CONSTITUTIVE MODELING OF NONLINEAR DAMPING MATERIALS

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Abstract

Many of the damping materials that have been developed recently are used under transient conditions at levels of strain where their response is highly nonlinear (e.g., to improve the seismic performance of buildings and other structures in earthquake-prone regions). Transient analysis of these nonlinear materials cannot be done using Fourier or other transform methods; real time analysis is needed. In this study these materials are treated as viscoelastic and their nonlinear constitutive relations are constructed in the time domain using internal variables to account for inelastic behavior and damage. This is an approach which is popular in modern constitutive theory and has many advantages. It leads to a formulation in terms of a system of first-order nonlinear ordinary differential (or differential-functional) equations, which is computationally convenient. Furthermore, if the analyst has physical insights into the micromechanical processes that are producing the inelastic deformation and damage, then the internal variable formulation provides a means for mathematically expressing this information in the structure of the constitutive relation. Some prototypes are constructed and employed to predict hysteresis loops.

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INTRODUCTION

The use of polymeric materials in engineering systems is ubiquitous. The large variety of desirable properties these materials possess makes them beneficial in many kinds of applications. In particular, because of their dissipative properties, some of them are employed to damp the motions of mechanical and structural systems which are undergoing either steady-state or transient vibrations. Frequently, to properly design a system employing such materials for this purpose, it is essential to have a quantitative description of their thermomechanical properties. As a matter of analytical and computational convenience, these materials are usually idealized, with good accuracy, to be viscoelastic. Several different constitutive characterizations of viscoelastic materials are possible. Generally, that one is chosen which is the most convenient for the task at hand.

The standard model for the viscoelastic response of damping materials has been the complex modulus formulation. This model, which is very popular and well understood, has proved to be an accurate and convenient technique for the vast majority of damping analyses. The complex modulus model readily allows for the incorporation of test results from steady state sinusoidal testing, permitting frequency dependence and temperature dependence to be included in a relatively simple way. It is easy to use in predicting the response of damped systems to disturbances with well defined frequencies. When it is applied to the transient response of impulsively loaded or seismically loaded systems, the prediction of the response is made using transform techniques which are intrinsically linear.

However, in certain applications the material may be used at quite large strains where the assumption of linear response is unwarranted. An example is the use of damping materials in the form of add-on viscoelastic damping devices to improve the seismic performance of sub-standard buildings and other structures in earthquake prone regions. If viscoelastic dampers were used to rehabilitate these structure, it might be possible to economically improve their dynamic behavior so that they could conform with the performance requirements of current seismic building codes. The dampers could considerably reduce the number of cycles of large deformation to be sustained by the structure and could also reduce the level of inter-story drift. Both would contribute to the ability of the structure to survive moderate earthquakes with negligible damage and survive large earthquakes without collapse.

The verification of the design of such a rehabilitated structure will require time history analysis that includes modeling of the viscoelastic dampers to permit transient, large strain response. The subsequent transient analysis of the nonlinear viscoelastic materials in the add-on damping devices that will be required cannot be performed using Fourier or other transform methods; real time analysis must be used. It is also known from testing that the first application of a sinusoidal load to a previously unloaded damping device produces, particularly at large strains, hysteresis loops that are larger than subsequent loops. The material in the device reaches a steady state only after several cycles of loading. Clearly, this is a manifestation of nonlinear action. More specifically, this kind of behavior is associated with damage that is being done to the material by the deformation to which it has been previously subjected. In seismic applications, it could be crucial to include this transitory nonlinear behavior in predicting the response of the damped structure. It is for problems of this kind that the models described in this paper have been developed.

In the sequel, the construction of constitutive relations for nonlinear polymers (which, of course, includes linear behavior as a special case) in the time domain using internal variables to account for inelastic behavior will be considered. Such relations can be applied to describe the strain (or stress) response to transient and oscillatory stress (or strain) excitations. This is an approach which is popular in modern constitutive theory and has many advantages. It leads to a

formulation in terms of a system of first order (usually nonlinear) ordinary differential (or differential-functional) equations. From the point of view of computation, this is convenient. Furthermore, if the analyst has physical insights of the micromechanical processes that are producing the inelastic deformation, then the internal variable formulation provides a means for expressing this information in the mathematical structure of the constitutive relation. Thus, this methodology is particularly useful for introducing damage mechanisms into constitutive equations.

The paper ends by considering some examples of constitutive relations constructed by the internal variable formulation. Numerical integration of the system of nonlinear ordinary differential (or differential-functional) equations representing these models is employed to predict the hysteresis loops that would be produced if they were subjected to a sinusoidal oscillating strain input. It is seen that when appropriate modeling is achieved, hysteresis loops produced by the numerical simulations are close in form to those observed in tests performed on commercially available viscoelastic dampers.

DERIVATION OF CONSTITUTIVE EQUATIONS FOR A NONLINEAR POLYMER VIA INTERNAL VARIABLES

A powerful methodology for constructing constitutive relations for nonlinear viscoelastic materials is the use of internal variables. This method starts from first principles of mechanics of continua and non-equilibrium thermodynamics to deduce a general mathematical structure of the constitutive relations which does not violate any physical principles. This general structure can then be specialized to obtain a large variety of possible behavior.

