TRUNCATION ERRORS IN NATURAL FREQUENCIES AS COMPUTED BY THE METHOD OF COMPONENT MODE SYNTHESIS*

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The intent of this investigation is to provide a method for estimating the errors in computed natural frequencies of a structure attributable to truncation of the generalized coordinate system. The study is confined to those mass and stiffness matrix arrangements associated with a particular method of analysis described elsewhere (References 1 and 2) as the method of component mode synthesis. An equation is developed which permits estimates to be made of these errors. Application to an example shows that the frequency corrections are not very accurate because of approximations involved in the analysis. Despite this there appears to be good correlation between the calculated errors and the accuracy of computed modes. Hence, the method appears to be useful in providing criteria of modal accuracy. Further applications to various structures are required in order to establish confidence levels for these criteria.

INTRODUCTION

All methods for the vibration analysis of real structures are approximate methods in the sense that an actual structure having an infinite number of degrees of freedom is represented by a model having a finite number. In lumped parameter methods the physical aspect of the structure is altered in the modeling process by representing it as a finite number of rigid masses and massless elastic elements. In modal methods the possible virtual displacements of the structure are limited to those that can be defined by a finite number of displacement modes. In either case the effect of the approximation is to introduce errors in the computed natural frequencies and modes of vibration. This paper is concerned with the problem of estimating errors introduced by use of a particular modal method described as the method of component mode synthesis (References 1 and 2).

In this method the properties and modes of the structural system are synthesized from those of the components of substructures that make up the system. Displacements related to each component are defined by a set of modal coordinates for which the modes are selected in three categories: rigid-body modes, constraint modes, and fixed-constraint normal modes. The term "rigid-body mode" is self-explanatory. Constraint modes are defined as those produced by relaxing in turn each redundant constraint acting on the component. Fixed-constraint normal modes are used to define displacements of the component relative to its system of constraints. The three categories are designated by use of the letters R, C, and N, respectively. As described in References 1 and 2, a transformation is derived which

*Research described in this paper was supported, in part, by the Jet Propulsion Laboratory, California Institute of Technology, under Contract No. NAS 7-100, sponsored by the National Aeronautics and Space Administration.

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connects component coordinates and system coordinates. By use of this transformation, mass and stiffness matrices for the system are constructed from those for the separate components. Using these system matrices, a matrix equation for the free vibration of an undamped elastic structure is written. For the purposes of this paper it is convenient to consider the rigid-body and constraint modes together as a set of "basic" displacement modes designated by the letter $\mathbf{B}$. The resulting equation is

$$\begin{bmatrix} M^{BB} & M^{BN} \\ M^{NB} & M^{NN} \end{bmatrix} \begin{bmatrix} q_o^B \\ q_o^N \end{bmatrix} = \lambda_o \begin{bmatrix} K^{BB} & 0 \\ 0 & K^{NN} \end{bmatrix} \begin{bmatrix} q_o^B \\ q_o^N \end{bmatrix} \quad (1)$$

where: M^{BB} is the mass submatrix associated with the basic modes

M^{NN} is the diagonal mass submatrix associated with the fixed-constraint normal modes

M^{BN}, M^{NB} are mass submatrices related to coupled modes

K^{BB}, K^{NN} are stiffness submatrices

q_o^B, q_o^N are generalized displacements of the basic and normal modes, respectively

$\lambda_o = \dfrac{1}{\omega_o^2}$ is an eigenvalue of the system

ω_o is a natural frequency of the system

The number of basic modes is determined by the geometry of the system and, in particular, by the system of interconnections among the components. The number of fixed-constraint normal modes is infinite so that the matrices are truncated by selecting a finite number of them for the analysis. The truncation errors, which are the subject of this paper, are those related to the truncation of the mass and stiffness matrices in Equation 1.

It is well known that the accuracy of the computed modes and frequencies resulting from any approximate method will be improved, in general, by increasing the number of degrees of freedom. Therefore, an effective method of estimating the truncation errors associated with Equation 1 is to increase in successive solutions the number of fixed-constraint normal modes and compare the resulting eigenvalues and eigenmodes. In general, they will approach their correct values asymptotically as will be seen by plotting them as is done in Figure 1. The practical drawback to this procedure very often lies in computer limitations. In analyzing large and complex structures it is often found that the selection of even a limited number of fixed-constraint normal modes taxes the computer, leaving no further capacity for exploring the question of accuracy by the above technique. Thus, a procedure for estimating the errors associated with a given solution without repeating the solution for higher order systems can be useful. In the following section an equation is developed that permits such an evaluation to be made using only the results of a single solution.

ANALYSIS

To begin the development it is supposed that Equation 1 has been solved using a certain number of fixed-constraint normal modes and that this solution has yielded a set of eigenvalues λ_{o_i} and eigenvectors $\left\{q_o^B, q_o^N\right\}_i$ $(i = 1,2,3,\cdots)$. The fixed-constraint normal modes in this solution are designated by use of the letter $\mathbf{N}$ (upper case). The accuracy of the solution can be improved by including an additional set of normal modes which are distinguished from the original set by use of the letter n (lower case). Equation 1 is modified accordingly and the new equation appears as follows.

$$\begin{bmatrix} M^{BB} & M^{BN} & M^{Bn} \\ M^{NB} & M^{NN} & 0 \\ M^{nB} & 0 & M^{nn} \end{bmatrix} \begin{bmatrix} q^B \\ q^N \\ q^n \end{bmatrix} = \lambda \begin{bmatrix} K^{BB} & 0 & 0 \\ 0 & K^{NN} & 0 \\ 0 & 0 & K^{nn} \end{bmatrix} \begin{bmatrix} q^B \\ q^N \\ q^n \end{bmatrix} \qquad (2)$$

The eigenvalues λ_i and eigenvectors $\{q^B, q^N, q^n\}_i$ $(i=1,2,3,\text{---})$ are distinguished from those which result from Equation 1 by dropping the subscript o. For convenience Equation 2 is rewritten in the following manner.

