BASIS FOR ELEMENT INTERCHANGEABILITY IN FINITE ELEMENT PROGRAMS

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The concept of characteristic matrices of any finite element for elastostatic analysis is presented. To demonstrate the concept the characteristic matrices of a simple beam element are derived in the Appendix. This approach permits complete interchangeability of different types of elements between independently developed analysis programs irrespective of the analysis method or basic element assumption. The analogy between element derivation procedures and the equality of principal matrices are developed. A wide spread adoption of this concept would have far reaching implications in, facilitating communications, exchanging of new developments, and speed-up practical applications in the field of finite elements.

INTRODUCTION

The element research and development over the past decade has been largely restricted to element stiffness matrices based on deformation assumptions. However, over the past few years element flexibility matrices based on stress assumptions have appeared with increasing frequency in the published literature. It is therefore appropriate to show the analogy between element derivation procedures and matrices and their interchangeability in the force, displacement and combined methods of analysis.

The data required to incorporate any element into an elastostatic analysis system, force, displacement or combined method, can be conveniently organized into four characteristic matrices. These matrices define the elastic behavior, the spatial assembly into the structure, and the required output information for each element. This concept was used for the force method. [1,2] Further characteristic matrices appear when extending the concept to vibration and non-linear analysis. The method of element development is independent of the method used to form the equations for the complete structure. Certain analogies which exist in the finite element methods, including methodology and elements, have been reported. [3,4,5,6,7,8] In the present paper an analogy is developed which covers the element derivation process for both stress and strain assumptions within the element. The formal presentation of this analogy demonstrates that it is possible to achieve complete interchangeability of the characteristic matrices of any type of element between the major methods of finite element analysis. This work complements the review of methodology relationships in Reference 2.

1. RELATIONS BETWEEN FINITE ELEMENT METHODS

Three basic finite element methods of analysis are usually distinguished, namely, force, displacement and combined (Lagrangian Multiplier) method. In the force method the nodal equilibrium equations are written in terms of the generalized element forces. These forces are solved first and then the nodal displacements are determined. In the displacement method the nodal equilibrium equations are written in terms of the generalized nodal displacements. These displacements are solved first and then the element forces are determined. In the combined method the system of equations to be solved contains both the element forces and nodal displacements simultaneously as one set of unknown variables. The full solution is obtained in one pass. A full derivation of the relations between the three methods was given in Reference 2. The equations relevant to the present work will be summarized subsequently.

In the force method the nodal equilibrium equations, in all unconstrained degrees of freedom, are expressed in terms of the assembled generalized element for variables, $\{F\}$, that is,

$$[E]\{F\} = \{L\} \tag{1}$$

where

$\{L\}$ = vector of external applied generalized nodal forces expressed in the global system.

$[E]$ = matrix which transforms the element force variables into equivalent nodal forces in the global system and allocates them to the appropriate node.

The total element deformations, $\{d\}$, corresponding to the element force variables and initial deformations, are given by

$$\{d\} = \{d_e\} + \{d_o\} = [D]\{F\} + \{d_o\} \tag{2}$$

where

$[D]$ = matrix of assembled element natural flexibility matrices.

$\{d_o\}$ = initial, stress-free element deformations (thermal, etc.)

$\{d_e\}$ = element stress-related deformations.

In the displacement method of analysis the nodal equilibrium equations, in all unconstrained degrees of freedom, are expressed in terms of the nodal displacements, $\{r\}$, expressed in the global system, that is,

$$[K]\{r\} = \{L\} + [a]^T\{F_o\} \tag{3}$$

where

$[K]$ = structural stiffness matrix which transforms the nodal displacements into equivalent nodal forces in the global system and allocates them to the appropriate node.

$\{F_o\}$ = initial element force variables due to initial strains

The element forces are given by

$$\{F\} = [k]\{d_e\} \tag{4}$$

where,

$[k]$ = matrix of assembled element non-singular stiffness matrices.

The continuity conditions, relating the independent element deformations to nodal displacements are given by

$$[a]\{r\} = \{d\} = \{d_e\} + \{d_o\} \tag{5}$$

where,

$[a]$ = transformation matrix relating the nodal displacements and the total independent element deformations $\{d\}$.

In References 2 and 9 it was shown that

$$[a] = [E]^T \tag{6}$$

and

$$[k] = [D]^{-1} \tag{7}$$

The structural stiffness matrix is given by,

$$[K] = [a]^T[k][a] = [E][D]^{-1}[E]^T \tag{8}$$

Substituting Equation (6) into (5) gives,

$$\{d_e\} = [E]^T\{r\} - \{d_o\} \tag{9}$$

Using Equation (2) and (9)

$$[D]\{F\} + \{d_o\} - [E]^T\{r\} = \{0\} \tag{10}$$

Writing Equations (1) and (10) into a single system yields the Combined Method.[2]

$$\left[\begin{array}{c|c} D & -E^T \\ \hline -E & 0 \end{array}\right]\left[\begin{array}{c} F \\ \hline r \end{array}\right] = -\left[\begin{array}{c} d_o \\ \hline L \end{array}\right] \tag{11}$$

or, in a slightly different form, using Equations (6) and (7),

$$\left[\begin{array}{c|c} I & -ka \\ \hline -a^T & 0 \end{array}\right]\left[\begin{array}{c} F \\ \hline r \end{array}\right] = -\left[\begin{array}{c} kd_o \\ \hline L \end{array}\right] \tag{12}$$

In the combined method the forces and nodal displacements are evaluated simultaneously in one pass.

2. CONCEPT OF CHARACTERISTIC ELEMENT MATRICES

Following this preliminary review of finite element methods, the paper will now concentrate on the common aspects of element derivation and formulation, using both strain and stress fields as the basic assumptions.

It will be shown that a set of characteristic matrices is required to incorporate any type of finite (structural) element into an analysis system based on either the force, displacement or combined method. This concept was discussed for the force method in References 1 and 2. The work of many authors has contributed to the formulation of this unified approach.

Characteristic matrices for elastostatic analysis are developed in this paper, namely;

1. The Element Natural Elastic Matrix (Stiffness and Flexibility).
2. The Initial Stress-Free Deformation Vector (Thermal).
3. The Element Assembly Matrix.
4. The Element Output Matrix.

Further characteristic matrices, to be considered in later work, are the Consistent Element Mass [16] and Inverse Mass [17,18] Matrices for vibration analysis, and the Geometric Stiffness Matrices [16] for various types of large order deflection analysis.

For any element, regardless of the analysis method, a force-deformation relationship is needed, which leads to the first two characteristic matrices, that is, the element natural elastic matrix and initial stress-free deformation vector. This relationship can be written in either of the following two forms,

$$\{F_m\} = [k_m]\{d_m\} - \{F_{mo}\} \tag{13}$$

or

$$\{d_m\} = [D_m]\{F_m\} + \{d_{mo}\} \tag{14}$$

where

$[k_m]$ = element natural stiffness matrix (non-singular)

$[D_m]$ = element natural flexibility matrix (non-singular)

$\{F_m\}$ = independent generalized elastic forces for element m.

$\{d_m\}$ = independent generalized deformations for element m

$\{F_{mo}\},\{d_{mo}\}$ = initial values of force and deformations.

The matrices $[k]$ and $[D]$ of Equations (7) and (2) are assembled from the element natural stiffness and flexibility matrices for all elements, respectively.

It is important for this parallel study of strain and stress elements and their interchangeability, that the generalized element deformation and force variables, $\{d_m\}$ and $\{F_m\}$ respectively, do not include the zero strain and zero stress states, and are thus independent quantities. The relationships of Equations (13) and (14) are derived in a local reference system for the element. The choice of this system is important in two ways; not only can this simplify the derivation but it usually helps the understanding of the physics of the element. The variables and reference system should represent the behavior of the element in a realistic 'natural' way. This philosophy has long been expounded by Professor Argyris.

The use of independent variables is the essential requirement for the present issue of interchangeability. Equation (13) is the form to be used in the displacement method, and Equation (14) that in the force method. Either relation can however, be derived based on strain or on stress assumptions.

It should be realized that if the element deformations $\{d_m\}$ are chosen as independent variables in Equations (13) and (14), then $\{F_m\}$ are forces corresponding to these deformations (in character and "line of action"), and vice versa.

The force-deformation relations for any type of element will be derived using virtual principles and certain basic assumptions, namely:

1. The assumption of a deformation or strain field within the element, which must satisfy continuity, but the corresponding stresses do no necessarily satisfy equilibrium. This is the basis of strain, displacement, compatible and Hybrid II type elements.
2. The assumption of a force or stress field within the element, which must satisfy equilibrium, but the corresponding strains will not necessarily satisfy continuity. This is the basis of stress, force, equilibrium and Hybrid I type elements.

Regardless of which assumption is made, the resulting force-deformation relation can be used for any of the finite element methods.

The <u>third characteristic matrix</u> is the element assembly matrix

To assemble the structural stiffness matrix in the displacement method the independent strain variables $\{d_m\}$ for each element have to be transformed into equivalent nodal displacements $\{\Delta_m\}$ in the global system, including rigid body freedoms, using the transformation (assembly) matrix $[a_m]$, that is,

$$\{d_m\} = [a_m]\{\Delta_m\} \tag{15}$$

To assemble the equilibrium equations in the force method the independent force variables $\{F_m\}$ for each element have to be transformed into equivalent nodal forces $\{Q_m\}$ in the global system, including the dependent forces (static supports), using the transformation (assembly) matrix $[E_m]$, that is,

$$[E_m]\{F_m\} = \{Q_m\} \tag{16}$$

The fourth characteristic matrix is the element output matrix.

In all methods of analysis the output data include generalized stress and/or strain quantities within each discrete element, it is therefore appropriate and convenient to define an element output matrix, $[S_m]$. This will relate the independent variables to the required output information which is suitable for structural engineering design.

3. THE STRAIN ELEMENT

3.1 FORCE-DEFORMATION RELATION

A generalized total strain field, $\{\varepsilon_\varepsilon\}$, is expressed in terms of a set of independent deformation variables, $\{d_{m\varepsilon}\}$, by the equation

$$\{\varepsilon_\varepsilon\} = [T_{\varepsilon d}]\{d_{m\varepsilon}\} \tag{17}$$

where the j^{th} column of $[T_{\varepsilon d}]$ gives the strain distribution, which satisfies compatibility, for a unit value of $d_{m\varepsilon j}$. The set of deformation variables does not contain the zero strain state (rigid body displacements) as is the case in many displacement element derivations.

To obtain the strain field in terms of the element deformation variables it is usual to assume displacement functions expressed by the following equation:

$$\{u\} = [T_{uB}]\{B\} \tag{18}$$

where,

$\{u\}$ = generalized displacements in the local axes system.

$\{B\}$ = function coefficients, the linear terms (such as, b_1, b_2x, b_3y) correspond to the zero strain state.

Appropriate differentiation of these functions, see Table 1, results in the total strain field of Equation (17). The deformation variables, $\{d_{m\varepsilon}\}$, and the coefficients $\{B\}$ are directly related by the equation

$$\{d_{m\varepsilon}\} = [T_{dB}]\{B\} \tag{19}$$

Applying the principle of virtual deformations to the j^{th} deformation variable establishes the relation

$$\tilde{d}_{m\varepsilon j}\, F_{m\varepsilon j} = \int_V \{\tilde{\varepsilon}_\varepsilon\}_j^T \{\sigma\}\, dV \tag{20}$$

where,

$d_{m\varepsilon j}$ = the j^{th} independent element deformation variable.