Consider an arbitrary portion of a solid body made of a polymeric material. The arbitrary portion of the body occupies some region R of space and has a boundary surface B. The unit vector normal to the surface B at a generic point on B is denoted by n. The traction vector acting across B is given by t, the heat flow vector across B by h, the body force vector per unit of volume by f, and the particle velocity vector by v. It is assumed that the local thermodynamic state of the material exists and is uniquely given by the strain tensor ϵ (with Cartesian components ϵ_{ij}, $i,j = 1,2,3$), the entropy per unit volume S, and a set of internal variables q_α ($\alpha = 1, \ldots n$) which are represented by the column matrix q. The internal variables are used to introduce the influence of microstructural effects into the constitutive relations. For example, for general materials, they may be components of the internal strain or dislocation loop density. Or, they might be the extent of phase transformation or chemical reaction; concentration of lattice imperfections; parameters characterizing changes in molecular conformation, grain size, hardening, damage, coiling and bending of long chain molecules in polymers; etc. Thus, the local internal energy per unit of volume, U, is written as

$$U = U(\epsilon, S, q) \tag{1}$$

Now, consider the balance of thermal and mechanical energy for the arbitrary portion of the body. (In the following, attention is restricted to infinitesimal deformation theory. The results obtained here may be extended to large deformation theory by introducing proper measures of strain and stress and interpreting all quantities defined as per unit of volume as per unit of volume in the reference state.) The rate of work done on the body by the surface forces is

$$\int_B t \cdot v \, ds \tag{2}$$

The rate of work done by the body forces is

$$\int_R \boldsymbol{f}\cdot\boldsymbol{v}\,ds \tag{3}$$

and the rate of energy added to the body because of heat flow is

$$-\int_B \boldsymbol{h}\cdot\boldsymbol{n}\,ds \tag{4}$$

where the usual dot notation has been employed for the inner product. Assuming no internal heat sources, the sum of these energy rates must be equal to the rate of change of the total energy of the body. The total energy of the body is the sum of the internal energy and the kinetic energy. Thus,

$$\frac{d}{dt}\int_R \left(U + \frac{1}{2}\rho\boldsymbol{v}\cdot\boldsymbol{v}\right)dV = \int_B [\,\boldsymbol{t}\cdot\boldsymbol{v} - \boldsymbol{n}\cdot\boldsymbol{h}]\,ds + \int_R \boldsymbol{f}\cdot\boldsymbol{v}\,dV \tag{5}$$

where ρ is the mass density of the material.

Utilizing the balance of mass principle, the left-hand side of this equation may be written as

$$\frac{d}{dt}\int_R \left(U + \frac{1}{2}\rho\boldsymbol{v}\cdot\boldsymbol{v}\right)dV = \int_R (\dot{U} + \rho\dot{\boldsymbol{v}}\cdot\boldsymbol{v})\,dV \tag{6}$$

where a superposed dot represents differentiation with respect to time. Recalling that the traction vector may be written in terms of the symmetric stress tensor $\boldsymbol{\sigma}$ (with Cartesian components σ_{ij}) as

$$\boldsymbol{t} = \boldsymbol{n}\cdot\boldsymbol{\sigma} \tag{7}$$

the first integral on the right-hand side of Eq. (5) takes the form

$$\int_B [\boldsymbol{t}\cdot\boldsymbol{v} - \boldsymbol{n}\cdot\boldsymbol{h}]\,ds = \int_B \boldsymbol{n}\cdot[\boldsymbol{\sigma}\cdot\boldsymbol{v} - \boldsymbol{h}]\,ds \tag{8}$$

The right-hand side of Eq. (8) can be converted to a volume integral by use of the divergence theorem, yielding

$$\int_B \boldsymbol{n}\cdot[\boldsymbol{\sigma}\cdot\boldsymbol{v} - \boldsymbol{h}]\,ds = \int_R \nabla\cdot[\boldsymbol{\sigma}\cdot\boldsymbol{v} - \boldsymbol{h}]\,dV \tag{9}$$

where ∇ is the gradient operator. Thus, Eq. (5) can be written as

$$\int_R [(\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{f} - \rho\dot{\boldsymbol{v}})\cdot\boldsymbol{v} - (\dot{U} - \sigma_{ij}\dot{\epsilon}_{ij} + \nabla\cdot\boldsymbol{h})]\,dV = 0 \tag{10}$$

where the summation convention is in effect, and use has been made of the fact that the scaler inner product of a symmetric tensor with an antisymmetric tensor is zero.

From the balance of linear momentum, the quantity within the first set of parentheses vanishes. Since the balance of energy holds for any region $\boldsymbol{R}$ of the body, then assuming that the

integrand is continuous, it must vanish at every point in the region occupied by the body. This gives the local equation of balance of energy (the first law of thermodynamics) as

$$\dot{U} = \sigma_{ij}\dot{\epsilon}_{ij} - \nabla\cdot h = \frac{\partial U}{\partial \epsilon_{ij}}\dot{\epsilon}_{ij} + \frac{\partial U}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial U}{\partial S}\dot{S} \tag{11}$$

or

$$\left(\sigma_{ij} - \frac{\partial U}{\partial \epsilon_{ij}}\right)\dot{\epsilon}_{ij} - \frac{\partial U}{\partial q_\alpha}\dot{q}_\alpha - \frac{\partial U}{\partial S}\dot{S} - \nabla\cdot h = 0 \tag{12}$$

