# EXAMINATION OF BOUNDARY CONDITIONS FOR SIXTH-ORDER DAMPED BEAM THEORY

by
Ralph E. Tate
LTV Aircraft Products Group
Dallas, TX

**ABSTRACT**

The purpose of sixth-order beam theory is to include the effects of core shearing due to extentional deformation in terms of the transverse displacements. The constraint to eliminate the extentional motion reduces a twelfth-order system of equations into a single sixth-order equation.

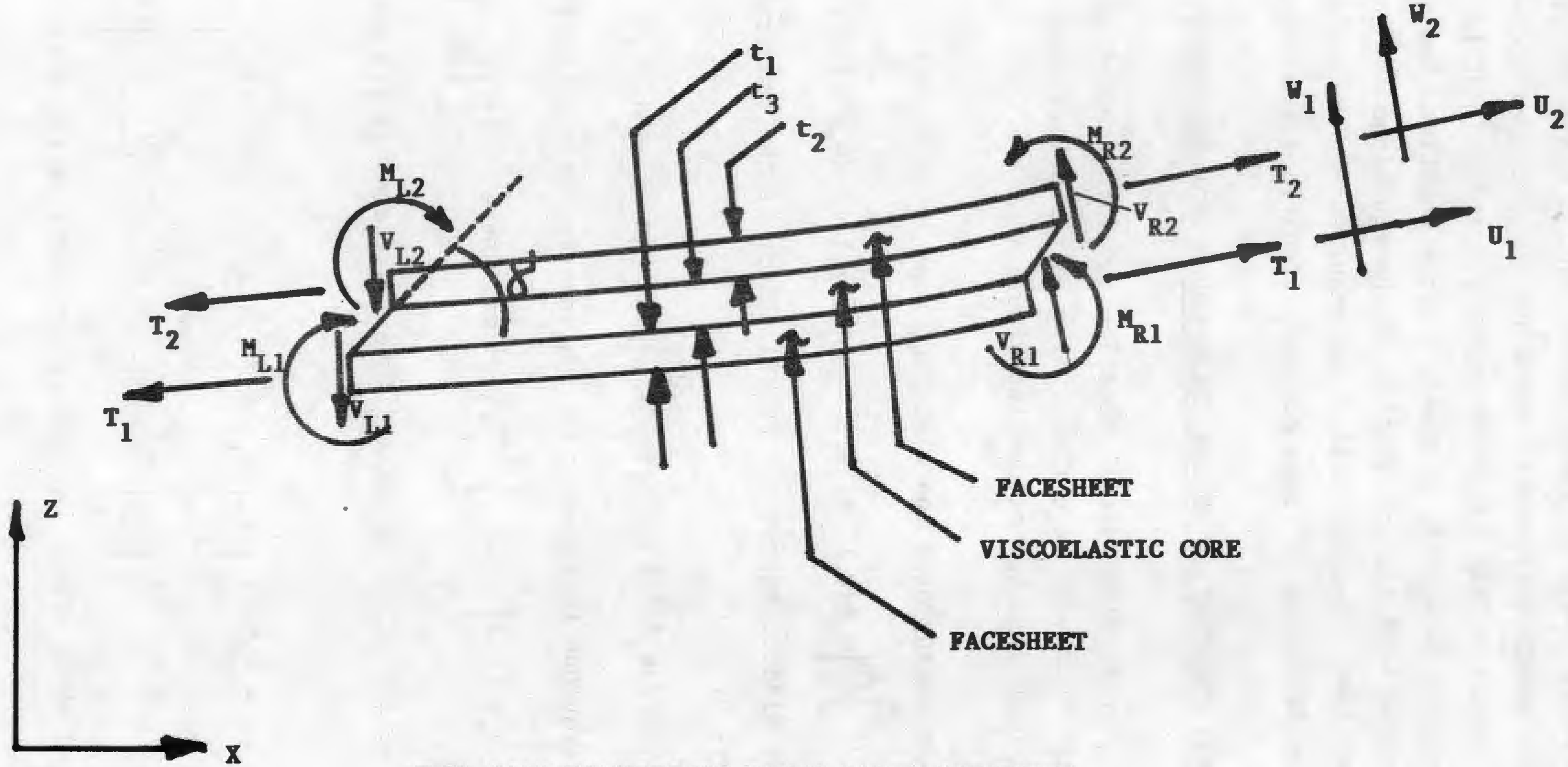
Since boundary conditions are necessary to completely specify the solution of partial differential equations, the author purposes to use this forum to present a detailed derivation of the sixth-order equation of motion using energy method techniques. The boundary conditions follow naturally as a consequence of the energy method formulation. The author show how two "natural" boundary conditions are lost, and must be replaced by two "kinematic" boundary conditions. The author interprets the boundary conditions and their consequences in the analysis of damped beams.

## 1.0 INTRODUCTION

Usage of constrained-layer damping composites for sound and vibration suppression originated in the 1950's, perhaps earlier[k]; however, the underlying theory was based on $4^{th}$-order beam theory and presumed no extensional deformation of the laminate. In order to include the extensional flexibility, several authors developed $6^{th}$-order beam and plate theories to describe the dynamic behavior of damped composite laminates [a,b,c,g,h,i]. The purpose of the $6^{th}$-order theories is to include the effects of core shearing due to the extension of the face sheets, in terms of the transverse bending displacements. The constraint to eliminate the extensional motion causes the equations of motion to be of $6^{th}$-order.

