DAMPING RATIO ESTIMATES FROM AUTOCORRELATION FUNCTIONS

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ABSTRACT

The paper deals with the possible overestimation of the damping ratio, when evaluated from autocorrelation functions in the time domain, because of a bias caused by a triangular window.

Some theoretical considerations permitted to evaluate a lower bound for the sampling period over which it has been possible to estimate the damping ratio with acceptable errors and therefore to limit the effects of the above said bias.

Several numerical examples singled out its possible effects on the modal parameter estimations and gave quantitative evaluations of it. Firstly numerical data regarded single degree of freedom systems in function of the sampling period, afterwards two modes have been considered. This last example is also presented with a high random noise added to the original impulse response, and that because the autocorrelation permits to clean up the noisy signal. In addition when more modes are present in the baseband the problem could become critical: in fact the mode with the highest time constant and therefore, in general, the one with the lowest frequency, is exposed to bias errors which, in case of a large oversampling, become completely unacceptable.

Nomenclature

f_n	natural frequency (Hz)
h[n]	sampled impulse response
m	lag index
n	index of samples
w[n]	sampled triangular window
E	expectation value
M	maximum number of lag points
N	number of points contained in the impulse response
R	residue magnitude
T_s	sampling period (s)
$(T_s)_{lim}$	limit sampling period (s)
$\rho_h[m]$	sampled autocorrelation function relative to h[n]
$\hat{\rho}_h[m]$	estimated autocorrelation sequence
ϵ_f	error in the natural frequency estimation
ϵ_R	error in the residue magnitude estimation
ϵ_ζ	error in the damping ratio estimation
ζ	viscous damping ratio
σ	decay rate (rad/s)
τ_c	time constant (s)
τ_{lim}	limit time (s)
ω_d	damped angular frequency (rad/s)
ω_n	natural angular frequency (rad/s)

1. Introduction

Correlation functions play an important role in diverse areas of science and technology, in particular they are commonly utilized in Modal Analysis to obtain Frequency Response Functions (FRFs) [1 to 4]. Besides, Autocorrelation Functions (AFs), derived from Impulse Responses (IRs), could be used to get modal parameters, i.e. natural frequencies and damping factors, directly in the time domain. This approach can turn out to be useful when the impulse responses are corrupted by a very high additive random noise [5], as it happens for data gathered from flight tests; in fact, evaluating autocorrelation functions, it is possible to remove an uncorrelated noise present in the original impulse response.

Autocorrelation functions are generally estimated by a relationship that introduces a bias consisting in a triangular window around the origin of the time axis.

For a sampled signal, the maximum time lag - given by the sampling period times the number of points whereof the signal is shifted with respect to the origin - where the autocorrelation is estimated from the available data, must satisfy some constraints. It not only ought to be of the order of one tenth of the data block length in order to avoid instability in Power Spectral Density estimates [6], but it must be also chosen in such a way that the exponentially decaying envelope of the autocorrelation is smaller than the contribution of the triangular window. So if the number of the lag points is determined in function of the available number of the total points of the original IR, the sampling period, in addition to the Shannon condition, that determines an upper limit, must also satisfy a lower limit, which permits to estimate the damping ratio with acceptable errors.

In this paper the evaluation of the above said parameter is derived, for each mode, from the instantaneous envelope and phase of the autocorrelation function, which in turn are obtained via the Hilbert transform [7,8].

2. Theoretical considerations

In order to discuss our subject, let us consider the impulse response - sampled over N points - of a real mode derived from a single degree of freedom system:

$$h[n] = R\, e^{-\sigma(nT_s)} \sin[\omega_d (nT_s)] \qquad 0 \le n \le (N-1) \qquad (1)$$

where n is the sample index, T_s is the sampling period, σ is the decay rate and ω_d is the damped angular frequency.

The autocorrelation sequence can be estimated by the following relationship:

$$\hat{\rho}_h[m] = \left(\frac{1}{N}\right) \sum_{n=0}^{N-m-1} h[n]\, h[n+m] \qquad (2)$$

valid for $0 \le m \le (N-1)$, where m indicates the number of lag points of which one sequence is shifted with respect to the other (the time lag is therefore given by mT_s).

Since the autocorrelation is an even function:

$$\rho_h[-m] = \rho_h[m] \qquad (3)$$

and in addition we are only interested in the sequence relative to $m > 0$, Eq. (2) is completely sufficient for our purpose.