If T is taken to be the local absolute temperature, then the second law of thermodynamics can be written as the local Clausius-Duhem inequality

$$\dot{S} + \nabla\cdot(h/T) \geq 0 \tag{13}$$

Combining Eqs. (12) and (13) gives

$$\left(T - \frac{\partial U}{\partial S}\right)\dot{S} + \left(\sigma_{ij} - \frac{\partial U}{\partial \epsilon_{ij}}\right)\dot{\epsilon}_{ij} - \frac{\partial U}{\partial q_\alpha}\dot{q}_\alpha - \frac{1}{T}h\cdot\nabla T \geq 0 \tag{14}$$

Assuming that the $\dot{q}_\alpha$ are independent of $\dot{S}$ and $\dot{\epsilon}$, and recalling that $\dot{S}$ and $\dot{\epsilon}$ can be specified arbitrarily, their coefficients must vanish, so that

$$T = \frac{\partial U}{\partial S} \tag{15}$$

$$\sigma_{ij} = \frac{\partial U}{\partial \epsilon_{ij}} \tag{16}$$

and

$$\frac{\partial U}{\partial q_\alpha}\dot{q}_\alpha + h\cdot\nabla T/T \leq 0 \tag{17}$$

But, since U and $\dot{q}_\alpha$ are independent of ∇T, it follows from Eq. (17) that

$$\frac{\partial U}{\partial q_\alpha}\dot{q}_\alpha \leq 0 \tag{18}$$

Now, the left-hand side of Eq. (18) represents that part of the rate of change of U that is due to the changes in the internal variables. Thus, Eq. (18) states that *internal processes reduce the internal energy.*

It is convenient at this point to introduce an alternative thermodynamic potential by the use of a Legendre transformation from the internal energy. The Gibbs free energy (or the complementary free energy) G is given by

$$G = \sigma_{ij}\epsilon_{ij} - U + TS = G(\sigma, T, q) \tag{19}$$

with, from Eqs. (15)-(17), the properties that

$$S = \frac{\partial G}{\partial T} \tag{20}$$

$$\epsilon_{ij} = \frac{\partial G}{\partial \sigma_{ij}} = \epsilon_{ij}(\sigma, T, q) \tag{21}$$

$$\frac{\partial G}{\partial q_\alpha} \dot{q}_\alpha \geq 0 \tag{22}$$

The pair of equations (20) and (21) is a set of thermomechanical constitutive equations for the material. Constitutive equations must also be specified for the internal variables; then Eq. (22) is a restriction on the form that these constitutive equations can take.

The rate of dissipation of energy by the material, d (power per unit of volume), is given by the difference between the rate of work done by the stresses, $\sigma_{ij}\dot{\epsilon}_{ij}$, and the rate of change of the internal energy (at constant entropy), $\dot{U}$. Thus

$$d = \sigma_{ij}\dot{\epsilon}_{ij} - \dot{U} = \sigma_{ij}\dot{\epsilon}_{ij} - \frac{\partial U}{\partial \epsilon_{ij}}\dot{\epsilon}_{ij} - \frac{\partial U}{\partial q_\alpha}\dot{q}_\alpha \tag{23}$$

Using (16), (19), and (22), this gives

$$d = \frac{\partial G}{\partial q_\alpha}\dot{q}_\alpha \geq 0 \tag{24}$$

Thus, the inequality expressed in (22) is just the statement that the rate of dissipation is non-negative, as is to be expected on physical grounds.

The total strain rate may be obtained by taking the time derivative of (21):

$$\dot{\epsilon}_{ij} = \frac{\partial^2 G}{\partial \sigma_{ij}\partial \sigma_{kl}}\dot{\sigma}_{kl} + \frac{\partial^2 G}{\partial \sigma_{ij}\partial q_\alpha}\dot{q}_\alpha + \frac{\partial^2 G}{\partial \sigma_{ij}\partial T}\dot{T} \tag{25}$$

The tensors $\partial^2 G/\partial\sigma_{ij}\partial\sigma_{kl}$ and $\partial^2 G/\partial\sigma_{ij}\partial T$ are, respectively, the instantaneous elastic compliance tensor and the instantaneous thermal strain tensor. In the sequel, attention will be restricted to *isothermal processes* so that $\dot{T} = 0$, and then (91) becomes

$$\dot{\epsilon}_{ij} = \alpha_{ijkl}\dot{\sigma}_{kl} + \frac{\partial^2 G}{\partial \sigma_{ij}\partial q_\alpha}\dot{q}_\alpha \tag{26}$$

where $\alpha_{ijkl} = \partial^2 G/\partial\sigma_{ij}\partial\sigma_{kl}$ is the instantaneous compliance tensor. It is natural to interpret the first term on the right-hand side of (26) as the elastic strain rate, and the second term as the inelastic strain rate.

It is an observed experimental fact that the instantaneous compliances are largely independent of the internal variables, i.e., that

$$\frac{\partial}{\partial q_\alpha}(a_{ijkl}) = \frac{\partial^3 G}{\partial q_\alpha \partial \sigma_{ij}\partial \sigma_{kl}} = 0 \tag{27}$$

The structure of G can be found by integrating (27) to obtain

$$G(\sigma,q;T) = G^*(\sigma;T) + b_{ij}(q;T)\sigma_{ij} + c(q;T) \tag{28}$$

where, for isothermal responses, the temperature T plays the role of a simple parameter. For simplicity of notation, explicit mention of T will be suppressed in what follows.

