FOREWORD

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The work reported herein was conducted by S. Eggwertz and G. Lindsjö under the guidance of Bo K. O. Lundberg, Director, FFA.

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ABSTRACT

A study is made of the probability of collapse of a fail-safe structure, consisting of a number of parallel members, subjected to a random load spectrum. In the individual members a fatigue crack is first initiated and failure of the members occurs due to a heavy load on the weakened members. The probability of element failure is obtained by a combination of the probabilities of crack initiation and of meeting a load exceeding the residual strength of the member. The probability of consecutive element failures is deduced from the probability of failure of the individual members. Collapse occurs when all members are broken, or, in practice, after a critical number of element failures. The probability of collapse of the assembly during the whole service life is the sum of the probabilities of all the inspection intervals.

A numerical procedure for calculating the probability of collapse has been developed and evaluations have been made for an assembly of six identical, parallel members. Diagrams of the probability of collapse P versus the service life time T_L have been plotted in figs. 3 - 8 for various lengths of regular inspection intervals, assuming different values of the crack initiation and strength reduction parameters introduced.

Some preliminary fatigue testing of assemblies with six members has been carried out in order to study the validity of basic assumptions in the theoretical investigation.

PUBLICATION REVIEW

This report has been reviewed and is approved.

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1. INTRODUCTION

There seems to be a rather common agreement since a few years ago [1] that the probability of a fatal failure during the service life of an aircraft due to fatigue of the structure should not exceed a very low figure. A value of around 10^{-5} has been considered. To ascertain such a low probability of failure for an ordinary safe-life structure will either make it necessary to carry out a very large number of fatigue tests or to apply an extraordinarily high safety factor on the results from a few fatigue tests. For economical reasons these two ways are only feasible for smaller parts of an aircraft structure. As is also commonly agreed most of the structure must consequently be designed fail-safe, which means that the structure is so built that a minor damage, e.g. a crack, will be detected and repaired before it has caused an appreciable reduction of the ultimate strength of the whole structure. The crack detection might be effected at special inspections carried out at predetermined intervals. It is also possible that some rather obvious damage, as a visible partial failure, will be detected with certainty without any inspections. The fail-safe properties are thus normally achieved by a combination of design features and inspections. The fact that a structure is fail-safe does not imply, however, that its service life can be extended indefinitely. One reason is that the repair rate will be so high when the various elements of the structure approach their mean fatigue lifes, that the operation of the aircraft can no longer be made profitable. But even if the economic repair-rate limit has not been reached, the probability of total failure due to a heavy gust on the weakened structure, which is always increasing with service time utilized, may exceed the value mentioned above, if the inspection intervals are not made uneconomically short.

Consequently it is necessary to evaluate the probability of failure of a fail-safe structure, taking into account the crack initiation, crack development and decrease of the residual strength of the cracked structure as well as the inspection procedure and the load spectrum. This has been attempted in some earlier publications [2, 3], where the fatigue properties from full-scale testing of whole structures, such as an aircraft wing, have been assumed to be known. Since full-scale fatigue testing is an extremely costly and time-consuming undertaking there is, so far, very little information available in the literature. Several laboratories are now carrying out full-scale testing and the situation might therefore soon be radically improved as long as one is only considering mean values of test results, while the

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scatter will probably still be a rather unknown quantity. Although the amount of testing necessary to assess a low probability of failure is considerably less for a fail-safe structure, than for a safe-life component, one may question if the designer will ever be able to rely entirely on a pure full-scale approach.

Another way of attacking the fatigue safety problem for a fail-safe structure would be to divide a large component into a number of small elements, the fatigue properties of which are rather cheap and easy to determine in a satisfactory way from a statistical point of view. The interaction between these elements would then be analyzed mathematically. Some structures consist of a number of discrete elements which are identified without difficulties, but in most aircraft components the division into elements is not so obvious. If the number of different elements, which contribute to the risk of fatigue failure, is very large, the amount of numerical calculations and fatigue testing necessary might possibly be so immense that the element approach cannot be effected in practice. This difficulty could possibly be overcome by persuading the designer to build up his components from a large number of identical elements, which would probably also be useful for the production.

The probability of collapse of structures consisting of smaller members has been treated earlier disregarding crack propagation and inspections [4, 5, 6, 7] . The aim of this report is to demonstrate how a rather simple built-up structure can be analyzed with the effects of strength reduction due to crack propagation and regular inspections taken into account. As an example, is chosen an assembly of six identical, parallel elements subjected to random gust loads. The numerical calculations are carried on to produce a diagram of the probability of total failure as a function of service life for various lengths of inspection intervals.

2. SYMBOLS

A B C	coefficients obtained from eq. (6), used for computing load redistribution factor c .
a b	coefficients obtained from eqs. (4) and (5), used for computing load redistribution factor c .
c c_j	load redistribution factor, j indicating position of element considered

c_{oj}	coefficient for computing load redistribution factor c; for load-carrying element $c_{oj} = 1$, for broken element $c_{oj} = 0$
F_ν	probability of one sequence of element failures during inspection interval no. ν
F_Q	distribution function of variable Q
f_Q	frequency function dF_Q/dQ
$G(y)$	conditional probability of element failure due to a high load exceeding the residual strength of the wing before a time y
$g(y)$	frequency function $dG(y)/dy$
H	expected number of times per hour that a load amplitude s_a is exceeded; H_u is number of times the ultimate load is exceeded
H_o	parameter of load spectrum, eq. (7); $H_o = 0.2$ chosen for all numerical calculations
h	parameter of load spectrum, eq. (7); $h = 20$ chosen for all numerical calculations
h_i	generalized parameter h including load redistribution, eq. (13)
i	number of element failures having occurred, or cracks initiated
j	position of element
K	coefficient used in relationships between real service time and fictitious fatigue service time and fictitious strength reduction service time; $K_{12} = c_j^\alpha$, where j applies to the position of the element that will fail as no. 2, provided that the first failure has occurred
k k_{3i}	coefficients computed from coefficient K, eqs. (28) and (29); the first index 3 implies that the real service time to be computed is T_3, until the third failure, whereas the index i refers to the various parts of T_3, k_{3i} being coefficient of x_i, y_i or z_i, $i = 1, 2, 3$

ℓ_i ℓ	parameter of quasitruncated, normal frequency function, eqs. (45) and (52); ℓ is obtained from eq. (55)
m	total number of parallel elements
n	total number of inspection intervals
n_u	ultimate design load factor
P	probability of collapse of the assembly
P_ν	probability of collapse of the assembly within inspection interval ν
P_c	probability of crack initiation, eq. (14)
P_u	probability of static failure, eq. (9)
p_c	frequency of crack initiation, eq. (33)
Q_3	real service time until failure from beginning of inspection interval ν ; Q_{3a} is time to crack initiation and Q_{3b} crack propagation time, eqs. (36) - (39)
q	critical number of element failures
R	parameter of strength reduction curve, eq. (16), indicating the inverse rate of strength reduction
$r_{i\nu}$	maximum frequency of crack initiation (crack no. i) at the end of inspection interval ν, eqs. (39) and (63)
S	load level
S_a S_{ai}	load amplitude; S_{ai} is amplitude after i failures have occurred, i = 0 being the original condition
S_{mi}	mean load
S_u	ultimate load
S_o	original static margin
s_{ai}	load amplitude, normalized with respect to S_o, eq. (1)

s_{ot}	normalized static margin at the time t after crack initiation, eq. (16)
T T_i	service time until failure, $T = T_c + t$; T_i applies to element failure no. i
T_c	service time until crack initiation
T_L	limit service time
$T_{\nu-1}$ T_ν	service time until beginning and end of inspection interval ν
T_{c50}	time to initiation of crack corresponding to median value of ${}^{10}\log T_c$
t	service time from crack initiation, crack propagation time
t_{insp}	interval between inspections
u_i	variable in the convolution of Q_{3a}, eq. (43)
x	fictitious fatigue service time, eq. (15)
y	fictitious strength reduction service time, eq. (17)
z	fictitious service time until failure, eq. (19)
α	constant introduced in the relationship between x and T_c, eq. (15)
β	constant introduced in the relationship between y and t, eq. (17)
ϑ	substitution introduced in the convolution of Q, eq. (59)
$\varkappa$	substitution introduced in the convolution of Q, eq. (60)
μ	parameter of normal frequency function of Q_{3b}, eqs. (54) and (56)
μ_c	parameter of normal distribution of ${}^{10}\log T_c$, corresponding to T_{c50}, eq. (14)
μ_i	parameter of normal frequency function of $\bar{g}(y_i)$, eqs. (45) and (48)

ν	number of inspection interval
σ	parameter of normal frequency function of Q_{3b}, eqs. (54) and (57)
σ_c	parameter of normal distribution of $^{10}\log T_c$, standard deviation, eq. (14)
σ_i	parameter of normal frequency function of $\bar{g}(y_i)$, eqs. (45) and (51)
ϕ φ	normal distribution function and frequency function, respectively.

Subscripts

c	crack
i	element failure no. i, or crack no. i
j	element with position no. j
L	limit
0	i = 0, initial state of load distribution
ν	inspection interval no. ν.

3. BASIC ASSUMPTIONS

A simple structure consisting of m parallel members is considered (Fig. 1). These members, which will be called elements no. 1, 2,..., in the following, are assumed to be identical and consist of a rather narrow sheet of aluminium alloy provided with some sort of notch, e.g. a central hole. The structure is subjected to a random load spectrum giving originally a load level S in each element. It is assumed that the probability distribution of service life until crack initiation for an individual element under the same load spectrum has been determined with a high confidence from separate fatigue testing. It is also assumed that the crack propagation time from crack initiation to failure under 1 g load and the simultaneous strength decrease rate is known from such testing, the scatter being neglected.

After the first crack has appeared, it is either possible that cracks are initiated in one or several of the other elements before the first crack has propagated so far that the first element failure occurs, or that this failure takes place without the presence of any other crack. It may also happen that the propagation of the first crack is slow and that the second or even a later crack results in the first failure. A cracked element will still carry its original share of the total load until the stresses cause very large plastic strains in parts adjacent to the crack. To simplify the analysis it has been presumed that no load redistribution among the members of the structure takes place until the first element failure.

An element failure is the result of a comparatively heavy load acting on an element weakened due to a fatigue crack. The probability of such an occurrence is obtained by combining the probabilities of the two events of crack initiation and a loading exceeding the residual strength. The broken element can no longer transfer its original load. If the magnitude and action line of the resultant load is kept unaltered, the load levels of the remaining elements will be changed. It has been assumed that all the parallel elements are connected with pin-joints in both ends to rigid clamping blocks, which are considered infinitely stiff. It is consequently easy to calculate the new loads of the remaining elements. On the basis of experimental results available, simple relationships regarding the influence of the load level on the time to crack initiation and on the strength decrease rate, have been adopted. This makes it possible to analyze also the probability of the consecutive failures. The total collapse of the assembly takes place when all the elements have failed. The probability of such a collapse is thus obtained as a combination of all the individual failures. In practice all the remaining elements will fail simultaneously when a "critical number" of failures has been reached, the critical number being dependant on both the design of the structure and the loading.

The structural assembly is inspected at regular intervals. Immediately after an inspection there is no visible crack present, since all elements with cracks have been replaced at the inspection. This is supposed to imply also that no strength decrease has taken place. At the beginning of the inspection interval no. ν the original elements have been subjected to fatigue loading during $T_{\nu-1}$ hours. This is not true for the new elements, which have replaced cracked and failed elements. The replacement effect is neglected, however, as a first approximation. By adding the probabilities of failure of the structure for the various intervals $\nu = 1,\ldots,n$, the probability of failure for the whole service life T_L is finally obtained.

4. LOAD SPECTRUM AND LOAD DISTRIBUTION AMONG THE ELEMENTS

The structure is subjected to a random loading, e.g. gust loads. It may be assumed for simplicity, in the case of gust loading, that the load has a constant mean value, the 1-g load, while the amplitude varies from very small values which occur frequently, to extremely high values which are rare. In the undamaged structure the resulting mean load in each element is denoted S_{mo}, the load amplitude S_{ao} and the ultimate load S_u

$$S_o = S_u - S_{mo}$$

is thus the original static margin in 1 g level flight. It is convenient to normalize the amplitude with respect to S_o

$$s_{ao} = S_{ao}/S_o \tag{1}$$

implying that $s_{ao} = 1$ gives failure in the undamaged element.

After a number of i element failures have occurred the mean load and the amplitude will be changed by a load redistribution factor c

$$S_{mi} = c\ S_{mo} \qquad S_{ai} = c\ S_{ao} \qquad s_{ai} = c\ s_{ao} \tag{2}$$

For the broken elements $c = 0$. The other elements have normally different values each, which depend also upon the positions of the broken elements. In the case of m parallel elements, under the conditions mentioned in chapter 3 and loaded only in the elastic region, the c-factor of element no. j has been obtained in Appendix A

$$c_j = (a + bj)c_{oj} \tag{3}$$

where

$$a = \frac{m\left[C - B(m+1)/2\right]}{A\left[C - B(m+1)/2\right] - B\left[B - A(m+1)/2\right]} \tag{4}$$

$$b = -\frac{m\left[B - A(m+1)/2\right]}{A\left[C - B(m+1)/2\right] - B\left[B - A(m+1)/2\right]} \tag{5}$$

$c_{oj} = 1$ for load-carrying element

$c_{oj} = 0$ for broken element

and

$$A = \sum_{j=1}^{m} c_{oj} \qquad B = \sum_{j=1}^{m} j\, c_{oj} \qquad C = \sum_{j=1}^{m} j^2\, c_{oj} \qquad (6)$$

Table I gives the c-factor for all possible combinations of one and two failed elements in an assembly of six parallel members.

The lower, frequently occurring, load amplitudes cause the main part of the fatigue damage which will eventually result in visible cracks and strength decrease. It is not intended to discuss in this paper how a random loading of an aircraft structure can be analyzed. Extensive research work is now going on in this field by means of power spectral techniques, but the possibilities of evaluating fatigue properties by such procedures are still limited [8, 9]. It is proposed only that the various members of the structural assembly be fatigue tested under either a representative random load spectrum or some program loading, or even a constant amplituding loading which may be shown to be equivalent to the random loading. The small element fatigue testing, which should not include very high amplitudes occurring less frequently than some ten times in a normal aircraft service life, is thus assumed to have provided reliable data of crack initiation and crack propagation.

High amplitudes, usually severe gusts, always give rise to the ultimate failure of an element weakened by a fatigue crack. The present situation with regard to statistical data of extreme value gusts does not seem to allow a very reliable assessment of the probability distribution, especially not for new and future aircraft. It is therefore considered to be fair to choose a simple exponential relationship which is in agreement with available data for "thunderstorm" gusts [10, 11]. The expected number of times H per hour that a load amplitude s_{ao} will be exceeded is expressed as

$$H = H_o \exp(-h\, s_{ao}) \qquad (7)$$

where H_o and h are two parameters. The value of H_o depends to a large extent on the flight plan and the route. For a modern transport aircraft under normal conditions $H_o = 0.2$ may be chosen as a representative average. The parameter h varies with the design stress level, the relative equivalent air speed and the relative weight of the aircraft [12]. An ultimate design load factor $n_u = S_u/S_{mo} = 3.75$ and a reduced speed of $0.75\ V_c$ in rough air yield approximately a value of $h = 20$.

There seems to be some dependence between the magnitudes of consecutive gust loads, in that a high load is usually followed immediately by several high loads. Since this dependence has not been established statistically, it is neglected in this report, which is unfortunately an unconservative approximation.

