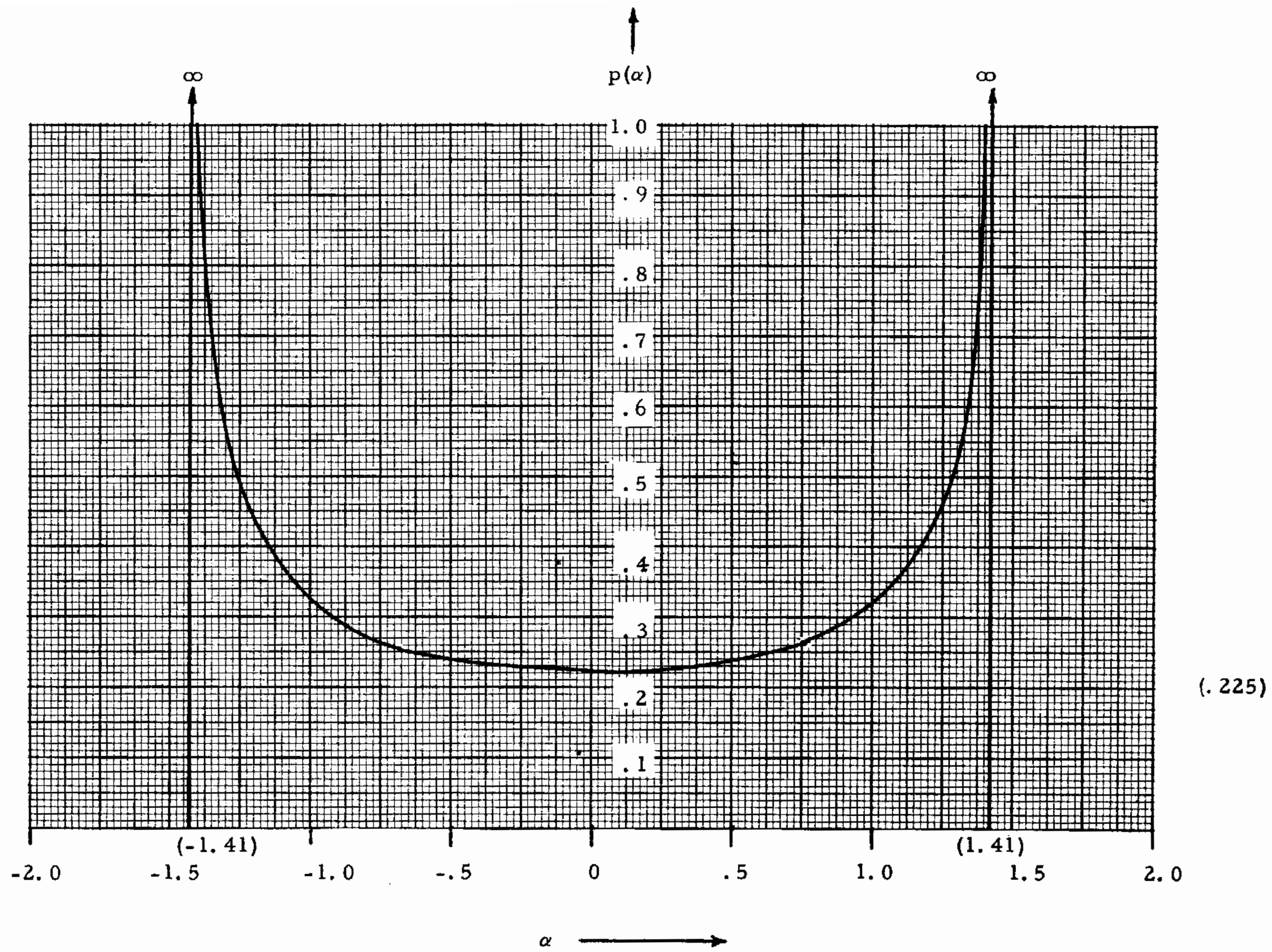


Figure 4. Instantaneous Amplitude Probability Density Function of a Sine Wave

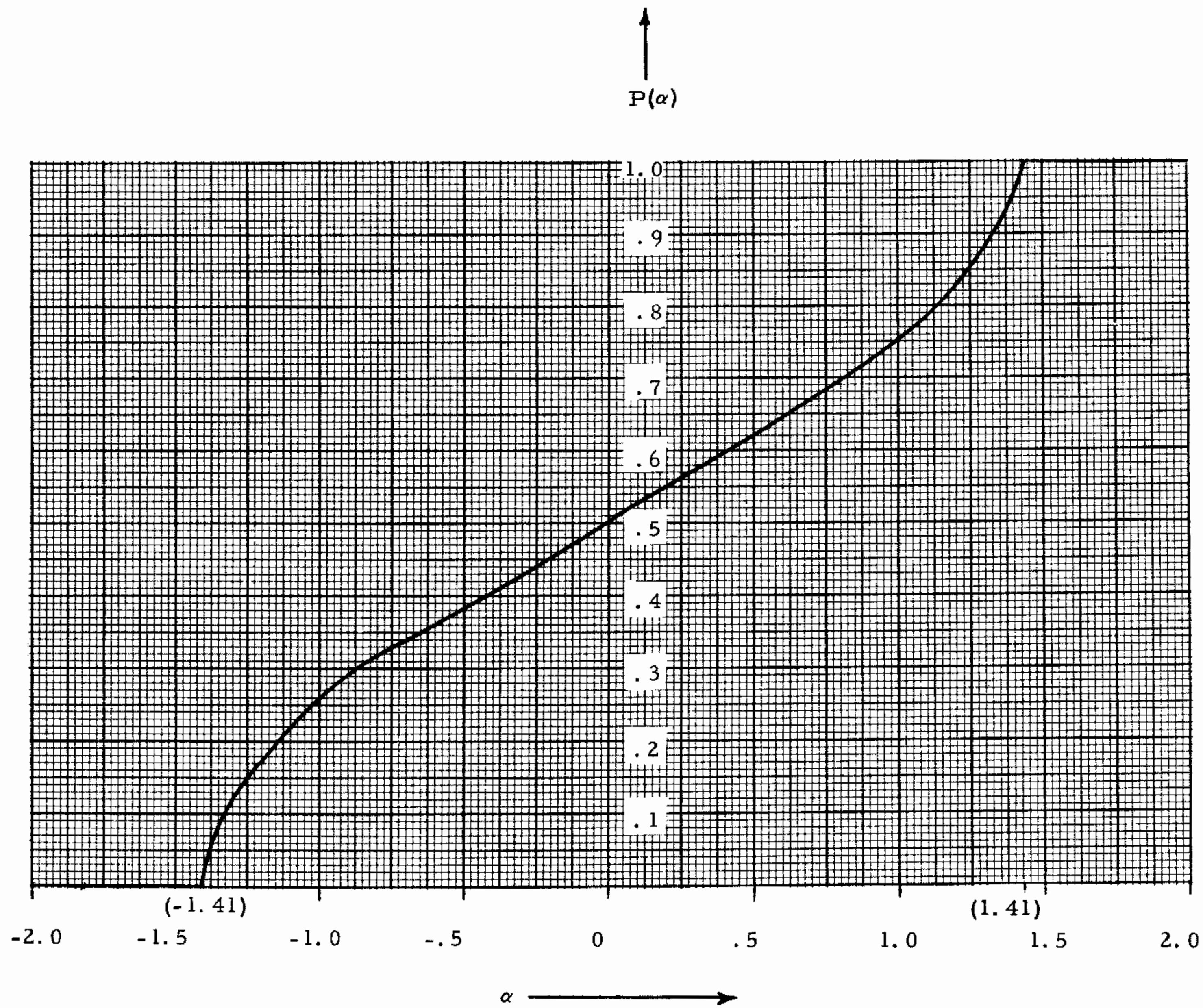


Figure 5. Instantaneous Amplitude Probability Distribution Function of a Sine Wave

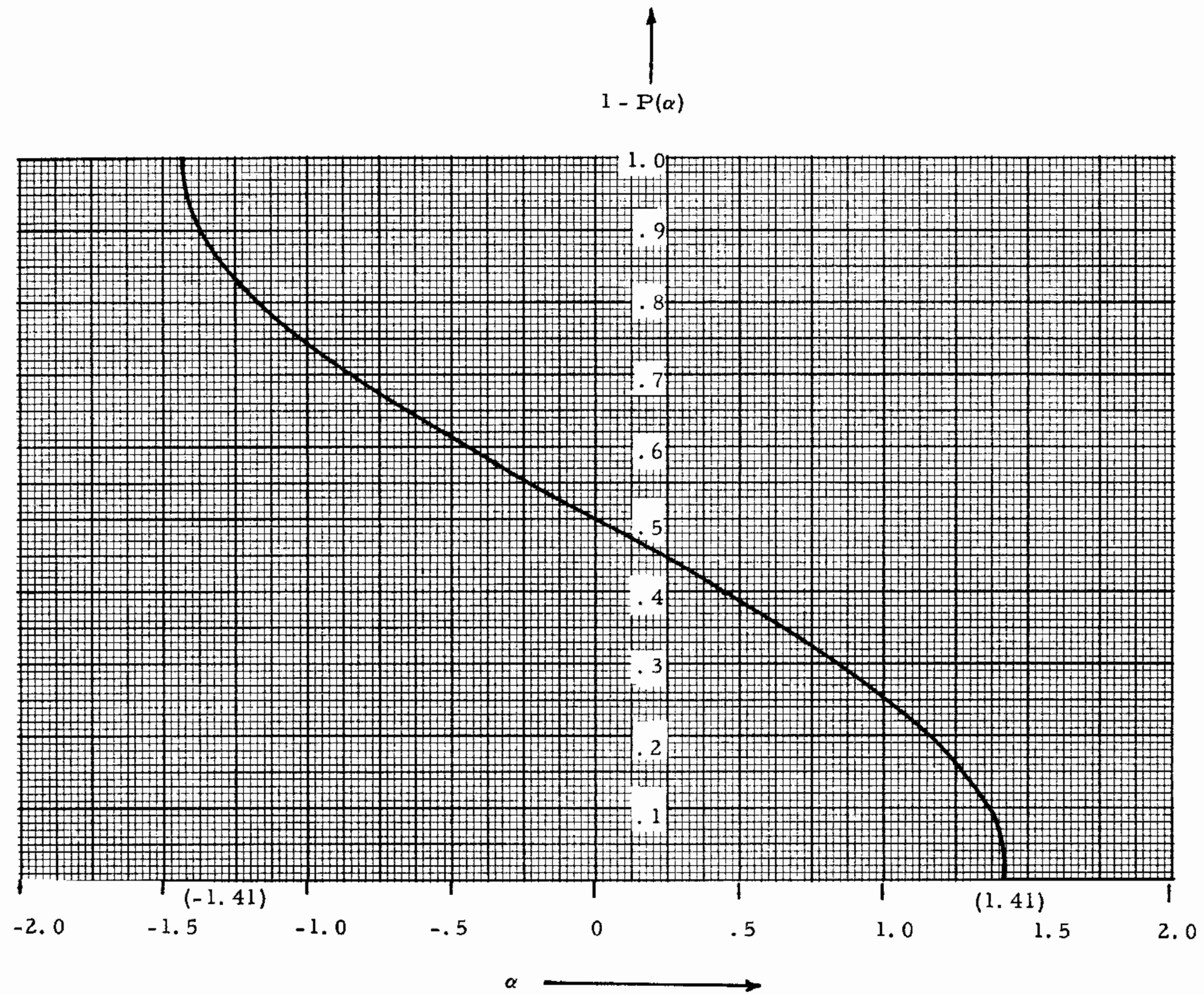


Figure 6. Negative Instantaneous Amplitude Probability Distribution Function of a Sine Wave

