A GENERAL APPROACH TO PROBLEMS OF PLASTICITY AND LARGE DEFORMATION USING ISOPARAMETRIC ELEMENTS

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A unified formulation of large deformation, large strain and plasticity problems is given. Lagrangian forms are preferred but an alternative Eulerian system is given. Various solution schemes are discussed and a program capable of dealing with alternatives is outlined. Isoparametric formulation is shown not only to possess merits of economy but to be highly versatile in the Lagrangian and Eulerian formulations of materially and geometrically nonlinear problems.

INTRODUCTION

Recent years have seen a growing interest in the application of the finite element method to nonlinear problems of structural mechanics. The three classes of problems which have been usually approached separately are:

(1) geometrically large deformation associated with small, elastic, strain

(2) geometrically large deformation associated with finite strains

and

(3) nonlinear material properties

Further, differing methods of formulation and of the solution of the nonlinear system have again introduced another classification. Most of these solution processes fall into two broad classes of

(1) incremental procedures where a 'marching' type of approach is used and equilibrium path is only approximately followed, with equilibrium checks occasionally introduced, and

(2) iterative procedures in which equilibrium is approached at all stages of computation.

The different approaches to the formulation and solution of nonlinear problems have been often confusing and, we believe, have led to misunderstanding on occasion. Several notable attempts at introduction of some degree of precision in the definition of formulation and solution have been made by Marcal (Reference 1 and 2).

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Oden (References 3 and 4), Haisler et al (Reference 5) and others (References 6, 7, and 8), and similar formulation was achieved in our Institute in 1968. In this paper we shall pursue this problem again and attempt to present

(a) the essentials of the formulation which has a degree of generality and conforms to the matrix-finite element schemes

(b) a generalized solution process which can use the advantages of several possible alternatives at will

and

(c) the advantages of isoparametric elements which appear to be naturally suited to this broad class of problems

2. A general formulation of equilibrium equations in finite element context

Let

$$\mathbf{x} \equiv [x,y,z]^T \tag{1}$$

define the rectangular coordinates of a material point P in a body shown in Figure 1 before deformation. If this point is displaced by

$$\mathbf{u} \equiv [u,v,w]^T \tag{2}$$

measured relative to the fixed frame of reference, its new coordinates will be

$$\bar{\mathbf{x}} = \mathbf{x} + \mathbf{u} \tag{3}$$

Further, using the usual finite element approximation we shall take the displacements as defined by suitable shape functions $\mathbf{N}$ of coordinates x, and a set of nodal parameters $\boldsymbol{\delta}$, element by element, as

$$\mathbf{u} = \mathbf{N}\boldsymbol{\delta} \tag{4}$$

If $\bar{\mathbf{p}}$ denotes the surface forces per unit area of the deformed body, and $\mathbf{q}$ the body forces per unit mass, then a simple application of the virtual work principle yields, by equating the external and internal work, the approximate equilibrium conditions as

$$\int_{\bar{V}} \bar{\rho}\, \mathbf{q}^T d\mathbf{u}\, d\bar{V} + \int_{\bar{A}} \bar{\mathbf{p}}^T d\mathbf{u}\, d\bar{A} = \int_{\bar{V}} \bar{\boldsymbol{\sigma}}^T d\bar{\boldsymbol{\epsilon}}\, d\bar{V} \tag{5}$$

in which $\bar{V}$, and $\bar{A}$ refer to volumes and areas of the deformed body, $\bar{\rho}$, is the density in the deformed state and $\bar{\boldsymbol{\sigma}}$ and $d\bar{\boldsymbol{\epsilon}}$ refer to vector forms of the Eulerian (real) stress and deformation increment (see Appendix I, Equation (A15) in the distorted coordinates $\bar{\mathbf{x}}$.

Alternatively, we may rewrite Equation (5) in terms of variables referred to the original, undistorted, coordinates and now obtain

$$\int_V \rho \mathbf{q}^T d\mathbf{u}\, dV + \int_A \left[\frac{d\bar{A}}{dA}\bar{\mathbf{p}}^T\right] d\mathbf{u}\, dA = \int_V \boldsymbol{\sigma}^T d\boldsymbol{\epsilon}\, dV \tag{7}$$

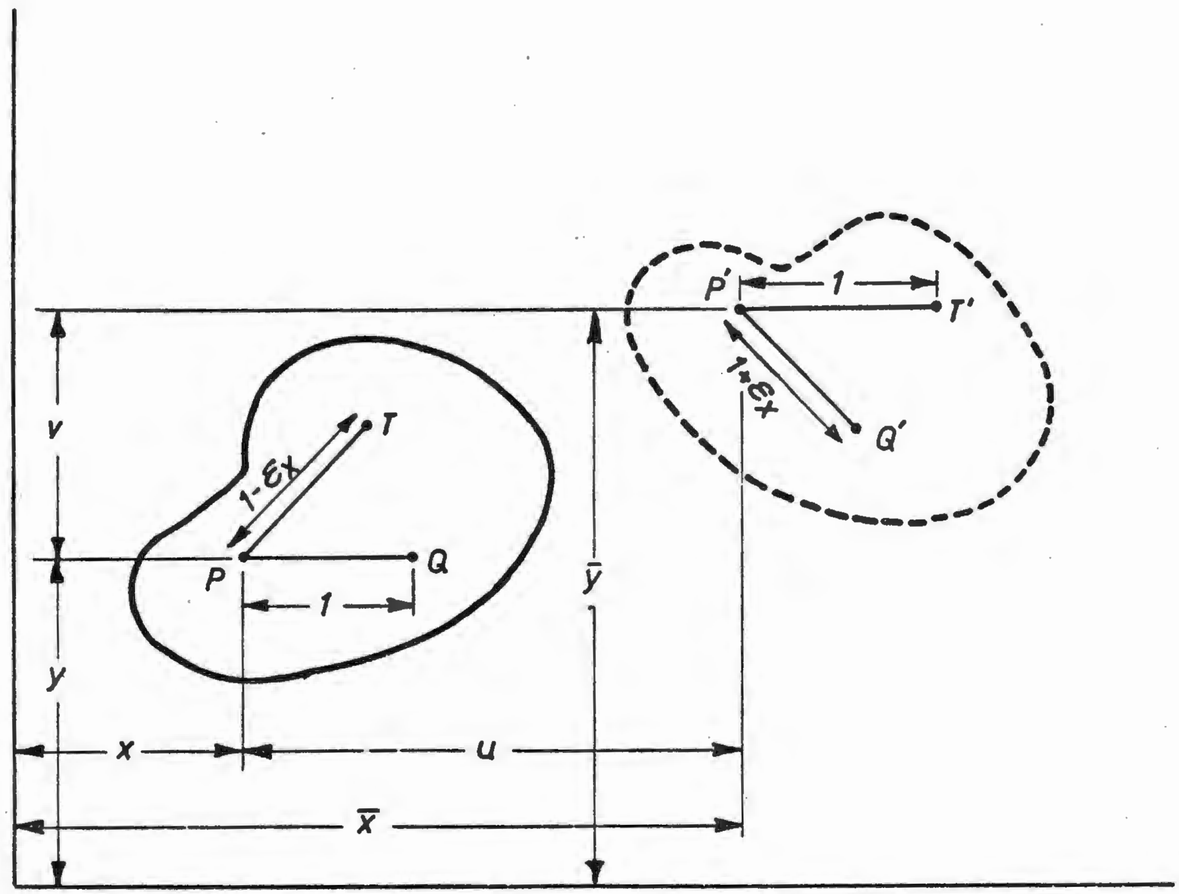

FIG. 1 DEFINITION OF LAGRANGIAN AND EULERIAN CO-ORDINATES AND STRAINS (EXACT ONLY IN INFINITESIMAL CASES)

in this $\boldsymbol{\sigma}$ stands for the Piola-Kirchhoff stress, and $d\boldsymbol{\epsilon}$ for the increment of Green's strain, both referred to as the Lagrange formulation and both again written in vector form.

The equivalence of the two work statements and the derivation of the appropriate definitions is relegated to the Appendix where appropriate matrix forms are given. We shall content ourselves by giving here only the general statements applicable in finite element analysis. Thus, from the above definition we can write always

$$d\bar{\boldsymbol{\epsilon}} = \bar{\mathbf{B}}^{\mathrm{o}} d\boldsymbol{\delta} \tag{8}$$

for the increment of Eulerian deformation using updated coordinates or

$$d\boldsymbol{\epsilon} = \mathbf{B} d\boldsymbol{\delta} \tag{9}$$

for the increment of Green's strain in Lagrangian (original) coordinates and obtain by substitutions in Equations 4, and 8 or 9 respectively

$$\bar{\boldsymbol{\psi}} \equiv \bar{\mathbf{R}} - \int_{\bar{V}} \bar{\mathbf{B}}^{\mathrm{o}T} \bar{\boldsymbol{\sigma}} \, d\bar{V} = \mathbf{0} \tag{10}$$

or

$$\boldsymbol{\psi} = \mathbf{R} - \int_V \mathbf{B}^T \boldsymbol{\sigma} \, dV = \mathbf{0} \tag{11}$$

as the equilibrium equation in Eulerian and Lagrangian systems respectively. Here

$$\mathbf{R} = \bar{\mathbf{R}} = \int_{\bar{V}} \bar{\rho} \mathbf{N}^T \mathbf{q} \, d\bar{V} + \int_{\bar{A}} \mathbf{N}^T \bar{\mathbf{p}} \, d\bar{A}$$

$$= \int_V \rho \mathbf{N}^T \mathbf{q} \, dV + \int_A \mathbf{N}^T \bar{\mathbf{p}} \frac{d\bar{A}}{dA} \, dA \tag{12}$$

gives the equivalent external nodal forces while the second term in Equations 10 and 11 can be interpreted as the internal force reactions.

In general both $\mathbf{R}$ and $\mathbf{B}$ depend on the displacement parameters $\boldsymbol{\delta}$ and as the stress may be a nonlinear function of strain, the set of equations implied in Equations 10 and 11 is nonlinear, and special solution methods will have to be used. The solution of these equations for the displacement parameters, $\boldsymbol{\delta}$, gives the solution of the basic problem.

In both forms, Equations 10 or 11, the 'residuals' $\boldsymbol{\psi}$ or $\bar{\boldsymbol{\psi}}$ can be visualized as nodal forces required to bring the assumed displacement pattern into nodal equilibrium and therefore these should be reduced to as nearly zero as possible. This physical concept is useful with calculations, which at times are so complex as to elude simple interpretation.

3. Outline of the solution of the solution of the nonlinear equations.

The Lagrangian Equation 11 or its Eulerian equivalent 10 can be solved in a variety of ways. If the loaas are proportional and are represented by some proportionality parameter, λ, then we can write either of the equations in the form

$$\boldsymbol{\psi} \equiv \boldsymbol{\psi}(\delta,\lambda) = \mathbf{Q}(\delta).\lambda - \mathbf{F}(\delta) = \mathbf{0} \tag{13}$$

with $\mathbf{F}$ standing for 'internal forces' equilibrating the external loads and $\mathbf{Q}$ is the value of $\mathbf{R}$ for unit λ. $\boldsymbol{\psi}$, will be called the residual force vector. For solution of Equation 13 two alternative processes can be used:

<u>Incremental solution</u>

The Equation 13 is considered as a function of one parameter λ and by differentiation we have

$$\frac{d\boldsymbol{\psi}}{d\lambda} = -\lambda \mathbf{K}_{\lambda} \frac{d\boldsymbol{\delta}}{d\lambda} + \mathbf{Q} - \mathbf{K}_{T} \frac{d\boldsymbol{\delta}}{d\lambda} = \mathbf{0} \tag{14}$$

where

$$d\mathbf{Q} = -\mathbf{K}_{\lambda}\, d\boldsymbol{\delta} \tag{15}$$

defines what may be called the 'initial load' stiffness matrix (which will be zero with conservative loading) and

$$d\mathbf{F} = \mathbf{K}_{T} d\boldsymbol{\delta} \tag{16}$$

gives the well known tangential stiffness matrix.

Taking small increments of the parameter λ one can write

$$\Delta\boldsymbol{\delta} = (\mathbf{K}_{T} + \mathbf{K}_{\lambda})^{-1}\, \mathbf{Q}\Delta\lambda \tag{17}$$

and starting from known initial equilibrium conditions solution can be incremented.

Clearly, though this process has been used by many investigators, very small increments of λ will be required not to diverge from the equilibrium condition. Such numerical refinements as the Runge–Kutta process etc. increases the accuracy of this, essentially marching, process but suffer from the same possibility of divergence.

An alternative to the incremental solution is presented by: iterative, Newton–Raphson solution.

Here the load parameter λ is considered as fixed and the solution of Equation 13 is attained iteratively. Noting that now $d\lambda \to 0$ we have by Equation 14

$$d\boldsymbol{\psi} = -(\mathbf{K}_{\lambda} + \mathbf{K}_{T}\lambda)^{-1}\, d\boldsymbol{\delta} \tag{18}$$

and if the n^{th} iterate $\boldsymbol{\delta}_n$ gives a nonzero residual force $\boldsymbol{\psi}_n$, the next iterate becomes, using the Newton–Raphson method

$$\boldsymbol{\delta}_{n+1} = \boldsymbol{\delta}_n + \Delta\boldsymbol{\delta}_n \tag{19}$$

with

$$\Delta\boldsymbol{\delta}_n = -(\mathbf{K}_{\lambda} + \mathbf{K}_{T})^{-1}\, \boldsymbol{\psi}_n \tag{20}$$

A modification frequently used to avoid repeated recalculation and inversion of a continuously varying stiffness matrix is to keep this matrix unchanged during several successive iterations. Indeed, on occasion, convergence can be combined with economy replacing the matrix $(\mathbf{K}_\lambda + \mathbf{K}_T)$ by $\mathbf{K}_o$ matrix corresponding to small deflection and elastic behaviour. Indeed, this is the basis of so called 'initial stress process' introduced by Zienkiewicz, Valliappan, and King in the context of plasticity (Reference 9).

In the Newton–Raphson process and even more so in its modifications there is no guarantee that convergence will always be achieved, or indeed that when it is achieved that the correct solution is obtained if multiple equilibrium states are possible.

It is thus most expedient to combine the various advantages of the alternative approaches in a combined algorithm.

Combined algorithm

In this the load parameter λ is incremented in several finite steps. In each step the first approximation to the solution is obtained by the incremental process using Equation 17 and the value of $\mathbf{K}_T$ pertaining to the initial conditions (or the value last found in previous step). ($\mathbf{K}_\lambda$ is not included as its value is generally small). Considering this value as the first approximation, the residual $\boldsymbol{\psi}_o$ is evaluated and iteration then commenced in one of the following ways.

(a) using Equations 18 – 20 and continuously updating $\mathbf{K}_T$

(b) proceeding as in (a) but updating the matrix once only and keeping it subsequently constant in the increment

(c) proceeding as in (a) but using the original value of the matrix at the initial conditions of the increment

(d) following the procedure (c) but replacing Equation 20 by an accelerator which gives

$$\Delta\boldsymbol{\delta}_n = \boldsymbol{\alpha}\, \mathbf{K}_T^{-1} \boldsymbol{\psi}_n \tag{21}$$

where $\boldsymbol{\alpha}$ is a diagonal matrix modifying suitably the correction by using 'accumulated experience'. Various methods of arriving at such an acceleration have been proposed and one such technique was devised by Nayak and shown to be particularly effective (References 10, 11, and 12).

A program written on the basis of providing such options is not difficult to arrive at and gives, as its extreme cases, both the simple, incremental process and the Newton–Raphson method. For each class of problem one of the options described provides the optimum solution scheme and has to be determined by experience. For a great majority of problems, (b) and (d) offer the best economy using typically 8–12 load parameter increments.

Haisler et al (Reference 5) make an assessment of nine solution combinations presented above in the context of a few numerical examples.

4. Explicit forms of Lagrangian formulation

Before a finite element solution can be attempted, a more explicit presentation of the various terms arising in the previous two sections is necessary. This will be here presented in the context of a full three dimensional solid with specialization to axisymmetric and plane problems given in Appendix II.

Strain - displacement relationships

The Green's strain in vector notation can be written using the engineering definitions as

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left[\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\right] \end{aligned} \tag{22}$$

with

$$\boldsymbol{\epsilon} = [\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{yz}, \gamma_{zx}, \gamma_{xy}]^T = \boldsymbol{\epsilon}^o + \boldsymbol{\epsilon}^L \tag{23}$$

where $\boldsymbol{\epsilon}^o$ is the usual linear, infinitesimal, strain vector (Reference 13) and $\boldsymbol{\epsilon}^L$ is the nonlinear contribution. The nonlinear part is conveniently written as

$$\underset{(6x1)}{\boldsymbol{\epsilon}^L} = \frac{1}{2}\begin{bmatrix} \boldsymbol{\theta}_x^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\theta}_y^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\theta}_z^T \\ \mathbf{0} & \boldsymbol{\theta}_z^T & \boldsymbol{\theta}_y^T \\ \boldsymbol{\theta}_z^T & \mathbf{0} & \boldsymbol{\theta}_x^T \\ \boldsymbol{\theta}_y^T & \boldsymbol{\theta}_x^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta}_x \\ \boldsymbol{\theta}_y \\ \boldsymbol{\theta}_z \end{bmatrix} = \frac{1}{2}\mathbf{A}\boldsymbol{\theta} \tag{24}$$

$$= (6x9) \times (9x1)$$

where $\boldsymbol{\theta}_x = \left[\frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial x} \quad \frac{\partial w}{\partial x}\right]^T$ etc.