If S_{ao} equals or exceeds the static margin S_o, failure will occur. In normalized loads this condition for failure is written

$$s_{ao} \geq s_o = 1 \tag{8}$$

It is now possible to compute, as an example, the probability of a static failure of an element which has not suffered any strength decrease due to fatigue. During a service time t the probability $P_u(t)$ is obtained [6, 13]

$$P_u(t) = 1 - e^{-H_u t} \tag{9}$$

where H_u can be obtained from eq. (7) introducing $s_{ao} = 1$. Considering that $H_u \ll 1$ eq. (8) can be simplified

$$P_u(t) = H_u\ t = H_o\ t\ \exp(-h) \tag{9a}$$

Introducing the values $H_o = 0.2$, $h = 20$ and $t = 1$ hour gives the failure rate per hour

$$F = 0.2\ \exp(-20) = 4.1 \times 10^{-10}$$

The probability of failure during a service life of around 25,000 hours would thus be 10^{-5}.

In this example, just as in the following treatment the statistical variation of the ultimate load of the element has been neglected. The influence of this variation can be taken into account by methods available [6, 7], but it is estimated to be less important in comparison with the rather large uncertainties connected with the gust load spectrum assumed.

If the load level is changed by a factor c according to eq. (2), this implies that both the amplitude and the mean load will be changed by this factor, while the static margin is not proportional to the load level. Thus

$$s_i = (S_u - S_{mi})/(S_u - S_{mo}) = \frac{S_u/S_{mo} - c}{S_u/S_{mo} - 1} = \frac{n_u - c}{n_u - 1} \tag{10}$$

Failure occurs when

$$s_{ai} \geq s_i \tag{11}$$

Eq. (7) can be generalized including a redistribution factor c [12]

$$H = H_o \exp(-h\, s_{ai}/c) \tag{7a}$$

Introducing eqs. (10) and (11) into (7a) gives

$$H_u = H_o \exp\left[-h(n_u - c)/c(n_u - 1)\right] \tag{12}$$

or

$$H_u = H_o \exp(-h_i) \tag{12a}$$

where

$$h_i = \frac{h}{c} \cdot \frac{n_u - c}{n_u - 1} \tag{13}$$

5. FATIGUE CRACK INITIATION AND REDUCTION OF ULTIMATE STRENGTH OF THE ELEMENTS

Results from fatigue testing indicate that the probability distribution of the life in number of cycles or hours is approximately log-normal, although it may be just as reasonable to assume some other similar distribution, as far as we know at present. In the first place one should know the moment when a crack starts to cause a measurable strength decrease in an element. This moment is likely to occur slightly before the crack can be detected at an ordinary airline inspection. Very little experimental evidence is available. It is thought to be acceptable, however, for a structure of 2024 aluminium alloy to assume that the moment of incipient strength decrease coincides with the moment of detectable crack length, which moment will be called the crack initiation. The time to crack initiation T_c is now taken to be log-normal, i.e.

$$P_c = \phi\left(\frac{{}^{10}\log T_c - \mu_c}{\sigma_c}\right) \tag{14}$$

where μ_c and σ_c are the mean value and the standard deviation of ${}^{10}\log T_c$. The mean value varies considerably with stress level, stress concentration and material. The standard deviation seems to be usually within the region 0.1 - 0.3 [14, 15]. In the following numerical examples $T_{c50} = 10^{\mu_c} = 50{,}000$ hours and $\sigma_c = 0.1$ and 0.2 have been chosen as representative values.

The said parameters μ_c and σ_c apply to the original load level of the elements. When the load is increased by a factor c according to eq. (2), the time to crack initiation is reduced. This fact can also be expressed by the introduction of a fictitious fatigue service time x which is proportional to a power α of the load redistribution factor c

$$x = T_c c^{\alpha} \tag{15}$$

When various load levels are included, the parameter x should replace T_c, and x is thus assumed to have a log-normal distribution according to eq. (14). Numerical evaluations of available test results from constant amplitude testing seem to indicate that α normally has a value between 2 and 4, with an average of around 3, which has been used in the following calculations.

Many investigations have recently been devoted to the crack propagation in metal sheets and stiffened panels both under constant and variable amplitude loading. A combined analytical and empirical approach may soon result in a rather reliable and general method of estimating the crack rate [16]. If such a method is supplemented by extensive testing of the residual strength, especially at short crack lengths, together with theoretical studies [17], the experimental background for determining the fatigue life of fail-safe structures would be rather complete. In this analysis the simple assumption is made that the residual static margin of an element decreases linearly with the time of service t from the initiation of the critical crack. The residual static margin, at the time t, in an assembly where no element has failed, is denoted s_{ot}

$$s_{ot} = 1 - t/R \tag{16}$$

where R is a parameter of the dimension hours, indicating the inverse rate of strength reduction. A study of a non-linear strength reduction curve has revealed that the influence of a realistic deviation from a straight line is not so important for short inspection intervals [3].

The strength reduction parameter R is obviously a stochastic variable. The scatter in R is less than for the time to crack initiation and further the crack propagation time is probably rather short in comparison with the whole service life for the narrow sheet elements under consideration. The value of R has therefore been assumed to be constant in the analysis. In the numerical evaluations it has been given different values from 3,000 to 15,000 hours.

When the load level is changed due to an element failure this will affect the residual static margin in two ways, partly according to eq. (10) and partly because the crack propagation time will be influenced. The latter fact can be taken into account by introducing, in analogy with eq. (15), a fictitious strength reduction service time y

$$y = t\ c^{\beta} \tag{17}$$

The expected number of times per hour of actual service time a gust load amplitude exceeds the residual static margin can thus be expressed

$$H_u = H_o \exp\left[-h_i(1 - y/R)\right] \tag{18}$$

The value of β should be rather close to that of α according to test results. It is assumed in the numerical calculation that $\beta = \alpha = 3$.

6. CONSECUTIVE FAILURES IN AN ASSEMBLY DURING ONE INSPECTION INTERVAL

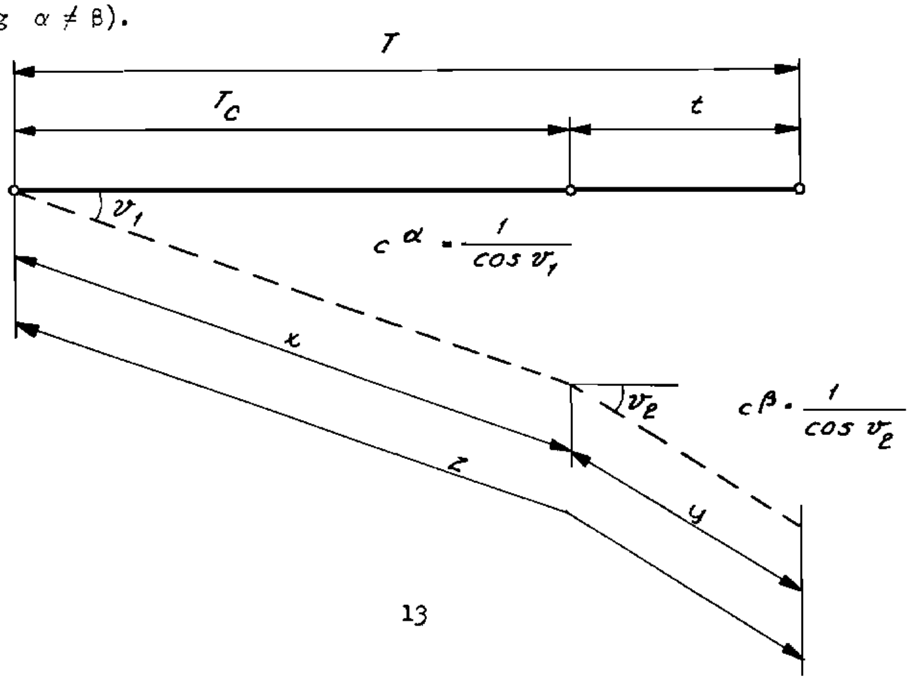
In chapter 5 the fictitious service times x and y were introduced in eqs. (15) and (17) as directly proportional to the crack initiation time T_c and the crack propagation time t respectively. Now if the factor of proportionality is the same in both cases, i.e. $\alpha = \beta$, it is obviously possible to write the relationship between the sums $T = T_c + t$ and $z = x + y$.

$$z = K\ T \tag{19}$$

where K is a coefficient depending on the load redistribution factor c.

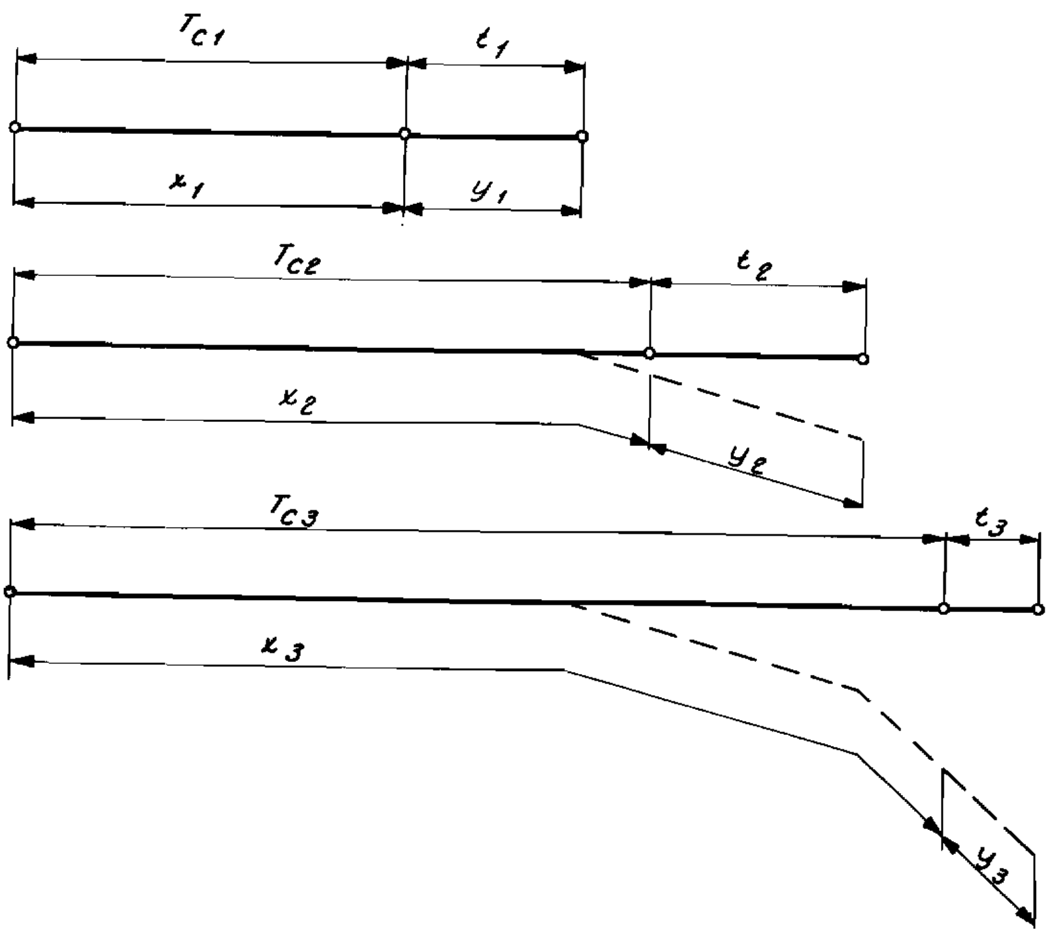
$$K = c^{\alpha} = c^{\beta}.$$

The following diagram illustrates the relations between the time variables (assuming $\alpha \neq \beta$).



Considering the consecutive crack initiations and failures of a number of parallel elements, there are a large number of possible sequences of occurrences. The first crack initiation c 1 may either be followed by the first element failure f 1 or the second crack initiation c 2 . There are three different sequences in which one can arrive at two failures and fifteen sequences to three failures, and so on (it is postulated that the sequence of failures is always in the right order f1, f2, f3, while the order of the crack initiations may change, e.g. c1, c2, c3, or c2, c1, c3).

The case of three consecutive failures will now be studied, assuming first the following sequence c1, f1, c2, f2, c3, f3, which is also demonstrated in the diagram below (assuming $\alpha = \beta$).



The service time T_3 until the third failure can be determined by writing

$$x_1 = T_{c1} \tag{20}$$

$$x_2 = T_1 + K_{12}(T_{c2} - T_1) \tag{21}$$

where $K_{12} = c_j^{\alpha}$.
The factor c_j applies to the position j of the element with the second failure, provided failure no. 1 has occurred. Further

$$x_3 = T_1 + K_{13}(T_2 - T_1) + K_{23}(T_{c3} - T_2) \tag{22}$$

where K_{13} and K_{23} are constants analogous to K_{12}, j denoting here the position of the element that will fail as no. 3, provided that only the first, and both the first and the second failures, respectively, have taken place. The following relations hold for the crack propagation time.

$$y_1 = t_1 \tag{23}$$

$$y_2 = K_{12}\, t_2 \tag{24}$$

$$y_3 = K_{23}\, t_3 \tag{25}$$

From the six equations (20) - (25) T_1, T_2 and T_3 can now be evaluated

$$T_1 = x_1 + y_1 = z_1 \tag{26}$$

$$T_2 = T_{c3} + t_2 = x_1 + y_1 - \frac{1}{K_{12}}(x_1 + y_1) + \frac{1}{K_{12}}(x_2 + y_2) = z_1\left(1 - \frac{1}{K_{12}}\right) + z_2\frac{1}{K_{12}} \tag{27}$$

$$T_3 = T_{c3} + t_3 =$$

$$= (x_1 + y_1)\left(1 - \frac{1}{K_{12}} - \frac{1}{K_{23}} + \frac{K_{13}}{K_{12}K_{23}}\right) + (x_2 + y_2)\left(\frac{1}{K_{12}} - \frac{K_{13}}{K_{12}K_{23}}\right) + (x_3 + y_3)\frac{1}{K_{23}} =$$

$$= z_1\left(1 - \frac{1}{K_{12}} - \frac{1}{K_{23}} + \frac{K_{13}}{K_{12}K_{23}}\right) + z_2\left(\frac{1}{K_{12}} - \frac{K_{13}}{K_{12}K_{23}}\right) + z_3\frac{1}{K_{23}} =$$

$$= k_{31} z_1 + k_{32} z_2 + k_{33} z_3 \tag{28}$$

Since it has been assumed that the change in load distribution among the members due to crack initiation is negligible, it follows that if $\alpha = \beta$, the same eqs. (26) - (28) will result for the other possible sequences of crack initiations and failures.

The time until the fourth failure T_4 can be obtained analogously

$$T_4 = z_1\left(1 - \frac{1}{K_{12}} - \frac{1}{K_{23}} + \frac{K_{13}}{K_{12}\,K_{23}} - \frac{1}{K_{34}} + \frac{K_{14}}{K_{12}\,K_{34}} - \frac{K_{13}\,K_{24}}{K_{12}\,K_{23}\,K_{34}} + \frac{K_{24}}{K_{23}\,K_{34}}\right) +$$

$$+ z_2\left(\frac{1}{K_{12}} - \frac{K_{13}}{K_{12}\,K_{23}} - \frac{K_{14}}{K_{12}\,K_{34}} + \frac{K_{13}\,K_{24}}{K_{12}\,K_{23}\,K_{34}}\right) +$$

$$+ z_3\left(\frac{1}{K_{23}} - \frac{K_{24}}{K_{23}\,K_{34}}\right) + z_4\,\frac{1}{K_{34}} =$$

$$= k_{41}\,z_1 + k_{42}\,z_2 + k_{43}\,z_3 + k_{44}\,z_4 \tag{29}$$

where K_{14}, K_{24} and K_{34} include the c-factors of the element which fails as no. 4, when the first, the first and the second, and the first, second and third failures, respectively, have occurred. The following failures 5, 6,... m give similar formulas, although the length of the expressions for the coefficients k will increase very rapidly.