$\{\varepsilon_\varepsilon\}_j = f(d_{m\varepsilon j})$, the corresponding j^{th} total strain field.

$F_{m\varepsilon j}$ = element force corresponding to $d_{m\varepsilon j}$

$\{\sigma\}$ = function of $\{\varepsilon_e\} = \{\varepsilon_\varepsilon\} - \{\varepsilon_o\}$

V = Volume of element

$\sim$ indicates virtual quantities

Subscript ε indicates quantities derived from strain assumptions.

If there are α independent deformation variables then there are α linear equations in the form of Equation (20), therefore the following system of equations results,

$$\left\lceil \tilde{d}_{m\varepsilon} \right\rfloor \{F_{m\varepsilon}\} = \int_V [\tilde{\varepsilon}_\varepsilon]^T \{\sigma\}\, dV \tag{21}$$

where $\left\lceil \tilde{d}_{m\varepsilon} \right\rfloor$ is a diagonal matrix of virtual deformation variables.

Using Equation (17), the complete set of virtual strain systems is given by,

$$[\tilde{\varepsilon}_\varepsilon] = [T_{\varepsilon d}]\left\lceil \tilde{d}_{m\varepsilon} \right\rfloor \tag{22}$$

Substituting Equation (22) into (21) gives,

$$\left\lceil \tilde{d}_{m\varepsilon} \right\rfloor \{F_{m\varepsilon}\} = \left\lceil \tilde{d}_{m\varepsilon} \right\rfloor \int_V [T_{\varepsilon d}]^T \{\sigma\}\, dV \tag{23}$$

or,

$$\{F_{m\varepsilon}\} = \int_V [T_{\varepsilon d}]^T \{\sigma\}\, dV \tag{24}$$

Equation (24) defines the true element force $F_{m\varepsilon j}$ corresponding to the chosen deformation variable $d_{m\varepsilon j}$, such that the product $d_{m\varepsilon j} . F_{m\varepsilon j}$ is the work done by the j^{th} deformation.

The stresses $\{\sigma\}$ are a function of the stress-related strains $\{\varepsilon_e\}$ which are the difference between total and initial strains, that is,

$$\{\sigma\} = \{\sigma(\varepsilon_e)\} = \{\sigma(\varepsilon_\varepsilon - \varepsilon_o)\} \tag{25}$$

Refer to Figure 1.

For linear stress-strain behavior, the stresses are,

$$\{\sigma\} = [T_{\sigma\varepsilon}]\{\varepsilon_e\} = [T_{\sigma\varepsilon}](\{\varepsilon_\varepsilon\} - \{\varepsilon_o\}) \tag{26}$$

where, $[T_{\sigma\varepsilon}]$ expresses the classical stress-strain relations.

Therefore, substituting Equations (17) and (26) into (24), yields

$$\{F_{m\varepsilon}\} = \left(\int_V [T_{\varepsilon d}]^T [T_{\sigma\varepsilon}][T_{\varepsilon d}]\, dV\right)\{d_{m\varepsilon}\} - \int_V [T_{\varepsilon d}]^T [T_{\sigma\varepsilon}]\{\varepsilon_o\}\, dV \tag{27}$$

or

$$\{F_{m\varepsilon}\} = [k_{m\varepsilon}]\{d_{m\varepsilon}\} - \{F_{m\varepsilon o}\} \tag{28}$$

where

$$\{F_{m\varepsilon o}\} = \int_V [T_{\varepsilon d}]^T [T_{\sigma\varepsilon}]\{\varepsilon_o\}\, dV \tag{29}$$

are the initial (fictitious) element forces which would produce the same element deformations as the stress-free initial strains $\{\varepsilon_o\}$ (thermal) and,

$$[k_{m\varepsilon}] = \int_V [T_{\varepsilon d}]^T [T_{\sigma\varepsilon}][T_{\varepsilon d}]\, dV \tag{30}$$

is the natural stiffness matrix based on strain assumptions.

This natural stiffness matrix, as well as the initial forces, are for a strain (displacement) element as well as a hybrid element based on an internal strain field.

To obtain the external initial element deformations $\{d_{m\varepsilon o}\}$ due to the initial strains $\{\varepsilon_o\}$, it is only necessary to visualize that the same deformations could be produced elastically, thus the corresponding forces are,

$$\{F_{m\varepsilon o}\} = [k_{m\varepsilon}]\{d_{m\varepsilon o}\} \tag{31}$$

Hence,

$$\{d_{m\varepsilon o}\} = [k_{m\varepsilon}]^{-1}\{F_{m\varepsilon o}\} \tag{32}$$

where $\{F_{m\varepsilon o}\}$ is given by Equation (29) and $[k_{m\varepsilon}]$ by Equation (30). In Equation (29) the function for $\{\varepsilon_o\}$ is not specified, and may actually be given numerically. $\{d_{m\varepsilon o}\}$ is the initial stress free deformation vector (second characteristic matrix).

Using Equation (32), Equation (28) can now be written as,

$$\{F_{m\varepsilon}\} = [k_{m\varepsilon}]\Big(\{d_{m\varepsilon}\} - \{d_{m\varepsilon o}\}\Big) = [k_{m\varepsilon}]\{d_{m\varepsilon e}\} \tag{33}$$

where $\{d_{m\varepsilon e}\}$ are the stress related deformations.

3.2 STRAIN ELEMENT ASSEMBLY MATRIX

The element nodal displacements in the local system, $\{\delta_m\}$, including rigid body freedoms, can be expressed in terms of the coefficients $\{B\}$ by substituting the proper nodal coordinates into Equation (18), that is,

$$\{\delta_m\} = [T_{\delta B}]\{B\} \tag{34}$$

Solving for $\{B\}$ gives,

$$\{B\} = [T_{\delta B}]^{-1}\{\delta_m\} \tag{35}$$

Substituting from Equation (35) into (19) results in,

$$\{d_{m\varepsilon}\} = [T_{d\delta\varepsilon}]\{\delta_m\} \tag{36}$$

where

$$[T_{d\delta\varepsilon}] = [T_{dB}][T_{\delta B}]^{-1} \tag{37}$$

= element displacement assembly matrix in the local system based on strain assumptions.

The relation between the corresponding force variables and nodal forces $\{q_m\}$ in the local element system, including dependent nodal forces (static supports), can be found by applying the principle of virtual displacements. Therefore,

$$[\tilde{\delta}_m]\{q_m\} = [\tilde{d}_{m\varepsilon}]^T\{F_{m\varepsilon}\} \tag{38}$$

From Equation (36)

$$[\tilde{d}_{m\varepsilon}] = [T_{d\delta\varepsilon}][\tilde{\delta}_m] \tag{39}$$

Hence, substituting from Equation (39) into (38)

$$\{q_m\} = [T_{qF\varepsilon}]\{F_{m\varepsilon}\} \tag{40}$$

where,

$$[T_{qF\varepsilon}] = [T_{d\delta\varepsilon}]^T \tag{41}$$

= element force assembly matrix in the local system based on strain assumptions.

When assembling the elements,the nodal forces and displacements for each element are transformed into the global system, denoted by $\{Q_m\}$ and $\{\Delta_m\}$ respectively, using a simple coordinate transformation matrix, $[T_c]$, that is,

$$\{Q_m\} = [T_c]\{q_m\} \tag{42}$$

$$\{q_m\} = [T_c]^T\{Q_m\} \tag{43}$$

and similarly,

$$\{\Delta_m\} = [T_c]\{\delta_m\} \tag{44}$$

$$\{\delta_m\} = [T_c]^T\{\Delta_m\} \tag{45}$$

Therefore, using the coordinate transformations of Equations (45) and (42) in Equations (36) and (40), respectively,

$$\{d_{m\varepsilon}\} = [a_{m\varepsilon}]\{\Delta_m\} \tag{46}$$

where,

$$[a_{m\varepsilon}] = [T_{d\delta\varepsilon}][T_c]^T \tag{47}$$

= element displacement assembly matrix in the global system.

and

$$\{Q_m\} = [E_{m\varepsilon}]\{F_{m\varepsilon}\} \tag{48}$$

where

$$[E_{m\varepsilon}] = [T_c][T_{qF\varepsilon}] = [a_{m\varepsilon}]^T \tag{49}$$

= element force assembly matrix in the global system.

3.3 STRAIN ELEMENT OUTPUT MATRIX

In a strain element the corresponding stress field is defined by Equation (26), that is,

$$\{\sigma_e\} = [T_{\sigma\varepsilon}]\{\varepsilon_e\} = [T_{\sigma\varepsilon}]\left(\{\varepsilon_\varepsilon\} - \{\varepsilon_o\}\right) \tag{50}$$

Substituting from Equation (17) into (50) gives,

$$\{\sigma_e\} = [T_{\sigma\varepsilon}]\left([T_{\varepsilon d}]\{d_{m\varepsilon}\} - \{\varepsilon_o\}\right) \tag{51}$$

It appears however that the real stress distribution in the element cannot be dependent on the quite arbitrary function of $\{\varepsilon_o\}$, but rather more appropriately on the element forces given by Equation (33).

In the case of a simple axial element the stress can only be constant, irrespective of the complexity of the initial strain distribution $\{\varepsilon_o\}$. This is contrary to Equation (51). This equation, though mathematically derived from previous relationships, gives a result which is inconsistent in a physical sense. The stresses can only be those which are consistent with the element forces. The elastic deformations are given by,

$$\{d_{m\varepsilon e}\} = \{d_{m\varepsilon}\} - \{d_{m\varepsilon o}\} \tag{52}$$

Where the deformation quantities are defined by Equations (32) and (46). Therefore, using the same transformation as in Equation (17) to give the elastic strains, that is,

$$\{\varepsilon_e\} = [T_{\varepsilon d}]\{d_{m\varepsilon e}\} \tag{53}$$

the stress distribution within the element is given by

$$\{\sigma_e\} = [T_{\sigma\varepsilon}]\{\varepsilon_e\} = [T_{\sigma\varepsilon}][T_{\varepsilon d}]\{d_{m\varepsilon e}\} \tag{54}$$

In the displacement method programs the stress output is evaluated using the computed element elastic deformations. Therefore, based on the specific output requirements and using Equation (54), the stress output is given by,

$$\{\sigma_{m\varepsilon}\} = [S_{m\varepsilon}]\{d_{m\varepsilon e}\} \tag{55}$$

where,

$\{\sigma_{m\varepsilon}\}$ = vector of required output data for a strain element

$[S_{m\varepsilon}]$ = element output matrix for a strain element.

From Equation (33),

$$\{F_{m\varepsilon}\} = [k_{m\varepsilon}]\left(\{d_{m\varepsilon}\} - \{d_{m\varepsilon o}\}\right)$$

or

$$\{F_{m\varepsilon}\} = [k_{m\varepsilon}]\{d_{m\varepsilon e}\} \tag{56}$$

Hence, from Equation (56)

$$\{d_{m\varepsilon e}\} = [k_{m\varepsilon}]^{-1}\{F_{m\varepsilon}\} \tag{57}$$

Substituting from Equation (57) into (55) gives the stress ouput in terms of force variables, that is,

$$\{\sigma_{m\varepsilon}\} = [S_{m\varepsilon}][k_{m\varepsilon}]^{-1}\{F_{m\varepsilon}\} \tag{58}$$

The element output matrix will now be denoted by $[S_{m\varepsilon}]$ with the appropriate subscripts of F or D to denote in which method it is to be used (force or displacement.