Rewriting, in the usual manner the approximate expression for displacements Equation 4 as

$$u = \Sigma N_i u_i, \quad v = \Sigma N_i v_i, \quad w = \Sigma N_i w_i, \tag{25}$$

we have after substituting into the expression for $\boldsymbol{\epsilon}^o$, and differentiation

$$d\boldsymbol{\epsilon}^o = \mathbf{B}^o \, d\boldsymbol{\delta}, \quad \boldsymbol{\delta}_i = [u_i v_i w_i]^T \tag{26}$$

with the well known small displacement matrix given by a typical component submatrix for node i

$$\mathbf{B}_i^o = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \end{bmatrix} \tag{27}$$

Differentiation of $\boldsymbol{\epsilon}^L$ yields

$$d\boldsymbol{\epsilon}^L = \frac{1}{2} d\mathbf{A}\boldsymbol{\theta} + \frac{1}{2}\mathbf{A} d\boldsymbol{\theta} \tag{28}$$

which due to the structure of the matrices involved becomes simply

$$d\boldsymbol{\epsilon}^L = \mathbf{A} d\boldsymbol{\theta} \tag{29}$$

The manipulation of Equation (29) is made easy by an interesting property of $\mathbf{A}$ and $\boldsymbol{\theta}$ (Reference 10).

It is easy to verify that if

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is a (9 x 1) arbitrary vector then

$$d\mathbf{A}\mathbf{x} = \begin{bmatrix} d\boldsymbol{\theta}_x^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & d\boldsymbol{\theta}_y^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & d\boldsymbol{\theta}_z^T \\ \mathbf{0} & d\boldsymbol{\theta}_z^T & d\boldsymbol{\theta}_y^T \\ d\boldsymbol{\theta}_z^T & \mathbf{0} & d\boldsymbol{\theta}_x^T \\ d\boldsymbol{\theta}_y^T & d\boldsymbol{\theta}_x^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{x}_2^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{x}_3^T \\ \mathbf{0} & \mathbf{x}_3^T & \mathbf{x}_2^T \\ \mathbf{x}_3^T & \mathbf{0} & \mathbf{x}_1^T \\ \mathbf{x}_2^T & \mathbf{x}_1^T & \mathbf{0} \end{bmatrix} d \begin{bmatrix} \boldsymbol{\theta}_x \\ \boldsymbol{\theta}_y \\ \boldsymbol{\theta}_z \end{bmatrix} \tag{a}$$

Thus

$$\mathbf{dA\theta} = \mathbf{A}\mathrm{d}\boldsymbol{\theta}$$

Similarly if a 6 x 1, vector

$$\mathbf{y} = [y_1, y_2 \ldots\ldots y_6]^T$$

then,

$$\mathrm{d}\mathbf{A}^T\mathbf{y} = \begin{bmatrix} d\boldsymbol{\theta}_x & \mathbf{0} & \mathbf{0} & \mathbf{0} & d\boldsymbol{\theta}_z & d\boldsymbol{\theta}_y \\ \mathbf{0} & d\boldsymbol{\theta}_y & \mathbf{0} & d\boldsymbol{\theta}_z & \mathbf{0} & d\boldsymbol{\theta}_x \\ \mathbf{0} & \mathbf{0} & d\boldsymbol{\theta}_z & d\boldsymbol{\theta}_y & d\boldsymbol{\theta}_x & \mathbf{0} \end{bmatrix} \mathbf{y} \quad \text{(b)}$$

$$= \begin{bmatrix} y_1\mathbf{I}_3 & y_6\mathbf{I}_3 & y_5\mathbf{I}_3 \\ & y_2\mathbf{I}_3 & y_4\mathbf{I}_3 \\ \text{SYM} & & y_3\mathbf{I}_3 \end{bmatrix} \mathrm{d} \begin{bmatrix} \boldsymbol{\theta}_x \\ \boldsymbol{\theta}_y \\ \boldsymbol{\theta}_z \end{bmatrix}$$

Use of this second property given by (b) will be made in Equation 39 .

Now, the (9 x 1) vector $\boldsymbol{\theta}$ in Equation 24 can be written as

$$\boldsymbol{\theta} = \mathbf{G}\boldsymbol{\delta} = [G_1, \ldots G_i \ldots] \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_i \\ \vdots \end{bmatrix} \tag{30}$$

where

$$\mathbf{G}_i = \begin{bmatrix} \mathbf{I}_3 \dfrac{\partial N_i}{\partial x} \\ \mathbf{I}_3 \dfrac{\partial N_i}{\partial y} \\ \mathbf{I}_3 \dfrac{\partial N_i}{\partial z} \end{bmatrix}$$

and, $\mathbf{I}_3$, is a 3 x 3 identity matrix.

Substituting Equation 30 into Equation 29 we have finally

$$\mathrm{d}\boldsymbol{\epsilon}^L = \mathbf{B}^L \mathrm{d}\boldsymbol{\delta} \tag{31}$$

with

$$\mathbf{B}^L = \mathbf{AG}$$

Thus the strain displacement matrix **B** of Equation 9 now becomes

$$\mathbf{B} = \mathbf{B}^{o} + \mathbf{B}^{L}$$

It can be observed that the total strain of Equation 23 can now be written, using Equations 24, 30, and 32, as

$$\boldsymbol{\epsilon} = (\mathbf{B}^{o} + \frac{1}{2}\mathbf{B}^{L})\ \boldsymbol{\delta} \tag{33}$$

This relation was used by Oden (References 3 and 4) in obtaining a nonsymmetric, secant, stiffness matrix.

For computational purposes, it is convenient to obtain **B** explicitly by multiplying out the appropriate terms in Equation 31. The expression for full three dimensional analysis thus becomes

$$\mathbf{B}_i^{L} = \left[\begin{array}{c|c|c}
\frac{\partial u}{\partial x}\cdot\frac{\partial N_i}{\partial x} & \frac{\partial v}{\partial x}\cdot\frac{\partial N_i}{\partial x} & \frac{\partial w}{\partial x}\cdot\frac{\partial N_i}{\partial x} \\ \hline
\frac{\partial u}{\partial y}\cdot\frac{\partial N_i}{\partial y} & \frac{\partial v}{\partial y}\cdot\frac{\partial N_i}{\partial y} & \frac{\partial w}{\partial y}\cdot\frac{\partial N_i}{\partial y} \\ \hline
\frac{\partial u}{\partial z}\cdot\frac{\partial N_i}{\partial z} & \frac{\partial v}{\partial z}\cdot\frac{\partial N_i}{\partial z} & \frac{\partial w}{\partial z}\cdot\frac{\partial N_i}{\partial z} \\ \hline
\frac{\partial u}{\partial z}\cdot\frac{\partial N_i}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial N_i}{\partial z} & \frac{\partial v}{\partial z}\cdot\frac{\partial N_i}{\partial y} + \frac{\partial v}{\partial y}\cdot\frac{\partial N_i}{\partial z} & \frac{\partial w}{\partial z}\cdot\frac{\partial N_i}{\partial y} + \frac{\partial w}{\partial y}\cdot\frac{\partial N_i}{\partial z} \\ \hline
\frac{\partial u}{\partial x}\cdot\frac{\partial N_i}{\partial z} + \frac{\partial u}{\partial z}\cdot\frac{\partial N_i}{\partial x} & \frac{\partial v}{\partial x}\cdot\frac{\partial N_i}{\partial z} + \frac{\partial v}{\partial z}\cdot\frac{\partial N_i}{\partial x} & \frac{\partial w}{\partial x}\cdot\frac{\partial N_i}{\partial z} + \frac{\partial w}{\partial z}\cdot\frac{\partial N_i}{\partial x} \\ \hline
\frac{\partial u}{\partial y}\cdot\frac{\partial N_i}{\partial x} + \frac{\partial u}{\partial x}\cdot\frac{\partial N_i}{\partial y} & \frac{\partial v}{\partial y}\cdot\frac{\partial N_i}{\partial x} + \frac{\partial v}{\partial x}\cdot\frac{\partial N_i}{\partial y} & \frac{\partial w}{\partial y}\cdot\frac{\partial N_i}{\partial x} + \frac{\partial w}{\partial x}\cdot\frac{\partial N_i}{\partial y}
\end{array}\right] \tag{34}$$

To obtain the tangential stiffness matrix of Equation 16 we find the increment of the internal force term of Equation 11

$$d\left(\int_V \mathbf{B}^T\boldsymbol{\sigma}\, dV\right) = \int_V (d\mathbf{B}^T\boldsymbol{\sigma} + \mathbf{B}^T d\boldsymbol{\sigma})\, dV \equiv \mathbf{K}_T d\boldsymbol{\delta} \tag{35}$$

As $\mathbf{B}^o$ does not vary with $\boldsymbol{\delta}$ we have by taking the variation of Equation 32

$$d\mathbf{B}^T = d[\mathbf{B}^L]^T = \mathbf{G}\ d\ \mathbf{A}^T \tag{36}$$

With the notation defining the tangential material property we can write generally

$$d\boldsymbol{\sigma} = \mathbf{D}_T\ d\epsilon \tag{37}$$

and therefore, after some transformation utilizing the properties of $\boldsymbol{\theta}$ matrices described in Equation (b), we arrive at

$$\mathbf{K}_T = \int_V (\mathbf{G}^T\mathbf{M}\mathbf{G} + \mathbf{B}^T\mathbf{D}_T\mathbf{B})\, dV \tag{38}$$

in which

$$\mathbf{M} = \begin{bmatrix} \sigma_x \mathbf{I}_3, & \tau_{xy}\mathbf{I}_3, & \tau_{xz}\mathbf{I}_3 \\ & \sigma_y\mathbf{I}_3, & \tau_{yz}\mathbf{I}_3 \\ \text{SYM} & & \sigma_z\mathbf{I}_3 \end{bmatrix} \tag{39}$$

is a 9 x 9 matrix arising from a rearrangement of the stress terms.

In this derivation we have implied the constitutive law for the material relating directly increments of the Piola-Kirchhoff stresses with increments of the Lagrangian strain. For small strain elasticity this definition is obviously most convenient and indeed the matrix $\mathbf{D}_T$ is the familiar one of elastic constants referred to original coordinates. For large strain elasticity this formulation again can be used as in general energy expressions and are specified in terms of Lagrangian strain (viz. Mooney material) and coefficients of $\mathbf{D}_T$ now become simply the derivatives at the strain energy with respect to the various strain components.

For plasticity involving small strain, the terms of $\mathbf{D}_T$ are well identified (Reference 9) and their derivation will be given in Section 7. For large strain, plasticity constitutive relationships are more complex and the discussion of these is relegated to Appendix IV.

All the ingredients for the numerical calculations are now available for the Lagrangian approach and it will be observed, by comparison with the next section that this is generally most convenient. In particular, specification of anisotropy is easy as this is always referred to axes coinciding with the undeformed structure.

One word should be added concerning the interpretation of stresses. While these for all cases of small strain correspond with real stresses, when large strains used stresses occur in the computation are the Piola–Kirchhoff definitions and require a transformation of the true (Eulerian) stress.

5. Explicit form of Eulerian formulations

Now we continuously update the coordinates to the new system $\bar{\mathbf{x}}$ which now defines the independent variables. The displacements $\bar{\mathbf{u}}$ are still referred to the same space directions. Strains are now determined by the derivatives of displacements with respect to the updated coordinates and are given explicitly as

$$\bar{\epsilon}_x = \frac{\partial u}{\partial \bar{x}} - \frac{1}{2}\left[\left(\frac{\partial u}{\partial \bar{x}}\right)^2 + \left(\frac{\partial v}{\partial \bar{x}}\right)^2 + \left(\frac{\partial w}{\partial \bar{x}}\right)^2\right]$$

$$\bar{\gamma}_{xy} = \frac{\partial u}{\partial \bar{y}} + \frac{\partial v}{\partial \bar{x}} - \left[\frac{\partial u}{\partial \bar{x}} \cdot \frac{\partial u}{\partial \bar{y}} + \frac{\partial v}{\partial \bar{x}} \cdot \frac{\partial v}{\partial \bar{y}} + \frac{\partial w}{\partial \bar{x}} \cdot \frac{\partial w}{\partial \bar{y}}\right] \tag{40}$$

etc.

With the shape functions N_i now being given in updated coordinates, $\bar{\mathbf{x}}$, i.e.

$$\mathbf{N}_i \equiv \mathbf{N}_i\,(\bar{x},\bar{y},\bar{z}) \tag{41}$$

we can write

$$\bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\epsilon}}^o + \bar{\boldsymbol{\epsilon}}^L \tag{42}$$

with

$$\bar{\boldsymbol{\epsilon}}^L = -\frac{1}{2}\,\bar{\mathbf{A}}\bar{\boldsymbol{\theta}} \tag{43}$$

where $\bar{\mathbf{A}}$ and $\bar{\boldsymbol{\theta}}$ have the same form as in Equation 24 but with differentiation referred now to $\bar{\mathbf{x}}$ system of coordinates.

The strains can now be determined for all large displacements; and, if strains are small, we can find stresses as the Eulerian stress definition coincides with that of conventional stress if the material is isotropic. If anisotropy of any kind is present, however, a difficulty arises immediately as the strain directions are now those of the global axes the material properties have to be specified in rotated coordinates. Eulerian formulation is thus not particularly well suited to the study of anisotropic situations.

To determine the residual forces of Equation 10 we need to determine

$$\bar{\mathbf{B}} \equiv \bar{\mathbf{B}}^o \tag{44}$$

where $\bar{\mathbf{B}}^o$ is given by Equation 27 with derivatives of the shape function being now taken with respect to $\bar{\mathbf{x}}$ system.

This very simple fact presents one of the advantages of the Eulerian formulation and providing $\bar{\boldsymbol{\sigma}}$ is known the residuals are determined very simply. Indeed the process is identical with that of small displacement formulation providing the coordinates are adjusted.

The derivation of the tangential stiffness matrix now presents a more complex problem. Proceeding as before we find the increment of the internal force term of Equation 10 i.e.

$$dF = d(\int_{\overline{V}} \overline{B}^{oT} \overline{\sigma}\, d\overline{V}) = \int_{\overline{V}} (d\overline{B}^{o.T} \overline{\sigma}\, d\overline{V} + $$

$$+ \quad \overline{B}^{oT} d\overline{\sigma}\, d\overline{V} \;+\; \overline{B}^{oT} \overline{\sigma}\, d(d\overline{V})) \tag{45}$$

noting that now all terms, being functions of $\overline{\mathbf{x}}$, depend on $\mathbf{u}$ and hence on the nodal parameters $\boldsymbol{\delta}$. Before proceeding further we must define the stress-strain relationahips. These are complicated by the fact that increments of Eulerian (Cauchy) stress and strain cannot be related by a constitutive relation. It is therefore necessary to introduce Jaumann stress increments (see Appendix I). Denoting this stress by $\overline{\boldsymbol{\sigma}}_J$ we can write the relation between the two stresses as

$$d\overline{\boldsymbol{\sigma}}_J = d\overline{\boldsymbol{\sigma}} + d\mathbf{T}_{\sigma}\overline{\boldsymbol{\sigma}} \tag{46}$$

and the constitutive relationship must be given by

$$d\overline{\boldsymbol{\sigma}}_J = \overline{\mathbf{D}}_T d\overline{\mathbf{e}} \tag{47}$$

where, for an isotropic material the tangential matrix of constants the same as are used in Equation 37 but in general this now referred to a direction changing with each increment.

Substituting into Relationship 45 we find that this can be written as

$$d\mathbf{F} = (\overline{\mathbf{K}}_{\sigma} + \overline{\mathbf{K}}_{o})\, d\boldsymbol{\delta} \tag{48}$$

in which often the predominant term is

$$\overline{\mathbf{K}}_{o} \quad \int_{\overline{V}} \overline{\mathbf{B}}^{o}\, \overline{\mathbf{D}}_T \overline{\mathbf{B}}^{o}\, d\overline{V} \tag{49}$$

The expression $\overline{\mathbf{K}}_{\sigma}$ is usually very small numerically and we find can be safely omitted from the calculations. In Appendix III we discuss some of the complexities of its form but as the equilibrium check is performed exactly, we conclude that the approximate tangential matrix in its very simple form is adequate for practical use.