Although the relationship above mentioned is frequently utilized, in fact it provides a true autocorrelation sequence (the matrix formed with its elements results to be always positive semidefinite), this estimate is biased by the triangular window:

$$w[m] = \left[1 - \frac{m}{N}\right] \tag{4}$$

As for N→∞ the window is identically equal to the unit, the autocorrelation estimate (2) results asymptotically unbiased [9,10]. Thus the expectation of the estimated sequence is given by the product of the actual autocorrelation times the window:

$$E\left\{\hat{\rho}_h[m]\right\} = \rho_h[m] \cdot w[m] \tag{5}$$

If the maximum lag index M is small enough in comparison with N, the estimate given by Eq. (2) is an acceptable approximation of $\rho_h[m]$ and, under proper conditions [5], it can be written as follows:

$$\hat{\rho}_h[m] \cong \left[\frac{1}{NT_s}\right] \frac{R^2 \omega_d e^{-\sigma(mT_s)}}{4\sigma\sqrt{\sigma^2 + \omega_d^2}} \cdot \cos\left[\omega_d(mT_s) - \arctan\left[\frac{\sigma}{\omega_d}\right]\right] \tag{6}$$

When the viscous damping model is adopted, the decay rate and the damped angular frequency can be written as follows:

$$\sigma = \zeta\omega_n \tag{7}$$

and

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} \tag{8}$$

where ζ is the damping ratio, thus relation (6) reduces to:

$$\hat{\rho}_h[m] \cong \left[\frac{1}{NT_s}\right] \frac{R^2\sqrt{1-\zeta^2}\, e^{-\sigma(mT_s)}}{4\zeta\omega_n} \cdot \cos\left[\omega_d(mT_s) - \arctan\left[\frac{\sigma}{\omega_d}\right]\right] \tag{9}$$

Besides, if the damping ratio is small such that its square value is negligible with respect to the unit, relationship (9) is further simplified:

$$\hat{\rho}_h[m] \cong \frac{R^2}{4NT_s\sigma} e^{-\sigma(mT_s)} \cdot \cos\left[\omega_d\left(mT_s\right)\right] \tag{10}$$

in fact, the damped and the natural frequencies are practically equal and for the above said position:

$$\arctan\left[\frac{\zeta}{\sqrt{1-\zeta^2}}\right] \cong \arctan(\zeta) \cong 0 \tag{11}$$

Actually, owing to the presence of the bias mentioned above, the decaying of the function results greater than the one due to the exponential term appearing in relations (6), (9) and (10). Because our interest is devoted to estimate the decay rate, or better the damping ratio, it is necessary that the contribution of the triangular window is negligible with respect to the one of the exponential decay. A limit time (τ_{lim}) can be derived from the following relationship (Fig.1):

$$e^{-\sigma\tau_{lim}} = \left[1-\frac{\tau_{lim}}{T}\right] \tag{12}$$

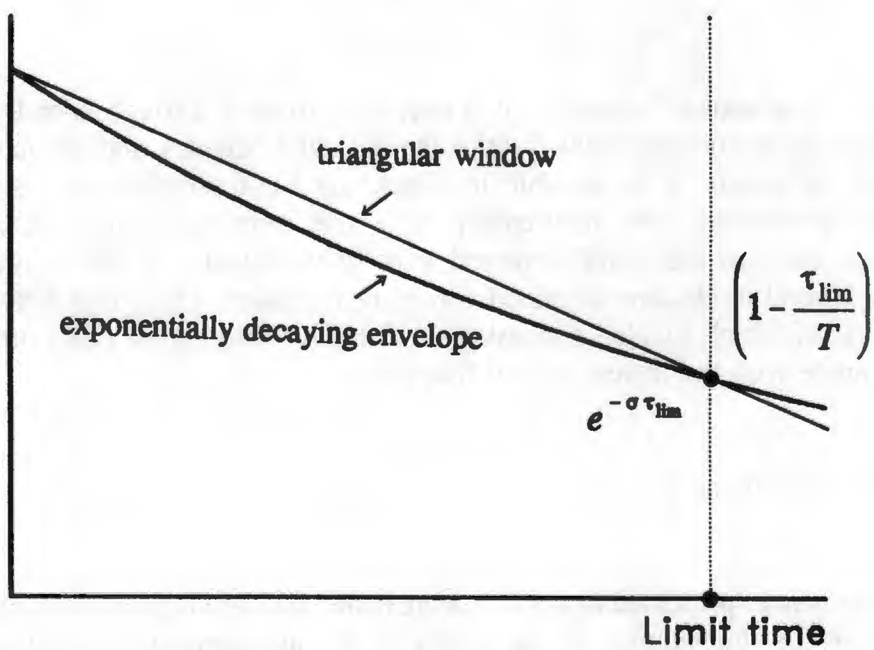

Figure 1 - Limit time evaluated at the maximum time lag.