Dowell[h] and Miles[c] derive the laminate equations using an energy method approach to obtain the equations of motion. Dowell then retains terms only to $4^{th}$-order, since the adhesive shear layer is assumed stiff. The resulting boundary conditions are those found for $4^{th}$-order beam and plate theory. Dowell's formulation is useful in evaluating the interlaminar shear in fiber composites. Miles' obtains the $6^{th}$-order equations as a side discussion to validate his model; he does not elaborate on the boundary conditions required for solution for the $6^{th}$-order system. Miles study proceeds to thickness effects on damping.

Mead[a,b] and Abdulhadi[g] derive the equations of motion from a standard strength of materials perspective. This approach does not directly yield the boundary conditions as part of the formulation. Abdulhadi also does not articulate the boundary conditions necessary for solution: simply supported boundary conditions are presumed. Mead develops the boundary conditions and discusses the solutions for various boundary conditions. Maynor [j] numerically evaluated the effect of Mead's boundary conditions on loss factor estimates. He also observed that Abdulhadi deleted two boundary conditions in obtaining his solution. Essentially, Abdulhadi's equations of motion are equivalent to the $4^{th}$-order RKU equations [k]. Maynor delineated the difficulties and limitations in using the $6^{th}$-order equations.




FREE-BODY DIAGRAM FOR THREE-LAYER COMPOSITE

In the attached paragraphs, the $6^{th}$-order partial differential equation is obtained using energy methods. Subsequently, the boundary conditions are obtained. The limiting procedure shows how two "natural" boundary conditions are lost. Thus, two "kinematic" boundary conditions must be specified, that further reduce the generality of the $6^{th}$-order equation. The basic outline of the paper begins with the derivation of the equations of motions following Miles' assumptions, then validating the required boundary conditions used by Mead. The ramifications of those boundary conditions are discussed.

## 2.0 VARIATIONAL FORMULATION OF THE SIXTH-ORDER BEAM EQUATIONS

The sixth-order differential equation governing the vibration of a three layer sandwich beam will be derived using a variational approach. The beam geometry is depicted in the preceeding Figure.

The kinetic energy of a the vibrating beam is given by:

$$T = 1/2 \int_0^L \left[ m_1[(\dot{w}_1)^2 + (\dot{u}_1)^2] + m_2[(\dot{w}_2)^2 + (\dot{u}_2)^2] \right] dx. \qquad (2\text{-}1)$$

Similarly the elastic energy due to deflection of the constraining skin materials is given by:

$$V_e = 1/2 \int_0^L \left[ (EI)_1 (w_1'')^2 + (EI)_2 (w_2'')^2 + (EA)_1 (u_1')^2 + (EA)_2 (u_2')^2 \right] dx. \qquad (2\text{-}2)$$

The strain energy due to shearing of the adhesive core material is[d]:

$$V_s = 1/2 \int_0^L Gb \int_0^{t_3} \gamma^2 \, d\tau \, dx, \text{ where } \quad \gamma = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}. \qquad (2\text{-}3)$$

The shear traction at the upper and lower surfaces of the adhesive is found to be:

$$\gamma_1 = w_1' + \left(\frac{t_1}{2t_3}\right) w_1' + \left(\frac{t_2}{2t_3}\right) w_2' + \left(\frac{u_1 - u_2}{t_3}\right). \qquad (2\text{-}4a)$$

$$\gamma_2 = w_2' + \left(\frac{t_2}{2t_3}\right) w_2' + \left(\frac{t_1}{2t_3}\right) w_1' + \left(\frac{u_1 - u_2}{t_3}\right). \qquad (2\text{-}4b)$$

The distributed shear strain throughout the adhesive thickness is:

$$\gamma = \gamma_1 + \tau \left(\frac{(\gamma_1 - \gamma_2)}{t_3}\right) . \qquad (2\text{-}5)$$

After substitution and integration over the thickness, the strain energy due to shearing of the adhesive is found to be:

$$V_s = 1/2 \int_0^L Gbt_3 \left\{\gamma_1\gamma_2 + \frac{(\gamma_2 - \gamma_1)^2}{3}\right\} dx \qquad (2\text{-}6)$$

In order to apply Hamilton's Principle, the total energy in the vibrating beam is given as $Q=T-V_e-V_s+W_{nc}$, where

$$Q= \int_0^L F(\dot{u}_1,\dot{u}_2,\dot{w}_1,\dot{w}_2,u_1,u_2,w_1,w_2,u_1',u_2',w_1',w_2',w_1'',w_2'';x,t)\, dx. \qquad (2\text{-}7)$$