G^* is independent of q_α, and thus can be thought of as the elastic part of the complementary free energy. Thus

$$\epsilon_{ij} = \frac{\partial G}{\partial \sigma_{ij}} = \frac{\partial G^*}{\partial \sigma_{ij}} + b_{ij}(q) = \epsilon_{ij}^{el} + \epsilon_{ij}^{inel} \tag{29}$$

and

$$\begin{aligned} d = \frac{\partial G}{\partial q_\alpha}\dot{q}_\alpha &= \left(\frac{\partial b_{ij}}{\partial q_\alpha}\dot{q}_\alpha\right)\sigma_{ij} + \frac{\partial c}{\partial q_\alpha}\dot{q}_\alpha \\ &= \dot{\epsilon}_{ij}^{inel}\sigma_{ij} + \frac{\partial c}{\partial q_\alpha}\dot{q}_\alpha \end{aligned} \tag{30}$$

If in Eq. (26) the range of α is restricted so that $\alpha = 1$, and if q_1 is interpreted in such a manner that the last term on the right hand side of (26) is taken to be the inelastic strain rate of the material, then (26) can be looked upon as the constitutive relation of a nonlinear Maxwell model. With that interpretation, and with the assumption that all of the dissipation is caused by the inelastic strain rate, then $c(q)$ can be neglected. Then, Eq. (26) reduces to

$$\dot{\epsilon}_{ij} = a_{ijkl}(\sigma)\dot{\sigma}_{kl} + \lambda_{ij}^{\alpha}(q)\dot{q}_\alpha \tag{31}$$

and

$$d = \left(\lambda_{ij}^{\alpha}(q)\dot{q}_\alpha\right)\sigma_{ij} \geq 0 \tag{32}$$

where

$$\lambda_{ij}^{\alpha}(q) = \frac{\partial b_{ij}(q)}{\partial q_\alpha} \tag{33}$$

For more general modeling, where Eq. (26) has an interpretation not restricted to a Maxwell model, $c(q)$ can *not* be neglected, and the dissipation rate is given by the more general expression (24) or (30), rather than the more restrictive one, (32).

It remains to specify the constitutive equations on the internal variables. Usually, these are taken as rate equations, or equations of evolution; that is, the time derivative of each internal variable is determined by the present state. In the simplest form, it is only the thermodynamic state variables (i.e., those which determine any one of the thermodynamic potentials U or G) which determine the rate; for example,

$$\dot{q}_\alpha = f_\alpha(\sigma,T,q), \quad \alpha = 1,2,\ldots n \tag{34}$$

For isothermal processes, T is just a parameter, and its appearance in f_α will be suppressed in the sequel. Thus, the constitutive relations for the material (undergoing an isothermal process) which relate stress, strain, and the internal variables are given by Eqs. (31) and (34), with the restriction given by (32). If attention is restricted to a uniaxial stress-strain relation, these equations reduce to

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E(\sigma)} + \lambda^{\alpha}(q)\dot{q}_{\alpha} \tag{35}$$

$$\dot{q}_{\alpha} = f_{\alpha}(\sigma,q) \qquad \alpha = 1,2,\ldots,n \tag{36}$$

with

$$\sigma\left[\lambda^{\alpha}(q)\dot{q}_{\alpha}\right] \geq 0 \tag{37}$$

(or the more general dissipation constraint, Eq. (24)) where $E(\sigma)$ is the instantaneous elastic modulus.

A type of equation more general than (34) can be used for the constitutive equations of the internal variables. For example, instead of the rate of evolution of the internal variables depending only on the current value of the thermodynamic state variables, they can be taken to depend also on the past history of these variables. Such a relationship would be expressed by means of a differential-functional equation, which is just a generalization of Eq. (34)

$$\dot{q}_{\alpha}(t) = \mathop{\mathscr{F}_{\alpha}}_{\tau=-\infty}^{\tau=t} [\sigma(\tau),T(\tau),q(\tau)], \qquad \alpha = 1,2,\ldots n \tag{38}$$

where $\mathscr{F}$ is a *functional* rather than an ordinary function. For the special case of a uniaxial stress-strain relationship for a material undergoing an isothermal process, Eq. (36) would become

$$\dot{q}_{\alpha} = \mathop{\mathscr{F}_{\alpha}}_{\tau=-\infty}^{\tau=t} [\sigma(\tau),q(\tau)], \qquad \alpha = 1,2,\ldots n \tag{39}$$

In practice, special cases of (39) can also be useful. Equation (36) is one such case. Another is when $\mathscr{F}$ is taken to be a functional of q but an ordinary function of σ. Use will be made of that form in the next section.

If the uniaxial form of the constitutive equations is considered, with α restricted to 1, we get from Eqs. (35)-(37)

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E(\sigma)} + \lambda(q)f(\sigma,q) \tag{40}$$

$$\dot{q} = f(\sigma,q) \tag{41}$$

with

$$\sigma[\lambda(q)\dot{q}] \geq 0 \tag{42a}$$

or the more general form, from (24),

$$d = \frac{\partial G}{\partial q}\,\dot{q} \geq 0 \tag{42b}$$

For simple forms of λ and f, this constitutive equation can be interpreted as that associated with a nonlinear Maxwell material or model, as mentioned previously, where $E(\sigma)$ is the stress-dependent modulus of elasticity of the nonlinear spring, and $\lambda(q)f(\sigma,q)$ is the nonlinear inelastic strain rate of the dashpot. If $E(\sigma)$ and $\lambda(q)$ are taken to be constant ($E(\sigma) = E$, $\lambda(q) = \lambda$), and $f(\sigma,q) = \sigma$, then the model is a *linear* Maxwell body with modulus of elasticity E and

viscosity = $1/\lambda$ (i.e., fluidity = λ). Notice that for the linear case, the constraint that $d \geq 0$ requires λ to be nonnegative.