6. Special features of the isoparametric formulation

The formulation and numerical integration of the isoparametric element has been widely discussed and is given in detail elsewhere (References 13, 14, and 15). Summarizing succinctly,we note that the Equation 4 which represents the parametric variation of displacements is written in component form as

$$u = \Sigma N_i u_i, \quad v = \Sigma N_i v_i, \quad w = \Sigma N_i w_i \tag{50}$$

where u_i, v_i and w_i are the parameters associated with the nodes, and N_i are scalar shape functions associated with a curvilinear coordinate system ξ, η, ζ

$$N_i \equiv N_i(\xi, \eta, \zeta) \tag{51}$$

The relationships of the curvilinear and cartesian coordinates are given in an identical form with

$$\xi = \Sigma N_i \xi_i \qquad \eta = \Sigma N_i \eta_i \qquad \zeta = \Sigma N_i \zeta_i$$

and

$$x = \Sigma N_i x_i \;, \qquad y = \Sigma N_i y_i \;, \qquad z = \Sigma N_i z_i \tag{52}$$

with x_i, y_i and z_i being the nodal coordinates in the global, cartesian system.

The derivatives of N_i with respect to the two coordinate systems are related by

$$\left\{\frac{\partial N_i}{\partial x}\right\}_i = \mathbf{J}_c^{-1} \left\{\frac{\partial N_i}{\partial \xi}\right\}_i \tag{53}$$

where

$$\left\{\frac{\partial N_i}{\partial \xi}\right\}_i = \left[\frac{\partial N_i}{\partial \xi}, \quad \frac{\partial N_i}{\partial \eta}, \quad \frac{\partial N_i}{\partial \zeta}\right]^T$$

$$\left\{\frac{\partial N}{\partial x}\right\}_i \quad \left[\frac{\partial N_i}{\partial x} \quad \frac{\partial N_i}{\partial y}, \quad \frac{\partial N_i}{\partial z}\right]^T \tag{54}$$

and

$$\mathbf{J}_c = \left[\frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)}\right]$$

is the 3 x 3 Jacobian coordinate transformation matrix. This has to be evaluated at each integrating point and inverted to obtain the Cartesian derivatives.

It is of interest to observe the structure of this Jacobian. On substitution of Relationship 52, we observe that it can be written as

$$\mathbf{J}_c = \begin{bmatrix} \mathbf{G}_\xi^T \mathbf{X} & \mathbf{G}_\eta^T \mathbf{X} & \mathbf{G}_\zeta^T \mathbf{X} \\ \mathbf{G}_\xi^T \mathbf{Y} & \mathbf{G}_\eta^T \mathbf{Y} & \mathbf{G}_\zeta^T \mathbf{Y} \\ \mathbf{G}_\xi^T \mathbf{Z} & \mathbf{G}_\eta^T \mathbf{Z} & \mathbf{G}_\zeta^T \mathbf{Z} \end{bmatrix} \tag{56}$$

with

$$\mathbf{G}_\xi^T = \left[\frac{\partial N_1}{\partial \xi}, \quad \frac{\partial N_2}{\partial \xi}, \; \ldots \; \frac{\partial N_i}{\partial \xi} \; \ldots\right]$$

$$\mathbf{X}^T = [x_1, \; x_2, \; \ldots, \quad x_i, \ldots] \tag{57}$$

etc.

This form allows the most convenient numerical evaluation and is useful to note in the context of programming.

When proceeding to evaluate the **B** matrices in the context of Lagrangian analysis, it is useful to observe that the separation into small and large strain components is not, in fact, convenient. Thus adding Expressions 27' and 34 we have on noting that $\bar{x} = x+u$, $\bar{y} = y+v$, $\bar{z} = z+w$ the form

$$\mathbf{B}_i = \left[\begin{array}{cccccc}
\frac{\partial \bar{x}}{\partial x} \frac{\partial N_i}{\partial x} & & \frac{\partial \bar{y}}{\partial x} \frac{\partial N_i}{\partial x} & & \frac{\partial \bar{z}}{\partial x} \frac{\partial N_i}{\partial x} \\
\hline
\frac{\partial \bar{x}}{\partial y} \frac{\partial N_i}{\partial y} & & \frac{\partial \bar{y}}{\partial y} \frac{\partial N_i}{\partial y} & & \frac{\partial \bar{z}}{\partial y} \frac{\partial N_i}{\partial y} \\
\hline
\frac{\partial \bar{x}}{\partial z} \frac{\partial N_i}{\partial z} & & \frac{\partial \bar{y}}{\partial z} \frac{\partial N_i}{\partial z} & & \frac{\partial \bar{z}}{\partial z} \frac{\partial N_i}{\partial z} \\
\hline
\frac{\partial \bar{x}}{\partial z} \frac{\partial N_i}{\partial y} + \frac{\partial \bar{x}}{\partial y} \frac{\partial N_i}{\partial z} & & \frac{\partial \bar{y}}{\partial z} \frac{\partial N_i}{\partial y} + \frac{\partial \bar{y}}{\partial y} \frac{\partial N_i}{\partial z} & & \frac{\partial \bar{z}}{\partial z} \frac{\partial N_i}{\partial y} + \frac{\partial \bar{z}}{\partial y} \frac{\partial N_i}{\partial z} \\
\hline
\frac{\partial \bar{x}}{\partial x} \frac{\partial N_i}{\partial z} + \frac{\partial \bar{x}}{\partial z} \frac{\partial N_i}{\partial x} & & \frac{\partial \bar{y}}{\partial x} \frac{\partial N_i}{\partial z} + \frac{\partial \bar{y}}{\partial z} \frac{\partial N_i}{\partial x} & & \frac{\partial \bar{z}}{\partial x} \frac{\partial N_i}{\partial z} + \frac{\partial \bar{z}}{\partial z} \frac{\partial N_i}{\partial x} \\
\hline
\frac{\partial \bar{x}}{\partial y} \frac{\partial N_i}{\partial x} + \frac{\partial \bar{x}}{\partial x} \frac{\partial N_i}{\partial y} & & \frac{\partial \bar{y}}{\partial y} \frac{\partial N_i}{\partial x} + \frac{\partial \bar{y}}{\partial x} \frac{\partial N_i}{\partial y} & & \frac{\partial \bar{z}}{\partial y} \frac{\partial N_i}{\partial x} + \frac{\partial \bar{z}}{\partial x} \frac{\partial N_i}{\partial y}
\end{array}\right] \qquad (58)$$

With $\frac{\partial N_i}{\partial x}$ etc. already known, the derivatives $\frac{\partial \bar{x}}{\partial x}$ etc. have now to be found.

These again are components of a Jacobian matrix

$$\mathbf{J} \equiv \left[\frac{\partial\ (\bar{x},\bar{y},\bar{z})}{\partial\ (x,y,z)}\right] \qquad (59)$$

which we shall call the deformation Jacobian matrix and which is given by a similar form to the coordinate Jacobian matrix already established.

$$
\mathbf{J} = \begin{bmatrix} \mathbf{G}_x^T \overline{\mathbf{X}} & \mathbf{G}_y^T \overline{\mathbf{X}} & \mathbf{G}_z^T \overline{\mathbf{X}} \\ \mathbf{G}_x^T \overline{\mathbf{Y}} & \mathbf{G}_y^T \overline{\mathbf{Y}} & \mathbf{G}_z^T \overline{\mathbf{Y}} \\ \mathbf{G}_x^T \overline{\mathbf{Z}} & \mathbf{G}_y^T \overline{\mathbf{Z}} & \mathbf{G}_z^T \overline{\mathbf{Z}} \end{bmatrix} \tag{60}
$$

where
$$
\mathbf{G}_x^T = \left[\frac{\partial N_1}{x} \quad \frac{\partial N_2}{x}, \ldots\ldots, \frac{\partial N_i}{x}\right], \ldots \tag{61}
$$

and $\mathbf{X}^T = [\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \ldots\ldots, \bar{\mathbf{x}}_i \ldots]$ etc.

in which $\bar{x}_i = x_i + u_i$ represent simply the updated nodal coordinates.

As the vectors $\mathbf{G}_x$ etc. given by Equation 61 have already been calculated, it is a simple matter to calculate the new coordinates and thus all the terms of the deformation; Jacobian matrix.

With the value of the **B** matrix known, all the residual forces can now be found and also the second part of the tangential matrix given by Equation 38 evaluated.

In evaluating this part of the stiffness matrix, the economics conventional in linear analysis which take account of zero terms cannot, in general, now be used (Reference 10). However, the evaluation of the first term of the tangential stiffness relation (the initial stress matrix given by Equation 38)automatically results in many such zeros, as will be noted from the Expressions 30 and 39, in which the identity matrix figures prominently. Here it is most convenient to do the multiplications explicitly. It can then be simply verified that the typical terms become simply

$$
\begin{aligned}
\mathbf{G}_i^T \mathbf{M} \mathbf{G}_j \equiv \mathbf{I}_3 \Big(& \sigma_x \frac{\partial N_i}{\partial x} \cdot \frac{\partial N_j}{\partial x} + \sigma_y \frac{\partial N_i}{\partial y} \cdot \frac{\partial N_j}{\partial y} \\
& + \sigma_z \frac{\partial N_i}{\partial z} \cdot \frac{\partial N_j}{\partial z} + \tau_{xy} \left(\frac{\partial N_i}{\partial x} \cdot \frac{\partial N_j}{\partial y} + \frac{\partial N_i}{\partial y} \cdot \frac{\partial N_j}{\partial x} \right) \\
& + \tau_{yz} \left(\frac{\partial N_i}{\partial y} \cdot \frac{\partial N_j}{\partial z} + \frac{\partial N_i}{\partial z} \cdot \frac{\partial N_j}{\partial y} \right) + \tau_{zx} \left(\frac{\partial N_i}{\partial z} \cdot \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial x} \cdot \frac{\partial N_j}{\partial z} \right)
\end{aligned} \tag{62}
$$

This particular structure of the initial stress matrix is of considerable importance when numerical integration is used where, for the basic element stiffness the full multiplication of submatrices has to be repeated at each integrating point. The number of operations is vastly reduced if the advantage of $\mathbf{I}_3$ in the explicit multiplication carried out in Equation 62 is used. For instance, in a solid isoparametric element, the number of operations at each Gauss point is 1830 if full multiplication is carried out, while with the explicit expression only 210 operations are needed.

For the <u>Eulerian formulation</u>, the calculations proceed in a parallel manner. Indeed it will be noted that the transition from one system to the other is particularly simple in the isoparametric formulation.

Now the shape functions are implied as functions of the updated coordinates $\bar{\mathbf{x}}$ or their curvilinear equivalents $\boldsymbol{\xi}$.

If the nodal coordinates are continuously updated the calculation follow precisely the same pattern as used in small displacements - infinitesimal strain analysis where evaluation of $\bar{\mathbf{B}}^o$, volume elements and of the approximate stiffness matrix of Equation 49 is concerned.

The coordinate transformation Jacobian is now found from Equations 53 to 57 by replacing x_i with $\bar{x}_i$ etc., as is implied in the previous statements.

Thus, for instance

$$\bar{\mathbf{J}}_c = \begin{bmatrix} \mathbf{G}_\xi^T \bar{\mathbf{X}} & \mathbf{G}_\eta^T \bar{\mathbf{X}} & \mathbf{G}_\zeta^T \bar{\mathbf{X}} \\ \mathbf{G}_\xi^T \bar{\mathbf{Y}} & \mathbf{G}_\eta^T \bar{\mathbf{Y}} & \mathbf{G}_\zeta^T \bar{\mathbf{Y}} \\ \mathbf{G}_\xi^T \bar{\mathbf{Z}} & \mathbf{G}_\eta^T \bar{\mathbf{Z}} & \mathbf{G}_\zeta^T \bar{\mathbf{Z}} \end{bmatrix} \tag{63}$$

with $\bar{\mathbf{X}}$ etc., defined by Equation 61 .

The matrix $[\mathrm{B}^o]$ is simply given by substitution of $\bar{\mathbf{X}}$ coordinate with Equation 27 i.e.

$$\bar{\mathbf{B}}_i^o = \begin{bmatrix} \dfrac{\partial N_i}{\partial \bar{x}}, & 0 & 0 \\ 0 & \dfrac{\partial N_i}{\partial \bar{y}}, & 0 \\ 0 & 0 & \dfrac{\partial \bar{N}_i}{\partial \bar{z}} \\ 0 & \dfrac{\partial N_i}{\partial \bar{z}} & \dfrac{\partial N_i}{\partial \bar{y}} \\ \dfrac{\partial N_i}{\partial \bar{z}} & 0 & \dfrac{\partial N_i}{\partial \bar{x}} \\ \dfrac{\partial N_i}{\partial \bar{y}} & \dfrac{\partial N_i}{\partial \bar{x}} & 0 \end{bmatrix} \tag{64}$$

with the derivatives given similarly to Equation 53.

$$\left[\frac{\partial N_i}{\partial \bar{x}}\right]_i = \bar{\mathbf{J}}_c^{-1} \left[\frac{\partial N_i}{\partial \xi}\right]_i \tag{65}$$

The only outstanding difference from simply pursuing a small strain analysis with continually updated coordinates, is in the determination of strain and stress components as already pointed out in the previous section where the Jaumann stress increment was introduced.

The case of a linear elastic and isotropic material with small strains but large deformation is of particular interest. Now one can establish the Eulerian stress $\bar{\boldsymbol{\sigma}}$ directly from the strains by the usual relation.

$$\bar{\boldsymbol{\sigma}} = \bar{\mathbf{D}}\,\bar{\boldsymbol{\epsilon}} \tag{66}$$

where $\bar{\boldsymbol{\epsilon}}$ is the Eulerian strain found by Equation 40 and in which $\bar{\mathbf{D}}$ is the matrix of usual elastic constants. These strains defined in Expression 40 are most conveniently found by using a new deformation Jacobian

$$\bar{\mathbf{J}} \equiv \left[\frac{\partial\,(x,y,z)}{\partial\,(\bar{x},\bar{y},\bar{z})}\right] \tag{67}$$

which is found by Expressions 60 to 61 with the bar symbols interchanged. Now vectors X listing the original coordinates are of importance. The same expressions used for evaluating Lagrangian strains are used with the above substitution to obtain Eulerian strains. i.e.

$$\bar{\mathbf{J}} = \begin{bmatrix} \bar{\mathbf{G}}_x\,\mathbf{x}, & \bar{\mathbf{G}}_y\,\bar{\mathbf{X}}, & \ldots\ldots \\ \ldots\ldots\ldots, & \ldots\ldots\ldots, & \ldots\ldots \\ \ldots\ldots\ldots, & \ldots\ldots\ldots & \ldots\ldots \end{bmatrix} \tag{68}$$

$$\bar{\mathbf{B}}_i = \begin{bmatrix} \frac{\partial x}{\partial \bar{x}} \quad \frac{\partial N_i}{\partial \bar{x}}, & \ldots\ldots \\ \ldots\ldots\ldots, & \ldots\ldots \\ \ldots\ldots\ldots, & \ldots\ldots \\ \ldots\ldots\ldots, & \ldots\ldots \\ \text{etc.} & \\ \text{as in } 58 & \end{bmatrix} \tag{69}$$

7. <u>The tangential modulus matrix for plasticity</u>

In previous sections we have introduced the concept of the tangential modulus matrix $\mathbf{D}_T$. For elastic problems this is well defined. For various nonlinear constitutive relations this matrix is perhaps not obvious and on occasion may not be available.

The case of plastic deformation is of particular interest and here indeed the specification of material behaviour is most conveniently given by this matrix and the corresponding deformation Equation 37 i.e.

$$d\boldsymbol{\sigma} = \mathbf{D}_T\, d\boldsymbol{\epsilon} \tag{37}$$

in which Lagrangian strains are raised. Although the matrix has been derived earlier (Reference 9) in context of small strain, it is convenient to present here a more compact derivation (References 10 and 12) as this will be used in the numerical examples following and is also applicable for large displacements (Reference 16).

Dividing the total increment of strain into its elastic and plastic components we can write

$$d\boldsymbol{\epsilon} = d\boldsymbol{\epsilon}^{e} + d\boldsymbol{\epsilon}^{p} \tag{70}$$

The yield surface is generally given as

$$F\,(\sigma\ ,K) = 0 \quad \text{and the plastic potential}$$
$$Q\,(\boldsymbol{\sigma}, K) = 0 \tag{71}$$

where $\boldsymbol{\sigma}$ now stood for the Piola—Kirchhoff stress are κ for a hardening parameter. As elastic strains associative with plasticity are always small, we can write

$$d\boldsymbol{\epsilon}^{e} = \mathbf{D}\ d\boldsymbol{\sigma} \tag{72}$$

For plastic strains the normality rule requires that

$$d\boldsymbol{\epsilon}^{p} = d\lambda\ \overset{*}{\mathbf{a}} \tag{73}$$

where

$$\mathbf{a}^{T} = \left[\frac{\partial Q}{\partial \sigma_{x,}} \ldots\ldots\right]$$

and $d\lambda$ is an undetermined positive proportionality constant.