that - in function of the sampling period, of the maximum number of lag points and of the total number of samples - becomes:

$$e^{-\sigma M T_s} = \left[1 - \frac{M}{N}\right] \tag{13}$$

Performing the natural logarithm, it is possible to derive a relation between the system time constant ($\tau_c = 1/\sigma$) and τ_{lim} :

$$\frac{\tau_{lim}}{\tau_c} = -\ln\left[1 - \frac{M}{N}\right] \tag{14}$$

Therefore the limit time ought to be in the order of $[(1/10)\tau_c]$ or less, supposing M much lower than N, so that the contribution of the triangular window is small enough. Thus the limit sampling period, evaluated for the maximum value of the lag points, is given by:

$$(T_s)_{lim} = -\left[\frac{\tau_c}{M}\right] \ln\left[1 - \frac{M}{N}\right] \tag{15}$$

Since τ_c is unknown "a priori", it is necessary to have a rough knowledge at least of its order of magnitude, anyway once the natural frequency and the damping ratio have been estimated, it is possible to check the limit sampling, in fact if $(T_s)_{lim}$, evaluated introducing into relationship (15) the estimated time constant, is not sufficiently less than the sampling period actually employed, a longer T_s must be used.

When several modes are contained within the baseband, $(T_s)_{lim}$ is constrained by the highest time constant, that is - supposing the damping ratio almost equal for each mode - by the mode with the lowest natural frequency.

3. Numerical tests

Data sequences presented in the following numerical simulations are formed by 4096 points, whereas the number of lag points of the autocorrelation function has been chosen equal to 512 in order to get the Hilbert transform by a standard FFT software.

All the impulse responses considered hereafter have the same amplitude: R=10, even when two modes are presented.

In Tab.1 estimates from the autocorrelation of an impulse response representing a real mode with a very low natural frequency and a light damping ratio (the input modal parameters are: f_n=0.71 Hz and ζ=0.003) are shown:

Table 1 - Modal parameter estimates from a highly truncated impulse response versus the sampling period.

T_s (s)	R	$\lvert\epsilon_R\rvert$ (%)	f_n	$\lvert\epsilon_f\rvert$ (%)	ζ	$\lvert\epsilon_\zeta\rvert$ (%)
0.150	10.00047	0.47 10^{-2}	0.71	0.46 10^{-3}	0.30002 10^{-2}	0.59 10^{-2}
0.100	10.00142	0.14 10^{-1}	0.71	0.47 10^{-3}	0.30007 10^{-2}	0.23 10^{-1}
0.050	10.04965	0.50	0.71	0.62 10^{-3}	0.30406 10^{-2}	1.35
0.030	10.32771	3.28	0.71	0.17 10^{-3}	0.33171 10^{-2}	10.57
0.020	10.80397	8.04	0.71	0.34 10^{-3}	0.39330 10^{-2}	31.10
0.015	11.25559	12.56	0.70996	0.51 10^{-2}	0.47019 10^{-2}	56.73
0.010	11.95407	19.54	0.70973	0.39 10^{-1}	0.64287 10^{-2}	114.29

Since the baseband is 2 (Hz), the sampling period must be less than 0.195 (s), because a sampling factor equal to 2.56 has been considered. On the other hand T_s should be greater than the limit time 0.0195 (s), derived from Eq.(15). Actually up to T_s=0.050 (s) errors on ζ are negligible, on the contrary for decreasing sampling periods higher and higher damping ratios have been obtained, owing to the presence of the triangular window.

The amplitude R has been achieved through estimates of the decay rate and of initial values of the autocorrelation function, Eq.(10).

Natural frequencies have been instead evaluated with immaterial errors for all the cases presented.

The truncation at the end of the maximum time lag does not affect the modal parameter estimates, in fact they have been carried out in the time domain with the Hilbert approach [7,8] (Appendix).

The same case is also shown in Tab.2, where estimates have been carried out in the frequency domain, using 400 spectral lines, with the commercial software SMS Modal 3.0 [11]:

Table 2 - Damping ratio estimations in the frequency domain.

T_s (s)	0.150	0.100	0.050	0.030	0.020	0.015	0.010
ζ	0.00300	0.00300	0.00305	0.00337	0.00404	0.00490	0.00703
$\|\epsilon_\zeta\|$ (%)	-	-	1.67	12.33	34.67	63.33	134.33

Errors on the damping ratio are in agreement with the ones obtained by the time approach: the FRF results to be biased, in fact the triangular window, due to the uncertainty principle [12,13], broadens the peaks and therefore an overestimation of ζ occurs. For the first two sampling periods no errors could be evaluated because of the limited number of decimal digits provided by the software outputs.

Amplitudes have not been reconstructed because they are not only altered by the errors on the decay ratio estimates, but they are also modified by the effects due to the truncation of the autocorrelation function at the end of its observation time [14].

Another example, with a greater decay rate ($f_n=8.25$ Hz and $\zeta=0.01$), is presented in Tab.3:

Table 3 - Estimates from a higher decay rate impulse response.