$$\begin{bmatrix} M^{oo} & M^{on} \\ M^{no} & M^{nn} \end{bmatrix} \begin{bmatrix} q^o \\ q^n \end{bmatrix} = \lambda \begin{bmatrix} K^{oo} & 0 \\ 0 & K^{nn} \end{bmatrix} \begin{bmatrix} q^o \\ q^n \end{bmatrix} \qquad (3)$$

where

$$M^{oo} = \begin{bmatrix} M^{BB} & M^{BN} \\ M^{NB} & M^{NN} \end{bmatrix}$$

$$M^{on} = \begin{bmatrix} M^{Bn} \\ 0 \end{bmatrix}, \quad M^{no} = \begin{bmatrix} M^{nB} & 0 \end{bmatrix}$$

$$q^o = \begin{bmatrix} q^B \\ q^N \end{bmatrix}$$

$$K^{oo} = \begin{bmatrix} K^{BB} & 0 \\ 0 & K^{NN} \end{bmatrix}$$

Equation 3 is written as two matrix equations.

$$M^{oo} q^o + M^{on} q^n = \lambda K^{oo} q^o \qquad (4)$$

$$M^{no} q^o + M^{nn} q^n = \lambda K^{nn} q^n \qquad (5)$$

Equation 5 is solved for q^n, giving

$$q^n = (\lambda K^{nn} - M^{nn})^{-1} M^{no} q^o \qquad (6)$$

This is substituted into Equation 4 to yield

$$\left(M^{oo} + M^{on} (\lambda K^{nn} - M^{nn})^{-1} M^{no}\right) q^o = \lambda K^{oo} q^o \qquad (7)$$

Comparing Equation 7 with Equation 1 it is seen that the effect of the added fixed-constraint normal modes is to alter the original mass matrix by adding an incremental matrix δM^{oo}, where

$$\delta M^{oo} = M^{on} \left(\lambda K^{nn} - M^{nn}\right)^{-1} M^{no} \tag{8}$$

This incremental matrix may be expanded as follows.

$$\begin{aligned} \delta M^{oo} &= \left[\begin{array}{c|c} \delta M^{BB} & \delta M^{BN} \\ \hline \delta M^{NB} & \delta M^{NN} \end{array}\right] \\ &= \left[\begin{array}{c} M^{Bn} \\ \hline 0 \end{array}\right] (\lambda K^{nn} - M^{nn})^{-1} \left[\begin{array}{c|c} M^{nB} & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c} M^{Bn} (\lambda K^{nn} - M^{nn})^{-1} M^{nB} & 0 \\ \hline 0 & 0 \end{array}\right] \end{aligned} \tag{9}$$

From this result it is evident that the foregoing statement may be amended to state the following: the effect of the added fixed-constraint normal modes is to alter only the original mass submatrix M^{BB} by adding an incremental matrix δM^{BB}, where

$$\delta M^{BB} = M^{Bn} (\lambda K^{nn} - M^{nn})^{-1} M^{nB} \tag{10}$$

Therefore, Equation 7 may be rewritten as follows.

$$\left[\begin{array}{c|c} M^{BB} + \delta M^{BB} & M^{BN} \\ \hline M^{NB} & M^{NN} \end{array}\right] \left[\begin{array}{c} q^{B} \\ \hline q^{N} \end{array}\right] = \lambda \left[\begin{array}{c|c} K^{BB} & 0 \\ \hline 0 & K^{NN} \end{array}\right] \left[\begin{array}{c} q^{B} \\ \hline q^{N} \end{array}\right] \tag{11}$$

This equation is expanded into the following two equations.

$$(M^{BB} + \delta M^{BB})\, q^{B} + M^{BN} q^{N} = \lambda K^{BB} q^{B} \tag{12}$$

$$M^{NB} q^{B} + M^{NN} q^{N} = K^{NN} q^{N} \tag{13}$$

The solutions to Equations 1 and 11 are compared by letting

$$\left.\begin{aligned} q^{B} &= q_{o}^{B} + \delta q^{B} \\ q^{N} &= q_{o}^{N} + \delta q^{N} \\ \lambda &= \lambda_{o} + \delta\lambda \end{aligned}\right\} \tag{14}$$

Substituting Equation 14 in Equations 12 and 13, expanding, and subtracting out the original equation, leads to the following pair of equations.

$$\begin{aligned} (M^{BB} - \lambda_{o} K^{BB})\, \delta q^{B} + M^{BN} \delta q^{N} &= (\delta\lambda K^{BB} - M^{BB})\, q_{o}^{B} \\ &\quad + (\delta\lambda K^{BB} - \delta M^{BB})\, \delta q^{B} \end{aligned} \tag{15}$$

$$\mathbf{M}^{NB} \quad \delta\mathbf{q}^{B} + (\mathbf{M}^{NN} - \lambda_o \mathbf{K}^{NN})\ \delta\mathbf{q}^{N} = \delta\lambda\ \mathbf{K}^{NN}\ \mathbf{q}_o^{N} + \delta\lambda \mathbf{K}^{NN}\ \delta\mathbf{q}^{N} \tag{16}$$

The second terms in the right-hand sides of these two equations are of second order and will be dropped at this point in the analysis. Into Equation 16 is substituted the following relationship which is derived from Equation 1.

$$\mathbf{q}_o^{N} = -\ (\mathbf{M}^{NN} - \lambda_o\ \mathbf{K}^{NN})^{-1}\ \mathbf{M}^{NB}\ \ \mathbf{q}_o^{B} \tag{17}$$

The resulting equation is substituted into Equation 15 to yield the following.

$$\left[(\mathbf{M}^{BB} - \lambda_o\ \mathbf{K}^{BB}) - \mathbf{M}^{BN}\ (\mathbf{M}^{NN} - \lambda_o\ \mathbf{K}^{NN})^{-1}\ \ \mathbf{M}^{NB}\right]\delta\mathbf{q}^{B}$$

$$= \delta\lambda\left[\mathbf{K}^{BB} + \mathbf{M}^{BN}\ (\mathbf{M}^{NN} - \lambda_o\ \mathbf{K}^{NN})^{-1}\ \mathbf{K}^{NN}\ \ (\mathbf{M}^{NN} - \lambda_o \mathbf{K}^{NN})^{-1}\ \mathbf{M}^{NB}\right]\mathbf{q}_o^{B} \tag{18}$$

$$- \delta\mathbf{M}^{BB}\ \ \mathbf{q}_o^{B}$$

Returning again to Equation 1 it is seen that it may be expressed in the form

$$\left[(\mathbf{M}^{BB} - \lambda_o\ \mathbf{K}^{BB}) - \mathbf{M}^{BN}\ (\mathbf{M}^{NN} - \lambda_o \mathbf{K}^{NN})^{-1}\mathbf{M}^{NB}\right]\ \mathbf{q}_o^{B} = \mathbf{0} \tag{19}$$

where the right-hand side represents a null column matrix. The matrix in square brackets in Equation 19 is symmetrical and is identical to that on the left-hand side of Equation 18. If the matrix product in Equation 19 is transposed the result is a null row matrix. Therefore, if Equation 18 is premultiplied by the transposed vector $\mathbf{q}_o^{B^T}$, yielding a scalar equation, its left-hand side will be identically zero. The resulting scalar equation is put in the form below.