Therefore, for the displacement method, Equation (55),

$$[S_{m\varepsilon D}] = [S_{m\varepsilon}] \tag{59}$$

and the force method, Equation (58),

$$[S_{m\varepsilon F}] = [S_{m\varepsilon}][k_{m\varepsilon}]^{-1} \tag{60}$$

The relationship between the two forms of output matrices is given by comparison of Equations (59) and (60), that is,

$$[S_{m\varepsilon F}] = [S_{m\varepsilon D}][k_{m\varepsilon}]^{-1} \tag{61}$$

In the case of the combined method the form of the output matrix can be either of the two forms depending on the choice made by the individual developing the program.

4. THE STRESS ELEMENT

4.1 FORCE-DEFORMATION RELATION

A generalized total stress field, $\{\sigma_\sigma\}$, is expressed in terms of a set of independent force variables, $\{F_{m\sigma}\}$, in the form

$$\{\sigma_\sigma\} = [T_{\sigma F}]\{F_{m\sigma}\} \tag{62}$$

where, the j^{th} column of $[T_{\sigma F}]$ gives the stress distribution, which must satisfy equilibrium, for a unit value of $F_{m\sigma j}$. The set of force variables does not contain the zero stress state. Such a generalized stress-field can be assumed directly, subject to the constraint that it satisfies stress-equilibrium conditions. Alternatively such a stress field can be obtained by assuming stress-functions $\{\Phi\}$, that is,

$$\{\Phi\} = [T_{\Phi B}]\{B\} \tag{63}$$

where

$\{B\}$ = function coefficients, the linear terms (such as, b_1, b_2x, b_3y) correspond to the zero stress state.

The stress field (which satisfies equilibrium) is then obtained by appropriate differentiation of the stress functions as indicated in Table 1.

The independent force variables, $\{F_{m\sigma}\}$, and the coefficients, $\{B\}$, are then directly related by,

$$\{F_{m\sigma}\} = [T_{FB}]\{B\} \tag{64}$$

Applying the principle of virtual forces to the j^{th} force variable establishes the relation

$$\tilde{F}_{m\sigma_j}\, d_{m\sigma j} = \int_V \{\tilde{\sigma}_\sigma\}_j^T \{\varepsilon\}\, dV \tag{65}$$

where

$F_{m\sigma j}$ = the j^{th} independent element force variable

$\{\sigma_\sigma\}_j = f(F_{m\sigma j}\}$ the corresponding j^{th}, stress field

$d_{m\sigma j}$ = element deformation corresponding to $F_{m\sigma j}$

$\{\varepsilon\}$ = $\{\varepsilon_o\}+\{\varepsilon_e\}$ total generalized strains corresponding to $\{\sigma\}$

$\{\varepsilon_o\}$ = initial strains, free thermal expansions, lack of fit, etc. Stress-free initial strains.

$\{\varepsilon_e\} = \{\varepsilon(\sigma)\}$ stress related strain

V = Volume of element

$\sim$ indicates virtual quantities

Subscript σ indicates quantities relative to an element based on stress assumptions.

If there are β independent force variables then there are β linear equations in the form of Equation (65), therefore, the following system of equations results.

$$\left[\tilde{F}_{m\sigma}\right]\{d_{m\sigma}\} = \int_V [\tilde{\sigma}_\sigma]^T\{\varepsilon\}\, dV \tag{66}$$

where, $\left[\tilde{F}_{m\sigma}\right]$ is a diagonal matrix of virtual force variables. Using Equation (62), the complete set of virtual stress systems is given by,

$$[\tilde{\sigma}_\sigma] = [T_{\sigma F}]\left[\tilde{F}_{m\sigma}\right] \tag{67}$$

Substituting Equation (67) into (66) gives,

$$\left[\tilde{F}_{m\sigma}\right]\{d_{m\sigma}\} = \left[\tilde{F}_{m\sigma}\right]\int_V [T_{\sigma F}]^T\{\varepsilon\}\, dV \tag{68}$$

or,

$$\{d_{m\sigma}\} = \int_V [T_{\sigma F}]^T\{\varepsilon\}\, dV \tag{69}$$

Equation (69) defines the actual element deformation, $d_{m\sigma j}$, corresponding to a chosen force variable, $F_{m\sigma j}$. This deformation is that through which the force variable does complementary work but is caused by the total strain distribution, $\{\varepsilon\}$. As a non-trivial example, if the force variable is a pressure or stress (lb/in.^{2}) then the corresponding deformation is a volumetric displacement (in.^{3}).

The complete strains can be expressed in terms of the stresses and the initial strains

$$\{\varepsilon\} = \{\varepsilon_o\} + \{\varepsilon_e\} \tag{70}$$

For linear stress-strain behavior (see also Equation (26))

$$\{\varepsilon_e\} = [T_{\varepsilon\sigma}]\{\sigma_\sigma\} \tag{71}$$

where,

$$[T_{\varepsilon\sigma}] = [T_{\sigma\varepsilon}]^{-1} \tag{72}$$

Therefore, introducing Equations (70) and (71) into (69),

$$\{d_{m\sigma}\} = \int_V [T_{\sigma F}]^T\{\varepsilon_o\}\, dV + \left(\int_V [T_{\sigma F}]^T[T_{\varepsilon\sigma}][T_{\sigma F}]\, dV\right)\{F_{m\sigma}\} \tag{73}$$

or,

$$\{d_{m\sigma}\} = \{d_{m\sigma o}\} + \{d_{m\sigma e}\} \tag{74}$$

where,

$$\{d_{m\sigma o}\} = \int_V [T_{\sigma F}]^T\{\varepsilon_o\}\, dV \tag{75}$$

= corresponding initial deformations (thermal)

$$\{d_{m\sigma e}\} = [D_{m\sigma}]\{F_{m\sigma}\} \tag{76}$$

and

$$[D_{m\sigma}] = \int_V [T_{\sigma F}]^T[T_{\varepsilon\sigma}][T_{\sigma F}]\, dV \tag{77}$$

= natural flexibility matrix based on stress assumptions.

This natural flexibility matrix, as well as the initial deformations, are for a stress element as well as for a Hybrid I element (Reference 10, Equation 8) based on an internal stress field.

4.2 STRESS ELEMENT ASSEMBLY MATRIX

Corresponding to each independent force variable there is an equilibrated set of boundary force distributions which are directly obtained from the values at the boundaries of the assumed internal stress field. For the i^{th} boundary

$$\{N_{bi}\} = [T_{N\sigma i}]\{\sigma_\sigma\} \tag{78}$$

where,

$\{N_{bi}\}$ = generalized forces on the i^{th} element boundary.

Substituting from Equation (62) into (78) gives

$$\{N_{bi}\} = [T_{NFi}]\{F_{m\sigma}\} \tag{79}$$

where,

$$[T_{NFi}] = [T_{N\sigma i}][T_{\sigma F}] \tag{80}$$

= boundary force distributions for unit values of the independent force variables.

The boundary forces, $\{N_{bi}\}$, are used, in one form or another, to form the equilibrium equations for the assembled structure. In the usual formulation of nodal equilibrium it is necessary to establish a set of nodal forces which are equivalent to the boundary forces. In stress elements this can be achieved by integrating the boundary forces and distributing them to the nodes while still maintaining equilibrium. The element nodal forces which are equivalent to the i^{th} boundary forces, $\{q_{mi}\}$, are given by,

$$\{q_{mi}\} = [T_{qNi}]\{N_{bi}\} \tag{81}$$

and contain dependent forces (static supports). Substituting Equation (79) into (81) and summing for all boundaries results in,

$$\{q_m\} = \left(\sum_{i=1}^{b} [T_{qNi}][T_{NFi}]\right)\{F_{m\sigma}\} \tag{82}$$

or,

$$\{q_m\} = [T_{qF\sigma}]\{F_{m\sigma}\} \tag{83}$$

where,

$$[T_{qF\sigma}] = \sum_{i=1}^{b} [T_{qNi}][T_{NFi}] \tag{84}$$

$\{q_m\}$ = equivalent equilibrated nodal forces

b = number of boundaries.

A more systematic procedure is to use the principle of virtual displacements to find the equivalent nodal forces which are in equilibrium with the boundary forces. This requires that a set of kinematically possible boundary displacement functions be established. These are not consistent with the assumed stress field. This procedure is the basis of Pian's Hybrid I elements.[10,11] For the i^{th} boundary,

$$\{u_{bi}\} = [T_{u\delta i}]\{\delta_m\} \tag{85}$$

where,

$\{\delta_m\}$ = nodal displacements in the local element system including rigid body freedoms.

$\{u_{bi}\}$ = boundary displacement functions.

Application of the principle of virtual displacements, for virtual nodal displacements, results in,

$$\lceil\tilde{\delta}_m\rfloor\{q_m\} = \sum_{i=1}^{b}\int_{S_i}[\tilde{u}_{bi}]^T\{N_{bi}\}\,dS_i \tag{86}$$

Substituting into Equation (86) from (79) and (85),

$$\lceil\tilde{\delta}_m\rfloor\{q_m\} = \lceil\tilde{\delta}_m\rfloor\left(\sum_{i=1}^{b}\int_{S_i}[T_{u\delta i}]^T[T_{NFi}]\,dS_i\right)\{F_{m\sigma}\} \tag{87}$$

where,

S_i = surface variable on the i^{th} boundary.

This gives,

$$\{q_m\} = [T_{qF\sigma}]\{F_{m\sigma}\} \tag{88}$$

where,

$$[T_{qF\sigma}] = \sum_{i=1}^{b}\int_{S_i}[T_{u\delta i}]^T[T_{NFi}]\,dS_i \tag{89}$$

The $[T_{qF\sigma}]$ matrix is referred to as the element force assembly matrix in the local system based on stress assumptions. This matrix transforms the element force variables into an equivalent set of nodal forces in the local element system including dependent forces.

The relation between the corresponding deformations and nodal displacements in the local element system can be found by applying the principle of virtual forces for virtual force variables. Therefore,

$$\lceil\tilde{F}_{m\sigma}\rfloor\{d_{m\sigma}\} = \lceil\tilde{F}_{m\sigma}\rfloor\left(\sum_{i=1}^{b}\int_{S_i}[T_{NFi}]^T[T_{u\delta i}]\,dS_i\right)\{\delta_m\} \tag{90}$$

Hence,

$$\{d_{m\sigma}\} = [T_{d\delta\sigma}]\{\delta_m\} \tag{91}$$

where,

$$[T_{d\delta\sigma}] = \sum_{i=1}^{b} \int_{S_i} [T_{NFi}]^T [T_{u\delta i}]\, dS_i \tag{92}$$

= element displacement assembly matrix in the local system based on stress assumptions.

Referring to Equations (89) and (92) it can be seen that,

$$[T_{d\delta\sigma}] = [T_{qF\sigma}]^T \tag{93}$$

In other words, the matrix which relates the element nodal displacements in the local system, including rigid body freedoms, to the element deformations corresponding to the force variables is the transpose of the element local assembly matrix.

In some cases the two methods of obtaining equivalent nodal forces will be identical, as is the case for the rectangular membrane element given in Reference 10, with the stress field:

$$\sigma_x = a_1 + a_2 y$$
$$\sigma_y = a_3 + a_4 x$$
$$\tau_{xy} = a_5$$

Substituting from Equation (88) into (42) gives,

$$\{Q_m\} = [E_{m\sigma}]\{F_{m\sigma}\} \tag{94}$$

where,

$$[E_{m\sigma}] = [T_c][T_{qF\sigma}] \tag{95}$$

= element force assembly matrix in the global system based on stress assumptions.

