THE HEAT CONDUCTION AND THERMAL STRESS ANALYSIS BY THE FINITE ELEMENT METHOD

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There are many engineering problems which require thermal stress analysis, because the objects to be treated are subject to thermal load. In order to solve the problems, it is necessary, to know the stationary and transient temperature distribution prior to the stress analysis. Fortunately, it is possible to apply the finite element method to the analysis of heat conduction, as to the stress analysis.

In this paper, we studied a method of analysis and its accuracy for heat conduction and stress by means of the finite element method using the triangular element with three and six nodes.

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SECTION I

SUMMARY

The analysis of heat conduction and temperature distribution is an important problem just as stress analysis is in the engineering field. The problem is described mathematically by partial differential equations which are solved numerically by digital computer by replacing them with difference equations based on an orthogonal lattice system.

But this calculation method is not efficient for the problems, in which the boundary has an arbitrary shape, because the orthogonal lattice system is not suitable for it. The matrix method or the finite element method are preferable in this respect, because we can use an arbitrary shaped element. The finite element method is usually used for the stress analysis of plate and solid. But they are also effective for the continuum analysis of all kinds, for instance, heat conduction, hydrodynamics, electro magnetic field and so on.

In this paper we studied a calculation method of transient heat conduction, stationary temperature distribution and thermal stress by means of the finite element method, and prepared computer programs which have general purpose applications.

It is necessary for us to study the calculation method with high accuracy for the saving of calculation time and memory space of the computers. Formerly, in the analysis of heat conduction and temperature distribution by means of the finite element method, triangular element with three nodes which give the linear temperature distribution in an element was used. This method has sufficient accuracy for the analysis of stationary temperature distribution when there is no heat exchange in the given domain. In other cases, however for instance, when there is heat loss due to the temperature rise or heat input or output due to the heat transfer on the surface of a plate, the temperature distribution is not a harmonic function, so that a linear temperature distribution in an element does not exist.

If so, the heat balance in the element cannot be kept, and hence the temperature distribution of more than second order polynomials must be used instead. Concentrated heat capacity and heat transfer were used for the analysis by the finite element method, but this method lacks accuracy due to the assumption of concentration of heat capacity and heat transfer. In this paper, we tried some test calculations to study the accuracy of various kinds of elements related to the above theory. For this purpose, elements of regular triangle and equilateral right triangle were used. Test calculation results obtained by the concentrated

and uniformly distributed heat capacity and heat transfer were compared with the theoretical value, and we came to the conclusion that both numerical calculations are not accurate, whereas mean value of them gives nearly correct value.

Calculations based on the regular triangle are stable, but those of equilateral right triangle are disturbed by the difference in the node valency. Regular triangle element system has the valency of six at all nodes, while in the other element system, node 1 has the valency of eight and node two has four as shown in Figure 16, resulting in disturbance of calculation. For the remedy of this defect, a modified system is introduced as stated in detail in Section III. The modified element system gives fairly accurate calculation results as shown in Figure 15. The modified element system with six nodes has more accuracy, but in most cases when special accuracy is not needed, modified element system with three nodes may be used with sufficient accuracy.

Transient heat conduction calculation method by the finite element method was also studied. Laplace transformation, inverse Laplace transformation and modal analysis technique were applied in this paper.

Finally, we studied calculation method of thermal stress using the triangular element with six nodes.

SECTION II

FUNDAMENTAL EQUATION OF HEAT CONDUCTION

The differential equation of heat conduction of thin plate in a given domain is given in the following:

$$Ch\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\lambda h\frac{\partial\theta}{\partial x} + \frac{\partial}{\partial y}\lambda h\frac{\partial\theta}{\partial y} + \alpha_a(\Theta_a - \theta) + q_a \tag{1}$$

and the boundary conditions

$$\theta = \theta_b = \text{given} \tag{2}$$

$$\lambda\frac{\partial\theta}{\partial\eta} = \alpha_b(\Theta_b - \theta) + q_b \tag{3}$$

By Laplace transformation Equation 1, we have

$$Ch(-\theta_o + S\theta) = \frac{\partial}{\partial x}\lambda h\frac{\partial\theta}{\partial x} + \frac{\partial}{\partial y}\lambda h\frac{\partial\theta}{\partial y} + \alpha_a(\Theta_a - \theta) + q_a$$

or

$$\left\{\frac{\partial}{\partial x}\lambda h\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\lambda h\frac{\partial}{\partial y} - (\alpha + ChS)\right\}\theta = -(q_a + \alpha_a\Theta_a + Ch\theta_o) \tag{4}$$

where θ_o are initial temperature.

To solve the above equation by means of the finite element method, it is necessary to find the functional which gives the above differential equation by variation.

In accordance with the stress analysis, we introduce the following temperature strain energy V and the temperature virtual work $\delta' W$ as follows:

$$V = \frac{1}{2}\iint_e \left[\lambda h\left\{\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2\right\} + (\alpha_a + ChS)\theta^2\right] dx\,dy + \frac{1}{2}\oint_s \alpha_b h\theta^2\,ds \tag{5}$$

$$\delta' W = \iint_e (q_a + \alpha_a\Theta_a + Ch\theta_o)\,\delta\theta\,dx\,dy + \oint_s h(q_b + \alpha_b\Theta_b)\,\delta\theta\,ds \tag{6}$$

Just like the case for the stress analysis, we have the following variation equation:

$$\delta V - \delta' W = \oint_s h \left\{ \lambda \frac{\partial \theta}{\partial \eta} - \alpha_b (\Theta_b - \theta) - q_b \right\} \delta\theta \, ds$$

$$- \iint_e \left[\left\{ \frac{\partial}{\partial x} \lambda h \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \lambda h \frac{\partial}{\partial y} - (\alpha_a + ChS) \right\} \theta \right.$$

$$\left. + q_a + \alpha_a \Theta_a + Ch\theta \right] \delta\theta \, dx\, dy = 0 \tag{7}$$

The above equation gives an identical differential equation with the Equation 4 and the boundary conditions 2 and 3. Hence we are able to use the Equation 7 as the fundamental equation of the finite element method applied to the heat conduction analogous to the stress analysis.

When the temperature distribution in an element is given by

$$\theta = \sum_m P_m (x, y)\, \theta_m \tag{8}$$

where $P_m(x,y)$ are the shape function and θ_m the node temperature, we have the following temperature stiffness matrices:

$$V = \frac{1}{2} \sum_{m,n} K_{mn}\, \theta_m\, \theta_n \tag{9}$$

$$K_{mn} = \iint_e \left\{ \lambda h (P_{mx} P_{nx} + P_{my} P_{ny}) + (ChS + \alpha_a) P_m P_n \right\} dx\, dy$$

$$+ \oint_s \alpha_b\, h P_m P_n \, ds \tag{10}$$

and the effective heat source

$$Q_m = \iint_e (q_a + \alpha_a \Theta_a + Ch\theta_o) P_m \, dx dy$$

$$+ \oint_s h (q_b + \alpha_b \Theta_b) P_m \, ds \tag{11}$$

Summarizing them for the total system, we have

$$V = \frac{1}{2} \sum_r K_{ij}\, \theta_i\, \theta_j \tag{12}$$

$$\delta' W = \sum_r Q_i\, \delta\theta_i \tag{13}$$

and finally we obtain the equations of heat conduction for the finite element method as follows:

$$\frac{\partial V}{\partial \theta_i} - Q_i = \sum_j K_{ij}\, \theta_i - Q_i = 0 \tag{14}$$

SECTION III

ANALYSIS OF STATIONARY TEMPERATURE DISTRIBUTION

In this study, we used triangular elements with three and six nodes, and the area coordinate which are defined as

$$\zeta_m = A_m / A = a_m + b_m x + C_m y \qquad (m=1,2,3) \tag{15}$$

Where A is the area of triangular element and A_m is shown in Figure 1.

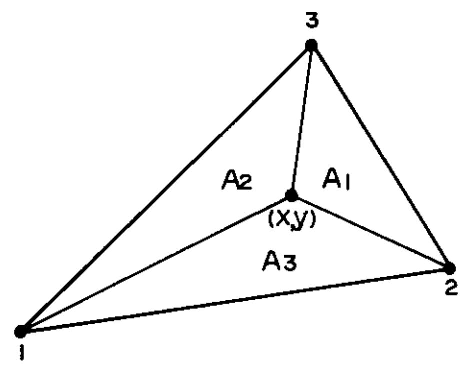


Figure 1

Differentiation of area coordinate is given by the following equations

$$\frac{\partial \zeta_m}{\partial x} = b_m \ , \qquad \frac{\partial \zeta_m}{\partial y} = C_m \tag{16}$$

Integration of temperature strain energy and virtual work are easily given by using area coordinate. An element 1-2-3 in x-y plane is transformed into the corresponding element $1' \ 2' \ 3'$ in area coordinate ζ_1 , ζ_2 plane as shown in Figure 2. The area integration referred to the element 1, 2, 3 can be written as

$$\iint_e f \, dx \, dy = \alpha \ (= 2A) \iint_e f \, d\zeta_1 \, d\zeta_2$$

Therefore, if we define

$$f = \zeta_1^{m_1} \ \zeta_2^{m_2} \ \zeta_3^{m_3}$$

as the basic functions, because the arbitrary functions are given as the polynomials of area coordinate, we have the following integration formulas

$$\frac{1}{A}\iint \zeta_1^{m_1}\,\zeta_2^{m_2}\,\zeta_3^{m_3}\,dx\,dy = 2\int_0^1 d\zeta_1 \int_0^{1-\zeta_1} \zeta_1^{m_1}\,\zeta_2^{m_2}\,\zeta_3^{m_3}\,d\zeta_1\,d\zeta_2$$

$$= \frac{2m_1!\,m_2!\,m_3!}{(m_1+m_2+m_3+2)!} \tag{17}$$

where m_1, m_2, m_3 are arbitrary integers.

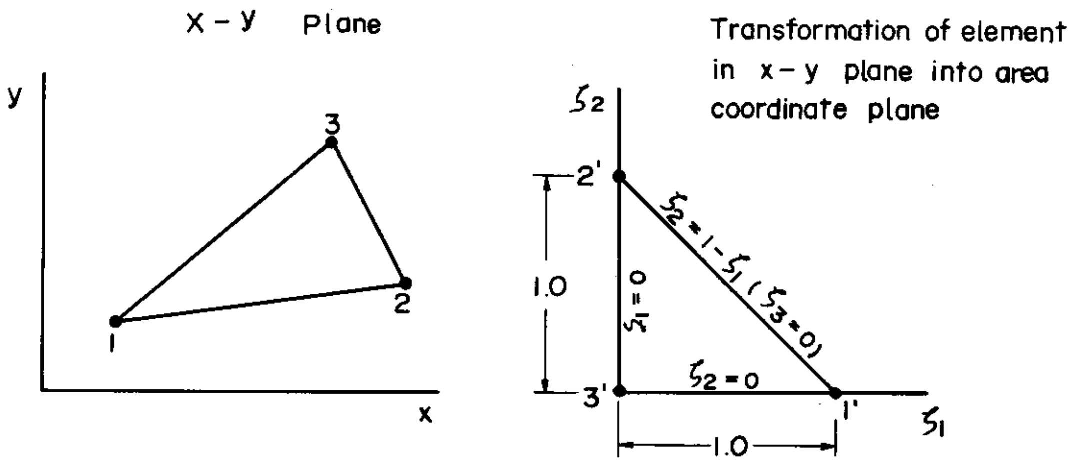


Figure 2

TRIANGULAR ELEMENT WITH THREE NODES

In this element, shape function $P_m(x,y)$ are the same as the area coordinate $\zeta_m(x,y)$ and the temperature distribution in the element is given by

$$\theta = \sum_m \zeta_m \theta_m \tag{18}$$

Therefore, the temperature stiffness matrix is

$$K_{mn} = A\,(\lambda h\,\psi_{mn} + a_a\,\mu_{mn}) + a_b\,h\,\nu_{mn} \tag{19}$$

where

$$\psi_{mn} = \frac{1}{A}\iint (b_m b_n + C_m C_n)\,dx\,dy \tag{20}$$

$$\mu_{mn} = \frac{1}{A}\iint \mu(x,y)\,\zeta_m\zeta_n\,dx\,dy\,,\qquad \frac{1}{A}\iint \mu(x,y)\,dx\,dy = 1 \tag{21}$$

$$\nu_{mn} = \oint_s \zeta_m\zeta_n\,ds \tag{22}$$

and the effective heat source is

$$Q_m = \iint (q_a + \alpha_a \Theta_a)\zeta_m \, dx\,dy + \oint_s h(q_b + \alpha_b \Theta_b)\zeta_m \, ds \tag{23}$$

Generally λh and α_a are the function of x, y in the element, but for the simplification we may assume a uniform distribution without large loss of accuracy. Formerly lumped heat capacity ch and heat transfer coefficient α_a were assumed in the analysis of transient heat conduction and stationary temperature distribution, but they were accompanied by the loss of accuracy as shown in Figure 19.

For the development of calculation method, we introduced the following four kinds of elements which contain respective distribution function μ (x, y) of ch and α_a namely μ (x, y) = 1 for the uniform distribution, $\mu(x,y) = \mu_m \zeta(x-x_m, y-y_m)$ for the concentrated constant system where $\delta(x,y)$ is the two-dimensional delta function, and $\mu(x,y) = (1 + \mu_m \delta(x-x_m, y-y_m))/2$ for the mean constant system.