If $E(\sigma) = E$, a constant, $\lambda(q) = 1$, and $f(\sigma,q) = \sigma/E\tau - q/\tau_1$, where E, τ and τ_1 are positive constants, then the resulting model is a standard linear solid with the constitutive relation

$$E\left(\dot{\epsilon} + \frac{1}{\tau_1} G\right) = \dot{\sigma} + \frac{1}{\tau} \sigma \tag{43}$$

where τ_1 and τ are the retardation and relaxation times of the model, respectively. Thus, it is seen that the constitutive relations given by (40) and (41), with the constraint (42) can lead to a rich variety of linear and nonlinear viscoelastic models. By utilizing more than one internal variable, this richness can be greatly increased, as is also true if functionals rather than functions are used in (41). In the next section, some numerical illustrations of the constitutive equations (40) and (41) are given.

EXAMPLES

The examples described below are simulations of a hysteresis test in which the imposed excitation is a sinusoidal oscillatory strain history. The material model chosen to illustrate applications of the theoretical methodology described above is a nonlinear Maxwell body with a viscosity that can degrade, depending on the maximum magnitude of the inelastic strain (the maximum magnitude of strain in the dashpot). This model displays a variety of behavior, depending on the values of the parameters controlling the magnitude of the fluidity (which is the reciprocal of the viscosity), the elastic modulus, and the amplitude and frequency of the imposed strain history. For certain values of the parameters, a steady-state is established quickly, yielding unchanging hysteretic loops. Other values produce a stress response that takes many cycles before a steady-state is reached. This gives a stress-strain diagram in which the hysteresis loops continually change until a constant loop evolves.

The model employed is described by Eqs. (40) and (41) with $\lambda(q) = 1$ and

$$f[\sigma,q] = F[\sigma(t),q(t)] \cdot \left|\frac{\sigma}{E}\right|^c \cdot \left(\frac{\sigma}{E}\right) \tag{44}$$

where $F[\sigma,q]$ may be thought of as the fluidity of the nonlinear dashpot of the Maxwell model. The viscosity is taken to decrease (i.e., the dashpot becomes damaged) with the maximum magnitude of the strain it experiences. Thus, the fluidity is an increasing function of the maximum absolute value of the inelastic strain. In particular, in this study the fluidity function was taken to be

$$F[\sigma(t),q(t)] = \max_{0\leq\tau\leq t} \left[F_1 + F_0\left|\frac{q(\tau)}{q_o}\right|^b\right] \tag{45}$$

where b, c, F_o, F_1 and q_o are constants. These constants were given different values in the course of the investigation to see how they affected the response of the model to a sinusoidal oscillatory test (i.e., a hysteresis test) with $\epsilon(t) = A \sin t/T$, where A is the maximum strain amplitude imposed and $2\pi T$ is the period of the imposed oscillation.

It should be noted that by choosing $F_o = 0$, damage is eliminated. The response of the model to the sinusoidal input was obtained by a numerical integration of the first order system of Eqs. (40) and (41). Quiescence is assumed to exist for $t < 0$.

For the first example, the parameters were chosen such that $A = 1$, $T = 1$, $F_o = 0$, $F_1 = 0.8$, $q_o = 1$, $b = 0$ and $c = 0$. This corresponds to a linear Maxwell material with no damage. The normalized results are shown in Fig. 1. Because this model is linear, the normalized results are independent of the magnitude of A. The form of the results is typical of linear dissipative behavior. The stress leads the strain, and, after a brief transient passes (which takes less than a full cycle), is of sinusoidal form. The hysteresis loops quickly reach a steady-state and they all fall on a single tilted ellipse. Because there is no damage, the fluidity remains constant.

In the second example, $A = 1$, $T = 1$, $F_o = 0$, $F_1 = 0.8$, $q_o = 1$, $b = 0$ and $c = 1$. This corresponds to a nonlinear Maxwell material with no damage. The normalized results are shown in Fig. 2. Because this model is nonlinear, the normalized results depend on the magnitude of A. Again the stress leads the strain, and, after a transient of less than a full period, attains a steady-state form. However, it is not sinusoidal. The hysteresis loops again quickly reach a steady-state and collapse onto a single closed, convex curve, which is clearly not an ellipse.

If, in the above example, A is increased to 4, then the results in Fig. 3 are obtained. They are similar to those shown in Fig. 2, but the nonlinear features are more apparent. The greater nonlinear action which occurs in this model when the maximum amplitude of the imposed sinusoidal oscillatory strain, A, is increased from 4 to 1, results in a more distorted shape for the stress history, a greater phase shift between the stress and the strain, and a steady-state hysteresis loop which has associated with it a greatly decreased stiffness.