T_s (s)	R	$\|\epsilon_R\|$ (%)	f_n	$\|\epsilon_f\|$ (%)	ζ	$\|\epsilon_\zeta\|$ (%)
0.020	10.00123	$12.33\ 10^{-1}$	8.24959	$0.50\ 10^{-2}$	$0.10001\ 10^{-1}$	$0.14\ 10^{-1}$
0.010	10.00031	$0.31\ 10^{-2}$	8.24959	$0.50\ 10^{-2}$	$0.10001\ 10^{-1}$	$0.90\ 10^{-2}$
0.005	10.00083	$0.83\ 10^{-2}$	8.24958	$0.50\ 10^{-2}$	$0.10002\ 10^{-1}$	$0.17\ 10^{-1}$
0.001	10.21345	2.13	8.25043	$0.52\ 10^{-2}$	$0.10544\ 10^{-1}$	5.44
0.0008	10.28766	2.88	8.24636	$0.44\ 10^{-1}$	$0.10940\ 10^{-1}$	9.40
0.0005	10.54330	5.43	8.28293	0.34	$0.12641\ 10^{-1}$	26.41
0.0003	12.76663	27.67	8.20648	0.53	$0.21966\ 10^{-1}$	119.66

In this case a baseband of 10 (Hz) has been considered, therefore the Shannon sampling period is equal to T_s=0.039 (s), on the contrary the limit sampling time equals 0.0005 (s). As in the first Table, errors on ζ increase as the sampling time lowers and unacceptable values have been obtained for T_s equal or less than the limit value. The order of magnitude of ϵ_ζ is similar to the one gained for the first example.

In the next Table two modes in the baseband of 10 (Hz) have been considered; modal parameter estimates achieved from autocorrelation functions of impulse responses with the same amplitudes and the same natural frequencies of the previous examples, but with the damping ratio equal to 0.003 for both the modes, are shown (Tab.4):

Table 4 - Modal parameter evaluations for two modes present in the same baseband (10 Hz) versus the sampling period.

T_s	mode	R	$\|\epsilon_R\|$ (%)	f_n	$\|\epsilon_f\|$ (%)	ζ	$\|\epsilon_\zeta\|$ (%)
0.035	1st	10.19364	1.94	0.71	-	$0.31820\ 10^{-2}$	6.07
	2nd	10.00335	$0.33\ 10^{-1}$	8.24995	$6.06\ 10^{-4}$	$0.30006\ 10^{-2}$	$0.21\ 10^{-1}$
0.030	1st	10.32731	3.27	0.71	-	$0.33169\ 10^{-2}$	10.56
	2nd	10.01256	0.13	8.24993	$8.49\ 10^{-4}$	$0.30025\ 10^{-2}$	$0.85\ 10^{-1}$
0.025	1st	10.52615	5.26	0.71	-	$0.35444\ 10^{-2}$	18.15
	2nd	10.00393	$0.39\ 10^{-1}$	8.24990	$1.21\ 10^{-3}$	$0.30009\ 10^{-2}$	$0.30\ 10^{-1}$
0.020	1st	10.80372	8.04	0.71	-	$0.39328\ 10^{-2}$	31.10
	2nd	10.00082	$0.82\ 10^{-2}$	8.24989	$1.33\ 10^{-3}$	$0.30003\ 10^{-2}$	$0.88\ 10^{-2}$
0.015	1st	11.26584	12.66	0.70996	$5.63\ 10^{-3}$	$0.47096\ 10^{-2}$	56.99
	2nd	9.97821	0.22	8.24984	$1.94\ 10^{-3}$	$0.29935\ 10^{-2}$	0.22
0.010	1st	11.97794	19.78	0.70973	$3.80\ 10^{-2}$	$0.64517\ 10^{-2}$	115.06
	2nd	9.95462	0.45	8.25002	$2.42\ 10^{-4}$	$0.29848\ 10^{-2}$	0.51

For sampling period up to the limit value 0.0195 (s), due to the first mode (i.e. the one with the lowest decay rate), errors on ζ increase to about 30% for the first mode, whereas they remain negligible for the second mode. In any case errors on natural frequencies are always irrelevant and therefore the ones on the amplitude are practically related to the errors on the correspondent ζ's. Obviously for shorter T_s's, errors on the damping ratios of the first mode result higher and higher, whereas for the second mode they always remain small. Due to the presence of more modes in the baseband, and since the Hilbert transform approach works on the single mode, a suitable filter must be applied. In particular, an adaptable cosine tapered filter has been used: its width has been chosen taking into account the shift of the peaks in the frequency domain, owing to the different sampling periods. Estimates from the previous two modes, when they are added to an uncorrelated random noise, with zero mean and standard deviation equal to 50% of the common impulse response amplitude, are shown in Tab.5 (Fig.2 shows the time history relative to T_s=0.02, whereas its autocorrelation is presented in Fig.3):

Table 5 - Effect of a high random noise on parameter estimates of the modes presented in Tab.4.