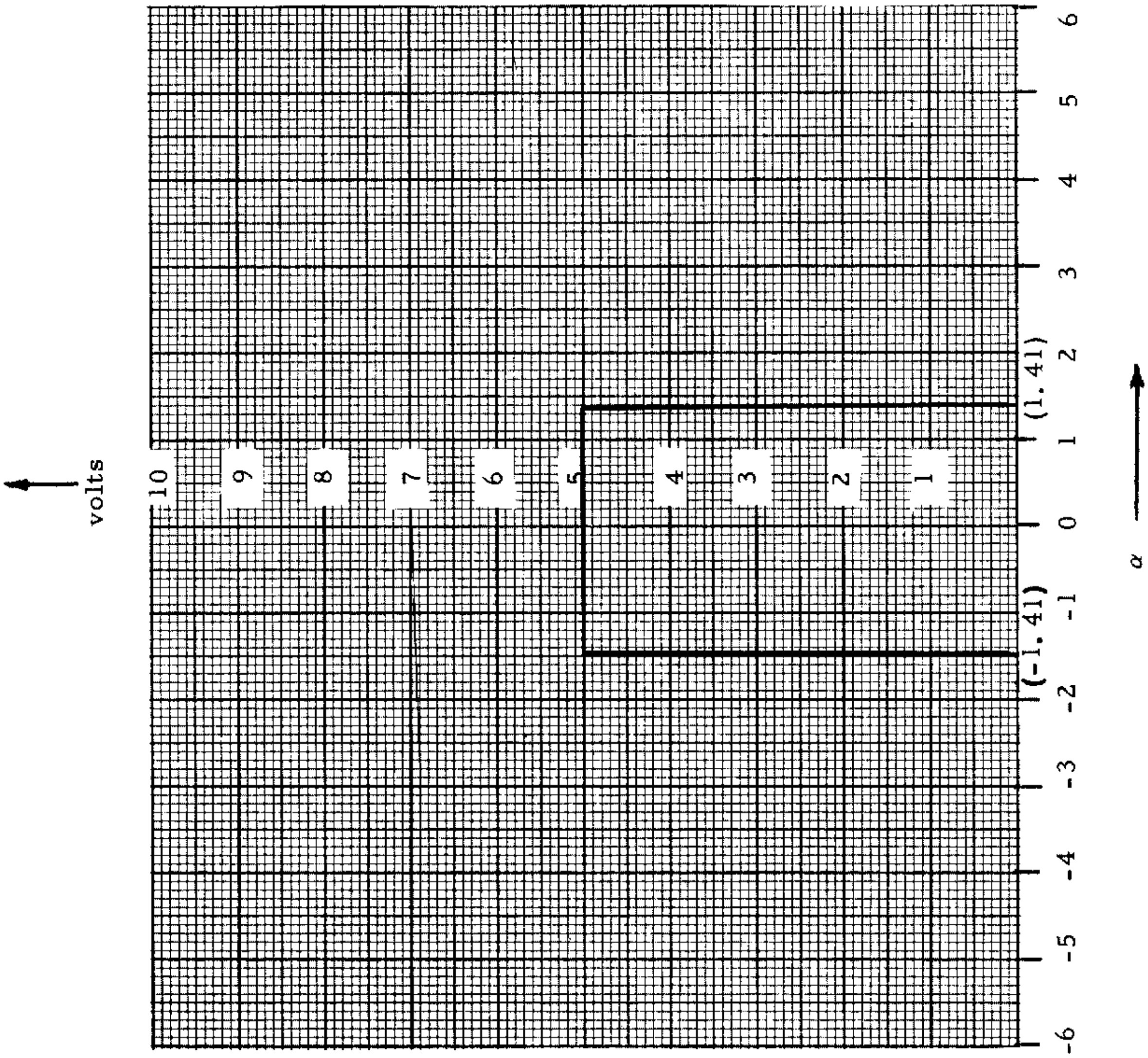


Figure 7. Expected Total Number of Threshold Crossings per Unit Time for a 1 KC Sine Wave (2 KC is full scale)

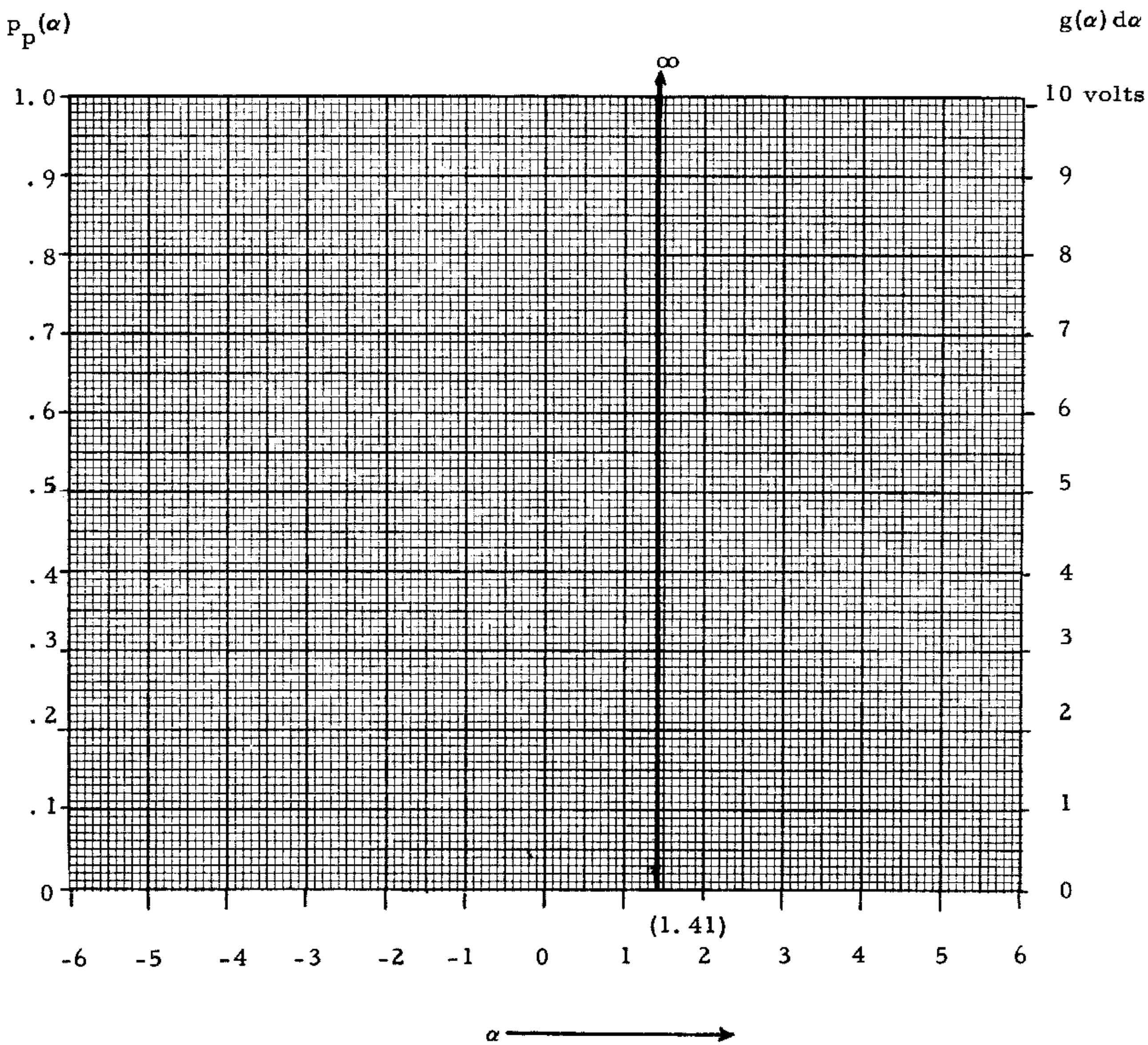


Figure 8. Peak Value Probability Density Function or Expected Number of Maxima per Unit Time in a Narrow Amplitude Window Function for a Sine Wave

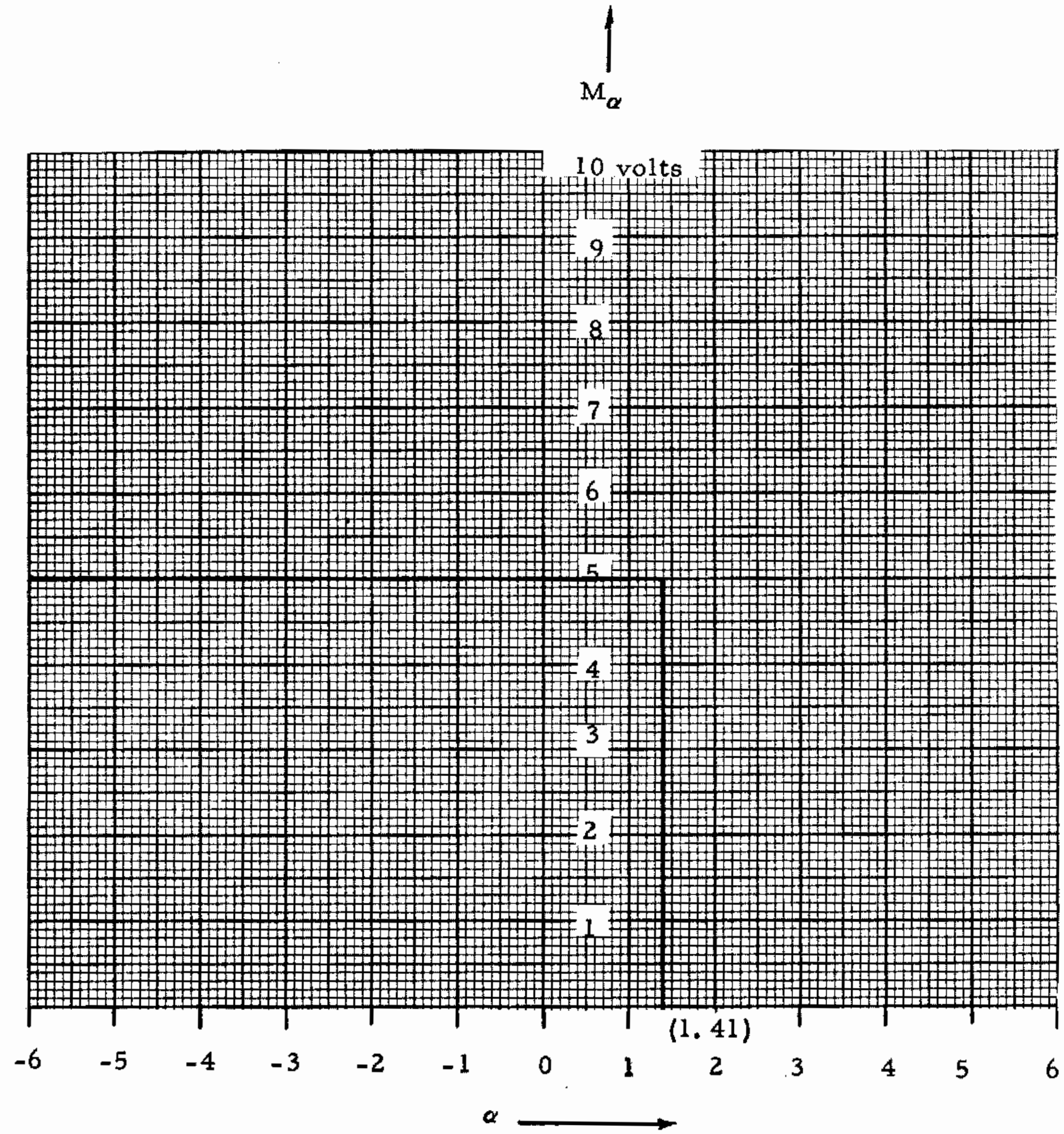


Figure 9. Expected Number of Maxima per Unit Time in Excess of a Given Level for a 1 KC Sine Wave (2 KC = Full Scale)