During plastic deformation the stress remains on the yield surface and thus

$$d\ \mathbf{F}\ = \mathbf{a}^{T}\ \ d\boldsymbol{\sigma} - \mathbf{A}\ d\ \lambda = 0$$

where

$$\mathbf{a}^{T} = \left[\frac{\partial F}{\partial \sigma_{x,}} \ldots\ldots\right]$$

and

$$A = -\ \frac{1}{d\lambda}\ \frac{\partial F}{\partial \kappa}\ d\kappa$$

For associative plasticity $\mathbf{a} = \overset{*}{\mathbf{a}}$ and if the yield surface is defined in terms of a uniaxial given stress A is equal to the slope of the uniaxial stress - plastic strain; curve.

Substituting Equation 72 and 73 with Equation 70 we have

$$d\boldsymbol{\epsilon}\ = d\lambda\overset{*}{\mathbf{a}} + \mathbf{D}^{-1}\ d\boldsymbol{\sigma} \tag{75}$$

and on premultiplying by $\mathbf{a}^T\mathbf{D}$ and using Equation 75 to eliminate $d\boldsymbol{\sigma}$

$$\mathbf{a}^T\mathbf{D}\, d\boldsymbol{\epsilon} = \mathbf{A} d\lambda + \mathbf{a}^T\mathbf{D}\mathbf{a}^* \, d\lambda \qquad (76)$$

From above $d\lambda$ can be found. Substitution of this into Equation 75 gives now the tangential matrix

$$\mathbf{D}_T = \mathbf{D} - \mathbf{D}_p \qquad (77)$$

with

$$\mathbf{D}_p = \mathbf{D}\mathbf{a}^* \, \mathbf{D}\mathbf{a}^T / (A + \mathbf{a}^T\mathbf{D}\mathbf{a}^*)$$

This matrix can be evaluated explicitly at all steps of the numerical computation providing $d\lambda$ is a positive quantity (if not purely elastic unloading takes place). It is thus necessary to compute $d\lambda$ at all stages of calculation.

In Reference 12 various yield surfaces are described in detail and the discussion of these in here is unnecessary. It is of interest to observe that the tangential modulus matrix loses its symmetry in the case of non-associative laws and therefore some additional computations or difficulties arise in such a case.

To illustrate the applicability of the processes outlined, several examples are quoted.

8.1 Thick cantilever – large elastic deformation – Figure 2

Here Lagrangian formulation is used and five parabolic isoparametric elements approximate to the beam. Linear elastic laws are used despite the appreciable strain due to the thick shape of the cantilever.

Comparison with solution with an analysis Bisshopp and Drucker (Reference 17) is given although the latter does not take the shear deformation or Poisson's ratio into account.

It is of interest to remark that two (and three) dimensional approaches of the kind outlined here have been used with success for much thinner sections and indeed, give an alternative approach to plate and shell problems (Reference 18). If thin sections are used, however, a reduction of integration order must be used to avoid spurious shear stiffness (Reference 19). Indeed, such reduced integration orders are of general applicability and improve overall performance as will be shown later.

8.2 Shallow spherical cap – large elastic deformation – Figure 3

The example is now axisymmetric and again treated as a simple solid body of revolution with 12 parabolic elements. Comparison is given with results of Haisler et al. (Reference 5) and shows good agreement with their most precise solution despite the fact this rather large increment was used here (all increments are shown in an accompanying figure).

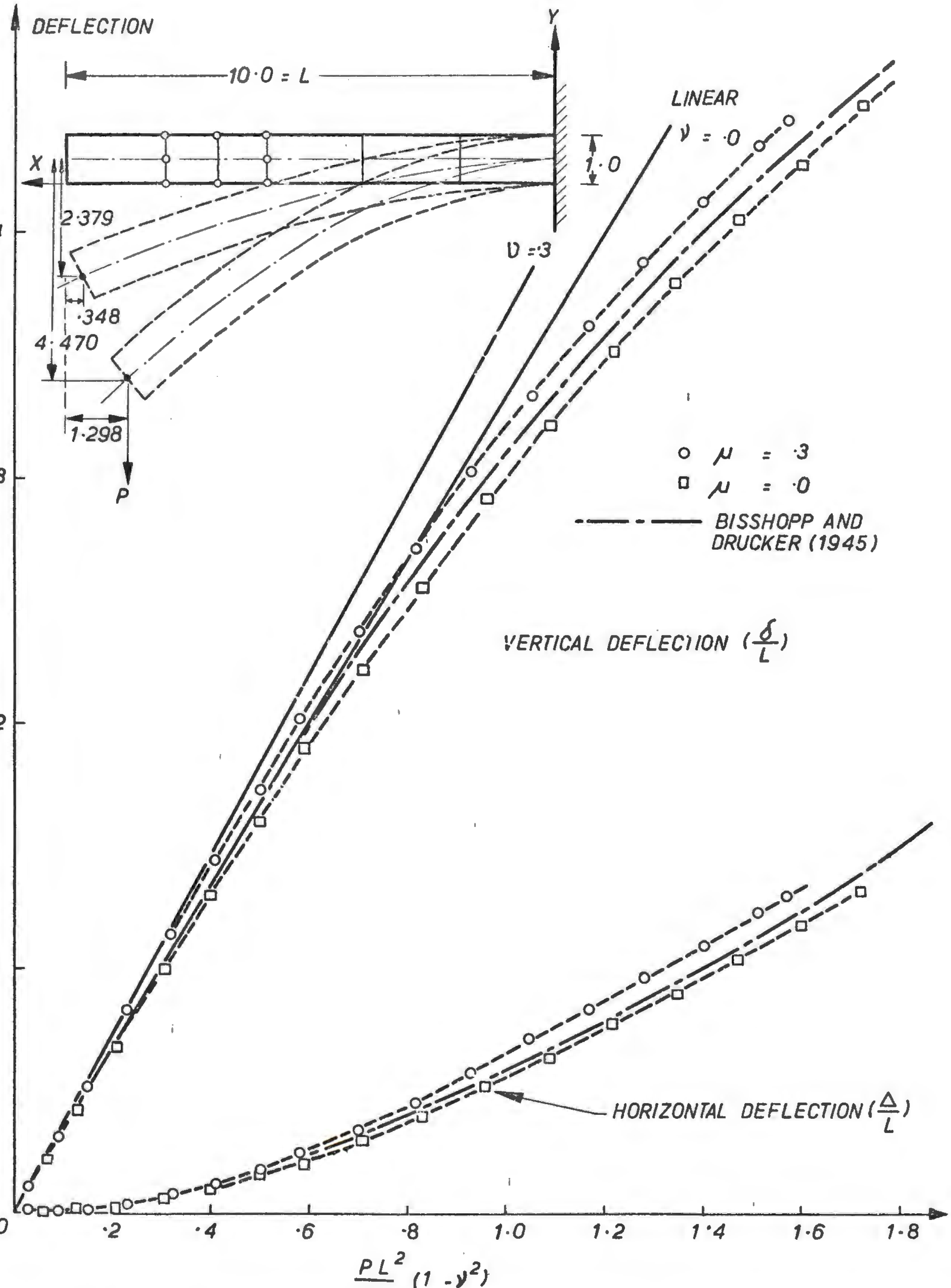

FIG. 2 LARGE DEFLECTIONS OF A CANTILEVER BEAM (ELASTICA PROBLEM)

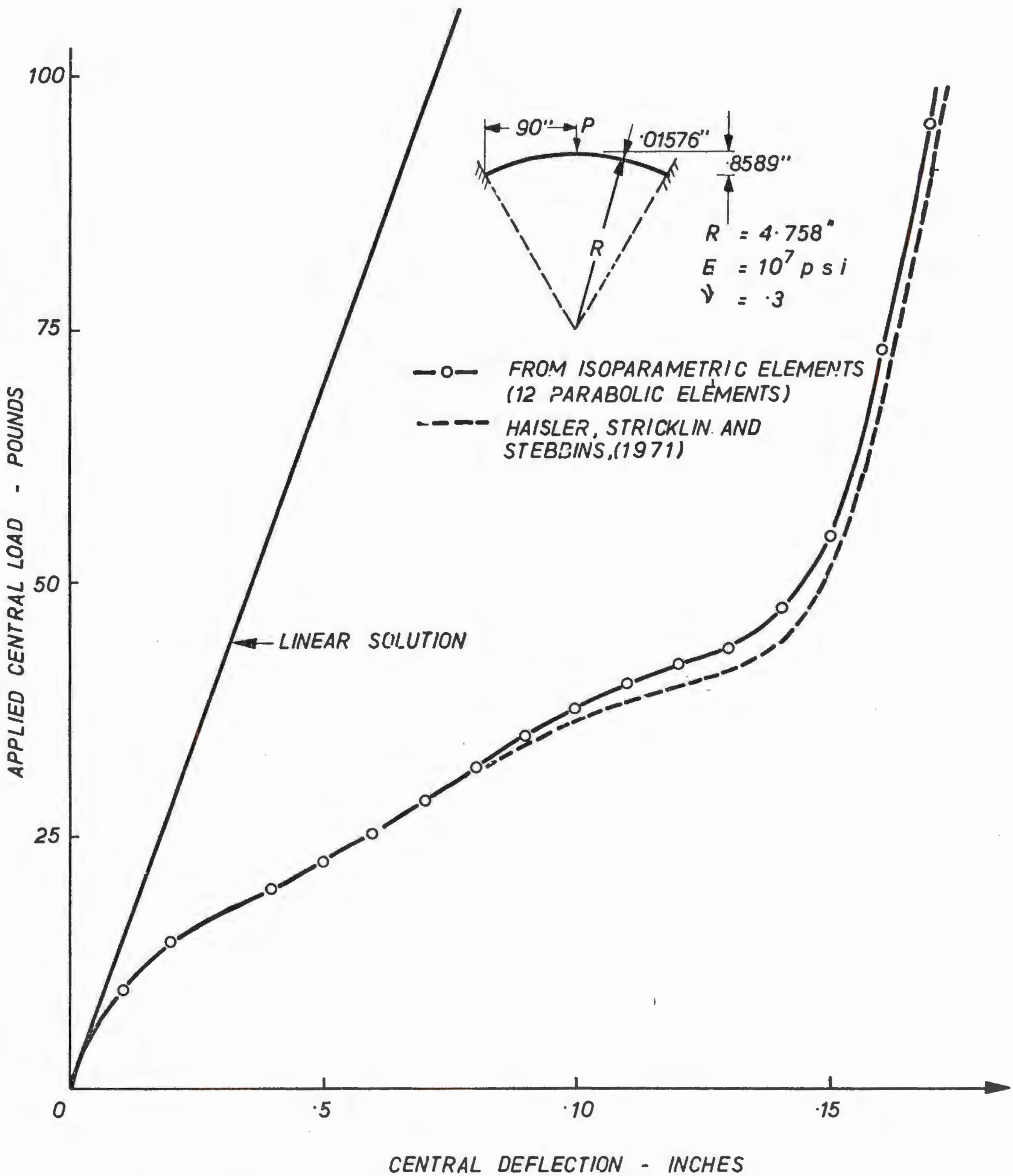

FIG. 3 LOAD DEFLECTION CURVE - SHALLOW SPHERICAL CAP

It was found in this and indeed other similar examples that the number of iterative steps in each increment is reduced considerably if displacement rather than load is incremented. This indeed is the only way of achieving solutions beyond or at the peak of load as seen in the next examples.

Displacement incrementation presents no difficulties for a single load — with several proportional loads an artifice requiring two solutions at each step must be introduced and this mitigates somewhat against the advantages of displacement increments (References 20 and 21).

8.3 Shallow arch – large elastic deformation — Figures 4 and 5

In this problem displacement of the central point was incremented to allow a study of the snap through behaviour. Excellent comparison with results quoted by Biezeno and Grammel (Reference 22) is achieved.

8.4 Bellows — Figure 6

This axisymmetric example is given to illustrate a practical application in which the nonlinear load deformation characteristics needed to be predicted.

8.5 Thick cylinder — Elastic-plastic Behaviour — Figures 7 and 8a, b, and c

This fairly trivial problem is solved to compare in some detail the efficiency of the program with every solution. Also, the differences between this application of Von Mises and Tresca yield conditions are highlighted.

Three parabolic isoparametric elements are used and integration is carried out for 4 x 4, 3 x 3, and 2 x 2 Gauss points. It is worth remarking that the latter given in this case has no detectible difference in deformation but shows a considerable improvement in stresses.

Comparison is made with results derived by Hodge and White (Reference 23).

8.6 Axisymmetric Extrusion — Plastic 'Failure' — Figures 9 and 10

An axisymmetric problem of extrusion is approached here via the solution of a fictitious elastic-plastic one of the same configuration. The development of plastic zones shown, as the displacement of the plunger is incremented, is only of academic interest. The "collapse" load at which deformation progresses without a further load increase does however coincide with the steady state extrusion process.

For this computation Johnson and Mellor (Reference 27) have obtained a solution by an approximate process. It is remarkable how closely the finite element solution approaches their predicted experimental results despite the poorness of the mesh of parabolic elements employed.