The criterion for collapse of the whole assembly between two inspections $\nu-1$ and ν is that all the m elements fail within the inspection interval

$$T_{\nu-1} < T_m < T_\nu \tag{30}$$

or

$$T_m - T_{\nu-1} < t_{insp} \tag{30a}$$

where

$$t_{insp} = T_\nu - T_{\nu-1} \tag{31}$$

The probability of total failure within interval no. ν may thus be written $P(T_m - T_{\nu-1} < t_{insp})$.

When the number m of parallel elements is large, it is not necessary to calculate the time for the higher orders of failure, since the time between the consecutive failures tend to decrease rather rapidly. There is in practice a critical number of failures q which involves immediate failure of the whole assembly. The critical number obviously depends on the magnitude of the mean load and the load amplitude spectrum applied. Under the assumptions made in the numerical evaluations it has been found both theoretically and experimentally that q = 2-3 for an assembly of six parallel elements.

The first failure f1 can occur in any one of the m parallel elements with the same probability. The second failure may take place in any of the remaining m-1 elements, although it is most likely that elements adjacent to the first failure will be fractured next, since they are more heavily loaded than the other elements. The further failures will follow, almost inevitably in the vicinity of the elements already fractured.

In the case of $m = 6$, there are altogether $6 \times 5 = 30$ possible sequences of two failures and $6 \times 5 \times 4$ sequences of three failures. Table I gives the load redistribution factors c_j for the remaining elements according to eqs. (3) - (6) after one and two failures. Due to the symmetry of the elements it is only necessary to show half of the possible combinations. It is obvious, however, that some of the sequences listed, which cause rather small load concentrations, do not contribute so much to the total probability of collapse, and may therefore be neglected. The most important sequences of two failures are (1,2), (1,3), (2,1), (2,3), (3,1), (3,2), and the corresponding ones on the other side of th axis of symmetry. The numbers within brackets denote the positions of the elements that fail. The third failures will occur for the said sequences in the elements with the positions nos. 3,2, 3,1, 2,1, respectively.

Each sequence of failures corresponds to a set of coefficients K_{12}, K_{13} and K_{23}, obtained by rising c_j to the third power, and of the resulting coefficients k_{31}, k_{32}, k_{33} according to eq. (28). When computing the probability of total failure of the assembly, the contributions of all the important sequences must be added together.

7. PROCESS OF CONVOLUTION

In the preceding chapter the service time T_q until the critical number of element failures has occurred, was expressed as a linear function of the fictitious service times x and y until the first and the subsequent crack initiations and failures, the frequency functions of which times are assumed to be known. It is now possible to write the distribution function of T as a product of $2q$ integrals [18], where x and y are positive variables.

$$P(T_q \leq T) = \int_0^T p_c(k_{q1}x_1)d(k_{q1}x_1) \int_0^{T-k_{q1}x_1} g(k_{q1}y_1)d(k_{q1}y_1) \dots \int_0^{T-k_{q1}x_1-k_{q1}y_1-\dots k_{qi-1}x_{i-1}-k_{qi-1}y_{i-1}} p_c(k_{qi}x_i)d(k_{qi}x_i) \times$$

$$\times \int_0^{T-kq_1x_1-k_{q1}y_1-\dots-k_{qi}x_i} g(k_{qi}y_i)d(k_{qi}y_i) \dots \int_0^{T-\sum_1^{q-1} k_{qi}x_i - \sum_1^{q-1} k_{qi}y_i} p_c(k_{qq}x_q)d(k_{qq}x_q) \int_0^{T-\sum_1^{q} k_{qi}x_i - \sum_1^{q-1} k_{qi}y_i} g(k_{qq}y_q)d(k_{qq}y_q) \quad (32)$$

The frequency function of the fictitious fatigue service time x_i until the crack initiations nos. 1, 2,..., i,..., q is log-normal according to eq. (14), i.e.

$$p_c(k_{qi}\ x_i) = \frac{1}{k_{qi}\ x_i\ \sigma_c \sqrt{2\pi}} \exp\left[-\frac{\left({}^{10}\log k_{qi}\ x_i - \mu_c\right)^2}{2\ \sigma_c^{\ 2}}\right] \qquad (33)$$

Starting from the failure rate expressed in eq. (18) the distribution function of the fictitious strength reduction service time y_i until the failures nos. 1, 2,..., i,... q can be obtained as [2]

$$G(y_i) = 1 - \exp\left[\frac{RH_o}{h_i}\left(1 - \exp\frac{h_i\ y_i}{R}\right)\exp\ (-h_i)\right] \qquad (34)$$

and thus the frequency function

$$g(y_i) = \frac{d\ G}{dy_i} = H_o\ \exp\left[-h_i(1 - \frac{y_i}{R}) + \right.$$

$$\left. + \frac{RH_o}{h_i}\left(1 - \exp\frac{h_i\ y_i}{R}\right)\exp\ (-h_i)\right] \qquad (35)$$

If eqs. (33) and (35) are introduced in the distribution function of eq. (32), this cannot be integrated in a closed form. In order to make a numerical calculation practically feasible a number of approximations have therefore to be introduced.

In order to be able to study the influence of inspections it is necessary to investigate each inspection interval separately. It is convenient therefore to write according to eq. (28), introducing $q = 3$,

$$T_3 - T_{\upsilon-1} = k_{31}(x_1 + y_1) + k_{32}(x_2 + y_2) +$$

$$+ k_{33}(x_3 + y_3) - T_{\upsilon-1} = Q_3 \qquad (36)$$

and

$$Q_3 = Q_{3a} + Q_{3b} \qquad (37)$$

where

$$Q_{3a} = k_{31}\ x_1 + k_{32}\ x_2 + k_{33}\ x_3 - T_{\upsilon-1} \qquad (38)$$

$$Q_{3b} = k_{31}\ y_1 + k_{32}\ y_2 + k_{33}\ y_3$$

First Q_{3a} is convoluted separately. Since $k_{31} + k_{32} + k_{33} = 1$ it is possible to rewrite Q_{3a}

$$Q_{3a} = k_{31} x_1 + k_{32} x_2 + k_{33} x_3 - T_{\nu-1}(k_{31} + k_{32} + k_{33}) =$$

$$= k_{31}(x_1 - T_{\nu-1}) + k_{32}(x_2 - T_{\nu-1}) + k_{33}(x_3 - T_{\nu-1}) \qquad (38a)$$

If the inspection interval t_{insp} is not too long compared to the whole service life of the aircraft structure, the frequency function of x can be assumed to be constant over the whole interval and equal to the maximum value at the end of the interval $T = T_\nu$, adopting the principle that all approximations introduced should, if possible, overestimate the probability of failure

$$p_c(x_{i\nu}) = r_{i\nu} \qquad (39)$$

within the limits $T_{\nu-1} < x_{i\nu} < T_\nu < T_{\nu-1} + \frac{1}{r_{i\nu}}$. Eqs. (20) - (22) give the values of x_1, x_2 and x_3. To simplify the analysis the following conservative assumptions are made

1. When computing $x_{2\nu}$ the first failure is assumed to have occurred immediately after the inspection, i.e. $T_1 = T_{\nu-1}$. Consequently the following value of $x_{2\nu}$, at the end of the interval ν, should be introduced when computing $r_{2\nu}$ according to eq. (39)

$$x_{2\nu} = T_{\nu-1} + K_{12}\, t_{insp} \qquad (40)$$

2. When computing x_3 the first and the second failures are assumed to have occurred at the beginning of the inspection interval $T_1 = T_2 = T_{\nu-1}$ and thus

$$x_{3\nu} = T_{\nu-1} + K_{23}\, t_{insp} \qquad (41)$$

Further

$$x_{1\nu} = T_{\nu-1} + t_{insp} \qquad (42)$$

The frequency function of

$$u_i = k_{3i}(x_i - T_{\nu-1}) \qquad (43)$$

is now $r_{i\nu}/k_{3i}$ if u_i is between the limits 0 and $k_{3i}/r_{i\nu}$. The distribution function of Q_{3a} is then easily integrated

$$F_{Q_{3a}}(Q_{3a}) = \int_0^{Q_{3a}} \frac{r_{1\nu}}{k_{31}}\, du_1 \int_0^{Q_{3a}-u_1} \frac{r_{2\nu}}{k_{32}}\, du_2 \int_0^{Q_{3a}-u_1-u_2} \frac{r_{3\nu}}{k_{33}}\, du_3 =$$

$$= \frac{r_{1\nu}\, r_{2\nu}\, r_{3\nu}}{6k_{31}\, k_{32}\, k_{33}}\, Q_{3a}^3 \qquad (44)$$

In the case $k_{31} = 0$, i.e. $u_1 = 0$, the distribution function of Q_{3a} is obtained as

$$\int_0^{Q_{3a}} \frac{r_{2\nu}}{k_{32}}\, du_2 \int_0^{Q_{3a}-u_2} \frac{r_{3\nu}}{k_{33}}\, du_3 = \frac{r_{2\nu}\, r_{3\nu}}{2k_{32}\, k_{33}}\, Q_{3a}^2 \qquad (44a)$$

If $k_{31} = k_{32} = 0$ the distribution function is analogously

$$\frac{r_{3\nu}}{k_{33}}\, Q_{3a} \qquad (44b)$$

The sum Q_{3b} is then convoluted replacing $g(y)$ of eq. (35) by a quasi-truncated normal frequency function

$$\bar{g}(y_i) = \frac{\ell_i}{\sigma_i\sqrt{2\pi}} \exp\left[-(y_i - \mu_i)^2/2\sigma_i^2\right] \qquad (45)$$

The three parameters ℓ_i, μ_i and σ_i are determined uniquely in terms of h_i, H_o and R from the requirements

1. Both frequency functions, eqs. (35) and (45) have the same maximum value for the same time y_i (two conditions)

2. Both functions have the same value for $y_i = 0$.

A closer study of the frequency functions reveals that $\bar{g}(y_i) \geq g(y_i)$ as long as $y_i \leq \mu_i$ if the requirements mentioned are satisfied. For $y_i > \mu_i$ the risk of failure is so big that in practice the inspection interval will never be allowed to reach such a high value.

The maximum of $g(y_i)$ is obtained by differentiating with respect to y_i. It occurs at

$$y_{i\,max} = \frac{R}{h_i}\ {}^{e}\log \frac{h_i}{H_o\, R} + R \qquad (46)$$

and the natural logarithm of the maximum value is

$$^{e}\log g(y_i)_{max} = {}^{e}\log \frac{h_i}{R} - 1 + \frac{R\,H_o}{h_i}\exp(-h_i) \tag{47}$$

Requirement 1. thus gives

$$\mu_i = \frac{R}{h_i}\,{}^{e}\log \frac{h_i}{R\,H_o} + R \tag{48}$$

$$\begin{aligned} ^{e}\log \bar{g}(\mu_i) &= {}^{e}\log \ell_i - {}^{e}\log \sigma_i - {}^{e}\log \sqrt{2\pi} = \\ &= {}^{e}\log \frac{h_i}{R} + \frac{R\,H_o}{h_i}\exp(-h_i) - 1 \end{aligned} \tag{49}$$

According to requirement 2. further

$$^{e}\log \ell_i - {}^{e}\log \sigma_i - {}^{e}\log \sqrt{2\pi} - \frac{\mu_i^2}{2\,\sigma_i^2} = {}^{e}\log H_o - h_i \tag{50}$$

A combination of eqs. (49) and (50) yields

$$\sigma_i = \mu_i \Big/ \sqrt{2\left[{}^{e}\log \frac{h_i}{R\,H_o} + h_i + \frac{R\,H_o}{h_i}\exp(-h_i) - 1\right]} \tag{51}$$

Finally ℓ_i is solved from eq. (49)

$$^{e}\log \ell_i = {}^{e}\log \frac{h_i}{R} + \frac{R\,H_o}{h_i}\exp(-h_i) - 1 + {}^{e}\log \sqrt{2\pi} + {}^{e}\log \sigma_i \tag{52}$$

It should be observed that μ_i is not the mean and σ_i is not the standard deviation of the truncated distribution.

The frequency function of $k_{3i}\,y_i$ has thus been replaced by a quasi-truncated normal frequency function

$$\bar{g}(k_{3i}y_i) = \frac{\ell_i}{k_{3i}\sigma_i\sqrt{2\pi}}\exp\left[-(k_{3i}y_i - k_{3i}\mu_i)^2/2k_{3i}^2\sigma_i^2\right] \tag{53}$$

The frequency function of $Q_{3b} = \sum_{i=1}^{3} k_{3i}y_i$ is then known [18] to have a slightly lower value than the following normal frequency function (Appendix B)

$$f_{Q_{3b}}(Q_{3b}) = \frac{\ell}{\sigma\sqrt{2\pi}}\exp\left[-(Q_{3b} - \mu)^2/2\sigma^2\right] \tag{54}$$

where

$$\ell = \prod_{i=0}^{2} \ell_i \tag{55}$$

$$\mu = \sum_{i=1}^{3} k_{3i}\, \mu_{i-1} \tag{56}$$

$$\sigma^2 = \sum_{i=1}^{3} k_{3i}^2\, \sigma_{i-1}^2 \tag{57}$$

The distribution function of Q_3 according to eq. (37), can now obtained by convolution of Q_{3a} and Q_{3b} employing eqs. (44) and (54).

$$F_{Q_3}(Q_3) = \int_0^{Q_3} F_{Q_{3a}}(Q_3 - Q_{3b})\, f_{Q_{3b}}(Q_{3b})\, dQ_{3b} =$$

$$= \frac{\ell\, r_{1\upsilon}\, r_{2\upsilon}\, r_{3\upsilon}}{6\, k_{31}\, k_{32}\, k_{33}} \int_0^{Q_3} (Q_3 - Q_{3b})^3\, \frac{1}{\sigma}\varphi\left(\frac{Q_{3b} - \mu}{\sigma}\right) dQ_{3b} \tag{58}$$

where $\frac{1}{\sigma}\varphi\left(\frac{Q_{3b} - \mu}{\sigma}\right)$ is the frequency function of Q_{3b}.
With the substitutions

$$\vartheta = (Q_3 - \mu)/\sigma \tag{59}$$

$$\varkappa = -\mu/\sigma \tag{60}$$

eq. (58) can be brought under the following form (Appendix C)

$$F_{Q_3}(Q_3) = \frac{\sigma^3\, \ell\, r_{1\upsilon}\, r_{2\upsilon}\, r_{3\upsilon}}{6\, k_{31}\, k_{32}\, k_{33}} \Big\{(\vartheta^3 + 3\vartheta)\left[\phi(\vartheta) - \right.$$

$$\left. -\phi(\varkappa)\right] + (\vartheta^2 + 2)\varphi(\vartheta) + (-3\vartheta^2 + 3\vartheta\varkappa - \varkappa^2 - 2)\varphi(\varkappa)\Big\} \tag{61}$$

The notations $\phi(\vartheta)$ and $\phi(\varkappa)$ have been introduced for the normal distribution functions of ϑ and $\varkappa$, while $\varphi(\vartheta)$ and $\varphi(\varkappa)$ are the frequency functions of the same variables.

Introducing the length of the inspection interval t_{insp} in eq. (61) will thus give the probability of total failure during the inspection interval under consideration with the assumed sequence of element failures.