1. (C3) Concentrated constant system element with three nodes

$$\mu_{mn} = \begin{bmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \mu_3 \end{bmatrix}$$

$$\mu_1 = \mu_2 = \mu_3 = \frac{1}{3}$$

$$\mu_m = A_m^* / A \qquad (m = 1, 2, 3)$$

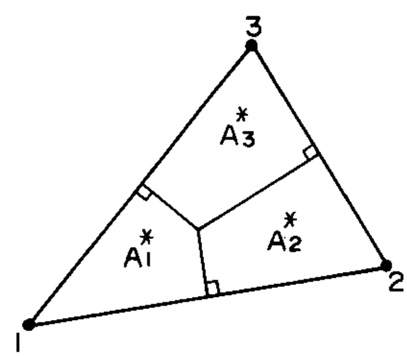


Figure 3

2. (D3) Uniformly distributed constant system element with three nodes

$$\mu_{mn} = \frac{1}{A} \iint \zeta_m \zeta_n \, dx\,dy = \frac{1}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

3. (E3) Mean constant system element with three nodes

$$\mu_{mn} = \frac{1}{2} \left(\begin{bmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \mu_3 \end{bmatrix} + \frac{1}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \right)$$

4. (M3) Modified mean constant system element with three nodes

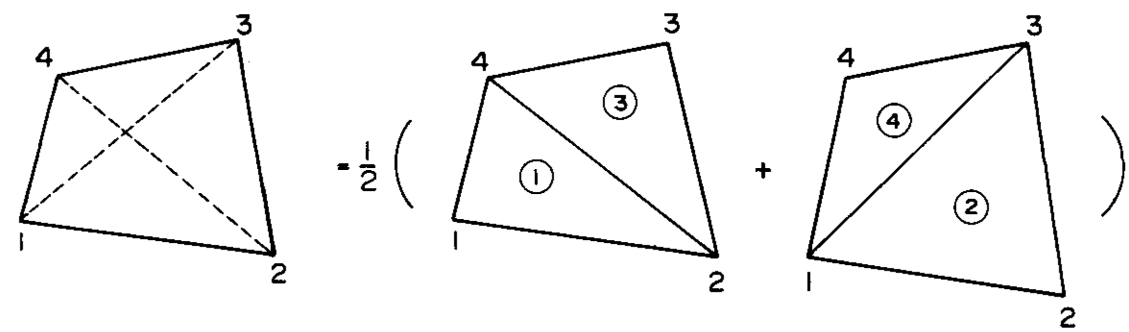
For the purpose of elimination of disturbance in calculation caused by the coexistence of nodes which have a different number of valency, for instance, in Figure 16 node 2 has four valency whereas node 1 has eight valency. Difference relation for node 1 and node 2 are not the same as shown in Figure 13, therefore when there are different kinds of nodes, which have different valency, the calculations are disturbed as shown in Figure 16. For the remedy of this difficulty, there is introduced a modified element as follows.

Two triangular elements 1, 2, 3 and 1-3-4 which have the side 1-3 in common, shown in Figure 4, are also divided into another set of triangular element, namely elements 1-2.4 and 2-3.4. The stiffness matrix referring to the rectangular element 1-2.3.4 K_{ij} is given by

$$K_{ij} = (K^1_{ij} + K^2_{ij} + K^3_{ij} + K^4_{ij})/2$$

where K^r_{ij} is the stiffness matrix referring to the rth triangular element.

Modified elements are shown as Figure 4.



Modified element

Figure 4

In the above figure (r) denotes the element number.

TRIANGULAR ELEMENT WITH SIX NODES

This element is shown in Figure 5. The node 4.5.6 are the middle points of the sides 2-3, 3.1 and 1,2, respectively.

The shape functions of this element are

$$P_1 = \zeta_1(\zeta_1 - \zeta_2 - \zeta_3)$$
$$P_2 = \zeta_2(-\zeta_1 + \zeta_2 - \zeta_3)$$
$$P_3 = \zeta_3(-\zeta_1 - \zeta_2 + \zeta_3)$$
$$P_4 = 4\zeta_2\zeta_3$$
$$P_5 = 4\zeta_3\zeta_1$$
$$P_6 = 4\zeta_1\zeta_2 \qquad (24)$$

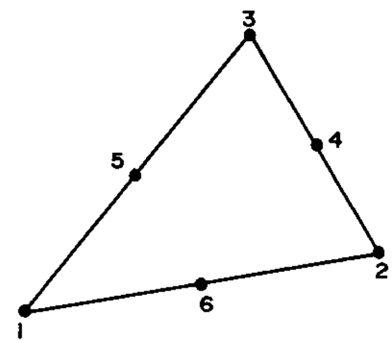


Figure 5

and the temperature distribution in the element is given by

$$\theta = \sum_{m=1}^{6} P_m \theta_m \qquad (25)$$

The temperature stiffness matrix K_{mn} is

$$K_{mn} = \iint \left\{ \lambda h (P_{mx} P_{nx} + P_{my} P_{ny}) + \alpha_a \mu(x,y) P_m P_n \right\} dx\, dy$$
$$+ \sum_{\ell=1} \int h \alpha_{b\ell} P_m P_n \, ds_\ell$$
$$= A\left(\frac{\lambda h}{12} \psi_{mn} + \alpha_a \mu_{mn}\right) + \sum_{\ell=1}^{3} h \alpha_{b\ell} \nu_{\ell mn}$$

The μ_{mn} matrices are given as follows in reference to the distribution function

1. (C6)

$$\mu_{mn} = \begin{bmatrix} \mu_1 & & & & & \\ & \mu_2 & & & & \\ & & \mu_3 & & & \\ & & & \mu_4 & & \\ & & & & \mu_5 & \\ & & & & & \mu_6 \end{bmatrix}$$

$\mu_m = A_m^* / A$

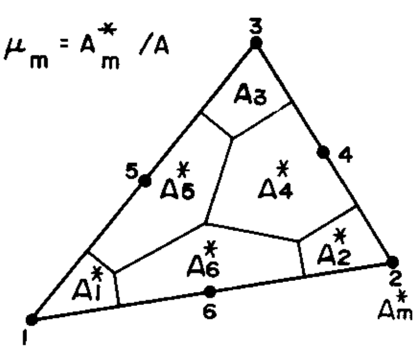


Figure 6

2. (D6)

$$\mu_{mn} = \frac{1}{180}\begin{bmatrix} 6 & -1 & -1 & -4 & & \\ -1 & 6 & 1 & & -4 & \\ -1 & -1 & 6 & & & -4 \\ -4 & & & 32 & 16 & 16 \\ & -4 & & 16 & 32 & 16 \\ & & -4 & 16 & 16 & 32 \end{bmatrix}$$

3. (E6)

$$\mu_{mn_{M6}} = (\mu_{mn_{C6}} + \mu_{mn_{D6}})/2$$

where

$$\psi_{mn} = \begin{bmatrix} 12d_1 & -4\ell_3 & -4\ell_2 & & 16\ell_2 & 16\ell_3 \\ & 12d_2 & -4\ell_1 & 16\ell_1 & & 16\ell_3 \\ & & 12d_3 & 16\ell_1 & 16\ell_2 & \\ & & & f & 32\ell_3 & 32\ell_2 \\ & \text{Sym.} & & & f & 32\ell_1 \\ & & & & & f \end{bmatrix}$$

$$d_1 = b_1^2 + c_1^2 \quad , \quad d_2 = b_2^2 + c_2^2 \quad , \quad d_3 = b_3^2 + c_3^2$$

$$\ell_1 = b_2\, b_3 + c_2\, c_3 \; , \quad \ell_2 = b_3\, b_1 + c_3\, c_1 \; , \quad \ell_3 = b_1\, b_2 + c_1\, c_2$$

$$f = 16\,(d_1 + d_2 + d_3)$$

$$\nu_{\ell mn} = \frac{S_\ell}{30}\, \Pi_{\ell mn}$$

$$\Pi_{1mn} = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 4 & 2 \\ 2 & 2 & 16 \end{bmatrix} \qquad \Pi_{2mn} = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 4 & 2 \\ 2 & 2 & 16 \end{bmatrix}$$

$$\Pi_{3mn} = \begin{bmatrix} 4 & -1 & & & & 2 \\ -1 & 4 & & & & 2 \\ & & & & & \\ & & & & & \\ & & & & & \\ 2 & 2 & & & & 16 \end{bmatrix}$$

SECTION IV

STUDY ON THE ACCURACY OF CALCULATION

We conducted a series of studies on the accuracy of calculation by the finite element method of various kinds which are stated in the previous paragraph. The regular triangle and the isosceles right triangle elements were choosen for the systematic study of accuracy. For both elements, calculation formula by the finite element method are turned into the difference equations referring to the nodes as follows.

REGULAR TRIANGLE ELEMENT WITH THREE NODES

Let a side length of a regular triangle be a. We have then the following temperature stiffness matrix from the Equation 19.

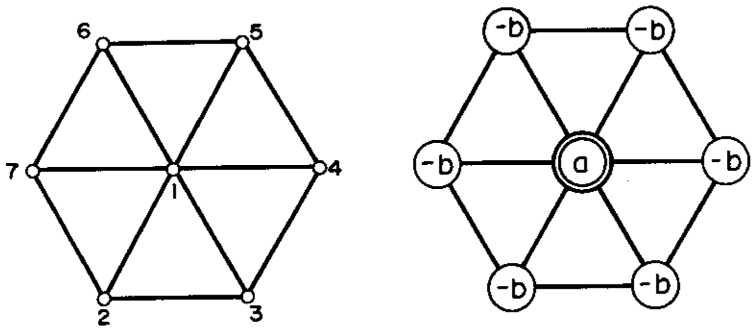
$$K_{mn} = A(\lambda h\, \psi_{mn} + \alpha_a\ \mu_{mn}) = \frac{2\lambda h A}{3a^2}\left(\phi_{mn} + \frac{3}{2}\alpha\ \mu_{mn}\right)$$

where $\alpha = a^2\alpha_a/\lambda h$ and

$$\phi_{mn} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\mu_{mn} = \begin{cases} \frac{1}{3}\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} & \text{for (C3)} \\ \frac{1}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} & \text{for (D3)} \\ \frac{1}{24}\begin{bmatrix} 6 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 6 \end{bmatrix} & \text{for (E3)} \end{cases}$$

From the above equations, we have the following difference equation referring to node 1 as shown in Figure 7.



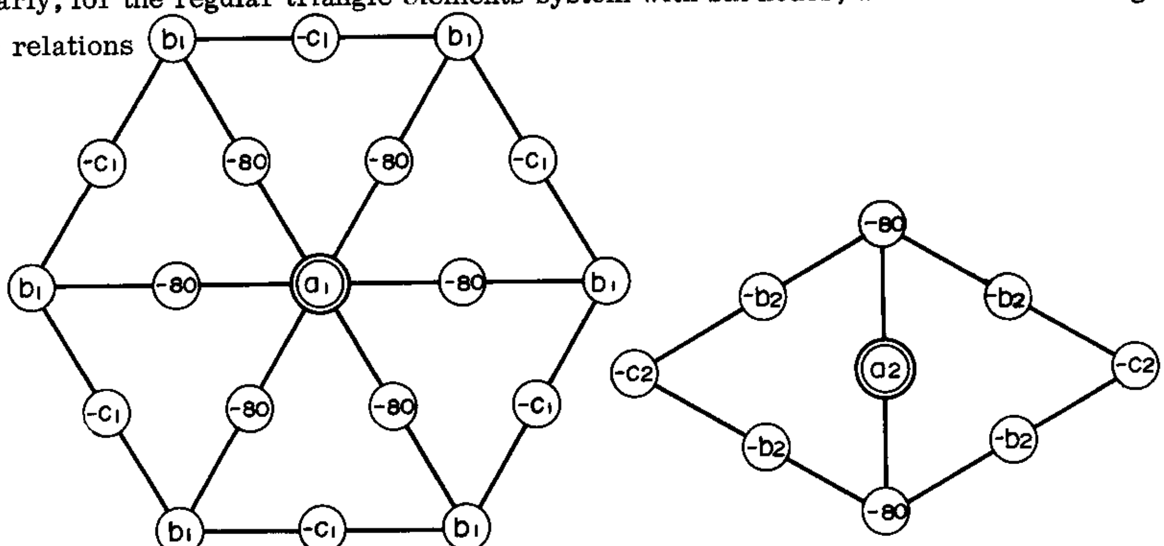
	a	b
C 3	96 + 24 α	16
D 3	96 + 12 α	16 − 2 α
E 3	96 + 18 α	16 − α

Figure 7

$$a\theta_1 - b(\theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 + \theta_7) = 0$$

REGULAR TRIANGULAR ELEMENT WITH SIX NODES

Similarly, for the regular triangle elements system with six nodes, we have the following difference relations



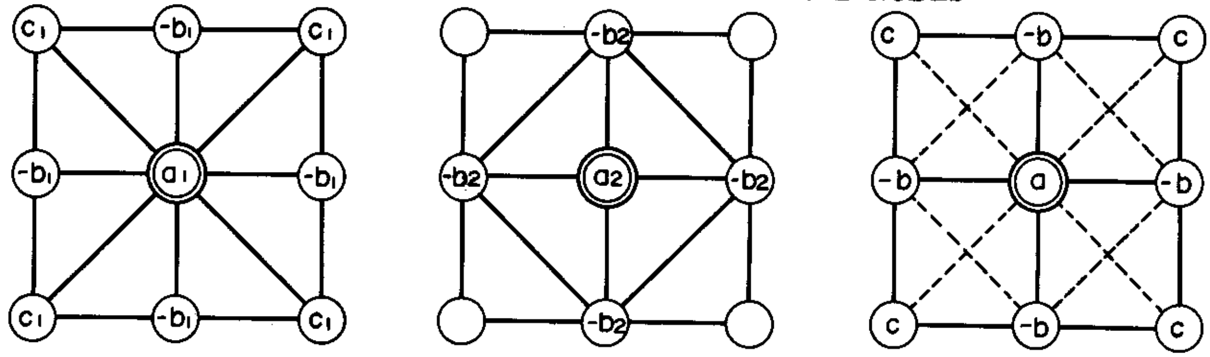
	a_1	b_1	c_1
C 6	360 + 90 α	20	
D 6	360 + 36 α	20 − 2 α	4 α
E 6	360 + 63 α	20 − α	2 α

	a_2	b_2	c_2
C 6	480 + 90 α	80	
D 6	480 + 64 α	80 − 16 α	4 α
E 6	480 + 77 α	80 − 8 α	2 α