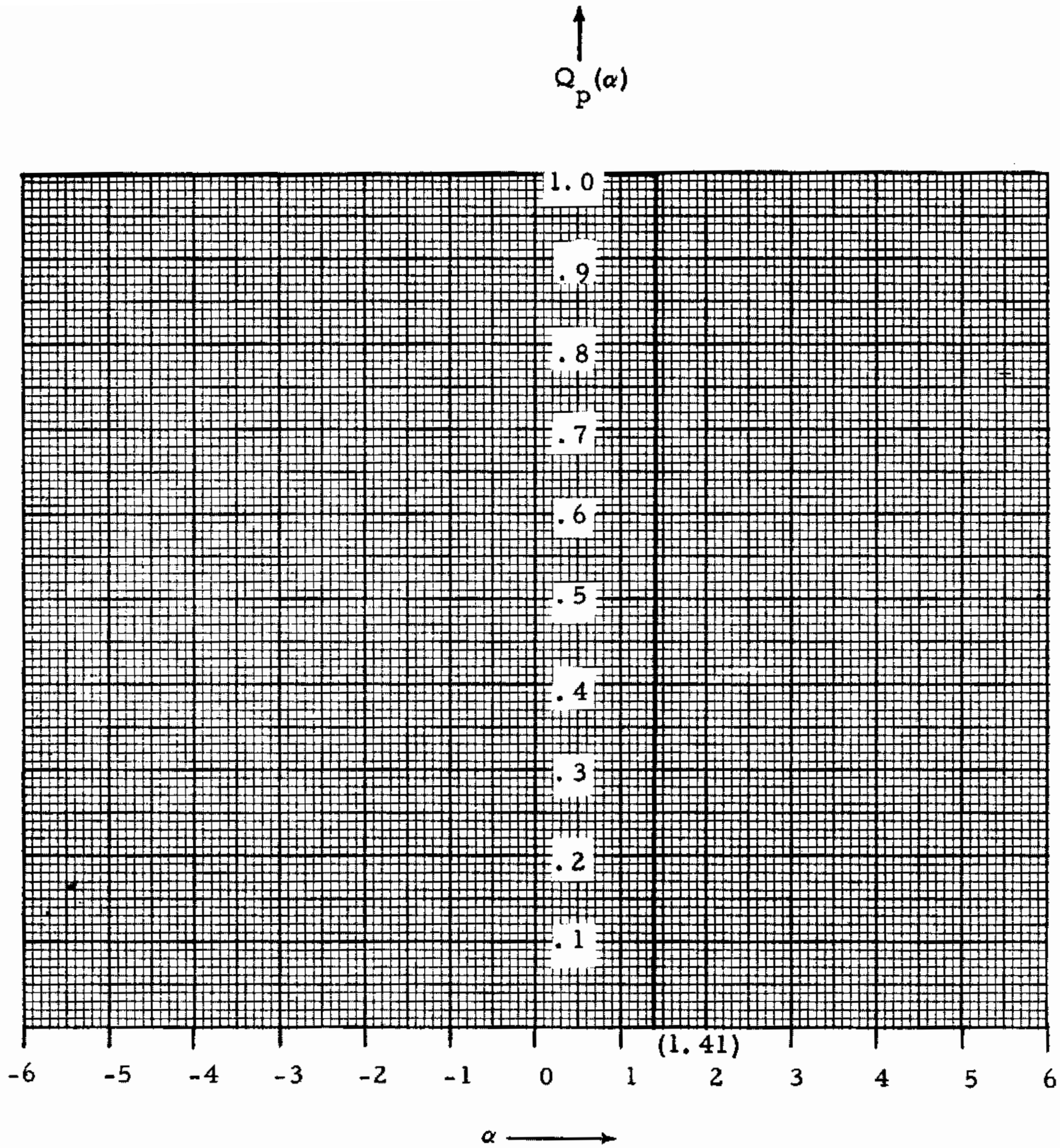


Figure 10. Negative Peak Value Probability Distribution Function of a Sine Wave

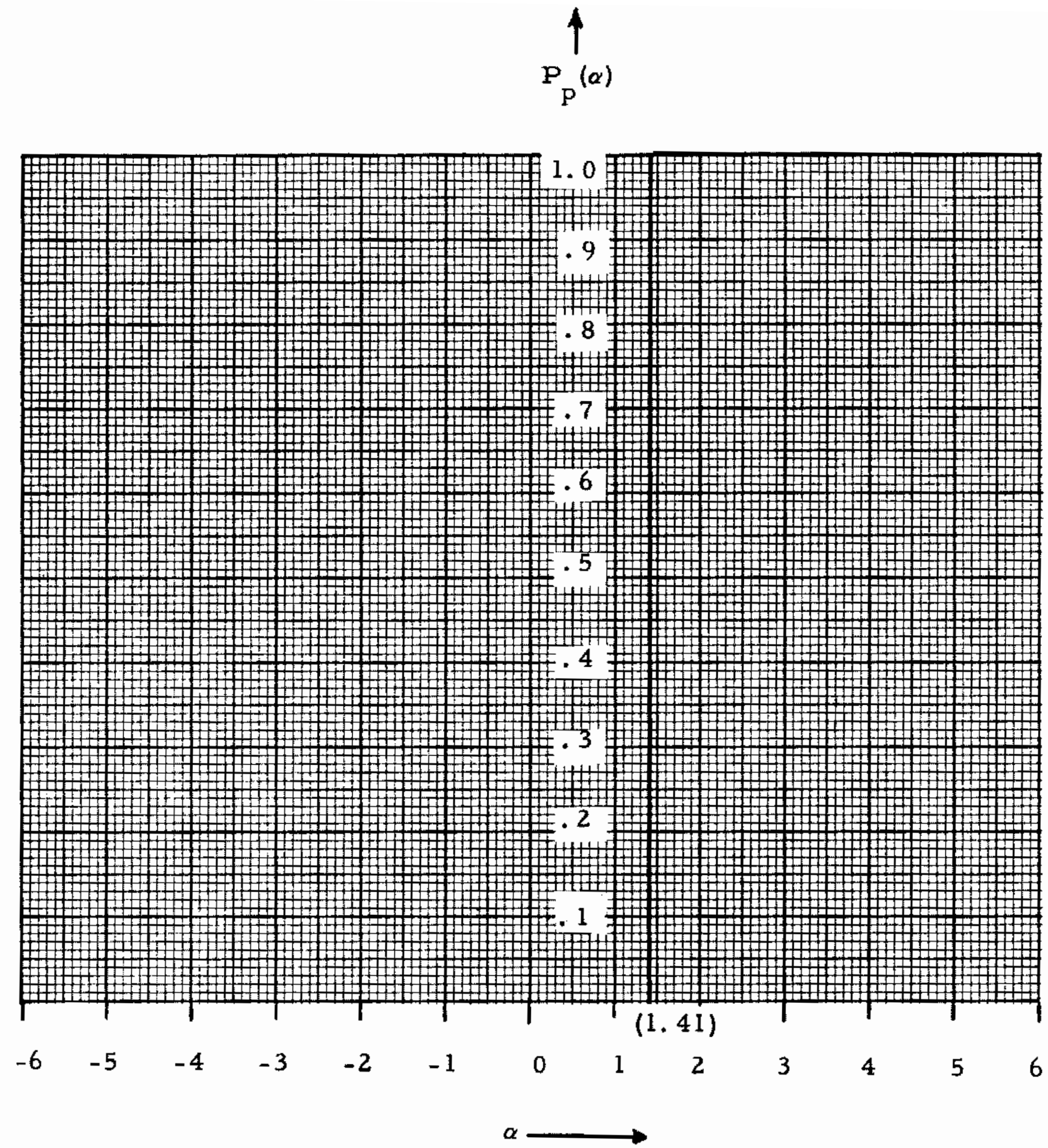


Figure 11. Positive Peak Value Probability Distribution Function for a Sine Wave

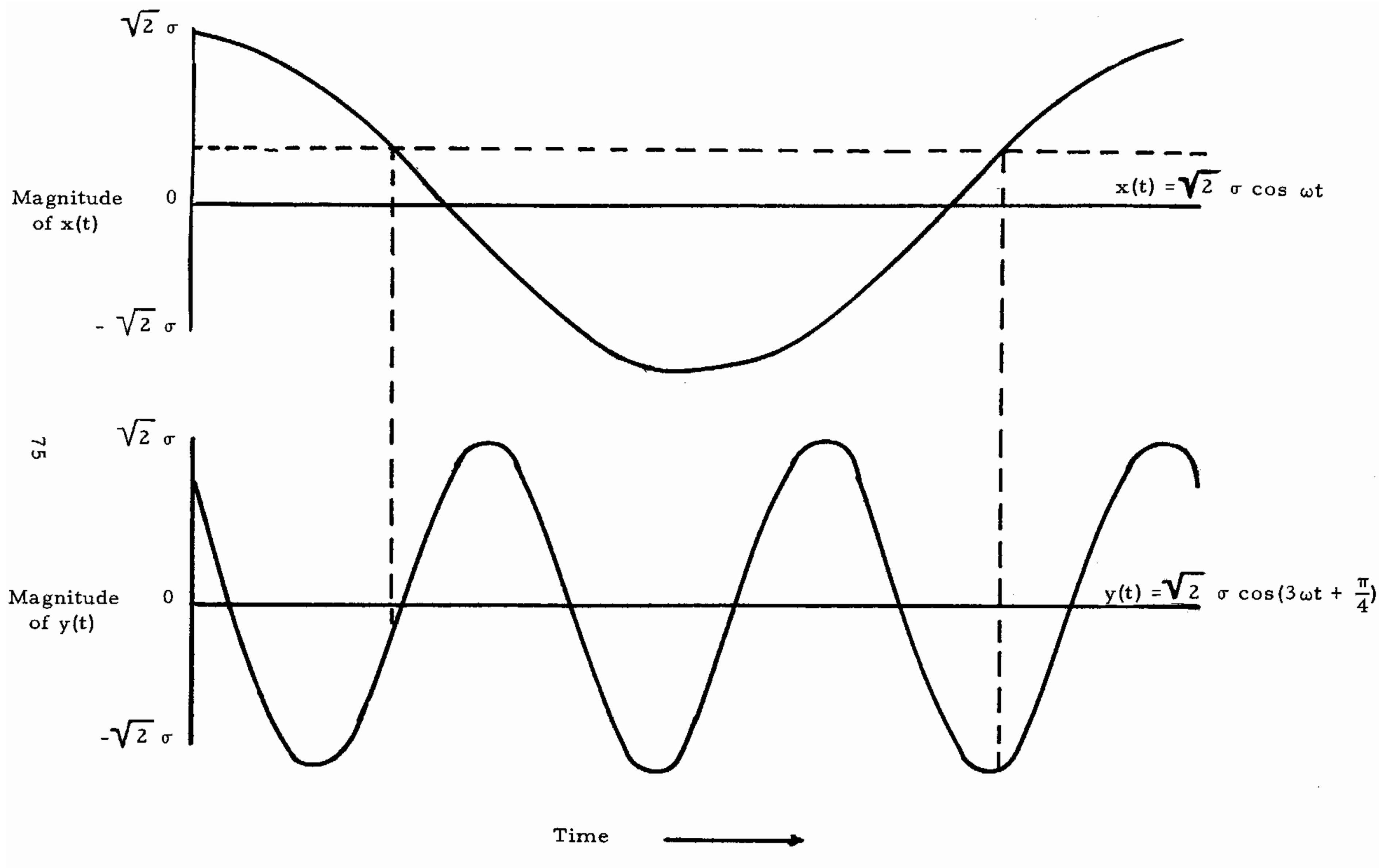


Figure 12. Time History of Cosine Waves x(t) and y(t)