In the last example, $A = 1$, $T = 1$, $F_o = 5$, $q_o = 1$, $b = 4$ and $c = 1$. This is a nonlinear Maxwell material with damage. It is seen that due to the damage, the fluidity increases over several cycles before a steady-state value is attained. This results in a stress history which leads the strain, has a non-sinusoidal form, and which takes several cycles before it attains a steady-state periodic behavior. The hysteresis loops are far from being elliptical in form, and do not collapse onto a single closed, convex curve until the passage of several cycles. This is due to the damage. These results are of the type actually observed in hysteresis tests performed in the laboratory on many types of polymers undergoing large strain cycles.

CONCLUDING REMARKS

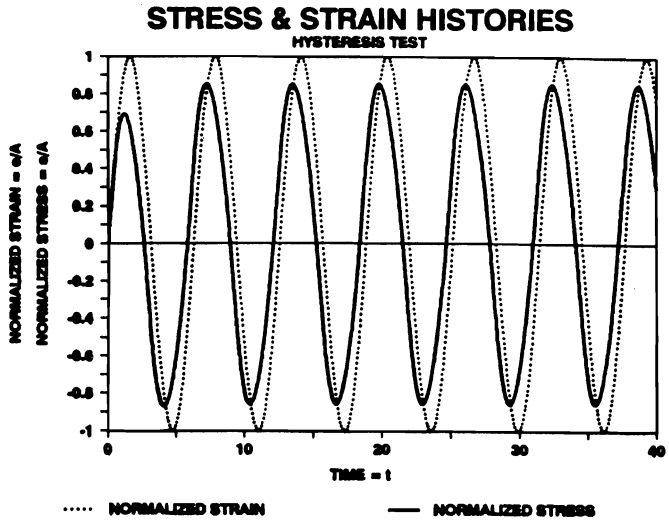
In order to determine the response of a nonlinear polymer to transient excitation, a proper constitutive relation is required. In this paper the method of internal variables was explored as a means of providing such a relation. It is seen that this is a very rich approach. By the use of this method, constitutive relations valid for transient phenomena could easily be generated that predict the kind of behavior for a sinusoidal oscillatory test (hysteresis test) that is actually observed in the laboratory.

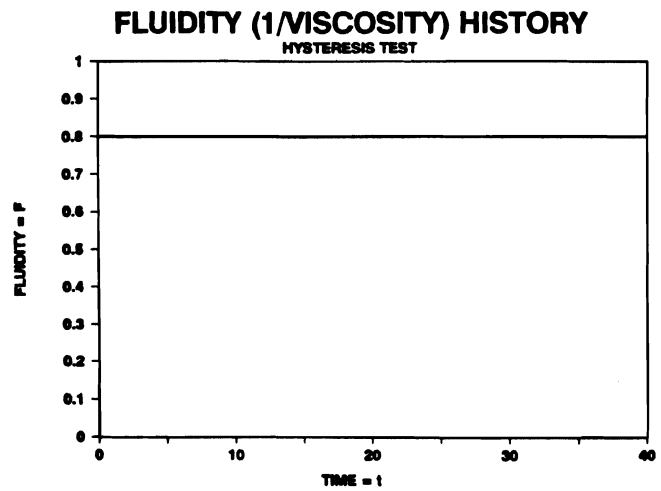
A fundamental practical problem associated with this method is the determination of the forms of the functions and the values of the parameters appearing in those forms. This leads to the study of inverse problems or identification. How from sinusoidal oscillatory tests can these forms and parameters be determined? Indeed, what other types of tests should be utilized to that end? A good deal of research has been done in the general area of inverse and identification

problems, and there is a large body of literature available. This material should be perused to determine what portions of it are relevant to the problem at hand.

Many problems associated with modeling by means of internal variables need further investigation. The use of more than a single internal variable should be examined. Employment of internal variables to represent damage of the material and the introduction of the formal concepts of damage mechanics should be attempted. In the constitutive equations of the internal variables, the use of functional (rather than function) relationships should be explored. Attempts should be made to introduce physics at the microstructural level as a basis for developing the constitutive equations governing the evolution of the internal (and damage) variables. The implementation of the methodology of internal variables should be broadened to include thermal effects (heat generation and conduction, and thermal expansion). Finally, an extension to deal with three-dimensional states of stress and strain ought to be considered; this would probably introduce the notion of loading surfaces (similar to those employed in plasticity theory) in order to ensure positive dissipation.

It is evident that much work remains to be done in this interesting and practical problem area.