$$\delta\lambda = \frac{\mathbf{q}_o^{B^T}\ \ \delta\ \mathbf{M}^{BB}\ \ \mathbf{q}_o^{B}}{\mathbf{q}_o^{B^T}\ \ \left[\mathbf{K}^{BB} + \mathbf{M}^{BN}\ (\mathbf{M}^{NN} - \lambda_o\ \mathbf{K}^{NN})^{-1}\ \mathbf{K}^{NN}\ (\mathbf{M}^{NN} - \lambda_o\ \mathbf{K}^{NN})^{-1}\ \mathbf{M}^{NB}\right]\ \mathbf{q}_o^{B}} \tag{20}$$

This equation permits an estimate to be made of the change in any one of the eigenvalues of Equation 1 resulting from the use of additional fixed-constraint normal modes beyond those used in Equation 1. These additional modes contribute to the determination of $\delta\mathbf{M}^{BB}$ according to Equation 10. The $\delta\lambda$ resulting from Equation 20 is not exact because of the second-order terms dropped from Equations 15 and 16. The importance of this point will be noted in connection with the results of an example to be discussed later.

Equation 20 does not lend itself to computation readily as it stands. The reason is that $\delta\mathbf{M}^{BB}$ depends upon an unknown λ rather than the known λ_o. Therefore, the expression for this quantity given in Equation 10 will be altered by working on the matrix $(\lambda\,\mathbf{K}^{nn} - \mathbf{M}^{nn})$. This is a diagonal matrix whose i^{th} diagonal element is $\lambda K_i^{nn} - M_i^{nn}$. It is noted that

$$K_i^{nn} = \frac{1}{\lambda_{n_i}} M_i^{nn} \tag{21}$$

where:

$$\lambda_{n_i} = \frac{1}{\omega_{n_i}^{2}}$$

ω_{n_i} is the i^{th} natural frequency of a component with fixed constraints. Hence, the i^{th} element of the matrix is

$$\lambda K_i^{nn} - M_i^{nn} = \frac{\lambda - \lambda_{n_i}}{\lambda_{n_i}} M_i^{nn}$$

The i^{th} element of the inverse matrix $(\lambda \mathbf{K}^{nn} - \mathbf{M}^{nn})^{-1}$

is

$$\frac{\lambda_{n_i}}{\lambda - \lambda_{n_i}} M_i^{nn^{-1}} = H_i \; M_i^{nn^{-1}}$$

Substituting $\lambda_0 + \delta\lambda$ for λ the number H_i in the above equation is expressed as follows.

$$\begin{aligned} H_i &= \frac{\lambda_{n_i}}{\lambda_0 - \lambda_{n_i} + \delta\lambda} \\ &= \frac{\lambda_{n_i}}{(\lambda_0 - \lambda_{n_i})} - \frac{\lambda_{n_i}}{(\lambda_0 - \lambda_{n_i})^2} \delta\lambda + \text{higher order terms in } \delta\lambda \end{aligned} \tag{22}$$

The diagonal matrix $\mathbf{H}$ of the elements H_i is given below where the higher order terms are neglected.

$$\mathbf{H} = \mathbf{H}_0 - \delta\lambda \; \mathbf{L}_n \tag{23}$$

where:

$$\mathbf{H}_0 = \left[\frac{\lambda_n}{\lambda_0 - \lambda_n} \right] \quad \text{a diagonal matrix.}$$

$$\mathbf{L}_n = \left[\frac{\lambda_n}{(\lambda_0 - \lambda_n)^2} \right] \quad \text{a diagonal matrix.}$$

The quantity $\delta \mathbf{M}^{BB}$ is expressed as

$$\delta \mathbf{M}^{BB} = \mathbf{M}^{Bn} \, (\mathbf{H}_0 - \delta\lambda \, \mathbf{L}_n) \, \mathbf{M}^{nn^{-1}} \, \mathbf{M}^{nB} \tag{24}$$

Using the relationship

$$K_i^{NN} = \frac{1}{\lambda_{N_i}} M_i^{NN} \tag{25}$$

the quantity in the denominator on the right side of Equation 20 can be simplified in a similar manner. The matrix in the square brackets is written as

$$\left[K^{BB} + M^{BN} \; L_N \; M^{NN^{-1}} \; M^{NB} \right]$$

where:

$$L_N = \left\lceil \frac{\lambda_N}{(\lambda_o - \lambda_N)^2} \right\rfloor \quad \text{a diagonal matrix.}$$

When this expression and Equation 24 are inserted into Equation 20 the expression for $\delta\lambda$ becomes the following one.

$$\delta\lambda = \frac{q_o^{B^T} \; M^{Bn} \; H_o \; M^{nn^{-1}} \; M^{nB} \; q_o^B}{q_o^{B^T} \left[K^{BB} + M^{BN} \; L_N \; M^{NN^{-1}} M^{NB} + M^{Bn} \; L_n \; M^{nn^{-1}} \; M^{nB} \right] q_o^B} \tag{26}$$

This equation represents the end point of the analysis. In summary it is noted that the matrices K^{BB}, M^{NN}, M^{BN}, M^{NB} and the values λ_N are available inasmuch as they were evaluated in order to write Equation 1 which is solved to yield the original set of eigenvalues and eigenmodes. The additional matrices M^{nn}, M^{Bn}, M^{nB} together with the quantities λ_n must be evaluated in order to use the equation. These will make use of as many additional fixed-constraint normal modes as will make a significant contribution to $\delta\lambda$. Having established these matrices the $\delta\lambda$ for a particular mode is computed by simply inserting the λ_o and q_o^B corresponding to that mode. Thus, values of $\delta\lambda$ can be determined for each mode of vibration included in the solution to Equation 1. The significance of the $\delta\lambda$ so obtained will be discussed following the example of the next section.