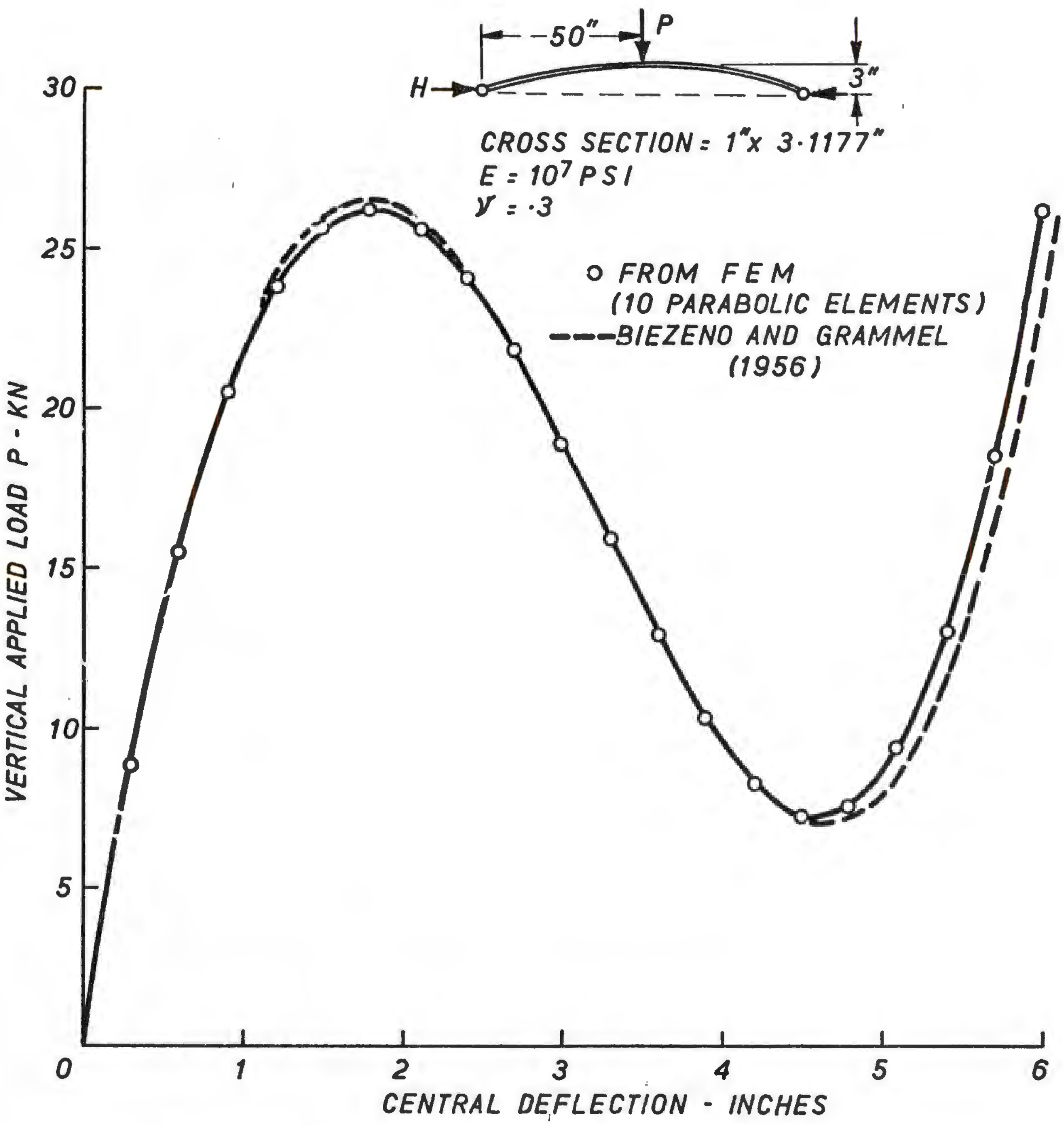

FIG. 4 A PINNED CIRCULAR SHALLOW ARCH
LOAD - CENTRAL DEFLECTION PLOT

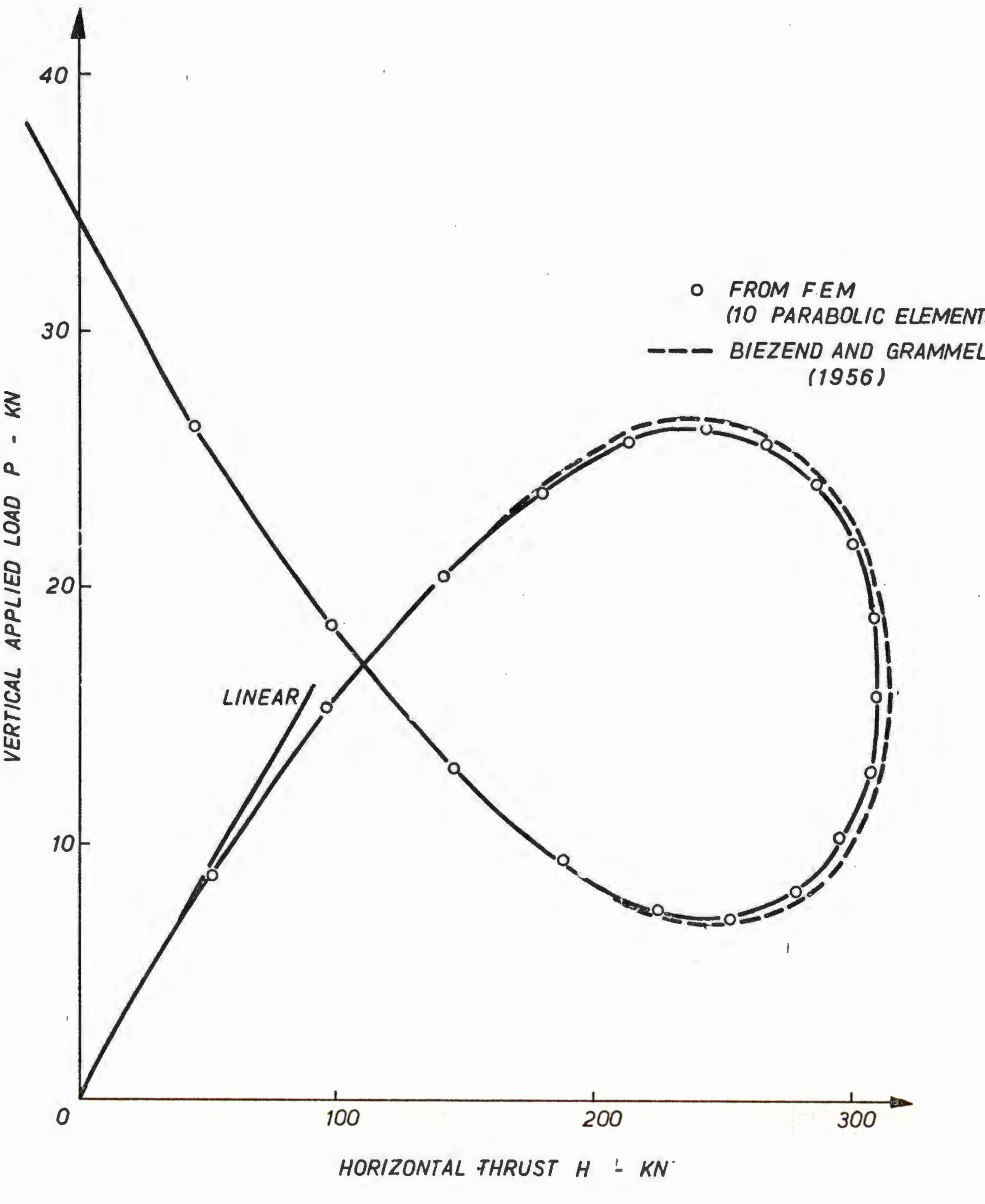

FIG. 5 A PINNED CIRCULAR SHALLOW CAP
APPLIED LOAD - HORIZONTAL THRUST CURVE

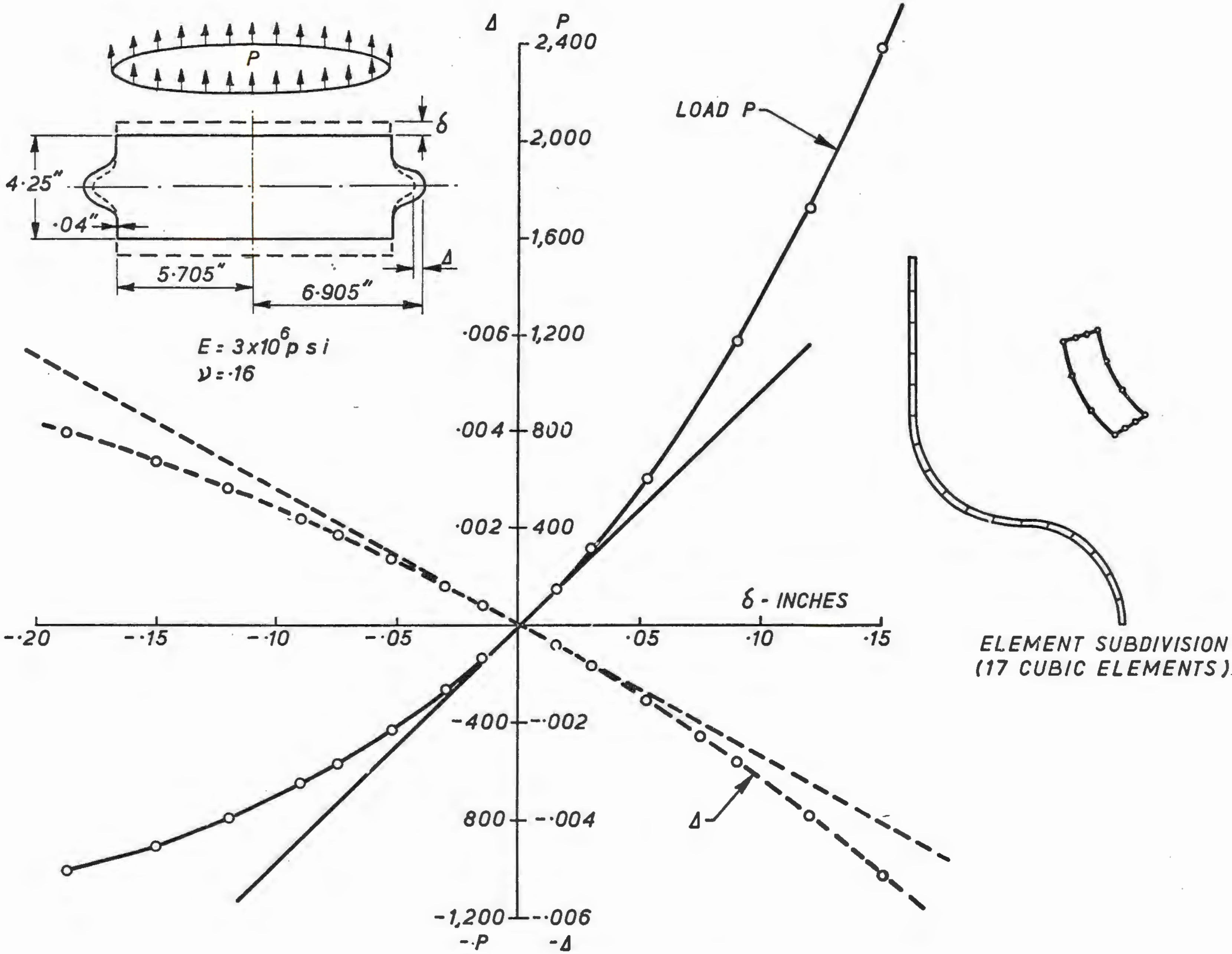

FIG 6 A BELLOWS JUNCTION. LOAD-DEFLECTION CURVES

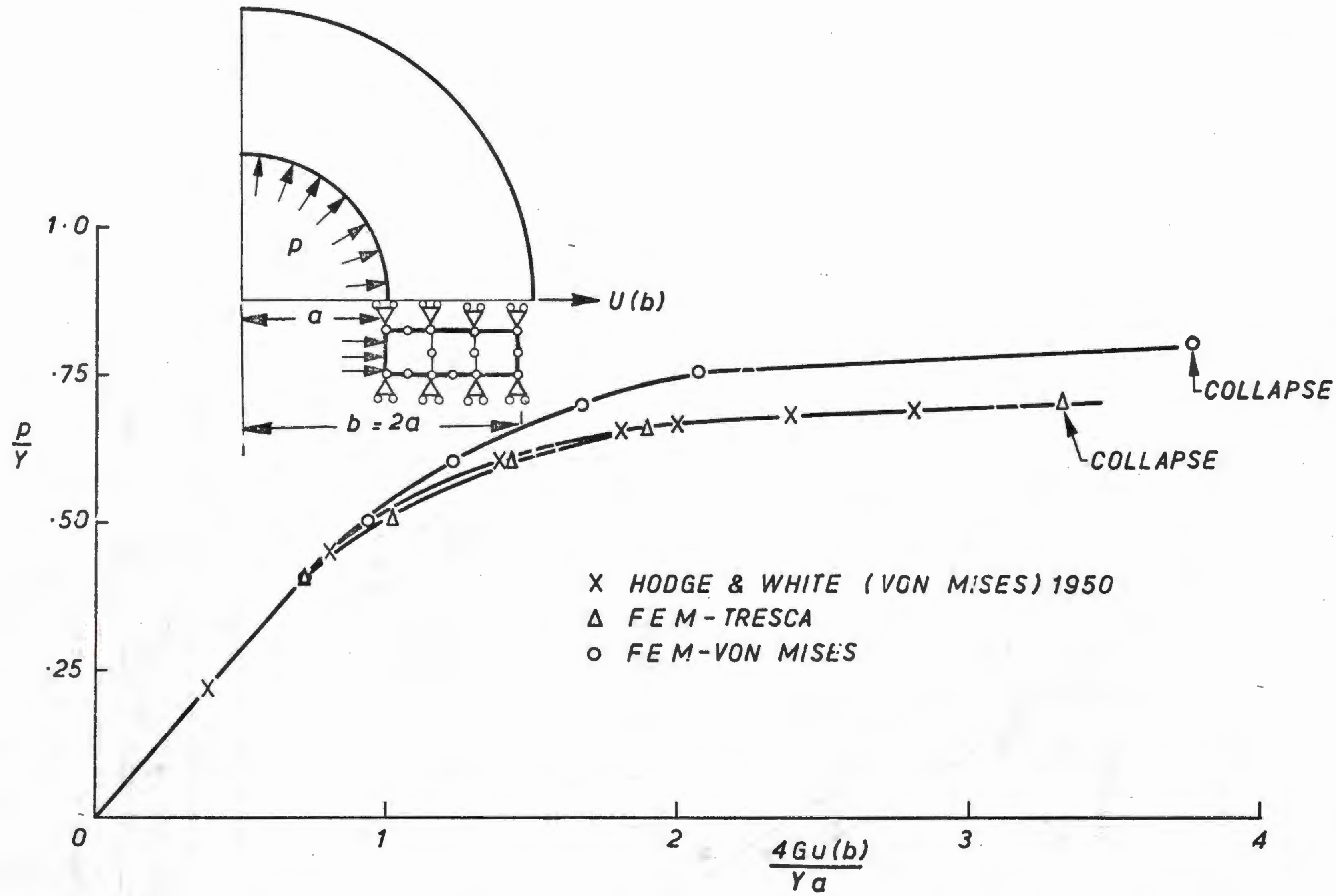

FIG. 7 THICK CYLINDER-(NO AXIAL STRAIN) NO DIFFERENCE OBTAINED FOR 2x2, 3x3 OR 4x4 GAUSS POINTS
Y-UNIAXIAL YIELD STRESS, G-SHEAR MODULUS