8. PROCEDURE FOR NUMERICAL CALCULATIONS

As a result of the analysis in chapter 7, a procedure for calculating the probability of failure of the assembly has been established including the following steps.

a. Analyse the possible sequences of failures and determine the critical number of failures q by studying the c-factors obtained from eqs. (3) - (6). In the case of six elements (see table I) $q \leq 3$ and the essential sequences of failing elements are (numbers within brackets denote the positions of the failing elements)

(1, 2, 3) (1, 3, 2) (2, 1, 3)

(2, 3, 1) (3, 1, 2) (3, 2, 1)

and the corresponding ones on the other side of the axis of symmetry.

b. Compute for each essential sequence of failures the coefficients k which define the relationship between the fictitious and the real service time. For $q = 3$

$$k_{31} = 1 - \frac{1}{K_{12}} - \frac{1}{K_{23}} + \frac{K_{13}}{K_{12}\,K_{23}} \tag{62a}$$

$$k_{32} = \frac{1}{K_{12}} - \frac{K_{13}}{K_{12}\,K_{23}} \tag{62b}$$

$$k_{33} = \frac{1}{K_{23}} \tag{62c}$$

where $K = c_j^{\,3}$.

The indices 12 and 13 of K refer to c_j-values of the elements which are going to fail as nos. 2 and 3 respectively, after failure no. 1 has taken place. The index 23 analogously means that a c_j should apply to the element failing no. 3 after two elements have failed.

c. Compute for the essential sequences the frequency of crack initiations nos. 1, 2, 3 (i) at the end of the inspection interval ν, i.e. $T = T_\nu = T_{\nu-1} + t_{insp}$

$$r_{i\nu} = p_c(x_{i\nu}) = \frac{1}{x_{i\nu}\,\sigma_c\sqrt{2\pi}} \exp\left[-\frac{\left({}^{10}\log x_{i\nu} - \mu_c\right)^2}{2\,\sigma_c^{\,2}}\right] \tag{63}$$

where μ_c and σ_c are constants, assumed to be known for the elements under consideration, and

$$x_{1\nu} = T_{\nu-1} + t_{insp} \qquad (64a)$$

$$x_{2\nu} = T_{\nu-1} + K_{12}\, t_{insp} \qquad (64b)$$

$$x_{3\nu} = T_{\nu-1} + K_{23}\, t_{insp} \qquad (64c)$$

d. Compute for the said essential sequences h_i according to eq. (13)

$$h_i = \frac{h}{c_j} \cdot \frac{n_u - c_j}{n_u - 1} \qquad (13a)$$

where h is a gust load parameter and n_u the ultimate design load factor. The c_j-factor should apply to the element failing as no. 2 after the first failure, i = 1, and to failing element no. 3 for i = 2. Compute further

$$\mu_i = \frac{R}{h_i}\, {}^{e}\log \frac{h_i}{R\, H_o} + R \qquad (48)$$

$$\sigma_i = \mu_i \Big/ \sqrt{2\left[{}^{e}\log \frac{h_i}{RH_o} + h_i + \frac{RH_o}{h_i} \exp(-h_i) - 1\right]} \qquad (51)$$

and

$${}^{e}\log \ell_i = {}^{e}\log \frac{h_i}{R} + \frac{RH_o}{h_i} \exp(-h_i) - 1 + {}^{e}\log \sqrt{2\pi} + {}^{e}\log \sigma_i \qquad (52)$$

H_o is the other gust load parameter while R is the strength reduction parameter. Finally for q = 3 obtain.

$$\mu = \mu_o\, k_{31} + \mu_1\, k_{32} + \mu_2\, k_{33} \qquad (56)$$

$$\sigma = \sqrt{\sigma_o^{\,2}\, k_{31}^2 + \sigma_1^{\,2}\, k_{32}^2 + \sigma_2^{\,2}\, k_{33}^2} \qquad (57a)$$

$$\ell = \ell_o\, \ell_1\, \ell_2 \qquad (55)$$

e. If the length of the inspection interval is t_{insp} service hours the following parameters should be computed

$$\vartheta = (t_{insp} - \mu)/\sigma \qquad (59a)$$

$$\varkappa = -\mu/\sigma \qquad (60)$$

Read in a Gaussian probability table

$$\varphi(\vartheta) = \frac{1}{\sqrt{2\pi}} \exp(-\vartheta^2/2)$$

$$\Phi(\vartheta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\vartheta} \exp(-\vartheta^2/2) d\vartheta$$

and also $\varphi(\varkappa)$, $\Phi(\varkappa)$.

f. Compute the probability of failure of the assembly during one inspection interval ν , assuming a certain sequence of element failures.

$$F_\nu(t_{insp}) = \frac{\sigma^3 \, \ell \, r_{1\nu} \, r_{2\nu} \, r_{3\nu}}{6k_{31} \, k_{32} \, k_{33}} \Big\{ (\vartheta^3 + 3\vartheta) \left[\Phi(\vartheta) - \Phi(\varkappa)\right] + (\vartheta^2 + 2)\varphi(\vartheta) + (-3\vartheta^2 + 3\vartheta\varkappa - \varkappa^2 - 2)\varphi(\varkappa) \Big\} \quad (61a)$$

Eq. (61a) is not valid for the rather unrealistic cases $k_{31} \doteq 0$, $k_{32} = 0$ or $k_{31} = k_{32} = 0$. Compare eqs. (44), (44a) and (44b).

g. Make the corresponding computations for all other important sequences of element failures. The sum of the contributions of all sequences gives the probability of failure of the assembly during the inspection interval ν .

$$P_\nu = \sum_{\text{all sequences}} F_\nu \quad (65)$$

h. Repeat the procedure for all the other inspection intervals 1,...n. Add the contributions of the inspection intervals

$$P = \sum_{\nu=1}^{n} P_\nu \quad (66)$$

It should be observed that ϑ and $\varkappa$ depend on the length of the inspection interval only and not on the number ν of the interval.

Numerical evaluations have been carried out on an electronic computer, using the procedure outlined above, for an assembly of six identical, parallel elements. The following parameters were chosen

Crack initiation:	$T_{c\,50}$	= 50,000 hours
	σ_c	= 0.1 and 0.2
Strength reduction:	R	= 3,000, 6,000, 10,000, 15,000 hours

Heavy gust spectrum: $H_o = 0.2$
$h = 20$

Inspection intervals: t_{insp} = 1,000, 2,000, 3,000, 4,000, 5,000, 6,000, 7,000, 8,000, 10,000 hours

Ultimate design load factor $n_u = 3.75$

The computations of the probability of failure have been proceeded until a maximum limit service life T_L of the aircraft of 50,000 hours. The results are presented in figs. 3 - 9, the first six of which give the probability of total failure P versus T_L for various lengths of inspection intervals, assuming different values of the standard deviation σ_c and the strength reduction parameter R. Fig. 9 shows the influence of a variation of R for one length of the inspection interval t_{insp} = 4,000 hours and $\sigma_c = 0.1$, at a limit service time T_L = 30,000 hours.

9. PRELIMINARY TESTING

The theoretical treatment of an assembly of parallel members needs experimental support. A reliable proof that the method advanced gives adequate distribution functions of the service time to collapse, would require a very extensive fatigue testing program. Such a program is being considered. So far, some preliminary testing has just been finished. Although the results have not yet been analyzed in detail, a brief account will be given.

The test specimens which had a central, circular hole as a notch (fig. 2), were manufactured from a sheet of 2024-T3 aluminium alloy (Swedish Specification 3526-38) with the dimensions 1000 x 3000 x 1.5 mm. The total number of specimens was around 200, from which 24 assemblies with six in each were chosen by a random procedure. Further 24 test pieces were selected for single-specimen testing. The fatigue testing of the assemblies was carried out in a 6 tons Schenck horizontal pulsator and the testing of the single specimens in a 2 tons FFA resonance machine, both at a speed of 2,000 - 3,000 c/min. Due to lack of equipment for random load testing a constant amplitude loading was applied. The fluctuating tension load had a mean of 300 kg and an amplitude of 150 kg per sheet specimen. Since the net area of the specimen section is 30 mm^2, the stress cycle was thus 10 ± 5 kg/mm^2 (14,200 ± 7,100 psi) as long as no redistribution of the loading had taken place in the assemblies.

The maximum error of the load amplitude in the Schenck machine is believed to be ± 15 kg, i.e. less than 2 per cent of the total amplitude 900 kg, while the 2 tons machine gives an error of less than ± 5 kg. The load distribution in the assembly was checked by strain-gauge measurements on each element. The loads in the elements were originally adjusted by means of turning the excentric bolts, which formed the pin-joints at one end. The adjustment proved to be rather laborious due to the interaction between the members of the statically indeterminate assembly. No element, however, was allowed to have an amplitude differing more than ± 5 kg from the mean amplitude of the six specimens. The load distribution was fairly constant among the elements until the first element failure had occurred, when a load redistribution took place. In most of the assemblies tested this was in good agreement with the loads computed according to Appendix A. The generation and propagation of cracks was watched by means of a magnifying-glass.

The material properties of the test specimens were determined on three coupons taken from various parts of the sheet. The following mean values were obtained.

$$\sigma_{0.2} = 31.9 \text{ kg/mm}^2 \quad (45{,}400 \text{ psi})$$
$$\sigma_u = 44.3 \text{ kg/mm}^2 \quad (63{,}000 \text{ psi})$$
$$E = 7300 \text{ kg/mm}^2 \quad (10.4 \times 10^6 \text{ psi})$$

The test results from the single specimens are shown in table II, ordered with the shortest fatigue life (to failure) first and then increasing number of cycles. Below the table the logarithmic mean values and standard deviations have been calculated for the crack detection and failure. The test values have also been plotted on a Gaussian probability paper in fig. 10, computing the probability by $P = M/(N + 1)$, where M is the fatigue life order number and $N = 24$ the total number of specimens tested. Straight lines obtained from the calculated values of table II have also been drawn to show the fit of the lognormal distribution. The mean of ${}^{10}\log T_c$ corresponds to a life to crack detection of 240,000 cycles. The fatigue life to failure has its logarithmic mean at 300,000 cycles. The standard deviation of ${}^{10}\log T_c$ is about 0.09 and of ${}^{10}\log T$ slightly less, 0.075. The number of cycles between crack initiation and failure is thus on the average about 60,000 with some tendency to increase with increasing number of cycles to crack initiation. Since failure always occurs when the residual static margin is

$$s_{ot} = 5/(44{,}3 - 10) = 0.146$$

the average of R may be obtained from eq. (16)

$$60{,}000/R = 1 - 0.146 = 0.854$$
$$R = 70{,}000 \text{ cycles.}$$

Instead of assuming a constant value of R, a better approximation would probably be to introduce a linear variation of R with the number of cycles to crack initiation. It should be noted, however, that fig. 10 does not give corresponding test points together, from crack initiation and failure of the same test specimen.

Table III gives the test results from the assemblies, which have not been ordered after the length of their fatigue lives. Assembly no. 24 has been omitted from the table since the loading had obviously not been within the limits stated above. As the crack which was first detected, did not cause the first element failure more than in about half of the assemblies, the detection of the critical crack has also been recorded. In assembly no. 23 two elements with the positions 3 and 4 (symmetrical) failed at the same time, whereas the collapse of the assembly took place 4,000 cycles later. In all the other assemblies total failure occurred at the same time as the second element failure, i.e. the critical number of failures was normally $q = 2$. The number of cycles between the first element failure and the collapse of the assembly was in most cases only a few per cent of the total number of cycles to collapse.

Fig. 11 shows the test results from the assemblies plotted on a Gaussian probability paper. If the number of cycles to the first (or critical) crack and to failure have lognormal distributions

$$1 - \Phi\left(\frac{{}^{10}\log T_c - \mu_c}{\sigma_c}\right)$$

and analogously for T_1, the distribution functions for the assemblies are, for the first crack

$$1 - \left[1 - \Phi\left(\frac{{}^{10}\log T_c - \mu_c}{\sigma_c}\right)\right]^6$$

and analogously for the number of cycles to the first element failure. These distribution curves, which are not lognormal, are also presented in fig. 11. They seem to be in fairly good agreement with the experimental points. A statistical χ^2-test gave the result that there is no significant deviation of the experimental values from the distribution calculated from the single-specimen test results.

The distribution function of the life length until collapse of the assembly can be calculated using, in principle, the numerical procedure described above, if the load spectrum is substituted by a constant load amplitude. It should not be necessary to go further than to element failure no. 2. Since no inspections have been performed and the inspection interval thus includes the whole life length, some approximations will have to be modified. This analysis and the comparison with test results has not yet been completed.

10. DISCUSSION

The theoretical study presented above should be regarded as a preliminary attempt to treat in detail the safety problem of a built-up fail-safe structure. Only a very simple assembly consisting of six parallel elements has been considered. The fact that the numerical procedure given in chapter 8, is still far from simple, does not look too promising for the future development of the method to suit complex structures. It seems to be possible, however, to make a more general treatment of the problem, which would reduce the numerical work and make it practically feasible to evaluate also aircraft structural components of quite realistic appearance. The results obtained in this report are of importance mainly because they prove that the probability of collapse of a fail-safe structure consisting of parallel elements can actually be calculated taking into account rather detailed conditions regarding service and structural behaviour. The diagrams of figs. 3 - 9 also show the influence of a variation of some parameters, such as the inspection interval and the strength reduction parameter.

Regarding the basic assumptions a number of simplifications have been introduced, although the aim was to treat the problem under rather realistic conditions. The most important of these simplifications are:

1. Linear strength reduction with time.
2. Constant value of strength reduction parameter R.
3. No variation of initial ultimate strength and static margin, and thus according to point 2, no variation of residual strength at any time t after crack initiation.
4. Detection of cracks only at regular inspections.
5. Effect of replacement of cracked or fractured elements on crack initiation neglected.

Simplification no. 1 has been studied in an earlier report [3], where it was concluded that the actual strength reduction curve should preferably be used, if it is known from experimental or theoretical investigations. In the numerical calculations, when carried out on an electronic computer, a non-linear strength reduction does not involve too much additional work or complications. The linear relationship, assumed in this report, is thought to be realistic for some structural elements and the results obtained should be valid for all sorts of elements when the inspection interval is shorter than some 20 per cent of the strength reduction parameter R.

Points nos. 2 and 3 are connected with each other. The scatter of the initial ultimate strength of an element made of structural steel or aluminium alloys seems to be rather well established, the coefficient of variation ranging from 0.015 to 0.05 [19]. The residual strength or static margin at a certain service time, or number of cycles, after the crack initiation is obviously also subjected to some variation, which cannot yet be assessed from available test data. Instead of considering scatter of the residual strength, one can also treat the crack propagation time from initiation until complete fracture as a stochastic variable. This would mean that the strength reduction parameter R is also a variable and not a constant as was assumed in this analysis. There seems to be a tendency in the test results, presented in chapter 9, towards a longer crack propagation time for elements with longer time to crack initiation, implying that introduction of some relationship between R and T_c might be necessary. The influence of a variation of the static margin, e.g. by varying the parameter R, which will, no doubt, complicate the analysis considerably, has not yet been studied.

Larger cracks and element failures can often be detected also between the particular inspections, where all fatigue-sensitive areas of the structure are scrutinized taking it apart or using the technical equipment necessary to find small cracks in hidden positions. This fact can be taken into account by introducing in the analysis one more type of inspection in which it is assumed that all cracks involving a certain rather large strength reduction, are discovered. Such simplified inspections, which could be rather frequent, possibly daily, may reduce the probability of collapse considerably. It should be kept in mind, however, that an inspection procedure must have a reliability very close to 100 per cent. Such inspections are expensive, even if they only aim at finding rather extensive fatigue damage. It is not obvious. therefore, that the operator will find it economical to undertake the frequent inspections. If cracks or failures are found only occasionally, the favourable effect can hardly be estimated.