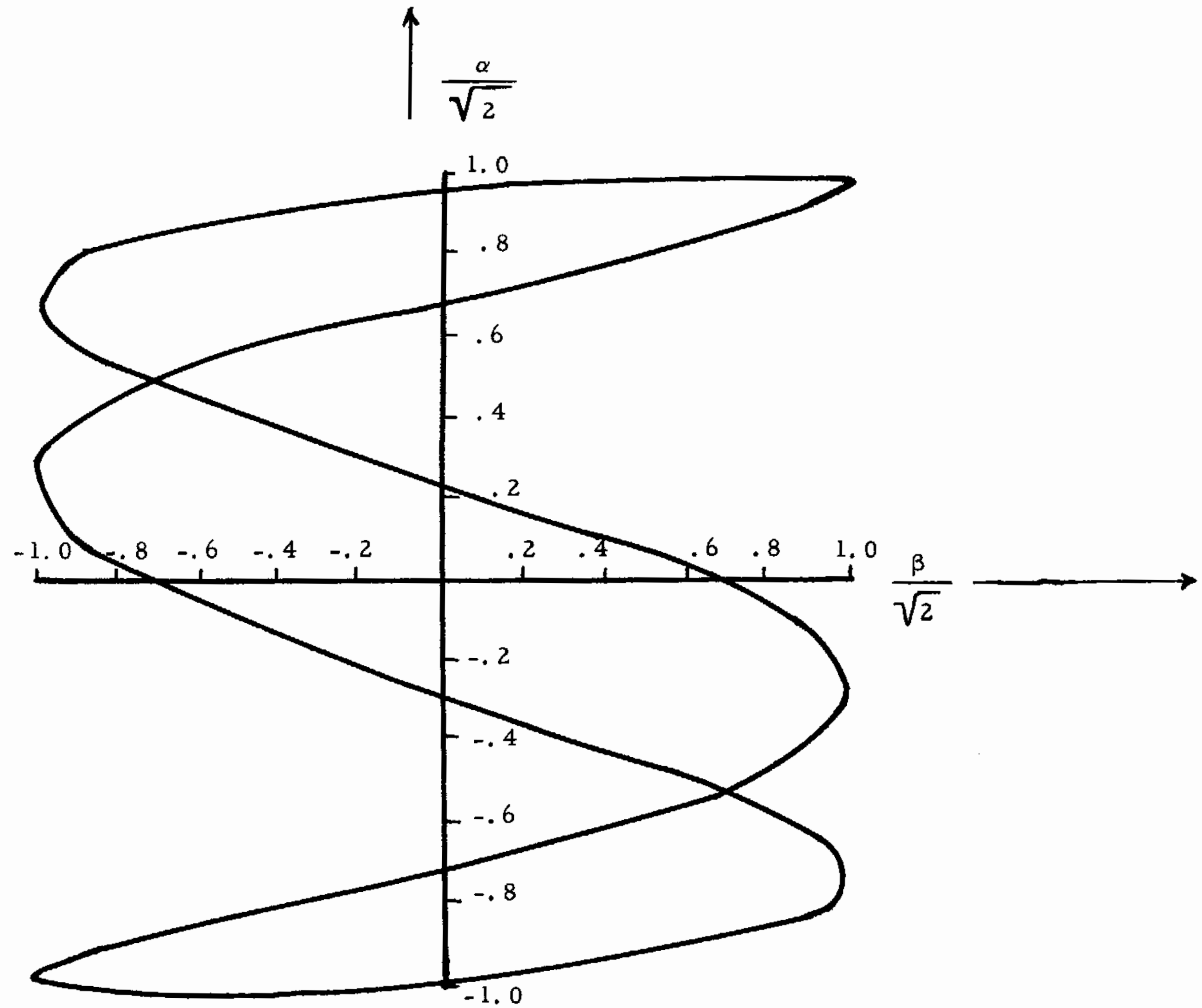


Figure 13. Locus of the Joint Probability for $x(t) = \sqrt{2}\,\sigma_x \sin \omega t$ and $y(t) = \sqrt{2}\,\sigma_y \sin(3\omega t + \frac{\pi}{4})$