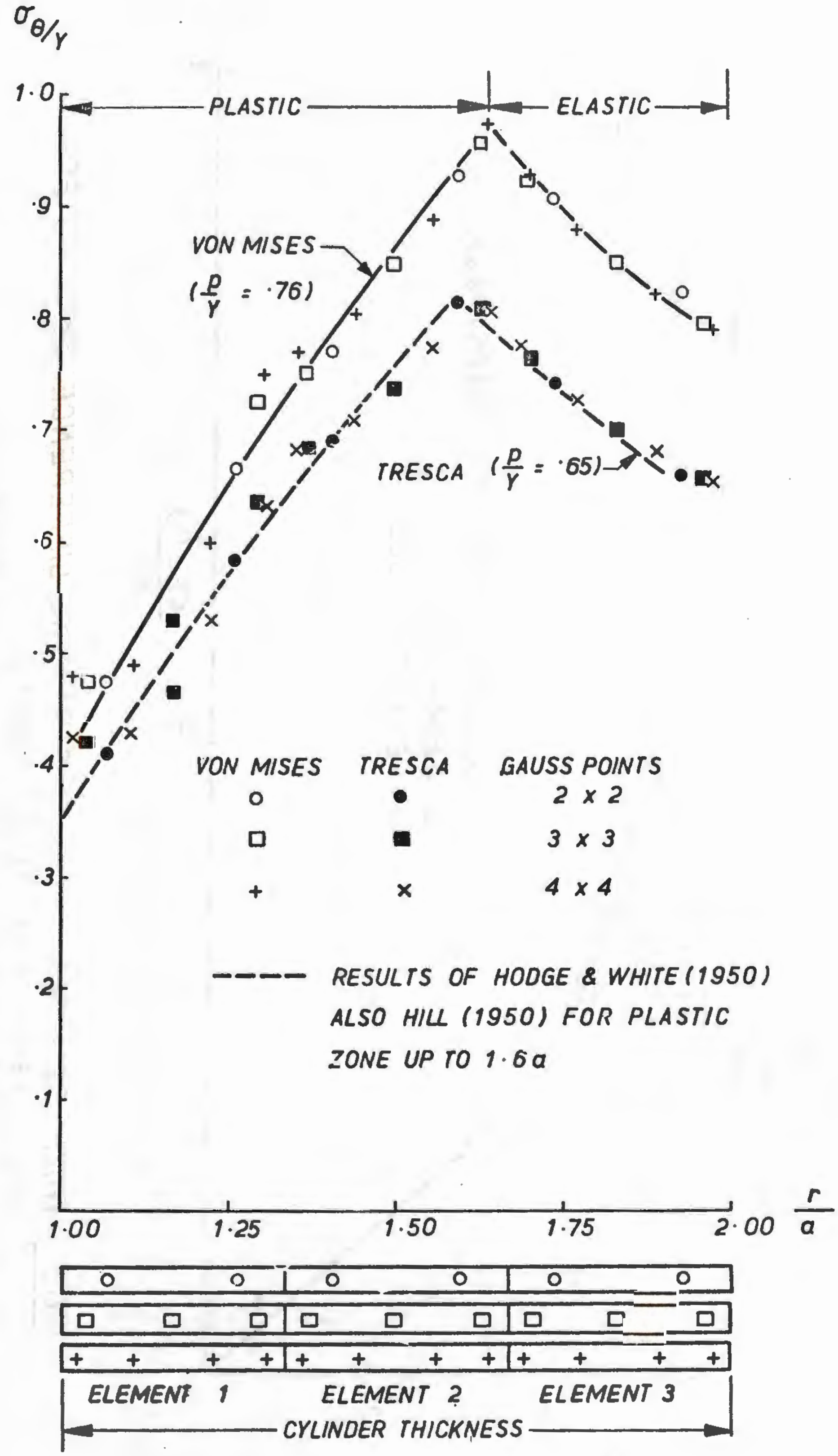

FIG. 8a THICK CYLINDER (NO AXIAL STRAIN)
HOOP STRESS DISTRIBUTION

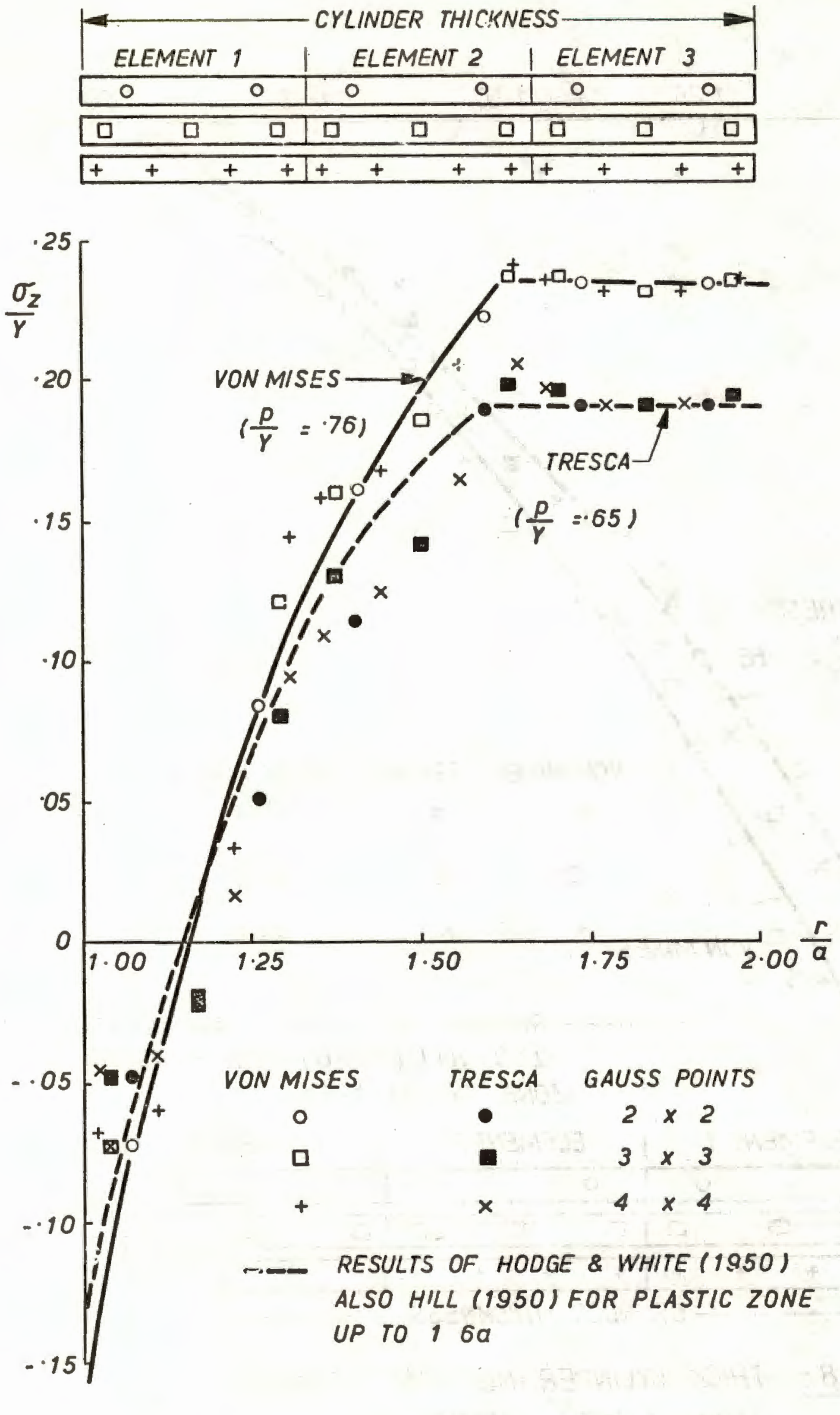

FIG. 8b THICK CYLINDER (NO AXIAL STRAIN)
AXIAL STRESS DISTRIBUTION

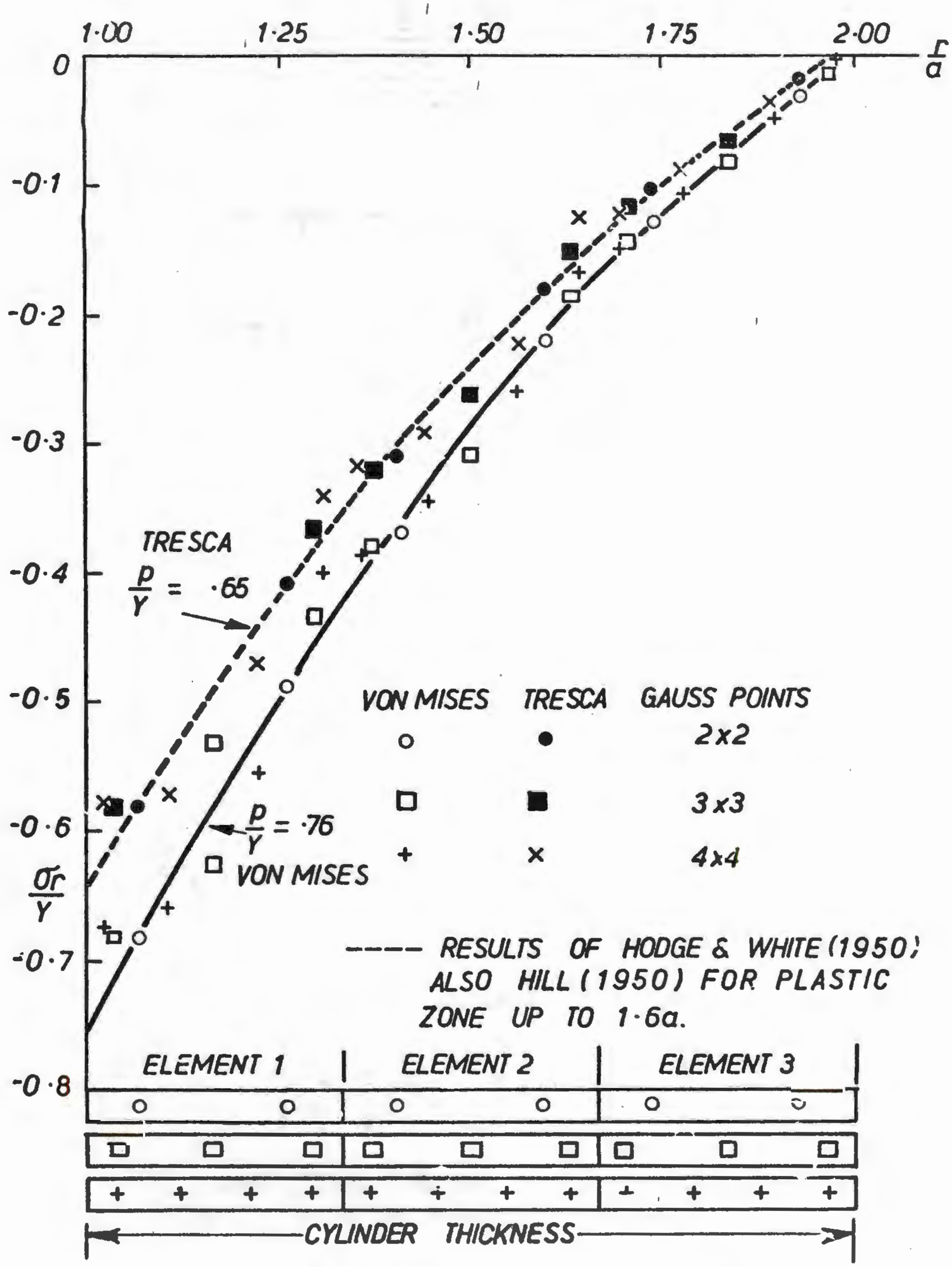

FIG. 8c THICK CYLINDER (NO AXIAL STRAIN)
RADIAL STRESS DISTRIBUTION.

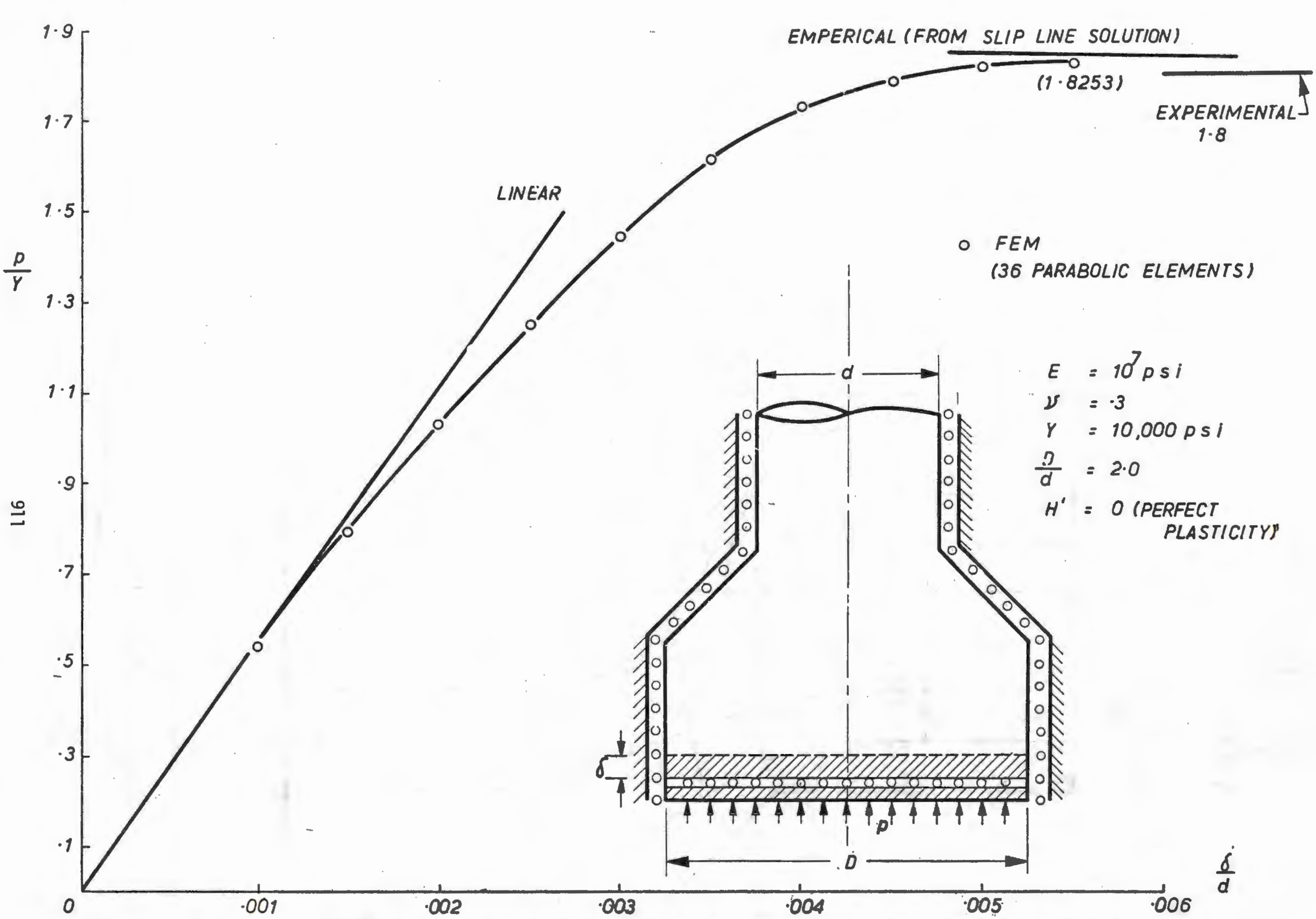

FIG. 9 AN AXISYMETRIC EXTRUSION PROBLEM, MEAN PRESSURE - DISPLACEMENT CURVE

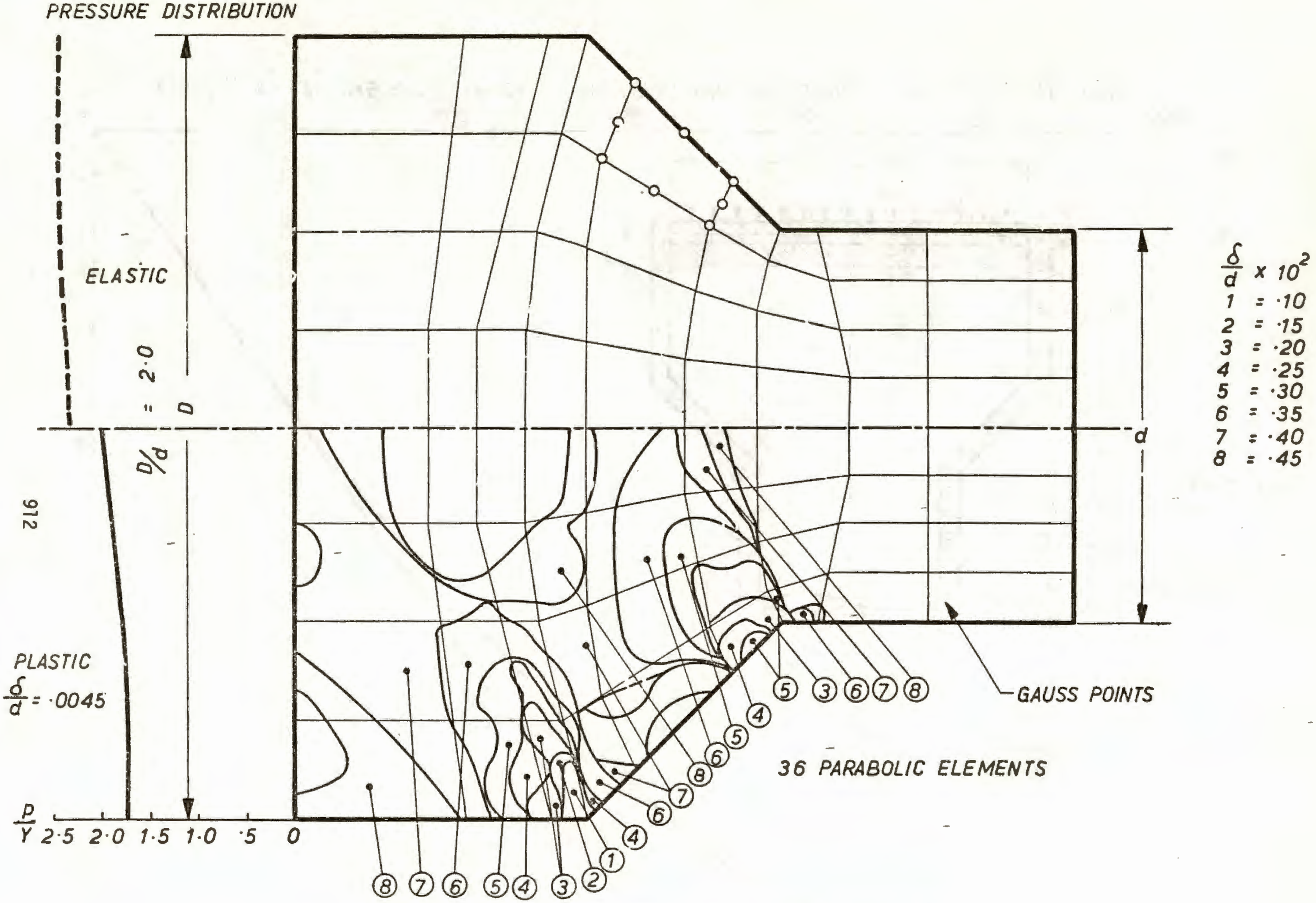

FIG. 10 AN AXISYMMETRIC EXTRUSION PROBLEM SPREAD OF PLASTIC ZONES AND PRESSURE DISTRIBUTION

This example incidently indicates one of the advantages of isoparametric elements in plasticity. The "chequerband" pattern of plastic zone spread familiar in the use of constant strain triangles is substituted by a smooth progression of Gauss points at which yield is observed.

Concluding Remarks

The formulation given in this paper shows that in essence large deformations present few additional difficulties. The calculation steps in isoparametric form are standard and follow a well established pattern. The cases of infinitesimal and finite strain differ only in the form of the constitutive relation established.

While it is more conventional to describe all the steps in terms of tensor notation, the recasting of these in matrix form is computationally advantageous and presents no difficulties.

In all the problems given here, we have used a generalized continuum approach and by the use of Green or Almansi strain deformations have avoided all approximations involved in the introduction of oriented bodies and as beams, plates, or shells. However, as shown in Reference 13 the specialization to such cases presents no difficulties providing we are prepared to accept restrictions of small rotations. With this restriction, approximate relations conventionally used are introduced and the formulation in essence remains unaltered.

Several solution techniques have been introduced into one program and the choice between alternatives left to the user. It is almost impossible to determine the one which at all times is most expeditious, and therefore the choice is essential.

In plasticity problems as stated earlier (Reference 9) we find (Reference 10) that modified Newton-Raphson (initial stress) techniques are usually most economic especially if used with an efficient accelerator. Indeed, if strain-softening problems are dealt with such methods are the only ones applicable as tangential matrix may become indefinite.

In problems involving significant deformation, the same techniques may lead to very slow or non-evident convergence and here the alternative of varying the tangential matrix in each computation step is often essential.

It is thus important to "keep all options open" in nonlinear problems and indeed it appears that other solution processes as yet unexplored may in future provide further alternatives.

We have not discussed in detail here the question of deciding at what stage of iteration a solution is 'sufficiently accurate'.

In earlier work an absolute maximum magnitude of error was often specified. We find in general that a norm of residual forces or of displacement changes provides a more convenient estimate. The question as to the permissible magnitude of such norms needs further investigation and at the moment these are specified by intuitive reasoning.

The introduction of the residual force concept is, we find, an essential feature of any reasonable approach to nonlinear problems.

The merits of isoparametric element forms as the basis of nonlinear analysis have been brought out in the text. The accuracy of representation in linear elastic analysis seems to carry over to nonlinear situations.

In three dimensional problems, their use is essential to linear and nonlinear analysis alike.

Although we have shown relatively simple quadrilateral forms only in the examples given - the use of hierarchial elements (Reference 28) is worth while in complex situations.

The numerical integration used invariably makes introduction to isoparametric elements of any nonlinear formulation easy. Indeed, the main question at the present time concerns not the numerical methodology but the lack of sufficiently broad constitutive relationships to cover many existing materials.