T_s	mode	R	$\|\epsilon_R\|$ (%)	f_n	$\|\epsilon_f\|$ (%)	ζ	$\|\epsilon_\zeta\|$ (%)
0.020	1st	9.97203	0.28	0.71019	$2.63\ 10^{-2}$	$0.32681\ 10^{-2}$	8.94
	2nd	12.32280	23.22	8.25263	$3.19\ 10^{-2}$	$0.34667\ 10^{-2}$	15.96
0.015	1st	11.54107	15.41	0.71043	$6.06\ 10^{-2}$	$0.48112\ 10^{-2}$	60.37
	2nd	9.32304	6.77	8.25823	$9.98\ 10^{-2}$	$0.24305\ 10^{-2}$	18.98
0.010	1st	11.68375	16.84	0.70958	$5.92\ 10^{-2}$	$0.60516\ 10^{-2}$	101.72
	2nd	8.69748	13.03	8.24198	$9.72\ 10^{-2}$	$0.24436\ 10^{-2}$	18.55

In this case the effect of the added noise, on the damping ratio estimations, is especially significant for the second mode because its impulse response is more rapidly damped out. Besides, even if the residues of the two modes are equal, the initial amplitudes of the autocorrelation functions - derived by filtering - result to be completely different: in fact - for the sake of simplicity - considering each mode independently and not taking account of the cross-correlation terms, autocorrelation amplitudes are inverse functions, being all the other values common, of the relative decay rates and so the second mode could have a much smaller amplitude than the first one:see Figs.4 and 5 for the filtered autocorrelation functions and Figs.6 and 7 for the relative envelopes.

4. Conclusions

In the use of an approach based on autocorrelation functions of impulse responses in order to obtain - in the time domain - the modal parameters, a possible source of error in the damping ratio estimation is connected with a bias due to a triangular window positioned around the time axis origin. Owing to this bias, it is necessary to evaluate - for example at the maximum time lag where the autocorrelation function is calculated - a minimum sampling period (lower limit), suitable to analyze the signal and in particular to estimate the decay rate and consequently the damping ratio. Although the value of this sampling period is a function of the unknown signal time constant, nevertheless it is possible, starting from an its first estimate, to update the value of the sampling period and so to eliminate bias errors. In this way, the peculiar advantages deriving from the use of autocorrelation functions - especially if evaluated from highly noisy impulse responses - to estimate modal parameters, and in particular damping ratios, can be thoroughly exploited.

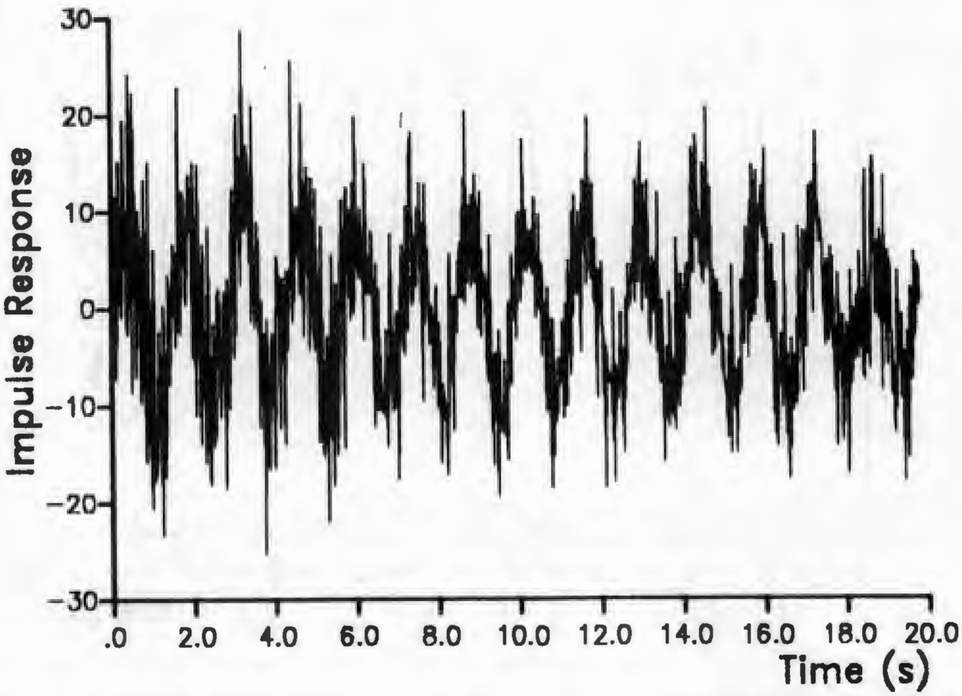

Figure 2 - Impulse response of Tab.5, relative to T_s=0.02 (s), contaminated by a random noise.