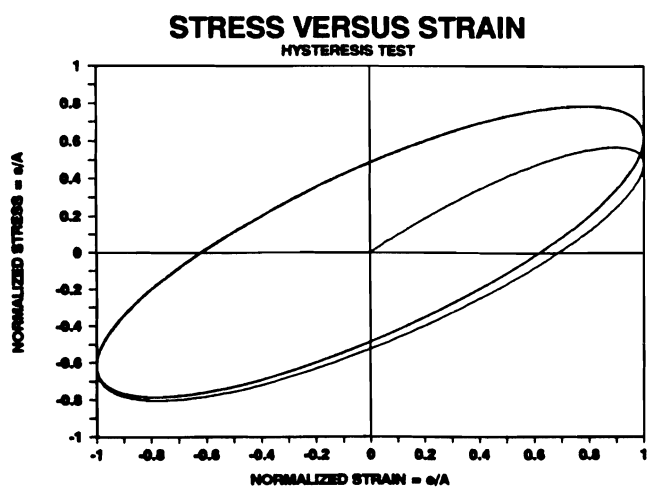
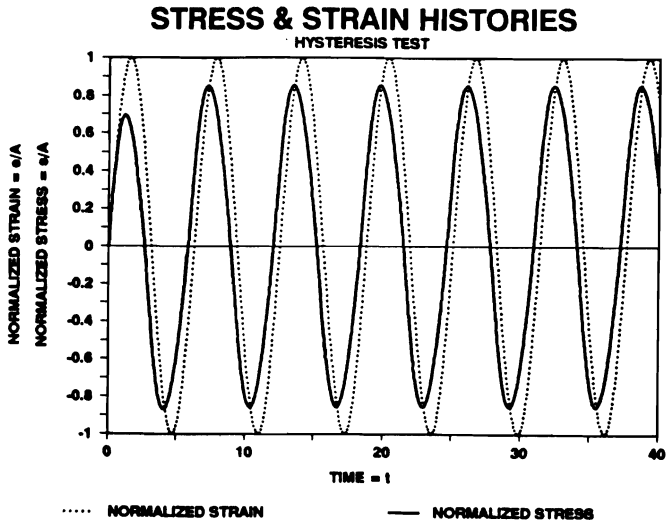
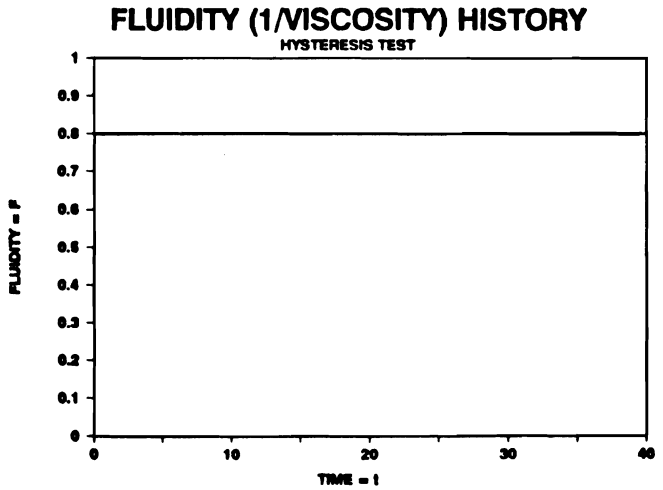


FIGURE 1: HYSTERESIS TEST ON A LINEAR MAXWELL MATERIAL





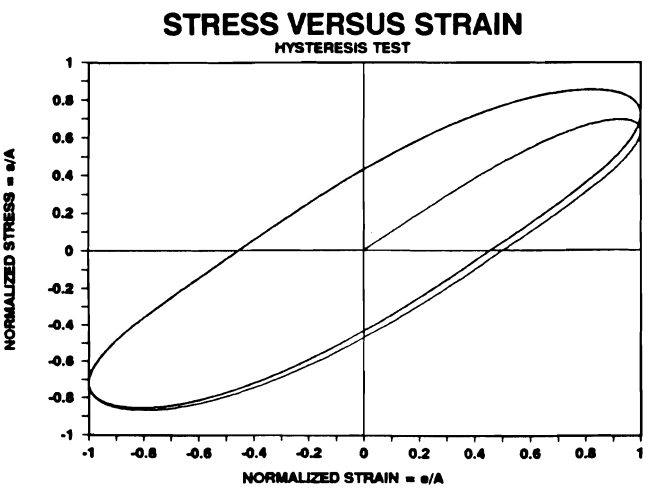
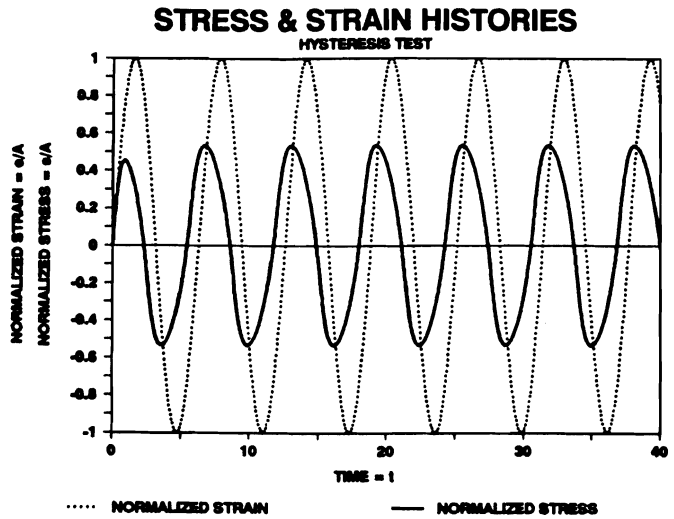
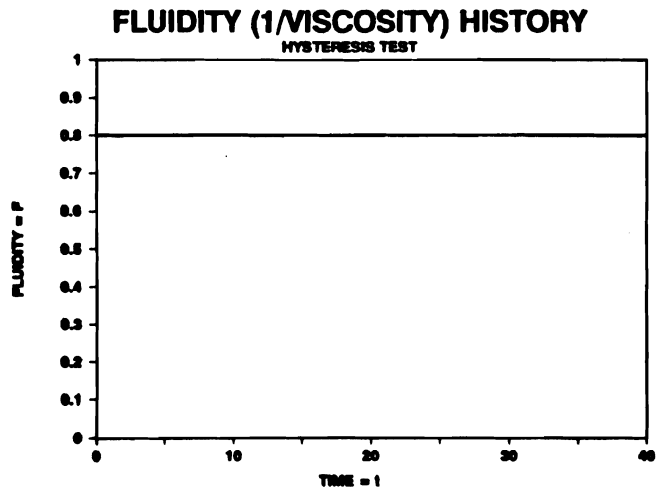