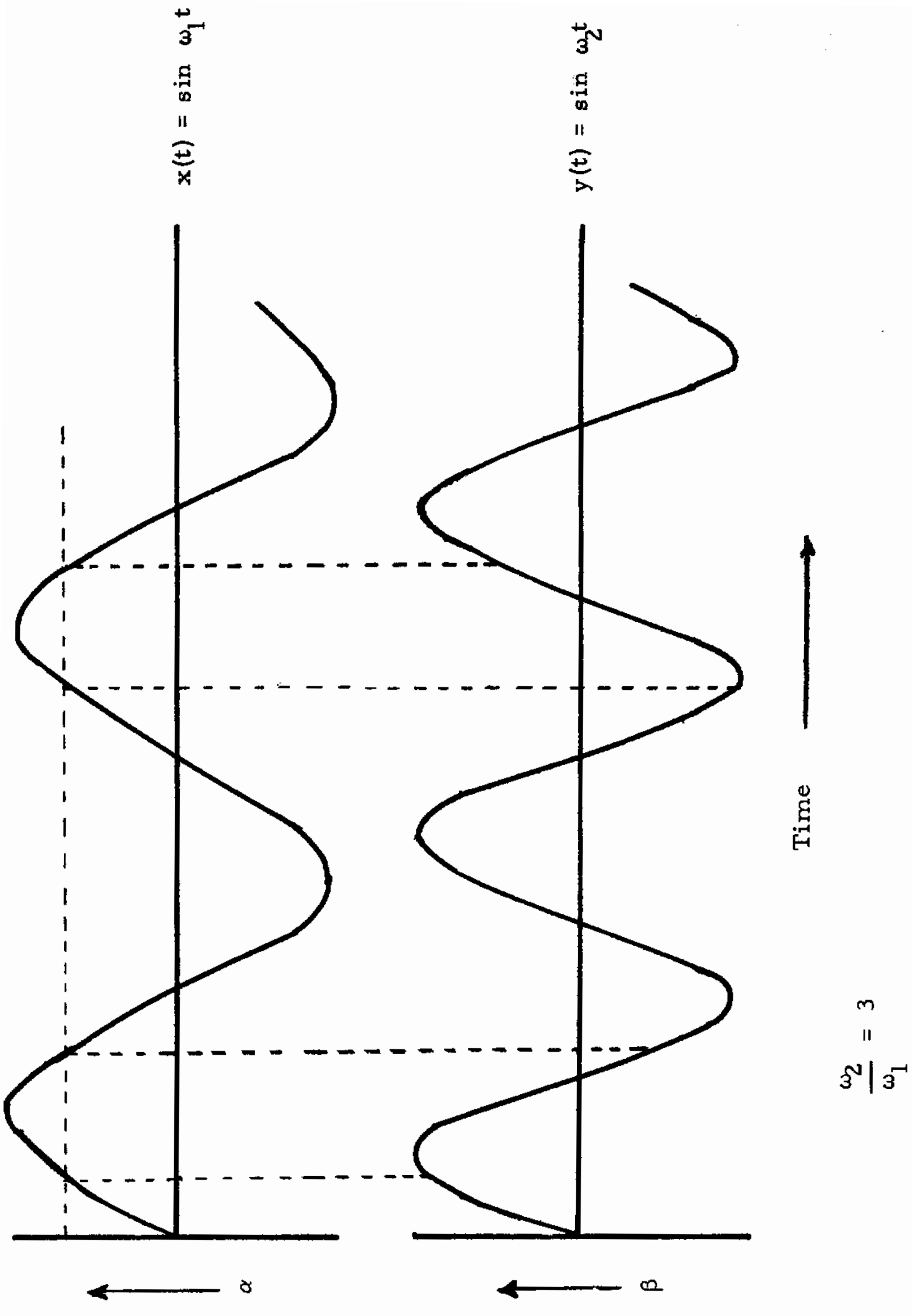


Figure 14. Time History of Two Rationally Related Sine Waves

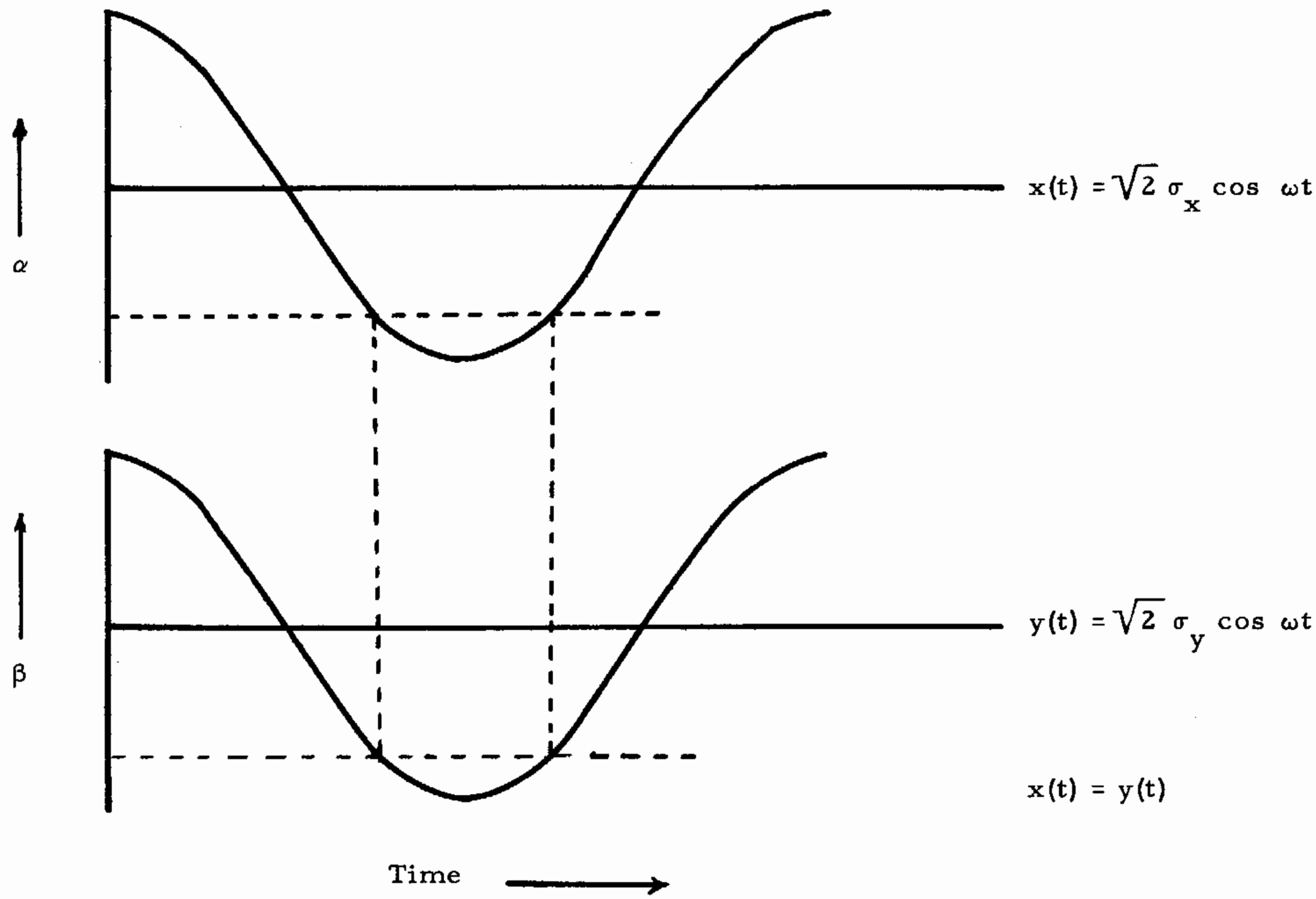


Figure 15. Time History of Two Identical Cosine Waves

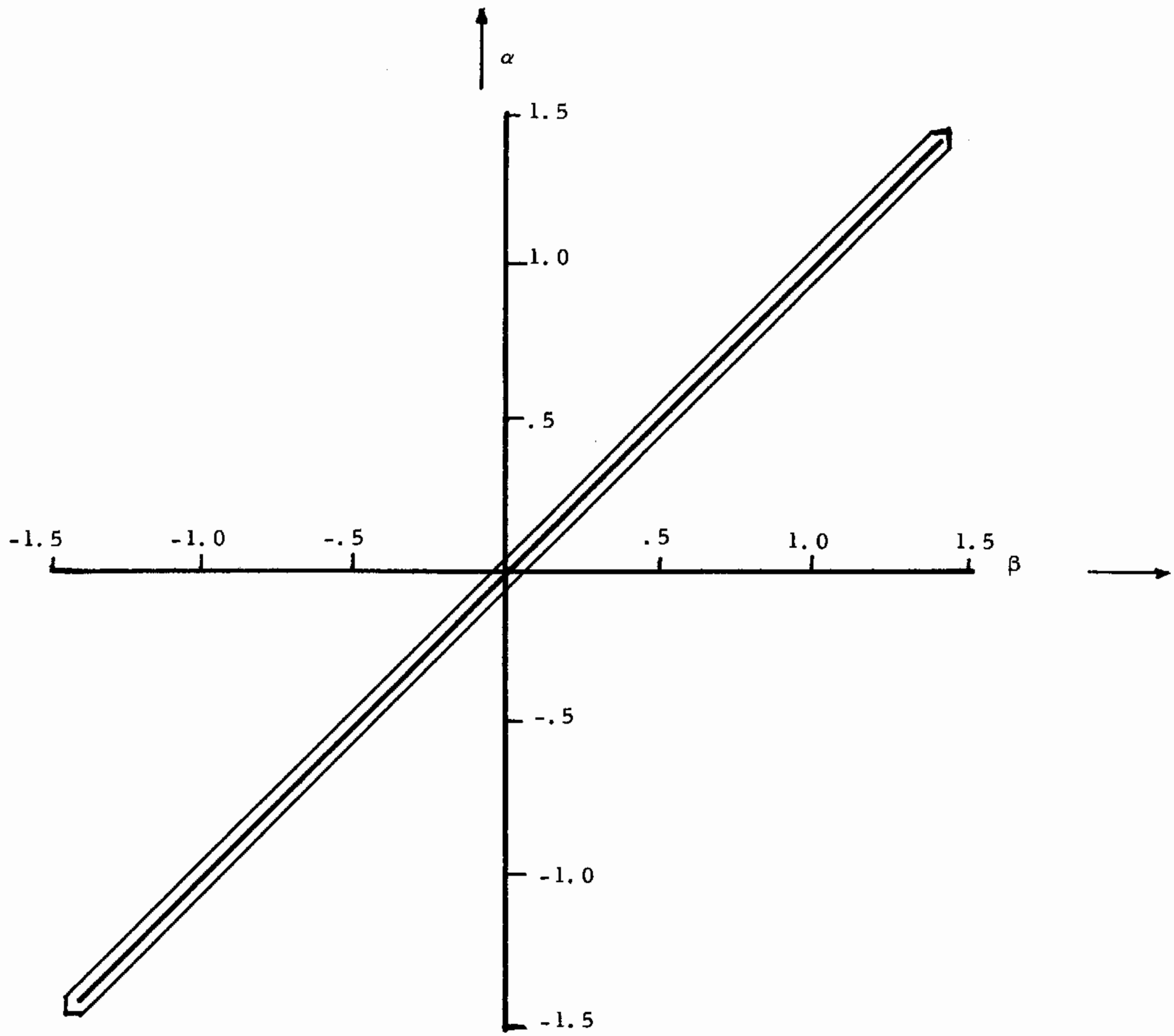


Figure 16. Locus of the Joint Probability Density Function of Two Identical Sine Waves