Substituting Equation (45) into (91) gives,

$$\{d_{m\sigma}\} = [a_{m\sigma}]\{\Delta_m\} \tag{96}$$

where,
$$[a_{m\sigma}] = [T_{d\delta\sigma}][T_c]^T = [E_{m\sigma}]^T \tag{97}$$

= element displacement assembly matrix in the global system based on stress assumptions.

To establish equilibrium models, 5,6,14/ that is, force continuity or boundary force equilibrium between adjacent element boundaries, the boundary force distributions should be equilibrated directly. These must be resolved into a suitable reference system which will also result in an element assembly matrix. In this case the concept of a node must be considered in a more general sense.

4.3 STRESS ELEMENT OUTPUT MATRIX

In a stress element the stress field is given by Equation (62), that is,

$$\{\sigma_\sigma\} = [T_{\sigma F}]\{F_{m\sigma}\} \tag{98}$$

This equation gives the stress distributions within the stress element in terms of the local element coordinates and generalized force variables. In the force method programs the stress output is evaluated directly from the computed element force variables, $\{F_{m\sigma}\}$, by means of Equation (62), using $[T_{\sigma F}]$ at specific locations. The stress output is given by,

$$\{\sigma_{m\sigma}\} = [S_{m\sigma}]\{F_{m\sigma}\} \tag{99}$$

where

$\{\sigma_{m\sigma}\}$ = vector of required output data for a stress element

$[S_{m\sigma}] = [T_{\sigma F}]$ at specified locations within the element.

To use this element in the displacement method, obtain from Equations (74) and (76),

$$\{F_{m\sigma}\} = [D_{m\sigma}]^{-1}\left(\{d_{m\sigma}\} - \{d_{m\sigma o}\}\right) = [D_m]^{-1}\{d_{m\sigma e}\} \tag{100}$$

Hence, substituting from Equation (100) into (99), the stress output in terms of deformations is given by,

$$\{\sigma_{m\sigma}\} = [S_{m\sigma}][D_{m\sigma}]^{-1}\left(\{d_{m\sigma}\} - \{d_{m\sigma o}\}\right) \tag{101}$$

or

$$\{\sigma_{m\sigma}\} = [S_{m\sigma}][D_{m\sigma}]^{-1}\{d_{m\sigma e}\} \tag{102}$$

The element output matrix will now be denoted by $[S_{m\sigma}]$ with subscripts F or D to denote in which method it is to be used (force or displacement). Therefore for the force method

$$[S_{m\sigma F}] = [S_{m\sigma}] \tag{103}$$

and for the displacement method

$$[S_{m\sigma D}] = [S_{m\sigma}][D_{m\sigma}]^{-1} \tag{104}$$

The relationship between the two forms of output matrices is,

$$[S_{m\sigma D}] = [S_{m\sigma F}][D_{m\sigma}]^{-1} \tag{105}$$

In the case of the combined method either form can be used as desired.

5. SUMMARY AND PRACTICAL SIGNIFICANCE OF CHARACTERISTIC ELEMENT MATRICES

The concept of characteristic element matrices has considerable practical significance. If every computer program for structural analysis was written on the basis of characteristic matrices for any element, complete interchangeability of elements between the various systems could be obtained. This would be irrespective of the analysis method (force, displacement, combined) or the basic element assumption (stress, strain). An analysis system of this nature has been developed at the Lockheed-California Company. Consider now a program based on the displacement method for static analysis which was written to accept four characteristic matrices for an element (elastic, initial deformation, assembly, output). Now let anyone who develops a stress element, write the subroutine for this element to return the characteristic matrices $[D_{m\sigma}]$, $\{d_{m\sigma o}\}$, $[E_{m\sigma}]$ and $[S_{m\sigma F}]$. This subroutine could then be used directly to introduce the new stress element into the displacement method system by changing a few cards and using a subroutine of the form shown in Figure 2(a). Accompanying the element subroutine would be a report of the element derivation and a check example. The converse situation of developing a strain element and having a program based on the force method would require an element subroutine of the form shown in Figure 2(b). The complete interchangeability of element matrices is shown in Figure 3.

A great many publications have appeared on elements for use in finite element programs. Various types of elements have been developed and published in many international journals and conference proceedings. However, the use of these elements in practice is limited mainly because analysis programs have not been written to readily and rapidly add new elements. In addition, the various programs using even the same method have different ways of introducing elements. The concept of element characteristic matrices presents a unified scheme for element introduction and interchangeability in finite element programs, which would facilitate communications and lead to a wider and more rapid use of new element developments. The concept is demonstrated in the Appendix.

6. ELEMENT SINGULAR STIFFNESS MATRIX

Most researchers and practitioners using the displacement method adopt singular stiffness matrices for an element. It is therefore appropriate to show how this matrix is formed simply using its natural stiffness matrix and its assembly matrix.

The energy within an element has the quadratic form

$$U_m = \frac{1}{2} \lfloor \Delta_m \rfloor [k_{mg}] \{\Delta_m\} = \frac{1}{2} \lfloor d_m \rfloor [k_m] \{d_m\} \tag{106}$$

where,

$[k_{mg}]$ = element singular stiffness matrix in the global system.

In the case of a strain element,

$$\{d_m\} = \{d_{m\varepsilon}\} = [a_{m\varepsilon}]\{\Delta_m\} \tag{107}$$

and

$$[k_m] = [k_{m\varepsilon}] \tag{108}$$

Substituting from Equations (107) and (108) into (106) gives the element singular stiffness matrix,

$$[k_{mg}] = [k_{m\varepsilon g}] = [a_{m\varepsilon}]^T [k_{m\varepsilon}] [a_{m\varepsilon}] \tag{109}$$

In the case of a stress element,

$$\{d_m\} = \{d_{m\sigma}\} = [E_{m\sigma}]^T \{\Delta_m\} \tag{110}$$

and

$$[k_m] = [D_{m\sigma}]^{-1} \tag{111}$$

Substituting from Equations (110) and (111) into (106) gives,

$$[k_{mg}] = [k_{m\sigma g}] = [E_{m\sigma}][D_{m\sigma}]^{-1}[E_{m\sigma}]^T \tag{112}$$

Stress Elements	Strain Elements
Assumed Airy type stress functions	Assumed displacement functions
1. Axial elements (local axis x) $\{\Phi\} = \Phi_1$	1. Axial elements (local axis x) $\{u\} = u$, when u is the displacement in the x-direction
2. Membrane elements (local axes x, y) $\{\Phi\} = \Phi_1$	2. Membrane elements (local axes x, y) $\{u\} = \{u\ v\}$, where u and v are the in-plane displacements in the x and y directions respectively
3. Plate bending elements (local axes x, y, z) $\{\Phi\} = \{\Phi_1 \Phi_2\}$	3. Plate bending elements (local axes x, y, z) $\{u\} = w$, where w is the displacement in the z-direction (normal to element plane)
4. Solid elements (local axes x, y, z) $\{\Phi\} = \{\Phi_1 \Phi_2 \Phi_3\}$	4. Solid elements (local axes x, y, z) $\{u\} = \{u\ v\ w\}$, where u, v and w are the displacements in the x, y and z directions respectively
Generalized stress field	Generalized strain field
1. Axial elements $\sigma_x = \frac{\partial \Phi_1}{\partial x}$	1. Axial elements $\epsilon_x = \frac{\partial u}{\partial x}$
2. Membrane elements $\sigma_x = \frac{\partial^2 \Phi_1}{\partial y^2}$ $\sigma_y = \frac{\partial^2 \Phi_1}{\partial x^2}$ $\tau_{xy} = - \frac{\partial^2 \Phi_1}{\partial x \partial y}$	2. Membrane elements $\epsilon_x = \frac{\partial u}{\partial x}$ $\epsilon_y = \frac{\partial v}{\partial y}$ $\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$
3. Plate bending elements $M_x = \frac{\partial \Phi_2}{\partial y}$ $M_y = \frac{\partial \Phi_1}{\partial x}$ $M_{xy} = \frac{1}{2}\left(\frac{\partial \Phi_1}{\partial y} + \frac{\partial \Phi_2}{\partial x}\right)$	3. Plate bending elements $\frac{1}{R_x} = \frac{\partial^2 w}{\partial x^2}$ $\frac{1}{R_y} = \frac{\partial^2 w}{\partial y^2}$ $\frac{1}{R_{xy}} = \frac{\partial^2 w}{\partial x \partial y}$
4. Solid elements $\sigma_x = \frac{\partial^2 \Phi_3}{\partial y^2} + \frac{\partial^2 \Phi_2}{\partial z^2}$ $\sigma_y = \frac{\partial^2 \Phi_1}{\partial z^2} + \frac{\partial^2 \Phi_3}{\partial x^2}$ $\sigma_z = \frac{\partial^2 \Phi_2}{\partial x^2} + \frac{\partial^2 \Phi_1}{\partial y^2}$ $\tau_{xy} = - \frac{\partial^2 \Phi_3}{\partial x \partial y}$ $\tau_{yz} = - \frac{\partial^2 \Phi_1}{\partial y \partial z}$ $\tau_{zx} = - \frac{\partial^2 \Phi_2}{\partial z \partial x}$	4. Solid elements $\epsilon_x = \frac{\partial u}{\partial x}$ $\epsilon_y = \frac{\partial v}{\partial y}$ $\epsilon_z = \frac{\partial w}{\partial z}$ $\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$ $\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$ $\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$

TABLE 1 FIELD-FUNCTION RELATIONS

	STRESS ELEMENT	STRAIN ELEMENT
BASIC RELATIONS	$d_{m\sigma} = D_{m\sigma} F_{m\sigma} + d_{m\sigma o}$ $Q_m = E_{m\sigma} F_{m\sigma}$ $\sigma_{m\sigma} = S_{m\sigma F} F_{m\sigma}$	$F_{m\varepsilon} = k_{m\varepsilon} d_{m\varepsilon} - F_{m\varepsilon o}$ $d_{m\varepsilon} = a_{m\varepsilon} \Delta_m$ $\sigma_{m\varepsilon} = S_{m\varepsilon D}(d_{m\varepsilon} - d_{m\varepsilon o})$
INTERCHANGEABLE RELATIONS	$F_{m\sigma} = k_{m\sigma} d_{m\sigma} - F_{m\sigma o}$ $d_{m\sigma} = a_{m\sigma} \Delta_m$ $\sigma_{m\sigma} = S_{m\sigma D}(d_{m\sigma} - d_{m\sigma o})$	$d_{m\varepsilon} = D_{m\varepsilon} F_{m\varepsilon} + d_{m\varepsilon o}$ $Q_m = E_{m\varepsilon} F_{m\varepsilon}$ $\sigma_{m\varepsilon} = S_{m\varepsilon F} F_{m\varepsilon}$
IDENTITIES	$k_{m\sigma} = D_{m\sigma}^{-1}$ $F_{m\sigma o} = D_{m\sigma}^{-1} d_{m\sigma o}$ $a_{m\sigma} = E_{m\sigma}^{T}$ $S_{m\sigma D} = S_{m\sigma F} D_{m\sigma}^{-1}$	$D_{m\varepsilon} = k_{m\varepsilon}^{-1}$ $d_{m\varepsilon o} = k_{m\varepsilon}^{-1} F_{m\varepsilon o}$ $E_{m\varepsilon} = a_{m\varepsilon}^{T}$ $S_{m\varepsilon F} = S_{m\varepsilon D} k_{m\varepsilon}^{-1}$