Difference relation of regular element with six nodes

Figure 8

THE ISOSCELES RIGHT TRIANGLE ELEMENT WITH THREE NODES



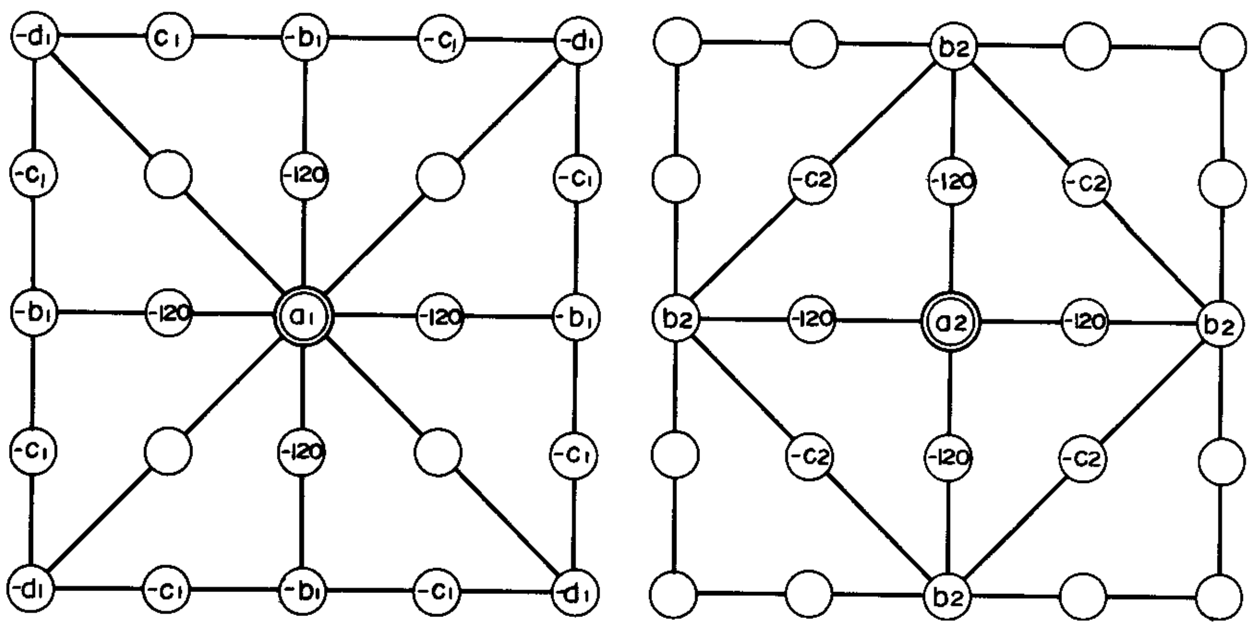
	a_1	b_1	c_1
$D_1 3$	$192+32\alpha$	$48-4\alpha$	4α
$E_1 3$	$192+40\alpha$	$48-2\alpha$	2α

	a_2	b_2
$D_2 3$	$192+16\alpha$	$48-4\alpha$
$E_2 3$	$192+32\alpha$	$48-2\alpha$

	a	b	c
C 3	$192+48\alpha$	48	
M 3	$192+36\alpha$	$48-2\alpha$	α

Figure 9

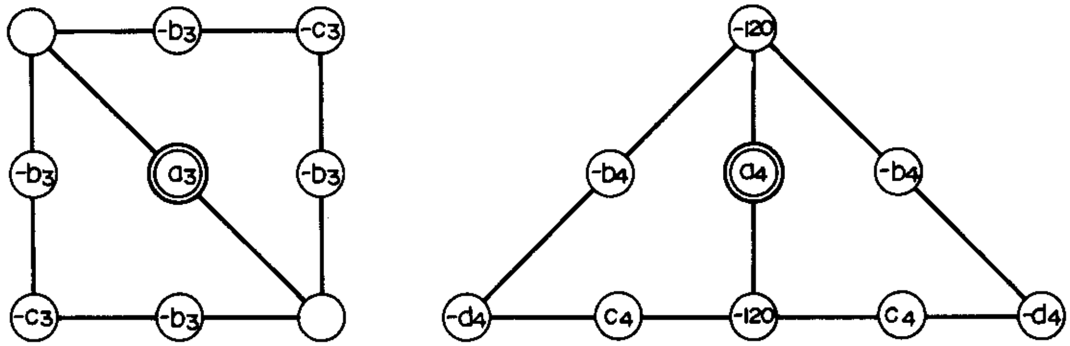
THE ISOSCELES RIGHT TRIANGLE ELEMENT WITH SIX NODES



	a_1	b_1	c_1	d_1
C 6	$360+90\alpha$	30		
$D_1 6$	$360+48\alpha$	$30-2\alpha$	4α	2α
$E_1 6$	$360+69\alpha$	$30-\alpha$	2α	α

	a_2	b_2	c_2
C 6	$360+90\alpha$	30	
$D_2 6$	$360+24\alpha$	$30-2\alpha$	4α
$E_2 6$	$360+57\alpha$	$30-\alpha$	2α

Figure 10a



	a3	b3	c3
C6	$480+90\alpha$	120	
D6	$480+64\alpha$	$120-16\alpha$	4α
E6	$480+77\alpha$	$120-8\alpha$	2α

	a4	b4	c4	d4
C6	$480+90\alpha$	120		
D6	$480+64\alpha$	$120-16\alpha$	16α	4α
E6	$480+77\alpha$	$120-8\alpha$	8α	2α

Figure 10b

The temperatures at the center of a regular triangular plate versus α or α^* are shown in Figures 11 and 12, $\alpha = a^2\alpha a/\lambda h$ and $\alpha^* = \ell^2 \alpha a/\lambda h$. The calculated temperature based on the mean constant system element is nearly the mean value of the concentrated and distributed constant system elements below $\alpha^* = 250$. Where $\ell = na$.

Figure 16 is a temperature distribution of a rectangular plate which is divided into 5 x 5 sub-rectangular plates. It gives unsymmetric temperature distribution caused by the coexistence of nodes of different valency. These disturbances are remedied by the modification of elements as shown in the same figure.

Figure 19 is the error percentage of the temperature at the center of rectangular plate. It is clear as shown in this figure that calculations based on the modified element and the element system with six nodes have high accuracy compared with others.

The reason why the temperature calculated by the mean constant system is about the mean temperature of concentrated and distributed constant systems is explained as follows. For simplicity, we treat one dimensional problem.

The equation of stationary heat conduction in nondimensional form is

$$\frac{d^2\theta}{dx^2} - \alpha\theta = 0$$

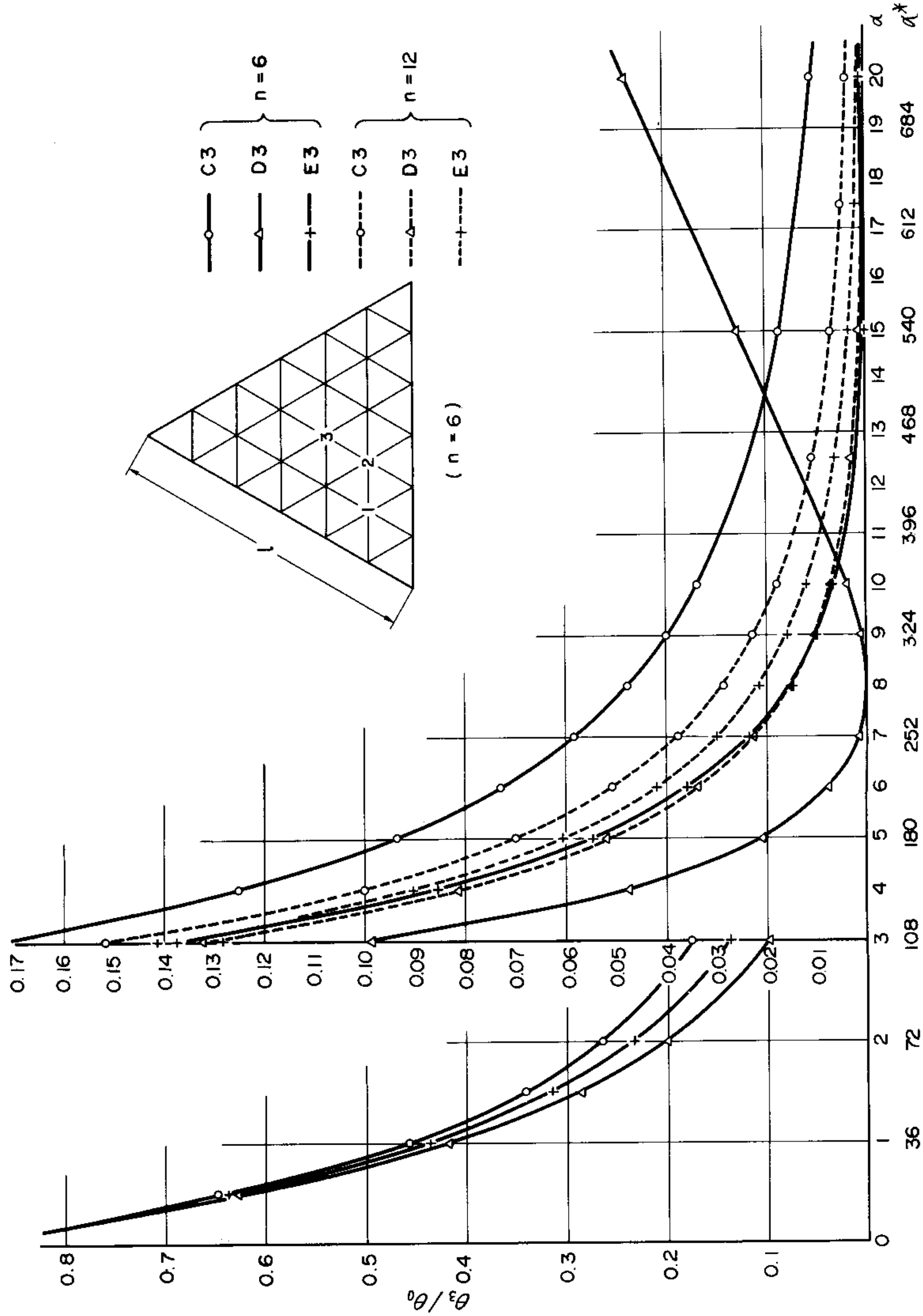


Figure 11

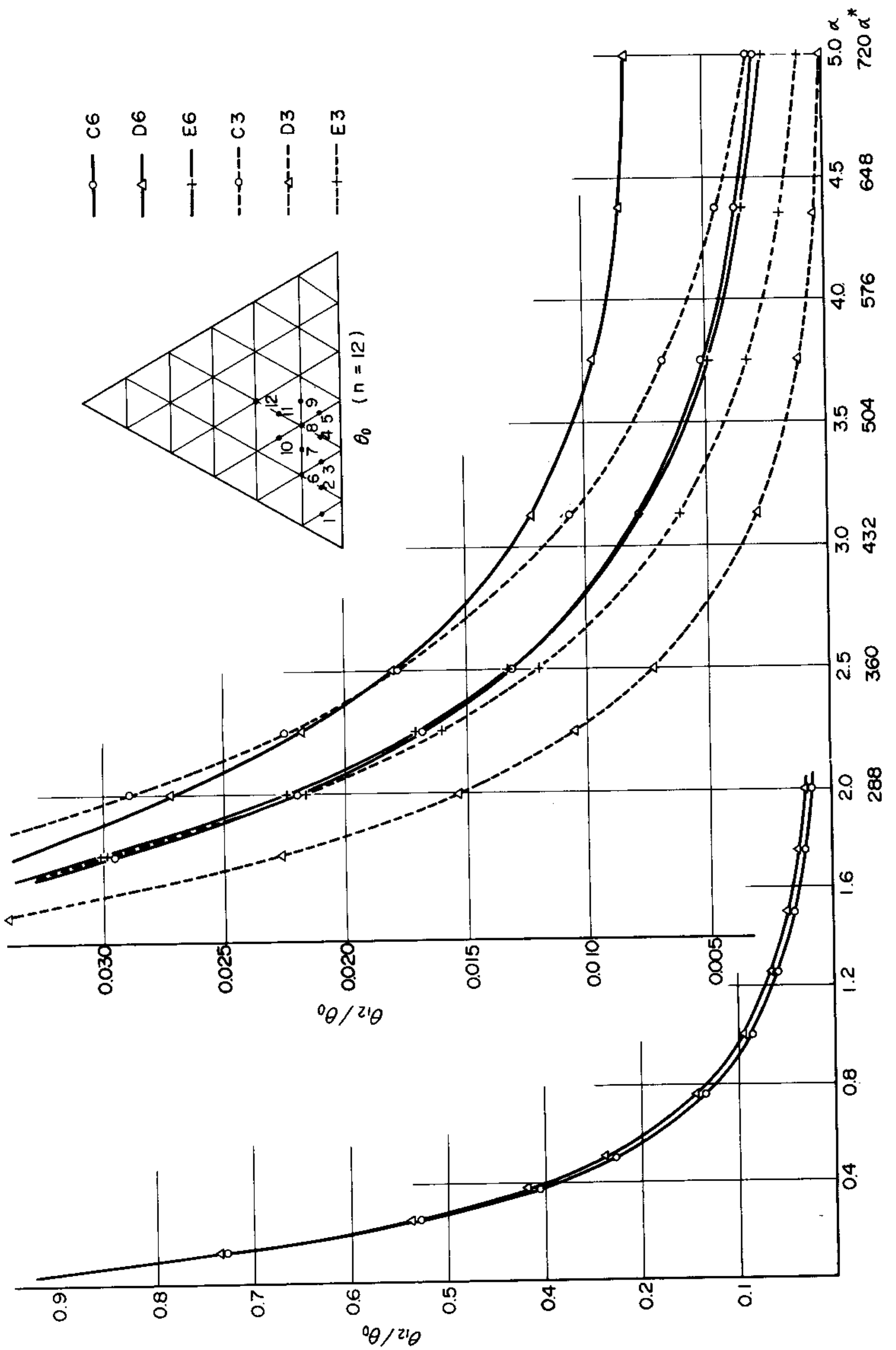

C6
D6
E6
C3
D3
E3
θ_0 (n = 12)
θ_{12}/θ_0
θ_{12}/θ_0
0.4 0.8 1.2 1.6 2.0 2.5 3.0 3.5 4.0 4.5 5.0 α
288 360 432 504 576 648 720 α*
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
0.005 0.010 0.015 0.020 0.025 0.030