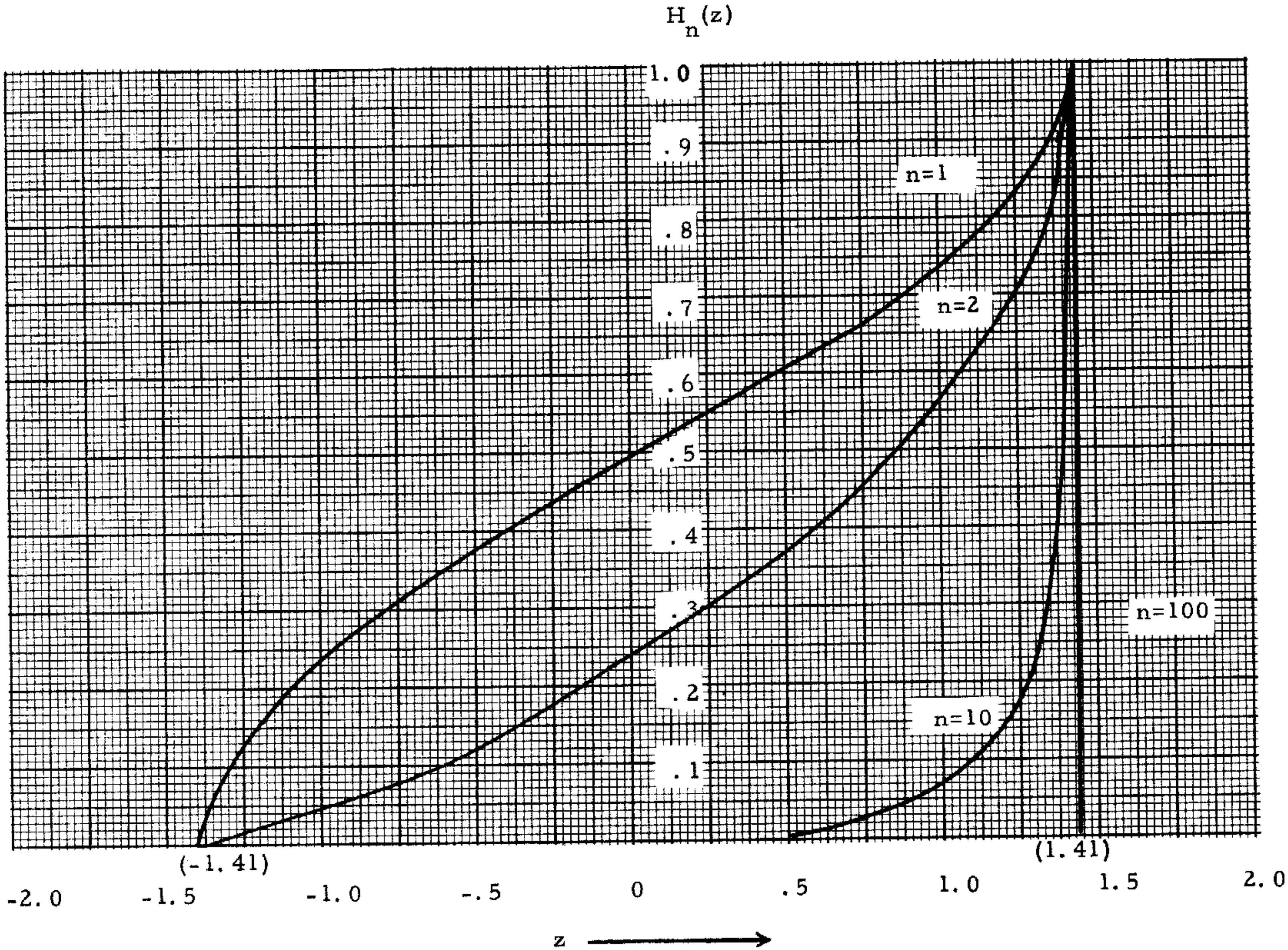


Figure 17. Extreme Value Distribution Functions of a Sine Wave

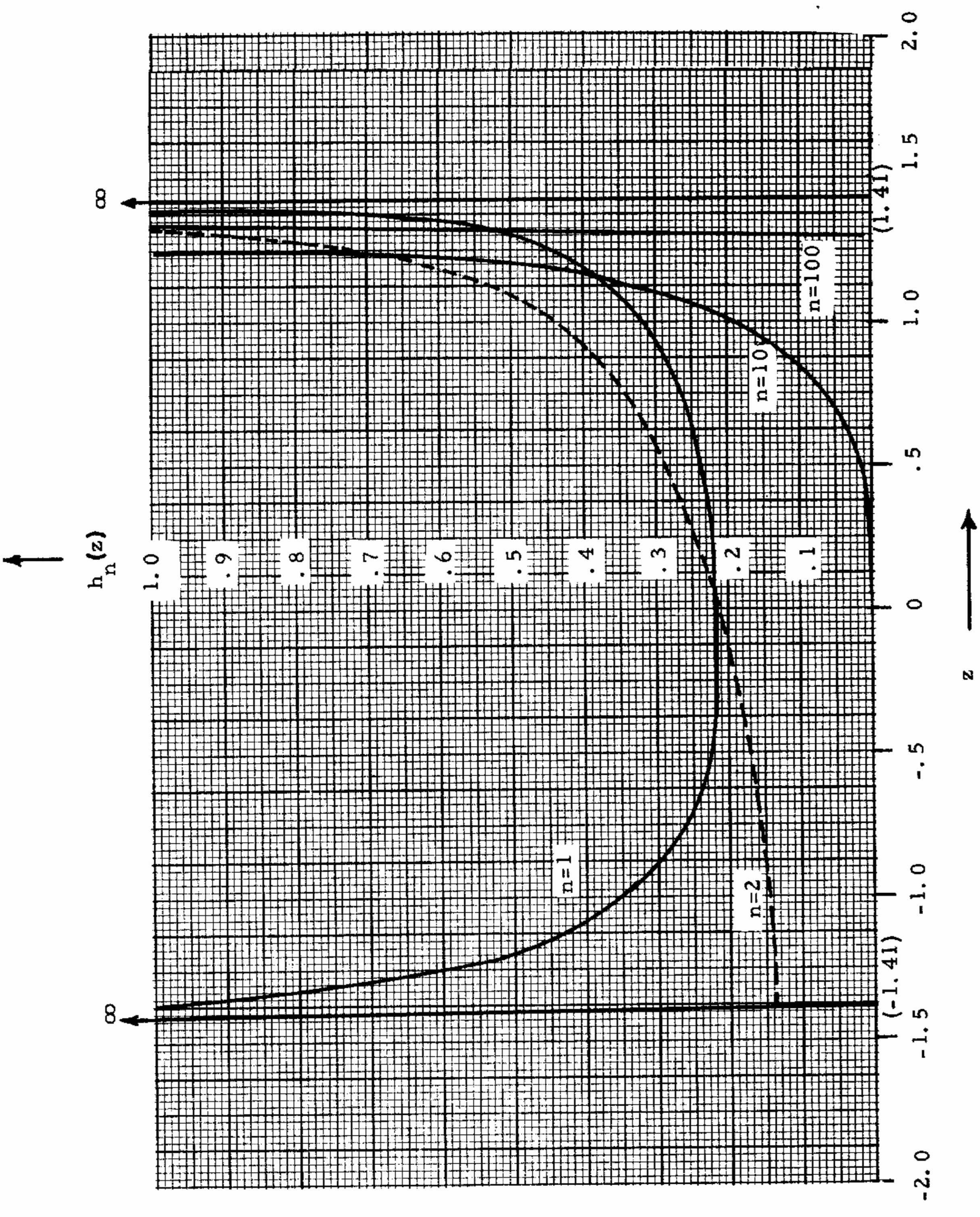


Figure 18. Extreme Value Density Functions of a Sine Wave

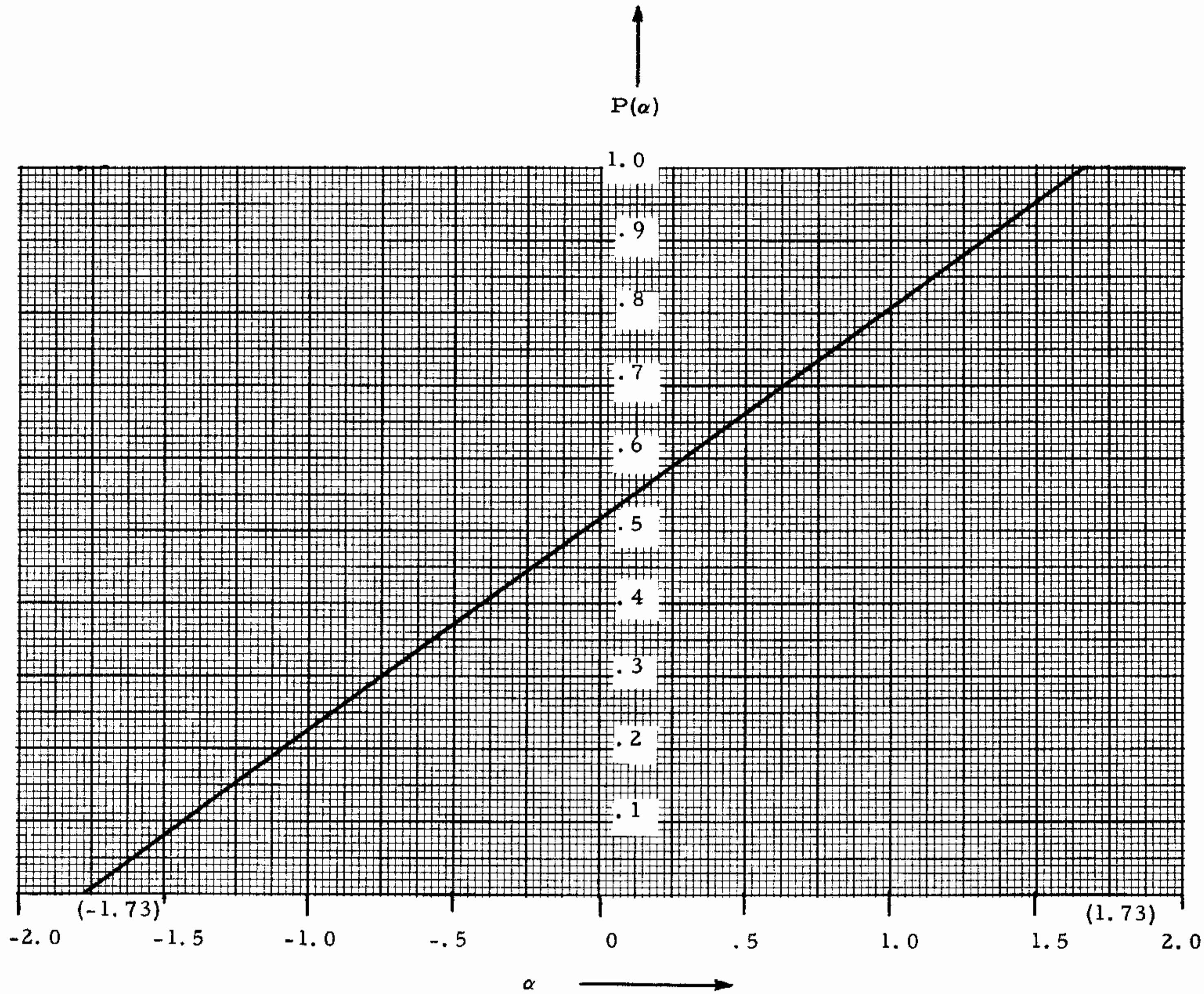


Figure 19. Instantaneous Amplitude Probability Distribution Function of a Triangular Wave

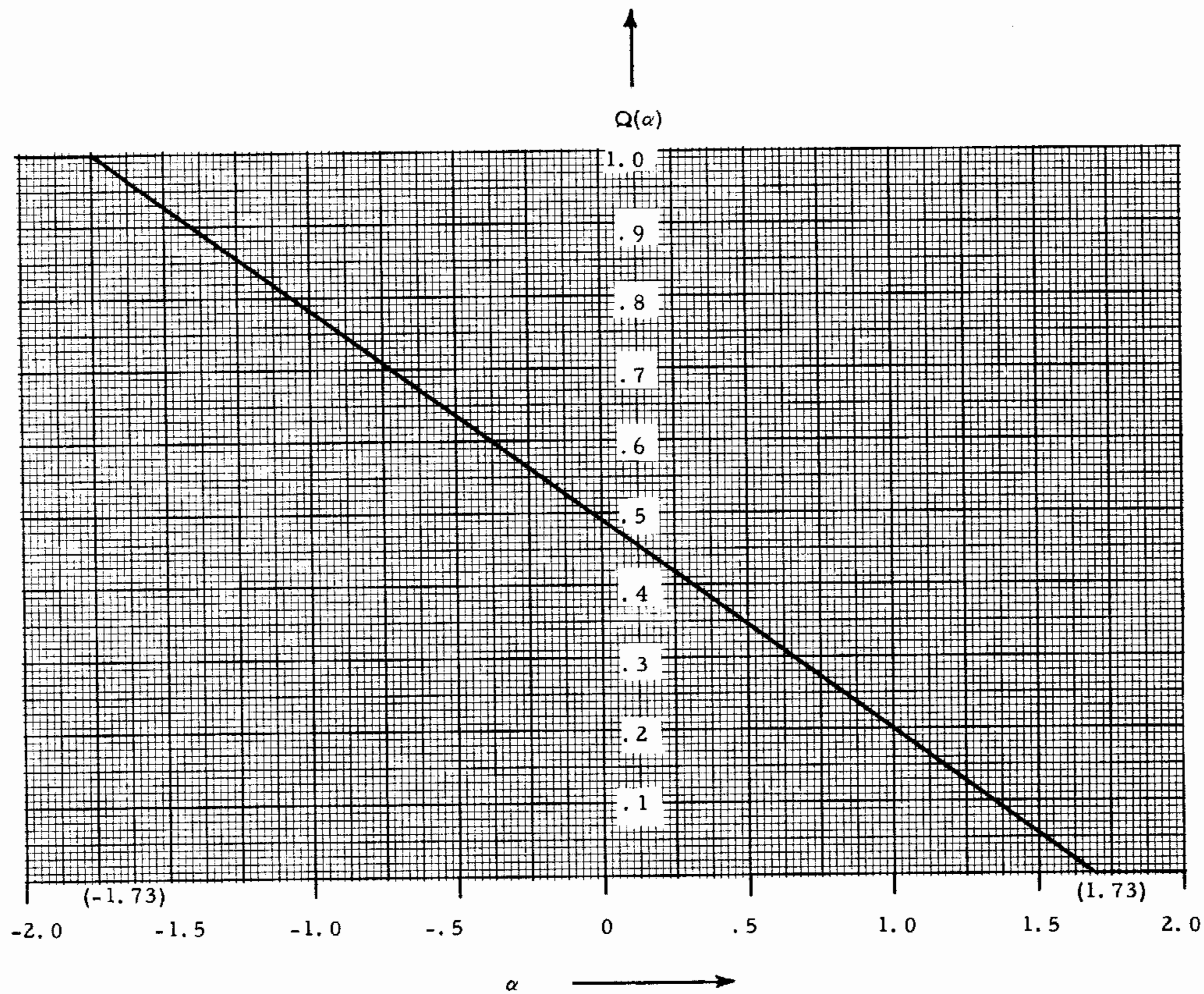


Figure 20. Negative Instantaneous Amplitude Probability Distribution Function of a Triangular Wave

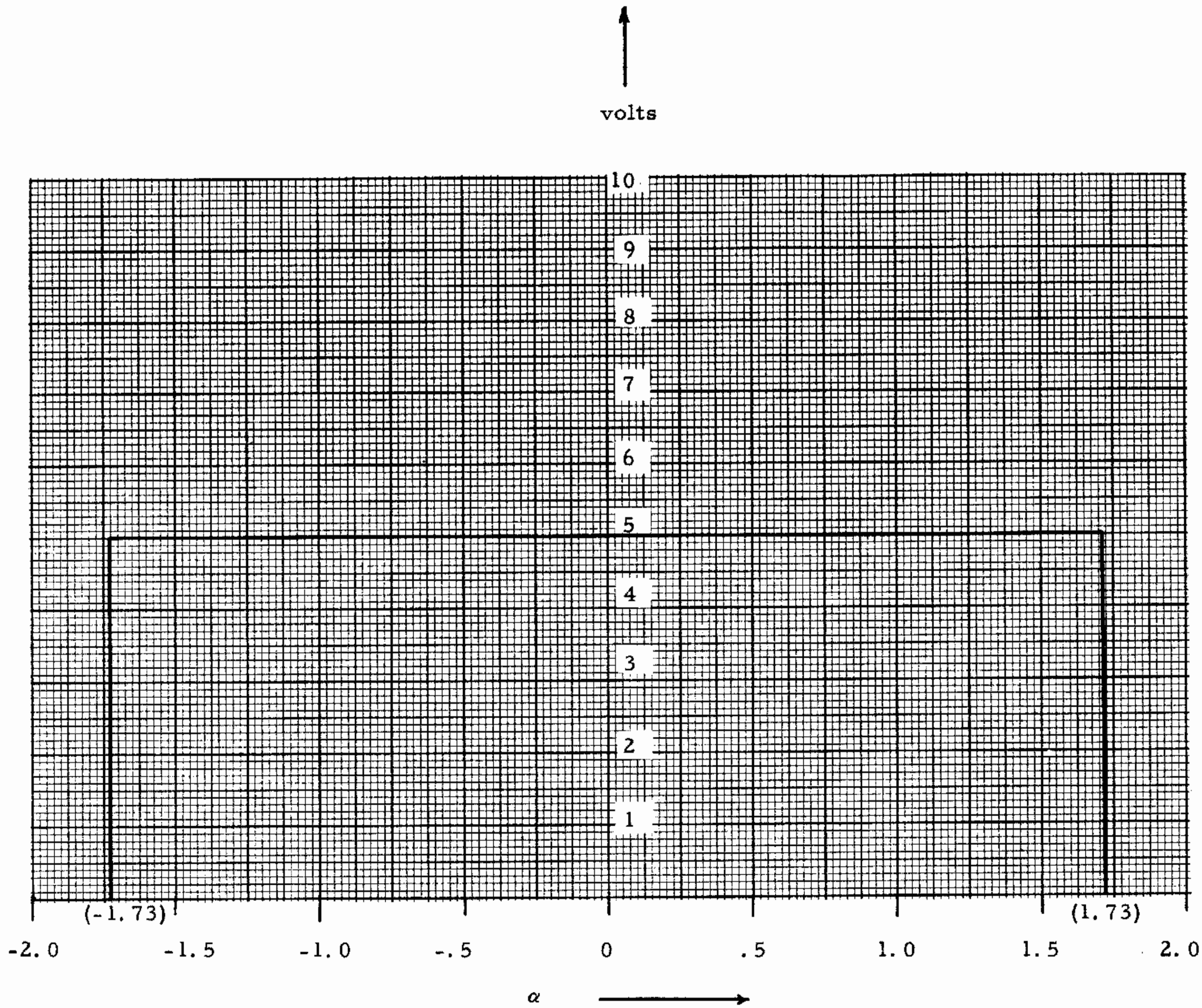


Figure 21. Expected Number of Total Threshold Crossings per Unit Time for a 1 KC Triangular Wave (Analyzer Set on 2 KC Position)

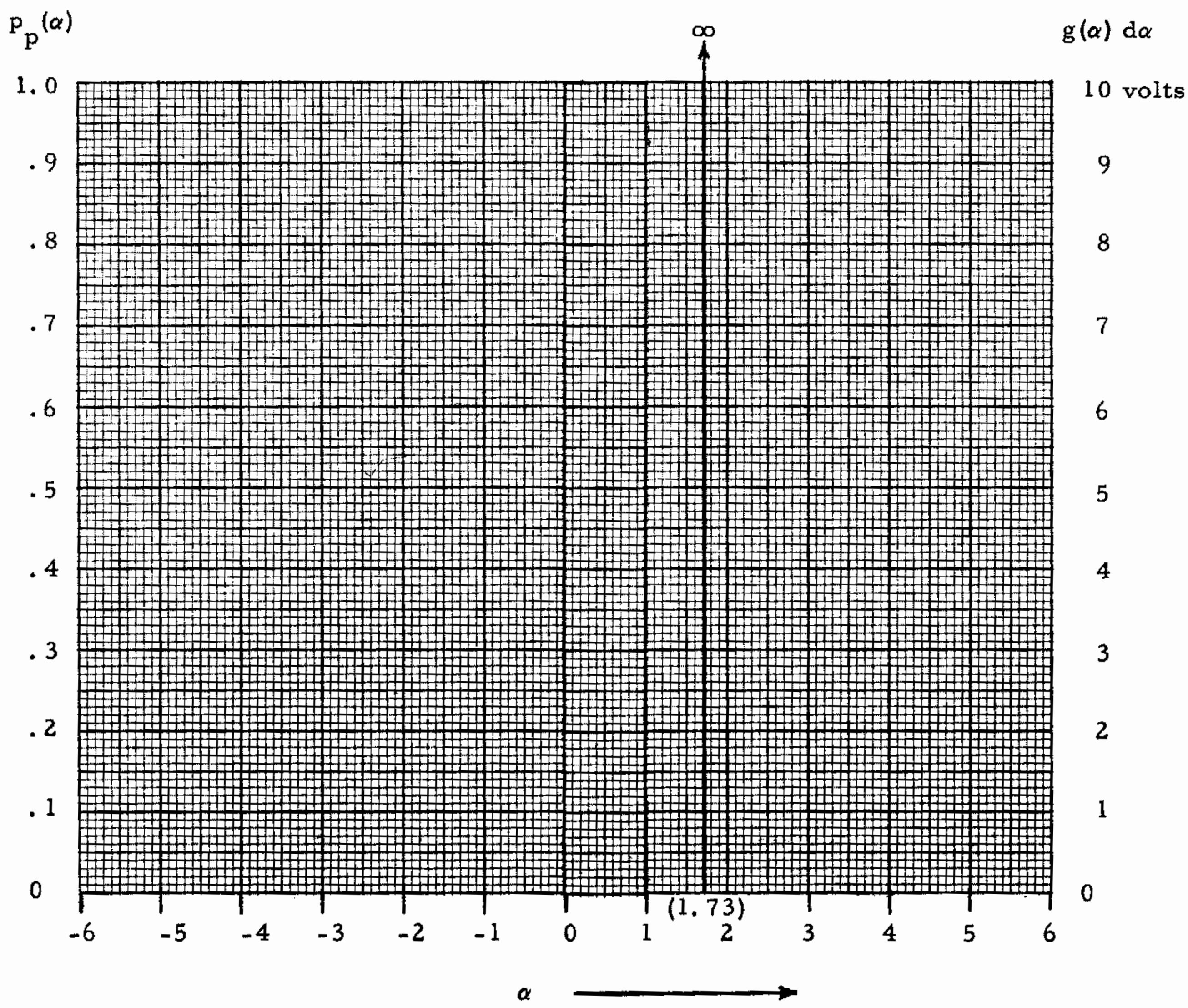


Figure 22. Peak Value Probability Density Function or Expected Number of Maxima Per Unit Time in a Narrow Amplitude Window Function for a Triangular Wave