TABLE 2 ELEMENT RELATIONSHIPS AND IDENTITIES

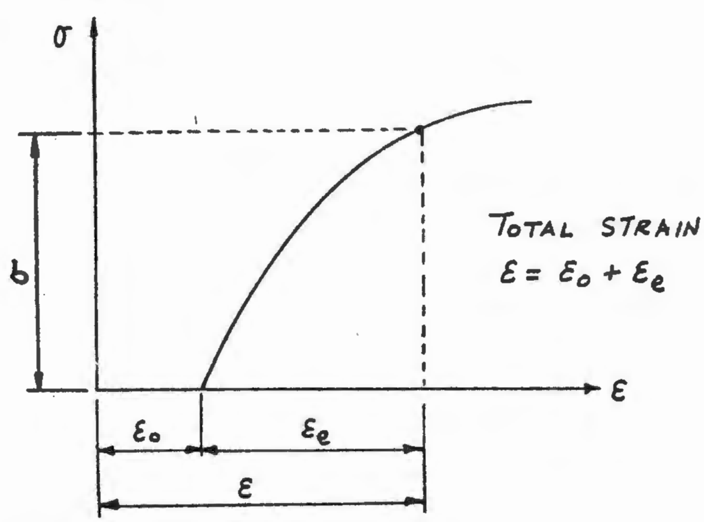

FIGURE 1 TOTAL STRESS-STRAIN RELATION

```
            SUBROUTINE DISEL (kmσ, Fmσo, amσ, SmσD)
COMMENT     ELEMENTS FOR DISPLACEMENT METHOD PROGRAM USING STRESS ELEMENTS
C           CALL THE FOLLOWING SUBROUTINE FOR A STRESS ELEMENT
            CALL STRESEL (Dmσ, dmσo, Emσ, SmσF)
C           CALCULATE THE IDENTIES
```

$$k_{m\sigma} = D_{m\sigma}^{-1}$$
$$F_{m\sigma o} = D_{m\sigma}^{-1}\, d_{m\sigma o}$$
$$a_{m\sigma} = E_{m\sigma}^{T}$$
$$S_{m\sigma D} = S_{m\sigma F}\, D_{m\sigma}^{-1}$$

```
            RETURN
            END
```

(a)

```
            SUBROUTINE FORSEL (Dmε, dmεo, Emε, SmεF)
C           ELEMENTS FOR FORCE METHOD PROGRAM USING STRAIN ELEMENTS
C           CALL THE FOLLOWING SUBROUTINE FOR A STRAIN ELEMENT
            CALL STRAEL (kmε, Fmεo, amε, SmεD)
C           CALCULATE THE IDENTITIES
```

$$D_{m\varepsilon} = k_{m\varepsilon}^{-1}$$
$$d_{m\varepsilon o} = k_{m\varepsilon}^{-1}\, F_{m\varepsilon o}$$
$$E_{m\varepsilon} = a_{m\varepsilon}^{T}$$
$$S_{m\varepsilon F} = S_{m\varepsilon D}\, k_{m\varepsilon}^{-1}$$

```
            RETURN
            END
```

(b)

FIGURE 2 TYPICAL SUBROUTINES FOR THE EXCHANGE OF ELEMENTS

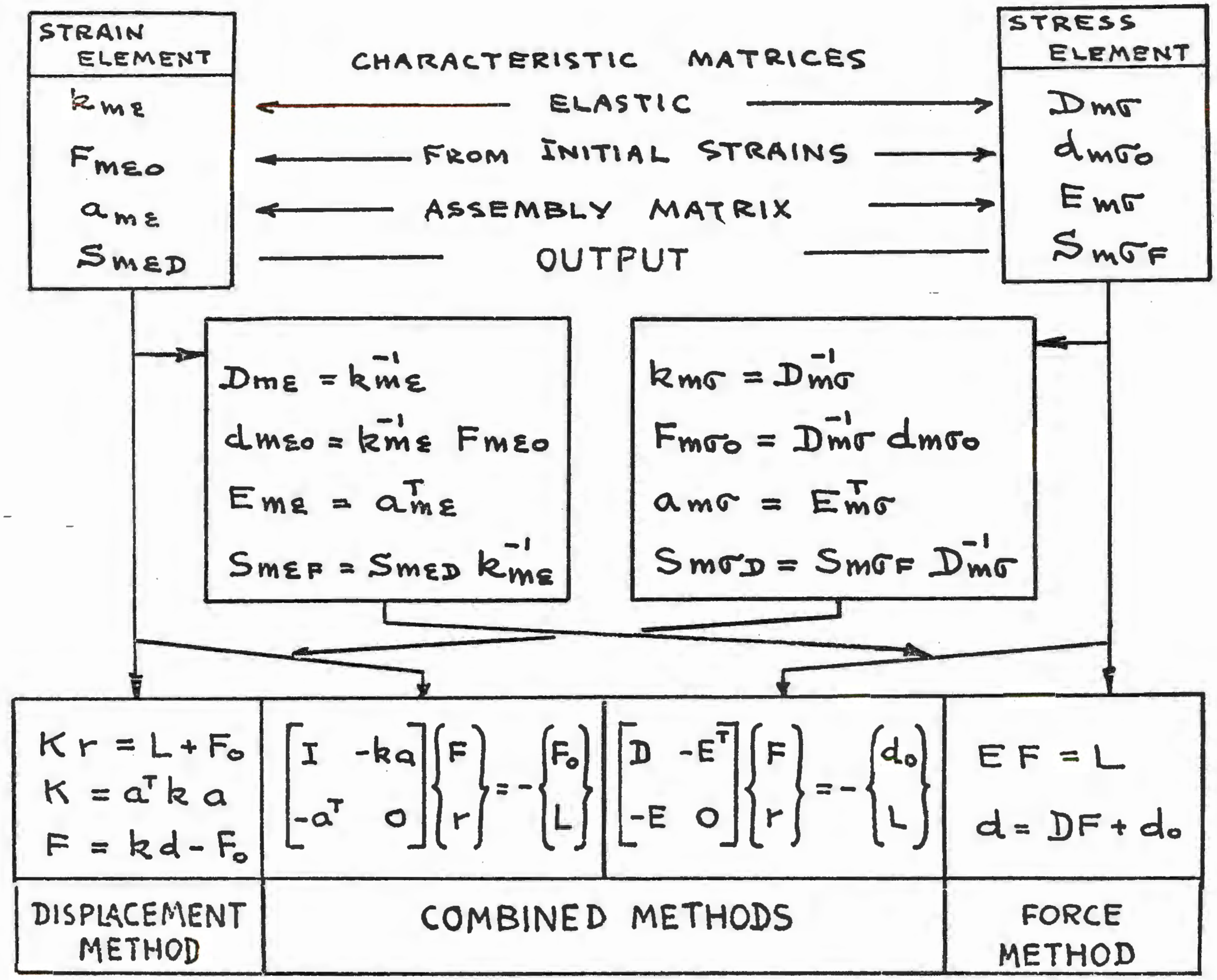

Figure 3. Interchange of Elements

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APPENDIX

The concept of characteristic matrices will be demonstrated using a two point beam, bending in one plane, based on both strain and stress assumptions. The required functions are expressed in a non-dimensional local coordinate system with the origin at mid-span. (See Figure A1). This choice of coordinate system, although not essential for the basic principle, is advantageous in the development.

A.1 STRAIN ELEMENT

The principal steps of Section 3 will now be followed.

Displacement function, Equation (18),

$$\{u\} = [T_{uB}]\{B\}$$

$$v = \lfloor 1 \quad \xi \quad \xi^2 \quad \xi^3 \rfloor \{b_1 \ldots b_4\} \qquad \text{A.1.1}$$

Displacement derivatives,

$$\frac{dv}{dx} = \frac{1}{a} \lfloor 1 \quad 2\xi \quad 3\xi^2 \rfloor \{b_2 \; b_3 \; b_4\} \qquad \text{A.1.2}$$

$$\frac{d^2v}{dx^2} = \frac{1}{a^2} \lfloor 2 \quad 6\xi \rfloor \{b_3 \; b_4\} \qquad \text{A.1.3}$$

Strain function, Equation (17),

$$\{\varepsilon_\varepsilon\} = [T_{\varepsilon d}]\{d_{m\varepsilon}\}$$

$$\varepsilon_x = -y\frac{d^2 v}{dx^2} = -\frac{y}{a^2}\lfloor 2 \quad 6\xi \rfloor \{b_3 \quad b_4\}$$

$$\varepsilon_x = \frac{-y}{a^2}\lfloor 1 \quad \xi \rfloor \{d_1 \quad d_2\} \qquad \text{A.1.4}$$

where d_1 and d_2 are the chosen independent deformation variables. Equation (19) becomes,

$$\{d_{m\varepsilon}\} = [T_{dB}]\{B\}$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad \text{A.1.5}$$

The natural strain modes of Equation (A.1.4) are shown in Figure A6.

Stress-strain behaviour, Equation (26)

$$\{\sigma\} = [T_{\sigma\varepsilon}]\{\varepsilon_e\}$$

$$[T_{\sigma\varepsilon}] = E \quad \text{(Modulus of Elasticity)} \qquad \text{A.1.6}$$

Force-deformation relation, Equation (28),

$$\{F_{m\varepsilon}\} = [k_{m\varepsilon}]\{d_{m\varepsilon}\} - \{F_{m\varepsilon o}\}$$

Initial element forces, Equation (29),

$$\{F_{m\varepsilon o}\} = \int_V [T_{\varepsilon d}]^T [T_{\sigma\varepsilon}]\{\varepsilon_o\}\, dV$$

$$\{F_{m\varepsilon o}\} = \int_V \left(-\frac{y}{a^2}\right)\begin{bmatrix} 1 \\ \xi \end{bmatrix} E\, \varepsilon_o(\xi, y)\, dV \qquad \text{A.1.7}$$

Often $\{\varepsilon_o\}$ is assumed in the same form as $\{\varepsilon\}$, however, Equation A.1.7 can be integrated numerically or, in the case of thermal strains, as part of the thermal program.

Natural stiffness matrix, Equation (30),

$$[k_{m\varepsilon}] = \int_V [T_{\varepsilon d}]^T [T_{\sigma\varepsilon}][T_{\varepsilon d}]\, dV$$

$$[k_{m\varepsilon}] = \int_V \left(-\frac{y}{a^2}\right)\begin{bmatrix} 1 \\ \xi \end{bmatrix} E \left(-\frac{y}{a^2}\right)\lfloor 1 \quad \xi \rfloor\, dA\, a\, d\xi$$

$$= \int_{-1}^{1} \frac{E}{a^3}\begin{bmatrix} 1 & \xi \\ \xi & \xi^2 \end{bmatrix}\left(\int_A y^2 dA\right) d\xi$$

$$[k_{m\varepsilon}] = \frac{2}{3}\left(\frac{EI}{a^3}\right)\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{A.1.8}$$

The simple form of $[k_{m\varepsilon}]$, that is the variables are uncoupled, should be noted. This results from the choice of natural coordinate axes and independent variables.