Applying Hamilton's Principle, the variational of the energy is minimized, that is:

$$\delta J = \delta \int_1^2 Q(.)\, dt = 0.$$

Hence, the differential of J(.) is:

$$\delta J= \int_1^2 \left(\int_0^L \left(\left[\frac{\partial F}{\partial \dot{u}_1} \delta\dot{u}_1 + \frac{\partial F}{\partial u_1} \delta u_1 + \frac{\partial F}{\partial u_1'} \delta u_1'\right]\right.\right. \qquad (2\text{-}8)$$

$$+ \left[\frac{\partial F}{\partial \dot{u}_2}\delta\dot{u}_2 + \frac{\partial F}{\partial u_2}\delta u_2 + \frac{\partial F}{\partial u_2'}\delta u_2'\right] + \left[\frac{\partial F}{\partial \dot{w}_1}\delta\dot{w}_1 + \frac{\partial F}{\partial w_1}\delta w_1 + \frac{\partial F}{\partial w_1'}\delta w_1' + \frac{\partial F}{\partial w_1''}\delta w_1''\right]$$

$$\left.\left.+ \left[\frac{\partial F}{\partial \dot{w}_2}\delta\dot{w}_2 + \frac{\partial F}{\partial w_2}\delta w_2 + \frac{\partial F}{\partial w_2'}\delta w_2' + \frac{\partial F}{\partial w_2''}\delta w_2''\right]\right)dx\right)dt . \qquad (2\text{-}8)$$

After integration by parts, the integral appears as:

$$\delta J= \int_1^2 \left(\int_0^L \left(\left[\frac{\partial F}{\partial u_1} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}_1}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial u_1'}\right)\right]\delta u_1 + \left[\frac{\partial F}{\partial u_2} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}_2}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial u_2'}\right)\right]\delta u_2\right.\right.$$

$$+ \left[\frac{\partial F}{\partial w_1} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{w}_1}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial w_1'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial w_1''}\right) - P_1(x,t)\right]\delta w_1$$

$$\left.+ \left[\frac{\partial F}{\partial w_2} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{w}_2}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial w_2'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial w_2''}\right) - P_2(x,t)\right]\delta w_2\right) dx$$

$$+ \left[\frac{\partial F}{\partial u_1'} - T_1\right]\delta u_1\Big|_{x=L} + \left[\frac{\partial F}{\partial u_2'} - T_2\right]\delta u_2\Big|_{x=L} + \left[\frac{\partial F}{\partial w_1'} - \frac{d}{dx}\left(\frac{\partial F}{\partial w_1''}\right) - V_{R1}\right]\delta w_1\Big|_{x=L}$$

$$- \left[\frac{\partial F}{\partial w_1'} - \frac{d}{dx}\left(\frac{\partial F}{\partial w_1''}\right) + V_{L1}\right]\delta w_1\Big|_{x=0} + \left[\frac{\partial F}{\partial w_2'} - \frac{d}{dx}\left(\frac{\partial F}{\partial w_2''}\right) - V_{R2}\right]\delta w_2\Big|_{x=L} \qquad (2\text{-}9)$$

$$- \left[\frac{\partial F}{\partial w_2{}'} - \frac{d}{dx}\left(\frac{\partial F}{\partial w_2{}''}\right) + V_{L2}\right]\delta w_2\Big|_{x=0} + \left[\frac{\partial F}{\partial w_1{}''} - M_{R1}{}'\right]\delta w_1{}'\Big|_{x=L} - \left[\frac{\partial F}{\partial w_1{}''} + M_{L1}{}'\right]\delta w_1{}'\Big|_{x=0}$$

$$+ \left[\frac{\partial F}{\partial w_2} - M_{R2}{}'\right]\delta w_2{}'\Big|_{x=L} - \left[\frac{\partial F}{\partial w_2{}''} + M_{L2}{}'\right]\delta w_2{}'\Big|_{x=0} - \left[\frac{\partial F}{\partial u_1{}'} + T_1\right]\delta u_1\Big|_{x=0} - \left[\frac{\partial F}{\partial u_2{}'} + T_2\right]\delta u_2\Big|_{x=0}\Bigg)\,dt.$$

Equating each variational term to zero then yields the equations of motion and the required "natural" (or force-type) boundary conditions. Thus, the complete system of equations is found to be:

$$\frac{\partial F}{\partial u_1} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}_1}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial u_1{}'}\right) = 0, \quad \frac{\partial F}{\partial u_2} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}_2}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial u_2{}'}\right) = 0,$$

$$\frac{\partial F}{\partial w_1} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{w}_1}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial w_1{}'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial w_1{}''}\right) = P_1(x,t), \qquad \text{(Eqns. 2-10)}$$

$$\text{and} \quad \frac{\partial F}{\partial w_2} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{w}_2}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial w_2{}'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial w_2{}''}\right) = P_2(x,t)$$