Those elements which have a detectable crack or are fractured at the inspections, are supposed to be replaced by virgin elements. During the earlier part of the service life of the assembly very few cracks occur and it is consequently a good approximation to assume that all elements have the same fatigue age as the assembly. At the limit service life some 10 per cent of the elements may have been replaced, on the average, in a normal design. The replacement effect can probably still be neglected without serious errors. Only when the probability of crack initiation in the elements is high, 0.5 or even more, it is obvious that the actual ages of the elements must be taken into account. This can easily be done in the numerical calculations. The replacement effect was not included in this analysis, since the aim has been to obtain as simple formulas as possible. The probability of failure computed is thus somewhat too high for long service lives.

In the mathematical treatment of the convolution problem the frequency functions of the time until crack initiation, as well as the time from crack initiation to element failure, have been replaced by other functions which simplify the integrations. It has thus been assumed a step function for the frequency of crack initiation, the function being constant within each inspection interval and equal to the actual value at the end of the interval. This approximation should be quite acceptable for short inspection intervals, whereas it is probably too much on the safe side when the whole service life consists of only a few inspection intervals. It is possible, of course, to refine the approximation by subdividing the inspection intervals into shorter steps. This will cause an increase of the volume of the numerical calculations, which is not so serious, since exactly the same program can be followed.

The replacement of the exponential frequency function of the time to element failure by a quasitruncated normal frequency function is permissible only between zero and the maximum of the functions. This implies that the procedure proposed is not quite generally applicable, which has caused same trouble in the numerical calculations carried out so far. The errors introduced have not yet been investigated for all values of the parameters chosen in the calculations, and some of the results where the accuracy is questionable have been omitted from the diagrams in figs. 3 - 8. It might prove to be more advantageous to use instead of the normal frequency function a Fourier series or some simple power series. The choice of replacement function should preferably be guided by the supply of suitable standard programs for the electronic computer utilized in the numerical evaluations.

The results of the computations performed, are shown in figs. 3 - 9. The first three of these diagrams give the probability of total failure P of the assembly versus service life time T_L until 50,000 hours for various lengths of inspection intervals, assuming the strength reduction parameter R to be 6,000, 10,000 and 15,000 hours respectively. All three diagrams have the same standard deviation for the logarithm (base 10) of the time to crack initiation, $\sigma_c = 0.1$. The next three diagrams, figs. 6 - 8, are analogous, the only difference being that the standard deviation is twice as big, $\sigma_c = 0.2$. Fig. 9 gives the variation of P with R for one length of inspection interval t_{insp} = 4,000 hours and a limit service time T_L = 30,000 hours, using the smaller standard deviation $\sigma_c = 0.1$. The other three gust load and crack initiation parameters, employed in the analysis, have the same values in all diagrams. The probabilities of total failure are extremely small at short service times, where the curves are very steep. This is expecially true for the small standard deviation in figs. 3 - 5. When approaching the logarithmic mean of the crack initiation, T = 50,000 hours, the curves are much flatter and the influence of a variation of the standard deviation is less pronounced. It is obvious that inspections form an effective means of reducing the probability. The inspection intervals should be considerably shorter than the strength reduction parameter R, however, to give an adequate safety level for a limit service time of some 30,000 hours. The influence of a variation of the parameter R is considerable, which can be studied by comparing mutually the diagrams in figs. 3 - 5 and also those in figs. 6 - 8, respectively, or directly in fig. 9. It is thus important to determine rather accurately the crack propagation time and the simultaneous strength reduction.

The preliminary testing with constant amplitude loading has not yet been analyzed in detail, using the numerical procedure. It can be observed from the results that the time from the first element failure until total collapse of the assembly is rather short compared to the whole life time. This might imply that a close study of the following failures nos. 2, 3 ... is not worth while. Since the testing has neither included inspections, nor random variable amplitude loading, general conclusions cannot be drawn at present. It is believed, however, that further theoretical and experimental investigations, carried out in close connection, will result in a simplified, although more adequate procedure.

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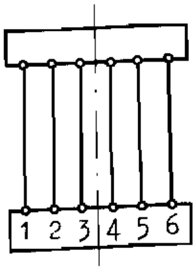
Table I. Load redistribution factors c_j computed from eqs. (3) - (6) for six parallel elements.



Failed elements in order	Load factor c_j in element no.					
	1	2	3	4	5	6
1	-	1.80	1.50	1.20	0.90	0.60
2	1.54	-	1.30	1.18	1.05	0.93
3	1.29	1.25	-	1.19	1.15	1.12
1, 2	-	-	3.30	2.10	0.90	-0.30
1, 3	-	2.66	-	1.63	1.11	0.60
1, 4	-	2.10	1.80	-	1.20	0.90
1, 5	-	1.80	1.63	1.46	-	1.11
1, 6	-	1.50	1.50	1.50	1.50	-
2, 1	-	-	3.30	2.10	0.90	-0.30
2, 3	2.24	-	-	1.50	1.29	1.07
2, 4	1.78	-	1.58	-	1.37	1.27
2, 5	1.50	-	1.50	1.50	-	1.50
2, 6	1.11	-	1.46	1.63	1.80	-
3, 1	-	2.66	-	1.63	1.11	0.60
3, 2	2.24	-	-	1.50	1.29	1.07
3, 4	1.50	1.50	-	-	1.50	1.50
3, 5	1.27	1.37	-	1.58	-	1.78
3, 6	0.90	1.20	-	1.80	2.10	-

Table II. Test results from single specimens, ordered with increasing number of cycles to failure.

Order number	Crack kc	Failure kc	Order number	Crack kc	Failure kc
1	137.0	213.6	13	265.5	308.9
2	184.5	221.3	14	275.7	324.7
3	193.0	236.0	15	286.8	326.3
4	204.8	245.1	16	215.6	330.5
5	193.0	254.4	17	227.0	337.6
6	241.4	264.7	18	259.0	338.4
7	214.2	267.1	19	246.0	339.5
8	236.6	275.6	20	257.5	341.0
9	198.0	280.0	21	312.0	352.7
10	228.9	285.2	22	338.9	374.5
11	264.0	288.7	23	307.3	389.0
12	279.7	306.9	24	290.0	402.5

First crack: mean $\mu_c = 5.379$, $T_{c50} = 239{,}000$

stand. dev. $\sigma_c = 0.089$

Failure: mean $^{10}\overline{\log T} = 5.477$, $T_{50} = 300{,}000$

stand. dev. $\sigma_{^{10}\log T} = 0.075$

Table III. Test results from assemblies with six elements.

Assembly No.	First crack pos.	First crack kc	Critical crack pos.	Critical crack kc	First failure pos.	First failure kc	Total failure kc
1	2	171.8	2	171.8	2	217.4	221.3
2	2	171.5	5	192.3	5	220.4	231.3
3	4	180.0	4	180.0	4	228.6	256.0
4	1	218.3	1	218.3	1	250.9	251.9
5	3	205.9	3	205.9	3	234.6	252.9
6	4	256.7	4	256.7	4	293.5	303.9
7	4	161.7	4	161.7	4	212.7	222.7
8	3	193.3	5	209.1	5	257.6	268.7
9	3	248.3	6	254.1	6	317.1	328.7
10	1	188.9	1	188.9	1	212.4	214.7
11	4	147.0	1	193.4	1	218.0	223.6
12	2	167.3	4	242.2	2	259.0	264.7
13	5	178.6	5	178.6	5	236.4	243.2
14	4	201.0	4	201.0	4	236.8	253.9
15	1	170.5	4	187.2	4	230.1	252.1
16	2	163.0	4	165.0	4	235.6	237.8
17	4	146.1	1	156.1	1	229.8	233.4
18	6	151.9	6	151.9	6	221.0	226.6
19	5	226.6	3	235.1	3	270.9	274.4
20	4	160.0	4	160.0	4	221.7	235.4
21	2	173.6	2	173.6	2	292.6	301.6
22	4	203.8	3	246.0	3	283.5	285.5
23	3	178.2	4	180.5	3,4	235.7	239.7

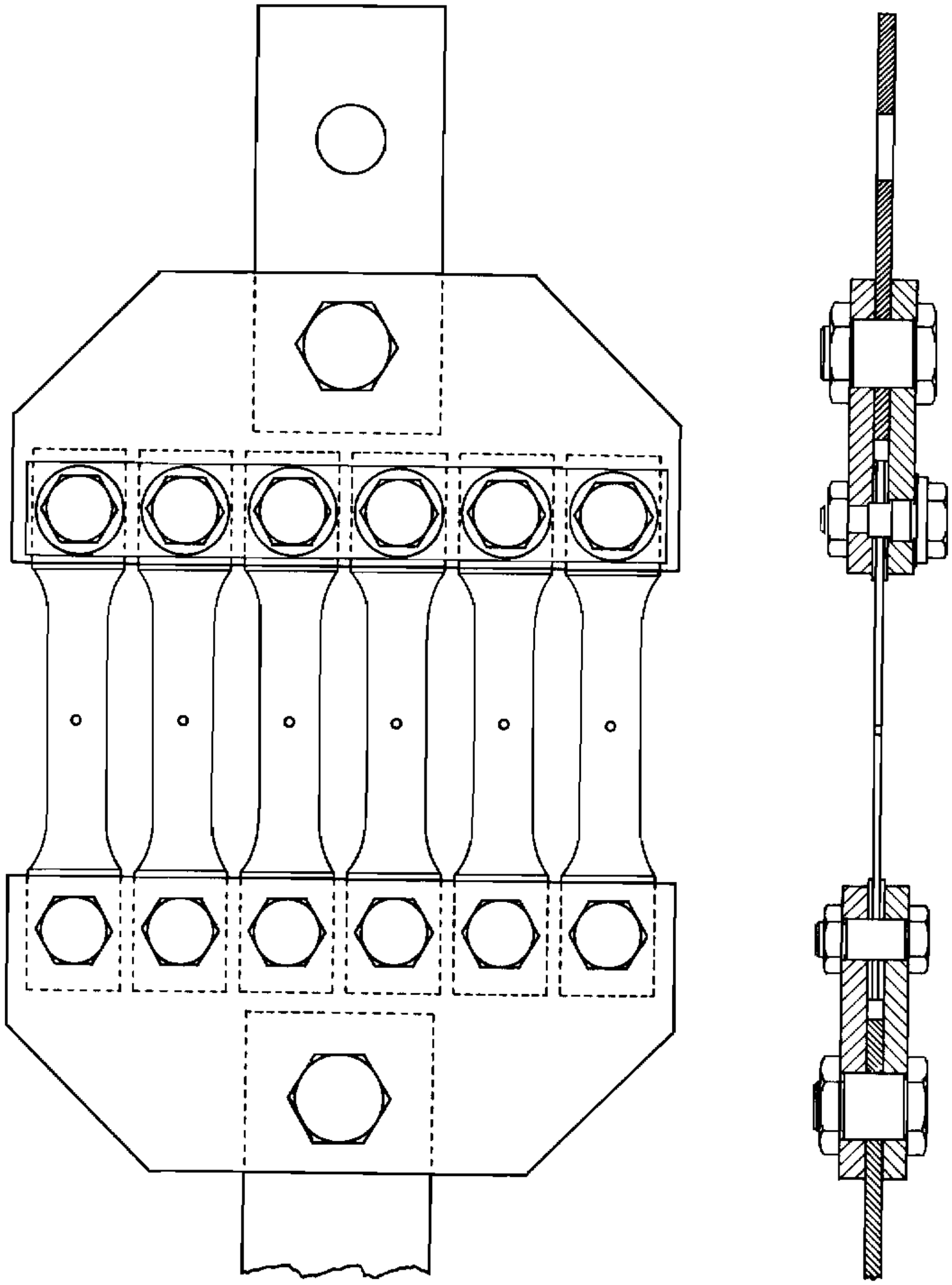


Fig. 1. Assembly of six parallel elements connected by pin-joints to rigid anchorage blocks in both ends.

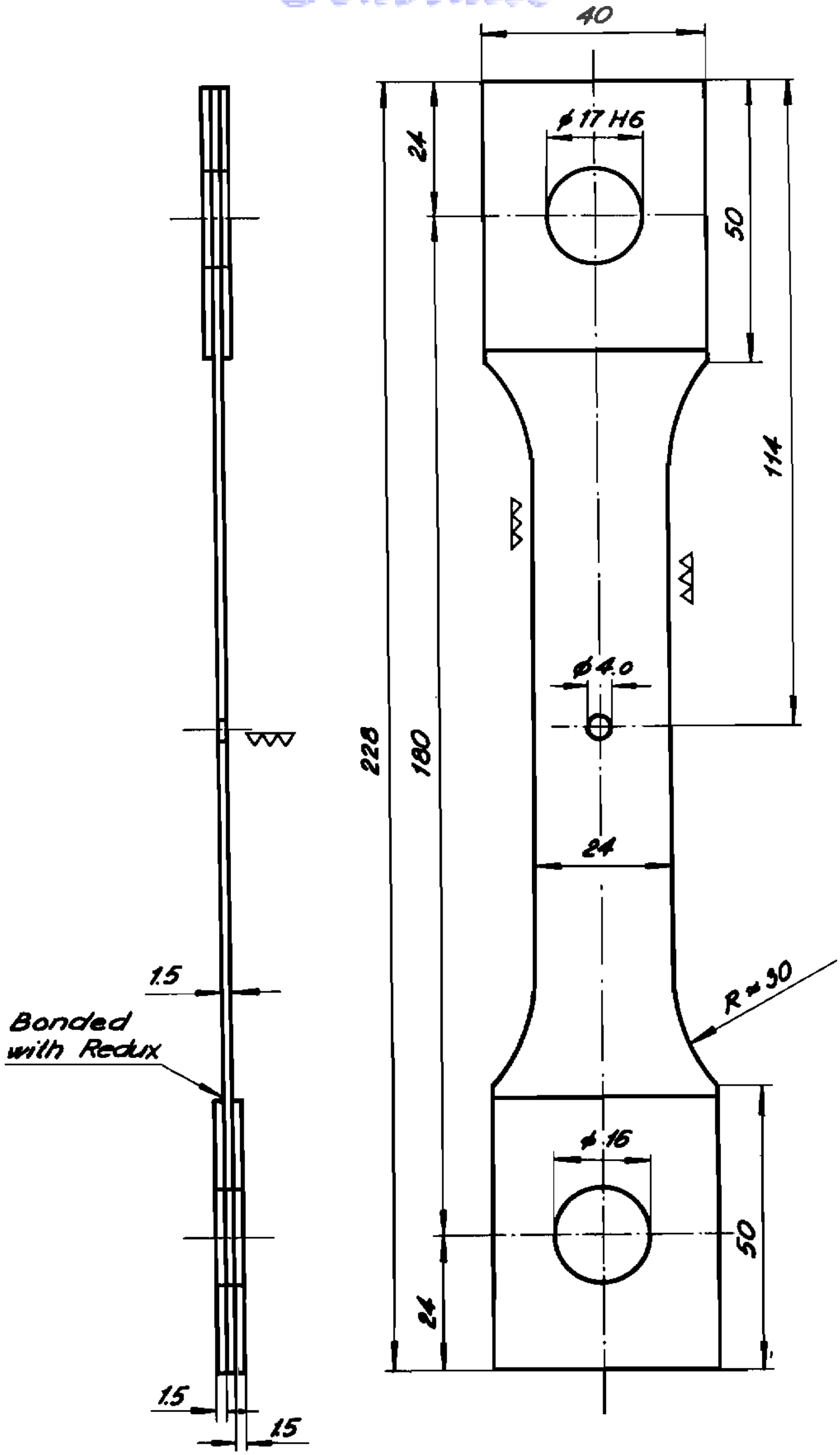


Fig. 2. Dimensions of test specimens (elements).

