

IMPROVEMENTS OF FINITE ELEMENT SOLUTIONS FOR  
STRUCTURAL AND NON STRUCTURAL APPLICATIONS

G. SANDER\* and P. BECKERS\*\*

Laboratoire de Techniques Aeronautiques et Spatiales

University of Liege, Belgium

The paper presents a new formulation which allows the derivation of finite element properties in terms of stress functions. It is applicable to both displacement and equilibrium models and leads for certain classes of problems to a substantial reduction in the number of unknowns. The choice of the stress functions as unknowns is identified with an automatic selection procedure of redundancies of minimal diffusion. Therefore this method can be interpreted as a force method in which the numerical search for the redundancies is avoided. It is shown to apply particularly well in certain non structural problems. Independently the effect of the introduction of internal degrees of freedom (bubble functions) in finite elements is evaluated. The conclusion is that a few of such modes can improve significantly the finite element properties and bring them practically in coincidence with assumed stress models.

\* Associate Professor

\*\* Assistant

## 1. INTRODUCTION.

The finite element method provides a powerful tool for structural analysis. Many problems of linear and non linear elasticity have been successfully solved and applications to other domains such as fluid mechanics and heat conduction have been proposed (Ref. 14). These analyses imply the solution of large systems of linear algebraic equations. If, in aerospace applications, the cost of these solutions is not often a limiting factor, especially when compared to the cost of preparing input data and presenting the output, the size of the problems that can be handled is still limited by the presently available computers and programmes. This limitation is due not only to the core storage size but also to the danger of round-off error propagation. Therefore a need exists for improved finite elements as well as improved algorithms of solution to minimize the size of the systems of equations to handle.

Most of the applications of the finite element method have been conducted using either conforming or non conforming finite elements while the algorithm of solution uses displacements as unknowns. Various attempts have been made to use other formulations. The force method which was proposed since the early days of the finite element method has always been handicapped by the need of a numerical search for the redundancies. The equilibrium elements, although their assumptions seem more appealing to the structural analyst, have received only few practical applications due to the higher number of unknowns they involve (Ref. 1,3,58). Therefore the possibilities offered by the dual analysis of a given problem, that is comparing displacement and equilibrium solutions, has remained unapplied.

The present paper proposes an alternative formulation of the finite element properties which uses a different set of variables : contour deformations can be used for displacement models and stress function modes for equilibrium models. In both cases the natural unknowns are stress function modes. Furthermore the choice of these unknowns can be interpreted as an automatic selection procedure of redundancies of minimal diffusion in the structure (Ref. 11). It follows that, when using these variables, the numerical search for the redundancies in the force method is avoided and the algorithm of solution becomes identical to that of the displacement method. The duality between the two sets of variables in both displacement and equilibrium models and the two solution methods becomes more complete.

A simple criterion is proposed to decide which formulation is the most economic. The problem of introducing the boundary conditions is examined for structural as well as for non structural applications.

Independently the effect of the introduction of internal degrees of freedom by bubble functions in the finite elements (Ref. 3,5) is evaluated. The conclusion is that a few of such modes can improve significantly the finite element properties and bring them practically in coincidence with assumed stress models.

## 2. FORMULATION FOR CONFORMING DISPLACEMENT MODELS

They are derived by application of the minimum total energy principle

$$\delta(U + P) = \delta\left(\frac{1}{2} \int_{\text{vol}} \epsilon' H \epsilon \, d\text{vol} - \int_{\Gamma_\sigma} \bar{t}' u \, d\Gamma_\sigma\right) = 0 \quad (1)$$

where  $U$  is the strain energy

$P$  is the potential energy (in which the body forces are not considered here)

$\epsilon$  is the column of the strain components

$H$  is a symmetric matrix of elastic coefficients

$t$  is the column of surface tractions on  $\Gamma_\sigma$

$u$  is the displacement vector

$\Gamma_\sigma$  is the part of the boundary where surface tractions are prescribed.

The prime denotes transposition and the bar prescribed values.

The principle may be applied only if the displacement field satisfies a priori the compatibility conditions :

$$\epsilon_{ij} = D_j u_i \quad \text{or in matrix form, } \epsilon = \partial u \quad \text{in the volume}$$

$$u = \bar{u} \quad \text{on the part } \Gamma_u \text{ of the boundary where the displacements are prescribed.}$$

The displacement field in a finite element is discretized in the form

$$u = M a \quad (2)$$

where  $M$  is a matrix of assumed modes and  $a$  the column of their  $n_a$  (unknown) intensities. The column of the strain tensor components is written

$$\epsilon = \partial M a \quad (3)$$

and the strain energy becomes

$$U = \frac{1}{2} \int_{\text{vol}} a' \partial M' H \partial M a \, d\text{vol} = \frac{1}{2} a' I a \quad (4)$$

When using the minimum total energy principle, the stresses are only defined by

$$\sigma = \frac{\partial U}{\partial \epsilon} \quad (5)$$

With the discretization adopted, the stress field is described by a set of parameters  $b$

$$b = \frac{\partial U}{\partial a} = I a = \int_{vol} \partial M' \sigma \, dvol \quad (6)$$

which are weighted averages of the stresses and yield therefore only a "weak" knowledge of the stress state in the element. From (6) the matrix  $I$  is clearly a generalized form of the stress-strain relations. A detailed description of the stresses is often derived from (3)

$$\sigma = H \partial M a \quad (7)$$

This distribution is however only a particular example of the stress fields satisfying (6) and in interpreting the stress output, it is worth to remember that any other stress field satisfying (6) is equally consistent. This means that stress distributions can be superimposed to (7) provided they yield a zero value to the generalized stresses (6).

The compatibility conditions in the volume are automatically satisfied by the assumptions (2). To insure conformity along the interfaces, a number  $n_c$  of interface displacement modes of intensities  $q_c$  has to be defined in such a way that the displacements along the element boundaries are uniquely defined. As the displacement field has to be unique at a vertex common to different elements,  $q_c$  always contains the local values of the displacement components at the vertices. In addition other local values, derivatives or integrals of displacements can be used as boundary modes along the interfaces. The expression of the boundary modes in terms of the parameters  $a_i$  is denoted

$$q_c = C a \quad (8)$$

where the matrix  $C$  is called the kinematic matrix of the element. It is not necessarily a square matrix and hence not necessarily invertible.

In the absence of body forces, the virtual work (V.W.) consist only in the work done by the prescribed surface tractions.

$$V.W. = \int_{\Gamma_\sigma} u' \bar{t} \, d\Gamma_\sigma = a' b = a' \left[ \int_{\Gamma_\sigma} M' \bar{t} \, d\Gamma_\sigma \right] \quad (9)$$

or alternatively

$$V.W. = q'_c g'_c = a' C' g_c \quad (10)$$

where  $g_c$  is the set of generalized loads conjugate to the interface modes. Equation (9) yields another expression for the generalized stresses  $b$  in terms



of surface tractions, while comparison of (9) and (10) shows that

$$b = C' g_c \quad (11)$$

The transpose of the kinematic matrix is a static matrix relating the generalized stress parameters to the applied loads.

The problem of the derivation of the finite element can now be presented as that of finding the minimum of the functional :

$$\left(\frac{1}{2} a' I a - q'_c g_c\right) \min \quad (12)$$

subject to the constraints (8). Three cases have now to be distinguished :

a)  $\underline{n_a = n_c}$

The number of parameters of the displacement field equals the number of interface connection modes. The matrix C is square, non singular. This allows to solve for the constraints. Inverting (3) and substituting into (12) yields

$$\left(\frac{1}{2} q'_c C^{-1} I C^{-1} q_c - q'_c g_c\right) \min$$

which furnishes the familiar stiffness relation,

$$g_c = K_c q_c = C^{-1} I C^{-1} q_c \quad (13)$$

between the interface modes and the corresponding generalized forces. The latter can now be interpreted, using (9) and (11)

$$g_c = C^{-1} \int_{\Gamma_\sigma} M' \bar{t} d\Gamma_\sigma \quad (14)$$

They are weighted averages of the prescribed surface tractions, giving again a "weak" knowledge of the stress state on  $\Gamma_\sigma$

b)  $\underline{n_a > n_c}$

In this case the homogeneous system

$$C a = 0 \quad (15)$$

has  $n_i = n_a - n_c$  non trivial solutions. Let  $s_i$  denote such solutions and group them in a matrix S.

$$S = (s_1, s_2, s_3 \dots s_{n_i})$$

The non zero solutions of (15) can be written

$$a = Sw$$

where  $w$  is any arbitrary vector. Such modes of the parameters  $a_i$  exist with zero displacement prescribed along the interfaces. For this reason they are sometimes called "bubble" modes of displacement. The constraints (3) can now be solved in the form

$$a = T q_c + S a_x \quad (16)$$

where  $Tq_c$  is a particular solution and  $Sa_x$  the complementary solution. In general the matrices  $T$  and  $S$  could be obtained numerically, by a Gauss Jordan elimination technique for instance. However this operation can always be avoided by one of the following methods.

Add to (8) a set of  $n_i$  internal values of the displacements  $q_i$  and form an enlarged connection matrix

$$q = \begin{vmatrix} q_c \\ q_i \end{vmatrix} = \begin{vmatrix} C \\ C_i \end{vmatrix} a = C^* a \quad (17)$$

$C^*$  is now non-singular and the general procedure described above can be applied. The displacements  $q_i$  can finally be eliminated by the standard condensation process.

Another simpler possibility arises when the displacement field can be a priori split in two parts, the first controlling the interface modes and the second ~~being~~ the "bubble" modes

$$u = M a = M_c a_c + M_b a_b \quad (18)$$

$$\text{In this case } q_c = C_c a_c \quad (19)$$

is invertible and the elimination of the modes  $a_b$  requires simply a condensation of the matrix  $I$ . The stiffness matrix turns out to be

$$K_c = C_c^{-1} [ I_{cc} - I_{cb} I_{bb}^{-1} I_{bc} ] C_c^{-1} \quad (20)$$

if  $I_{cc}$ ,  $I_{bb}$ ,  $I_{bc}$  are the submatrices of  $I$  corresponding to  $a_c$  and  $a_b$ . This latter procedure represents a significant saving of numerical work compared to the first one.

c)  $\underline{n_a} < n_c$

The constraints (3) cannot be inverted and this case was ruled out of previous formulations (Ref. 1). It is however possible to derive a finite element in this case.

Considering the static relation (11) the number  $n_b$  of generalized stress parameters  $b$  is smaller than the number  $n_g$  of generalized interface loads  $g_c$ .

This means that  $n_k = n_g - n_b$  non-zero solutions exist to the problem

$$C' g_c = 0 \quad (21)$$

These solutions correspond to combinations of loads applied along the interfaces which do not generate stresses in the element. By analogy with the terminology adapted for equilibrium models these modes might be called "spurious static modes".