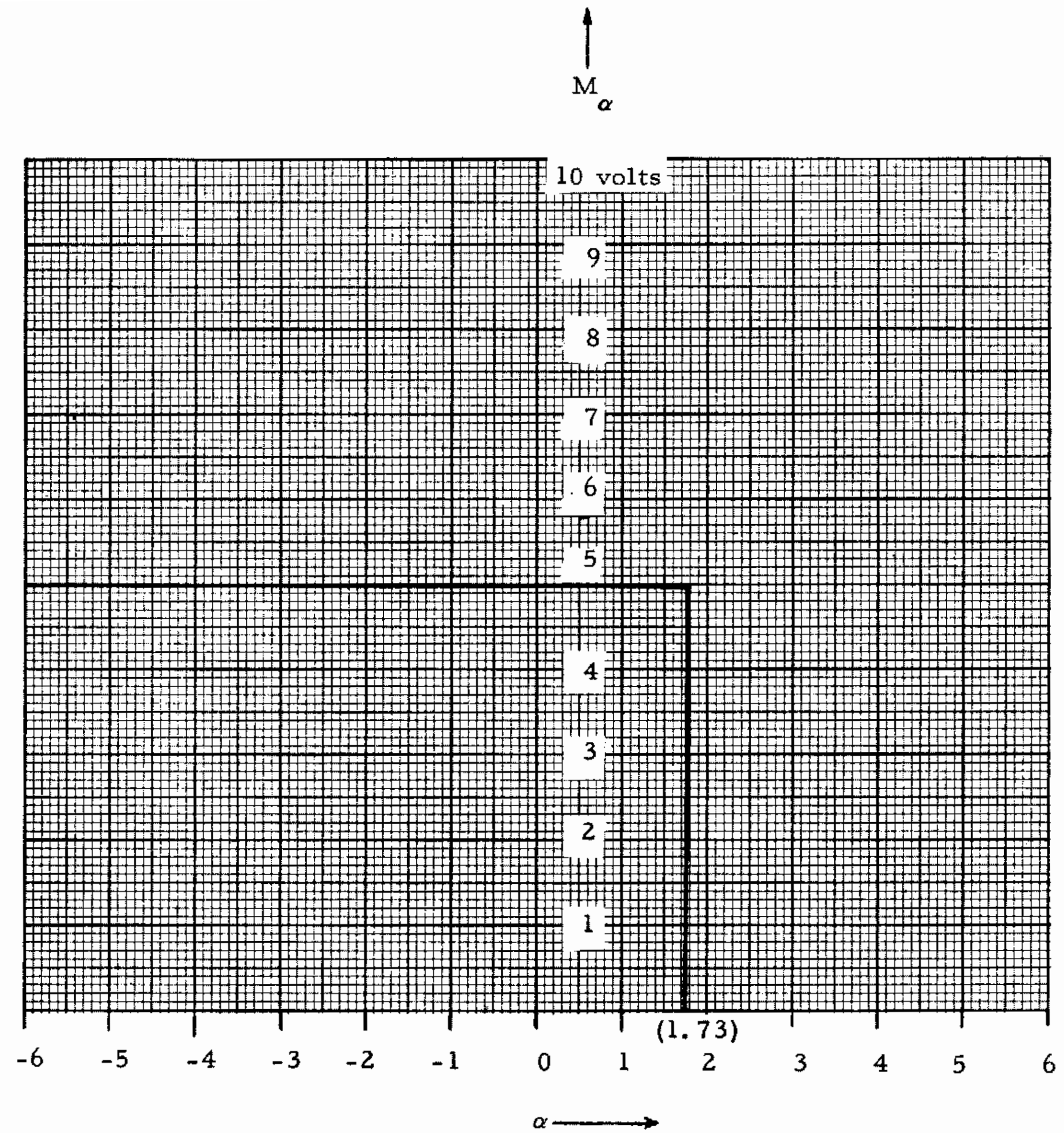


Figure 23. Expected Number of Maxima per Unit Time in Excess of a Given Level for a 1 KC Triangular Wave (2 KC = Full Scale)

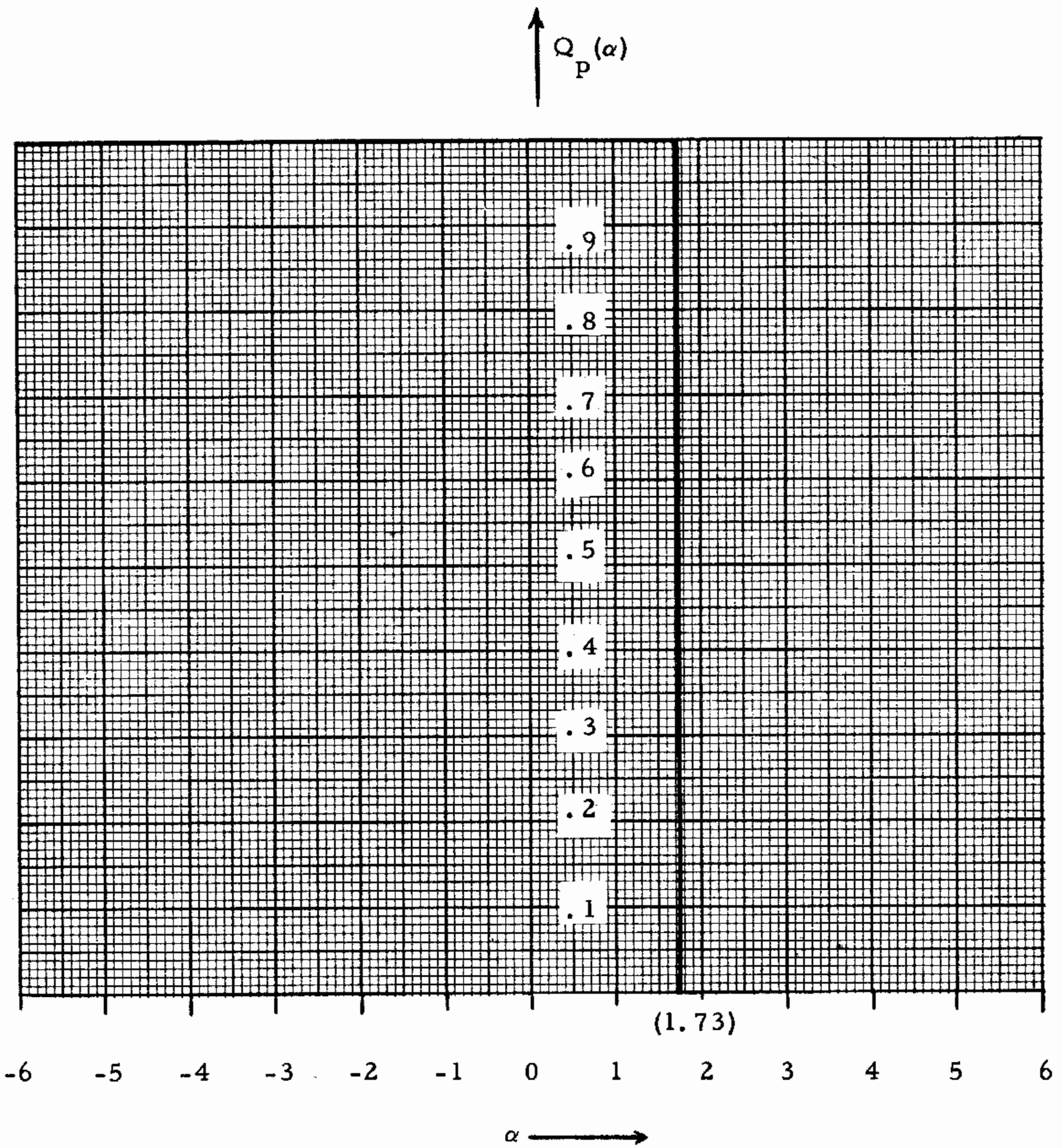


Figure 24. Negative Peak Value Probability Distribution Function of a Triangular Wave

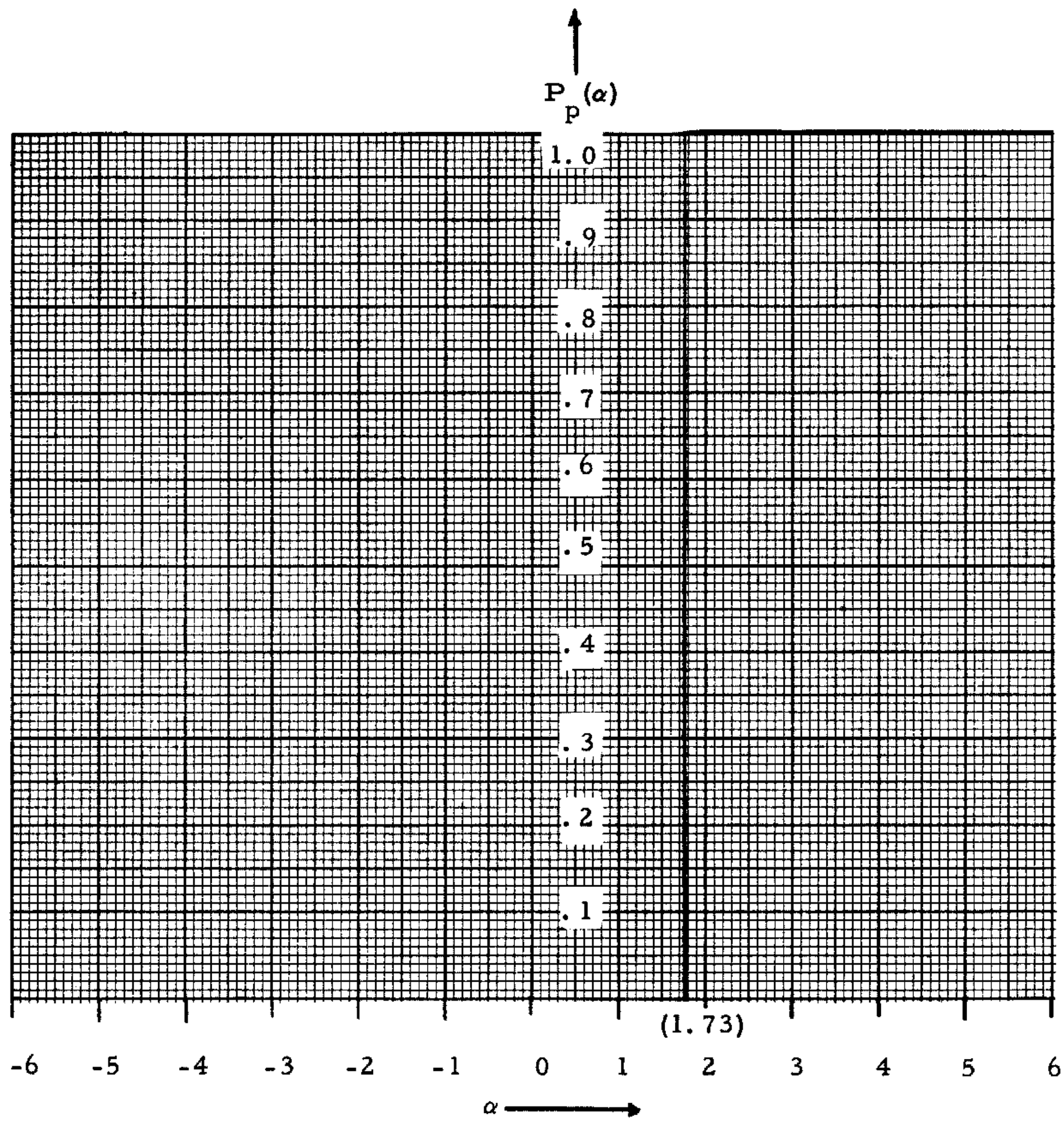


Figure 25. Positive Peak Value Probability Distribution Function of a Triangular Wave