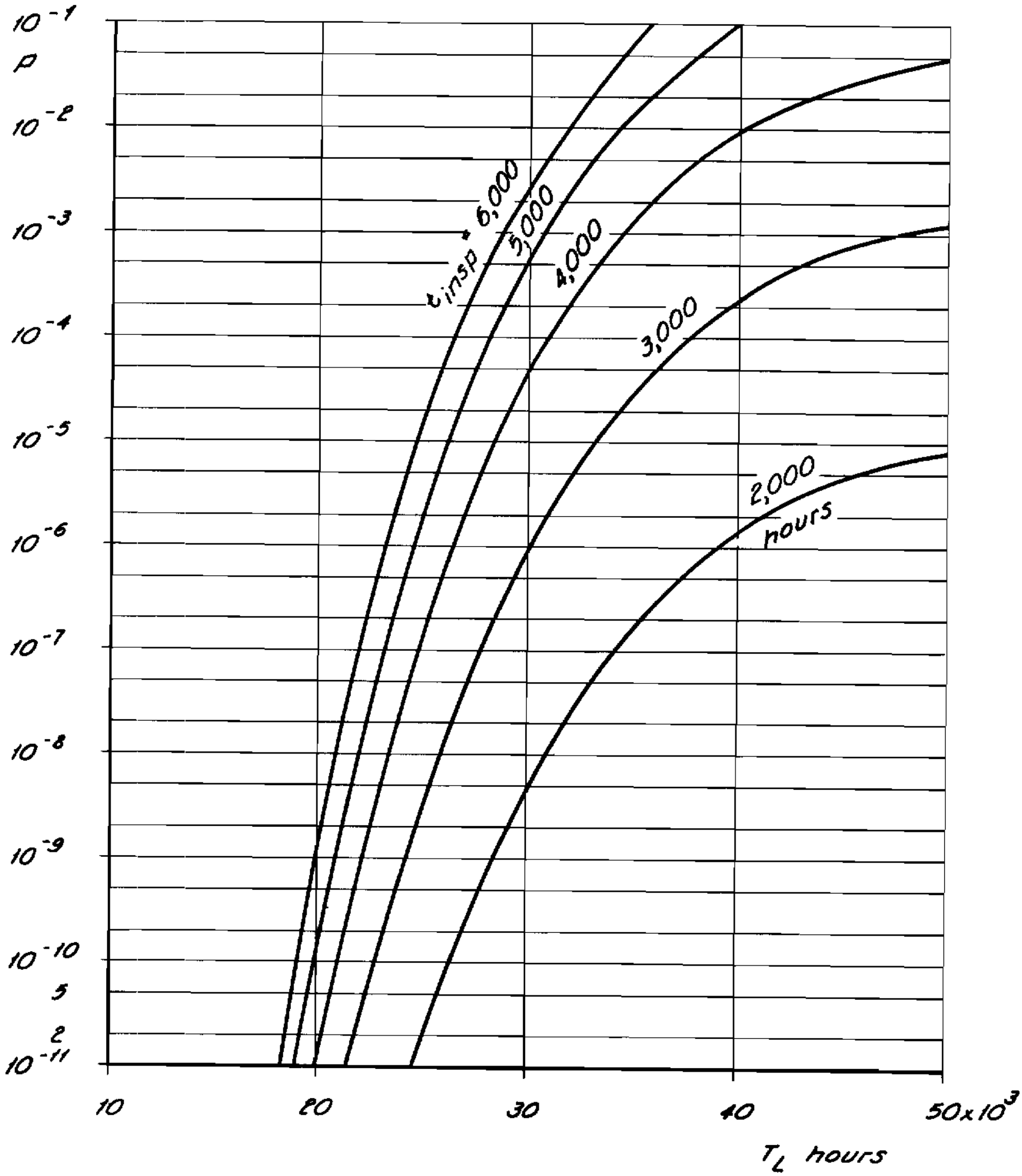


Fig. 3. Probability of collapse of assembly with six parallel elements versus service time for various lengths of inspection intervals from 2,000 to 6,000 hours. Strength reduction parameter R = 6,000 hours, standard deviation σ_o = 0.1. Other parameters assumed: T_{c50} = 50,000 hours, H_o = 0.2, h = 20.

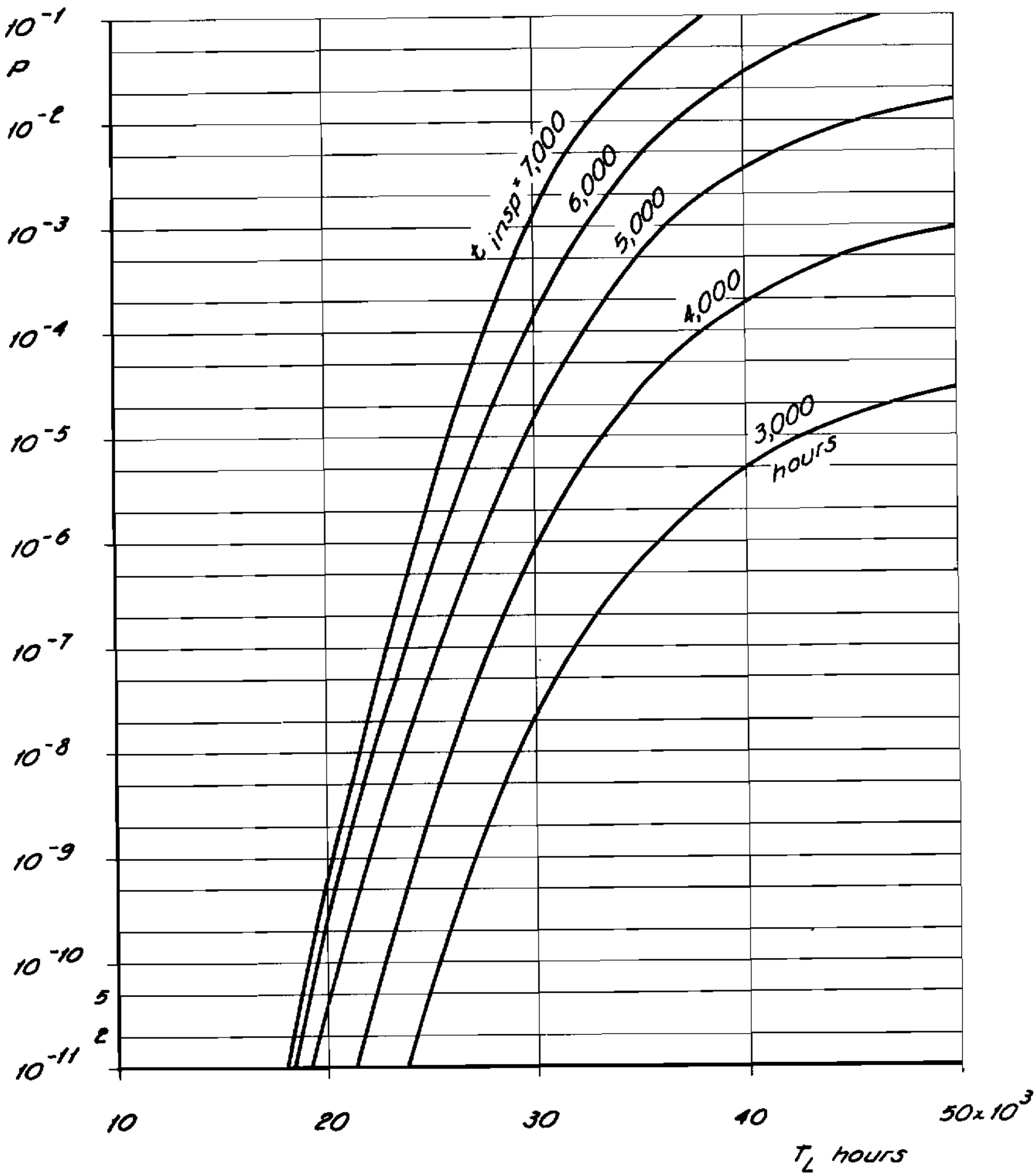


Fig. 4. Probability of collapse of assembly with six parallel elements versus service time for various lengths of inspection intervals from 3,000 to 7,000 hours. R = 10,000 hours, $\sigma_c = 0.1$; other parameters same as in fig. 3.

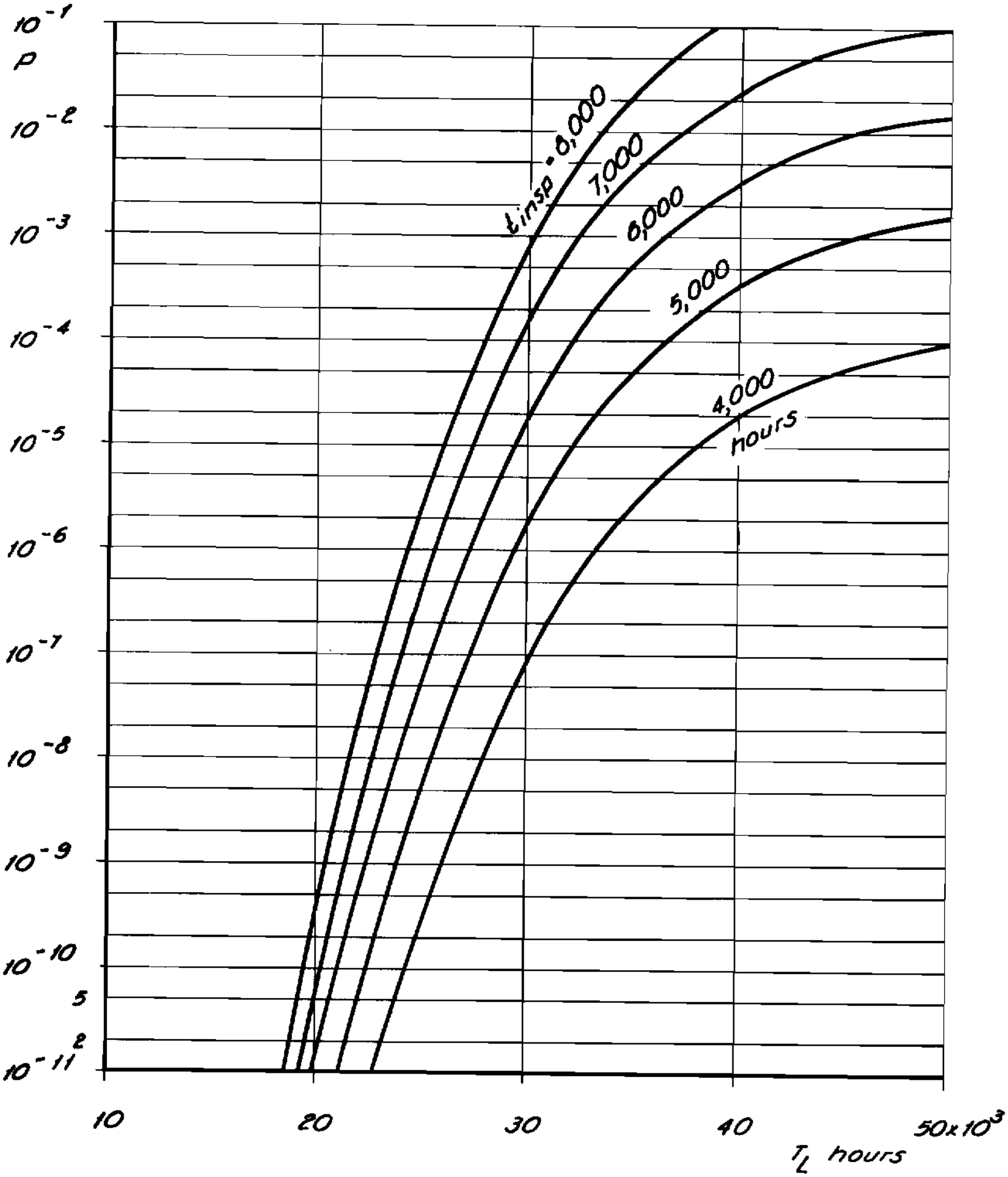


Fig. 5. Probability of collapse of assembly with six parallel elements versus service time for various lengths of inspection intervals from 4,000 to 8,000 hours. R = 15,000 hours, σ_c = 0.1; other parameters same as in fig. 3.

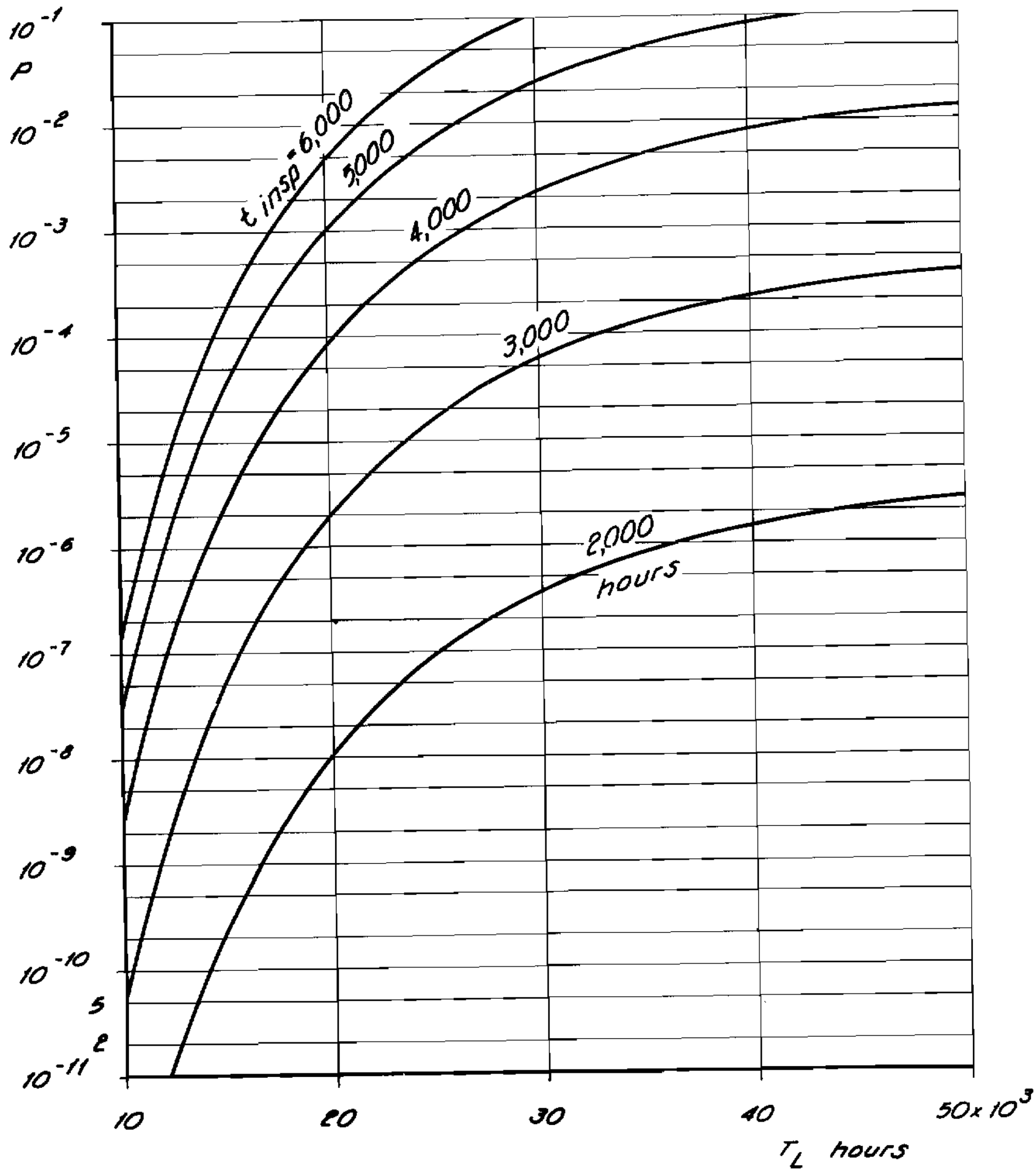


Fig. 6. **Probability of collapse of assembly with six parallel elements versus service time for various lengths of inspection intervals from 2,000 to 6,000 hours. R = 6,000 hours, σ_c = 0.2 ; other parameters same as in fig. 3.**