FIGURE 2: HYSTERESIS TEST ON A NONLINEAR MAXWELL MATERIAL, A = 1





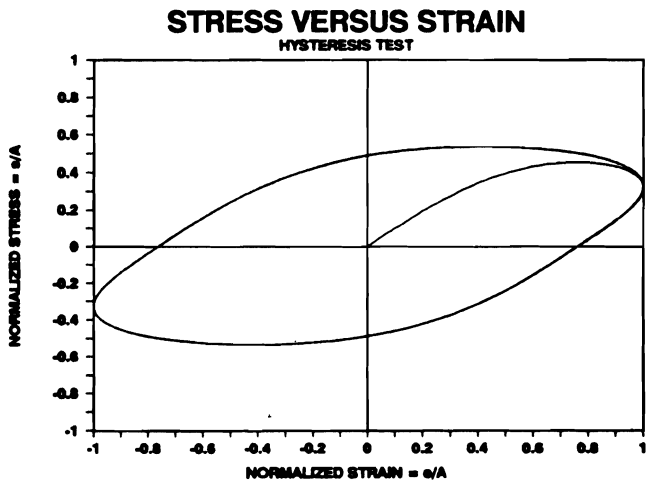
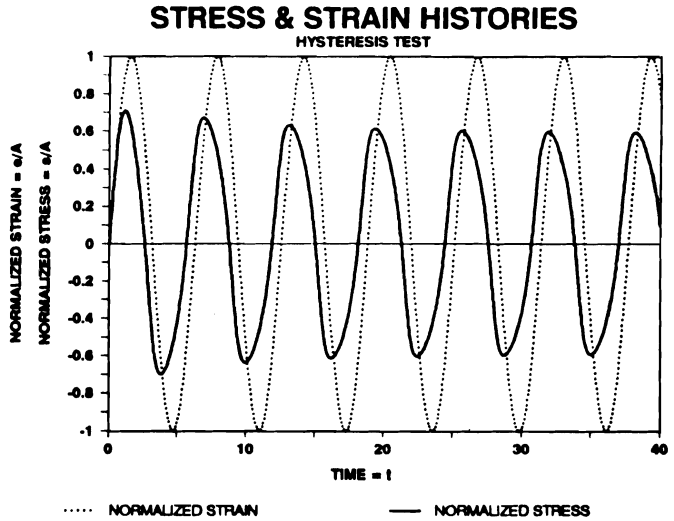
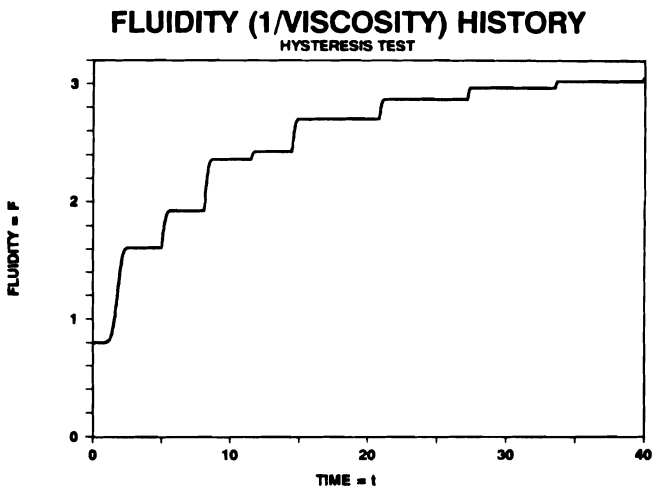


FIGURE 3: HYSTERESIS TEST ON A NONLINEAR MAXWELL MATERIAL, A = 4





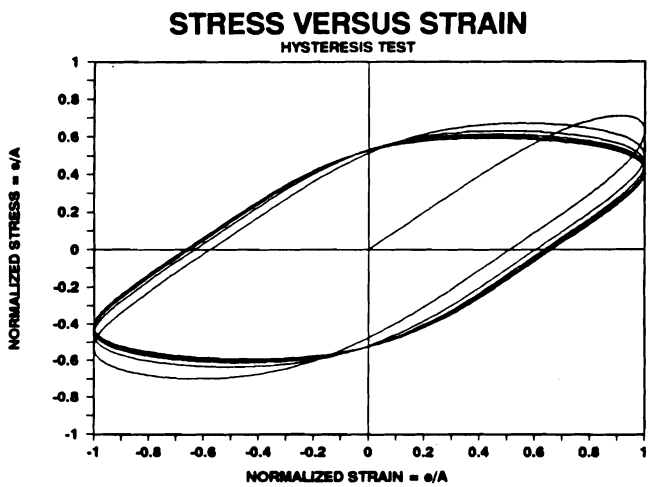


FIGURE 4: HYSTERESIS TEST ON A NONLINEAR MAXWELL MATERIAL WITH DAMAGE