To derive the finite element properties, let us split the constraints in two parts

$$\begin{aligned} q_c^* &= C^* a \\ q_c &= C_r a \end{aligned} \quad (22)$$

where  $q_c^*$  is a subset of  $q_c$  composed of  $n_a$  components arbitrarily selected while  $q_r$  forms the complementary subset of  $q_c^*$ . The matrix  $C^*$  is non-singular and (22) yields

$$a = C^{*-1} q_c^*$$

Substitution in the expression of the minimum principle yields

$$\left( \frac{1}{2} q_c^{*'} C^{*-1'} I C^{*-1} q_c^* - q_c^{*'} g_c^* - q_r' g_r \right) \min \quad (23)$$

with the remaining constraints

$$q_r = C_r C^{*-1} q_c^* \quad (24)$$

Introducing these remaining constraints in (23) affected by a column of lagrangian multiplier  $\lambda$ , it turns out that

$$\begin{aligned} K_c^* q_c^* &= g_c^* + C^{*-1'} C_r \lambda \quad \text{with} \quad K_c^* = C^{*-1'} I C^{*-1} \\ \partial_r &= \lambda \\ q_r &= C_r C^{*-1} q_c^* \end{aligned} \quad (25)$$



This procedure yields a stiffness matrix  $K_c^*$  which can be used if the satisfaction of the remaining constraints is achieved at the structural level. It appears often in practice (Ref. 2,3) that the remaining constraints can be eliminated in a group of finite elements. This allows the derivation of super-elements free of such constraints. This procedure justifies a posteriori and generalizes the derivation of certain previously developed elements like the conforming quadrilateral plate bending elements (Ref. 4,5) or its triangular version (Ref. 2,6). In fact a complete family of such elements exists up to any degree (Ref. 2).

The distinction between the three cases described above is based on the comparison between the numbers  $n_a$  and  $n_c$ . However it is possible that bubble modes and spurious static modes exist simultaneously in a model. Therefore it is more correct to base the distinction on the number of solutions of the homogeneous problems (15) and (21) and, if such solutions exist, apply the corresponding treatment. In general, a model will be free of remaining constraints if

$$n_a - n_i = n_c$$

There is another possible formulation for displacement models which although it appears less physical, can be useful in certain circumstances and allows a better understanding of the next section.

Consider the description of the strain field in (3). It does not depend upon the rigid body modes of the displacement field and therefore it can be described in terms of a reduced set  $a^*$  of parameters

$$\epsilon = \overline{\partial M} a^*$$

The relation between the two sets of parameters can be written in general

$$a^* = D a \quad (26)$$

but in some cases  $a^*$  is simply a subset of  $a$ . The strain energy is now written

$$U = \frac{1}{2} a^{*'} \left[ \int_{\text{vol}} \partial \bar{M}' \text{ if } \partial \bar{M} \text{ dvol} \right] a^{*} = \frac{1}{2} a^{*'} I^{*} a^{*} \quad (27)$$

where  $I^{*}$  is now non-singular.

Using (26) the relation with (4) is

$$I = D' I^{*} D$$

showing that  $I^{*}$  is a non singular kernel of  $I$ .

Instead of expressing the continuity of the displacements between elements in terms of displacement modes, one could adopt a system of contour deformation modes providing the same continuity. Let us denote by  $d_c$  the set of these contour deformation modes. They can be expressed in terms of the parameters of the deformation by

$$d_c = Q a^{*} \quad (28)$$

where  $Q$  is another kinematic matrix. The properties of the finite element are obtained by minimization of

$$\left( \frac{1}{2} a^{*'} I^{*} a^{*} - d_c' f_c \right) \min \quad (29)$$

subject to the constraints (23) and where  $f_c$  denotes the generalized contour forces conjugate of the contour deformation modes  $d_c$ .

Substituting (28) in (29) and using the fact that  $I$  is non singular, we obtain

$$\begin{aligned} I^{*} a^{*} &= a' f \\ d_c &= Q a^{*} = Q I^{*-1} Q' f = F f \end{aligned} \quad (30)$$

where  $F$  is now a flexibility matrix.

Apparently the discussion of the properties of the kinematic matrix  $Q$  is unnecessary in this formulation where we obtain in any case a flexibility matrix. However the difficulty is only hidden as will appear in the next section, where the same problem arises in the derivation of equilibrium elements.

Finally the following relations between the two formulations are easily obtained

$$d_c = Q a^* = Q D a = a D C^{-1} q_c = C^* q_c$$

where  $C^*$  is a square matrix relating the contour deformation modes to the interface displacement modes.

By virtual work we also have

$$V.W. = g'_c q_c = f'_c d_c = f'_c C^* q_c$$

$$g_c = C^{*'} q_c$$

### 3. FORMULATION FOR STRESS-DIFFUSING EQUILIBRIUM MODELS

Stress diffusing equilibrium models are derived by application of the minimum complementary energy principle

$$\delta(\psi + Q) = \delta\left(\frac{1}{2} \int_{vol} \sigma' H^{-1} \sigma \, dvol - \int_{\Gamma_u} t' \bar{u} \, d\Gamma_u\right) = 0$$

where  $\psi$  and  $Q$  denotes respectively the complementary stress energy and the potential energy, while  $\sigma$  is a column of the stress tensor components. The application of this principle implies the satisfaction of the equilibrium equations

$$D_j \sigma_{ij} + X_i = 0 \quad \text{in the volume}$$

$$t_i = l_j \sigma_{ji} = \bar{t}_i \quad \text{on } \Gamma_\sigma$$

( $l_j$  are the direction cosines of the outward normal).

In the following the case without body forces only will be considered for simplicity.

The part of the stress field satisfying the homogeneous equilibrium equations can be discretized by two different procedures :

- a) Each stress component is represented by a linear combination of assumed modes  $P_i(x_m)$  of intensity  $b_i$

$$\sigma = P b \quad (31)$$

These modes must be adjusted in such a way that the equilibrium equations in the volume are satisfied.

- b) The other procedure is relatively new and reveals extremely fruitful for the development of finite elements as well as it provides a new method of solution of the global problem.

The stress tensor can be derived from a set of stress functions

$$\sigma_{ij} = \epsilon_{imr} \epsilon_{jns} \phi_{rs, mn} \quad (ij \text{ not summed}) \quad (32)$$

where, in the three dimensional case  $\phi_{rs}$  is a symmetric tensor of stress functions, while  $\epsilon_{ijk}$  is the alternator symbol. Note that for membrane problems  $\phi_{rs}$  reduces to the Airy stress function and for plate bending problems to two stress functions (Ref. 7).

These stress functions can be approximated as a linear combination of modes  $M_i(rs)(x_m)$  of intensity  $a_i$

$$\phi_{rs} = \sum_k M_k(rs)(x_m) a_k$$

or, in matrix form

$$f = M a \quad (33)$$

The stress field derived according to (30) automatically satisfies the homogeneous equilibrium equations

$$\sigma = \partial^2 M a$$

The number of parameters  $a_i$  is always larger than that of the  $b_i$  by the number of integration constants of the system (30) which are the "rigid body modes" of the stress functions. In some cases  $b$  can be simply a subset of  $a$ , but in general the relation between the two sets is denoted

$$b = D a \quad (34)$$

The complementary strain energy  $\psi$  can be alternatively written

$$\psi = \frac{1}{2} \int_{vol} b' P' H^{-1} P b \, dvol = \frac{1}{2} b' J^* b \quad (35)$$

$$\psi = \frac{1}{2} \int_{vol} a' \partial^2 M' H^{-1} \partial^2 M a \, dvol = \frac{1}{2} a' J a$$

where  $J^*$  and  $J$  are two flexibility matrices.

The generalized strains can be defined by

$$\pi_b = \frac{\partial \psi}{\partial b} = J^* b = \int_{\text{vol}} P \epsilon' \, d\text{vol} \quad (36)$$

$$\pi_a = \frac{\partial \psi}{\partial a} = J a = \int_{\text{vol}} \partial^2 M \epsilon' \, d\text{vol} \quad (37)$$

They are weighted averages of strains and this corresponds only to a "weak" knowledge of the strain field. The parameters  $b_i$  are independent and each of them represents a non-zero contribution to the strain energy. The matrix  $J^*$  is therefore non-singular. As the parameters  $a_i$  contains the "rigid body modes" of the stress functions which do not contribute to the strain energy, the matrix  $J$  is singular. Using (33) the relation between the two is

$$J = D' J^* D \quad (38)$$

which shows that  $J^*$  is a non-singular kernel of  $J$ . In view of (36) and (37) both can be interpreted as generalized strain-stress relations.

Let us turn next to the equilibrium requirements along the boundary. They concern only the interfaces as we have ruled out the body forces. Using the first description of the stress field (29), the surface tractions can be expressed by

$$t = L \sigma = L P b \quad (39)$$

where  $L$  is a matrix of the direction cosines  $l_j$ . The surface traction modes described by  $t$  have to be uniquely determined by a set of  $n_c$  forces denoted  $g_c$ . These can be local values or derivatives, integrals or any combinations of them provided that along each edge the definition is unique.

The expression of these generalized forces in terms of the stress fields parameters is written

$$g_c = C b \quad (40)$$

where  $C$  is called the static matrix of the element. It is in general a rectangular matrix, but even if it is square, the relation (40) cannot be inverted since the generalized forces are a priori not independent since they satisfy the global equilibrium equations. Note that in contrast to the formulation of the displacement elements, the stress tensor has no unique value at a vertex. Therefore in pure equilibrium models, there are no variables associated with a vertex and the forces  $g_c$  are essentially interface variables.



The virtual work consists only in the work done by the prescribed displacements and can be written alternatively

$$V.W. = \int_{\Gamma_u} u' t d\Gamma_u = \int_{\Gamma_u} u' L P b d\Gamma_u = \pi'_b b \quad (41)$$

$$V.W. = q'_c g_c = q'_c C b \quad (42)$$

where  $q_c$  are the generalized (average) displacements conjugate to  $g_c$ . Comparison of the two expressions yields

$$\pi'_b = C' q_c \quad (43)$$

The transpose static matrix turns out to be a kinematic matrix relating the generalized (average) strains to the generalized interface displacements.

The problem of the derivation of the finite element reduces now to that of finding the minimum of the functional

$$\left( \frac{1}{2} b' J^* b - g'_c q_c \right) \min \quad (44)$$

subject to the constraints (40). It yields

$$J^* b = C' q_c \quad (45)$$

and as  $J^*$  is non singular

$$g_c = C b = C J^{*-1} C' q_c = K_c q_c \quad (46)$$

which is the familiar form of the stiffness matrix for an equilibrium element. This derivation of the stiffness matrix for equilibrium models in terms of generalized forces appears formally identical with the derivation of the flexibility matrix for conforming displacement elements in terms of contour deformations. In this former case it was apparently unnecessary to examine the properties of the kinematic matrix  $Q$  to obtain a flexibility matrix. Here also a stiffness matrix is obtained whatever the properties of the static matrix  $C$  are. The difficulty is however only hidden and the distinction between these cases is necessary if one wishes to use in practice the relations (46) or by analogy (30).