EXAMPLE

Equation 26 has been applied to a structure which is essentially a plane frame, the configuration of which is shown by the drawing in Figure 2. The structure is described in more detail in Reference 3 where the results of a vibration analysis using the Component Mode Method are given. The frame was intended originally as a conceptual representation of a three-tank launch vehicle system. The components are uniform beams instead of the more realistic fuel tanks and connecting structures but this in no way alters the essential features of the analysis. The obvious asymmetry was introduced deliberately to avoid the simplicity of symmetric and antisymmetric modes of vibration. In obtaining the results to be included in this paper the base of the central element of the structure is fixed and only vibrations in the plane of the structure are considered. The eight members or components that comprise the system are numbered in Figure 2. These members are considered to be axially rigid so that the modes of vibration involve beam deflections only. This is not an essential restriction on the analysis but is done only to simplify it.

The structure has eight basic modes which are shown in Figure 3. Three of these can be related to translations; two of which are vertical translations of members 3 and 4. The third is a lateral translation of the top of the frame. The remaining five are identified with rotations of the five free-frame junctions.

The fixed-constraint normal modes relate to beam vibration modes of the eight members. For all members, except member two, these are fixed-fixed beam modes. For member two they are fixed-free, or cantilever, beam modes.

Eigenvalue problems were formulated and solved for 16, 24, 32 and 40 degrees of freedom. The first of these cases involves the first fixed-constraint mode of each of the eight members in addition to the eight basic modes. The second case includes the first two fixed-constraint modes of each member and so on.

The eigenvalues, λ_N or λ_n, for all of the members through the first four modes are listed in Table I.

All of the eigenvalues and eigenmodes that resulted from these analyses are not listed in this paper but the more significant results are discussed later. The eigenvalues are plotted in Figure 1 for several significant modes in order to show how they approach their asymptotic values as the number of degrees of freedom is increased.

Equation 26 is applied to two general problems. In the first one the λ_o and q_o^B are taken from the 16-degree-of-freedom case and the additional fixed-constraint normal modes include all those up to the 40-degree-of-freedom case. The results are included in Table II where values of λ_o, $\delta\lambda$, $\lambda_o + \delta\lambda$, and λ_{40} are listed for the first twelve modes. λ_{40} means the true eigenvalues found by direct solution of the 40-degree-of-freedom eigenvalue problem.

In the second problem the λ_o and q_o^B are taken from the 24-degree-of-freedom case. Again, the extrapolation is to 40 degrees of freedom. The results for the first eighteen modes are listed in Table III.

RESULTS AND CONCLUSIONS

It is evident from an examination of Tables II and III that the values of $\delta\lambda$ when added to the corresponding λ_o do not necessarily give a better approximation to the true eigenvalues. For many of the values listed λ_o is a better approximation to the λ for the 40-degree-of-freedom case than is $\lambda_o + \delta\lambda$. This is clearly true for those cases in which $\delta\lambda$ is negative. The emergence of negative values from the analysis is unexpected since, for all modes, the eigenvalues progressively increase as the number of degrees of freedom is increased as shown by the curves in Figure 1. However, a careful study of the numerator in Equation 26 shows that it is quite possible to obtain negative quantities; first, through the matrix H_o where negative terms are obtained if $\lambda_o < \lambda_N$ and second, through the negative elements in the mass coupling matrices M^{Bn} and M^{nB}. Therefore, it is concluded that the approximations introduced into Equation 26 through the neglect of second order terms not only may lead to appreciable errors in $\delta\lambda$ but may lead also to negative values. Furthermore, it is observed that some values of $\delta\lambda$ are so small, particularly in the lower modes, that round off errors in the computations could lead to a change of sign.

Despite the conclusion expressed in the foregoing paragraph, it appears that Equation 26 can provide a useful criterion for judging the accuracy of eigenmodes and eigenvalues. In order to clarify this point Table IV is shown in which eigenvalues λ_o and eigenmodes q_o^B are listed for several of the critical modes using the 16- and 24-degree-of-freedom results. These are compared with results from the 40-degree-of-freedom case. To supply a criterion of merit for each mode the corresponding number $\left|\frac{\delta\lambda}{\lambda_o}\right|$, the absolute value of the ratio $\frac{\delta\lambda}{\lambda_o}$, is included. In this table several modes are given beginning with mode No. 7.

It is necessary to clarify the reasons for the selection of the particular modes included in Table IV. In order to do this Table V is shown in which all of the comparable mode numbers for the 16-, 24-, 32-, and 40-degree-of-freedom cases are listed. For example, the modes bearing the number 7 are basically the same mode for all cases. On the other hand mode 13 in the 16-degree-of-freedom case is comparable to mode 15 in the 24-degree-of-freedom case, mode 16 in the 32-degree-of-freedom case, and mode 17 in the 40-degree-of-freedom case. This mode is identified as 13-15-16-17. For each case the modes are listed in the order of descending eigenvalues λ_o . In the table, modes are typified as G or L modes; these are "general" or "localized" modes, respectively. A large number of localized modes exist in which the predominant modal amplitudes are localized in a single member which vibrates at a frequency very near to its own fixed-constraint natural frequency. The members that vibrate in this manner are the relatively flexible beams 5, 6, 7 and 8 which are loosely coupled to stiffer adjacent members. The components of the vectors q_o^B are small in these modes. The general modes are those in which the amplitudes of the basic coordinates predominate. These are the modes selected for study.

Returning to Table IV the number $\left|\frac{\delta\lambda}{\lambda_o}\right|$ is seen to have a correlation with mode accuracy. In mode 7 the eigenvectors are very accurate from an engineering point of view for both the 16- and the 24-degree-of-freedom cases, improving in the latter case. The values 0.37×10^{-2} and 0.37×10^{-3} for $\left|\frac{\delta\lambda}{\lambda_o}\right|$ indicate this progressive improvement. In mode 10-11-11 the vector for the 16-degree-of-freedom case is poor and that for 24 degrees of freedom is quite accurate. The values 1.465 and 0.61×10^{-2} indicate the relative accuracies of these modes. Similar comparisons may be made for the other modes tabulated.

This example indicates that values of $\left|\frac{\delta\lambda}{\lambda_o}\right|$ of order 10^{-2} or smaller are associated with very accurate modes. Those of order 10^{-1} indicate borderline cases, while higher values indicate poor modes. Mode 12 in the 24-degree-of-freedom case is an exception inasmuch as the number 0.19 would seem to indicate a somewhat inaccurate mode whereas the comparison with the 40-degree-of-freedom case shows it to be a rather good one.