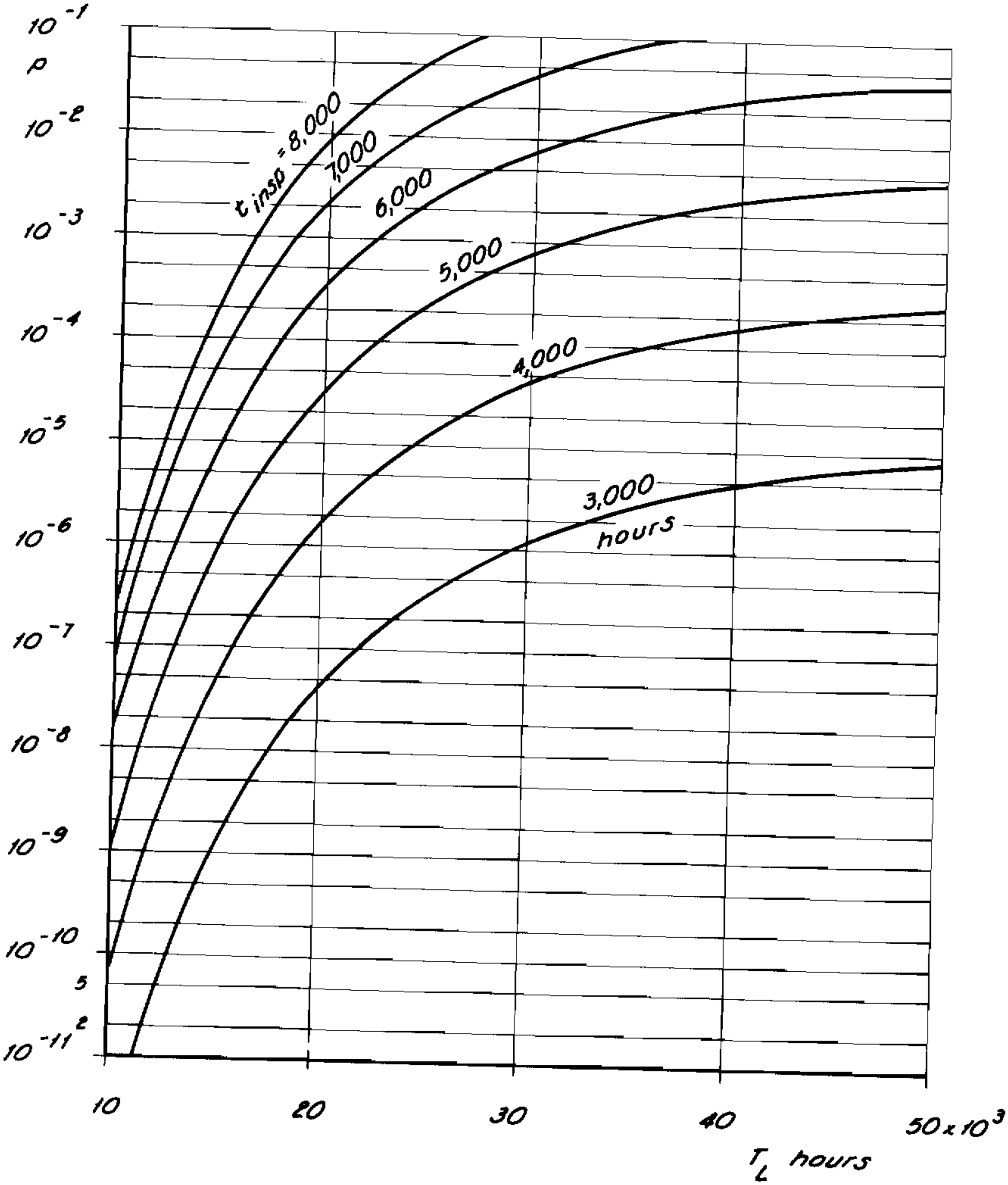


Fig. 7. Probability of collapse of assembly with six parallel elements versus service time for various lengths of inspection intervals from 3,000 to 8,000 hours. R = 10,000 hours, σ_0 = 0.2 ; other parameters same as in fig. 3.

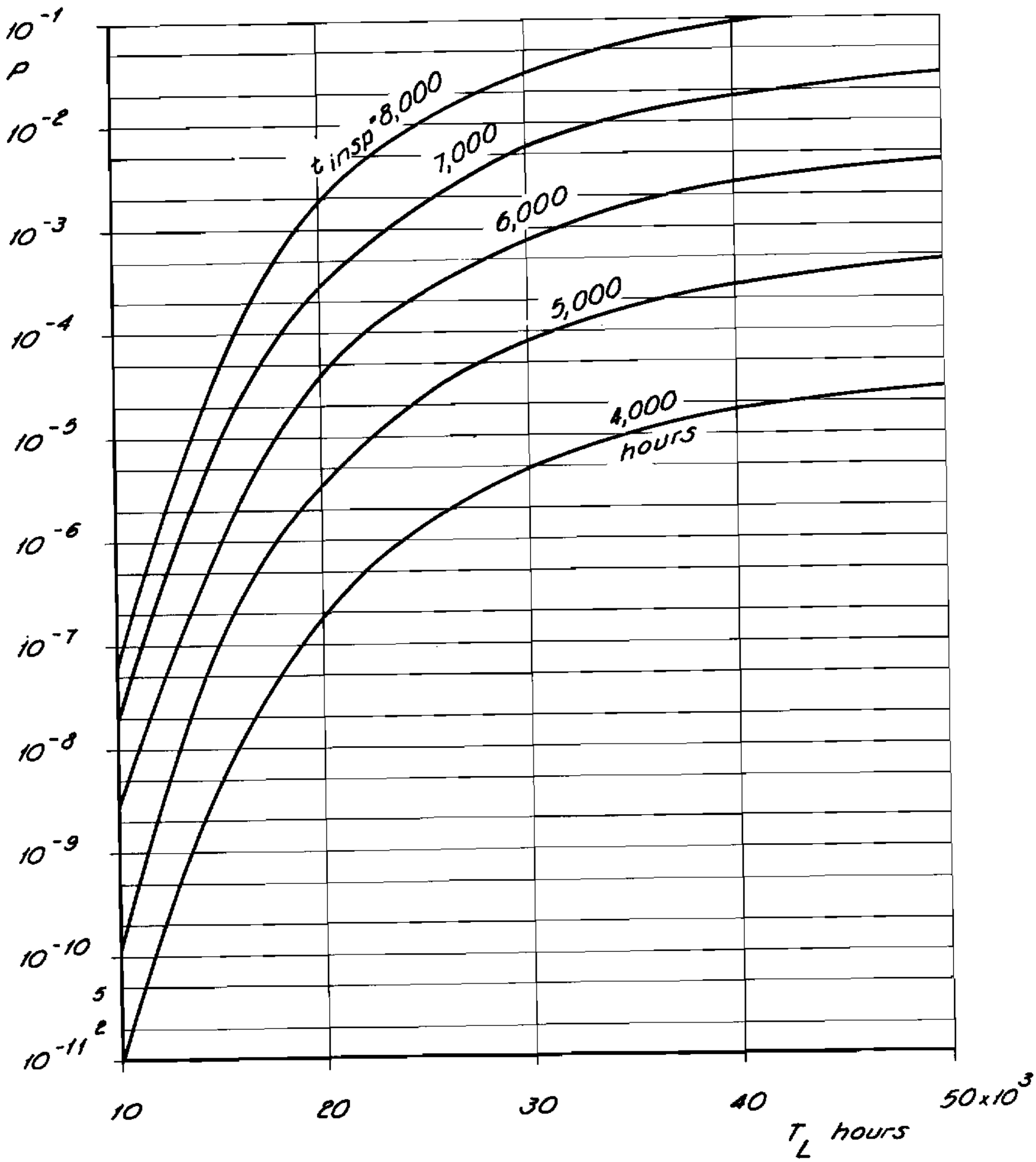


Fig. 8. Probability of collapse of assembly with six parallel elements versus service time for various lengths of inspection intervals from 4,000 to 8,000 hours. $R = 15{,}000$ hours, $\sigma_c = 0{,}2$; other parameters same as in fig. 3.

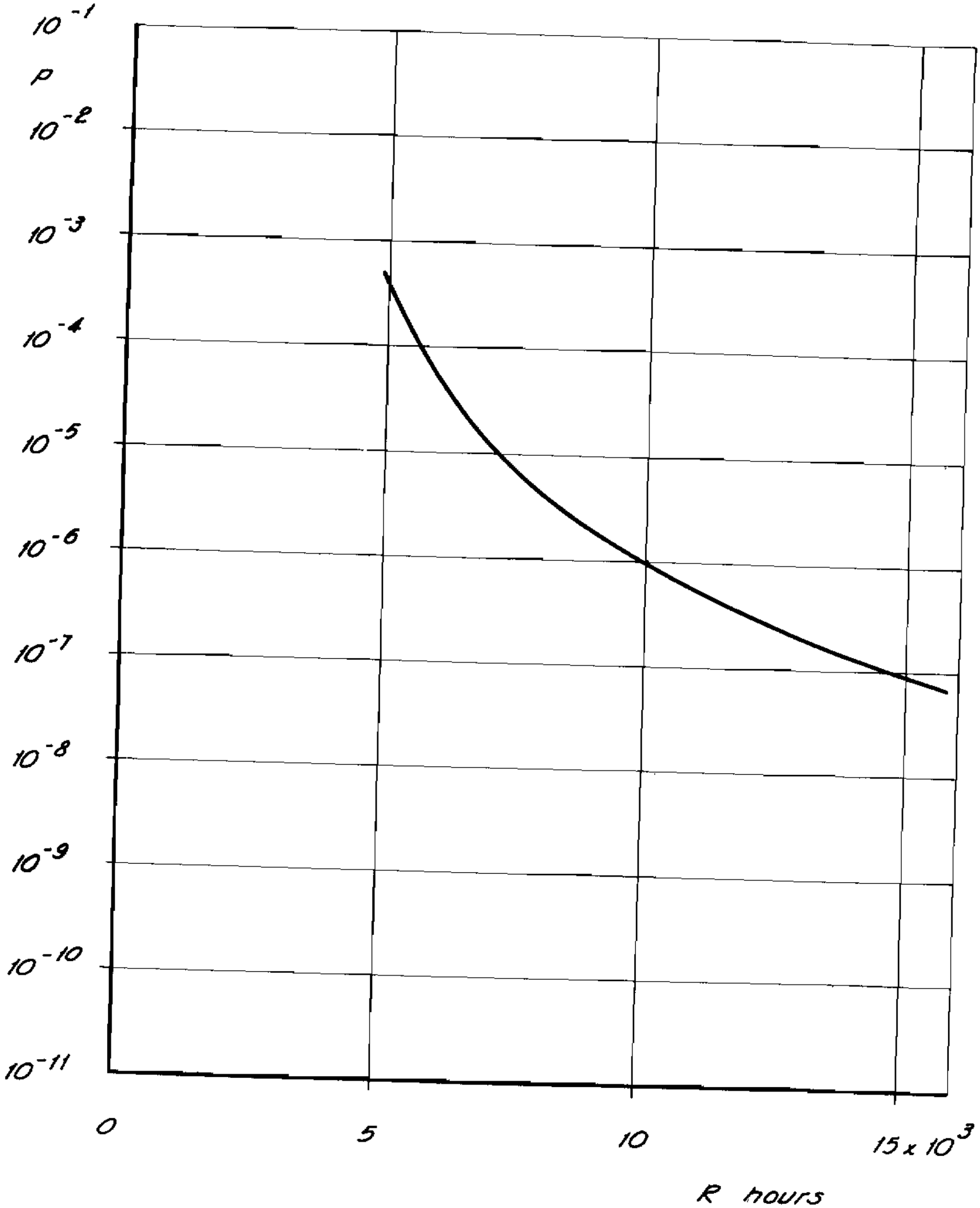


Fig. 9. Variation of probability of collapse of assembly with strength reduction parameter R. Other parameters assumed: T_{c50} = 50,000 hours, σ_c = 0.1, H_o = 0.2, h = 20, T_L = 30,000 hours, t_{insp} = 4,000 hours.

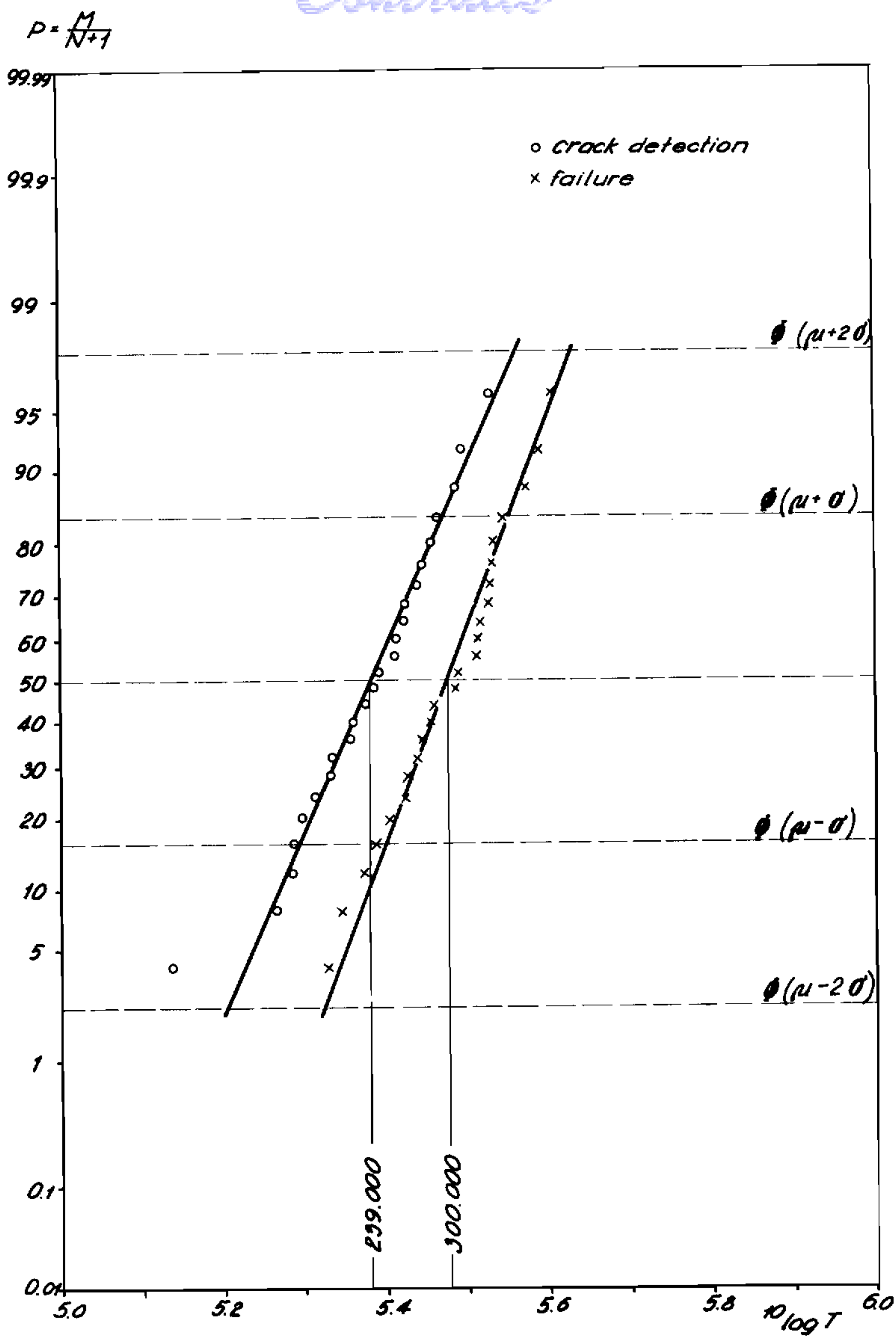


Fig. 10. Test results of crack detection and failure in single specimens, plotted on a Gaussian probability paper.