Consider first the static matrix defined in (40). The homogeneous system can have non trivial solutions representing internal stress modes which exist with zero prescribed surface tractions on the interfaces. These  $n_i$  modes are "bubble" stress modes of the element. It is however useless to define here internal generalized forces to determine these modes in a similar way to the internal displacements defined in (17). In fact the solution of the constraints (40) is not necessary and the bubble stress modes do not contribute to the complementary potential energy. Therefore the reduction of these modes, whatever they are, is automatically included in the expression (46).

Consider next the kinematic matrix defined in (43). The homogeneous system has always non-trivial solutions or, more physically, displacement modes of the interfaces which do not produce generalized strains. Among these are the rigid body modes of elements. If all these solutions in number  $n_k$  are grouped in a matrix  $Z$ , the solutions of the homogeneous system (43) can be expressed by

$$q_c = Z w \quad (47)$$

where  $w$  is any arbitrary vector of dimension  $n_k$ . The virtual work of such modes is zero

$$V.W. = g'_c q_c = g'_c Z w = 0 \quad (48)$$

As  $w$  is arbitrary, it implies  $n_k$  relations to be satisfied by the generalized forces

$$Z' g_c = 0 \quad (49)$$

If  $n_k$  equals the number  $n_r$  of rigid body modes these relations are simply the global equilibrium equations. If  $n_k$  is larger than  $n_r$  the stiffness matrix defined in (46) cannot be used unless the  $n_k - n_r$  remaining constraints are satisfied. The displacement modes corresponding to these remaining constraints have been called spurious kinematic modes (ref. 8).

Here again it appears often that such constraints can be eliminated in a group of elements, leading to a super-element free of such constraints. The procedure has been applied implicitly in the derivation of a certain number of equilibrium membrane elements (Ref. 2,5). The advantage of the present formulation is to show that the procedure is general and that the problem is strictly the same as in displacement elements. Finally it appears that the condition to fulfil to avoid remaining constraints (or spurious kinematic modes) is that

$$n_b - n_i = n_c - n_r \quad (50)$$

The strict similarity between the derivation of displacement and equilibrium elements becomes even more transparent if the second formulation, in terms of stress functions, is used. In that case the requirements of surface traction continuity has to be translated into requirements of stress function continuity. They certainly involve the continuity of the stress function local values but in addition, they eventually involve the continuity of some of its derivatives. Those requirements are especially simple to formulate in the case of membrane and plate bending problems if the Southwell slab ana-

logies are recalled (Ref. 7) and specific examples are given in the following. In general it requires to insure the single valuedness of  $m_c$  local value of the stress functions and its derivatives denoted  $f_c$ . These  $m_c$  stress function modes are equivalent to the  $n_c$  forces  $g_c$ . They can be expressed in terms of the intensities of the assumed stress function modes  $a_i$  by

$$f_c = T a \quad (51)$$

The matrix  $T$  is another static matrix of the element. Note that the  $m_c$  interface connection modes necessarily define the "rigid body" modes of the stress functions and that their number  $m_c$  is equal to  $n_c$ . In contrast with the first formulation, the generalized stress function modes  $f_c$  are not necessarily interface variables only, as the continuity often requires a single value of the stress function at a vertex.

The virtual work of the surface tractions can be alternatively written

$$V.W. = \int_{\Gamma_u} u' t d\Gamma_u = \int_{\Gamma_u} u' \partial^2 M a d\Gamma_u = \pi'_a a \quad (52)$$

$$V.W. = f'_c d_c = a' T' d_c$$

where  $d_c$  are generalized displacements conjugate of  $f_c$ . Comparison of the two expressions reveals that

$$\pi_a = T' d_c \quad (53)$$

which again indicates that the transpose static matrix is a kinematic matrix relating the strains  $\pi_a$  to the displacements  $d_c$ .

The properties of the finite element are now obtained by finding the minimum of

$$\left(\frac{1}{2} a' J a - f'_c d_c\right) \min \quad (54)$$

subject to the constraints (51). As  $J$  is singular, the procedure applied in the first formulation does not apply and the inversion of the relation (51) is necessary. Comparison of (54) and (12) shows that the situation is formally identical to that of deriving a displacement element if  $f_c$

plays the role of  $q_c$  and  $d_c$  that of  $g_c$ . The solution of the constraints (51) requests again the distinction between three cases.

a)  $\underline{n_a = n_c}$

In this case it turns out that

$$a = T^{-1} f_c$$

and  $d_c = T^{-1} J T^{-1} f_c = F_c f_c$  (55)

where  $F_c$  is the flexibility matrix of the element in terms of the stress function modes.

b)  $\underline{n_a > n_c}$

The homogeneous system

$$T a = 0 \quad (56)$$

has  $n_i$  non-trivial solutions which are internal or "bubble" stress function modes. The procedure for their elimination can be formally identical to that described for the displacement models.

c)  $\underline{n_a < n_c}$

The homogeneous system

$$T' a = 0 \quad (57)$$

has  $n_k$  non-trivial solutions corresponding to spurious kinematic deformation modes. A flexibility matrix can still be derived by selecting arbitrarily a subset of independent values of stress function modes to invert (51). In this case the flexibility matrix cannot be used without the  $n_k$  remaining constraints. Again these constraints can often be eliminated in a group of elements.

Finally the two formulations for the equilibrium elements are related to each other by the following relations

$$g_c = C b = C D T^{-1} f_c = C^* f_c \quad (58)$$

Note that although the matrix  $C^*$  is square it is always singular as the forces  $g_c$  are not independent. By virtual work

$$V.W. = g'_c q_c = f'_c C^{*'} q_c = f'_c d_c \quad (59)$$

it appears

$$d_c = C^{*'} q_c \quad (60)$$



which completes the analogy.

#### 4. DUALITY BETWEEN DISPLACEMENT AND EQUILIBRIUM MODELS

Before discussing the relative advantages of the two formulations, proposed above, their practical meaning in elasticity is illustrated in an example. For this purpose the formulation of the Southwell slab analogies is briefly recalled (Ref. 7).

A first analogy allows to show that the function used to describe the deflection  $w(x,y)$  of a Kirchhoff plate can be used as the Airy stress function  $\phi(x,y)$  of a membrane and that the continuity requirements for this function in order to derive a conforming displacement plate bending element are precisely the same as those required for the stress function in order to derive an equilibrium membrane element. Therefore the derivation of the flexibility matrix for an equilibrium element in the form required by the stress function formulation is identical to that of the stiffness matrix for a conforming plate bending element in the displacement formulation. The deflection has simply to be interpreted as the Airy stress function and the stress-strain relations substituted for the moment-curvature relations. At this time it may be better to replace the word displacement by deformation function in order to make the analogy more transparent. Note that as a by-product the first analogy has recently allowed to define new families of triangular and quadrangular equilibrium membrane elements and at the same time of conforming plate bending elements up to any degree, by forming super-elements composed of triangles (Ref. 2,3). In addition sub-equilibrium and hyper-equilibrium membrane elements are recognized as the analog respectively of non conforming and hyper-conforming plate bending elements.

A second analogy allows to show that the displacement functions  $u, v$  used in conforming membrane elements are precisely the stress functions required to derive equilibrium plate bending elements and that the continuity conditions of these two functions are also identical in the two problems. Therefore the families of conforming membrane elements of triangular and quadrangular shape recognized by many authors (Ref. 12) furnish immediately the flexibility matrices for equilibrium plate bending elements as required by the stress function method.

The two analogies are illustrated in figure 1. The elements (A) and (D) correspond to dual membrane elements, one conforming, the other equilibrium for which the choice of the unknowns leads to a stiffness matrix. (B) and (C) are dual plate bending elements for which the choice of the unknowns also yields a stiffness matrix. Another interpretation of the analogies is possible : (C) and (D) represent the same equilibrium membrane



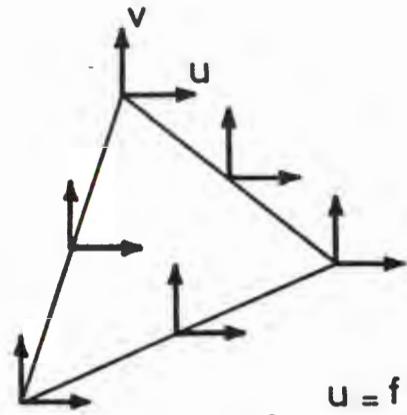
element formulated either in terms of stress function and leading to a flexibility matrix, or in terms of surface tractions and yielding a stiffness matrix. But (C) and (D) also represent the two formulations of the conforming plate bending element either in terms of deformation functions (displacements) or in terms of contour deformations. The same double interpretation holds for (A) and (B).

The interpretation of these contour deformations follows immediately by application of the Southwell analogies : in the case (B) a corner load corresponds to the variation of the angle at a vertex, a bending moment to the extension of the edge, etc (Ref. 7).

Finally the following interpretation of the local values of the stress functions or deformation functions is important for what follows. In a group of equilibrium membrane elements, say of type (C), formulated in terms of stress functions, the Airy stress function is single-valued along the interfaces in the absence of loads. Therefore a local value of the stress function represents a self stressing state (Ref. 11). This is easy to realize if one imagines the stress state produced by one single local value different from zero. By analogy a local value of the deformation function (a local displacement) can be interpreted as a self straining state. By virtual work considerations the variables conjugate of the stress function modes are generalized cuts while those conjugate of the contour deformations are stress functions.

# THE SOUTHWELL ANALOGIES

(A)



$$u = f(x, y)$$

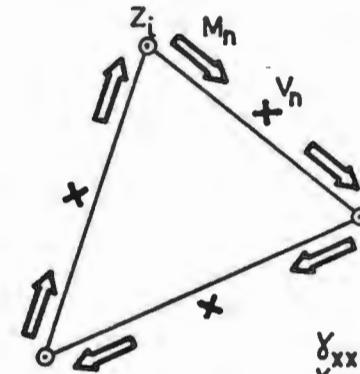
$$v = f(x, y)$$

(B)

$$\gamma_{xx} = u_{,y}$$

$$\gamma_{yy} = v_{,x}$$

$$\gamma_{xy} = -\frac{1}{2}(u_{,y} + v_{,x})$$



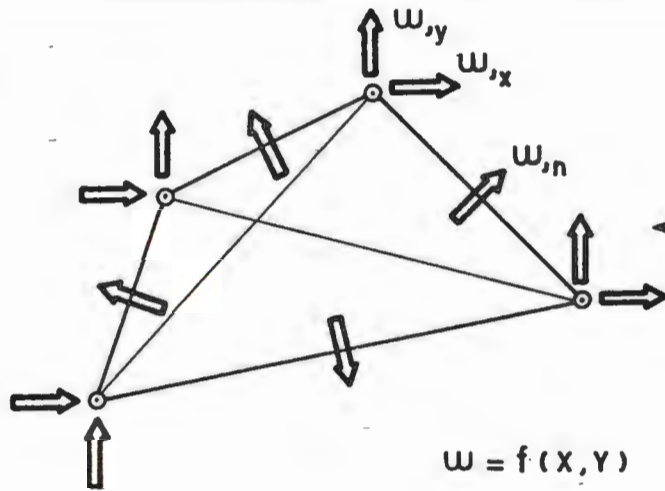
$$\gamma_{xx} = f_1(x, y)$$

$$\gamma_{yy} = f_2(x, y)$$

$$\gamma_{xy} = f_3(x, y)$$

325

(C)



$$w = f(x, y)$$

VARIABLES:

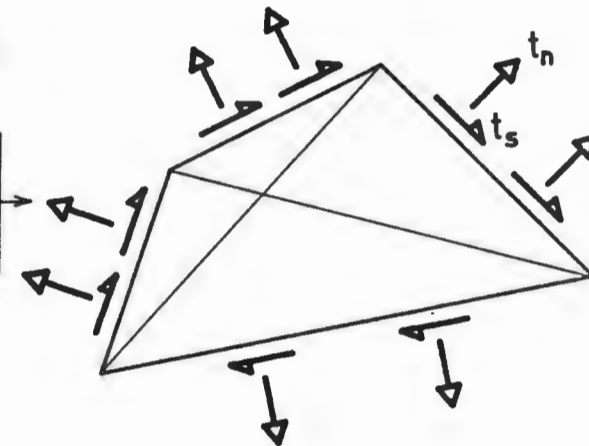
STRESS FUNCTION  
DEFORMATION FUNCTION

(D)

$$t_{xx} = w_{,yy}$$

$$t_{yy} = w_{,xx}$$

$$t_{xy} = -w_{,xy}$$



VARIABLES:

CONTOUR DEFORMATIONS  
SURFACE TRACTIONS

$$t_{xx} = f_1(x, y)$$

$$t_{yy} = f_2(x, y)$$

$$t_{xy} = f_3(x, y)$$

FIGURE 1

## 5. METHODS OF SOLUTION OF THE GLOBAL PROBLEM

Considering the strict analogy between the derivation of displacement and equilibrium models, the methods of solution of the global problem will be examined only for one category of models and the results translated for the other. As the equilibrium models are less well known, we shall concentrate on them in the following.

The complementary energy principle applied to an assemblage of equilibrium finite elements can be written, using the formulation (44) in terms of surface tractions

$$\left( \frac{1}{2} \sum_k b'_k J_k^* b_k - g' q \right) \min \quad (k = \text{element index}) \quad (61)$$

while using the stress function formulation (51)

$$\left( \frac{1}{2} \sum_k f'_k F_k f_k - d' f \right) \min \quad (62)$$

Assembling the elements is achieved by stating that along each interface the sum of the generalized loads equals the applied external loading

$$g = \sum_k L_k g_k = \sum_k L_k C_k b_k \quad (63)$$

where  $L_k$  are localizing or boolean matrices and  $C_k$  the static matrices defined in (40).

In the absence of interface loading, the stress functions are continuous across the interfaces, what is written

$$f_k = \bar{L}_k f \quad (64)$$

where  $\bar{L}_k$  are other localizing matrices. Along the loaded interfaces the relation (55) allows to express that the dislocations or jumps of the stress functions equal the loads

$$g = \sum_k g_k = \sum_k L_k C_k^* f_k \quad (65)$$

The solution of the minimization problem (61) or (62) is obtained by substitution of the constraints (63) or (64). In the first formulation we obtain the classical results

$$\sum_k J_k^* b_k = \sum_k C'_k L'_k q$$

$$g = \sum_k L_k g_k = \sum_k L_k C_k b_k = \left( \sum_k L_k C_k J_k^{*-1} C'_k L'_k \right) q = \sum_k L_k K_k L_k q = K_s q \quad (66)$$

corresponding to the displacement method applied to equilibrium finite elements. The unknowns are effectively interface displacement modes.  $K_s$  is the global stiffness matrix obtained by a suitable localizing of the element stiffnesses.

In the second formulation, we obtain by the same operation, neglecting first the eventually prescribed loads

$$\left( \sum_k \bar{L}'_k F_k \bar{L}_k \right) f = F_s f = d \quad (67)$$

where  $F_s$  is the structural flexibility matrix for the same assemblage of equilibrium elements. The unknowns are here interface stress function modes. As these have been interpreted as self-stressing states, this solution is in fact in the spirit of the force method, but in which the search for the redundancies is avoided. The second member  $d$  represents contour deformations or generalized cuts conjugate to the self stressing states.

The figure 2 translates the results obtained for equilibrium models into the corresponding ones for displacement models. With the help of the analogies described above, the figure is self explanatory. In the two formulations denoted force method it is important to remember that the search for the self stressing states is avoided by the particular choice of the stress functions as variables.

The choice for one method or the other should be based on the number of variables involved in the particular application considered. The same computer program can obviously be used to assemble the four systems of equations and for their solution.

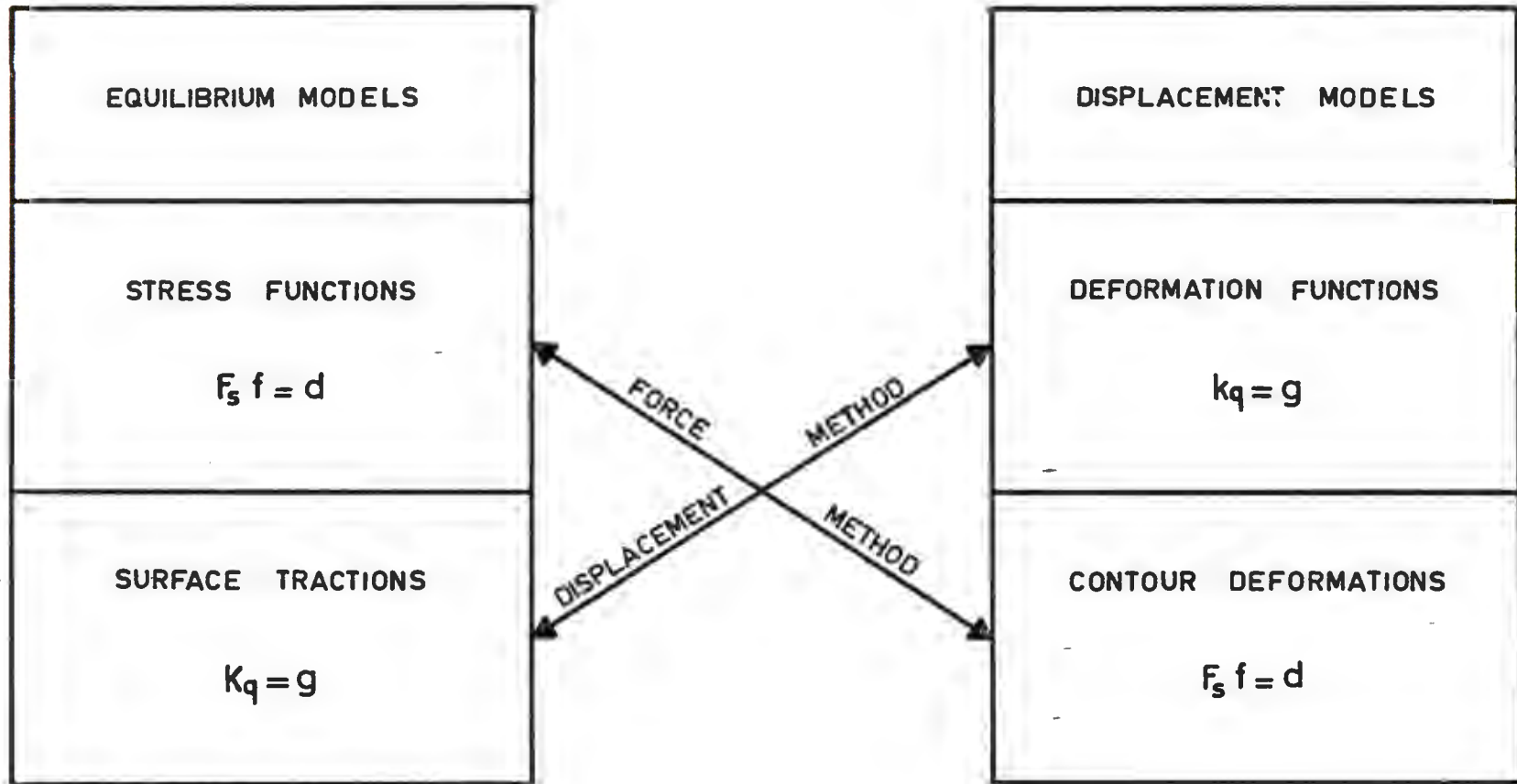


FIGURE 2



## 6. INTRODUCTION OF THE BOUNDARY CONDITIONS

The introduction of the boundary conditions has not been considered in the preceding formulation. These conditions can be prescribed forces or displacements; they can be zero or non-zero prescribed or the average over some region can be prescribed.

In the two forms of displacement methods, prescribed forces appear in the second member and prescribed displacements lead to a reduction of the number of unknowns. In case of prescribed average values of forces or displacement (like for instance a prescribed global shear force in a box beam) the corresponding linear constraints between the variables has to be written and added to the system of equations.

In the two new forms of force method, the boundary conditions have to be expressed in terms of stress functions or of contour deformations. When these conditions involve only the contour of the assemblage of finite elements the change of variables is relatively easy. However if loads are applied inside the structure, dislocations have to be introduced in the stress function field. This can be achieved in the following way.

In the expression (62) for the potential energy, we separate the terms pertaining to unloaded and loaded inter element boundaries and use for the latter their expression in terms of forces  $\bar{g}$  and displacements  $\bar{q}$

$$d' f = d'_u f_u + \bar{g}' \bar{q} \quad (\text{subscript } u \text{ stands for unloaded}) \quad (68)$$

Using then (57) and (63)

$$g = \sum_k L_k C_k^* f_k = \sum_k L_k C_k^* \bar{L}_k f = C_s^* f \quad (69)$$

where  $C_s^*$  is a global matrix translating the stress functional modes in terms of surface traction modes.

Substitution of (68) into (61) yields

$$\begin{vmatrix} F_s & C_s^* \\ C_s^* & 0 \end{vmatrix} \begin{vmatrix} f \\ -\bar{q} \end{vmatrix} = \begin{vmatrix} 0 \\ \bar{g} \end{vmatrix} \quad (70)$$

It appears therefore that the introduction of the boundary conditions corresponds to the addition of constraint equations in number proportional to the number of loaded interfaces. Hence it is obvious that if all the inter-

faces are loaded, the stress function formulation becomes unattractive.