Experience indicates that the method of component mode synthesis produces very accurate lower modes and very inaccurate higher modes with a rather abrupt transition between the two groups. In a sense this is a fortunate characteristic of the method because the number $\left|\frac{\delta\lambda}{\lambda_o}\right|$ then changes abruptly from very small values to very large ones in this transition so that there are relatively few doubtful modes.

To conclude it appears that the method developed in this paper can result in a criterion which may be used to distinguish, with reasonable engineering accuracy, between accurate and inaccurate modes. More work should be done on the example treated in the paper and the method should be extended to other examples in order to further test the validity of these conclusions.

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ACKNOWLEDGEMENT

The author wishes to acknowledge the assistance of Mrs. Carolyn Level and Miss Mary Harris of Jet Propulsion Laboratory in programming and computing.

TABLE I

FIXED-CONSTRAINT EIGENVALUES FOR BEAM ELEMENTS

MEMBER	MODE 1	MODE 2	MODE 3	MODE 4
1	$.19977469 \times 10^{-2}$	$.26291317 \times 10^{-3}$	$.68410542 \times 10^{-4}$	$.25035174 \times 10^{-4}$
2	$.50556673 \times 10^{-2}$	$.12872827 \times 10^{-3}$	$.16419083 \times 10^{-4}$	$.42757627 \times 10^{-5}$
3	$.19977469 \times 10^{-2}$	$.26291317 \times 10^{-3}$	$.68410542 \times 10^{-4}$	$.25035174 \times 10^{-4}$
4	$.84060359 \times 10^{-2}$	$.11062750 \times 10^{-2}$	$.28785499 \times 10^{-3}$	$.10534194 \times 10^{-3}$
5	$.78036988 \times 10^{-3}$	$.10270045 \times 10^{-3}$	$.26722828 \times 10^{-4}$	$.97793657 \times 10^{-5}$
6	$.55492975 \times 10^{-2}$	$.73031423 \times 10^{-3}$	$.19002928 \times 10^{-3}$	$.69542164 \times 10^{-4}$
7	$.31214797 \times 10^{-2}$	$.41080178 \times 10^{-3}$	$.10689150 \times 10^{-3}$	$.39117460 \times 10^{-4}$
8	$.49943673 \times 10^{-1}$	$.65728294 \times 10^{-2}$	$.17102635 \times 10^{-2}$	$.62587936 \times 10^{-3}$

TABLE II

EIGENVALUES FROM 16-DEGREE-OF-FREEDOM PROBLEM

MODE NO.	λ_o	$\delta\lambda$	$\lambda_o + \delta\lambda$	λ_{40}	$\left\lvert \frac{\delta\lambda}{\lambda_o} \right\rvert$
1	$.54936816 \times 10^{1}$	$.89374404 \times 10^{-6}$	$.54936825 \times 10^{1}$	$.54937116 \times 10^{1}$	$.1626 \times 10^{-6}$
2	$.11915032 \times 10^{1}$	$-.18035796 \times 10^{-5}$	$.11915013 \times 10^{1}$	$.11915047 \times 10^{1}$	$.1514 \times 10^{-5}$
3	$.57737046 \times 10^{0}$	$-.45010962 \times 10^{-5}$	$.57736596 \times 10^{0}$	$.57740881 \times 10^{0}$	$.7796 \times 10^{-5}$
4	$.49986493 \times 10^{-1}$	$.89677151 \times 10^{-5}$	$.4999546 \times 10^{-1}$	$.49986915 \times 10^{-1}$	$.1794 \times 10^{-3}$
5	$.40372534 \times 10^{-1}$	$.23265796 \times 10^{-3}$	$.40139876 \times 10^{-1}$	$.40381679 \times 10^{-1}$	$.5762 \times 10^{-2}$
6	$.13422468 \times 10^{-1}$	$.74899705 \times 10^{-4}$	$.13497368 \times 10^{-1}$	$.13436030 \times 10^{-1}$	$.5580 \times 10^{-2}$
7	$.77520230 \times 10^{-2}$	$-.28900896 \times 10^{-4}$	$.7723122 \times 10^{-2}$	$.77800668 \times 10^{-2}$	$.3728 \times 10^{-2}$
8	$.55554979 \times 10^{-2}$	$.87011445 \times 10^{-10}$	$.5555498 \times 10^{-2}$	$.55586010 \times 10^{-2}$	$.1566 \times 10^{-7}$
9	$.31148230 \times 10^{-2}$	$.58281787 \times 10^{-5}$	$.31206511 \times 10^{-2}$	$.31148910 \times 10^{-2}$	$.1871 \times 10^{-2}$
10	$.17961570 \times 10^{-2}$	$.26316675 \times 10^{-2}$	$.44278245 \times 10^{-2}$	$.26627479 \times 10^{-2}$	$.1465 \times 10^{1}$
11	$.13789289 \times 10^{-2}$	$-.42962576 \times 10^{-2}$	$-.29173286 \times 10^{-2}$	$.15458259 \times 10^{-2}$	$.3116 \times 10^{1}$
12	$.77916999 \times 10^{-3}$	$-.23303676 \times 10^{-9}$	$.77916975 \times 10^{-3}$	$.78004726 \times 10^{-3}$	$.2991 \times 10^{-6}$