Element nodal displacements in the local system, see Figure A1,

$$\delta_1 = (v)_{\xi=-1} \quad , \quad \delta_2 = \left(\frac{dv}{dx}\right)_{\xi=-1}$$

$$\delta_3 = (v)_{\xi=1} \quad , \quad \delta_4 = \left(\frac{dv}{dx}\right)_{\xi=1} \qquad \text{A.1.9}$$

Using Equations (A.1.1) and (A.1.2), Equation (34)

$$\{\delta_m\} = [T_{\delta B}]\{B\}$$

becomes

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & \frac{1}{a} & -\frac{2}{a} & \frac{3}{a} \\ 1 & 1 & 1 & 1 \\ 0 & \frac{1}{a} & \frac{2}{a} & \frac{3}{a} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad \text{A.1.10}$$

Hence, Equation (35),

$$\{B\} = [T_{\delta B}]^{-1}\{\delta_m\}$$

$$[T_{\delta B}]^{-1} = \frac{1}{4}\begin{bmatrix} 2 & a & 2 & -a \\ -3 & -a & 3 & -a \\ 0 & -a & 0 & a \\ 1 & a & -1 & a \end{bmatrix} \qquad \text{A.1.11}$$

Element displacement assembly matrix in the local system, Equations (36) and (37),

$$\{d_{m\varepsilon}\} = [T_{d\delta\varepsilon}]\{\delta_m\}$$

$$[T_{d\delta\varepsilon}] = [T_{dB}][T_{\delta B}]^{-1}$$

Therefore, using Equations (A.1.5) and (A.1.11),

$$[T_{d\delta\varepsilon}] = \frac{1}{4}\begin{bmatrix} 0 & -2a & 0 & 2a \\ 6 & 6a & -6 & 6a \end{bmatrix} \qquad \text{A.1.12}$$

Coordinate transformation, Equation (45),

$$\{\delta_m\} = [T_c]^T\{\Delta_m\}$$

Referring to Figures A1, A2 and A3,

$$[T_c] = \begin{bmatrix} (\vec{m}_1 \cdot \vec{n}_2) & 0 & 0 & 0 \\ (\vec{m}_2 \cdot \vec{n}_2) & 0 & 0 & 0 \\ 0 & (\vec{m}_3 \cdot \vec{n}_3) & 0 & 0 \\ 0 & 0 & (\vec{m}_1 \cdot \vec{n}_2) & 0 \\ 0 & 0 & (\vec{m}_2 \cdot \vec{n}_2) & 0 \\ 0 & 0 & 0 & (\vec{m}_3 \cdot \vec{n}_3) \end{bmatrix} \qquad \text{A.1.13}$$

where, $\vec{m}_1, \vec{m}_2, \vec{m}_3$ are unit vectors in the global system

and $\vec{n}_1, \vec{n}_2, \vec{n}_3$ are unit vectors in the local system.

$$\hat{n}_1 = \frac{\overrightarrow{V2} - \overrightarrow{V1}}{abs(\overrightarrow{V2}-\overrightarrow{V1})} = \frac{\overrightarrow{V12}}{abs(\overrightarrow{V12})}$$

$$\hat{n}_3 = \frac{\overrightarrow{V12} \times \overrightarrow{V13}}{abs(\overrightarrow{V12} \times \overrightarrow{V13})}$$

$$\hat{n}_2 = \hat{n}_3 \times \hat{n}_1$$

A.1.14

Element displacement assembly matrix in the global system, Equations (46) and (47),

$$\{d_{m\varepsilon}\} = [a_{m\varepsilon}]\{\Delta_m\}$$

$$[a_{m\varepsilon}] = [T_{ds\varepsilon}][T_c]^T$$

Using Equations (A.1.12) and (A.1.13)

$$[a_{m\varepsilon}] = \frac{1}{4}\begin{bmatrix} 0 & 0 & -2a(\hat{m}_3.\hat{n}_3) & 0 & 0 & 2a(\hat{m}_3.\hat{n}_3) \\ 6(\hat{m}_1.\hat{n}_2) & 6a(\hat{m}_2.\hat{n}_2) & 6a(\hat{m}_3.\hat{n}_3) & -6(\hat{m}_1.\hat{n}_2) & -6(\hat{m}_2.\hat{n}_2) & 6a(\hat{m}_3.\hat{n}_3) \end{bmatrix}$$

A.1.15

To obtain the output matrix determine the stress distributions within the element, Equation (54),

$$\{\sigma_e\} = [T_{\sigma\varepsilon}]\{\varepsilon_e\} = [T_{\sigma\varepsilon}][T_{\varepsilon d}]\{d_{m\varepsilon e}\}$$

$$\sigma_x = -y\left(\frac{E}{a^2}\right)\lfloor 1 \quad \xi \rfloor\{d_{m\varepsilon e}\} \qquad \text{A.1.16}$$

NOTE: $\{d_{m\varepsilon e}\} = \{d_{m\varepsilon}\} - \{d_{m\varepsilon o}\}$, Equation (52)

$$\{d_{m\varepsilon}\} = \{d_1 \quad d_2\}$$

$\{d_{m\varepsilon o}\} = [k_{m\varepsilon}]^{-1}\{F_{m\varepsilon o}\}$, Equation (32)

$\{F_{m\varepsilon o}\}$, Equation (29)

Stresses could also be directly calculated from the element nodal forces given by Equation (40).

Element output matrix for the displacement method, Equations (55) and (59),

$$\{\sigma_{m\varepsilon}\} = [S_{m\varepsilon}]\{d_{m\varepsilon e}\}$$

$$[S_{m\varepsilon D}] = [S_{m\varepsilon}]$$

$$\{\sigma_{m\varepsilon}\} = \{(\sigma_x)_{\substack{\xi=-1\\y=c}} \quad (\sigma_x)_{\substack{\xi=1\\y=c}}\} \qquad \text{A.1.17}$$

From Equation (A.1.16)

$$[S_{m\varepsilon D}] = -c\left(\frac{E}{a^2}\right)\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \text{A.1.18}$$

Summary of the strain element characteristic matrices for the displacement method,

$[k_{m\varepsilon}]$	Equation A.1.8,	Natural Stiffness Matrix
$\{F_{m\varepsilon o}\}$	Equation A.1.7,	Initial Force Vector
$[a_{m\varepsilon}]$	Equation A.1.15,	Assembly Matrix
$[S_{m\varepsilon D}]$	Equation A.1.18,	Output Matrix

The strain element characteristic matrices for the force method are obtained using the transformations given in Figure 3, that is,

$$[D_{m\varepsilon}] = [k_{m\varepsilon}]^{-1}$$, Natural Flexibility Matrix

$$\{d_{m\varepsilon o}\} = [k_{m\varepsilon}]^{-1}\{F_{m\varepsilon o}\}$$, Initial Deformation Vector

$$[E_{m\varepsilon}] = [a_{m\varepsilon}]^{T}$$, Assembly Matrix

$$[S_{m\varepsilon F}] = [S_{m\varepsilon D}][k_{m\varepsilon}]^{-1}$$, Output Matrix

A.2 STRESS ELEMENT

The principal steps of Section 4 will now be followed.

Stress field, Equation (62),

$$\{\sigma_\sigma\} = [\,T_{\sigma F}\,]\{F_{m\sigma}\}$$

$$\{M\} = M = \lfloor 1 \quad \xi \rfloor \{F_1 \quad F_2\} \qquad \text{A.2.1}$$

$$\{\sigma_\sigma\} = \sigma_x = -\frac{y}{I} M = -\frac{y}{I} \lfloor 1 \quad \xi \rfloor \{F_1 \quad F_2\} \qquad \text{A.2.2}$$

Alternatively, assuming a stress function, Equation (63),

$$\{\Phi\} = [\,T_{\Phi B}\,]\{B\}$$

$$\Phi = \lfloor 1 \quad \xi \quad y \quad \xi^2 \quad \xi y \rfloor \{b_1 \ldots b_5\} \qquad \text{A.2.3}$$

$$M = \frac{\partial \Phi}{\partial y} = \lfloor 1 \quad \xi \rfloor \{b_3 \quad b_5\} \qquad \text{A.2.4}$$

$$\sigma_x = -\frac{y}{I} \lfloor 1 \quad \xi \rfloor \{b_3 \quad b_5\} \qquad \text{A.2.5}$$

Equation (64),

$$\{F_{m\sigma}\} = [\,T_{FB}\,]\{B\}$$

becomes,

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_5 \end{bmatrix} \qquad \text{A.2.6}$$

The natural stress modes of Equation A.2.2 are shown in Figure A7.

Stress-strain behaviour, Equation (71),

$$\{\varepsilon_e\} = [T_{\varepsilon\sigma}]\{\sigma_\sigma\}$$

$$[T_{\varepsilon\sigma}] = \frac{1}{E} \qquad \text{A.2.7}$$

Force-deformation relation, Equations (74) and (76),

$$\{d_{m\sigma}\} = [D_{m\sigma}]\{F_{m\sigma}\} + \{d_{m\sigma o}\}$$

Initial element deformations, Equation (75),

$$\{d_{m\sigma o}\} = \int_V [T_{\sigma F}]^T \{\varepsilon_o\}\, dV$$

$$\{d_{m\sigma o}\} = \int_V \left(-\frac{y}{I}\right)\begin{bmatrix} 1 \\ \xi \end{bmatrix} \varepsilon_o(\xi, y)\, dV \qquad \text{A.2.8}$$

Natural flexibility matrix, Equation (77),

$$[D_{m\sigma}] = \int_V [T_{\sigma F}]^T [T_{\varepsilon\sigma}][T_{\sigma F}]\, dV$$

$$[D_{m\sigma}] = \int_V \left(-\frac{y}{I}\right)\begin{bmatrix} 1 \\ \xi \end{bmatrix} \frac{1}{E}\left(-\frac{y}{I}\right)\lfloor 1 \quad \xi \rfloor\, dA\, a\, d\xi$$

$$= \int_{-1}^{1} \frac{a}{EI^2}\begin{bmatrix} 1 & \xi \\ \xi & \xi^2 \end{bmatrix}\left(\int_A y^2 dA\right) d\xi$$

$$[D_{m\sigma}] = \frac{2}{3}\left(\frac{a}{EI}\right)\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{A.2.9}$$

Again, the simple form of $[D_{m\sigma}]$, resulting from the choice of coordinates and independent variables, should be noted.

Forces on the ith element boundary, Equation (79),

$$\{N_{bi}\} = [T_{NFi}]\{F_{m\sigma}\}$$

From Equation (A.2.1),

$$M = -\frac{I}{y}\sigma_x = \lfloor 1 \quad \xi \rfloor \{F_1 \quad F_2\} \qquad \text{A.2.10}$$

The shear is given by,

$$Q = -\frac{1}{a}\frac{dM}{d\xi} = \frac{I}{ay}\frac{d\sigma_x}{d\xi} = -\frac{1}{a}\lfloor 0 \quad 1 \rfloor\{F_1 \quad F_2\} \qquad \text{A.2.11}$$

See Figures A4 and A5.

For the first boundary (node i = 1),

$$\{N_{b1}\} = \begin{bmatrix} N_{M1} \\ N_{Q1} \end{bmatrix} = \begin{bmatrix} -(M)_{\xi=-1} \\ -(Q)_{\xi=-1} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & \frac{1}{a} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \qquad \text{A.2.12}$$

For the second boundary (node i = 2),

$$\{N_{b2}\} = \begin{bmatrix} N_{M2} \\ N_{Q2} \end{bmatrix} = \begin{bmatrix} (M)_{\xi=1} \\ (Q)_{\xi=1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -\frac{1}{a} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

A.2.13

The generalized forces on the boundaries have a positive vector sign convention.