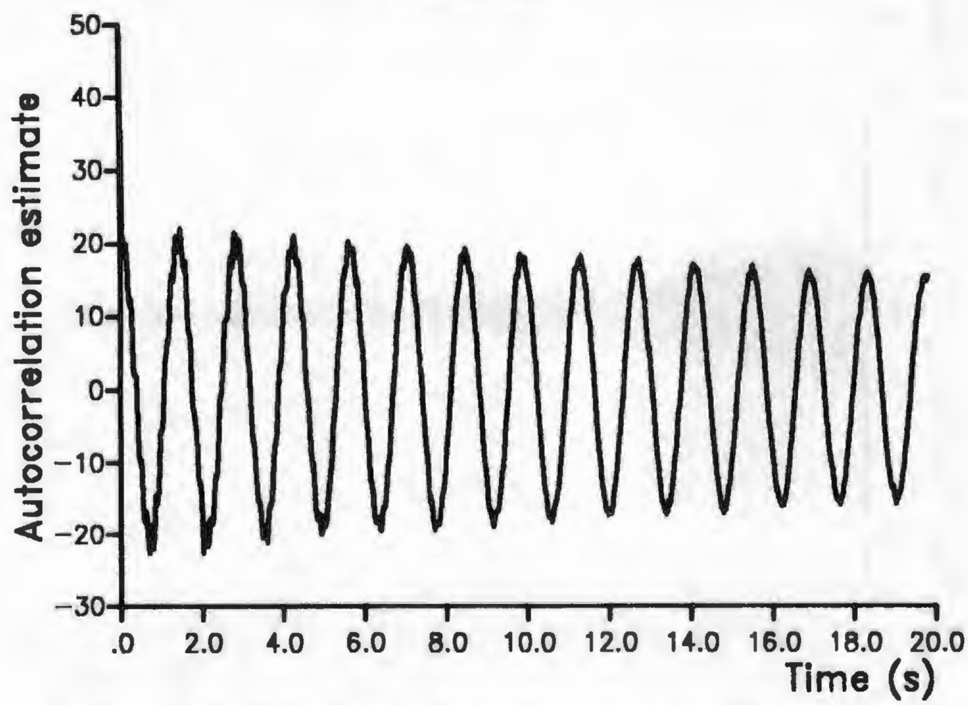

Figure 3 - Autocorrelation estimate of the time history presented in the previous Fig.

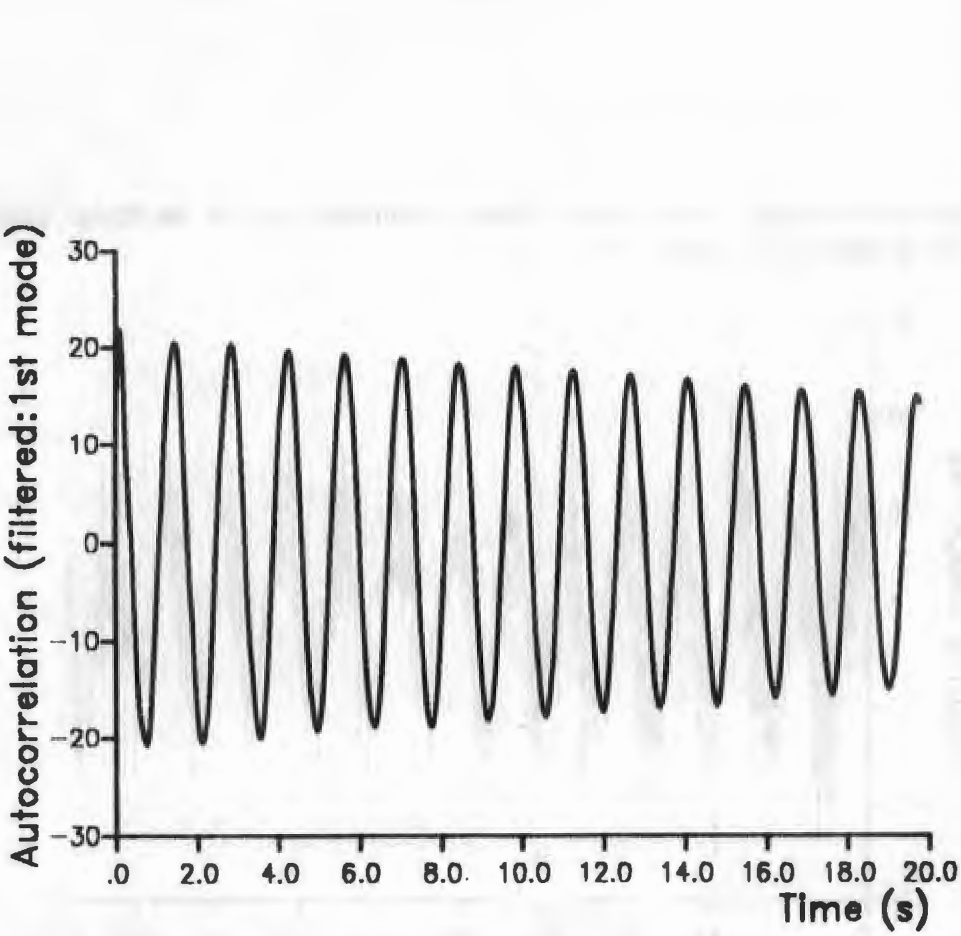

Figure 4 - Filtered autocorrelation function of the first mode.