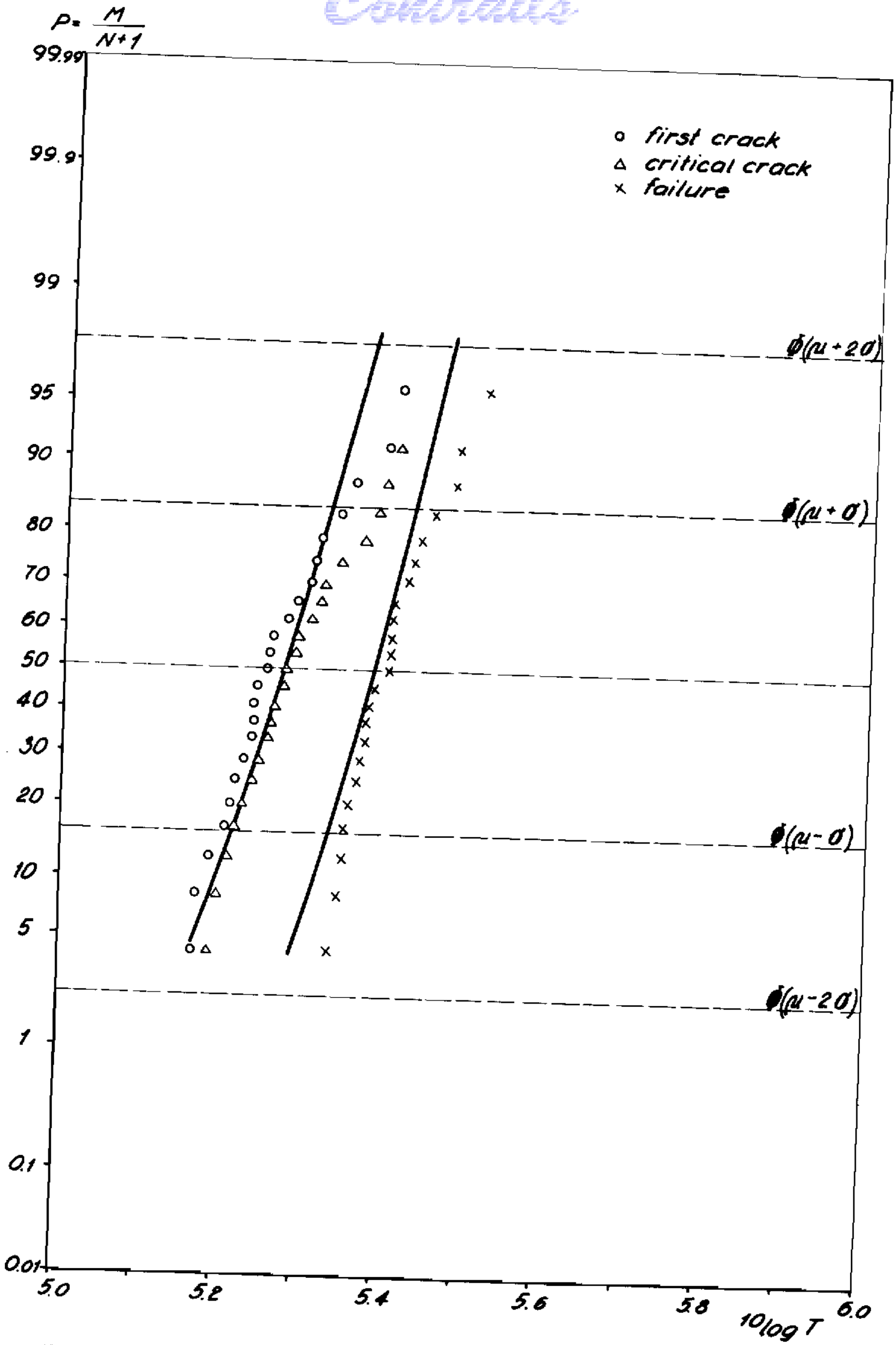
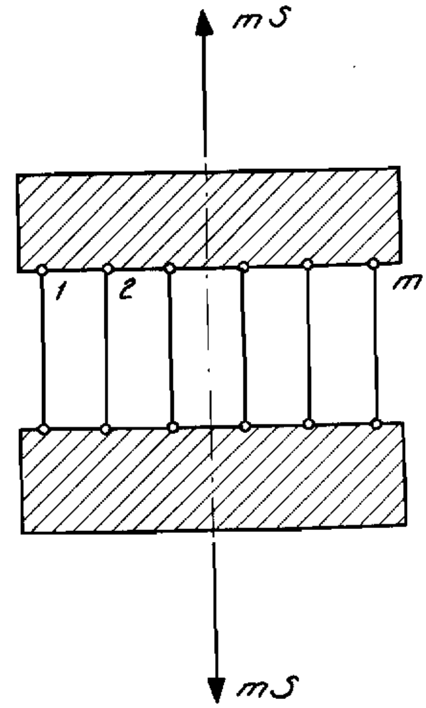


Fig. 11. Test results of crack detection and first element failure in assemblies with six parallel elements, plotted on a Gaussian probability paper. The curves have been derived from the single specimen tests.

APPENDIX A

LOAD DISTRIBUTION IN ASSEMBLY WITH BROKEN MEMBERS

In the analysis of the load distribution among the m parallel, identical members of the assembly it is assumed that the members are connected by pin-joints to rigid blocks in each end. The loading, which has a magnitude m S, is acting on the blocks. The action line is symmetrical with respect to and parallel with the axes of the members and remains so irrespective of the small rotations of the blocks due to failure of one or more members. The load on each member is thus S before any element failure has occurred. After failure a load redistribution takes place. In a member with the position number j the load is denoted c_j S , c_j being the load redistribution factor. It is further assumed that the elongations of all the members are proportional to their loading, i.e. the members are always within the elastic region.



The assumptions of rigid anchorage blocks and elastic behaviour of the members will obviously result in a linear variation of the load with the position j of the member.

$$c_j = (a + bj)\, c_{oj} \qquad \text{(A 1)}$$

where $c_{oj} = 0$, if member no. j has failed, and $c_{oj} = 1$, if it still carries the load corresponding to the elongation of the member. The constants a and b can be determined by two equilibrium equations.

The resultant force in the axial direction is still

$$S \left[a \sum_{j=1}^{m} c_{oj} + b \sum_{j=1}^{m} j\, c_{oj} \right] = m\, S \qquad \text{(A 2)}$$

The moment of the member forces with respect to the resultant force is equal to zero

$$S\,a\left(\sum_{j=1}^{m} j\,c_{oj} - \frac{m+1}{2}\sum_{j=1}^{m} c_{oj}\right) +$$

$$+ S\,b\left(\sum_{j=1}^{m} j^2\,c_{oj} - \frac{m+1}{2}\sum_{j=1}^{m} j\,c_{oj}\right) = 0 \qquad \text{(A 3)}$$

If the following notations are introduced

$$A = \sum_{j=1}^{m} c_{oj} \qquad B = \sum_{j=1}^{m} j\,c_{oj} \qquad C = \sum_{j=1}^{m} j^2\,c_{oj} \qquad \text{(A 4)}$$

eqs. (A2) and (A3) yield

$$a = \frac{m\left[C - B(m+1)/2\right]}{A\left[C - B(m+1)/2\right] - B\left[B - A(m+1)/2\right]} \qquad \text{(A 5)}$$

$$b = -\frac{m\left[B - A(m+1)/2\right]}{A\left[C - B(m+1)/2\right] - B\left[B - A(m+1)/2\right]} \qquad \text{(A 6)}$$

APPENDIX B

DISCUSSION OF QUASITRUNCATED NORMAL FREQUENCY FUNCTION INTRODUCED IN EQ.(45)

Every frequency function $f(x)$ must satisfy the condition

$$\int_{-\infty}^{+\infty} f(x)\, dx = 1 \tag{B 1}$$

This is the case with the ordinary normal distribution

$$\int_{-\infty}^{+\infty} \frac{1}{\sigma_i \sqrt{2\pi}} \exp\left[-\frac{(y_i - \mu_i)^2}{2\,\sigma_i^2}\right] dy_i = 1 \tag{B 2}$$

The quasitruncated normal frequency functions $\bar{g}$ of eqs. (45) and (53) include, however, a factor ℓ_i which has the effect that

$$\int_{-\infty}^{+\infty} \bar{g}(k_{3i}\, y_i)\, d(k_{3i}\, y_i) = \ell_i \tag{B 3}$$

To overcome this difficulty a constant E is defined by

$$\int_{-\infty}^{E} \bar{g}\,(k_{3i}\, y_i)\, d(k_{3i}\, y_i) = 1 \tag{B 4}$$

A truncated function $\bar{\bar{g}}$ is introduced

$$\bar{\bar{g}}\,(k_{3i}\, y_i) = \bar{g}\,(k_{3i}\, y_i) \qquad \text{for} \quad k_{3i}\, y_i \leq E$$

$$\bar{\bar{g}}\,(k_{3i}\, y_i) = 0 \qquad \text{for} \quad k_{3i}\, y_i > E$$

Since $\bar{\bar{g}}$ is a true frequency function, a convolution of Q_{3b} using $\bar{\bar{g}}$ will give a true frequency function, which is obviously smaller than the frequency function obtained when utilizing $\bar{g}$ $(0 \leq \bar{\bar{g}} \leq \bar{g})$

$$\int_{-\infty}^{+\infty} \bar{\bar{g}}\,(k_{31}\, y_1)\, d(k_{31}\, y_1) \int_{-\infty}^{+\infty} \bar{\bar{g}}\,(k_{32}\, y_1)\, \bar{\bar{g}}\,(Q_{3b} - k_{31}\, y_1 -$$

$$- k_{32}\, y_1)\, d(k_{32}\, y_2) \leq$$

$$\leq \int_{-\infty}^{+\infty} \bar{g}(k_{31}\ y_1)\ d(k_{31}\ y_1) \int_{-\infty}^{+\infty} \bar{g}(k_{32}\ y_1)\ \bar{g}(Q_{3b} - k_{31}\ y_1 -$$

$$- k_{32}\ y_1)\ d(k_{32}\ y_2) =$$

$$= \ell_o\ \ell_1\ \ell_2 \int_{-\infty}^{+\infty} \frac{1}{k_{31}\ \sigma_o \sqrt{2\pi}} \exp\left[-\frac{(k_{31}\ y_o - k_{31}\ \mu_o)^2}{2\ k_{31}^2\ \sigma_o^{\ 2}}\right] d(k_{31}\ y_o)\ \times$$

$$\times \int_{-\infty}^{+\infty} \frac{1}{k_{32}\ \sigma_1 \sqrt{2\pi}} \exp\left[-\frac{k_{32}\ y_1 - k_{32}\ \mu_1)^2}{2\ k_{32}^2\ \sigma_1^{\ 2}}\right] d(k_{32}\ y_1)\ \times$$

$$\times \int_{-\infty}^{+\infty} \frac{1}{k_{33}\ \sigma_2 \sqrt{2\pi}} \exp\left[-\frac{Q_{3b} - k_{31}y_o - k_{32}y_1 - k_{33}\mu_2)^2}{2\ k_{32}^2\ \sigma_2^{\ 2}}\right] d(k_{33}\ y_2) =$$

$$= \ell_o\ \ell_1\ \ell_2 \frac{1}{\sqrt{2\pi}\ \sqrt{k_{31}^2\ \sigma_o^{\ 2} + k_{32}^2\ \sigma_1^{\ 2} + k_{33}^2\ \sigma_2^{\ 2}}}\ \times$$

$$\times \exp\left[-\frac{(Q_{3b} - k_{31}\ \mu_o - k_{32}\ \mu_1 - k_{33}\ \mu_2)^2}{2(k_{31}^2\ \sigma_o^{\ 2} + k_{32}^2\ \sigma_1^{\ 2} + k_{33}^2\ \sigma_2^{\ 2}}\right] \qquad \text{(B 5)}$$

The last two membra of eq. (B5) follow from the definition of $\bar{g}$ and from the theorem that the sum of independent normally distributed variables is a normally distributed variable.

APPENDIX C

DEDUCTION OF DISTRIBUTION FUNCTION OF Q

According to eq. (58) the convolution procedure gives the following integral for the distribution function of the time to the third failure Q_3

$$F_{Q_3}(Q_3) = \frac{\ell\, r_{1\upsilon}\, r_{2\upsilon}\, r_{3\upsilon}}{6\, k_{31}\, k_{32}\, k_{33}} \int_0^{Q_3} (Q_3 - Q_{3b})^3 \frac{1}{\sigma}\varphi\left(\frac{Q_{3b} - \mu}{\sigma}\right) d\, Q_{3b} \qquad (C\ 1)$$

where φ is the frequency function of Q_{3b}.

Since

$$(Q_3 - Q_{3b})^3 = \left[-(Q_{3b} - \mu) + (Q_3 - \mu)\right]^3 =$$

$$= -(Q_{3b} - \mu)^3 + 3(Q_{3b} - \mu)^2 (Q_3 - \mu) -$$

$$- 3(Q_{3b} - \mu)(Q_3 - \mu)^2 + (Q_3 - \mu)^3 \qquad (C\ 2)$$

introduction of the following substitutions

$$\vartheta = (Q_3 - \mu)/\sigma \qquad (C\ 3)$$

$$\varkappa = - \mu/\sigma \qquad (C\ 4)$$

$$v = (Q_{3b} - \mu)/\sigma \qquad (C\ 5)$$

$$\frac{d\, Q_{3b}}{d\, v} = \sigma \qquad (C\ 6)$$

gives

$$F_{Q_3}(Q_3) = \frac{\ell\, r_{1\upsilon}\, r_{2\upsilon}\, r_{3\upsilon}}{6\, k_{31}\, k_{32}\, k_{33}} \int_{\varkappa}^{\vartheta} \frac{1}{\sigma}\varphi(v)\ \sigma^3\left[-v^3 + 3v^2\vartheta - 3v\vartheta^2 + \vartheta^3\right]\sigma\, dv$$

Now the following integration formulae are used

$$\int v^3\varphi(v)\, dv = - v^2\varphi(v) - 2\varphi(v) + C_1 \qquad (C\ 7)$$

$$\int v^2\varphi(v)\, dv = - v\varphi(v) + \Phi(v) + C_2 \qquad (C\ 8)$$

$$\int v\,\varphi(v)\,dv = -\,\varphi(v) + C_3 \tag{C 9}$$

$$\int \varphi(v)\,dv = \Phi(v) + C_4 \tag{C 10}$$

After rearrangement of the terms the distribution function is finally written

$$F_{Q_3}(Q_3) = \frac{\sigma^3\,\ell\,r_{1v}\,r_{2v}\,r_{3v}}{6\,k_{31}\,k_{32}\,k_{33}}\Big\{(\vartheta^3 + 3\vartheta)\left[\Phi(\vartheta) - \Phi(\varkappa)\right] + (\vartheta^2 + 2)\,\varphi(\vartheta) + (-\,3\vartheta^2 + 3\vartheta\varkappa - \varkappa^2 - 2)\,\varphi(\varkappa)\Big\} \tag{C 11}$$