In the formulation presented for the tangential stiffness matrix $\mathbf{K}_T$ of Equation 38 we notice that this is given in two parts — conveniently separating the initial stress stiffness matrix from the remainder. We would like to point out that it is possible to present the expression in a more compact way in which only 3 matrices are involved, i.e.

$$\mathbf{K}_T = \int_V \mathbf{G}^T \mathbf{D}_g \mathbf{G} \, dV \tag{78}$$

where $\mathbf{D}_g$ is a 9 x 9 "generalized modulus matrix". This form is more convenient in some computation and is given in Reference 10. We have, however, retained the 'split' form which is applicable also in case of 'oriented bodies' such as bars, plates, etc. (see Ch. 19, Reference 13).

The program described for plasticity allows the inclusion of any generalized yield criteria (Reference 12) by a simple change of four Fortran statement cards.

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APPENDIX I

THE MATRIX FORM OF FINITE DEFORMATION THEORY

1. Deformation and Strain

With $\mathbf{x} = [x\ y\ z]^T$ denoting undeformed rectangular coordinates, $\mathbf{u} = [u\ v\ w]^T$ the displacements measured in the same frame of reference, the coordinates of a deformed point become

$$\bar{\mathbf{x}} = [\bar{x}\ \bar{y}\ \bar{z}]^T = \mathbf{x} + \mathbf{u} \qquad \text{A1}$$

A line element ds with components $d\mathbf{x} = [dx\ dy\ dz]^T$ on deformation changes to a new length $d\bar{s}$ with components $d\bar{\mathbf{x}}$. We can write

$$d\bar{\mathbf{x}} = \mathbf{J}\, d\mathbf{x} \qquad \text{A2}$$

where

$$\mathbf{J} = \left[\frac{\partial(\bar{x},\bar{y},\bar{z})}{\partial(x,y,z)}\right] = \begin{bmatrix} \frac{\partial \bar{x}}{\partial x} & \frac{\partial \bar{x}}{\partial y} & \frac{\partial \bar{x}}{\partial z} \\ \frac{\partial \bar{y}}{\partial x} & \frac{\partial \bar{y}}{\partial y} & \frac{\partial \bar{y}}{\partial z} \\ \frac{\partial \bar{z}}{\partial x} & \frac{\partial \bar{z}}{\partial y} & \frac{\partial \bar{z}}{\partial z} \end{bmatrix} \qquad \text{A3}$$

is a Jacobian matrix defining the deformed state in some way with $[\bar{J}]$ being similarly given.

We note in passing that

$$\bar{\mathbf{J}} = \mathbf{J}^{-1} = \left[\frac{\partial(x,y,z)}{\partial(\bar{x},\bar{y},\bar{z})}\right]$$

from the definition A2 .

We can write the change of length measure as

$$\tfrac{1}{2}(d\bar{s}^2 - ds^2) = \tfrac{1}{2}(d\bar{\mathbf{x}}^T d\bar{\mathbf{x}} - d\mathbf{x}^T d\mathbf{x})$$

$$\equiv d\mathbf{x}^T \boldsymbol{\epsilon}\, d\mathbf{x} \equiv d\bar{\mathbf{x}}^T \bar{\boldsymbol{\epsilon}}\, d\bar{\mathbf{x}} \qquad \text{A4}$$

where

$$\boldsymbol{\epsilon} = \tfrac{1}{2}(\mathbf{J}^T\mathbf{J} - \mathbf{I}) \qquad \text{A5}$$

is the Lagrangian definition of strain (Green's strain 1839–42) and

$$\epsilon = \tfrac{1}{2}(\mathbf{I} - \bar{\mathbf{J}}^T\bar{\mathbf{J}}) \qquad \text{A6}$$

is the Eulerian definition of strain (Almansi strain 1911) (in both $\mathbf{I}$ is an identity (unit) matrix).

The physical measuring of the strains defined above is discussed in many text-books, but it is of importance for an engineer to realize that if strains are small (but displacements large), the first definition gives elongations and angular changes in reference to an element positioned in the undeformed body while the second refers to the same quantities of an element positioned in the deformed body.

Thus, for instance, in Figure 1 we illustrate for the small strain case the meaning of the components ϵ_x and $\bar{\epsilon}_x$.

By substituting Equation A1 into the expressions A5 and A6 the explicit engineering expressions for strain components, Equations 22 and 40 given earlier in the text are derived. In these we arrange the strains in a vector form ϵ or $\bar{\boldsymbol{\epsilon}}$ in the usual manner as the strains are symmetric.

At this stage it is worth noting that the computer evaluation of the Jacobian matrices is an economic way of arriving at the strain components as described in the text.

2. Changes of Geometry

If $d\mathbf{A} = [dA_x \; dA_y \; dA_z]^T$ defines an elementary area by its three projections (on planes perpendicular to x, y, z axes) in the undeformed coordinates, and if $d\bar{\mathbf{A}} = [d\bar{A}_x \; d\bar{A}_y \; d\bar{A}_z]^T$ refers to the same area in deformed coordinates then

$$d\bar{\mathbf{A}} = |J| \; \mathbf{J}^{T-1} \, d\mathbf{A}$$

and

$$d\mathbf{A} = |J|^{-1} \, \mathbf{J}^T \, d\bar{\mathbf{A}} \qquad \text{A7}$$

Similarly elementary volumes dV and $d\bar{V}$ are related by

$$d\bar{V} = |J| \; dV \qquad \text{A8}$$

For elaboration on the proofs, the reader is referred to Murnaghan (Reference 29).

3. Variation of Strain

Let us consider the variation of the strain Jacobian matrix for small changes of displacement. Now

$$d\mathbf{J} = \left[\frac{\partial\,(\bar{x}+du,\; \bar{y}+dv,\; \bar{z}+dw)}{\partial\,(x,\, y,\, z)}\right] - \left[\frac{\partial\,(\bar{x},\bar{y},\bar{z})}{\partial\,(x,y,z)}\right] \qquad \text{A9}$$

which on expanding gives

$$d\mathbf{J} = \overline{\mathbf{V}}_d \, \mathbf{J}$$

where $$\overline{\mathbf{V}}_d \equiv \left[\frac{\partial \, (du, dv, dw)}{\partial \, (\bar{x}, \bar{y}, \bar{z})} \right] \qquad \text{A10}$$

is variation of deformation.

As

$$\mathbf{J} \, \overline{\mathbf{J}} = \mathbf{I}$$, by taking its variation

we have

$$d\overline{\mathbf{J}} = - \overline{\mathbf{J}} \, \overline{\mathbf{V}} d \qquad \text{A11}$$

It is simple to show that the variation of Green's and Almansi strain matrices is given by

$$d\boldsymbol{\epsilon} = \mathbf{J}^T \, d\bar{\boldsymbol{\epsilon}} \, \mathbf{J} \qquad \text{A12}$$

and

$$d\bar{\boldsymbol{\epsilon}} = d\bar{\boldsymbol{\epsilon}} - \overline{\mathbf{V}}_d{}^T \, \bar{\boldsymbol{\epsilon}} + \bar{\boldsymbol{\epsilon}} \, \overline{\mathbf{V}}_d \qquad \text{A13}$$

with

$$d\bar{\boldsymbol{\epsilon}} = \tfrac{1}{2} (\overline{\mathbf{V}}_d + \overline{\mathbf{V}}_d{}^T) \qquad \text{A14}$$

$$d\bar{\boldsymbol{\omega}} = \tfrac{1}{2} (\mathbf{V}_d - \mathbf{V}_d{}^T) \qquad \text{A15}$$

4. Definitions of Stress

The natural definition of stresses is obviously the Eulerian one referring in the usual way to the forces per unit deformed areas.

Thus we can write

$$\bar{\boldsymbol{\sigma}} = \begin{bmatrix} \bar{\sigma}_x & \bar{\tau}_{xy} & \bar{\tau}_{xz} \\ \bar{\tau}_{xy} & \bar{\sigma}_y & \bar{\tau}_{yz} \\ \bar{\tau}_{xz} & \bar{\tau}_{yz} & \bar{\sigma}_z \end{bmatrix} \qquad \text{A16}$$

in matrix notation or simply rearrange in the usual vector form as $\bar{\boldsymbol{\sigma}}$.

The forces acting on an area $d\overline{\mathbf{A}}$ are given by the vector

$$d\overline{\mathbf{F}} = \bar{\boldsymbol{\sigma}} \, d\overline{\mathbf{A}} \qquad \text{A17}$$

For a Lagrangian definition of stress the situation is by no means so clear and various alternatives were proposed in the literature. Here we shall use the Piola–Kirchhoff (1833, 1852) definition.

This gives an equivalent force vector acting on an original, undeformed area dA by an expression identical to Equation A17, i.e.

$$d\overline{\mathbf{F}} = \boldsymbol{\sigma}\, d\mathbf{A} \qquad \text{A18}$$

and the actual force on the deformed area

$$d\overline{\mathbf{F}} = \mathbf{J}\, d\mathbf{F} \qquad \text{A19}$$

From Equations A18 and A7 we have immediately the stress transformation relation

$$\boldsymbol{\sigma} = |\mathrm{J}|\ \mathbf{J}^{-1}\, \overline{\boldsymbol{\sigma}}\, \mathbf{J}^{T-1} \qquad \text{A20}$$

which shows that the above stress definition is still symmetric and can be written in as a vector $\boldsymbol{\sigma}$.

5. Rate of Work

The rate at which work is being done internally or virtual work done during a displacement $d\mathbf{u}$ can be written in the usual way as in small displacement analysis

$$\int_{\overline{V}} (\overline{\boldsymbol{\sigma}}^{T}\, d\overline{\boldsymbol{\epsilon}})\, d\overline{V} \qquad \text{A21}$$

in which

$$d\overline{\mathbf{e}} = d \begin{bmatrix} \dfrac{\partial u}{\partial \overline{x}} \\[2ex] \dfrac{\partial v}{\partial \overline{y}} \\[2ex] \dfrac{\partial w}{\partial \overline{z}} \\[2ex] \dfrac{\partial w}{\partial \overline{y}} + \dfrac{\partial v}{\partial \overline{z}} \\[2ex] \dfrac{\partial u}{\partial \overline{z}} + \dfrac{\partial w}{\partial \overline{x}} \\[2ex] \dfrac{\partial v}{\partial \overline{x}} + \dfrac{\partial u}{\partial \overline{y}} \end{bmatrix} \qquad \text{A22}$$

corresponds to changes $d\overline{\mathbf{e}}$ and is identical to the expression for infinitesimal strains defined with respect to the instantaneous coordinate $\overline{\mathbf{x}}$.

Purely formal transformations transform the above Eulerian definition to the equivalent Lagrangian and which now is

$$\int_V (\boldsymbol{\sigma}^T d\boldsymbol{\epsilon})\ dV \qquad \text{A23}$$

This form easily recognizable by similarity with small strain and displacement analysis, is oblivious of small strain but large displacements are considered and $\boldsymbol{\sigma}$ represents the locally directed stresses dependent on $\boldsymbol{\epsilon}$. In the more general sense of finite strain it is the merit of the Piola–Kirchhoff definition that the form is preserved.

6. Constitutive Relations and Increments of Strain/Stress

The additiveness of stress/strain changes needs to be considered in general finite element analysis.

Consider first the question of thermal strains incrementing during the deformation.

In the case of Lagrangian definition such strains can be added to the total strain irrespective of whether the material is 'anisotropic' or isotropic. In the case of Eulerian strain definition isotropic strain can be added but if Figure 1 is considered it will be seen that anisotropic initial strains need to be transformed to account for the rotation of material axes.

If small strain linear elasticity is used to define the constitutive relation of the usual form

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon} \qquad \text{A24}$$

again we note that correct stresses will be obtained for any material in the Lagrangian definition but, that for anisotropy a rotation of axes is necessary in defining the **D** matrix when the Eulerian system is used.

The same remarks apply to large deformation, elasticity where effectively the **D** matrix is derived by differentiation of the strain-energy density with respect to strain components.

Here again, anisotropy will favour the use of the Lagrangian definition.

For more general constitutive relations, such as may be involved in plasticity, etc., we derive relationships between changes of strain and changes of deformation.

In Lagrangian formulation we note that the stress is always associated with direction of strains and we can use the form

$$d\boldsymbol{\sigma} = \mathbf{D}_T\, d\boldsymbol{\epsilon} \qquad \text{A25}$$

with elements $d\boldsymbol{\sigma}$ being simply additive. In plasticity, for instance, we have to refer the straining rate to some form of actual stress changes. However, it is clear from previous discussion that, physically, changes of Eulerian stress and strain cannot be related (as changes in these will occur by pure rigid body rotation). This necessitates the introduction of another type of stress rate – due to Jaumann (1911) and a relation of the type

$$d\boldsymbol{\sigma}_J = \overline{\mathbf{D}}_T\, d\bar{\mathbf{e}} \qquad \text{A26}$$

These changes of the Jaumann stress can be related to the changes of Eulerian stress components (in the additive sense) by writing in vector form

$$d\bar{\boldsymbol{\sigma}} = d\boldsymbol{\sigma}_J + d\,\mathbf{T}_\omega\, \bar{\boldsymbol{\sigma}} \qquad \text{A27}$$

where d $\mathbf{T}_\omega$ is defined by Equation A15, and is given by

$$d\,\mathbf{T}_\omega = \begin{bmatrix} 0 & 0 & 0 & 0 & -\bar{\omega}_y & \bar{\omega}_z \\ 0 & 0 & 0 & \bar{\omega}_x & 0 & -\bar{\omega}_z \\ 0 & 0 & 0 & -\bar{\omega}_x & \bar{\omega}_y & 0 \\ 0 & -\tfrac{1}{2}\bar{\omega}_x & \tfrac{1}{2}\bar{\omega}_x & 0 & -\tfrac{1}{2}\bar{\omega}_z & \tfrac{1}{2}\omega_y \\ \tfrac{1}{2}\bar{\omega}_y & 0 & -\tfrac{1}{2}\bar{\omega}_y & \tfrac{1}{2}\omega_z & 0 & -\tfrac{1}{2}\bar{\omega}_x \\ -\tfrac{1}{2}\bar{\omega}_z & \tfrac{1}{2}\bar{\omega}_z & 0 & -\tfrac{1}{2}\omega_y & \tfrac{1}{2}\bar{\omega}_x & 0 \end{bmatrix} \qquad \text{A28}$$

where $\bar{\omega}_x = d\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right)$ $\bar{\omega}_y = d\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)$ $\bar{\omega}_z = d\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$

APPENDIX II

PLANE AND AXISYMMETRIC CASES

Plane Strain / Stress:

The Green strain in vector form can be written as

$$\boldsymbol{\epsilon} = [\epsilon_x\ \epsilon_y\ \epsilon_{xy}]^T = \boldsymbol{\epsilon}^o + \boldsymbol{\epsilon}^L \qquad \text{B1}$$

where

$$\underset{3\times 1}{\boldsymbol{\epsilon}^L} = \tfrac{1}{2}\begin{bmatrix} \boldsymbol{\theta}_x^T & \boldsymbol{\theta}_x \\ \boldsymbol{\theta}_y^T & \boldsymbol{\theta}_y \\ 2\boldsymbol{\theta}_x^T & \boldsymbol{\theta}_y \end{bmatrix} = \tfrac{1}{2}\underset{3\times 4}{\begin{bmatrix} \boldsymbol{\theta}_x^T & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\theta}_y^T \\ \boldsymbol{\theta}_y^T & \boldsymbol{\theta}_x^T \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} \boldsymbol{\theta}_x \\ \boldsymbol{\theta}_y \end{bmatrix}} = \tfrac{1}{2}\,\mathbf{A}\boldsymbol{\theta} \qquad \text{B2}$$

The displacement gradients are arranged in vector form as

$$\underset{4\times1}{\boldsymbol{\theta}} = \begin{bmatrix} \boldsymbol{\theta}_x \\ \\ \boldsymbol{\theta}_y \end{bmatrix} \begin{bmatrix} \cdots & \mathbf{I}_2 \dfrac{\partial N_i}{\partial x} & \cdots \\ \\ \cdots & \mathbf{I}_2 \dfrac{\partial N_i}{\partial y} & \cdots \end{bmatrix} \underset{4\times n \quad n\times1}{\boldsymbol{\delta} = \mathbf{G}\boldsymbol{\delta}} \qquad \text{B3}$$

where $\boldsymbol{\theta}_x = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \end{bmatrix}^T$, $\boldsymbol{\theta}_y = \begin{bmatrix} \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{bmatrix}^T$ B4

and $\mathbf{I}_2$ is a 2x2 identity matrix.