Figure 12

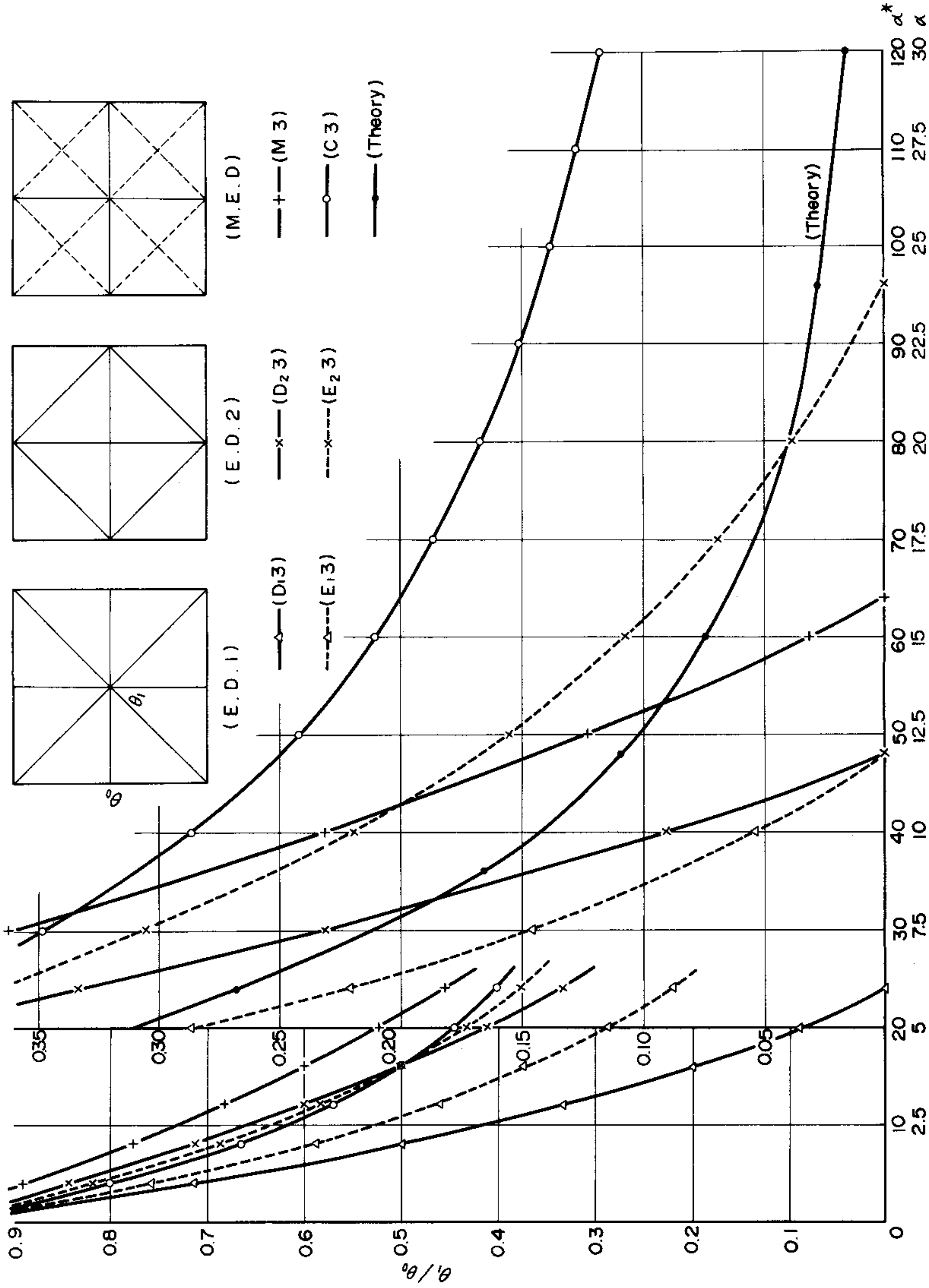

(E.D.1)
(E.D.2)
(M.E.D)
(D₁3)
(E₁3)
(D₂3)
(E₂3)
(M 3)
(C 3)
(Theory)
θ_1/θ_0
α^*
α

Figure 13

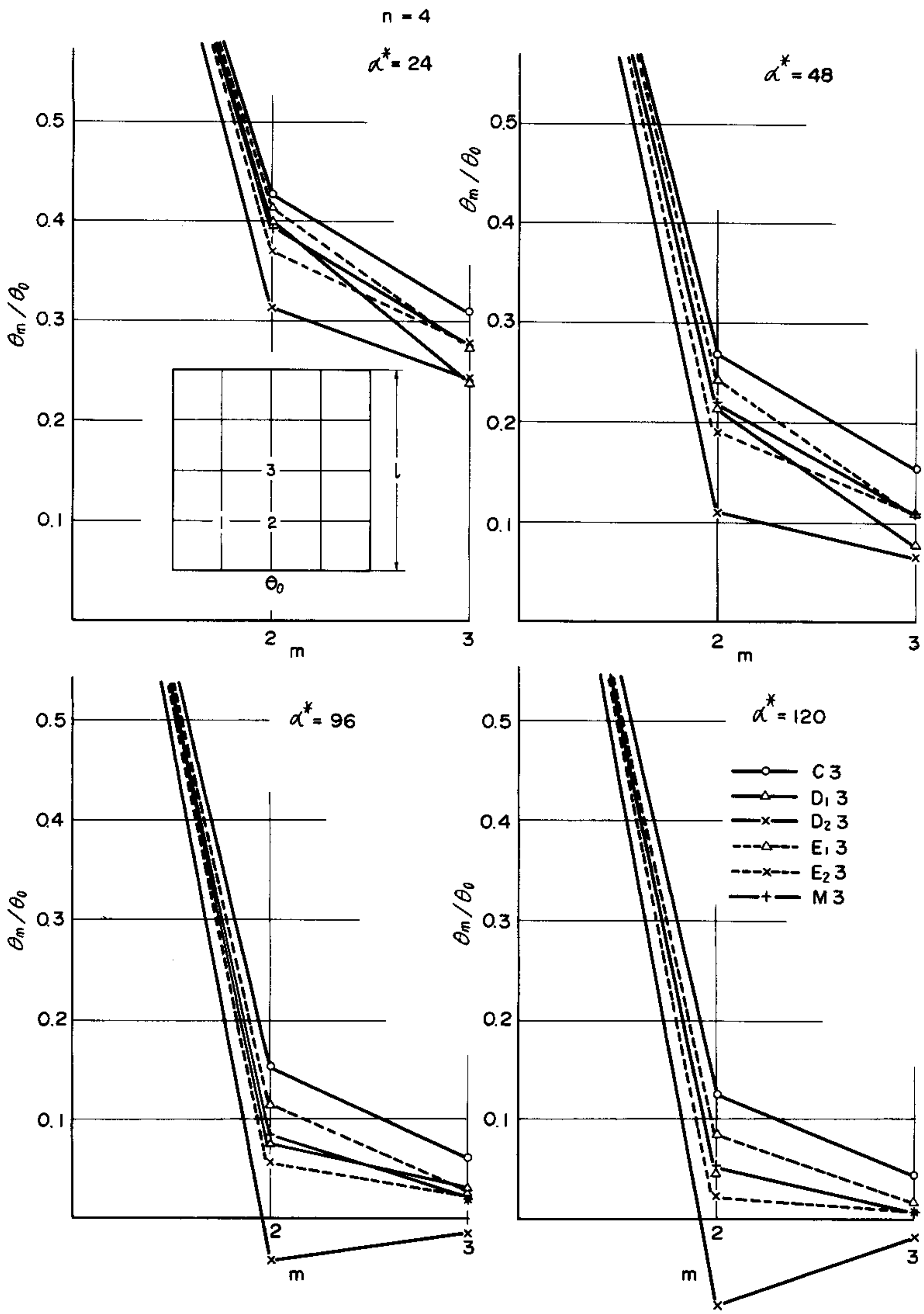

n = 4
α* = 24
α* = 48
α* = 96
α* = 120
θm / θ0
0.5
0.4
0.3
0.2
0.1
2
3
m
θ0
l
C 3
D1 3
D2 3
E1 3
E2 3
M 3