The system is subject to the following boundary conditions:

$$\frac{\partial F}{\partial u_1{}'} - T_1\Big|_{x=L} = 0, \quad \frac{\partial F}{\partial u_2{}'} - T_2\Big|_{x=L} = 0, \quad \frac{\partial F}{\partial w_1{}'} - \frac{d}{dx}\left(\frac{\partial F}{\partial w_1{}''}\right) - V_{R1}\Big|_{x=L} = 0,$$

$$\frac{\partial F}{\partial w_1{}'} - \frac{d}{dx}\left(\frac{\partial F}{\partial w_1{}''}\right) + V_{L1}\Big|_{x=0} = 0, \quad \frac{\partial F}{\partial w_2{}'} - \frac{d}{dx}\left(\frac{\partial F}{\partial w_2{}''}\right) - V_{R2}\Big|_{x=L} = 0, \qquad \text{(Eqns. 2-11)}$$

$$\frac{\partial F}{\partial w_2{}'} - \frac{d}{dx}\left(\frac{\partial F}{\partial w_2{}''}\right) + V_{L2}\Big|_{x=0} = 0, \quad \frac{\partial F}{\partial w_1{}''} - M_{R1}{}'\Big|_{x=L} = 0, \quad \frac{\partial F}{\partial w_1{}''} + M_{L1}{}'\Big|_{x=0} = 0,$$

$$\frac{\partial F}{\partial w_2{}''} - M_{R2}{}'\Big|_{x=L} = 0, \quad \frac{\partial F}{\partial w_2{}''} + M_{L2}{}'\Big|_{x=0} = 0, \quad \frac{\partial F}{\partial u_1} + T_1\Big|_{x=0} = 0, \quad \text{and} \quad \frac{\partial F}{\partial u_2} + T_2\Big|_{x=0} = 0.$$

The system is comprised of two fourth-order equations and two second-order equations. The twelve "natural" boundary conditions completely specify the solution. Hence, this set of differential equations is well-posed, as should be expected.

Next, the various partial derivatives of F(.) are derived:

$$\frac{\partial F}{\partial \dot{u}_1} = m_1\dot{u}_1, \quad \frac{\partial F}{\partial \dot{u}_2} = m_2\dot{u}_2, \quad \frac{\partial F}{\partial \dot{w}_1} = m_1\dot{w}_1, \quad \frac{\partial F}{\partial \dot{w}_2} = m_2\dot{w}_2, \quad \frac{\partial F}{\partial w_1{}''} = -(EI)_1 w_1{}'', \quad \frac{\partial F}{\partial w_2{}''} = -(EI)_2 w_2{}''$$

$$\frac{\partial F}{\partial u_1{}'} = -(EA)_1 u_1{}', \quad \frac{\partial F}{\partial u_2{}'} = -(EA)_2 u_2{}',$$

$$\frac{\partial F}{\partial u_1} = -(1/2)Gb\left\{(1+2\alpha_1)w_1{}' + (1+2\alpha_2)w_2{}' + 2\left(\frac{u_1-u_2}{t_3}\right)\right\},$$

$$\frac{\partial F}{\partial u_2} = (1/2)Gb\left\{(1+2\alpha_1)w_1{}' + (1+2\alpha_2)w_2{}' + 2\left(\frac{u_1-u_2}{t_3}\right)\right\},$$

$$\frac{\partial F}{\partial w_1'} = -(1/2)Gbt_3\left\{[2/3+2\alpha_1(1+\alpha_1)]w_1' + [-2/3+(1+\alpha_1)(1+\alpha_2)+\alpha_1\alpha_2]w_2' + (1+2\alpha_1)\left(\frac{u_1-u_2}{t_3}\right)\right\},$$ and

$$\frac{\partial F}{\partial w_2'} = -(1/2)Gbt_3\left\{[-2/3+(1+\alpha_1)(1+\alpha_2)+\alpha_1\alpha_2]w_1' + [2/3+2\alpha_2(1+\alpha_2)]w_2' + (1+2\alpha_2)\left(\frac{u_1-u_2}{t_3}\right)\right\}.$$

After substitution into the (2-10) and (2-11), the equations of motion for the system are obtained:

$$m_1\ddot{w}_1 + (EI)_1 w_1^{(4)} - (1/2)Gbt_3\left\{[2/3+2\alpha_1(1+\alpha_1)]w_1'' + [-2/3+(1+\alpha_1)(1+\alpha_2)+\alpha_1\alpha_2]w_2'' + (1+2\alpha_1)\left(\frac{u_1'-u_2'}{t_3}\right)\right\} = P_1(x,t), \quad (2\text{-}12a)$$

$$m_2\ddot{w}_2 + (EI)_2 w_2^{(4)} - (1/2)Gbt_3\left\{[-2/3+(1+\alpha_1)(1+\alpha_2)+\alpha_1\alpha_2]w_1'' + [2/3+2\alpha_2(1+\alpha_2)]w_2'' + (1+2\alpha_2)\left(\frac{u_1'-u_2'}{t_3}\right)\right\} = P_2(x,t), \quad (2\text{-}12b)$$

$$m_1\ddot{u}_1 - (EA)_1 u_1'' - (1/2)Gb\left\{(1+2\alpha_1)w_1' + (1+2\alpha_2)w_2' + 2\left(\frac{u_1-u_2}{t_3}\right)\right\} = 0, \quad (2\text{-}12c)$$

$$m_2\ddot{u}_2 - (EA)_2 u_2'' + (1/2)Gb\left\{(1+2\alpha_1)w_1' + (1+2\alpha_2)w_2' + 2\left(\frac{u_1-u_2}{t_3}\right)\right\} = 0. \quad (2\text{-}12d)$$