Finally, the strain displacement relationship is described by

$$d\boldsymbol{\epsilon} = \mathbf{B}\, d\, \boldsymbol{\delta}$$

where the submatrix for node i

$$\mathbf{B}_i = \left[\begin{array}{cc|cc} \dfrac{\partial \bar{x}}{\partial x} \dfrac{\partial N_i}{\partial x} & & \dfrac{\partial \bar{y}}{\partial x} \dfrac{\partial N_i}{\partial x} & \\ \dfrac{\partial \bar{x}}{\partial y} \dfrac{\partial N_i}{\partial y} & & \dfrac{\partial \bar{y}}{\partial y} \dfrac{\partial N_i}{\partial y} & \\ \dfrac{\partial \bar{x}}{\partial x} \dfrac{\partial N_i}{\partial y} + \dfrac{\partial \bar{x}}{\partial y} \dfrac{\partial N_i}{\partial x} & & \dfrac{\partial \bar{y}}{\partial y} \dfrac{\partial N_i}{\partial x} + \dfrac{\partial \bar{y}}{\partial x} \dfrac{\partial N_i}{\partial y} & \end{array}\right] \qquad \text{B5}$$

and the (4x4) $\mathbf{M}$ matrix required for 'initial stress stiffness'

$$\mathbf{K}_\sigma = \int_V \mathbf{G}^T \mathbf{M} \mathbf{G}\, dV$$

can be derived as

$$\mathbf{M} = \begin{bmatrix} \sigma_x \mathbf{I}_2 & \tau_{xy} \mathbf{I}_2 \\ \tau_{xy} \mathbf{I}_2 & \sigma_y \mathbf{I}_2 \end{bmatrix} \qquad \text{B6}$$

Axisymmetric Case:

The Green's strain in vector form in r, z, θ cylindrical coordinates is written as

$$\boldsymbol{\epsilon} = [\epsilon_r \ \epsilon_z \ \gamma_{rz} \ \epsilon_\theta]^T = \boldsymbol{\epsilon}^o + \boldsymbol{\epsilon}^L \qquad \text{B7}$$

where

$$\underset{4\times 1}{\boldsymbol{\epsilon}^{L}} = \frac{1}{2}\begin{bmatrix} \boldsymbol{\theta}_r^{T} & \boldsymbol{\theta}_r \\ \boldsymbol{\theta}_z^{T} & \boldsymbol{\theta}_z \\ 2\boldsymbol{\theta}_r^{T} & \boldsymbol{\theta}_z \\ & \left(\frac{u}{r}\right)^2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta}_r^{T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\theta}_z^{T} & \mathbf{0} \\ \boldsymbol{\theta}_z^{T} & \boldsymbol{\theta}_r^{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \left(\frac{u}{r}\right) \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta}_r \\ \boldsymbol{\theta}_z \\ \frac{u}{r} \end{bmatrix} = \frac{1}{2}\underset{4\times 5}{\mathbf{A}}\,\underset{5\times 1}{\boldsymbol{\theta}} \qquad \text{B8}$$

The displacement gradient $\boldsymbol{\theta}$ for an axisymmetric case now becomes a 5x1 vector and the submatrix $\mathbf{B}_i$ for node i in the strain displacement relationship can be derived as

$$\mathbf{B}_i = \left[\begin{array}{cc|cc} \frac{\partial \bar{r}}{\partial r} & \frac{\partial N_i}{\partial r} & \frac{\partial \bar{z}}{\partial r} & \frac{\partial N_i}{\partial r} \\ \frac{\partial \bar{r}}{\partial z} & \frac{\partial N_i}{\partial z} & \frac{\partial \bar{z}}{\partial z} & \frac{\partial N_i}{\partial z} \\ \multicolumn{2}{c|}{\frac{\partial \bar{r}}{\partial r}\frac{\partial N_i}{\partial z} + \frac{\partial \bar{r}}{\partial z}\frac{\partial N_i}{\partial r}} & \multicolumn{2}{c}{\frac{\partial \bar{z}}{\partial z}\frac{\partial N_i}{\partial r} + \frac{\partial \bar{z}}{\partial r}\frac{\partial N_i}{\partial z}} \\ \left(\frac{\bar{r}}{r}\right) & \frac{N_i}{r} & \multicolumn{2}{c}{0} \end{array}\right] \qquad \text{B9}$$

Finally the matrix $\mathbf{M}$ is modified to 5x5 matrix

$$\mathbf{M} = \begin{bmatrix} \sigma_r \mathbf{I}_2 & \tau_{rz}\mathbf{I}_2 & \mathbf{0} \\ \tau_{rz}\mathbf{I}_2 & \sigma_z \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \sigma_\theta \end{bmatrix} \qquad \text{B10}$$

APPENDIX III

EULERIAN FORMULATION

For the derivation incremental force displacement relationship in Eulerian form, if we consider increment of the internal force term of Equation 10 we have

$$d\left(\int_{\bar{V}} \bar{\mathbf{B}}^{oT}\,\bar{\boldsymbol{\sigma}}\,d\bar{V}\right) = \int_{\bar{V}} \left(d\bar{\mathbf{B}}^{oT}\,\bar{\boldsymbol{\sigma}}\,d\bar{V} + \bar{\mathbf{B}}^{oT}\,d\bar{\boldsymbol{\sigma}}\,d\bar{V} + \bar{\mathbf{B}}^{oT}\,\bar{\boldsymbol{\sigma}}\,d\,(d\bar{V})\right) \qquad \text{C1}$$

Substituting in Equation C1

$$d\bar{\boldsymbol{\sigma}} = \bar{\mathbf{D}}_T \, d\bar{\boldsymbol{\epsilon}} + d\mathbf{T}_{\omega} \, \bar{\boldsymbol{\sigma}} \text{ (from Equation A27)}$$

and

$$d(d\bar{V}) = (d\bar{\epsilon}_x + d\bar{\epsilon}_y + d\bar{\epsilon}_z) \, d\bar{V} \qquad \text{C2}$$

we have

$$d\left(\int_{\bar{V}} \bar{\mathbf{B}}^{o\,T} \bar{\boldsymbol{\sigma}} \, d\bar{V}\right) = \int_{\bar{V}} \bar{\mathbf{B}}^{o\,T} \bar{\mathbf{D}}_T \bar{\mathbf{B}}^{o} \, d\bar{V} \, d\boldsymbol{\delta}$$

$$+ \int_{\bar{V}} (d\bar{\mathbf{B}}^{o\,T} + \bar{\mathbf{B}}^{o\,T} d\mathbf{T}_{\omega} + (d\bar{\epsilon}_x + d\bar{\epsilon}_y + d\bar{\epsilon}_z) \, \bar{\mathbf{B}}^{o\,T} \bar{\boldsymbol{\sigma}} \, d\bar{V} \qquad \text{C3}$$

The first term on the right-hand side of the Equation C3 gives $\bar{\mathbf{K}}^o$ and the second term gives the initial stress stiffness matrix $\bar{\mathbf{K}}_{\sigma}$. The 6×6 matrix $d\mathbf{T}_{\omega}$ appearing in second term is defined by Equation A28 and $d\bar{\epsilon}_x$, $d\bar{\epsilon}_y$ — are given by Equation A15. Finally the variation $d\bar{\mathbf{B}}^{o\,T}$ can be derived by considering equation

$$\left[\frac{\partial N_i}{\partial x}\right]_i = \mathbf{J}^T \left[\frac{\partial N_i}{\partial \bar{x}}\right] \qquad \text{C4}$$

$$d\left[\frac{\partial N_i}{\partial x}\right] = 0 = d\mathbf{J}^T \left[\frac{\partial N_i}{\partial \bar{x}}\right] + \mathbf{J}^T d\left[\frac{\partial N_i}{\partial \bar{x}}\right]$$

therefore

$$d\left[\frac{\partial N_i}{\partial \bar{x}}\right] = -\,\mathbf{J}^{T-1} \, d\mathbf{J}^T \left[\frac{\partial N_i}{\partial \bar{x}}\right]$$

and as from Equation A10 $\mathbf{J}^{T-1} d\mathbf{J}^T = \bar{\mathbf{V}}_d^{\,T}$,

$$d\left[\frac{\partial N_i}{\partial \bar{x}}\right] = -\bar{\mathbf{V}}_d^{\,T} \left[\frac{\partial N_i}{\partial \bar{x}}\right] \qquad \text{C5}$$

From Equation 27 it can be easily seen that the variation $d\bar{\mathbf{B}}^{o\,T}$ can be easily obtained from C5 by rearranging terms. However, the rearrangement of the whole of second term on right-hand side of Equation C3 requires elaborate algebraic organization.

APPENDIX IV

LARGE STRAIN PLASTICITY

In the literature several attempts have been made to establish elastic plastic relationships at finite deformations and we shall restrict ourselves to papers by Hill (Reference 26) (1958), Green and Naghdi (Reference 16) (1965) and Lee (Reference

27) (1969). We shall classify them into two cases, (i) small strain with large rotation (References 16 and 26) and (ii) finite elastic-plastic strain (Reference 27). We shall briefly recollect some of these relationships in Lagrangian form and state the counterpoints in Eulerian form. First of all we shall consider the case of small elastic plastic strains with large rotations in which the total Green's strain can be divided into elastic and plastic components as considered by Green and Naghdi (16, section 5). Thus

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_e + \boldsymbol{\epsilon}_p \tag{D1(a)}$$

and so the increments of Green's strain

$$d\boldsymbol{\epsilon} = d\boldsymbol{\epsilon}_e + d\boldsymbol{\epsilon}_p \tag{D1(b)}$$

The yield criterion in terms of Piola–Kirchhoff stresses σ is expressed as (neglecting temperature effects)

$$F\,(\boldsymbol{\sigma}, \boldsymbol{\epsilon}_p, \kappa) = 0 \tag{D2}$$

Where κ is a hardening parameter defining subsequent surfaces corresponding to $\boldsymbol{\epsilon}_p$ and whole history. The usual restrictions for neutral states or unloading hold as in the classical plasticity theory. Also, the subsequent yield surfaces defined in general by

$$d\kappa = d\kappa\,(\boldsymbol{\sigma}, \boldsymbol{\epsilon}_p, d\boldsymbol{\sigma}, d\boldsymbol{\epsilon}_p)$$

is restricted to linear relation with $d\boldsymbol{\epsilon}_p$ and thus normality rule is derived as

$$d\boldsymbol{\epsilon}_p = d\lambda\,\mathbf{a} \tag{D3}$$

with

$$\frac{\partial F}{\partial \boldsymbol{\sigma}} = \mathbf{a}^T \tag{D4}$$

For infinitesimal elastic strains one can write

$$d\boldsymbol{\sigma} = \mathbf{D}\,d\boldsymbol{\epsilon}_e \tag{D5}$$

where **D** is the usual elasticity modulus matrix (Reference 13). Now the elastic plastic matrix can be derived in the same way (Reference 12).

$$d\lambda = (A + \beta)^{-1}\,(\mathbf{d}^T\,d\boldsymbol{\epsilon}) \quad \text{with } \mathbf{d} = \mathbf{D}\mathbf{a},$$

$$\beta = \mathbf{a}^T\,\mathbf{D}\mathbf{a} \quad \text{and A is arbitrary hardening constant and}$$

$$d\boldsymbol{\sigma} = \mathbf{D}_T\,d\boldsymbol{\epsilon} = (\mathbf{D} - (A + \beta)^{-1}\,\mathbf{d}\;\mathbf{d}^T) \tag{D6}$$

However, an important difference should be noticed at this stage as regards to yield criterion D2. Usually, the yield criterion is expressed in terms of true stresses $\bar{\boldsymbol{\sigma}}$ and for finite strain case with isotropic hardening, following Prager (Reference 31) one can write

$$F = f\,(\,|J|\,\bar{\boldsymbol{\sigma}}) - Y(\kappa) = 0 \tag{D7}$$

And from Equation A13 the true stresses are related to Piola–Kirchhoff stresses

$$|J| \bar{\boldsymbol{\sigma}} = \mathbf{J} \boldsymbol{\sigma} \mathbf{J}^T$$

and by transforming it into vectorial form, one can easily write

$$|J| \bar{\boldsymbol{\sigma}} = \mathbf{T} \boldsymbol{\sigma} \qquad \text{D8}$$

where 6 x 6 matrix **T** is function of deformation gradients. From Equations D7 and D8 one can derive

$$\frac{\partial F}{\partial \boldsymbol{\sigma}} = \mathbf{a}^T = \frac{\partial f}{\partial \bar{\boldsymbol{\sigma}}} \cdot \frac{\partial \bar{\boldsymbol{\sigma}}}{\partial \boldsymbol{\sigma}} = \bar{\mathbf{a}}^T \mathrm{T} \qquad \text{D9}$$

where $\bar{\mathbf{a}}^T = \dfrac{\partial f}{\partial \bar{\boldsymbol{\sigma}}}$

the particular case of Von Mises yield criterion for isotropic material is given by equation:

$$F = \sqrt{3} \; J_2^{\frac{1}{2}} - Y(\kappa) = 0$$

where J_2 is the second invariant of deviators of stress $\bar{\sigma}$ and $Y(\kappa)$ is the yield stress from uniaxial tests. And

$$\frac{\partial F}{\partial \boldsymbol{\sigma}} = \mathbf{a}^T = \sqrt{3} \frac{\partial J_2^{\frac{1}{2}}}{\partial \bar{\boldsymbol{\sigma}}} \frac{\partial \bar{\boldsymbol{\sigma}}}{\partial \boldsymbol{\sigma}} = \sqrt{3}\, \bar{\mathbf{a}}_2^{\,T} \mathbf{T}$$

The general derivations for various yield criteria are given in (Reference 12). And the hardening constant A can be determined from the slope of work hardening curve between Y and plastic work Wp. The plastic work Wp is obtained by integrating increments of plastic work per unit volume

$$dWp = \boldsymbol{\sigma}^T d\boldsymbol{\epsilon}_p \qquad \text{D10}$$

In Eulerian form one can again divide the strain rate given by Equation A13 into elastic and plastic parts:

$$d\bar{\mathbf{e}} = d\bar{\mathbf{e}}_e + d\bar{\mathbf{e}}_p \qquad \text{D11}$$

Now we shall use Equation D7 for yield criterion and restrict ourselves to isotropic materials to derive

$$d\bar{\boldsymbol{\sigma}}_J = \bar{\mathbf{D}}_T \, d\bar{\mathbf{e}} \qquad \text{D12}$$

and noting that relation with elastic component is

$$d\bar{\boldsymbol{\sigma}}_J = \bar{\mathbf{D}} \, d\bar{\mathbf{e}}_e \qquad \text{D13}$$

The essential difference with Lagrangian from Equations D5 and D6 is that we have to use Jaumann stress increments given by Equation A27 so that in the spatial form the constitutive Equations D12 and D13 are unaltered when the continuum is subjected to superimposed rigid body motions. After differentiating Equation D7 we have have to consider

$$\frac{\partial f}{\partial \bar{\boldsymbol{\sigma}}} \, d\bar{\boldsymbol{\sigma}}_J = \bar{\mathbf{a}}^T \, d\bar{\boldsymbol{\sigma}}_J = A d\lambda \tag{D14}$$

where A is again undetermined constant depending upon hardening parameter. Writing the normality rule

$$d\bar{\mathbf{e}}_p = d\lambda \bar{\mathbf{a}} \tag{D15}$$

we can again derive

$$d\lambda = (A + \bar{\beta})^{-1} (\bar{\mathbf{d}}^T \, d\bar{\mathbf{e}}) \tag{D16}$$

with $\bar{\mathbf{d}} = \bar{\mathbf{D}} \, \bar{\mathbf{a}}$ and $\bar{\beta} = \bar{\mathbf{a}}^T \, \bar{\mathbf{D}} \, \bar{\mathbf{a}}$

and

$$\bar{\mathbf{D}}_T = (\bar{\mathbf{D}} - (A + \bar{\beta})^{-1} \, \bar{\mathbf{d}} \, \bar{\mathbf{d}}^T) \tag{D17}$$

It may be noted that in arriving at D17 the transformation D8 used in Lagrangian form is now no longer necessary but instead we have to use Equations D12 and A27.

Lee (Reference 27) considers the case when both elastic and plastic strains are finite and so that Equation D1 is now no longer valid. In order to separate reversible and irreversible deformation we have to consider the problem of three configurations, initial, final and intermediate originally recognized by Sedov (1962). Let us denote the rectangular coordinates $(\bar{x}_p, \bar{y}_p, \bar{z}_p)$ of intermediate configuration corresponding to irreversible part. Then the deformation matrix.

$$\mathbf{J} = \left[\frac{\bar{x}, \bar{y}, \bar{z}}{x, y, z}\right] = \left[\frac{\bar{x}, \bar{y}, \bar{z}}{\bar{x}_p, \bar{y}_p, \bar{z}_p}\right] \left[\frac{\bar{x}_p, \bar{y}_p, \bar{z}_p}{x, y, z}\right]$$

$$= \mathbf{J}_e \, \mathbf{J}_p \tag{D18}$$

replaces Equation D1(a). Also, it will be wrong to consider Equations D1(b) and D11 as valid until one considers the elastic strain components infinitesimal i.e. $\mathbf{J}_e \triangleq \mathbf{I}$. The modification thus required for large strain cases is given in Reference 27 and will not be discussed any further.