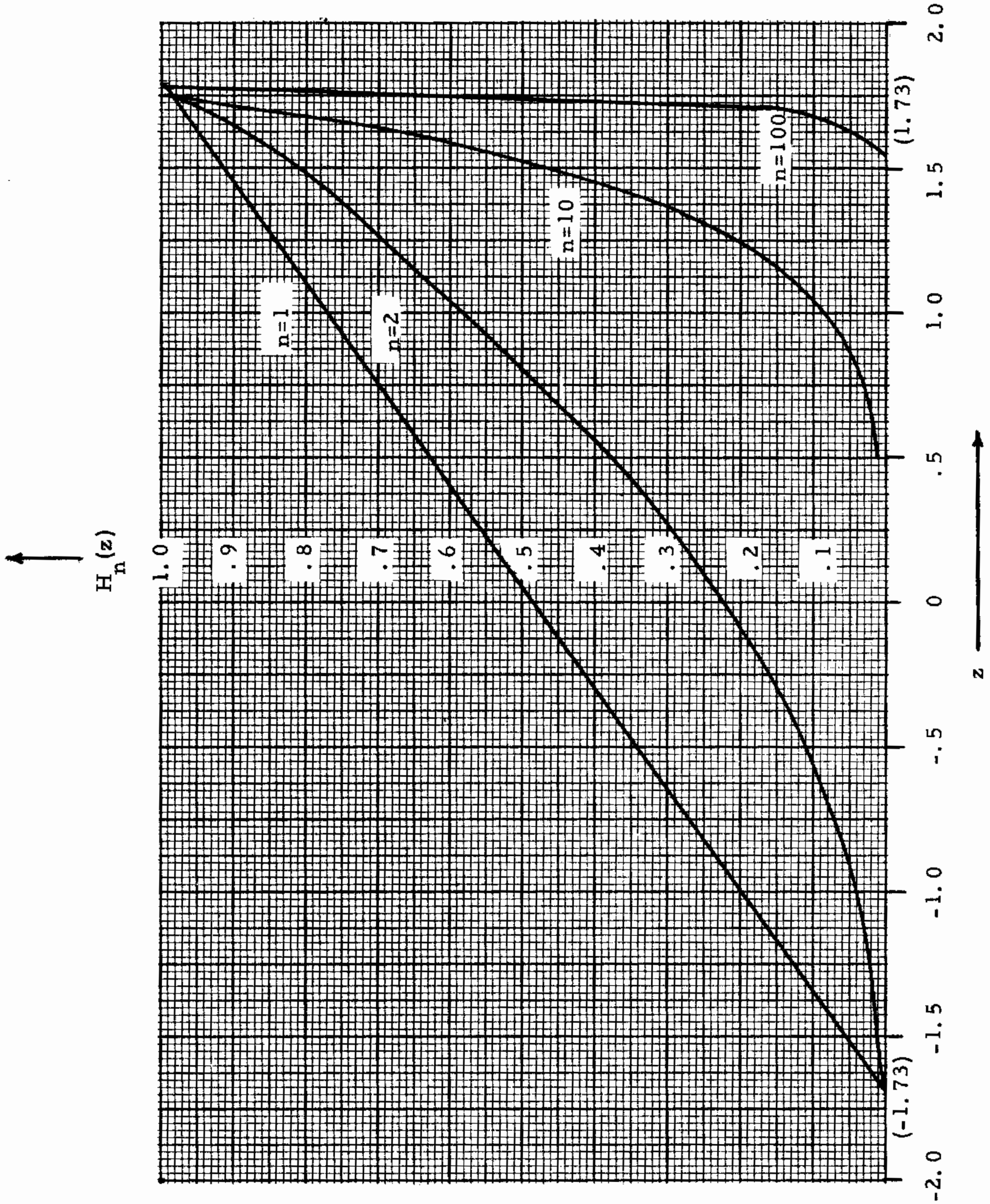


Figure 26. Extreme Value Distribution Functions for a Triangular Wave

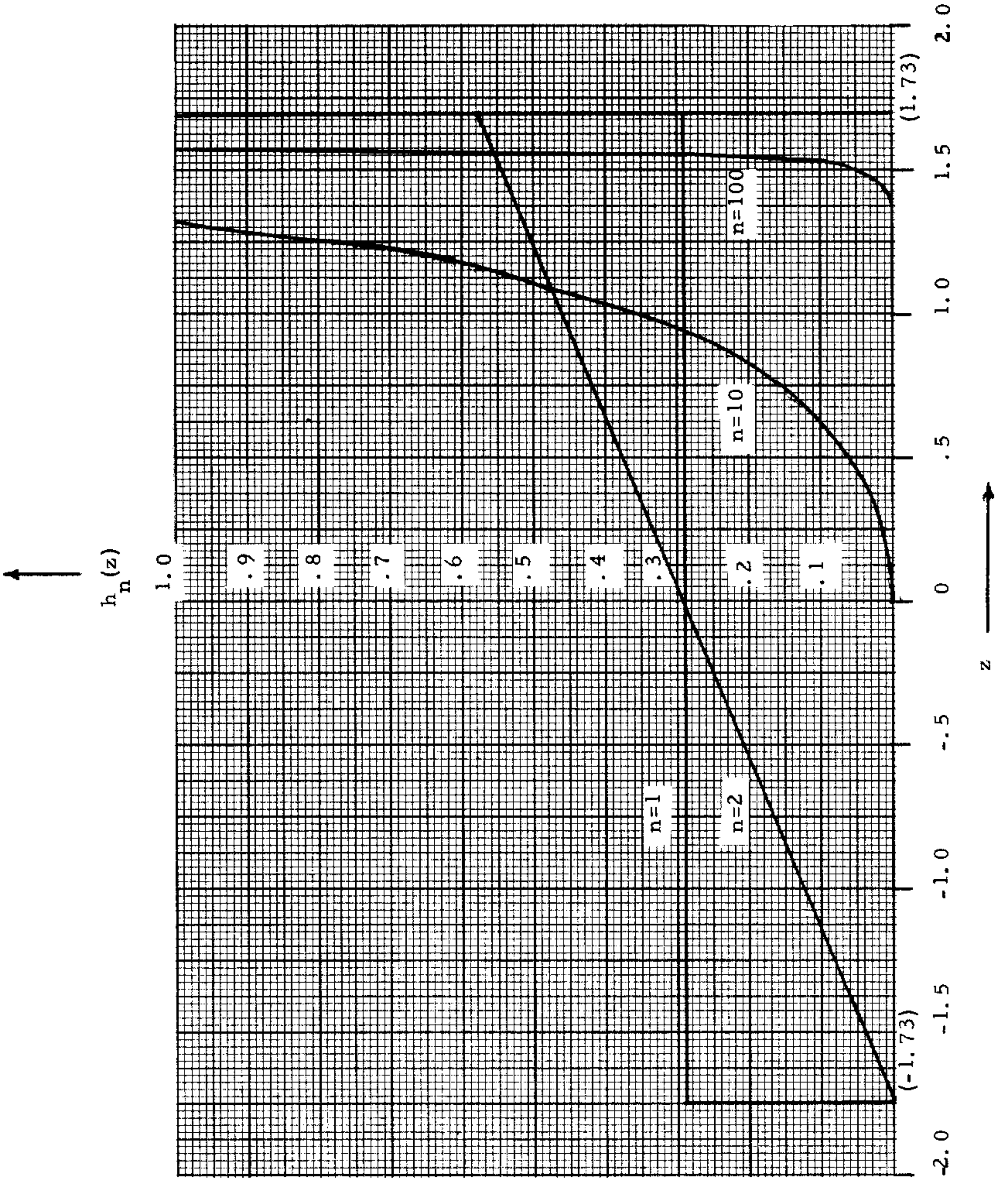


Figure 27. Extreme Value Density Functions for a Triangular Wave

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Measurement Analysis Corporation 10962 Santa Monica Boulevard Los Angeles, California 90025	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP N/A

3. REPORT TITLE

Evaluation Tests for Statistical Analyzers

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Final Report (one of several under this contract)

5. AUTHOR(S) *(Last name, first name, initial)*

R. D. Kelly

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1965	90	5

8a. CONTRACT OR GRANT NO. AF33(615)-1418 b. PROJECT NO. 1370 c. Task 137005 d.	9a. ORIGINATOR'S REPORT NUMBER(S) AFFDL-TR-65-93 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* MAC 403-05B

10. AVAILABILITY/LIMITATION NOTICES

No Limitation

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
None	AF Force Flight Dynamics Laboratory, (FDDS) Wright-Patterson AFB, Ohio 45433

13. ABSTRACT This report describes a series of tests designed to evaluate the performance of statistical analyzers. The types of analyses that these analyzers typically perform and that must be evaluated are: (1) Instantaneous amplitude probability density, (2) Instantaneous amplitude probability distribution, (3) Negative instantaneous amplitude probability distribution, (4) Peak value probability density, (5) Expected number of maxima per unit time, (6) Expected number (total, positive, or negative) of threshold crossings per unit time, (7) Joint instantaneous amplitude probability distribution, (8) Joint instantaneous amplitude probability density, (9) Extreme value density, (10) Extreme value distribution.

Tests with both periodic (sinusoidal and triangular) and random (broadband Gaussian, narrowband Gaussian, and clipped Gaussian) signal inputs are delineated for each of the above analysis modes. Tolerances on the output wave shapes of the periodic signal generators are described so that generators whose outputs will not contribute significantly to the measurement errors can be selected. It is suggested that the random test signals be recorded on magnetic tape so that the identical signals can be analyzed by the statistical analyzer and a digital computer. The digital computer analysis will accurately define the statistical properties of the actual test signal so that the problems associated with imperfections in the random noise generator and statistical uncertainty fluctuations can be avoided. The analytical derivation of all of the above statistical functions for sinusoidal input signals are included to illustrate the operating principles of this analyzer.

DD FORM 1 JAN 64 1473

UNCLASSIFIED

Security Classification

UNCLASSIFIED
Security Classification

14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
Analyzers						
Probability Density and Distribution						
Joint Density and Distribution						
Peak Probability Density						
Expected Number of Maxima per Unit Time						
Expected Number of Threshold Crossings per Unit Time						
Extreme Value Density and Distribution						
Evaluation Tests						
Errors						
Statistical Analyzers						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization (*corporate author*) issuing the report.

2*a*. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2*b*. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7*a*. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7*b*. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8*a*. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8*b*, 8*c*, & 8*d*. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9*a*. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9*b*. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers (*either by the originator or by the sponsor*), also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through

_______________________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through

_______________________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through

_______________________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring (*paying for*) the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, *or (U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.

UNCLASSIFIED
Security Classification