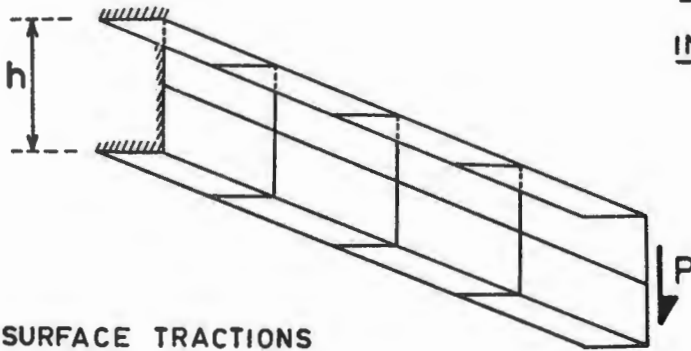
The introduction of the boundary conditions in terms of stress functions is illustrated by two examples in figures 3 to 8 for a cantilever U beam composed of membrane elements and for the bending of a centrally loaded point supported square plate.

Taking first the cantilever U beam, figure 3 represents the boundary conditions as they are used in a classical equilibrium analysis formulated in terms of generalized force and solved by a displacement method. These boundary conditions for the normal and tangential surface tractions  $n$ ,  $t$  are translated into boundary conditions expressed in terms of curvature of the stress function  $F(x,y)$  as illustrated by figure 4. On the same figure the arbitrary fixation of the "rigid body" modes of the stress function is also indicated. Note that  $F_x$  stands for  $\frac{\partial F}{\partial x}$  etc. From these boundary conditions it is easy to derive the set of constraints represented in figure 5. This operation is easily done by hand on the example but could also result from equation 69.

Turning next to the plate bending problem, the boundary conditions in terms of surface tractions are indicated in figure 6 and translated in terms of stress functions in figure 7 where the arbitrary fixation of the "rigid body" modes is also represented. One notes that the introduction of a concentrated load inside the plate requires to impose a dislocation of the stress functions  $u,v$ , along a line arbitrarily selected from the load to an external boundary. The angle of opening due to the dislocation is proportional to the load. The deformed shape of the stress function is indicated by the dashed lines. In figure 8 the detailed expressions of the constraints corresponding to the concentrated load are represented. The upper part of the figure indicates the general conditions to satisfy by the stress functions on both sides of the dislocation. The lower part of the figure illustrates the application in the case where the dislocation is bordered by triangular elements with a linear stress function field. It appears that the final form of the constraints expressed point by point is extremely simple and can be introduced by adding a certain number of relations of the same type between nodal values. This observation allows the systematic treatment of the constraints by the computer program.