Figure 14

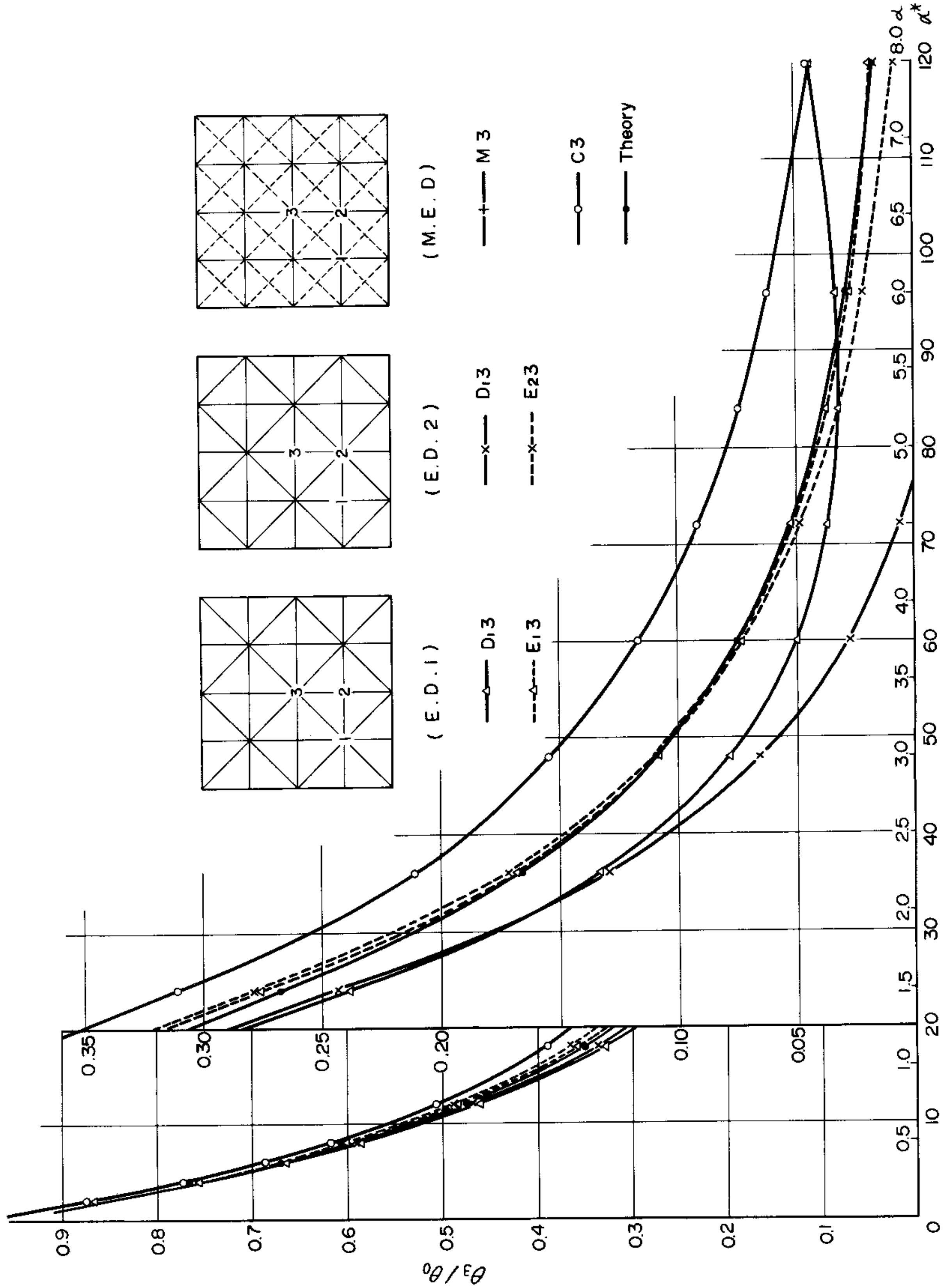

(E.D.1)
D,3
E,3
(E.D.2)
D,3
E2,3
(M.E.D)
M 3
C 3
Theory
θ_3/θ_0
α^*
8.0 α

Figure 15

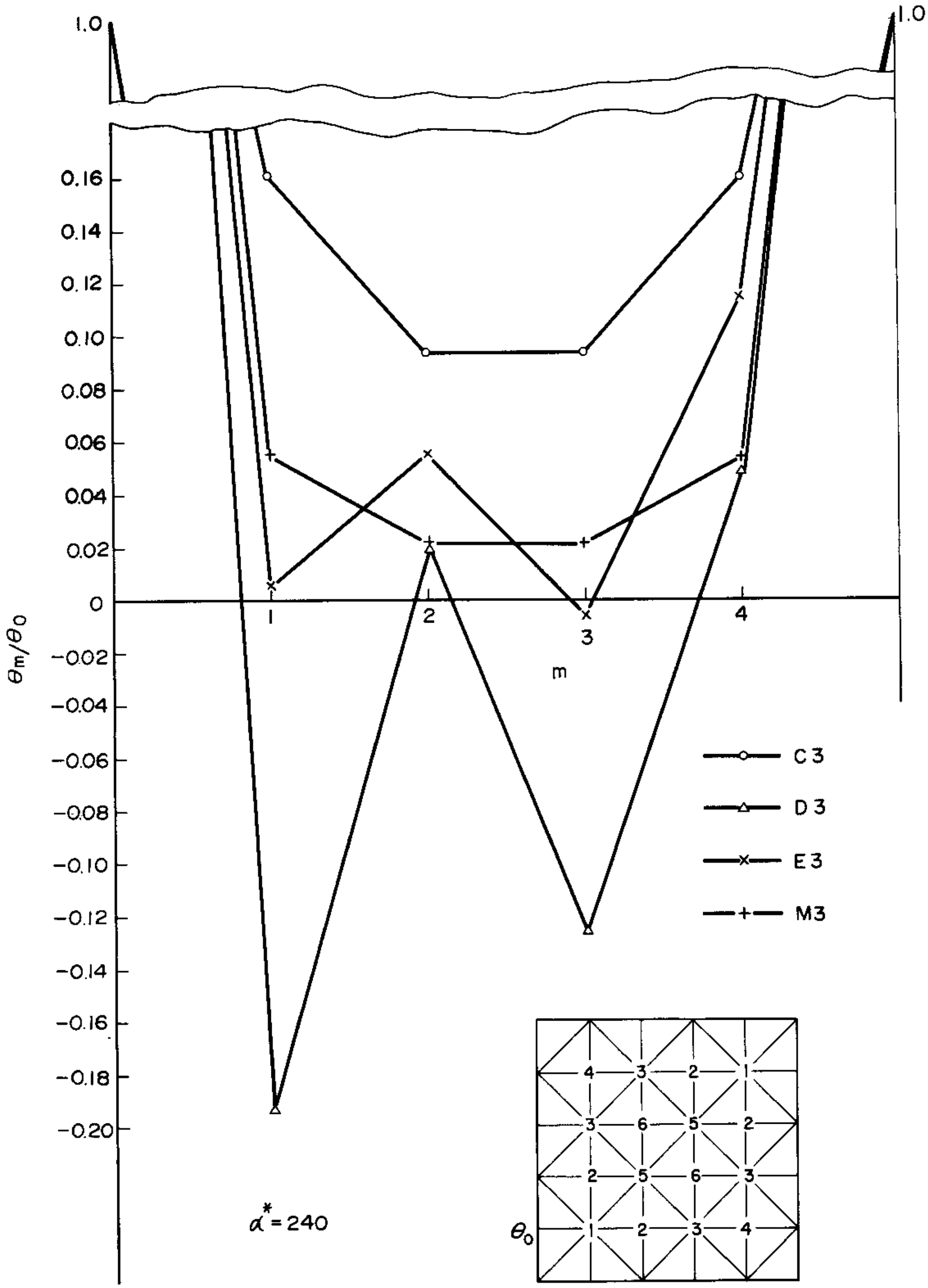

1.0
1.0
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0
−0.02
−0.04
−0.06
−0.08
−0.10
−0.12
−0.14
−0.16
−0.18
−0.20
θm/θ0
1
2
3
4
m
C3
D3
E3
M3
α* = 240
θ0

Figure 16

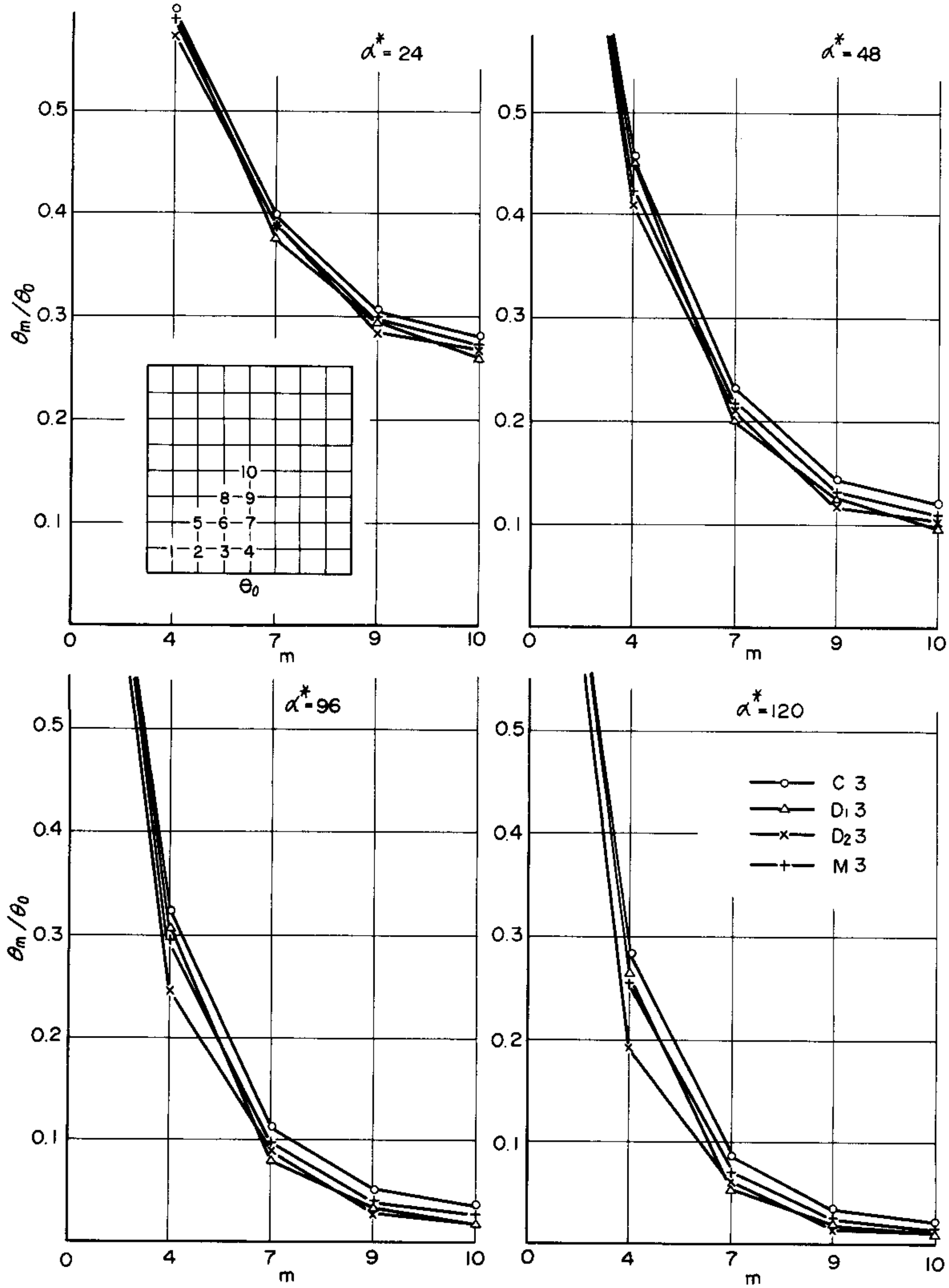
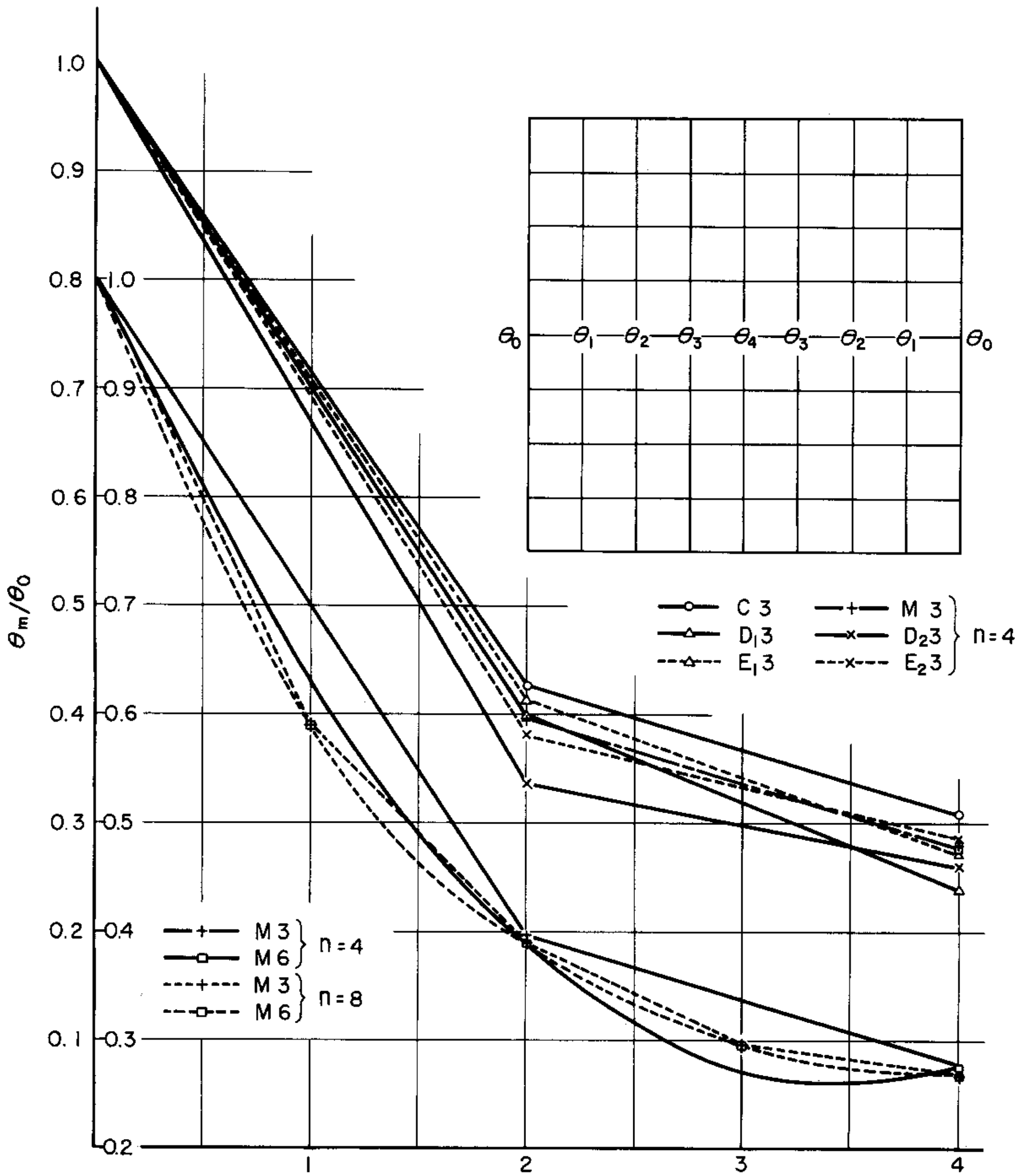

α* = 24
α* = 48
α* = 96
α* = 120
θm / θ0
m
0.5
0.4
0.3
0.2
0.1
0
4
7
9
10
1 2 3 4
5 6 7
8 9
10
θ0
C 3
D1 3
D2 3
M 3

Figure 17



θm/θ0
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
1
2
3
4
θ0 θ1 θ2 θ3 θ4 θ3 θ2 θ1 θ0
C 3
D1 3
E1 3
M 3
D2 3
E2 3
n = 4
M 3
M 6
n = 4
M 3
M 6
n = 8

Figure 18

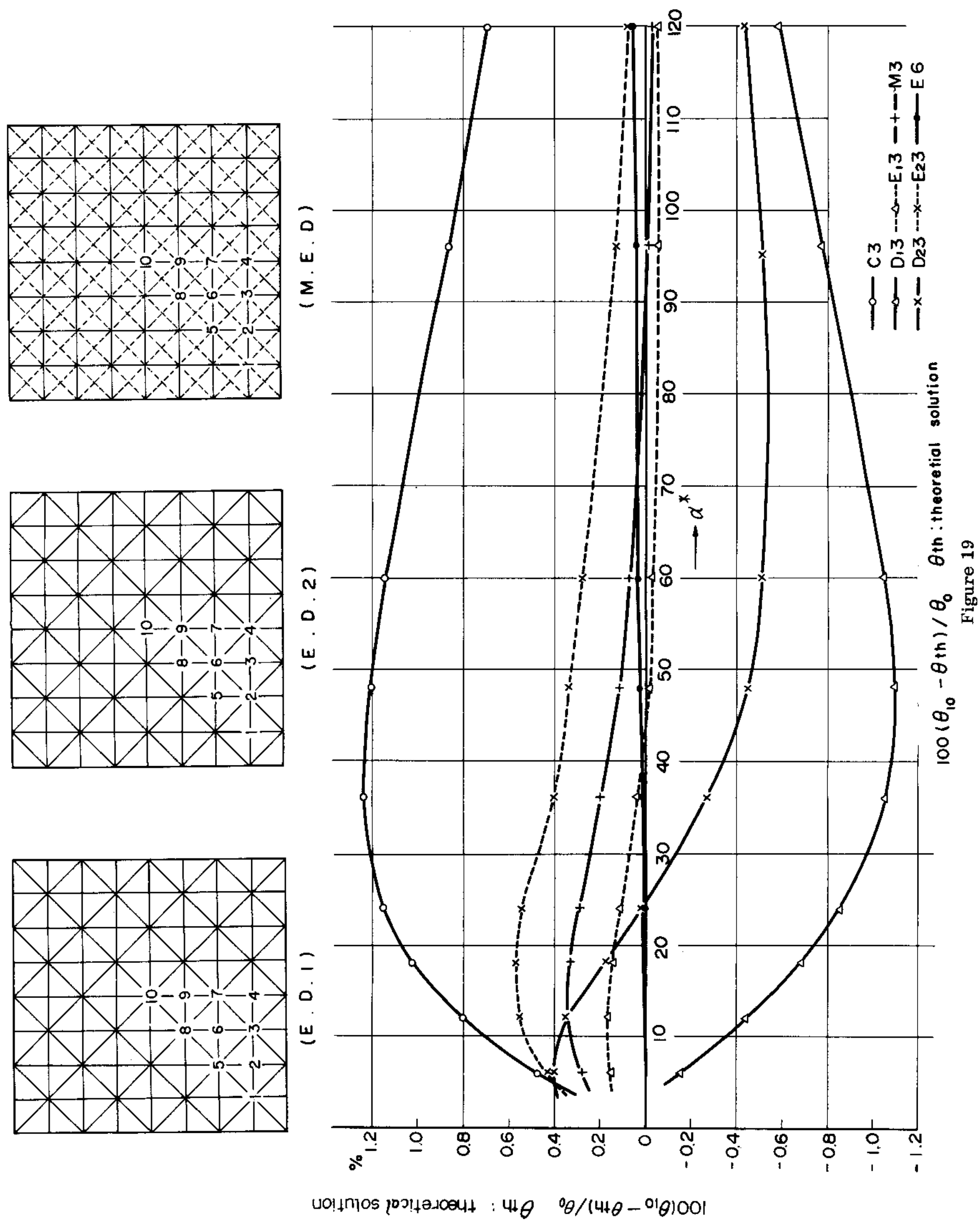

(E . D . 1)
(E . D . 2)
(M . E . D)
C3
D,3
E,3
M3
D23
E23
E6
α^*
%
$100(\theta_{10} - \theta_{th})/\theta_0$ θ_{th} : theoretical solution