Correspondingly, the boundary conditions become:

$$(EI)_1 w_1^{(3)} - (1/2)Gbt_3\left\{[2/3+2\alpha_1(1+\alpha_1)]w_1' + [-2/3+(1+\alpha_1)(1+\alpha_2)+\alpha_1\alpha_2]w_2' + (1+2\alpha_1)\left(\frac{u_1-u_2}{t_3}\right)\right\}\Bigg|_{x=L} = V_{R1} \quad (2\text{-}13a)$$

$$(EI)_1 w_1^{(3)} - (1/2)Gbt_3\left\{[2/3+2\alpha_1(1+\alpha_1)]w_1' + [-2/3+(1+\alpha_1)(1+\alpha_2)+\alpha_1\alpha_2]w_2' + (1+2\alpha_1)\left(\frac{u_1-u_2}{t_3}\right)\right\}\Bigg|_{x=0} = -V_{L1} \quad (2\text{-}13b)$$

$$(EI)_2 w_2^{(3)} - (1/2)Gbt_3\left\{[-2/3+(1+\alpha_1)(1+\alpha_2)+\alpha_1\alpha_2]w_1' + [2/3+2\alpha_2(1+\alpha_2)]w_2' + (1+2\alpha_2)\left(\frac{u_1-u_2}{t_3}\right)\right\}\Bigg|_{x=L} = V_{R2}. \quad (2\text{-}13c)$$

$$(EI)_2 w_2^{(3)} - (1/2)Gbt_3\left\{[-2/3+(1+\alpha_1)(1+\alpha_2)+\alpha_1\alpha_2]w_1' + [2/3+2\alpha_2(1+\alpha_2)]w_2' + (1+2\alpha_2)\left(\frac{u_1-u_2}{t_3}\right)\right\}\Bigg|_{x=0} = -V_{L2} \quad (2\text{-}13d)$$

$$(EI)_1 w_1'' - M_{L1} \Big|_{x=L} = 0 \quad (2\text{-}13e), \qquad (EI)_2 w_2'' - M_{L2} \Big|_{x=L} = 0 \quad (2\text{-}13f),$$

$$(EI)_1 w_1'' + M_{L1} \Big|_{x=0} = 0 \quad (2\text{-}13g), \qquad (EI)_2 w_2'' + M_{L2} \Big|_{x=0} = 0 \quad (2\text{-}13h),$$

$$(EA)_1 u_1' - T_1 \Big|_{x=L} = 0 \quad (2\text{-}13i), \qquad (EA)_2 u_2' - T_2 \Big|_{x=L} = 0 \quad (2\text{-}13j),$$

$$(EA)_1 u_1' + T_1 \Big|_{x=0} = 0 \quad (2\text{-}13k), \text{ and} \qquad (EA)_2 u_2' + T_2 \Big|_{x=0} = 0 \quad (2\text{-}13l).$$

The procedure of Miles and Reinhall[c] will be followed to reduce the system of equations to sixth-order. First, the two bending equations are added, then the two longitudinal equations are subtracted, respectively:

$$\begin{aligned} m_1\ddot{w}_1 &+ m_2\ddot{w}_2 + (EI)_1 w_1^{(4)} + (EI)_2 w_2^{(4)} \\ &-(1/2)Gbt_3\left\{[2/3+2\alpha_1(1+\alpha_1)]w_1'' + [-2/3+(1+\alpha_1)(1+\alpha_2)+\alpha_1\alpha_2]w_2'' + (1+2\alpha_1)\left(\frac{u_1'-u_2'}{t_3}\right)\right\} \\ &-(1/2)Gbt_3\left\{[-2/3+(1+\alpha_1)(1+\alpha_2)+\alpha_1\alpha_2]w_1'' + [2/3+2\alpha_2(1+\alpha_2)]w_2'' + (1+2\alpha_2)\left(\frac{u_1'-u_2'}{t_3}\right)\right\} \\ &= P_1(x,t) + P_2(x,t) = P(x), \text{ and} \end{aligned}$$

$$\begin{aligned} m_1\ddot{u}_1 &- m_2\ddot{u}_2 - (EA)_1 u_1'' + (EA)_2 u_2'' \\ &- (1/2)Gb\left\{(1+2\alpha_1)w_1' + (1+2\alpha_2)w_2' + 2\left(\frac{u_1-u_2}{t_3}\right)\right\} \\ &- (1/2)Gb\left\{(1+2\alpha_1)w_1' + (1+2\alpha_2)w_2' + 2\left(\frac{u_1-u_2}{t_3}\right)\right\} = 0. \end{aligned}$$

Now allowing $w_1 = w_2$, the equations reduce to:

$$D_t\, w^{(4)} - Gbt_3(1+\alpha_1+\alpha_2)^2 w^{(2)} - Gb(1+\alpha_1+\alpha_2)(u_1'-u_2') = P'(x) \text{ ,and} \quad (2\text{-}14a)$$

$$\left(\frac{u_1-u_2}{t_3}\right) = -(1+\alpha_1+\alpha_2)w' + \left(\frac{(EA)_1 u_1'' - (EA)_2 u_2''}{2Gb}\right). \quad (2\text{-}14b)$$