# CANTILEVER U BEAM

## BOUNDARY CONDITIONS IN EQUILIBRIUM ANALYSIS



### a) SURFACE TRACTIONS

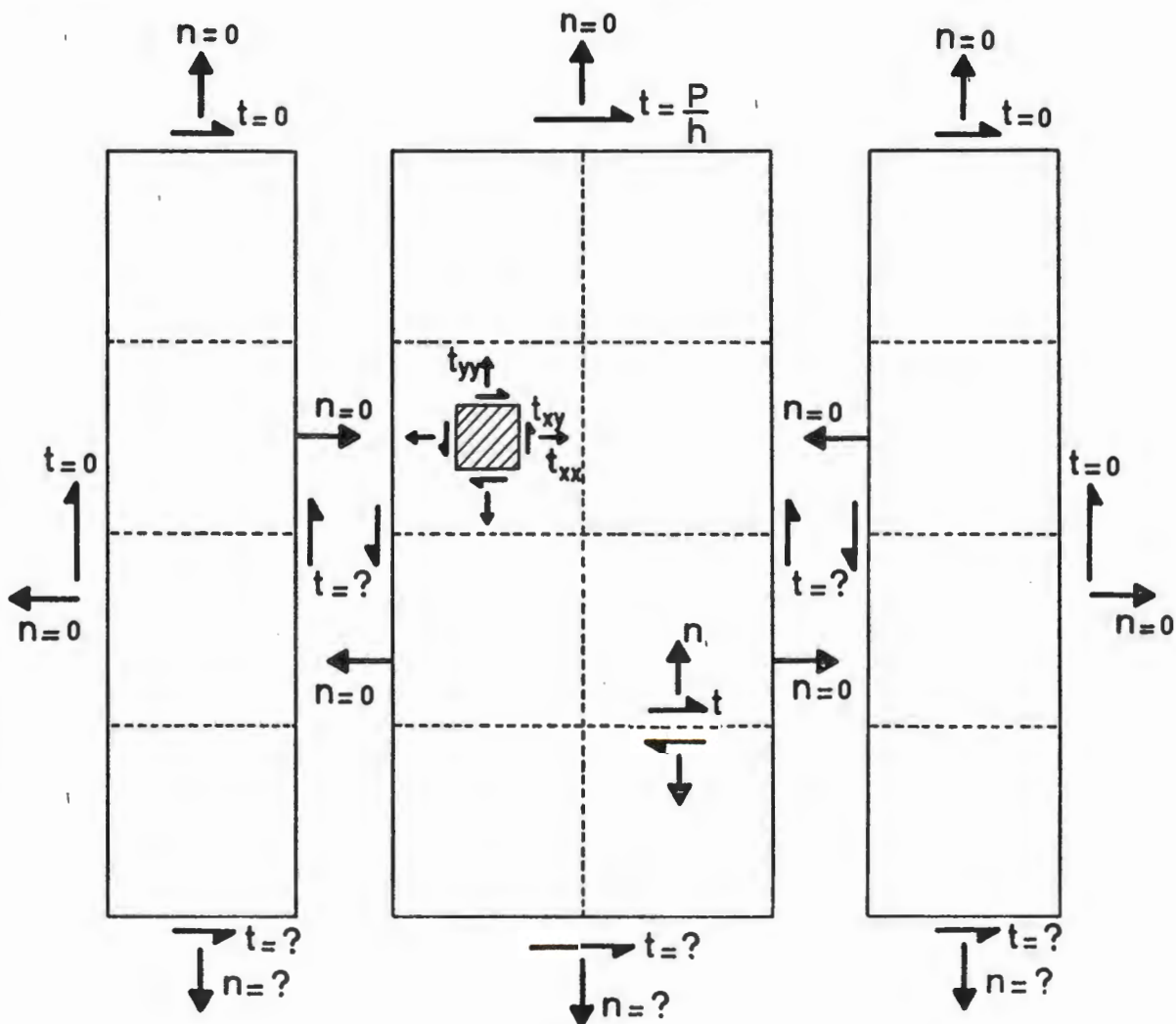


FIGURE 3

b) BOUNDARY CONDITIONS IN TERMS OF STRESS FUNCTION  $F(x,y)$

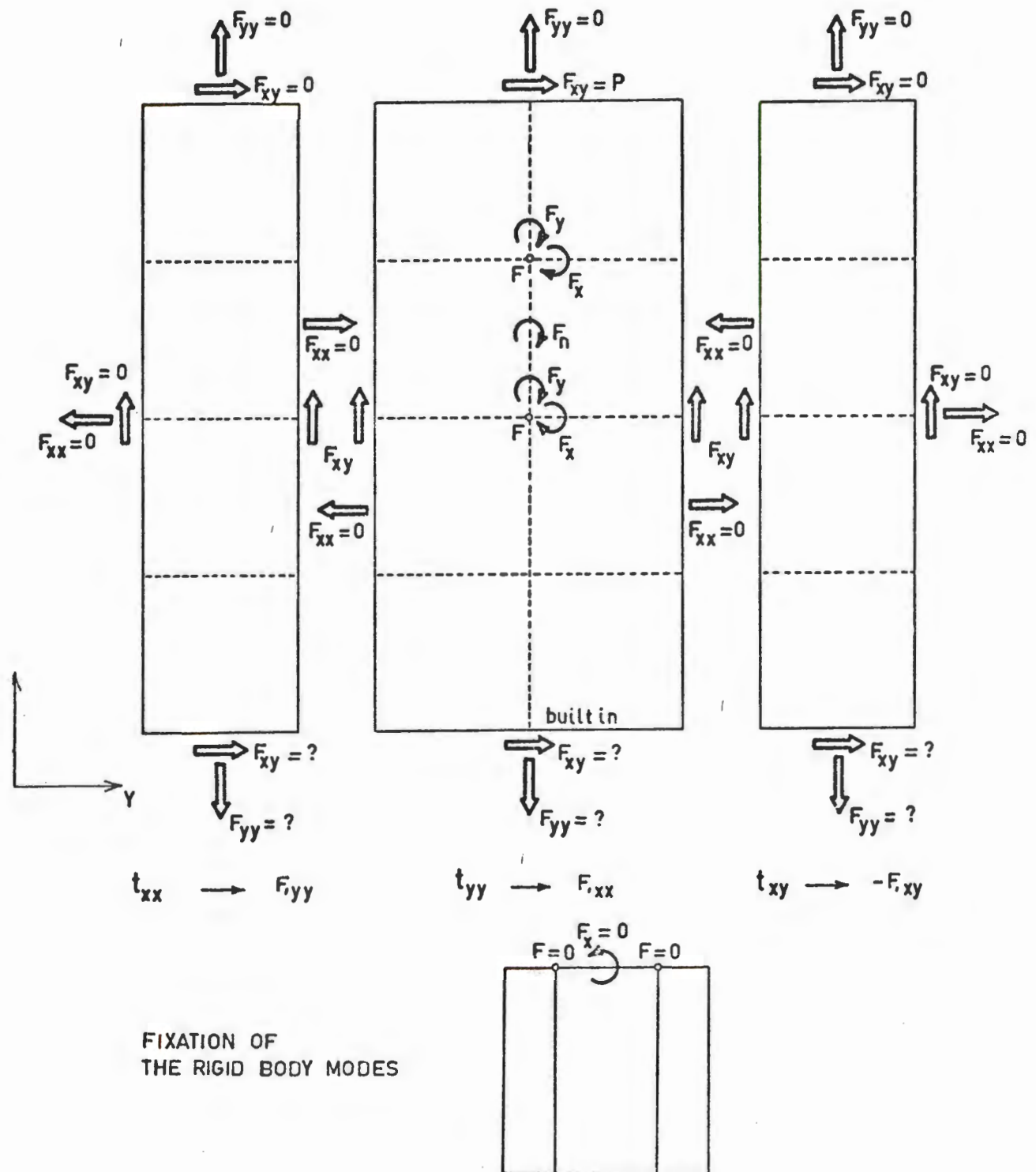


FIGURE 4

c) CONSTRAINTS IMPOSED TO THE STRESS FUNCTION  $F(x,y)$

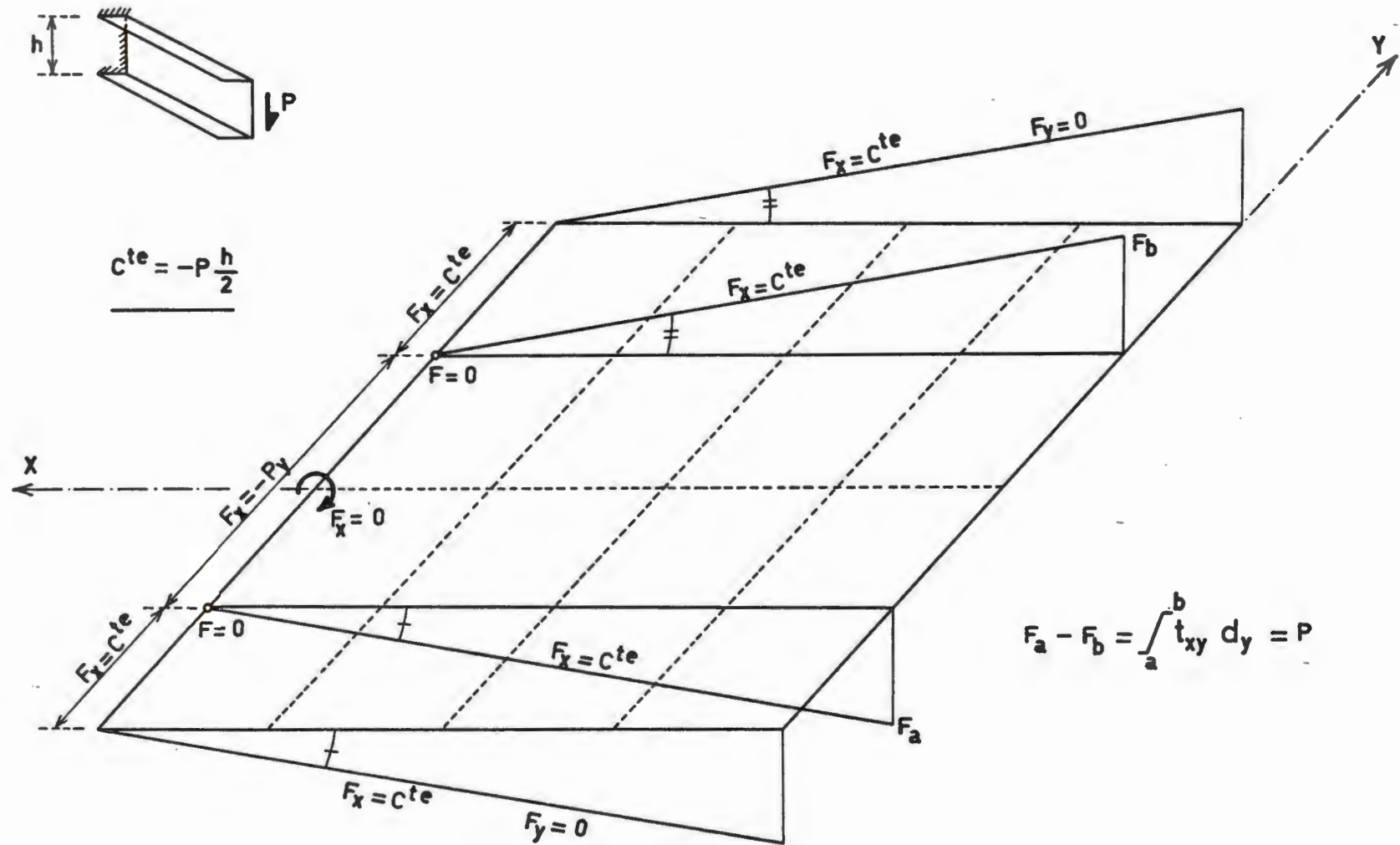
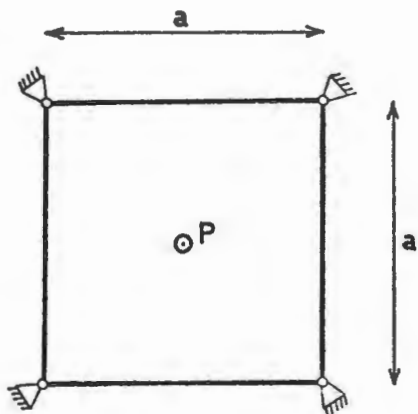


FIGURE 5



# POINT SUPPORTED SQUARE PLATE

## CENTRAL CONCENTRATED LOAD



$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = V_x$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = V_y$$

$$K_n = V_n + \frac{\partial M_{sn}}{\partial s}$$

$$Z_i = M_{sn_{i+\epsilon}} - M_{sn_{i-\epsilon}}$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = -P = 0$$

### a) BOUNDARY CONDITIONS IN TERMS OF SURFACE TRACTIONS

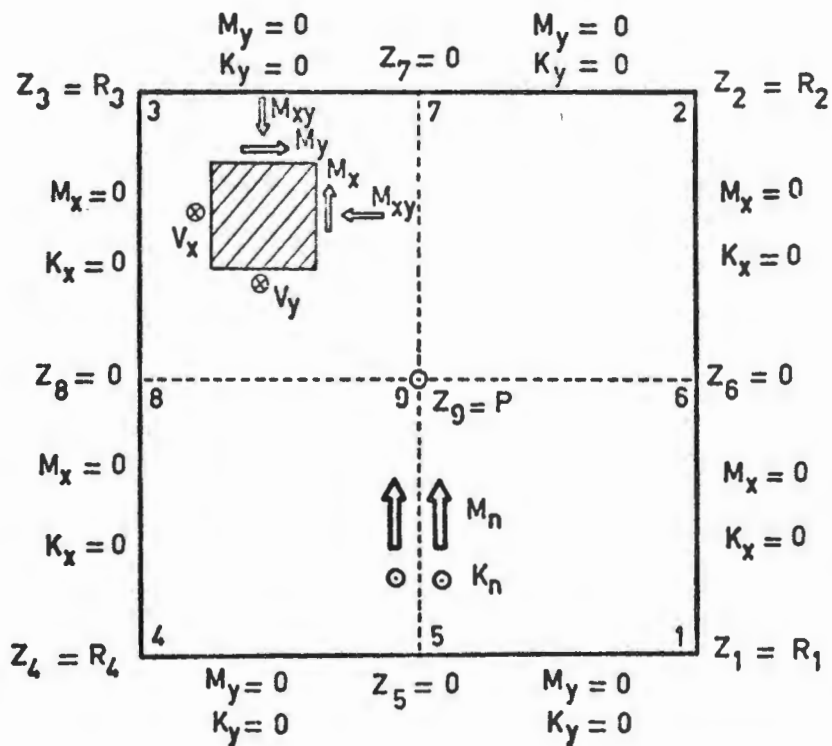


FIGURE 6

b) BOUNDARY CONDITIONS IN TERMS OF STRESS FUNCTIONS U AND V

$$M_x = \frac{\partial V}{\partial Y} = V_y$$

$$K_x = -U_{yy}$$

$$M_y = \frac{\partial U}{\partial X} = U_x$$

$$K_y = -V_{xx}$$

$$M_{xy} = -\frac{1}{2} (U_y + V_x)$$

$$Z_{jxy} = 2 M_{xyj-\xi} = U_{y_x=ij} + V_{x_y=jk}$$

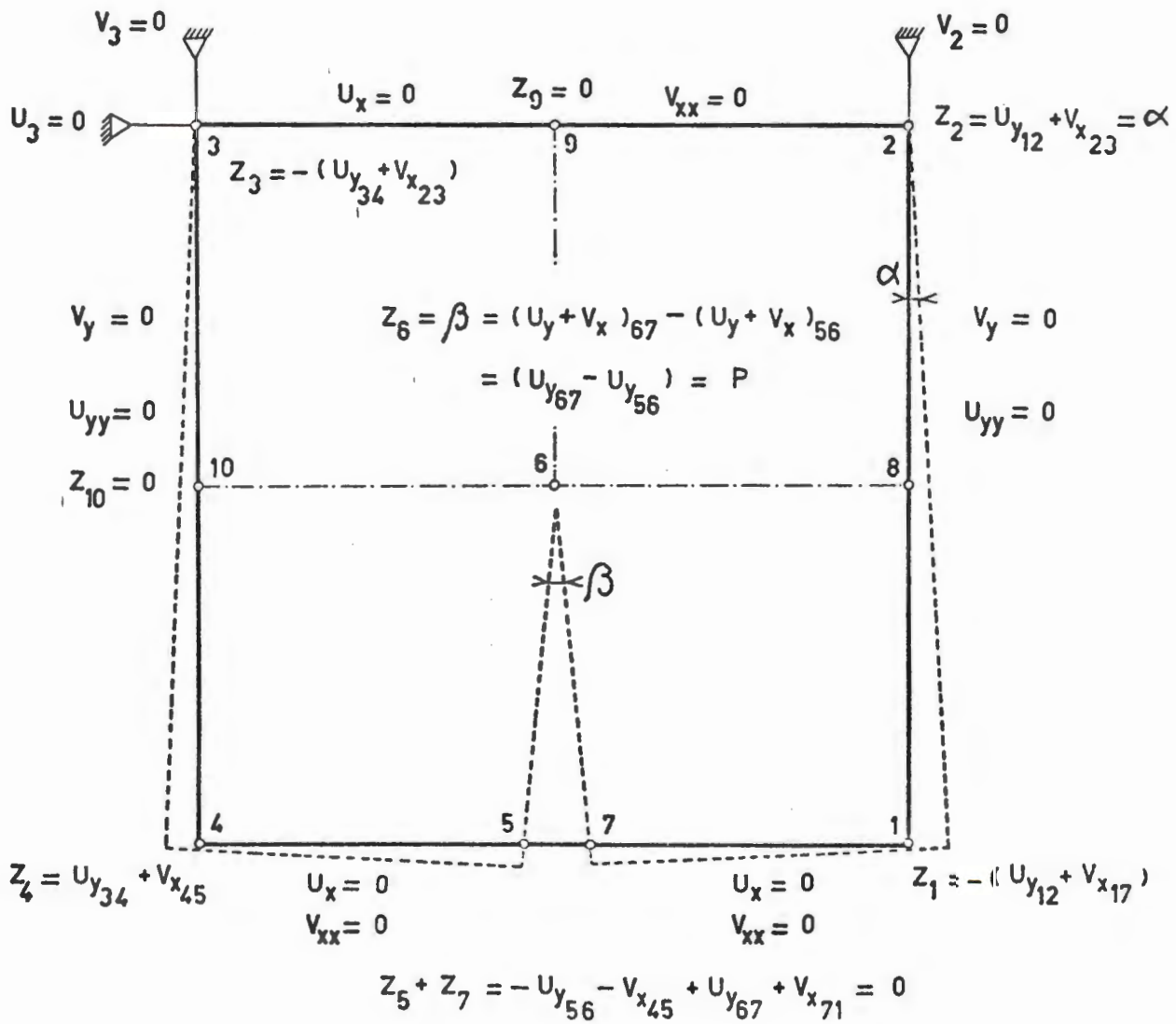
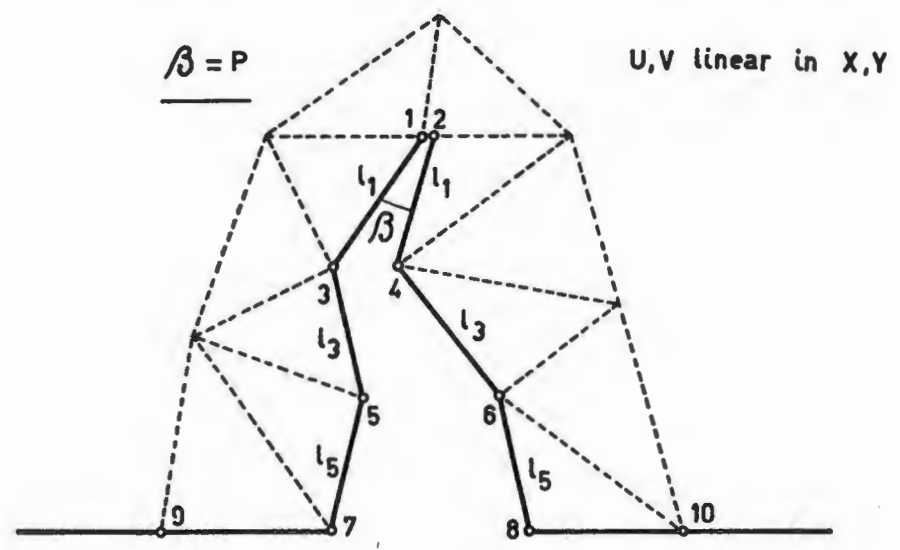
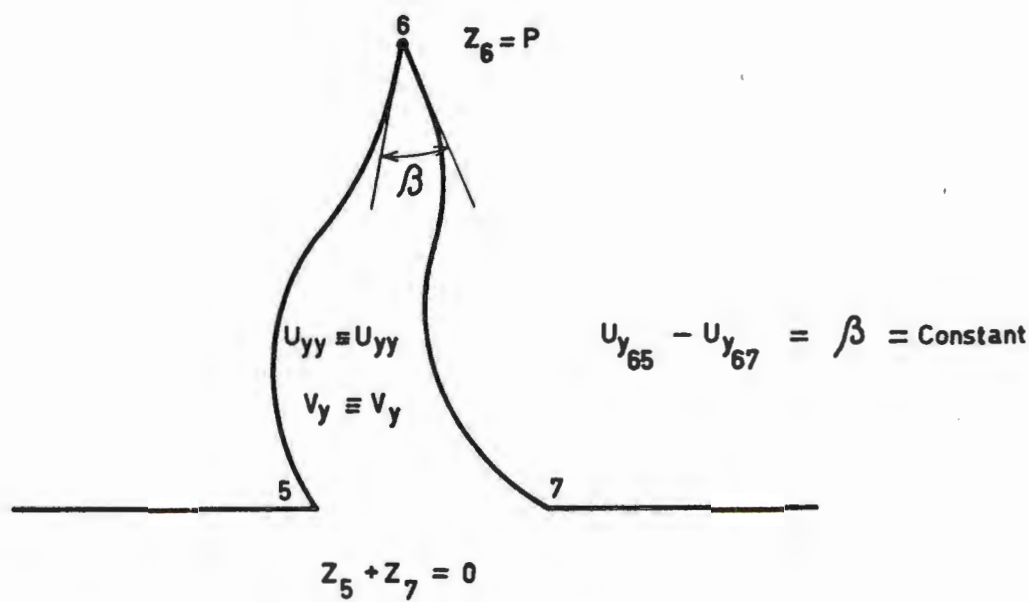