Figure 19

We assume a temperature distribution of

$$\theta = \begin{cases} \theta_0 + (\theta_1 - \theta_0)\, x & 0 < x < 1 \\ \theta_0 + (\theta_1 - \theta_0)(2 - x) & 1 < x < 2 \end{cases}$$

The temperature strain energy in the concentrated, distributed, and mean constant systems are

$$V_C = (\theta - \theta_0)^2 + \frac{\alpha}{2}(\theta_0^2 + \theta_1^2)$$

$$V_D = (\theta - \theta_0)^2 + \frac{\alpha}{3}(\theta_0^2 + \theta_0\theta_1 + \theta_1^2)$$

and

$$V_E = (\theta_1 - \theta_0)^2 + \frac{\alpha}{12}(5\theta_0^2 + 2\theta_0\theta_1 + 5\theta_1^2)$$

respectively, from which we have

$$\left(\frac{\theta_1}{\theta_0}\right)_C = \frac{2}{2+\alpha} \doteqdot 1 - \frac{\alpha}{2} + \frac{6}{24}\alpha^2 - \cdots\cdots$$

$$\left(\frac{\theta_1}{\theta_0}\right)_D = \frac{6-\alpha}{6+2\alpha} \doteqdot 1 - \frac{\alpha}{2} + \frac{4}{24}\alpha^2 - \cdots\cdots$$

$$\left(\frac{\theta_1}{\theta_0}\right)_E = \frac{12-\alpha}{12+5\alpha} \doteqdot 1 - \frac{\alpha}{2} + \frac{5}{24}\alpha^2 - \cdots\cdots$$

The theoretical value is

$$\left(\frac{\theta_1}{\theta_0}\right)_T = \cosh\sqrt{\alpha} + \frac{\sinh\sqrt{\alpha}}{\sinh 2\sqrt{\alpha}}\,(1 - \cosh 2\sqrt{\alpha})$$

$$\doteqdot 1 - \frac{\alpha}{2} + \frac{5}{24}\alpha^2 - \cdots$$

therefore the following relation

$$\left(\frac{\theta_1}{\theta_0}\right)_E = \left(\frac{\theta_1}{\theta_0}\right)_T = \frac{1}{2}\left\{\left(\frac{\theta_1}{\theta_0}\right)_C + \left(\frac{\theta_1}{\theta_0}\right)_D\right\}$$

is varified except for extremely large value of α .

SECTION V

TRANSIENT HEAT CONDUCTION

When we calculate the transient heat conduction, it is convenient to divide the temperature stiffness matrix K_{mn} into two parts as follows:

$$K^s_{mn} = \iint \left\{ \lambda h \left(P_{mx} P_{nx} + P_{my} P_{ny} + \alpha_a P_m P_n \right\} dx\, dy + \oint_s \alpha_b h P_m P_n \, dx dy \qquad (26)$$

$$K^t_{mn} = S \iint Ch\, P_m \; P_n \; dx\, dy \qquad (27)$$

where K^s_{mn}, K^t_{mn} are the stationary and transient temperature stiffness matrices, respectively.

Referring to the total domain, we have

$$K^s_{ij} = \sum_r K^s_{mn} = K_{ij} \qquad (28)$$

$$K^t_{ij} = \sum_r K^t_{mn} = S\; A_{ij} \qquad (29)$$

and the equation of heat conduction from the Equation 14

$$\sum \left(K_{ij} + S A_{ij} \right) \theta_j = Q_i \qquad (30)$$

The solution of the above equations are

$$\theta_i = \sum_j \frac{F_{ij}(S)}{F(S)} Q_j(S) \qquad (31)$$

where F(S) is the characteristic equation and $F_{ji}(S)$ are the cofactor about ij element

$$F(S) = \det \left(K_{ij} + S A_{ij} \right) = 0 \qquad (32)$$

and then $\theta_i(t)$ are given by the inverse Laplace transformation as follows:

$$\theta_i(t) = \mathcal{L}^{-1}\left[\sum_j \frac{F_{ji}(S)}{F(S)} Q_j(S) \right]$$

$$= \sum_m \sum_j \frac{F_{ji}(S_m)}{F'(S_m)} Q_j(S\;) \, e^{S_m t} \qquad (33)$$

This calculation, however, is very troublesome in practical application, and hence we used instead the modal analysis like that which is applied in vibration analysis.

Considering the symmetric property of K_{ij} and A_{ij} matrices, following orthogonal relations about normal functions are derived.

$$\sum_{ij} K_{ij}\,\theta_{mi}\,\theta_{nj} = \begin{cases} -S_m a_m & n = m \\ 0 & n \neq m \end{cases} \tag{34}$$

$$\sum_{ij} A_{ij}\,\theta_{mi}\,\theta_{nj} = \begin{cases} a_m & n = m \\ 0 & n \neq m \end{cases} \tag{35}$$

where θ_{mi} are normal function referring to the proper value S_m.

There is a following linear relation between node temperature θ_i and normal mode temperature θ^*_m as

$$\theta_i = \sum_m \theta_{mi}\,\theta^*_m \tag{36}$$

and inversely

$$\theta^*_m = \sum_i \theta^*_{mi}\,\theta_i \tag{37}$$

where

$$\theta^*_{mi} = \sum_j A_{ii}\,\theta_{mj}/a_m \tag{38}$$

Applying the above equations, we have the following normal mode heat conduction equation:

$$a_m\,(S - S_m)\,\theta^*_m(S) = Q^*_m(S) \tag{39}$$

where

$$Q^*_m(S) = \sum_i \theta_{mi}\,Q_i(S) \tag{40}$$

Equation 39 is a single variable equation referring to the m th mode and we can obtain the solution very easily.

The time domain solution of the Equation 39 is

$$\theta^*_m(t) = \mathcal{L}^{-1}\left[\frac{Q^*_m(S)}{a_m(S-S_m)}\right] = \frac{Q^*_m(S_m)}{a_m} e^{S_m t} \tag{41}$$

When the $\theta^*_m(S)$ is expressed by the polynomials in S^{-1} as

$$Q^*_m(S) = a_m\ \theta_{mo} + S^{-1} Q^*_{mo} + S^{-2} \dot{Q}^*_{m1} + \cdots \tag{42}$$

normal mode temperature is given by

$$\theta^*_m(t) = \theta^*_{mo}\ \zeta_{mo}(t) + Q^*_{mo}\zeta_{m1}(t) + Q^*_{m1}\ \zeta_{m2}(t) + \cdots \tag{43}$$

where

$$\zeta_{mo}(t) = \mathcal{L}^{-1}\left[\frac{1}{a_m(S-S_m)}\right] = \frac{1}{a_m} e^{S_m t} \tag{44}$$

$$\zeta_{m1}(t) = \mathcal{L}^{-1}\left[\frac{1}{a_m S(S-S_m)}\right] = \frac{1}{a_m S_m}(-1 + e^{S_m t}) \tag{45}$$

$$\zeta_{m2}(t) = \mathcal{L}^{-1}\left[\frac{1}{a_m S^2(S-S_m)}\right] = \frac{1}{a_m S_m^2}\left\{-(1+S_m t) + e^{S_m t}\right\} \tag{46}$$

$$\zeta_{m3}(t) = \mathcal{L}^{-1}\left[\frac{1}{a_m S^3(S-S_m)}\right] = \frac{2}{a_m S_m^3}\left\{-\left(1+S_m t + \frac{S_m^2 t^2}{2}\right) + e^{S_m t}\right. \tag{47}$$

$$\begin{aligned} Q^*_m(t) &= \mathcal{L}^{-1}\left[Q^*_m(S)\right] \\ &= Q_{mo}\,\delta(t) + \left(Q_{m1} + Q_{m2}\ t + Q_{m3}\frac{t^2}{2} + \cdots\right) u(t) \end{aligned} \tag{48}$$

where δ (t) and u(t) are the delta and step function, respectively.

When the normal temperature force $Q_m(t)$ are given by the polygonal line of time interval to as shown in Figure 20.

$$Q^*_m(t) = \left\{ \begin{array}{l} Q^*_{mo} + (Q^*_{m1} - Q^*_{mo})\tau \\ Q^*_{m1} + (Q^*_{m2} - Q^*_{m1})(\tau - 1) \\ Q^*_{m2} + (Q^*_{m3} - Q^*_{m2})(\tau - 2) \\ \cdots\cdots \end{array} \right. \tag{49}$$

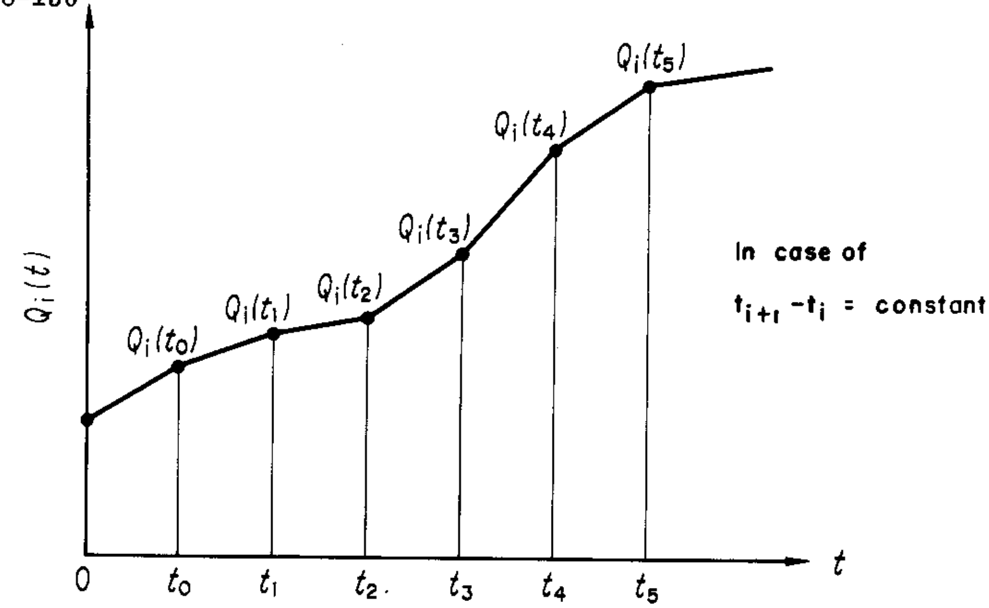


Figure 20

where $\tau = t/to$ and

$$\sum_i \theta_{mi} \sum_j A_{ij}\, \theta_{jo} = a_m \sum_i \theta^*_{mi}\theta_{io} = a_m \theta^*_{mo}$$

The solution from t = 0 to t_o is

$$\theta^*_m(t) = \theta^*_{mo}\, e^{S_m t} + Q^*_{mo}\, \zeta_{m1}(t) + \frac{(Q^*_{m1} - Q^*_{mo})}{t_o}\, \zeta_{m2}(t) \qquad (50)$$

From the above equation, we have the following cyclic equation

$$\begin{aligned} \theta^*_{m1} &= \theta^*_{mo}\, \zeta_{mo} + Q^*_{mo}\, \zeta_{m1} + (Q^*_{m1} - Q^*_{mo})\zeta_{m2} && 0 < t < t_o \\ \theta^*_{m2} &= \theta^*_{m1}\, \zeta_{mo} + (Q^*_{m1}\, \zeta_{m1} + (Q^*_{m2} - Q^*_{m1})\, \zeta_{m2} && t_o < t < t_1 \\ \theta^*_{m3} &= \theta^*_{m2}\, \zeta_{mo} + (Q^*_{m2}\, \zeta_{m1} + (Q^*_{m3} - Q^*_{m2})\zeta_{m2} && t_1 < t < t_2 \end{aligned} \qquad (51)$$

where

$$\zeta_{mo} = e^{S_m t_o}, \qquad \zeta_{m1} = (-1 + e^{S_m t_o}) / a_m S_m,$$

$$\zeta_{m2} = \left\{ -(1 + S_m t_o) + e^{S_m t_o} \right\} / a_m S_m^2 t_o$$

The temperature distribution due to a moving heat source such as in the case of welding is calculated by replacing it with the time dependent node heat source as follows.