$P'(x)=P(x) - (m_1 + m_2)\ddot{W}$. The effect of longitudinal inertia is also neglected. Since the each cross-section must remain balanced in tension, $(EA)_1 u_1' = -(EA)_2 u_2'$. Using this relation and after substituting (2-14b) into (2-14a) the former, the following equations are obtained:

$$D_t\, w^{(4)} - Gbt_3(1+\alpha_1+\alpha_2)^2 w^{(2)} - Gb(1+\alpha_1+\alpha_2)\left(\frac{1}{(EA)_1} + \frac{1}{(EA)_2}\right)(EA)_1 u_1' = P'(x), \quad (2\text{-}15a)$$

and

$$D_t\, w^{(4)} - t_3(EA)_1(1+\alpha_1+\alpha_2)u_1^{(3)} = P'(x). \quad (2\text{-}15b)$$

The equations can be greatly simplified using two scale factors[b,c]:

$$G' = \frac{Gb}{t_3}\left(\frac{1}{(EA)_1} + \frac{1}{(EA)_2}\right), \text{ and } Y = \frac{t_3^{\,2}}{D_t}\left(\frac{(EA)_1(EA)_2}{(EA)_1 + (EA)_2}\right); \text{ thus,} \quad (2\text{-}16)$$

$$w^{(4)} - G'Yw^{(2)} - (G't_3/D_t)(1+\alpha_1+\alpha_2)\, u_1' = P'(x)/D_t\,, \text{ and} \quad (2\text{-}17a)$$

$$w^{(4)} - [t_3(EA)_1/D_t](1+\alpha_1+\alpha_2)u_1^{(3)} = P'(x)/D_t\,. \quad (2\text{-}17b)$$

The final step is to eliminate $u_1$ from the equations. This is accomplished by taking the second partial with respect to "x" of the first equation and multiplying the second equation by G′, then subtracting:

$$w^{(6)} - (1+Y)G'\, w^{(4)} = \frac{1}{D_t}\left(\frac{\partial^2 P'}{\partial x^2} - G'P'\right), \text{or}$$

$$w^{(6)} - (1+Y)G'\, w^{(4)} + \frac{(m_1+m_2)}{D_t}\left(\frac{\partial^2 \ddot{w}}{\partial x^2} - G'\ddot{w}\right) = \frac{1}{D_t}\left(\frac{\partial^2 P}{\partial x^2} - G'P\right) \quad (2\text{-}18)//.$$

The corresponding reduction in the boundary conditions follows in the following section.

## 3.0 BOUNDARY CONDITIONS FOR THE SIXTH-ORDER BEAM EQUATIONS

The procedure for reducing combining the boundary conditions follows the same prescription as above. After adding the shear terms together, the boundary equations become ($w_1 = w_2$):

$$D_t\, w^{(3)} - Gbt_3(1+\alpha_1+\alpha_2)^2 w' - Gb(1+\alpha_1+\alpha_2)(u_1 - u_2)\Big|_{x=L} = V_R = V_{R1} + V_{R2},\ \text{and} \quad (3\text{-}1)$$

$$D_t\, w^{(3)} - Gbt_3(1+\alpha_1+\alpha_2)^2 w' - Gb(1+\alpha_1+\alpha_2)(u_1 - u_2)\Big|_{x=0} = -V_L = -V_{L1} - V_{L2}. \quad (3\text{-}2)$$

The following equation is valid throughtout the beam and can be shown to be equivalent to the extentional boundary conditions (after a lot of work):

$$\left(\frac{u_1 - u_2}{t_3}\right)\Bigg|_0^L = -(1+\alpha_1+\alpha_2)w' + \left(\frac{(EA)_1 u_1'' - (EA)_2 u_2''}{2Gb}\right)\Bigg|_0^L. \quad (3\text{-}3)$$

Since the procedure is identical for both equations, the derivation will proceed using only the first equation. After direct substitution of the extentional terms, the boundary condition becomes:

$$w^{(3)} - [t_3(1+\alpha_1+\alpha_2)/D_t](EA)_1 u_1'' \Big|_{x=L} = V_R/D_t. \quad (3\text{-}4)$$

Taking the second derivative of (3-1) yields:

$$w^{(5)} - G'Yw^{(3)} - [G't_3(1+\alpha_1+\alpha_2)/D_t](EA)_1 u_1'' \Big|_{x=L} = 0. \quad (3\text{-}5)$$

Eliminating the $u_1$ from the preceeding equation:

$$-w^{(5)} + (1+Y)G'\, w^{(3)} \Big|_{x=L} = (G'/D_t)\, V_R. \quad (3\text{-}6)$$

Thus, the set "natural" boundary conditions become[a,b]:

$$-w^{(5)} + (1+Y)G'\, w^{(3)} \Big|_{x=L} = (G'/D_t)\, V_R. \quad (3\text{-}7a)$$

$$-w^{(5)} + (1+Y)G'\, w^{(3)} \Big|_{x=0} = -(G'/D_t)\, V_L, \quad (3\text{-}7b)$$

$$D_t w'' - M_R' \Big|_{x=L} = 0 \ (3\text{-}7c), \quad \text{and} \quad D_t w'' + M_L' \Big|_{x=0} = 0 \ (3\text{-}7d).$$