The element nodal forces, in the local system, which are equivalent to the ith boundary forces, Equation (81),

$$\{q_{mi}\} = [T_{qNi}]\{N_{bi}\}$$

See Figures A1 and A5.

For the first boundary (node i = 1),

$$\{q_{m1}\} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} N_{M1} \\ N_{Q1} \end{bmatrix}$$

A.2.14

For the second boundary (node i = 2),

$$\{q_{m2}\} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} N_{M2} \\ N_{Q2} \end{bmatrix}$$

A.2.15

Element force assembly matrix in the local system, Equations (83) and (84),

$$\{q_m\} = [T_{qF\sigma}]\{F_{m\sigma}\}$$

$$[T_{qF\sigma}] = \sum_{i=1}^{b} [T_{qNi}][T_{NFi}]$$

Substituting from Equations (A.2.12) and (A.2.13) into (A.2.14) and (A.2.15) respectively, gives,

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & \frac{1}{a} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -\frac{1}{a} \end{bmatrix} \right) \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

that is,

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{a} \\ -1 & 1 \\ 0 & -\frac{1}{a} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

A.2.16

The boundary displacement functions for the hybrid approach are given for the ith boundary by Equation (85),

$$\{ u_{bi} \} = [T_{u\delta i}] \{ \delta_m \}$$

For the first boundary (node i = 1),

$$\{U_{b1}\} = \begin{bmatrix} \left(\frac{dv}{dx}\right)_1 \\ (v)_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \qquad \text{A.2.17}$$

For the second boundary (node i = 2),

$$\{U_{b2}\} = \begin{bmatrix} \left(\frac{dv}{dx}\right)_2 \\ (v)_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \qquad \text{A.2.18}$$

Element force assembly matrix in the local system, hybrid approach, Equations (88) and (89),

$$\{q_m\} = [T_{qF\sigma}]\{F_{m\sigma}\}$$

$$[T_{qF\sigma}] = \sum_{i=1}^{b} \int_{S_i} [T_{uSi}]^T [T_{NFi}] \, dS_i$$

Substituting from Equations (A.2.12), A.2.13), (A.2.17) and (A.2.18),

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{a} \\ -1 & 1 \\ 0 & -\frac{1}{a} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \qquad \text{A.2.19}$$

In this example it can be seen that the element force assembly matrix in the local system is the same using either the stress integration approach or the hybrid approach. This is because for this element the set of nodal forces is unique.

Element force assembly matrix in the global system, Equations (94) and (95),

$$\{Q_M\} = [E_{M\sigma}]\{F_{M\sigma}\}$$

$$[E_{M\sigma}] = [T_c][T_{qF\sigma}]$$

Using Equations (A.1.13) and (A.2.19),

$$[E_{M\sigma}] = \begin{bmatrix} 0 & \frac{1}{a}(\vec{m}_1 \cdot \vec{n}_2) \\ 0 & \frac{1}{a}(\vec{m}_2 \cdot \vec{n}_2) \\ -(\vec{m}_3 \cdot \vec{n}_3) & (\vec{m}_3 \cdot \vec{n}_3) \\ 0 & -\frac{1}{a}(\vec{m}_1 \cdot \vec{n}_2) \\ 0 & -\frac{1}{a}(\vec{m}_2 \cdot \vec{n}_2) \\ (\vec{m}_3 \cdot \vec{n}_3) & (\vec{m}_3 \cdot \vec{n}_3) \end{bmatrix} \quad \text{A.2.20}$$

To obtain the output matrix use the stress field given by Equation (98),

$$\{\sigma_\sigma\} = [T_{\sigma F}]\{F_{m\sigma}\}$$

that is,

$$\sigma_x = -\frac{y}{I} \lfloor 1 \quad \xi \rfloor \{F_1 \quad F_2\} \qquad \text{A.2.21}$$

Element output matrix for the force method, Equations (99) and (103),

$$\{\sigma_{m\sigma}\} = [S_{m\sigma F}]\{F_{m\sigma}\}$$

$$\{\sigma_{m\sigma}\} = \{ (\sigma_x)_{\substack{\xi=-1\\ y=c}} \quad (\sigma_x)_{\substack{\xi=1\\ y=c}} \} \qquad \text{A.2.22}$$

From Equation (A.2.22),

$$[S_{m\sigma F}] = -\frac{c}{I}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \text{A.2.23}$$

Summary of the stress element characteristic matrices for the force method,

$[D_{m\sigma}]$ Equation A.2.9, Natural Flexibility Matrix

$\{d_{m\sigma o}\}$ Equation A.2.8, Initial Deformation Vector

$[E_{m\sigma}]$ Equation A.2.20, Assembly Matrix

$[S_{m\sigma F}]$ Equation A.2.24, Output Matrix

The stress element characteristic matrices for the displacement method are obtained using the transformations given in Figure 3, that is,

$[k_{m\sigma}] = [D_{m\sigma}]^{-1}$, Natural Stiffness Matrix

$\{F_{m\sigma o}\} = [D_{m\sigma}]^{-1}\{d_{m\sigma o}\}$, Initial Force Vector

$[a_{m\sigma}] = [E_{m\sigma}]^{T}$, Assembly Matrix

$[S_{m\sigma D}] = [S_{m\sigma F}][D_{m\sigma}]^{-1}$, Output Matrix

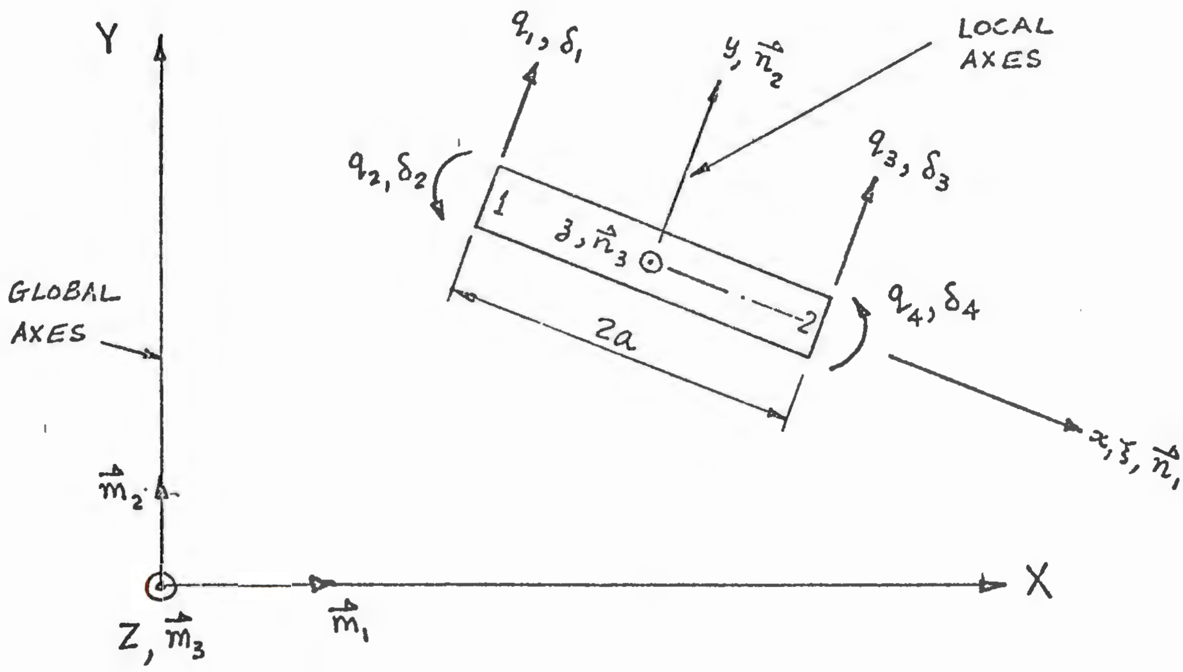


FIGURE A1 BEAM POSITIVE NODAL FORCES AND DISPLACEMENTS IN THE LOCAL SYSTEM.