Let us assume that a point heat source on the side line 1-2 of the element 1-2-3 shown in Figure 21 and 22 moves with a constant velocity V, the node heat sources q_1 and q_2 are given by the following temperature virtual work

$$\delta' W = \int q(s_3 t)\, \delta'\theta\, ds_3 = \int q \sum_{m=1}^{3} \zeta_m\, \delta\theta_m\, ds_3 = \sum_{m=1}^{3} q_m\, \delta\theta_m \qquad (52)$$

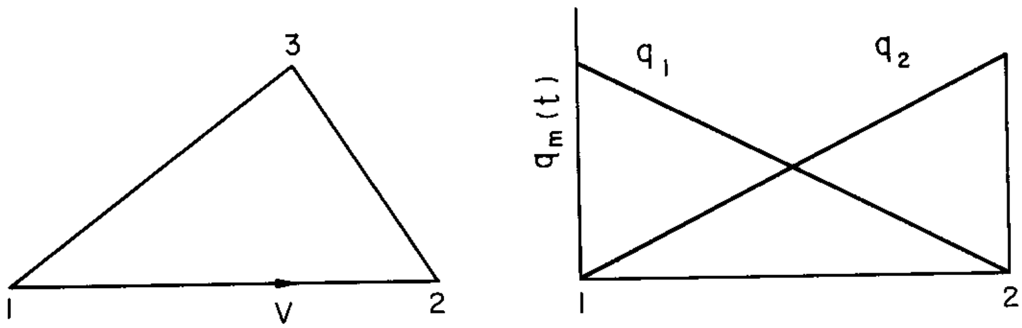


Figure 21

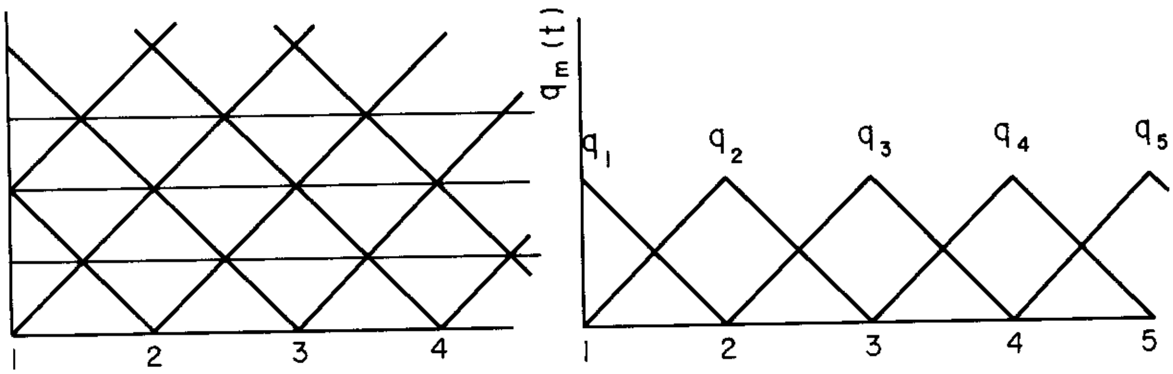


Figure 22

where

$$q(t) = q\,\delta(\zeta_2 - \tau) \qquad , \qquad \tau = Vt/s_3$$

$$q_1 = \int q(t)\zeta_1\, ds_3 = q(1-\tau)$$

$$q_2 = \int q(t)\,\zeta_2\, ds_3 = q\tau \tag{53}$$

Referring to the triangular element with six nodes, we have the following time dependent node heat sources $q_1 = (1-\tau)(1-2\tau)$, $q_2 = \tau(2\tau - 1)$, $q_6 = 4\tau(1-\tau)$

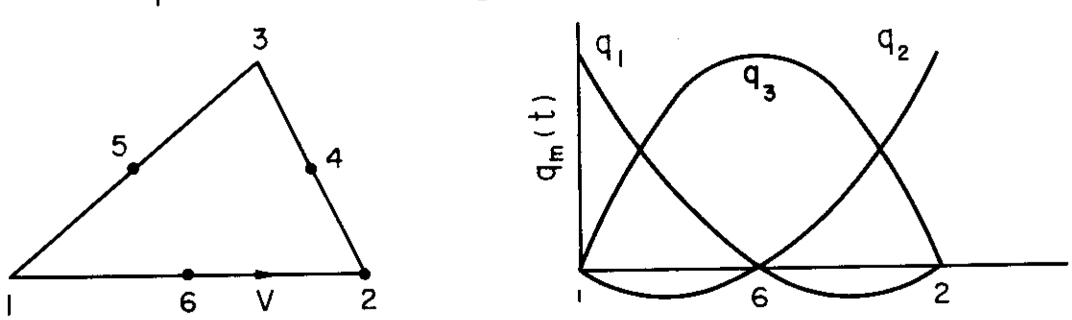


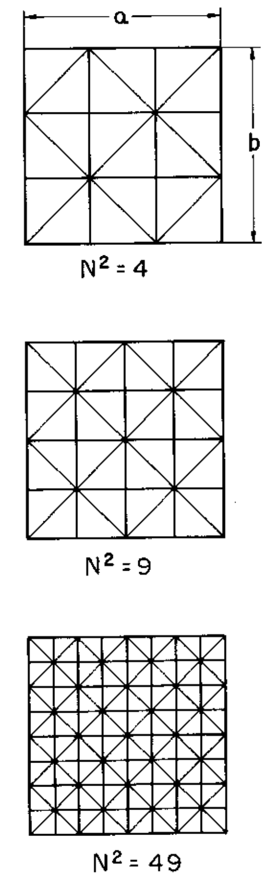
Figure 23

The accuracy of results by means of the above method may be independent of the time interval, it is, however, needed to obtain exact eigenvalues and eigenvectors and then to conform the sum of the eigenvectors to the exact temperature distribution as accurately as possible.

Table 1 shows the eigenvalues of the typical characteristic equations calculated by the various element system in the same manner such as analysis of stationary temperature distribution. The result of type C3 is equal to one of the difference equation. The result of type C3 gives lower value and type D3 higher than theoretical one.

Then the higher degree eigenvectors are required to obtain the accurate temperature distribution of the plate heated on the spot region, the test calculation was, therefore, done as shown in Figure 24.

degree \ N^2	(D 3)			(C 3)			(E 3)			Theory
	4	9	49	4	9	49	4	9	49	
1	2.774	2.540	2.305	1.932	2.069	2.189	2.280	2.281	2.245	2.254
2	6.420	5.791	4.649	2.836	3.500	4.088	3.935	4.368	4.350	4.335
3	12.734	8.553	7.328	6.309	4.708	6.513	8.432	6.781	6.859	6.936
4	20.077	10.843	8.916	7.209	4.882	6.840	10.567	6.309	7.773	7.803
5		12.041	10.099		5.417	8.270		7.221	9.113	9.012
6		24.139	15.379		11.346	10.976		15.752	12.498	12.485
7		27.892	15.683		11.518	10.104		15.694	12.533	12.658
8		30.974	16.265		12.780	12.441		18.281	14.137	14.739
9		46.210	19.878		14.179	13.834		21.666	16.797	16.819
10			23.186			13.300			17.144	17.340
11			25.709			14.007			17.449	18.901
12			26.009			15.729			20.331	20.288
13			30.164			16.794			22.315	23.582
14			34.750			16.757				
15			34.322			16.853				
16			39.114			19.383				
17										



$a : b = 2 : 3$

$\alpha = 1.2$, $\lambda h = 1.0$

$Ch = 10.0$, $\alpha_a = 0.0$

fixed boundary

Comparison of result of characteristic value by F.E.M. with that by theory

Table 1

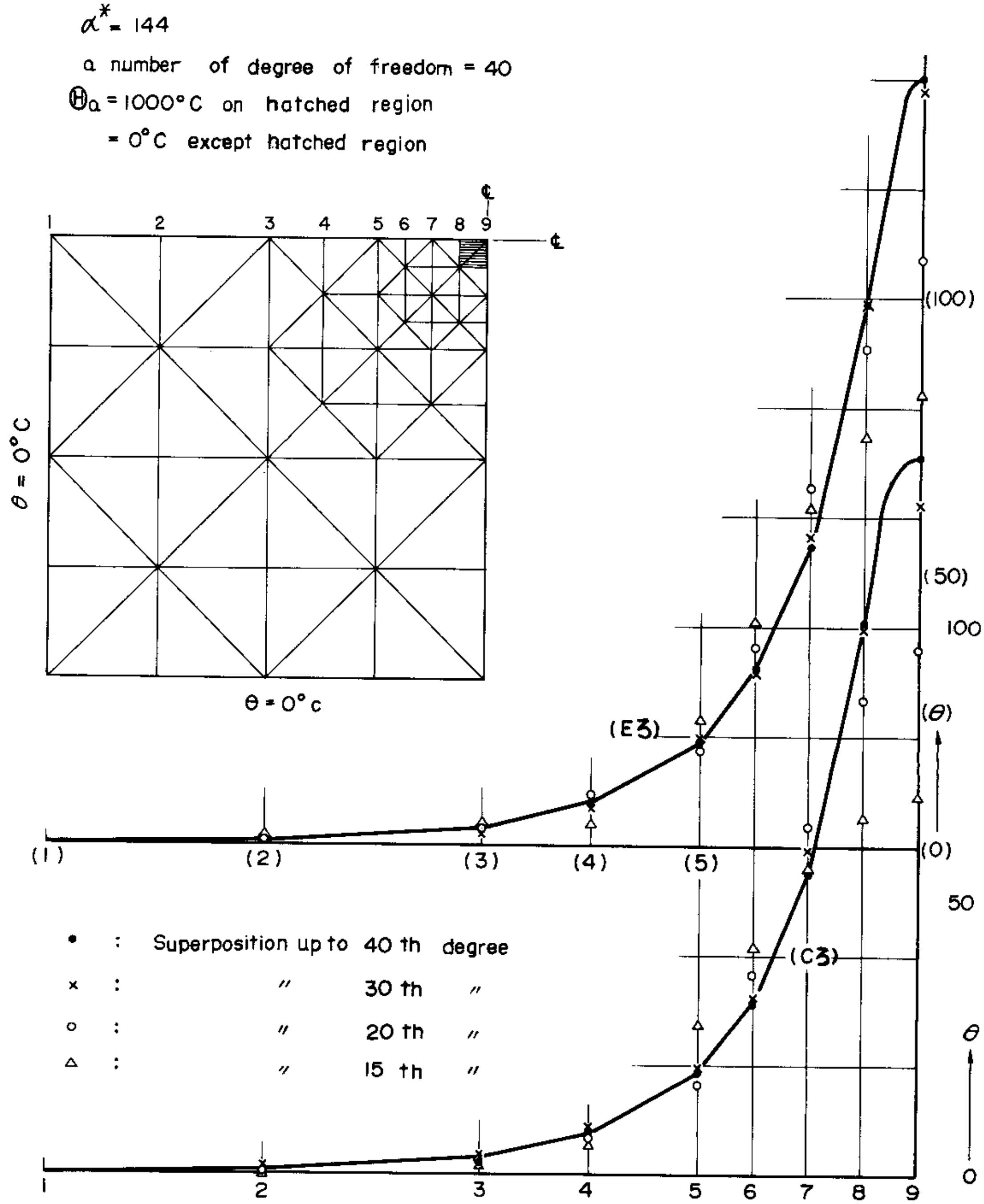

α* = 144
a number of degree of freedom = 40
Θa = 1000°C on hatched region
= 0°C except hatched region
θ = 0°C
θ = 0°c
℄
1 2 3 4 5 6 7 8 9
(100)
(50)
100
(θ)
(0)
50
(E3)
(C3)
θ
0
(1) (2) (3) (4) (5)
• : Superposition up to 40 th degree
× : 〃 30 th 〃
○ : 〃 20 th 〃
△ : 〃 15 th 〃

Figure 24

SECTION VI

THERMAL STRESS ANALYSIS

When there is a temperature distribution in a plate, the strain energy V is given by

$$V = \frac{1}{2} \iint a \left\{ \left(\frac{\partial u}{\partial x} - \alpha\theta \right)^2 + 2\nu \left(\frac{\partial u}{\partial x} - \alpha\theta \right) \left(\frac{\partial v}{\partial y} - \alpha\theta \right) + \left(\frac{\partial v}{\partial y} - \alpha\theta \right)^2 + \lambda \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right\} dx\,dy \tag{51}$$

where

$$a = Eh/(1-\nu^2), \qquad \lambda = (1-\nu)/2$$

The inplane stress distribution is solved by the finite element approach based on the following variational equation.

$$\begin{aligned} \delta V = \oint a \Big[& \Big\{ \left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} - \overline{1+\nu}\ \alpha\theta \right) \frac{dy}{ds} - \lambda \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \frac{dx}{ds} \Big\} \delta u \\ & + \Big\{ \lambda \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \frac{dy}{ds} - \left(\nu \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \overline{1+\nu}\ \alpha\theta \right) \frac{dx}{ds} \Big\} \delta v \Big] ds \\ & - \iint \Big[\Big\{ \frac{\partial}{\partial x} a \left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} - \overline{1+\nu}\ \alpha\theta \right) + \frac{\partial}{\partial y} \lambda a \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \Big\} \delta u \\ & + \Big\{ \frac{\partial}{\partial x} \lambda a \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} a \left(\nu \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \overline{1+\nu}\ \alpha\theta \right) \Big\} \delta v \Big] dx\,dy = 0 \end{aligned} \tag{52}$$

From the above Equation 52, we have the following equilibrium equation

$$\left. \begin{aligned} \frac{\partial X_{xx}}{\partial x} + \frac{\partial X_{xy}}{\partial y} &= (1+\nu)\, a\alpha\theta \\ \frac{\partial X_{yx}}{\partial x} + \frac{\partial X_{yy}}{\partial y} &= (1+\nu)\, a\alpha\theta \end{aligned} \right\} \tag{53}$$

where

$$X_{xx} = a \left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} \right), \qquad X_{xy} = \lambda a \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$X_{yy} = a \left(\nu \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

It is assumed that the temperature distribution in an element is expressed by

$$\theta = \sum_{m=1}^{3} \zeta_m \theta_m \tag{54}$$

and the displacement components u and v are

$$u = \sum_{m=1}^{6} P_m u_m \qquad v = \sum_{m=1}^{6} P_m v_m \tag{55}$$

where P_ms are the shape function of triangular elements with six nodes.