Analogously, the moment boundaries are evaluated. Again, right hand boundary alone will be evaluated, since the process for the left hand is identical. Thus, the moment equation equation becomes:

$$D_t w'' - (M_{R1} + M_{R2}) - \frac{(T_1 - T_2)}{2} t_3 (1+\alpha_1+\alpha_2) \Big|_{x=L} = 0. \tag{3-10}$$

After substituting for $T_1$ and $T_2$, the resulting equation is:

$$D_t w'' - M_R - \frac{[ (EA)_1 u_1' - (EA)_2 u_2' ]}{2} t_3 (1+\alpha_1+\alpha_2) \Big|_{x=L} = 0 \quad \text{, or} \tag{3-11a}$$

since $(EA)_1 u_1' = -(EA)_2 u_2'$ throughout the beam, the equation reduces to,

$$D_t w'' - M_R - (EA)_1 u_1' [ t_3 (1+\alpha_1+\alpha_2) ] \Big|_{x=L} = 0. \tag{3-11b}$$

Taking the second partial derivative with respect to "x" of (3-11a) yields:

$$D_t w^{(4)} - M_R - \frac{[ (EA)_1 u_1^{(3)} - (EA)_2 u_2^{(3)} ]}{2} t_3 (1+\alpha_1+\alpha_2) \Big|_{x=L} = 0 . \tag{3-12}$$

Substituting (3-3), this equation becomes:

$$D_t w^{(4)} - Gbt_3 (1+\alpha_1+\alpha_2)^2 w'' - Gb(1+\alpha_1+\alpha_2)(u_1' - u_2') \Big|_{x=L} = 0. \tag{3-13}$$

This can be re-written as:

$$D_t w^{(4)} - Gbt_3 (1+\alpha_1+\alpha_2)^2 w'' - Gb(1+\alpha_1+\alpha_2) \left( \frac{1}{(EA)_1} + \frac{1}{(EA)_2} \right) (EA)_1 u_1' \Big|_{x=L} = 0. \tag{3-14}$$

After applying the scale factors,

$$D_t w^{(4)} - G'Y\, w^{(2)} - [ G' t_3 (1+\alpha_1+\alpha_2)/D_t ] (EA)_1 u_1' = 0. \tag{3-15}$$

Eliminating $u_1$ using (3-11b), the moment boundary condition reduces to:

$$M_R = \frac{D_t}{G'} \left( -w^{(4)} + (1+G')\, w^{(2)} \right) \Big|_{x=L}. \tag{3-16}$$

Thus, the set of "natural" boundary conditions become:

$$-w^{(5)} + (1+Y)G' w^{(3)} \Big|_{x=L} = (G'/D_t) V_R, \tag{3-17a}$$

$$-w^{(5)} + (1+Y)G' w^{(3)} \Big|_{x=0} = -(G'/D_t) V_L, \tag{3-17b}$$

$$M_R = \frac{D_t}{G'} \left( -w^{(4)} + (1+G') w^{(2)} \right) \Big|_{x=L}, \text{ and} \tag{3-17c}$$

$$M_L = -\frac{D_t}{G'} \left( -w^{(4)} + (1+G') w^{(2)} \right) \Big|_{x=0}. \tag{3-17d}$$

These force-type boundary conditions agree with those obtained by Mead and Markus [a,b]. Only four "natural" boundary conditions now remain to specify the sixth-order equation. Thus, it is necessary to specify two addition "kinematic" constraints; otherwise, the problem is not well-posed. Representative "kinematic" constraints are:

<u>clamped-free</u>-

$$w_R = w_R' = 0 \text{ or } w_L = w_L' = 0, \tag{3-18a}$$

<u>simply supported</u>-

$$w_R = w_L = 0, \tag{3-18b}$$

<u>simple-roller</u>-

$$w_R = w_L' = 0 \text{ or } w_R' = w_L = 0, \text{ and} \tag{3-18c}$$

<u>no rotation</u>-

$$w_R' = w_L' = 0. \tag{3-18d}$$

Mead [a] discusses other exotic boundary conditions that are permutations of the above "natural" and "kinematic" end conditions through relaxing the various boundary tractions.

## 4.0 DISCUSSION/OBSERVATION

The equations of motion and associated boundary conditions for a three-layer composite laminate were derived in Section 2.0 (Eqns. 2-12a thru 2-13l). The "natural" or force type boundary conditions are a consequence of the energy method formulation [e,l]. That system of equations is of twelfth-order and the solution is completely specified by the "natural" boundary conditions; thus, the equations of motion are well-posed. Consequently, since the system solution is completely specified by its "natural" boundary conditions, the formulation can be employed in the analysis of built-up structures (eg. a finite element analysis), albeit cumbersome. The existence of all "natural" boundary conditions permits the universal satisfaction of internal compatibility conditions required in a finite element type solution. Miles and Reinhall [c] proceed to perform an assumed modes solution to examine the thickness deformations in a three-layer composite. Their studies showed that thickness deformation is an important damping mechanism , especially in higher order modes.