TABLE III

EIGENVALUES FROM 24-DEGREE-OF-FREEDOM PROBLEM

MODE NO.	λ_o	$\delta\lambda$	$\lambda_o + \delta\lambda$	λ_{40}	$\left\lvert \frac{\delta\lambda}{\lambda_o} \right\rvert$
1	$.54937066 \times 10^{1}$	$-.26075286 \times 10^{-6}$	$.54937063 \times 10^{1}$	$.54937116 \times 10^{1}$	$+.47463922 \times 10^{-7}$
2	$.11915044 \times 10^{1}$	$-.22912894 \times 10^{-8}$	$.11915043 \times 10^{1}$	$.11915047 \times 10^{1}$	$+.19230222 \times 10^{-8}$
3	$.57740236 \times 10^{0}$	$.41153074 \times 10^{-6}$	$.57740277 \times 10^{0}$	$.57740881 \times 10^{0}$	$.71272785 \times 10^{-6}$
4	$.49986646 \times 10^{-1}$	$.58900469 \times 10^{-6}$	$.49987234 \times 10^{-1}$	$.49986916 \times 10^{-1}$	$.11783241 \times 10^{-4}$
5	$.40376684 \times 10^{-1}$	$.40264720 \times 10^{-4}$	$.40416948 \times 10^{-1}$	$.40381680 \times 10^{-1}$	$.99722701 \times 10^{-3}$
6	$.13433935 \times 10^{-1}$	$.40327840 \times 10^{-4}$	$.13474263 \times 10^{-1}$	$.13436030 \times 10^{-1}$	$.30019380 \times 10^{-2}$
7	$.77776890 \times 10^{-2}$	$.29288290 \times 10^{-5}$	$.77806177 \times 10^{-2}$	$.77800668 \times 10^{-2}$	$.37656803 \times 10^{-3}$
8	$.65585969 \times 10^{-2}$	$.28436215 \times 10^{-9}$	$.65585972 \times 10^{-2}$	$.65586195 \times 10^{-2}$	$.43357162 \times 10^{-7}$
9	$.55585479 \times 10^{-2}$	$.15083896 \times 10^{-9}$	$.55585480 \times 10^{-2}$	$.55586011 \times 10^{-2}$	$.27136397 \times 10^{-7}$
10	$.31148870 \times 10^{-2}$	$-.12785558 \times 10^{-4}$	$.31021014 \times 10^{-2}$	$.31148910 \times 10^{-2}$	$+.41046619 \times 10^{-2}$
11	$.26581000 \times 10^{-2}$	$-.16343590 \times 10^{-4}$	$.26417563 \times 10^{-2}$	$.26627479 \times 10^{-2}$	$+.61485986 \times 10^{-2}$
12	$.15414339 \times 10^{-2}$	$-.29350459 \times 10^{-3}$	$.12479293 \times 10^{-2}$	$.15458259 \times 10^{-2}$	$+.19041010 \times 10^{0}$
13	$.77999099 \times 10^{-3}$	$-.52016942 \times 10^{-12}$	$.77999099 \times 10^{-3}$	$.78004726 \times 10^{-3}$	$+.66689158 \times 10^{-9}$
14	$.73227999 \times 10^{-3}$	$.64816619 \times 10^{-9}$	$.73228063 \times 10^{-3}$	$.73346655 \times 10^{-3}$	$.88513437 \times 10^{-6}$
15	$.58119699 \times 10^{-3}$	$-.29883504 \times 10^{-4}$	$.55131348 \times 10^{-3}$	$.58885463 \times 10^{-3}$	$+.51417169 \times 10^{-1}$
16	$.41061000 \times 10^{-3}$	$-.31607750 \times 10^{-11}$	$.41060999 \times 10^{-3}$	$.41061723 \times 10^{-3}$	$+.76977546 \times 10^{-8}$
17	$.34823100 \times 10^{-3}$	$.26235972 \times 10^{-4}$	$.37446696 \times 10^{-3}$	$.51106047 \times 10^{-3}$	$.75340713 \times 10^{-1}$
18	$.27207699 \times 10^{-3}$	$-.31752962 \times 10^{-2}$	$-.2903219s \times 10^{-2}$	$.34778677 \times 10^{-3}$	$+.11670580 \times 10^{2}$

TABLE IV

COMPARISON OF MODES

D.O.F.	16	24	40
MODE NO.	7	7	7
q_1	+.00287	.00324	.00321
q_2	+.00023	.00060	.00057
q_3	-.52524	-.51918	-.51875
q_4	+1.67437	1.65799	1.65685
q_5	+1.76997	1.81293	1.81341
q_6	-2.96549	-2.91038	-2.90910
q_7	+1.34514	1.30987	1.30998
q_8	+2.52851	2.56615	2.56597
λ_o	.007752	.007777	.007780
$\lvert\delta\lambda/\lambda_o\rvert$	.003728	.0003766	

D.O.F.	16	24	40
MODE NO.	10	11	11
q_1	.001306	-.00250	-.001516
q_2	.001518	-.00288	-.001906
q_3	-.35578	.03621	.036705
q_4	-1.01173	-.46122	-.46107
q_5	7.47475	6.17288	6.16907
q_6	6.67297	6.27754	6.27528
q_7	-7.01332	-6.14875	-6.14099
q_8	7.80957	6.06620	6.05723
λ_o	.001796	.002658	.002662
$\lvert\delta\lambda/\lambda_o\rvert$	1.465	.006111	

D.O.F.	16	24	40
MODE NO.	11	12	13
q_1	-.00340	-.00065	.00063
q_2	-.00253	.00011	.00138
q_3	-.72961	-.75978	-.75267
q_4	-.97624	-1.32521	-1.32263
q_5	-7.61991	-1.80925	-1.73934
q_6	-8.83026	-3.06402	-2.98507
q_7	8.36817	2.63315	2.68793
q_8	-6.54993	-.62956	-.70532
λ_o	.001379	.001541	.001546
$\lvert\delta\lambda/\lambda_o\rvert$	3.116	.1904	

TABLE IV (Cont.)

COMPARISON OF MODES

D.O.F.	16	24	40
MODE NO.	13	15	17
q_1	.00095	.00180	.00188
q_2	-.00055	-.00289	-.00293
q_3	.42496	.40418	.33650
q_4	.01765	-.07385	-.06214
q_5	-9.99031	-4.88114	-6.68466
q_6	5.62477	5.06781	2.65511
q_7	2.83647	3.38714	1.29971
q_8	11.86812	5.27840	6.98300
λ_0	.0003554	.0005812	.0005889
$\lvert\delta\lambda/\lambda_0\rvert$		.05142	

D.O.F.	16	24	40
MODE NO.	14	17	18
q_1	.00314	-.00006	.00137
q_2	.00306	.00128	.00078
q_3	.32670	-.20524	.27918
q_4	.04481	-7.73480	-.23594
q_5	17.71465	2.17839	7.90101
q_6	21.94244	.94535	11.27086
q_7	20.21166	1.41851	10.14683
q_8	-16.36834	-2.25961	-7.54563
λ_0	.0002736	.0003482	.0005111
$\lvert\delta\lambda/\delta_0\rvert$		.07534	

D.O.F.	16	24	40
MODE NO.	15	18	20
q_1	-.00129	.00331	-.00050
q_2	-.00175	.00318	.00091
q_3	-.03785	.19184	-.23985
q_4	-18.66878	1.50445	-7.86021
q_5	-.01539	18.94000	-.28916
q_6	-.24330	19.82006	-1.63468
q_7	-.14071	19.14806	-1.22981
q_8	-.15165	-18.48168	.20332
λ_0	.0001659	.0002721	.0003478
$\lvert\delta\lambda/\lambda_0\rvert$		11.67	