FIGURE 7



CONSTRAINTS

$$\text{I} \begin{cases} U_i = U_{i+1} + l_i \beta \\ v_i = v_{i+1} \end{cases} \quad i = 1, 3, 5$$

$$\text{II} \begin{cases} \frac{\Delta V_{97}}{l_{97}} - \frac{\Delta U_{57}}{l_{57}} = \frac{\Delta V_{8.10}}{l_{8.10}} - \frac{\Delta U_{68}}{l_{68}} & \text{or } Z_7 = Z_8 \\ \text{etc... for } (5,6) \quad (3,4) \quad (1,2) \end{cases}$$

FIGURE 8

## 7. CRITERION FOR SELECTING THE METHOD OF SOLUTION

The number of equations to be solved is obviously different in the two formulations available for displacement and equilibrium models as the topology of the connections is dependent of the choice of the variables. It is also dependent of the boundary conditions which can either decrease the number of equations (as for clamped boundaries in displacement methods) or increase it (as in case of additional constraints). A simple criterion that can be used to select the most efficient method is based on the hyperstaticity index. It is defined by :

$$\Delta = \frac{n_p - n_f}{n_f} \quad (70)$$

where  $n_p$  is the total number of strain or stress parameters in the structure and  $n_f$  is the number of free, unprescribed degrees of freedom in the formulation examined. This index is the ratio between the numbers of unknowns involved in the two formulations of the same problem. Therefore if the index  $\Delta$  is larger than 1., the formulation examined is more efficient than the other.

In the case of the displacement method, consider the global static matrix of the assembled structure (55), and suppose that all the lines corresponding to reactions against prescribed displacements have been suppressed. The number  $n_p$  of parameters  $b$  is larger than the number  $n_f$  of the free degrees of freedom (they are equal only if the structure is isostatic). This connection matrix  $C$  can be split in two parts corresponding to a particular solution and a set of self-stressing states

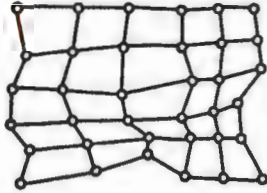
$$g = C b = C_1 b_1 + C_2 b_2$$

where  $C_1$  is a square matrix and  $C_2$  a rectangular matrix of dimension  $n_f \times (n_p - n_f)$ . It is clear that the displacement method is well adapted if  $n_f < (n_p - n_f)$  that is <sup>if</sup> the number of redundancies is larger than the number of unknown displacements.

An application of this criterion is illustrated in figure 9 for a membrane stretching problem idealized by a 5 x 5 grid. Displacement and equilibrium models of increasing degree of sophistication are considered.

THE HYPERSTATICITY INDEX FOR A MEMBRANE STRETCHING PROBLEM

5x5 GRID



25 ELEMENTS  
36 NODES  
60 INTERFACES

( 2 I.O.F PAR NODAL POINT )

	MODEL	d	$n_b$	$n_p$	FREE EDGES		CLAMPED EDGES	
					$n_f$	$\Delta$	$n_f$	$\Delta$
DISPLACEMENT		0	5	125	72	.810	32	2.91
		1	13	325	192	.720	112	1.90
		2	21	525	312	.700	192	1.79
EQUILIBRIUM		0	5	125	120	.07	80	.56
		1	13	325	240	.37	160	1.03
		2	21	525	360	.47	240	1.19

d = DEGREE OF THE STRESS OR STRAIN FIELD

$n_b$  = NUMBER OF PARAMETERS OF THE STRAIN OR STRESS FIELD

$n_p$  = TOTAL NUMBER OF STRAIN OR STRESS FIELD PARAMETERS

$n_f$  = NUMBER OF UNPRESCRIBED UNKNOWNNS

$\Delta$  = HYPERSTATICITY INDEX

FIGURE 9



The hyperstaticity index is established for the displacement method in two limit cases of boundary conditions : all the contour is either free or clamped. Had the index been established on the basis of the force method, the values of  $\Delta$  would have been the inverse of the present one.

It appears that the advantage of the displacement method is the largest for the clamped plate idealized by the simplest displacement element ( $\Delta=2.91$ ) while the force method is by far the best choice in the case of a free contour and the simplest equilibrium element ( $\Delta=.07$ ). Note that this element is very similar to the well known shear panel. Contrary to an established opinion, a force method would be more economic even for displacement models in the case of free boundaries. It is recalled that in the present context the search for the redundancies is avoided in the force method by the special choice of the variables.

From this example it appears that the choice of the solution method is strongly influenced not only by the type of element used, the mesh size and connectivity of the structure, but also by the boundary conditions. In the cases where these are such that constraints are to be added to the system of equations, their number should be introduced in the evaluation of the hyperstaticity index.

#### 8. NON-STRUCTURAL APPLICATIONS

The structural problems encountered in aerospace applications are characterized by the complexity of their geometry. Their physical subdivision is usually so fine that even the simplest elements yield already a good approximation of the displacements. The representation of the stress field is however relatively crude. In the boundary conditions one usually finds a large number of concentrated loads. These characteristics explain that for displacement models the displacement method has been exclusively used, while for bars and shear panels which are very similar to the simplest equilibrium elements, the classical force method has been preferred although it requires a numerical search for the redundancies. But usually the hyperstaticity index of the structures idealized by bars and shear panels is so small that some advantage remains even if the additional work of the Gauss-Jordan elimination is considered.

It has been shown in the preceding section that the new formulation of the force method proposed in this paper, as it avoids the search for the redundancies, can be more economic in a large class of problems. This is

especially true for analysis by equilibrium models which suffer in the displacement method of its relatively larger number of equations. The utilisation of these models has been very limited for this reason and this despite the fact that they yield often better stresses which are at the same time of easier interpretation. Comparison of the energy content at the element level as well as globally is also sometimes fruitful.

Nevertheless the practical use of the stress function formulation for equilibrium models still presents a difficulty which is peculiar to structural problems. That is the need for translating the boundary conditions from applied forces into local values and dislocations of the stress functions. This step is presently achieved by hand but the development of a simple and automatic algorithm is studied.

There is however a large class of problems, that might be called non-structural, to which the new force method can be readily applied. Among them plate bending and plane elasticity problems, fluid flows, heat conduction problems etc, can be mentioned. The boundary conditions met in such problems are easier to express in the variables required by the new force method while at the same time the domains studied are often less physically subdivided than in structural applications. The number of dislocations to introduce in the stress function field (or its equivalent) is therefore reduced.

The complete duality between finite element models and methods of solution is illustrated in the case of potential fluid flow and heat conduction problems in figure 10 which is self explanatory. More details can be found in Reference 13.

Note that passing from one set of variables to the other and from "displacement" to "equilibrium" models is much simpler in these applications, which involve scalar fields, than in elasticity, and only require formally the change of the constitutive equations. In particular the topology of the connection remains the same for displacement and equilibrium models in a given formulation.