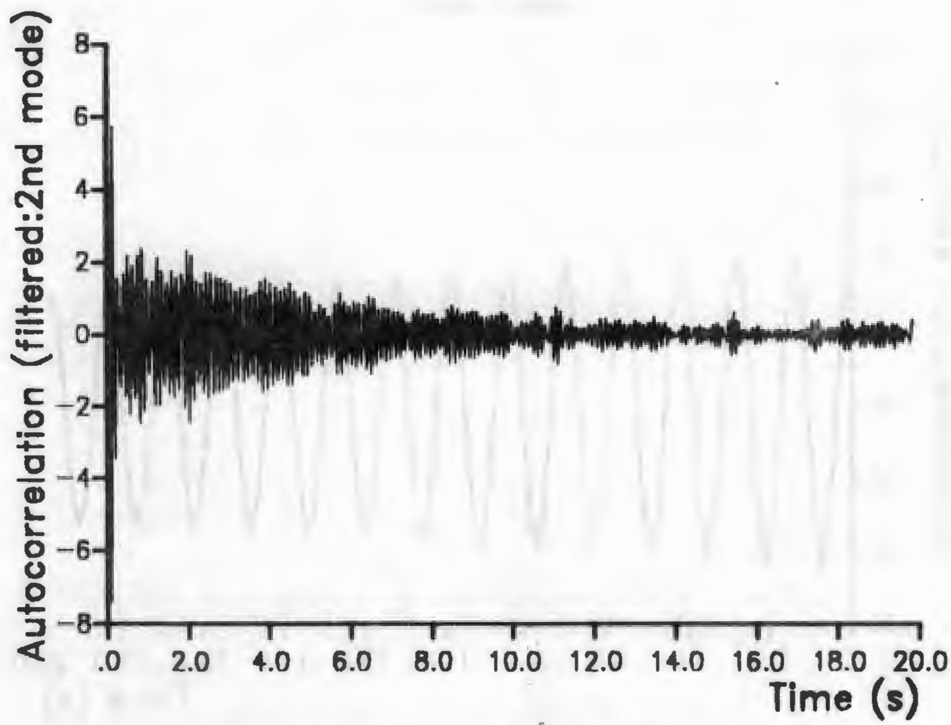

Figure 5 - Filtered autocorrelation function of the second mode.

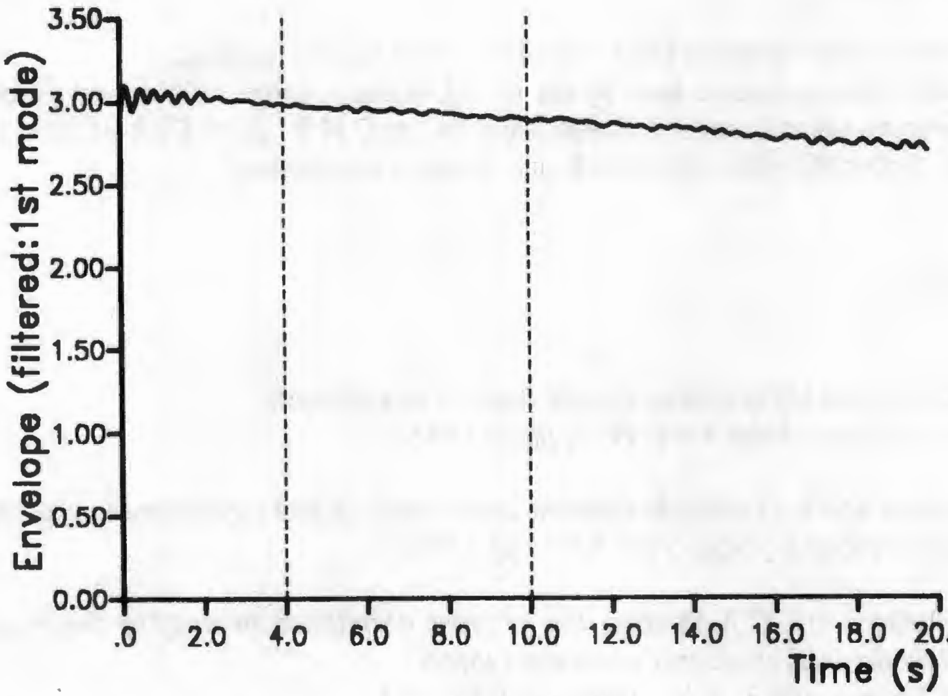

Figure 6 - Envelope relative to the first mode (semi-log plane):the vertical dashed lines contain the points used in the least squares regression.