Substituting the Equations 54 and 55 into the Equation 51, we have the following biquadratic form

$$\begin{aligned} V = \frac{1}{2} \iint a \Big\{ &\Big(\sum_{m=1}^{6} P_{mx} \xi_m - \alpha \sum_{m=1}^{3} \zeta_m \theta_m \Big) \\ &+ 2\nu \Big(\sum_{m=1}^{6} P_{mx} \xi_m - \alpha \sum_{m=1}^{3} \zeta_m \theta_m \Big)\Big(\sum_{m=7} P_{my} \xi_m - \alpha \sum_{m=1}^{3} \zeta_m \theta_m \Big) \\ &+ \Big(\sum_{m=7}^{12} P_{my} \xi_m - \sum_{m=1}^{3} \zeta_m \theta_m \Big)^2 + \lambda \Big(\sum_{m=1}^{6} P_{my} \xi_m + \sum_{m=7}^{12} P_{mx} \xi_m \Big)^2 \Big\} dx\,dy \\ = \frac{1}{2} \Big(&\sum_{m,n} K_{mn} \xi_m \xi_n - 2 \sum_m f_m \xi_m + q \Big) \end{aligned} \tag{56}$$

where

$$\xi_m = \begin{cases} u_m & m = 1 \sim 6 \\ v_{m-6} & m = 7 \sim 12 \end{cases}$$

$$P_m = P_{m-6} \qquad m = 7 \sim 12$$

In the above equation, $\mathbf{K}_{mn}$ is the stiffness matrix and is expressed as

1. $m = 1 \sim 6,\ n = 1 \sim 6$

$$\mathbf{K}_{mn} = \iint a \left(P_{mx} P_{nx} + \lambda P_{my} P_{ny} \right) dx\,dy$$

2. $m = 1 \sim 6,\ n = 7 \sim 12$

$$\mathbf{K}_{mn} = \iint a \left(\nu P_{mx} P_{ny} + \lambda P_{my} P_{nx} \right) dx\,dy$$

3. $m = 7 \sim 12,\ n = 7 \sim 12$

$$\mathbf{K}_{mn} = \iint a \left(P_{my} P_{ny} + \lambda P_{mx} P_{nx} \right) dx\,dy$$

and f_m is given as follows: Let

$$f_m = \sum_{n=1}^{3} f_{mn} \theta_n$$

we have then

1. $m = 1 \sim 6$

$$f_{mn} = (1+\nu)\iint a\,\alpha\,P_{mx}\,\zeta_n\,dx\,dy$$

2. $m = 7 \sim 12$

$$f_{mn} = (1+\nu)\iint a\,\alpha\,P_{my}\,\zeta_n\,dx\,dy$$

The above area integrals are easily evaluated by using the area integral formula or table shown in the Equation 17.

SECTION VII

CONCLUSION

From the results of this study outlined in this paper, we reached the following conclusion:

1. Accuracy of the calculations of the transient heat conduction and stationary temperature distribution by means of the finite element method based on the triangular elements with three or six nodes, is improved by assuming the mean system of concentrated and distributed constant.

2. Calculations based on the system of nearly regular triangular elements, all the nodes of which have the valency of six, are not disturbed by the method of element division.

3. On the contrary, calculations based on the nearly equilateral right triangular elements are disturbed by the coexistence of nodes, which have the valency of four or eight. This difficulty is remedied by the application of the modified element system, which is described in the previous paragraph.

4. Laplace transformation and nodal analysis method are effectively used for the analysis of the transient heat conduction problem.

5. Thermal Stress

Thermal stress due to the temperature distribution is calculated by means of the finite element method using the triangular elements with three nodes or six nodes. The strain or stress distribution in an element is constant when it is calculated by the elements with three nodes, whereas is linear by the elements with six nodes. Generally the latter is more accurate than the former when there is shear deformation.

Acknowledgment

We should like to express our appreciation to Prof. K. Washizu and Associate Prof. T. Kawai of University of Tokyo for their valuable advice and suggestions and also to Miss M. Ohishi for her cooperation in programming.

SECTION VIII

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APPENDIX

NUMERICAL CALCULATION OF LARGE MATRIX

One of the most important problems of the numerical calculation presented in the engineering field is: how effectively to solve large scale simultaneous linear equations? It is because a partial differential equation which appears in the engineering problems may be substituted approximately by a difference equation with a finite number of unknowns, which is described by simultaneous linear equations or characteristic equations as a general form. The finite element method is also a technique for reducing a continuous system to a discontinuous system.

For the reasons which have been described above, it would be easy to obtain a numerical solution of such problems with sufficient accuracy for practical use if the large scale simultaneous linear equations could be solved in economical computation time. The amount of the computation time for it may become, however, far greater than the upper limitation of the computational capability of today's supercomputers. Therefore, this approach never will be useful without the development of an efficient numerical calculation method.

Now, the number of operations of addition and multiplication which is needed to solve the linear equation with n unknowns may be described as a function of n as the following:

$$V = \frac{1}{3} n^3 + \frac{3}{2} n^2 \tag{57}$$

This is for the case of using the Gaussian elimination method, which gives the minimum amount of computation except for the iterative methods.

The required location is

$$S = n^2 + n \tag{58}$$

In the above two expressions, it is clear that the higher order terms of n should be preferably reduced as far as possible to make use of computer most effectively. It seems, however, to be almost impossible to realize it by a mathematical approach. But it seems possible by a physical approach, namely, to utilize the fact that the state of equilibrium at an arbitrary point is determined only by the neighboring boundary condition around the point.

Considering the form of matrix from this view point, the nonzero elements in the matrix should be arranged in the band form itself as shown in Figure 25. The matrix obtained would be generally symmetrical.

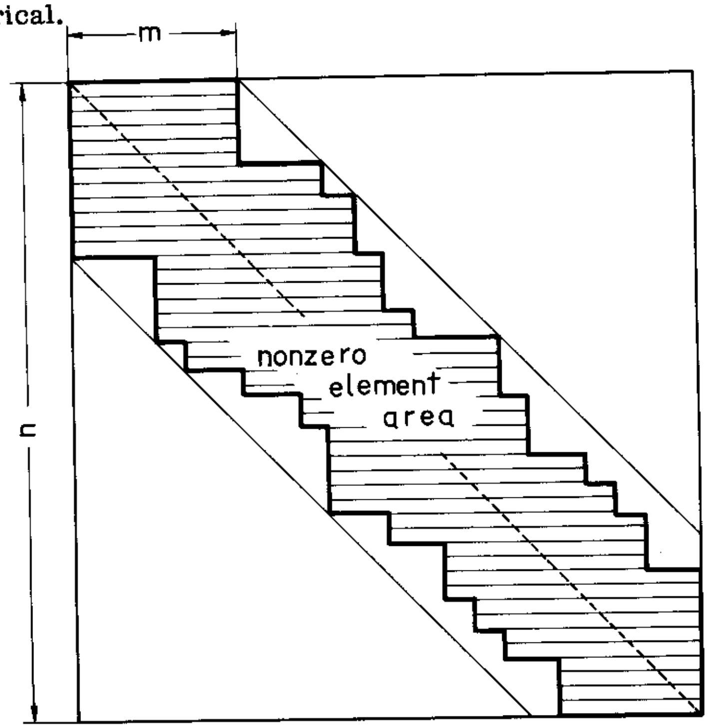

Figure 25

When the width of the band is m, the number of the operations to solve the linear equation with the band form matrix by the Gaussian method may become

$$V_B \doteqdot m^2 n - \frac{2}{3} m^3 \tag{59}$$

It should be noted that the Expression 59 is linear in n. Considering that m/n is practically about 0.1 at most, the ratio Expression 59 to 57 is

$$\frac{V_B}{V} \doteqdot 3\left(\frac{m}{n}\right)^2 - 2\left(\frac{m}{n}\right)^3 \doteqdot 0.03 \tag{60}$$

and the required location is

$$S_B = (m+1)\,n \tag{61}$$

The ratio Equation 61 to 58 is

$$\frac{S_B}{S} = \frac{m+1}{n+1} \doteqdot 0.1 \tag{62}$$

From the above results, it is seen that there is great merit in using this property of the matrix for the practical calculation.

And further the process of the reduction of matrix may be usefully separated into two parts. The first step is the conversion of the coefficient matrix to a triangular matrix, namely, the forward elimination. The number of operations required in this step is about

$$V_f = m^2 n - \frac{2}{3} m^3 \tag{63}$$

The second step is the conversion of the constant vector as the first step conversion. Then the triangular matrix is converted to the diagonal matrix and the conversion of the constant vector, namely, the backward elimination. The number of operations in this step is about

$$V_b = 2mn - m^2 \tag{64}$$

V_b shows the number of additional operations in the case of changing constant vector. The ratio V_b to $V_f + V_b$ is about

$$\frac{V_b}{V_f + V_b} \doteqdot \frac{2}{m} \tag{65}$$

In the next place, we will describe the calculation method of the solution of characteristic equations.

There are a number of approaches for solving characteristic equations. No effective method for a large matrix, however, has yet been presented.

The following method obtained after trying various approaches is fit for solving large scale characteristic equations. The characteristic equation is generally represented in matrix form by

$$\sum_j (\mathbf{K}_{ij} + s\mathbf{A}_{ij})\boldsymbol{\theta}_j = 0 \tag{66}$$

where, in the heat conduction problem, $\mathbf{K}_{ij}$ is defined as the heat conductivity matrix, $\mathbf{A}_{ij}$ the heat capacity matrix, and $\boldsymbol{\theta}_i$ the vector of temperature distribution.

Supposing $\boldsymbol{\theta}_{ni}$ the eigenvector corresponding to nth degree eigenvalue Sn of the solution of Equation 66

$$\sum_j \mathbf{K}_{ij} \boldsymbol{\theta}_{nj} + s_n \sum_j \mathbf{A}_{ij} \boldsymbol{\theta}_{nj} = 0 \tag{67}$$

On the other hand, Equation 67 is transposed by the symmetricity of the matrix as follows:

$$\sum_i \mathbf{K}_{ij} \boldsymbol{\theta}_{ni} + s_n \sum_i \mathbf{A}_{ij} \boldsymbol{\theta}_{ni} = 0 \tag{68}$$

From Equations 67 and 68

$$\sum_{ij} K_{ij}\,\theta_{mi}\,\theta_{nj} + S_n \sum_{ij} A_{ij}\,\theta_{mi}\,\theta_{nj}$$
$$= -S_m \sum_{ij} A_{ij}\,\theta_{mi}\,\theta_{nj} + S_n \sum_{ij} A_{ij}\,\theta_{mi}\,\theta_{nj}$$
$$= (S_n - S_m) \sum_{ij} A_{ij}\,\theta_{mi}\,\theta_{nj}$$
$$= 0 \tag{69}$$

when $S_m \neq S_n$, Equation 69 becomes

$$\sum_{ij} A_{ij}\,\theta_{mi}\,\theta_{nj} = 0 \tag{70}$$

From this result, an arbitrary vector θ_i may be written as the following:

$$\theta_i = \alpha_1 \theta_{1i} + \alpha_2 \theta_{2i} + \alpha_3 \theta_{3i} + \cdots \tag{71}$$

where

$$\alpha_m = \frac{\sum A_{ij}\,\theta_{mi}\,\theta_j}{\sum A_{ij}\,\theta_{mi}\,\theta_{mi}} \tag{72}$$

The above expression may be obtained by using Equation 70, that is

$$\sum_{ij} A_{ij}\,\theta_{mi}\,\theta_j = \sum_{ij} A_{ij}\,\theta_{mi} \sum_{k=1} \alpha_k\,\theta_{kj}$$
$$= \alpha_m \sum_{ij} A_{ij}\,\theta_{mi}\,\theta_{mj} \tag{73}$$

Now, suppose an arbitrary initial vector $\theta^{(0)}$ described as follows:

$$\theta_i^{(0)} = \sum_{m=1} \alpha_m\,\theta_{mi} \tag{74}$$

Substituting Equation 74 into Equation 66,

$$\sum_j A_{ij}\,\theta_i^{(0)} = \sum_j A_{ij} \sum_{m=1} \frac{1}{S_m}\alpha_m\,\theta_{mj}$$
$$= \sum_j K_{ij}\,\theta_j^{(1)} \tag{75}$$

Similarly,

$$\sum_j A_{ij}\,\theta_j^{(1)} = \sum_j A_{ij} \sum_{m=1} \left(\frac{1}{S_m}\right)^2 \alpha_m\,\theta_{mj}$$
$$= \sum_j K_{ij}\,\theta_j^{(2)} \tag{76}$$

After k times repetition, we obtain

$$\theta_i^{(k)} = \sum_{m=1} \left(\frac{1}{S_m}\right)^k \alpha_m \theta_{mi} \tag{77}$$

and then

$$\frac{\theta_i^{(k)}}{\theta_i^{(k+1)}} = S_1 \frac{\alpha_1 \theta_{1i} - \sum_{m=2} \left(\frac{S_1}{S_m}\right)^k \alpha_m \theta_{mi}}{\alpha_1 \theta_{1i} - \sum_{m=2} \left(\frac{S_1}{S_m}\right)^{k+1} \alpha_m \theta_{mi}} \tag{78}$$

Supposing $|S_1| < |S_2| < |S_3| < \cdots\cdots$, the minimum eigenvalue may be obtained after a number of times repetition, for

$$\lim_{k \to \infty} \cdot \frac{\theta_i^{(k)}}{\theta_i^{(k+1)}} = S_1 \tag{79}$$

And rth degree eigenvalue and vector may also be obtained in the same manner for the first eigenvalue by substituting arbitrary initial vector $\theta_i^{(o)}$ into

$$\theta_i^{(o)} - \sum_{m=1}^{r-1} \alpha_{mi} \theta_{mi} \tag{80}$$

Our method here is a modified form of the well-known iterative approach. This method, however, is fundamentally good in several respects, that is, the inversion of a large matrix is not needed, both the eigenvalue and eigenvector are successively obtained from the first degree to the higher ones and the solution may easily be obtained without the expertness in numerical calculation, and consequently, it is easy, generally to make a general purpose program.

Furthermore, the procedure of computation for the solution of the characteristic equation is approximately the same as that of the linear equation and it may be effectively used to do a program development.

Practically, the above-mentioned fact was profitably used in making the multipurpose application programs for the dynamical and critical load analysis of framed structure, named FRAME, which we developed in 1964.