# FORMULATIONS FOR POTENTIAL FLUID FLOWS AND HEAT CONDUCTION PROBLEMS

341

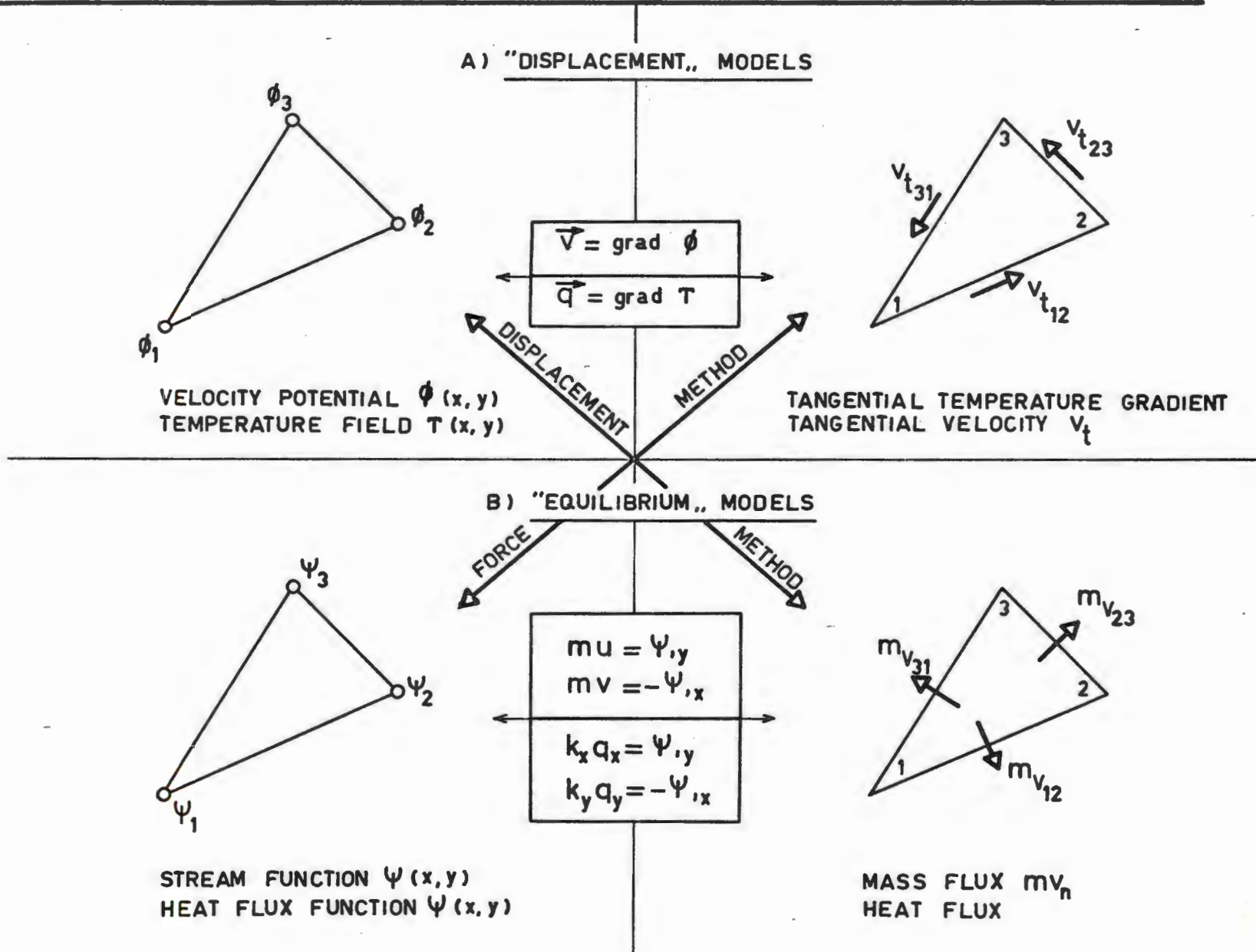


FIGURE 10

## 9. EFFECT OF INTERNAL DEGREES OF FREEDOM ON FINITE ELEMENT PROPERTIES.

In sections 2 and 3 the possibility of introducing internal displacement modes (bubble functions) has been mentioned. An attempt is made here to evaluate the effect of such modes on the finite element properties. Four membrane elements are used in this investigation :

- CTM denotes the classical Conforming Triangular Membrane element
- CPM denotes the Conforming Parallelogram Membrane element using a bilinear or biquadratic displacement field.
- CQM denotes the Conforming Quadrilateral Membrane super-element formed by assembling four triangles CTM and condensation of the internal displacements.
- HPM denotes the Hybrid Parallelogram Membrane element with assumed displacements along the boundaries and assumed stresses inside.

The displacement field in the three conforming elements is represented in the form

$$u = P_1(x,y) + f_2(x,y) \cdot P_2(x,y)$$

where  $P_1(x,y)$  is the classical polynomial which contains a number of parameters just equal to the number of generalized displacements on the contour of the element (6 or 12 for the triangles, 8 or 16 for the quadrangles).

$f_2(x,y) = 0$  is the product of the equations of the interfaces, that is a function vanishing on the boundaries

$P_2(x,y)$  is another polynomial representing bubble or internal displacement modes. The elements have been tested without bubble modes and then the degree of  $P_2$  has been increased from zero to four. Note that zero corresponds already to one internal mode.

In the hybrid element the contour modes are linear or quadratic while the stress field inside the element derives from an Airy stress function represented by a complete polynomial of degree ranging from 2 to 8. The stress field is therefore of degree 0 to 6.



In all cases the internal degrees of freedom have been eliminated by condensation. The elements are either square or equilateral triangle of side length equal to 1., the Young's modulus is  $10^4$ , the thickness is 1. and the Poisson's ratio is .3. All the eigenvalues of the stiffness matrices have been computed for each of these elements. The first three eigenvalues are zero and correspond to the rigid body modes. In addition in the hybrid element additional zero eigenvalues are found when the degree of the stress field is not sufficient. They correspond to spurious kinematic deformation modes (Ref. 3). The first non zero and the maximum eigenvalues are given in figure 11 together with the trace of the stiffness matrices.

It appears from these results that when the number of internal degrees of freedom grows, the conforming and hybrid elements converge, as expected by the theory, to the same values which correspond to the exact solution of a membrane with the constraints of linear or quadratic variation of the displacements prescribed along the edges. The hybrid elements yield eigenvalues which are very close of the exact solution even for the simplest stress field, that can be used without kinematic modes. The convergence of the other elements is very fast so that it is seen that a few bubble modes allow to satisfy almost exactly the equilibrium inside the element.

In conforming elements, an internal field which is one degree higher than that of the interfaces brings the major part of the possible improvement. It can affect predominantly the minimum eigenvalue in which case it will influence the global response of the structure or the maximum eigenvalue in which case it reveals more important for a correct stress representation inside the element. Note that the simplest triangular element is unaffected by such modes. Comparison of the numerical work necessary to build a NPM or a CPM element with a few bubble modes shows little difference.

The conclusion is that the solution with hybrid models or with conforming models containing a few internal modes is practically coincident. The choice of one model or the other has therefore little influence on the results which are governed only by the degree of the displacements along the contour.



EFFECT OF INTERNAL DEGREES OF FREEDOM

			linear interface displacements			quadratic interface displacements		
Model	d	n <sub>b</sub>	λ <sub>min</sub>	λ <sub>max</sub>	trace	λ <sub>min</sub>	λ <sub>max</sub>	trace
HPM	0		-	-	-	-	-	-
	1		75.973	285.71	745.35	-	-	-
	2		75.973	285.71	745.35	-	-	-
	3		78.638	285.71	750.68	31.058	395.87	2180.3
	4		78.638	285.71	750.68	31.401	396.86	2236.9
	5		78.855	285.71	751.12	31.401	396.86	2254.0
	6		78.856	285.71	751.12	31.408	398.74	2256.9
CQM	1	2	113.960	285.71	821.33	-	-	-
	2	10	80.953	285.71	755.31	38.978	412.62	2553.1
	3	26	73.997	285.71	751.40	31.554	401.04	2283.6
	4	50	73.925	285.71	751.26	31.417	399.21	2267.1
CPM	-	0	93.901	285.71	791.21	33.610	943.38	4114.3
	0	2	79.794	285.71	752.99	33.610	467.15	2604.5
	1	8	79.263	285.71	751.93	31.435	403.23	2313.7
	2	18	73.952	285.71	751.31	31.412	399.53	2273.0
	3	32	73.913	285.71	751.23	31.410	399.19	2265.4
	4	50	73.902	285.71	751.21	31.403	398.97	2263.8
CTM	1	0	95.940	343.62	593.40	-	-	-
	2	0	95.940	343.62	593.40	16.078	1022.9	2967.0
	3	2	95.940	343.62	593.40	15.777	857.65	2423.3
	4	6	95.940	343.62	593.40	15.629	843.26	2362.2
	5	12	95.940	343.62	593.40	15.601	841.42	2354.1

d = degree of the polynomial  $P_2(x,y)$  or, in HPM, degree of the stress field  
n<sub>b</sub> = number of condensed internal degrees of freedom

FIGURE 11

## REFERENCES

1. SANDER G. and FRAEIJIS de VEUBEKE B.  
"Upper and Lower Bounds to Structural Deformations by Dual Analysis in Finite Elements"  
USAF Tech. Report. - AFFDL-TR-66-199 - 1966
2. SANDER G.  
"Application of the Dual Analysis Principle"  
Proc. IUTAM Coll. on "High Speed Computing of Elastic Structures"  
Ed. FRAEIJIS de VEUBEKE, Liège, 1970 (in Press)
3. FRAEIJIS de VEUBEKE B., SANDER G. and BECKERS P.  
"Dual Analysis by Finite Elements - Linear and Nonlinear Applications"  
USAF Tech. Report - AFFDL - 72-93, 1973.
4. FRAEIJIS de VEUBEKE B.  
"A Conforming Finite Element for Plate Bending"  
Int. J. Solids and Struct., Vol. 4, 1968, Pergamon
5. SANDER G.  
"Dual Analysis of a Multiwek Swept Back Wing Model"  
Aircraft Engineering, Febr. 1968
6. CLOUGH R. and TOCHER J.  
"Finite Element Stiffness Matrices for Analysis of Plates in Bending"  
Proc. USAF Conf. DAYTON, Report AFFDL-TR-66-80, 1966
7. FRAEIJIS de VEUBEKE B. and ZIENKIEWICZ O.C.  
"Strain Energy Bounds in Finite Element Analysis by Slab Analogies"  
J. of Strain Analysis, Vol 2, n° 4, 1967
8. FRAEIJIS de VEUBEKE B.  
"Displacements and Equilibrium Models in the Finite Element Method"  
Chap. 9 in "Stress Analysis" Ed. Zienkiewicz, John Wiley, 1965
9. SANDER G.  
"Application de la Méthode des Eléments Finis à la Flexion des Plaques"  
Coll. des Publications Fac. Sc. Appl. - Univ. Liège, n° 15, 1969
10. KOSKO E.  
"The Equivalence of Force and Displacement Methods in the Analysis of Elastic Structures"  
Proc. USAF Conf. DAYTON, Ohio, AFFDL-TR-66-80, 1966
11. FRAEIJIS de VEUBEKE B.  
"Basis of a Well Conditioned Force Program for Equilibrium Models the Southwell Analogies"  
USAF Report AFFDL-TR-67-80, 1967

12. ZIENKIEWICZ, IRONS, ERGATOUDIS, HOLLAND, SCOT  
"Isoparametric and Associated Element Families for Two and Three Dimensional Analysis"  
Chap.13 in "Finite Element Method" Ed. Holand and Bell, TAPIR, 1969
13. BECKERS P.  
"Les fonctions de tensions dans la méthode des éléments finis"  
Dc. Thesis, Univ. Liège 1971  
To be published in Coll. Publ. Fac. Sc. Appl., Liège
14. ZIENKIEWICZ, O.C.  
"The finite element method in continuum mechanics"  
2nd edition Mac Graw Hill, 1971.