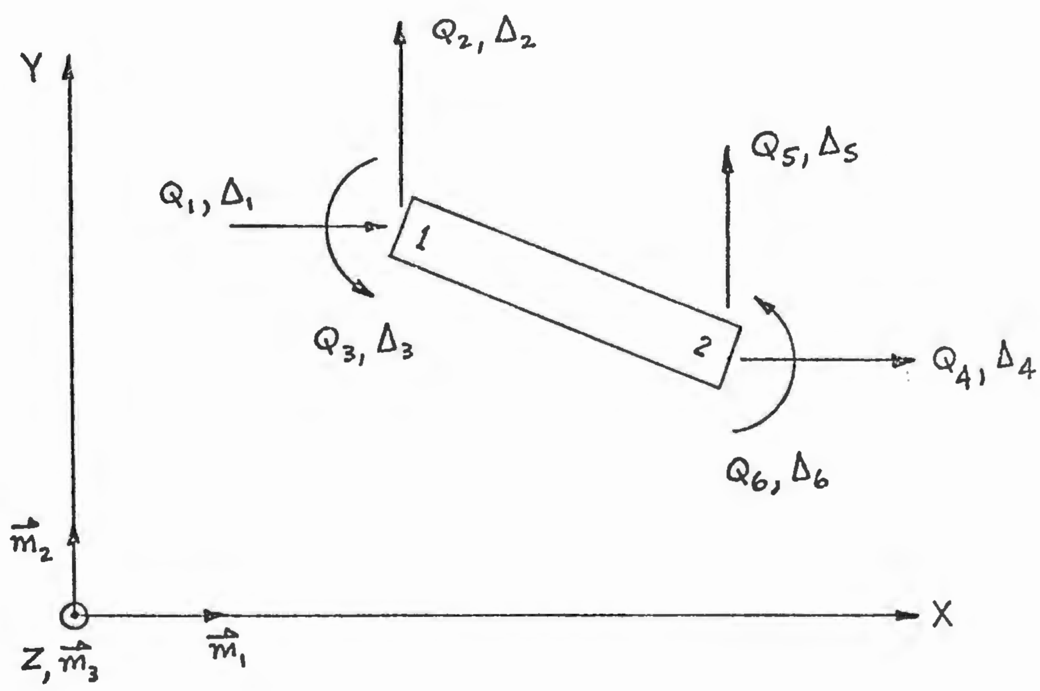


FIGURE A2 BEAM POSITIVE NODAL FORCES AND DISPLACEMENTS IN THE GLOBAL SYSTEM

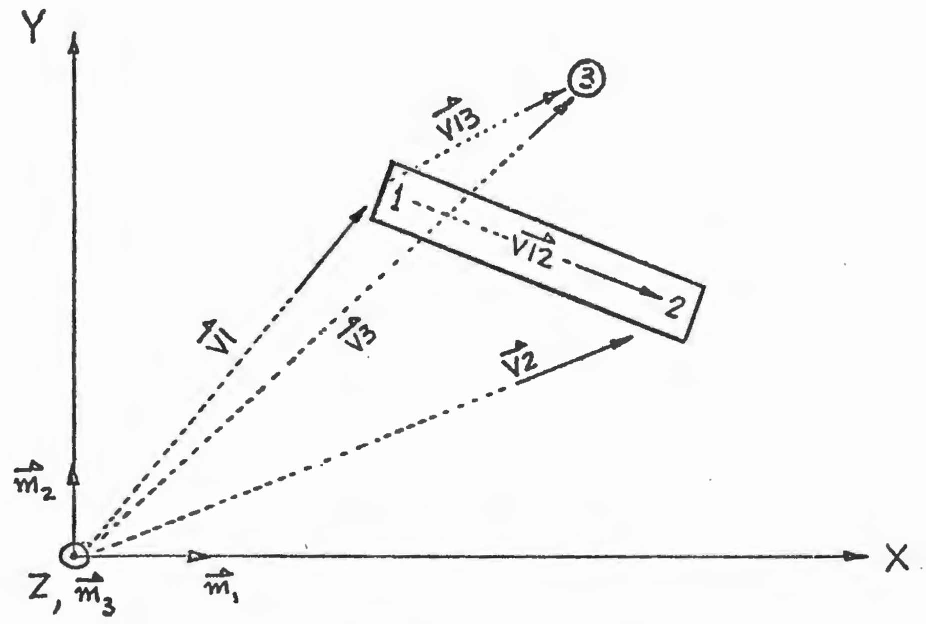


FIGURE A3 ELEMENT VECTORS

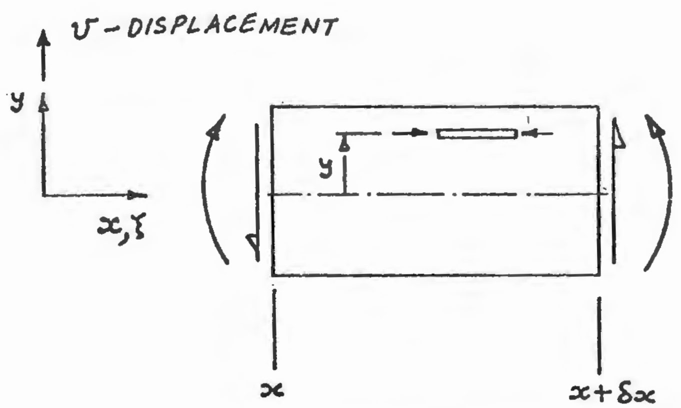


FIGURE A4 POSITIVE INCREMENTAL FORCES AND DISPLACEMENTS

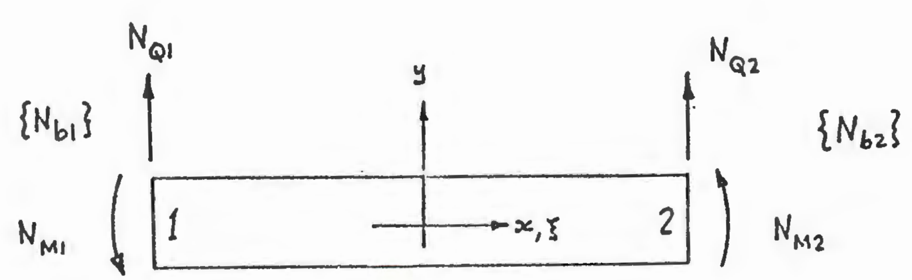
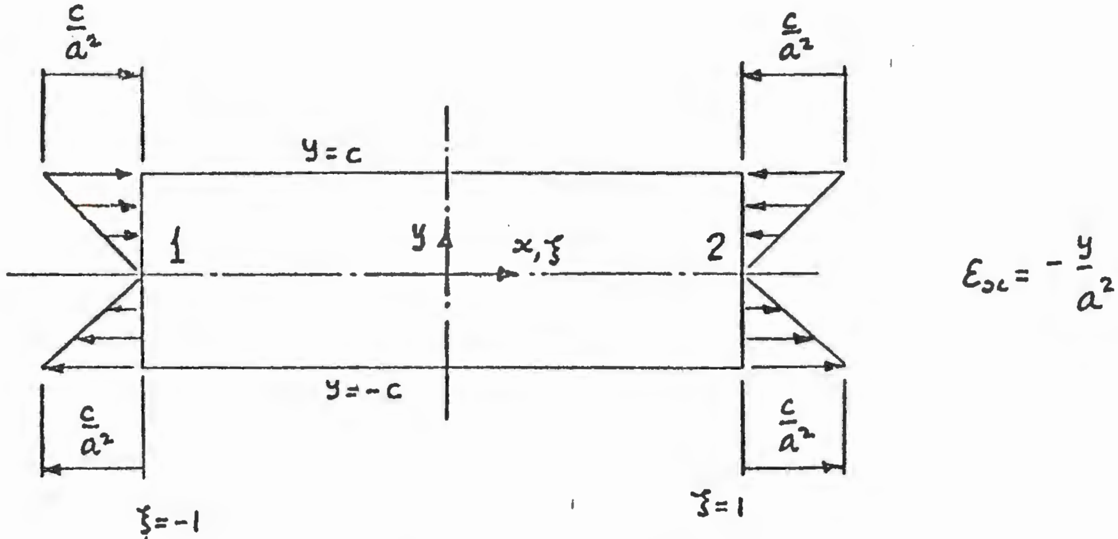
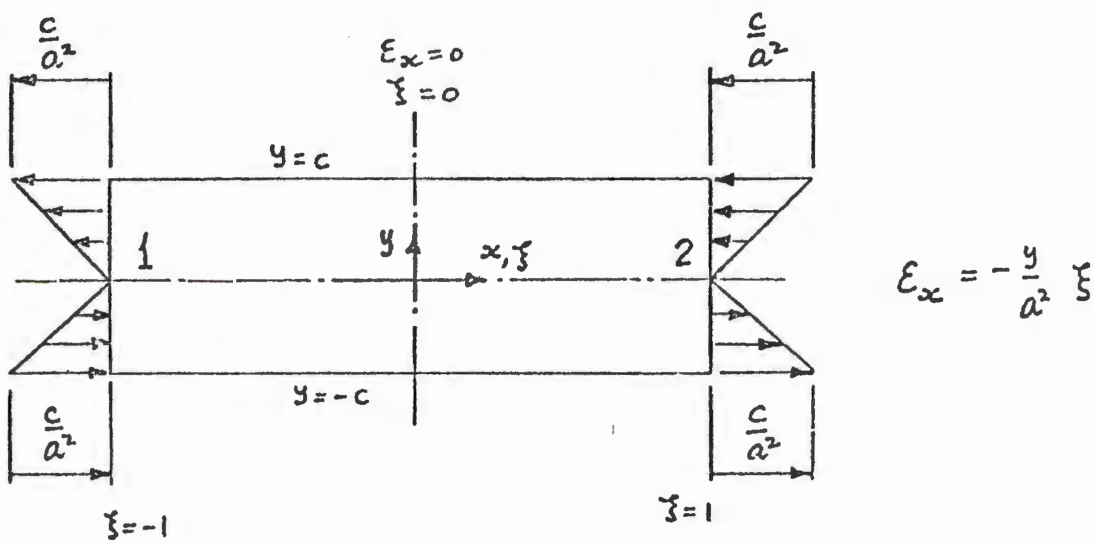


FIGURE A5 POSITIVE BOUNDARY FORCE DISTRIBUTIONS

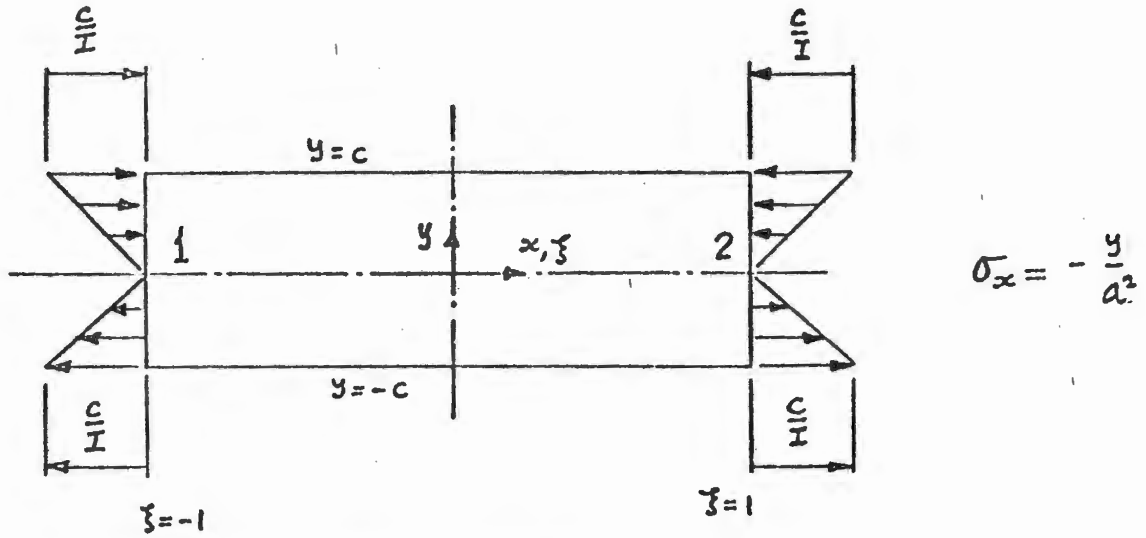


(a) FIRST INDEPENDENT STRAIN MODE, CONSTANT CURVATURE, $d_1 = 1$, $d_2 = 0$.

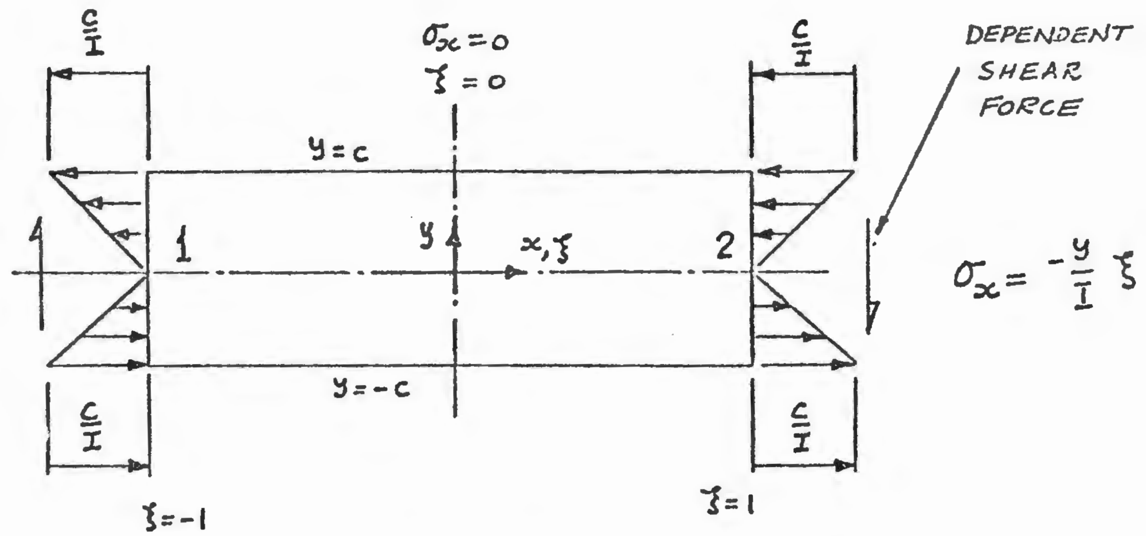


(b) SECOND INDEPENDENT STRAIN MODE, LINEAR CURVATURE, $d_1 = 0$, $d_2 = 1$.

FIGURE A6 NATURAL STRAIN MODES



(a) FIRST INDEPENDENT STRESS MODE, CONSTANT MOMENT, $F_1 = 1$, $F_2 = 0$.



(B) SECOND INDEPENDENT STRESS MODE, LINEAR MOMENT, $F_1 = 0$, $F_2 = 1$.

FIGURE A7 NATURAL STRESS MODES