TABLE V

COMPARISON OF MODE NUMBERS

DEGREES OF FREEDOM

TYPE	16	24	32	40
G	1	1	1	1
G	2	2	2	2
G	3	3	3	3
L	4	4	4	4
G	5	5	5	5
G	6	6	6	6
G	7	7	7	7
L		8	8	8
L	8	9	9	9
L	9	10	10	10
G	10	11	11	11
L			12	12
G	11	12	13	13
L	12	13	14	14
L		14	15	15
L				16
G	13	15	16	17
G	14	17	17	18
L		16	18	19
G	15	18	19	20
L			20	21
G	16	19	21	22
G		21	22	23
G		22	25	24
L			23	25
L		20	24	26
L				27
G		23	26	28
L				29
G		24	27	30
G			29	31
G			30	32
L			28	33
G			31	34
G			32	35
L				36
G				37
G				38
G				39
G				40

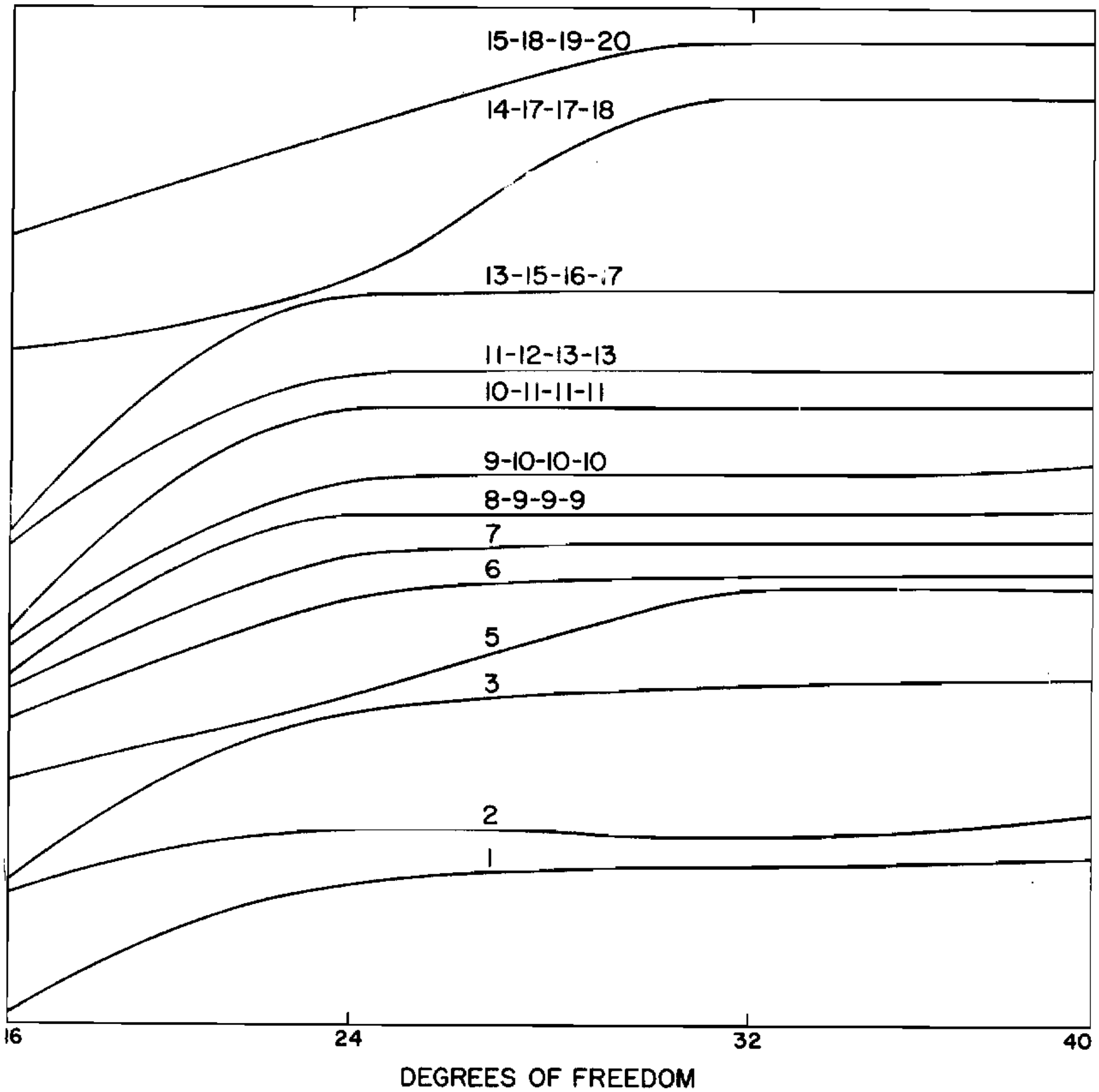

Figure 1. Eigenvalues Plotted Against Degrees of Freedom

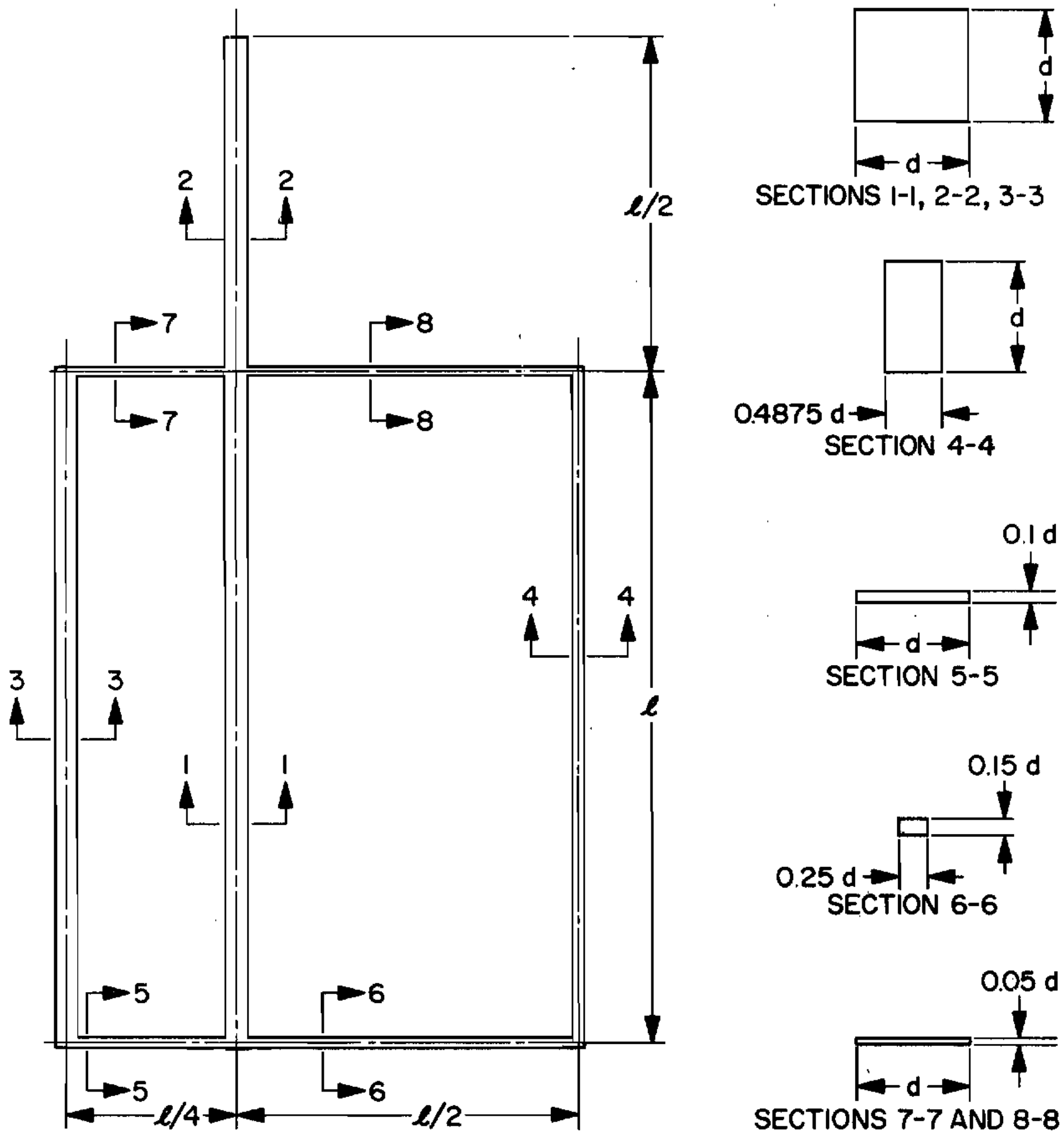

Figure 2. Frame Structure Treated in Example

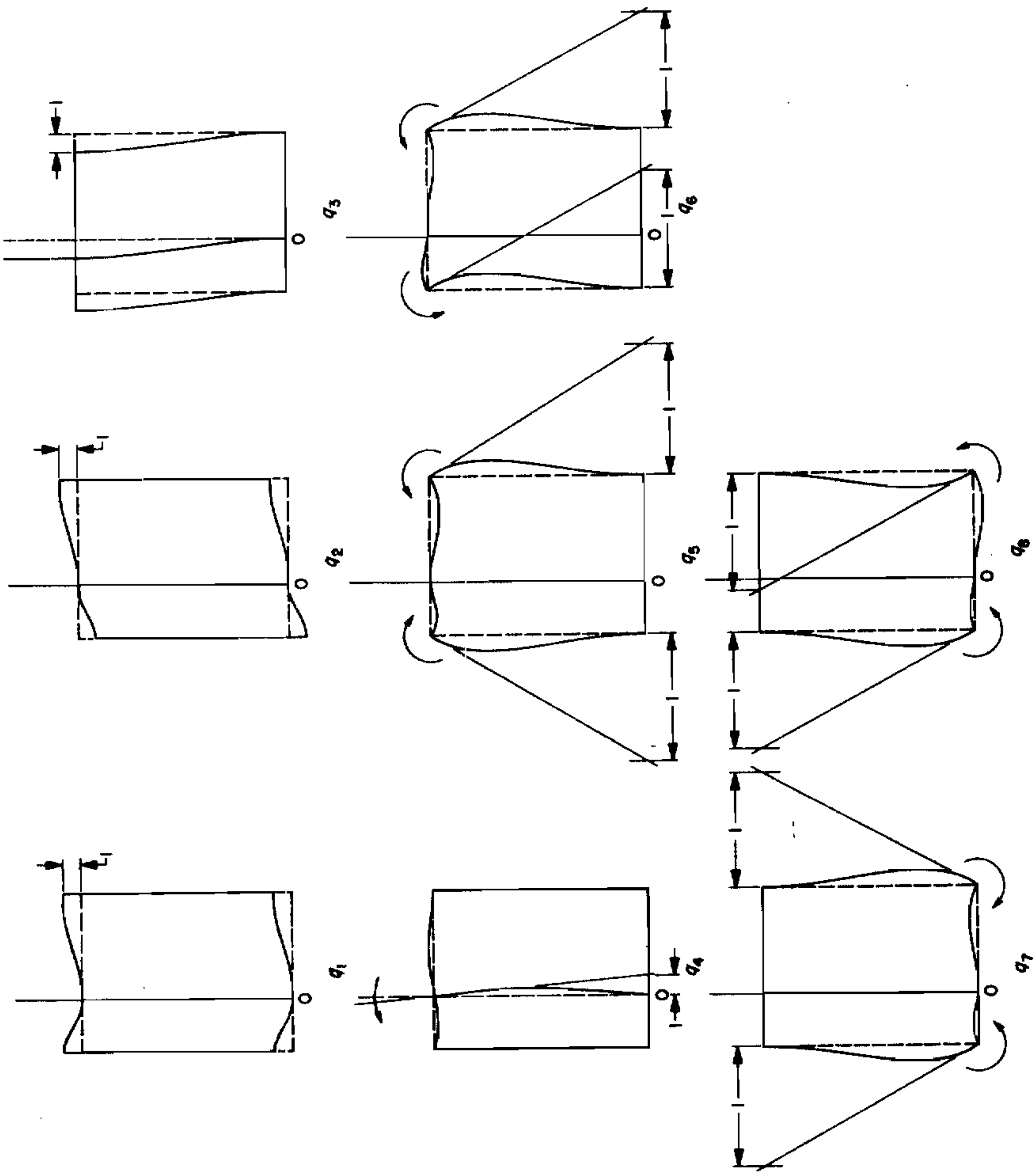

Figure 3. Generalized Basic Displacement Coordinates