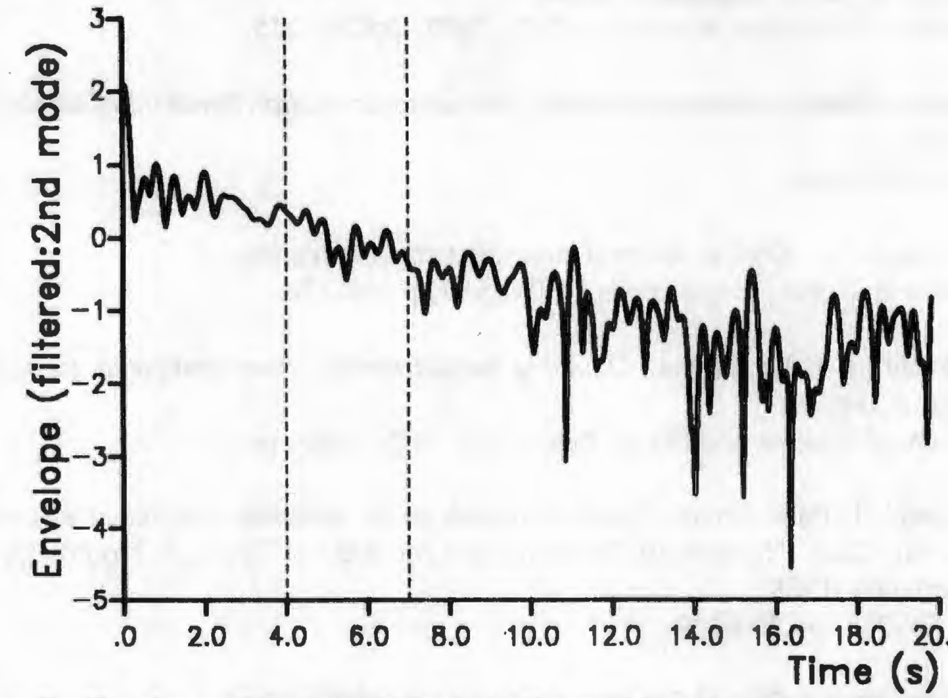

Figure 7 - Envelope of the second mode in the semi-log plane:the points used in the least squares regression are contained within the dashed lines.

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Appendix: Hilbert transform and modal parameter estimation

The time sequence (10) can be considered as an amplitude modulated signal with the carrier equal to the damped angular frequency and the modulating signal equal to the decaying exponential function: $[R^2/(4NT_s\sigma)] \exp(-\sigma m T_s)$. Since the spectrum of this last function is unbounded, relationship (10) and:

$$\hat{\rho}'_h[m] \cong \frac{R^2}{4NT_s\,\sigma}\, e^{-\sigma(mT_s)} \cdot \sin\left[\omega_d\left(mT_s\right)\right] \qquad \text{(A.1)}$$

are not strictly a pair of Hilbert transforms. Nevertheless, under proper conditions [8], the Bedrosian theorem [15,16] can be applied at least in the limit sense, thus the complex signal:

$$z[m] = \frac{R^2}{4NT_s\sigma} e^{-\sigma(mT_s)} \left\{ \cos\left[\omega_d\left(mT_s\right)\right] + j\sin\left[\omega_d\left(mT_s\right)\right] \right\} = \tag{A.2}$$

$$= \frac{R^2}{4NT_s\sigma} e^{-\sigma(mT_s)} e^{j\omega_d(mT_s)}$$

can be considered as analytic.

It is easy to recognize that the magnitude of z[m] represents the modulating function, whose decay rate could be directly estimated from the straight line:

$$\ln|z[m]| = \ln\left(\frac{R^2}{4NT_s\sigma}\right) - \sigma m T_s \tag{A.3}$$

whereas its argument is the instantaneous phase, the slope of which gives the damped angular frequency:

$$Arg\{z[m]\} = \omega_d m\, T_s \tag{A.4}$$

Consequently, it is possible to achieve the natural angular frequency:

$$\omega_n = \sqrt{\sigma^2 + \omega_d^2} \tag{A.5}$$

and the damping ratio:

$$\zeta = \frac{\sigma}{\omega_n} \tag{A.6}$$

The residue magnitude R can be evaluated introducing the known values and the estimate of σ, achieved from the envelope, into the initial amplitude of Eq.(10).

Good estimates of the parameters mentioned above can be gained performing least squares regressions on the two straight lines represented by Eqs.(A.3) and (A.4).