The twelfth-order system was reduced to a single sixth-order partial differential equation (2-18), as shown in Section 2.0. By a similar process, the "natural" boundary conditions are reduced to four in number (3-17a thru 3-17d). Both the boundary conditions and the sixth-order equations agree with those derived by Mead [a,b].

The point to be observed here is that only four "natural" boundary conditions remain to specify the solution of a sixth-order differential equations; that is, a deficit of two differential equations. By constraining the extentional degrees of freedom (3-3), two boundary conditions are lost. Thus, two geometric or "kinematic" boundary conditions must be specified for the solution to be well-posed. Several possible "kinematic" boundary conditions are provided in Section 3.0 (3-18a thru 3-18d) to augment the "natural" boundary conditions. Mead discusses other admissible boundary conditions [a, b].

Since the sixth-order partial differential equation cannot be completely specified by the "natural" boundary conditions, a complex built-up structure cannot be modelled. Only simple structures (eg. single span beams and plates) can be evaluated. For example, element compatibility conditions in a finite element formulation cannot be universally satisfied without the imposition of a "kinematic" constraint; thus, the type of structure evaluated is limited, that is a general sixth-order beam or plate finite element cannot be formulated.

Further, Mead demonstrated that all solutions to the sixth-order equation are complex-valued functions, with the sole exception of the case with simply supported boundaries. (The solution to the simply supported case is a real-valued function. This case can be further reduced to the standard RKU equations [c, k].) Thus, computationally, the sixth-order equation is effectively a twelfth-order system. No gain in computational effeciency is obtained.

The principal benefit derived from the sixth-order equation is when the relative extentional motion of the face sheets becomes significant, that is when one or both of the face sheets possess a low stiffness relative to the core shear stiffness. In this case, Maynor [j] has shown that numerical solution is neither particularly easy nor necessarily guaranteed. For the majority of engineering applications, a fourth-order (RKU) formulation is adequate to describe the dynamic behavior of damped laminate beams and plates [j].

## 5.0 SUMMARY

The author has presented a detailed derivation of the sixth-order beam equation and attendant boundary conditions. The author has shown how these boundary conditions naturally arise as a consequence of the variational energy method approach. The author shows how the boundary conditions vanish as a result of constraining the extentional motion of the face sheets, thereby requiring the imposition of "kinematic" constraints for a well-posed solution. These additional restraints restrict the types of structures which can be evaluated using the sixth-order equation. A useful modification to these boundary conditions is the inclusion of damping into the boundary conditions [m]. Inman has observed that such terms in the boundary conditions are important in the mechanics of line-of-sight/slewing or pointing/control applications of articulating structures.

## REFERENCES


a) DJ Mead and S Markus, "The Forced Vibration of a Three-Layer Damped Sandwich Beam with Arbitrary Boundary Conditions", **J. Sound and Vibration**, (1969) 10 (2), pp 163-175.

b) DJ Mead and S Markus, "Loss Factors and Resonant Frequencies of Encastre' Damped Sandwich Beams", **J. Sound and Vibration**, (1970) 12 (1), pp 99-112.

c) RN Miles and PG Reinhall, "An Analytical Model for the Vibration of Laminated Beams Including the Effects of Both Shear and Thickness Deformation in the Adhesive Layer", **J. of Vibration, Acoustics, Stress and Reliability in Design**, ASME, Vol. 108, January 1986.

d) SP Timoshenko and JM Goodier, **Mechanics of Materials**, McGraw-Hill, NY(1972).

e) HL Langhaar, **Energy Methods in Applied Mechanics**, John Wiley, NY(1962).

f) IM Gel'fand, SV Fomin and RA Silverman (tr.), **Calculus of Variations**, Prentice-Hall, Englewood Cliffs, NJ(1963).

g) F Abdulhadi, "Transverse Vibrations of Laminated Plates with Viscoelastic Layer Damping", **Shock and Vibration Bulletin**, 40(5), December 1969.

h) EH Dowell and M-J Yan, "Governing Equations for Vibrating Constrained-Layer Damping Sandwich Plates and Beams", **J. of Applied Mechanics**, ASME, Vol. 94, December 1972.

i) RA DiTaranto, "Theory of the Vibratory Bending for Elastic and Viscoelastic Finite Length Beams", **J. of Applied Mechanics**, ASME, Vol. 32, 1965.

j) JW Maynor, "INTERIM PROGRESS REPORT - The Effect of End Fixity on Loss Factors Obtained Using Abdulhadi's Equation", LTV 9-51240/7AVO-054, Dtd. 31 July 1987.

JW Maynor, "FINAL REPORT - The Effect of End Fixity on Loss Factors Obtained Using Abdulhadi's Equations", LTV 9-51240/7AVO-111, Dtd. 18 December 1987.

k) L Cremer, M Heckl, and EE Ungar(tr.), **Structure-Borne Sound**, 2nd Ed., Springer-Verlag, Berlin(1988).

l) RR Craig, **Structural Dynamics: An Introduction to Computer Methods**, Wiley, New York(1981).

m) DJ Inman, **Vibration with Control, Measurement, and Stability**, Prentice Hall, Englewood Cliffs, NJ (1989).